Stability analysis and control of discrete-time systems with delay

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Stability analysis and control
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Stability analysis and control
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Preliminaries

Summary

The research presented in this thesis considers the stability analysis and control of discrete-time systems with delay. The interest in this class of systems has been motivated traditionally by sampled-data systems in which a process is sampled periodically and then controlled via a computer. This setting leads to relatively cheap control solutions, but requires the discretization of signals which typically introduces time delays. Therefore, controller design for sampled-data systems is often based on a model consisting of a discrete-time system with delay. More recently the interest in discrete-time systems with delay has been motivated by networked control systems in which the connection between the process and the controller is made through a shared communication network. This communication network increases the flexibility of the control architecture but also introduces effects such as packet dropouts, uncertain time-varying delays and timing jitter. To take those effects into account, typically a discrete-time system with delay is formulated that represents the process together with the communication network, this model is then used for controller design

While most researchers that work on sampled-data and networked control systems make use of discrete-time systems with delay as a modeling class, they merely use these models as a tool to analyse the properties of their original control problem. Unfortunately, a relatively small amount of research on discrete-time systems with delay addresses fundamental questions such as: What trade-off between computational complexity and conceptual generality or potential control performance is provided by the different stability analysis methods that underlie existing results? Are there other stability analysis methods possible that provide a better trade-off between these properties? In this thesis we try to address these and other related questions. Motivated by the fact that almost every system in practice is subject to constraints and Lyapunov theory is one of the few methods that can be easily adapted to deal with constraints, all results in this thesis are based on Lyapunov theory.

In *Chapter 2* we introduce delay difference inclusions (DDIs) as a modeling class for systems with delay and discuss their generality and advantages. Furthermore, the two standard stability analysis results for DDIs that make use of Lyapunov theory, i.e., the Krasovskii and Razumikhin approaches, are considered. The Krasovskii approach provides necessary and sufficient conditions for stability while the Razumikhin approach provides conditions that are relatively simple to verify but conservative. An important observation is that the Razumikhin approach makes use of conditions that involve the system state only while those corresponding to the Krasovskii approach involve trajectory segments. Therefore, only the Razumikhin approach yields information about DDI trajectories directly, such that the cor-

responding computations can be executed in the low-dimensional state space of the DDI dynamics. Hence, we focus on the Razumikhin approach in the remainder of this thesis.

In *Chapter 3* it is shown that by considering each delayed state as a subsystem, the behavior of a DDI can be described by an interconnected system. Thus, the Razumikhin approach is found to be an exact application of the small-gain theorem, which provides an explanation for the conservatism that is typically associated with the Razumikhin approach. Then, inspired by the relation of DDIs to interconnected systems, we propose a new Razumikhin-type stability analysis method that makes use of a stability analysis result for interconnected systems with dissipative subsystems. The proposed method is shown to provide a trade-off between the conceptual generality of the Krasovskii approach and the computational convenience of the Razumikhin approach. Unfortunately, these novel Razumikhin-type stability analysis conditions still remain conservative.

Therefore, in *Chapter 4* we propose a relaxation of the Razumikhin approach that provides necessary and sufficient conditions for stability. Thus, we obtain a Razumikhin-type result that makes use of conditions that involve the system state only and are non-conservative. Interestingly, we prove that for positive linear systems these conditions are equivalent to the standard Razumikhin approach and hence, both are necessary and sufficient for stability. This establishes the dominance of the standard Razumikhin approach over the Krasovskii approach for positive linear discrete-time systems with delay.

Next, in *Chapter 5* the stability analysis of constrained DDIs is considered. To this end, we study the construction of invariant sets for DDIs. In this context the Krasovskii approach leads to algorithms that are not computationally tractable while the Razumikhin approach is, due to its conservatism, not always able to provide a suitable invariant set. Therefore, based on the non-conservative Razumikhin-type conditions that were proposed in Chapter 4, a novel invariance notion is proposed. This notion, called the invariant family of sets, preserves the conceptual generality of the Krasovskii approach while, at the same time, it has a computational complexity comparable to the Razumikhin approach. The properties of invariant families of sets are analyzed and synthesis methods are presented.

Then, in *Chapter 6* the stabilization of constrained linear DDIs is considered. In particular, we propose two advanced control schemes that make use of online optimization. The first scheme is designed specifically to handle constraints in a non-conservative way and is based on the Razumikhin approach. The second control scheme reduces the computational complexity that is typically associated with the stabilization of constrained DDIs and is based on a set of necessary and sufficient Razumikhin-type conditions for stability.

In *Chapter 7* interconnected systems with delay are considered. In particular, the standard stability analysis results based on the Krasovskii as well as the Razumikhin approach are extended to interconnected systems with delay using small-gain arguments. This leads, among others, to the insight that delays on the channels that connect the various subsystems can not cause the instability of the overall interconnected system with delay if a small-gain condition holds. This result stands in sharp contrast with the typical destabilizing effect that time delays have. The aforementioned results are used to analyse the stability of a classical power systems example where the power plants are controlled only locally via a communication network, which gives rise to local delays in the power plants.

A reflection on the work that has been presented in this thesis and a set of conclusions and recommendations for future work are presented in *Chapter 8*.

Basic notation and definitions

Sets and set operations

The following standard sets and set operations are considered:

$\mathbb{R}, \mathbb{R}_+, \mathbb{Z}, \mathbb{Z}_+$ The set of real numbers, of nonnegative reals, of integers and of non-negative integers;

$\Pi_{\geq c_1}, \Pi_{[c_1,c_2)}$ The sets $\{r \in \Pi \; : \; r \geq c_1\}$ and $\{r \in \Pi : c_1 \leq r < c_2\}$, where $(c_1, c_2) \in \mathbb{R} \times \mathbb{R}_{>c_1}$ and $\Pi \subseteq \mathbb{R}$;

$\mathbb{B}^n$ The closed unit disc $\{x \in \mathbb{R}^n : \|x\| \leq 1\}$ in $\mathbb{R}^n$;

$\mathbb{S}^h$ The h-times Cartesian product of $\mathbb{S} \subseteq \mathbb{R}^n$, i.e., $\mathbb{S} \times \ldots \times \mathbb{S}$, $h \in \mathbb{Z}_{\geq 1}$;

$\text{int}(\mathbb{S}), \partial\mathbb{S}, \text{cl}(\mathbb{S})$ The interior, boundary and closure of $\mathbb{S}$;

$\text{supp}(\mathbb{S}, y)$ The support function of a closed set $\mathbb{S}$ with respect to the vector $y \in \mathbb{R}^n$, i.e., $\max\{y^\top x : x \in \mathbb{S}\}$;

$M\mathbb{S}$ The set $\{Mx : x \in \mathbb{S}\}$ for any $M \in \mathbb{R}$ or $M \in \mathbb{R}^{m \times n}$;

$\mathbb{S}_1 \oplus \mathbb{S}_2$ The Minkowski addition of $\mathbb{S}_1 \subset \mathbb{R}^n$ and $\mathbb{S}_2 \subset \mathbb{R}^n$, i.e., $\{x + y \; : \; x \in \mathbb{S}_1, \; y \in \mathbb{S}_2\}$;

$\bigoplus_{i=1}^N \mathbb{S}_i$ The Minkowski addition of the sets $\{\mathbb{S}_i\}_{i \in \mathbb{Z}_{[1,N]}}$, where $\mathbb{S}_i \subset \mathbb{R}^n$ for all $i \in \mathbb{Z}_{[1,N]}$;

$\text{Com}(\mathbb{S})$ The family of non-empty compact subsets of $\mathbb{S}$.

- A *polyhedron* is a set obtained as the intersection of a finite number of half-spaces and a *polytope* is a compact polyhedron;

- A *C-set* is a compact and convex set that contains 0 and a *proper C-set* is a C-set with 0 in its interior.

Vectors, matrices and norms

The following definitions regarding vectors and matrices are used:

$1_n, I_n, 0_{n \times m}$ A vector in $\mathbb{R}^n$ with all elements equal to 1, the n-th dimensional identity matrix and a matrix in $\mathbb{R}^{n \times m}$ with all elements equal to 0;

$[x]_i, [A]_{i,j}, [A]_{:,j}$ The i-th component of $x \in \mathbb{R}^n$, the i, j-th entry of $A \in \mathbb{R}^{n \times m}$ and the j-th column of A, where $(i, j) \in \mathbb{Z}_{[1,n]} \times \mathbb{Z}_{[1,m]}$;

$\|x\|, \|A\|$ An arbitrary norm of $x \in \mathbb{R}^n$ and the induced norm of A, i.e., $\max\{\|Ax\| : x \in \mathbb{R}^n, \|x\| \leq 1\}$;

$\|x\|_p, \|x\|_\infty$ The p-norm, $p \in \mathbb{Z}_{\geq 1}$ and the infinity-norm of the vector x, i.e., $\left(\sum_{i=1}^n |[x]_i|^p\right)^{\frac{1}{p}}$ and $\max_{i \in \mathbb{Z}_{[1,n]}} |[x]_i|$, respectively;

$\text{sr}(A)$ The spectral radius of the matrix $A \in \mathbb{R}^{n \times n}$;

$\mathbf{x}, \mathbf{x}_{[c_1,c_2]}$ The sequences of vectors $\{x_{l'}\}_{l' \in \mathbb{Z}_+}$, with $x_{l'} \in \mathbb{R}^n$ for all $l' \in \mathbb{Z}_+$, and $\{x_l\}_{l \in \mathbb{Z}_{[c_1,c_2]}}$, with $x_l \in \mathbb{R}^n$ for all $l \in \mathbb{Z}_{[c_1,c_2]}$;

$\|\mathbf{x}\|$ The norm of the sequence $\mathbf{x}$ defined as $\sup\{\|x_l\| : l \in \mathbb{Z}_+\}$;

$\text{diag}(x)$ A matrix in $\mathbb{R}^{n \times n}$ with $[\text{diag}(x)]_{i,i} = [x]_i$ for all $i \in \mathbb{Z}_{[1,n]}$ and zero elsewhere;

$\text{col}(\{x_l\}_{l \in \mathbb{Z}_{[c_1,c_2]}})$ The vector $\begin{bmatrix} x_{c_1}^\top & \cdots & x_{c_2}^\top \end{bmatrix}^\top$, where $(c_1, c_2) \in \mathbb{Z} \times \mathbb{Z}_{>c_1}$;

$\text{diag}(A_1, \ldots, A_N)$ A block-diagonal matrix in $\mathbb{R}^{nN \times nN}$ with the matrices $\{A_i\}_{i \in \mathbb{Z}_{[1,N]}}$, $A_i \in \mathbb{R}^{n \times n}$ for all $i \in \mathbb{Z}_{[1,N]}$, on its diagonal and zero elsewhere;

$Z \succ 0, Z \succeq 0$ The symmetric matrix $Z \in \mathbb{R}^{n \times n}$ is positive definite and positive semidefinite;

$\lambda_{\max}(Z), \lambda_{\min}(Z)$ The largest and the smallest eigenvalue of the symmetric matrix Z;

$*$ The symmetric part of a matrix, i.e., $\left[\begin{smallmatrix} a & b^\top \\ b & c \end{smallmatrix}\right] := \left[\begin{smallmatrix} a & * \\ b & c \end{smallmatrix}\right]$.

Furthermore, given two matrices $A \in \mathbb{R}^{n_1 \times n_2}$ and $B \in \mathbb{R}^{m_1 \times m_2}$, with $m_1 \geq n_1$ and $m_2 \geq n_2$, let $[B]_{i:i+n_1-1, j:j+n_2-1} = A$ denote that A is a block in B, i.e.,

$$[B]_{i-1+k, j-1+l} := [A]_{k,l}, \quad \forall (k,l) \in \mathbb{Z}_{[1,n_1]} \times \mathbb{Z}_{[1,n_2]},$$

for any $(i,j) \in \mathbb{Z}_{[1,m_1-n_1+1]} \times \mathbb{Z}_{[1,m_2-n_2+1]}$.

Basic functions and classes of functions

The following definitions and classes of functions are distinguished:

$\mathrm{co}(\cdot)$ The convex hull;

$\alpha \circ \tilde{\alpha}(\cdot)$ The composition of $\alpha : \mathbb{R} \to \mathbb{R}$ with $\tilde{\alpha} : \mathbb{R} \to \mathbb{R}$, i.e., such that $\alpha \circ \tilde{\alpha}(r) := \alpha(\tilde{\alpha}(r))$ for all $r \in \mathbb{R}$;

$\alpha^k(\cdot)$ The k-times composition of α;

$f : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ A set-valued map from $\mathbb{R}^n$ to $\mathbb{R}^m$, i.e., $f(x) \subseteq \mathbb{R}^m$ for all $x \in \mathbb{R}^n$;

$\mathcal{K}, \mathcal{K}_\infty$ The class of all functions $\alpha : \mathbb{R}_{[0,a)} \to \mathbb{R}_+$, $a \in \mathbb{R}_{>0}$ that are continuous, strictly increasing and satisfy $\alpha(0) = 0$ and the class of all $\alpha \in \mathcal{K}$ with $a = \infty$ and such that $\lim_{r \to \infty} \alpha(r) = \infty$;

$\mathcal{K} \cup \{0\}$ The class of all functions α such that $\alpha \in \mathcal{K}$ or $\alpha : \mathbb{R}_+ \to \{0\}$;

$\mathcal{K}_\infty \cup \{0\}$ The class of all functions $\alpha \in \mathcal{K}_\infty$ or $\alpha : \mathbb{R}_+ \to \{0\}$;

$\mathcal{KL}$ The class of all continuous functions $\beta : \mathbb{R}_{[0,a)} \times \mathbb{Z}_+ \to \mathbb{R}_+$, $a \in \mathbb{R}_{>0}$ such that for each fixed $s \in \mathbb{Z}_+$, $\beta(r,s) \in \mathcal{K}$ with respect to r and for each fixed $r \in \mathbb{R}_{[0,a)}$, $\beta(r,s)$ is decreasing with respect to s and $\lim_{s \to \infty} \beta(r,s) = 0$.

- A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *sublinear* if $f(rx) = rf(x)$ and $f(x+y) \leq f(x) + f(y)$ for all $(x,y,r) \in (\mathbb{R}^n)^2 \times \mathbb{R}_+$;

- A map $g : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is called *homogeneous of order N* and *positively homogeneous of order N*, $N \in \mathbb{Z}_+$ if for all $(x,r) \in \mathbb{R}^n \times \mathbb{R}$ it holds that $g(rx) = r^N g(x)$ and $g(rx) = |r|^N g(x)$, respectively;

- g is called *$\mathcal{K}$-continuous with respect to zero* if there exists an $\alpha \in \mathcal{K}$ such that $\|x_1\| \leq \alpha(\|x_0\|)$ for all $x_0 \in \mathbb{R}^n$ and all corresponding $x_1 \in g(x_0)$;

- g is called *upper semicontinuous* if for each $x \in \mathbb{R}^n$ and $\varepsilon \in \mathbb{R}_{>0}$ there exists a $\delta \in \mathbb{R}_{>0}$ such that $g(y) \subseteq g(x) + \varepsilon \mathbb{B}^m$ for all $y \in \mathbb{R}^n$ satisfying $\|x - y\| \leq \delta$.

List of abbreviations

The following abbreviations are used throughout this thesis:

AGC	automatic generation control
BMI	bilinear matrix inequality
cLRF	control Lyapunov-Razumikhin function
DDE	delay difference equation
DDI	delay difference inclusion
FDE	functional differential equation
GAS	globally asymptotically stable
GES	globally exponentially stable
LF	Lyapunov function
LS	Lyapunov stable
LKF	Lyapunov-Krasovskii function
LMI	linear matrix inequality
LRF	Lyapunov-Razumikhin function
MAD	maximal admissible delay
MPC	model predictive control
NCS	networked control systems
ODE	ordinary differential equation
SDP	semidefinite programming

Chapter 1

Introduction

1.1 Time delays

This thesis discusses the stability analysis and control of models with time delays. Time delays arise due to the propagation of physical quantities over large distances and are frequently used to obtain relatively simple models of complex physical effects such as viscoelasticity, finite reaction rates and polymer crystallization. Moreover, actuators and sensors connected to networks and digital controllers introduce delays as well. Because of the wide variety of the above effects, models with time delays can be found in many different fields such as biology, chemistry, economics and mechanics, see, e.g., [77] for an extensive list of examples. Furthermore, time delays are sometimes [122] introduced intentionally in a dynamical system to obtain a response with certain desirable properties, e.g., the delayed resonator [48] makes use of an artificial delay to enhance a vibration absorber's performance with respect to its sensitivity to the excitation frequency.

To illustrate the widespread presence of time delays, consider the water temperature regulation problem in the shower, which we encounter almost every day and is a simple example of a dynamical system with a propagation delay.

Example 1.1 (Part I) When taking a shower one has to mix warm and cold water to obtain the right water temperature. However, when one adds, for example, more warm water to increase the perceived temperature of the water, it can take some time before this change is noticeable because the water takes some time to reach the showerhead. An impatient person who does not anticipate this delay, will increase the water temperature too much. This overshoot has to be corrected, but now impatience will cause the water temperature to become too low, etc. Fortunately, experience has told us how to deal with this delay and, hence, how to avoid large temperature deviations. The water temperature regulation problem in the shower is illustrated in Figure 1.1. □

1.2 Continuous-time models with delay

Functional differential equations (FDEs) [51,52] form the most common type of models that make use of time delays to describe the behavior of dynamical systems and are used in, e.g.,
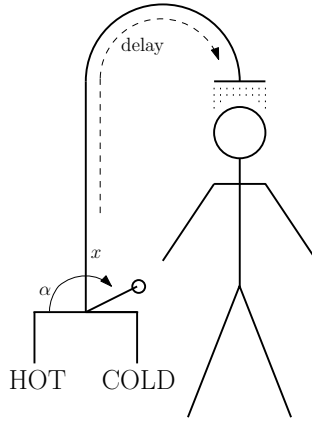
Figure 1.1: The water temperature regulation problem in the shower.

the comprehensive textbooks [48, 77, 79, 104, 109]. FDEs differ from ordinary differential equations (ODEs) in the sense that an ODE is an equation that connects the values of an unknown function and some of its derivatives for one and the same argument value, while an FDE connects the values of an unknown function and some of its derivatives for, in general, different argument values. As such ODEs are a subclass of FDEs. An FDE can map continuous functions on some bounded interval, dictated by the size of the delay, to a real vector space. In particular, given a continuous trajectory of the system under study, the FDE provides the derivative of this trajectory at the current time instant while making use of the current as well as delayed states. Hence, the dynamic behavior of the process is modeled using time delays. An FDE that makes use of only a finite number discrete delays is sometimes called a retarded differential equation.

To illustrate the use of dynamical models with delay in general and FDEs in particular, consider the water temperature regulation problem in the shower[1] again.

Example 1.1 (Part II) Let x_d be the desired value we would like the perceived water temperature to reach. Suppose that the change in water temperature at the mixer output, i.e., where the warm and cold water are mixed, is proportional to the change of the mixer angle α, which influences the ratio of warm and cold water only and not the flow of water, via some constant c. Let x denote the water temperature at the mixer output and let h denote the time it takes the water from the mixer output to reach the person's head. Assume that the rate of rotation of the handle is proportional to the deviation in water temperature from x_d perceived by the person via coefficient κ (which depends on the person's temperament). Because at time t the person feels the water temperature leaving the mixer at time $t - h$, we obtain $\dot{\alpha}(t) = -\kappa(x(t - h) - x_d(t))$, which implies that

$$\dot{x}(t) = -c\kappa(x(t - h) - x_d(t)), \quad t \in \mathbb{R}. \tag{1.1}$$

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[1]This example was borrowed from the excellent textbook [77].

Clearly, equation (1.1) is a description of the water temperature as a function of time and as such provides a model for the water temperature regulation problem in the shower. □

Typically, analyzing the stability of an ODE with a delay at its input or output is simpler than analyzing the stability of an FDE as classical stability analysis techniques such as the Nyquist stability criterion can be applied. However, Example 1.1, Part II illustrates the limitations as a modeling class of the combination of an ODE with a delay at its input or output, which is the simplest modeling class that makes use of delays. Indeed, observe that for the water temperature regulation problem in the shower the transfer function from the desired temperature x_d to the actual temperature x is

$$H(s) = \frac{c\kappa}{s + c\kappa e^{-hs}}.$$

This transfer function can not be modeled by an ODE with a delay at its input or output because such a delay appears in the numerator of the overall transfer function only. This indicates that, in general, state delays pose a challenge for the combination of ODEs with input or output delays only. Note that, for the current example, the open-loop transfer function from the mixer angle α to the water temperature x can be described by a transfer function with an input delay. Hence, it is possible to analyse the stability of the closed-loop system (1.1) using, e.g., the Nyquist stability criterion. On the other hand, when the system model has multiple inputs and outputs with different delays, is subject to time-varying or uncertain delays or the system is uncertain itself, these techniques do not apply. This indicates that FDEs form an important class of models that is a superclass of ODEs with input or output delays. Therefore, techniques to analyse the stability of FDEs are needed, even though such techniques are in general more complex than classical stability analysis methods for ODEs with input or output delays.

1.2.1   Stability analysis: Frequency-domain methods

Techniques that can be used to establish stability for ODEs do not apply to FDEs directly. For example, even for the linear FDE (1.1), it is not straightforward to determine its stability by simply computing the characteristic roots (i.e., solutions to the characteristic polynomial) of the system, as the linear FDE (1.1) has an infinite number of characteristic roots. Nevertheless, many frequency-domain methods for analyzing the stability of FDEs exist. Indeed, as a linear FDE is [122] asymptotically stable if and only if all solutions to the characteristic equation lie in the open left half-plane, tools to compute the solutions to this equation, e.g., using discretization methods [19] and the Lambert W function [147], have been well studied. For example, based on the aforementioned concepts, a wide variety of eigenvalue-based stability analysis results are provided in [104] for linear FDEs with delays and parameters that both take unknown but constant values in some predefined interval. Alternatively, matrix pencils [25, 109] and frequency sweeping methods [48] can also be used to obtain similar results for the same class of systems. Furthermore, a Padé approximation of the delay can be used in combination with standard robust control techniques to analyse the stability of linear FDEs as well, see [129].

However, when the FDE under study is nonlinear or affected by time-varying delays, the aforementioned frequency-domain stability analysis methods do not apply. For FDEs

with time-varying or distributed delays, approximation techniques [48] or integral quadratic constraints [69] can be used. Furthermore, for general nonlinear FDEs sufficient conditions for stability can be obtained based on the small-gain theorem or using structured singular values, see, e.g., [48] for both methods. Unfortunately, these conditions are conservative in most cases and verifying them can be very difficult and tedious, which makes them unattractive for practical applications.

To illustrate the use of frequency domain methods to analyse the stability of FDEs, let us consider the water temperature regulation problem in the shower again.

Example 1.1 (Part III) As the FDE (1.1) is equivalent to a single input single output transfer function, it is possible to analyse its stability by assessing the characteristic roots of (1.1), which can be computed via the Lambert W function [147]. In this case it suffices to consider [147] the first two branches of the Lambert W function only. The results of such a computation for $h = 0.5$, $c = 1$ and $\kappa = 1.5$ as well as $\kappa = 3.5$ are shown in Figure 1.2. Notice that for $\kappa = 1.5$ the first two characteristic roots (and therefore also all other char-
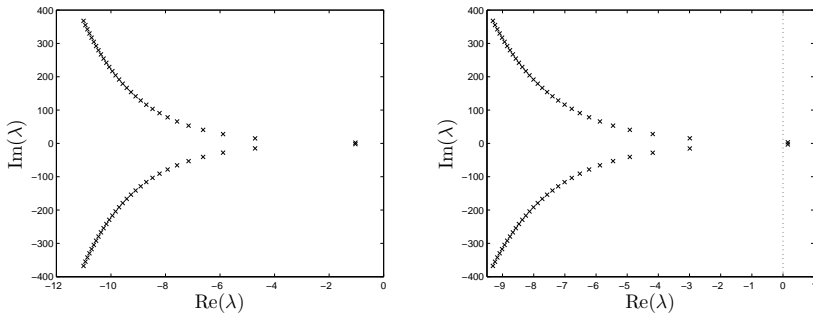
Figure 1.2: The characteristic roots of the FDE (1.1), computed via the Lambert W function, for $h = 0.5$, $c = 1$ and $\kappa = 1.5$ (left) as well as $\kappa = 3.5$ (right).

acteristic roots) lie in the left half-plane and hence that the FDE (1.1) is stable. However, for $\kappa = 3.5$ the first two characteristic roots lie in the right half-plane, which implies that the FDE (1.1) is unstable. As such it can be concluded that if a person is too aggressive and changes the ratio of warm and cold water α too quickly, the system becomes unstable. Indeed, this is the type of behavior one may have experienced in practice or otherwise would expect intuitively. Furthermore, it is interesting to note that the system without delay is stable for any positive c and κ, which indicates the nontrivial behavior that the delay introduces and emphasizes the importance of taking delays into account in the stability analysis. □

1.2.2 Stability analysis: Time-domain methods

An alternative category of stability analysis methods for FDEs makes use of conditions in the time domain. An important advantage of these methods over frequency-domain methods is that they can be modified such that it is possible to take constraints into account.

Furthermore, these methods have the potential to handle nonlinear FDEs with time-varying delays, which is a difficult task for most frequency-domain methods. Unfortunately, while for delay-free systems time-domain stability analysis methods are mostly based on the existence of a strictly decreasing energy function, called Lyapunov function (LF) [50], the classical Lyapunov theory does not apply straightforwardly to systems with delay. This is due to the fact that the influence of the delayed states can cause a violation of the monotonic decrease condition that a standard LF obeys. To solve this issue, two different approaches have been proposed, i.e., the Krasovskii and Razumikhin approaches.

The *Krasovskii approach* [79], or Lyapunov functionals approach, relies on the construction of a functional that decreases along all trajectories of the system. The advantage of this approach is that it can provide a set of necessary and sufficient conditions for stability. For example, Theorem 4.1.10 in [77] establishes that an FDE is globally asymptotically stable if and only if it admits a so-called Lyapunov-Krasovskii function (LKF). Furthermore, Theorem 5.19 in [48] establishes that if a linear FDE is globally exponentially stable, then it admits a quadratic LKF. This result was partially extended to FDEs with uncertain parameters in [76]. Moreover, necessary and sufficient conditions for the stability of linear FDEs with time-varying delay were presented in [98]. Unfortunately, except for some relatively simple cases, e.g., linear FDEs with a single delay term [76], the construction of the required functional is not straightforward. On the other hand, if one is satisfied with merely sufficient Krasovskii-type conditions, a wide variety of computationally tractable results is available, see, e.g., [48, 99, 109, 122] and the references therein.

The *Razumikhin approach* [51] on the other hand requires the construction of a function (rather than a functional) that does not need to decrease at all times, which makes this approach typically easier to apply. Indeed, motivated by the computational advantages of the Razumikhin approach, several stability analysis results have been developed using this technique, see, e.g., [20, 109]. However, the Razumikhin approach is based [133] on a type of small-gain condition for FDEs and as such it is inherently conservative. This is further illustrated by the fact that the Razumikhin approach can be considered [77] as a particular case of the Krasovskii approach. For example, it is known [75] that any quadratic Lyapunov-Razumikhin function (LRF) yields a particular quadratic LKF.

To illustrate the application of the Krasovskii and Razumikhin approaches, we consider the water temperature regulation problem in the shower again.

Example 1.1 (Part IV) Consider the FDE (1.1) and, to simplify the presentation, let us assume that $x_d(t) = 0$ for all t. To establish stability for the FDE (1.1) via the Krasovskii approach, we will use the functional

$$\bar{V}(\mathbf{x}_{[t-h,t]}) := px(t)^2 + q \int_{-h}^{0} \int_{\theta}^{0} x(t+\xi)^2 d\xi d\theta,$$

where $\mathbf{x}_{[t-h,t]}$ denotes the trajectory $x(t')$ for all $t' \in \mathbb{R}_{[t-h,t]}$ and where $(p,q) \in \mathbb{R}_{>0}^2$ are two constants that need to be chosen properly to guarantee certain properties. Firstly, if both p and q are strictly positive, then $\bar{V}$ is positive definite with respect to 0 and radially unbounded. Secondly, by choosing $p := c\kappa \in \mathbb{R}_{>0}$ and $q := 2 \in \mathbb{R}_{>0}$ it can be shown using the techniques presented in Chapter 5 of [48] that, if $h = 0.5$, $c = 1$ and $\kappa = 1.5$,

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then

$$\tfrac{d}{dt}\bar{V}(\mathbf{x}_{[t-h,t]}) < -\varepsilon\|\mathbf{x}_{[t-h,t]}\|_2^2,$$

for some $\varepsilon \in \mathbb{R}_{>0}$. Hence, it follows that the function $\bar{V}$ is strictly decreasing along all trajectories of the FDE (1.1), which implies that (1.1) is stable. However, the derivations required to show that the functional $\bar{V}$ is strictly decreasing along all trajectories of the FDE (1.1) are highly nontrivial. Moreover, for more complex linear and nonlinear FDEs it remains unclear how to choose the structure of the functional $\bar{V}$. These observations illustrate the complexity that is typically associated with the Krasovskii approach. On the other hand, the Razumikhin approach can also be used to establish stability for the FDE (1.1). To do so, the FDE (1.1) is transformed via a state transformation into a system with a distributed delay. For this new system a function of the form

$$V(y(t)) := py(t)^2,$$

for some $p \in \mathbb{R}_{>0}$ is constructed that satisfies a decrease condition. Above, $y(t)$ is the new state vector. While this task is relatively simple compared to finding the functional that was used for the Krasovskii approach, finding the right state transformation is not straightforward, see Section 5.3 in [48] for details. □

A graphical summary of the relations of the Krasovskii and Razumikhin approaches to the set of all globally asymptotically stable FDEs $\mathcal{S}$ is shown in Figure 1.3.
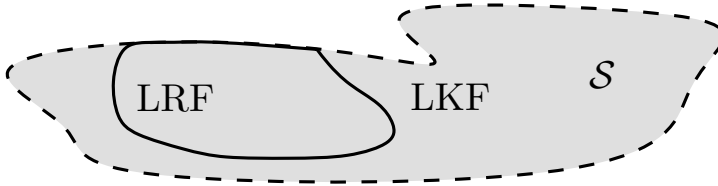
Figure 1.3: The set $\mathcal{S}$ (grey) consists of all globally asymptotically stable FDEs and is equivalent to the set of all FDEs that admit an LKF (- - -). Furthermore, the set of all FDEs that admit an LRF (——) forms a strict subset of $\mathcal{S}$.

1.3 Discrete-time models with delay

Discrete-time models with delay are also studied already for quite some time. This interest is motivated traditionally by sampled-data systems and more recently by networked control systems (NCS). Therefore, these classes of systems are discussed in what follows.

1.3.1 Sampled-data systems

Over the last decades the efficiency of computers has increased while at the same time their costs have decreased dramatically. As a consequence most modern controllers are designed for a discrete-time approximation of the continuous-time model and then implemented via

a computer. In this case, the controller for the discrete-time model also stabilizes the original continuous-time model if some mild assumptions are satisfied, see, e.g., [107, 108]. This practice, which is typically referred to as sampled-data control, has caused a demand for models of dynamical systems in the discrete-time domain, see, e.g., [2]. However, an inherent consequence of using a digital architecture to control a system is the presence of time delays in the control loop, e.g., due to the discretization of signals, and gives rise to sampled-data systems with delayed inputs [41]. Furthermore, when the continuous-time model is subject to time delays, it has to be modeled by a discrete-time model with delay as well. Unfortunately, even for linear systems with delay an exact discretization can, due to the presence of state delays, lead to an infinite series for which a closed-form expression is missing [67]. Moreover, state delays also give rise to an infinite input memory [33]. On the other hand, if the delay is a multiple of the sampling time and a system with state delay has a cascaded structure or the delay appears at the input or output of the system only, exact discretization is possible [1]. Moreover, for arbitrary linear continuous-time models with delay, sufficiently accurate approximations can be obtained using, e.g., the block pulse function approximation [24] or a finite series approximation [33, 67]. Furthermore, approximations of nonlinear models with arbitrary delay can be obtained using a standard forward Euler discretization [28] or using more complex discretization techniques [28, 152].

To illustrate the discretization of continuous-time models with delay, let us consider the water temperature regulation problem in the shower again and equation (1.1) in particular.

Example 1.1 (Part V) For the water temperature regulation problem in the shower a forward Euler discretization [28] with sampling time $T_s = 0.1h$ yields a discrete-time approximation of the continuous-time model with delay (1.1), i.e.,

$$x_{k+1} = -c\kappa T_s(x_{k-10} - x_{d,k}) + x_k, \quad k \in \mathbb{Z}_+. \tag{1.2}$$

To illustrate the effectiveness of this modeling approach, a simulation of the continuous-time model (1.1) and the discrete-time approximation (1.2) for $h = 0.5$, $c = 1$, $\kappa = 1.5$ and $x_d(t) = 38$ for all $t \in \mathbb{R}_+$ is considered here. Figure 1.4 shows a simulation from the initial
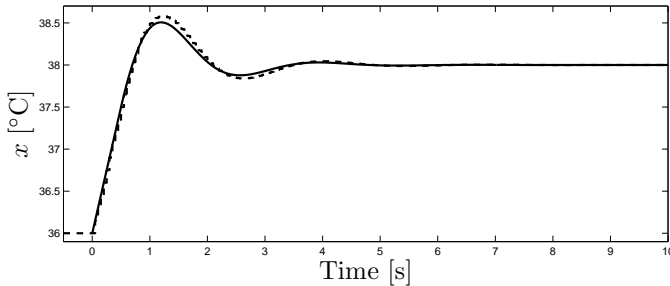
Figure 1.4: A simulation of the continuous-time model (1.1) (——) and the discrete-time model (1.2) (- - -) for the water temperature regulation problem in the shower.

condition $x(t) = 36$ for all $t \in \mathbb{R}_{[-h,0]}$. It can be seen from Figure 1.4 that the discrete-time model (1.2) provides a reasonable approximation of the original FDE (1.1). Moreover, it can be shown that for this particular sampling time, the discrete-time model (1.2) preserves the stability of the original continuous-time model. □

1.3.2 Networked control systems

There is a second important driver for the interest in discrete-time models of systems with delay. This second driver comes from the widespread application of NCS, which were identified to be one of the emerging key topics in control in a recent survey [106] on future directions in control. The distinguishing feature of NCS is that the connection between plant and controller is made through a shared communication network, such as it is the case in Figure 1.5. The introduction of the communication network brings several advantages,
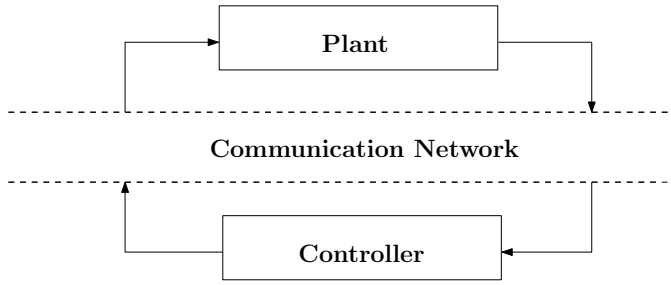
Figure 1.5: The typical setup of a networked control system.

most importantly a reduced amount or, when a wireless communication network is used, an almost complete absence of wiring. This greatly increases the flexibility and robustness of the control architecture and has led to the introduction of NCS in, e.g., automotive applications [18, 21], the mining industry [145], aircrafts [120] and robotics [102]. However, the communication network also brings specific additional challenges to controller design, such as the presence of uncertain time-varying delays, communication constraints, timing jitter, quantization errors and packet dropouts, see, e.g., the comprehensive NCS overviews [57, 135] and the recent textbook [10].

In this context, a wide variety of modeling approaches exists, but a unifying feature of most modeling methods is that the system and communication network are combined into a single model which, due to the packet based nature of the communication network, is in the discrete-time domain. For example, using polytopic over-approximation methods a linear system that is controlled over a communication network can be modeled as an uncertain discrete-time system with delay, see, e.g., [27, 46, 58]. Alternatively, the same setup can be modeled using a discrete-time switched system with delay [151] or as a stochastic discrete-time system with delay [35, 146] when the stochastic nature of the delay is taken into account. Furthermore, when communication protocols are taken into account a discrete-time switched system with delay [36] is obtained. For nonlinear systems that are controlled over a communication network, various approximate modeling techniques in the discrete-time domain exist, see, e.g., [113, 138], while an exact model was obtained using a hybrid sys-

tems approach in [53]. Hence, the widespread application of NCS further illustrates the importance of discrete-time models of systems with delay.

1.3.3 Stability analysis of discrete-time systems with delay

For the stability analysis of discrete-time systems with delay two different lines of research can be distinguished. The first line of research focuses on deriving tractable sufficient conditions for the stability of linear (and sometimes uncertain linear) systems with time-varying delay. For example, systems with uncertain stochastic delay were considered in, e.g., [35, 146], while systems with time-varying delay were considered in, e.g., [36, 61]. A somewhat broader class of systems, i.e., uncertain systems with time-varying delay, was discussed in [38, 42, 100]. Moreover, to obtain sharper results, some articles consider systems with time-varying delay with a bounded rate of variation, see, e.g., [43] and the references therein. In all of the aforementioned articles two important factors play a role, i.e., which method is the least conservative and which method provides the conditions that are simplest to verify. Unfortunately, theoretical bounds on the best possible performance with respect to either of these two properties are missing.

The second line of research focuses on the stability analysis of NCS. Most important in this case is how to obtain the best control performance using only a limited amount of resources. In this setting, the results in the literature differ most with respect to which network effects are taken into account. Indeed, the simplest case is to merely consider a network that introduces time-varying delay [22, 46] or timing jitter [58, 132]. Other articles also include packet dropout [27] or all of the aforementioned effects [26]. Moreover, in some cases communication constraints [36] or stochastic delays [35] are considered. As the aforementioned results all deal with a linear continuous-time model that is controlled over a communication network an exact discretization is possible, which implies that a discrete-time controller that stabilizes a discretization of the original system and communication network, also stabilizes the real NCS setup. The extension of these results to nonlinear systems is typically based on the reasoning that was developed for sampled-data systems in [107, 108], see, e.g., [113, 138] and the references therein.

Unfortunately, a relatively small amount of research on discrete-time systems with delay addresses fundamental questions such as: What trade-off between computational complexity and conceptual generality is provided by the stability analysis methods that underlie existing results? For what classes of systems do the relatively simple methods provide necessary conditions? Are there other stability analysis methods possible that provide a better trade-off between computational complexity and conceptual generality? How can constraints be taken into account via computationally tractable algorithms? In fact, such questions have thus far not been answered or answered only partially. Therefore, these issues are studied in this thesis.

1.4 Outline of the thesis

Motivated by the facts outlined in Section 1.3, discrete-time systems with delay are considered in this thesis. Furthermore, as almost every system in practice is subject to constraints and Lyapunov theory is uniquely suited for the stability analysis and control of systems that are subject to constraints, all results in this thesis will essentially be based on Lyapunov theory. While for the stability analysis of FDEs a wide variety of Lyapunov-based

techniques has been available for many years and their advantages are well known (see Section 1.2.2 for details), such an overview is missing for discrete-time systems. As such the main contributions of this thesis are to analyse the parallels for discrete-time systems to the standard stability analysis results for FDEs and to develop a wide variety of new techniques that do not suffer from the drawbacks that are inherently linked to the aforementioned parallels. During this process some of the fundamental questions that were posed above are answered. Throughout this thesis, to illustrate the practical implications of the results, these results are applied to a benchmark NCS example, i.e., a DC motor that is controlled over a communication network. The following topics are discussed:

In *Chapter 2* delay difference inclusions (DDIs) are introduced as a modeling class for discrete-time systems with delay and the generality and advantages of this modeling class are highlighted. Then, motivated by the fact that a comprehensive overview of stability analysis methods for discrete-time systems with delay based on Lyapunov theory is missing, the standard stability analysis results for this class of systems are considered. Among others, counterparts of the Krasovskii and Razumikhin approaches for DDIs are discussed. It is found that, like for continuous-time systems, the Krasovskii approach provides necessary and sufficient conditions while the Razumikhin approach provides conditions that are relatively simple to verify but conservative. Furthermore, the invariance notions that are related to these approaches and the stability analysis and stabilizing controller synthesis algorithms for linear DDIs that make use of these approaches are also presented.

An important observation in Chapter 2 is that the Razumikhin approach makes use of conditions that involve the system state only while the Krasovskii approach makes use of conditions involving trajectory segments. Therefore, only the Razumikhin approach yields information about DDI trajectories directly, such that the corresponding computations can be executed in the low-dimensional state space of the DDI dynamics. Motivated by this fact we focus on the Razumikhin approach in the remainder of the thesis. It is shown in *Chapter 3* that by considering each delayed state as a subsystem, a DDI can be considered as an interconnected system with a particular structure. Thus, the Razumikhin approach is found to be an exact application of the small-gain theorem, which provides an explanation for the conservatism that is typically associated with this approach. Then, we propose a new Razumikhin-type stability analysis method that makes use of a stability analysis result for interconnected systems with dissipative subsystems. The proposed method is shown to provide a trade-off between the conceptual generality of the Krasovskii approach and the computationally convenience of the Razumikhin approach. The stability analysis and stabilizing controller synthesis algorithms for linear DDIs that make use of this Razumikhin-type approach are also presented.

Unfortunately, these novel Razumikhin-type stability analysis conditions still remain conservative. Therefore, in *Chapter 4* we propose a relaxation of the Razumikhin approach that provides necessary and sufficient conditions for stability. Thus, we obtain a Razumikhin-type result that makes use of conditions that involve the system state only and are non-conservative. Unfortunately, currently even for linear DDIs, only the stability analysis problem that corresponds to these relaxed Razumikhin conditions can be solved via convex optimization algorithms. Interestingly, for positive linear systems the relaxed Razumikhin-type conditions in this chapter are proven to be equivalent to the standard Razumikhin approach and hence both are non-conservative. This establishes the dominance

of the Razumikhin approach over the Krasovskii approach for positive linear discrete-time systems with delay, in the sense that both approaches provide necessary and sufficient conditions for stability but only the Razumikhin approach yields relatively simple conditions that provide information about the system trajectories directly.

Next, in *Chapter 5* the stability analysis of constrained DDIs is considered. To this end, we study the construction of invariant sets, which are crucial for the analysis of constrained systems. In this context the Krasovskii approach leads to algorithms that are not computationally tractable while the Razumikhin approach is, due to its conservatism, not always able to provide a suitable invariant set. Therefore, we propose, based on the non-conservative Razumikhin-type conditions of Chapter 4, a novel invariance notion. This notion is called the invariant family of sets and ultimately leads to the derivation of computationally tractable methods for the construction of invariant sets for DDIs. The invariant family of sets, preserves the conceptual generality of the Krasovskii approach while, at the same time, it has a computational complexity comparable to the Razumikhin approach. The concept is accompanied by a considerable collection of synthesis solutions that can be solved via various convex optimization algorithms. Furthermore, the results are illustrated by simple examples that highlight some of the most important facts.

Then, in *Chapter 6* the stabilization of constrained linear DDIs is considered. In particular, we propose two advanced control schemes that make use of online optimization. The first scheme is designed specifically to handle constraints in a non-conservative way and is based on the Razumikhin approach. A detailed stability analysis of the resulting closed-loop system shows the advantages of this method. The second control scheme reduces the computational complexity that is typically associated with the stabilization of constrained DDIs and is based on a set of necessary and sufficient conditions for stability. This scheme makes use of a so-called state-dependent LRF and is able to handle constraints as well. In both cases, by exploiting properties of the Minkowski addition of polytopes and the structure of the developed control law, an efficient implementation can be attained.

In view of the close relationship of DDIs to interconnected systems that was established in Chapter 3, it seems natural to expect that stability analysis results for DDIs can be extended to interconnected systems with delay. Therefore, in *Chapter 7* this extension is considered. In particular, the standard stability analysis results based on the Krasovskii as well as the Razumikhin approach are extended to interconnected systems with delay using a small-gain theorem. This leads, among others, to the insight that delays on the channels that connect the various subsystems can not cause the instability of the overall interconnected system with delay if a small-gain condition holds. This result stands in sharp contrast with the typical destabilizing effect that time delays have. The aforementioned results are used to analyse the stability of a classical power systems example. In this example, the case is considered where the power plants are controlled only locally via a communication network, which gives rise to local delays in the power plants.

Finally, in *Chapter 8* we reflect on the work that has been presented in this thesis and provide a set of conclusions and recommendations for future work.

1.5 Summary of publications

The results that are presented in this thesis have appeared in publications that were written together with one or more co-authors. In all of these works, with two exceptions, the

promovendus has been the main contributer, as reflected by his position of first and corresponding author. In this section we indicate, chapter by chapter, where the results appeared originally and with whom they were derived.

Chapter 2 contains results that were presented in:

- R.H. Gielen, M. Lazar and I.V. Kolmanovsky (2010). "On Lyapunov theory for delay difference inclusions", in the proceedings of the American Control Conference, Baltimore, July, 2010 (invited paper);

- R.H. Gielen, M. Lazar and I.V. Kolmanovsky (2012). "Lyapunov methods for time-invariant delay difference inclusions". *SIAM Journal on Control and Optimization*, Vol. 50, No. 1, pp. 110-132.

Chapter 3 contains results that were presented in:

- R.H. Gielen, M. Lazar and A.R. Teel (2011). "On input-to-state stability of delay difference equations", in the proceedings of the IFAC World Congress, Milano, Italy, August, 2011;

- R.H. Gielen, A.R. Teel and M. Lazar (2013). "Tractable Razumikhin-type conditions for input-to-state stability analysis of delay difference inclusions". *Automatica*, Vol. 49, No. 2, in press.

Chapter 4 contains results that were presented in:

- R.H. Gielen, M. Lazar and S.V. Raković (2013). "Necessary and sufficient Razumikhin-type conditions for stability of delay difference equations". Submitted for publication to a journal.

Chapter 5 contains results that were presented in:

- R.H. Gielen, S.V. Raković and M. Lazar (2012). "Further results on the construction of invariant families of sets for linear systems with delay", in the proceedings of the IFAC Workshop on Time-delay Systems, Boston, June, 2012;

- S.V. Raković, R.H. Gielen and M. Lazar (2012). "Construction of invariant families of sets for linear systems with delay", in the proceedings of the American Control Conference, Montréal, Canada, June, 2012;

- S.V. Raković, R.H. Gielen and M. Lazar (2013). "Positively invariant families of sets for interconnected and time-delay systems". Submitted for publication to a journal.

Chapter 6 contains results that were presented in:

- R.H. Gielen and M. Lazar (2009). "Further results on stabilization of linear systems with time-varying delays", in the proceedings of the IFAC Workshop on Time Delay Systems, Sinaia, Romania, September, 2009;

- R.H. Gielen and M. Lazar (2009). "Stabilization of networked control systems via non-monotone control Lyapunov functions", in the proceedings of the IEEE Conference on Decision and Control, Shanghai, China, December, 2009;

- R.H. Gielen and M. Lazar (2010). "Stabilization of polytopic delay difference inclusions: Time-varying control Lyapunov functions", in the proceedings of the IEEE Conference on Decision and Control, Atlanta, December, 2010;

- R.H. Gielen and M. Lazar (2011). "Stabilization of polytopic delay difference inclusions via the Razumikhin approach". *Automatica*, Vol. 47, No. 12, pp. 2562-2570;

- R.H. Gielen and M. Lazar (2012). "Stabilisation of linear delay difference inclusions via time-varying control Lyapunov functions". *IET Control Theory and Applications*, Vol. 6, No. 12, in press.

Chapter 7 contains results that were presented in:

- R.H. Gielen, M. Lazar and A.R. Teel (2011). "Small-gain results for discrete-time networks of systems with delay", in the proceedings of the IEEE Conference on Decision and Control, Orlando, December, 2011;

- R.H. Gielen, M. Lazar and A.R. Teel (2012). "Input-to-state stability analysis for networks of difference equations with delay". *Mathematics of Control, Systems and Signals*, Vol. 24, pp. 33-54. Special issue on Robust Stability and Control of Large-Scale Nonlinear Systems;

- R.H. Gielen, R.M. Hermans, M. Lazar and A.R. Teel (2012). "Input-to-state stability analysis of power systems with delay", in the proceedings of the American Control Conference, Montréal, Canada, June, 2012.

Furthermore, a book chapter and a journal publication have appeared over the last few years that are closely related to the topics covered in this thesis but which have not been included in this thesis for brevity, i.e.:

- R. H. Gielen and M. Lazar (2012). On the construction of $\mathcal{D}$-invariant sets for linear polytopic delay difference inclusions. *Journal Européen des Systèmes Automatisés*, Vol. 46, No. 2-3, pp. 183-195;

- R. H. Gielen, M. Lazar and S. Olaru (2012). Set-induced stability results for delay difference equations. In *Time Delay Systems: Methods, Applications and New Trends*, Vol. 423 of *Lecture Notes in Control and Information Sciences*, pp. 73-84. Springer, Berlin, Germany.

Where appropriate, a reference to one or more of these articles has been included in this thesis for further reading.

Chapter 2

Delay difference inclusions and stability

In this chapter we introduce DDIs as a modeling class for discrete-time systems with delay and highlight the generality and advantages of this modeling class. Then, a comprehensive collection of stability analysis methods, based on Lyapunov theory, for DDIs is presented. In particular, necessary and sufficient conditions for stability of various classes of DDIs are obtained based on the Krasovskii approach. Furthermore, relatively simple but merely sufficient conditions for stability are derived via the Razumikhin approach. Next, the relation of both methods to each other and to certain types of set invariance properties is established. The chapter is completed by the corresponding stability analysis and stabilizing controller synthesis methods for linear DDIs and quadratic functions, which can be solved via semidefinite programming (SDP).

2.1 Introduction

As indicated in Section 1.3 the stability analysis of discrete-time systems with delay is an important topic in the field of control theory because of the role this class of systems plays in NCS and in sampled-data control. Moreover, as almost every system in practice is subject to constraints and Lyapunov theory is uniquely suited for the stability analysis and control of systems that are subject to constraints, we will essentially restrict our attention to Lyapunov theory. While for the stability analysis of FDEs a wide variety of Lyapunov-based stability analysis techniques has been available for many years and their advantages are well known (see Section 1.2.2 for details), such an overview is missing for discrete-time systems.

Indeed, it is not immediately clear how the Razumikhin and Krasovskii approaches are to be used for the stability analysis of discrete-time systems with delay. One of the most commonly used approaches [1] to stability analysis of this class of systems is to augment the state vector with all delayed states/inputs that affect the current state, which yields a standard discrete-time system of higher dimension. Thus, classical stability analysis methods for discrete-time systems (such as frequency-domain results) can be used to analyse the stability of the discrete-time system with delay. Similarly, time-domain methods for standard discrete-time systems that are based on Lyapunov theory, such as, e.g., [2, 73], become applicable. Such an LF for the augmented system provides an LKF for the original system with delay. Moreover, in [59] it was shown that all existing methods based on the Krasovskii approach provide a particular type of LF for the augmented system. As such, an

interpretation of the Krasovskii approach for discrete-time systems is readily available. Examples of controller synthesis methods based on this approach can be found in, among many others, [22, 26, 38, 42, 61, 78, 100, 132, 146]. However, converse results for the Krasovskii approach, such as the ones mentioned for continuous-time systems in Section 1.2.2, are missing. For the Razumikhin approach the situation is more complicated. The exact translation of this approach to discrete-time systems yields a set of so-called backward Razumikhin conditions [37, 149], which are very difficult to verify. An alternative interpretation of the Razumikhin approach for discrete-time systems was proposed in [92, 93], where the LRF was essentially required to be less than the maximum over its past values for the delayed states. Stability analysis and stabilizing controller synthesis methods based on the existence of an LRF can be found in, e.g., [91, 94]. Unfortunately, the relation between LKFs and LRFs has not yet been investigated. Moreover, it remains an open question whether there exist systems that are stable but do not admit an LRF.

Motivated by the fact that DDIs form a rich modeling class that can model both sampled-data systems and many types of NCS [46, 150, 153], this thesis focuses on DDIs, which are discrete-time systems with delay and a set-valued right-hand side. Apart from being a rich modeling class, DDIs also allow for the derivation of results that, when specialized to a specific subclass of DDIs, are equivalent to the results that can be derived for that subclass. As such to consider DDIs is a generalization that does not compromise the sharpness of the derived results when only a specific subclass is of interest. Therefore, in the first part of this chapter we focus on the introduction of DDIs as a modeling class and we provide a set of examples that illustrate how DDIs can be used to model NCS and sampled-data systems. Then, motivated by the fact that an overview of the corresponding counterpart of the Lyapunov methods for FDEs is missing, the purpose of the remainder of this chapter is to provide a comprehensive collection of Lyapunov methods for DDIs. To this end, firstly, using the augmented system, we prove several converse Lyapunov theorems for the Krasovskii approach. This is the first time that such converse theorems are established for discrete-time systems with delay. Secondly, for the Razumikhin approach, the results of [37] and [93] are extended to DDIs. Thirdly, via an example of a scalar linear system with delay that is stable but does not admit an LRF, it is shown that the existence of an LRF is a sufficient but not a necessary condition for stability of DDIs. Then, an LKF is constructed from an LRF and a set of additional assumptions is derived under which the converse is possible. Furthermore, it is shown that an LRF induces a set with certain contraction properties that are particular to systems with delay. On the other hand, an LKF is shown to induce a standard contractive set for the augmented system, similar to the contractive set induced by a classical LF. Using quadratic functions, stability analysis and stabilizing controller synthesis methods for linear DDIs in terms of LKFs as well as LRFs are proposed, which can be solved via SDP.

2.2 Delay difference inclusions

Consider the DDI

$$x_{k+1} \in F(\mathbf{x}_{[k-h,k]}), \quad k \in \mathbb{Z}_+, \tag{2.1}$$

where $\mathbf{x}_{[k-h,k]} \in (\mathbb{R}^n)^{h+1}$ is a sequence of (delayed) states, $h \in \mathbb{Z}_{\geq 1}$ is the maximal delay and $F : (\mathbb{R}^n)^{h+1} \rightrightarrows \mathbb{R}^n$ is a *set-valued* map with the origin as equilibrium point, i.e.,

$F(\mathbf{0}_{[-h,0]}) := \{0\}$. To guarantee the existence of solutions, the set $F(\mathbf{x}_{[-h,0]}) \subset \mathbb{R}^n$ is assumed to be non-empty for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$. This assumption is without loss of generality when studying stability of the origin or invariance of a set that contains the origin since one can always consider a system that introduces solutions that jump to the origin when the original system is undefined. DDIs of the form (2.1) are a rich modeling class that can provide models to analyse the properties of most types of sampled-data systems and NCS. This is illustrated by the following examples.

Example 2.1 In [42, 44, 59] the following system with uncertain time-varying delay was considered, i.e.,

$$x_{k+1} = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix} x_{k-\tau_k} + \begin{bmatrix} 0.8 & 0 \\ 0 & 0.97 \end{bmatrix} x_k, \quad k \in \mathbb{Z}_+,$$

where $x_k \in \mathbb{R}^2$ denotes the state at time $k \in \mathbb{Z}_+$ and where the size of the delay is determined (within a specified upper and lower bound) by an arbitrary sequence, i.e., $\tau : \mathbb{Z}_+ \to \mathbb{Z}_{[\underline{\tau},\overline{\tau}]}$ for some $(\underline{\tau}, \overline{\tau}) \in \mathbb{Z}_+ \times \mathbb{Z}_{\geq \underline{\tau}}$. This system can be modeled by the DDI (2.1) with

$$F(\mathbf{x}_{[-h,0]}) = \left\{ \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix} x_{-d} + \begin{bmatrix} 0.8 & 0 \\ 0 & 0.97 \end{bmatrix} x_0 : d \in \mathbb{Z}_{[\underline{\tau},\overline{\tau}]} \right\},$$

and $h = \overline{\tau}$. The time-varying delay in this example can, for example, be due to the discretization of signals and computation times in a sampled-data system, i.e., the delayed term is generated by a control signal that is updated at varying time intervals. $\qquad\square$

Example 2.2 (Part I) Consider a DC-motor that is controlled over a communication network, which is a benchmark example for NCS [135], see Figure 2.1. The communication
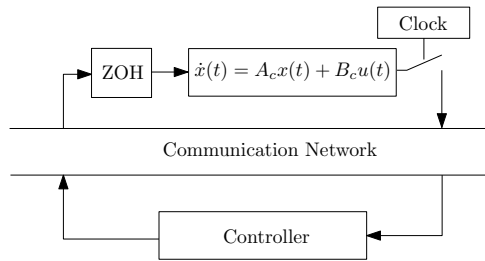
Figure 2.1: A linear system controlled over a communication network.

network introduces uncertain time-varying input and output delays, which yields

$$\begin{bmatrix} \dot{i}_a(t) \\ \dot{\omega}(t) \end{bmatrix} = \begin{bmatrix} -27.47 & -0.09 \\ 345.07 & -1.11 \end{bmatrix} \begin{bmatrix} i_a(t) \\ \omega(t) \end{bmatrix} + \begin{bmatrix} 5.88 \\ 0 \end{bmatrix} e_a(t) + \begin{bmatrix} 0 \\ 23474 \end{bmatrix} d_{\text{load}},$$

$$e_a(t) = u_k, \quad \forall t \in \mathbb{R}_{[t_k+\tau_k, t_{k+1}+\tau_{k+1})},$$

where i_a is the armature current, ω is the angular velocity of the motor, the armature voltage e_a is the input signal and d_{load} is a constant load torque. Furthermore, $t_k := kT_s$, $k \in \mathbb{Z}_+$ is the sampling instant, $T_s \in \mathbb{R}_+$ denotes the sampling period and $u_k \in \mathbb{R}^m$ is the control action generated at time $t = t_k$. Moreover, $\tau_k \in \mathbb{R}_{[0,\bar{\tau}]}$ denotes the sum of the input and output delay[1] at time $k \in \mathbb{Z}_+$ and $\bar{\tau} \in \mathbb{R}_{[0,T_s]}$ is the maximal delay induced by the network. It is assumed that $\bar{\tau} \leq T_s$, i.e., the delay is always smaller than or equal to the sampling period. The DC-motor is controlled via a static state-feedback control law, i.e., such that $u_k = Kx_k$ for all $k \in \mathbb{Z}_+$.

For constant load torques, a discretization of the DC-motor in closed loop with the controller and the communication network yields the DDI (2.1) with $h = 1$ and

$$F(\mathbf{x}_{[-1,0]}) = \left\{ \Delta(\tau)Kx_{-1} + (A_d + (B_d - \Delta(\tau))K)x_0 : \tau \in \mathbb{R}_{[0,\bar{\tau}]} \right\},$$

where $A_d = e^{A_c T_s}$, $B_d = \int_0^{T_s} e^{A_c(T_s - \theta)} d\theta B_c$ and $\Delta(\tau) = \int_0^\tau e^{A_c(T_s - \theta)} d\theta B_c$. The matrices A_c and B_c follow from the differential equation that describes the DC-motor. $\square$

It is interesting to observe that the sampled-data setting considered in [41] corresponds to the situation that is considered in Example 2.2. Hence, the DDI (2.1) can model this case as well. Note that, Examples 2.1 and 2.2 illustrate that while the DDI (2.1) is time-invariant, systems with uncertain time-varying delay can be modeled by the DDI (2.1). Similarly, uncertain systems with delay can also be modeled by the DDI (2.1).

Remark 2.1 For systems with a time-varying delay that is known, a more accurate model is given by a switched system with known switching signal [59]. Furthermore, if bounds on the rate of variation of an uncertain time-varying delay are known, a more accurate model can also be obtained [43]. In the conclusions of this thesis it is explained how the modeling framework that is used in this thesis can be extended to handle the latter kind of system. $\square$

Throughout this thesis, to obtain sharper results, we will sometimes focus on specific subclasses of the DDI (2.1). These classes are defined in what follows.

Definition 2.1 *(i)* The DDI (2.1) is called a *linear delay difference equation* (DDE) if $F(\mathbf{x}_{[-h,0]}) = \{\sum_{i=-h}^0 A_i x_i\}$ for some $(A_{-h}, \ldots, A_0) \in (\mathbb{R}^{n \times n})^{h+1}$; and *(ii)* the DDI (2.1) is called a *linear DDI* if $F(\mathbf{x}_{[-h,0]}) = \{\sum_{i=-h}^0 A_i x_i : (A_{-h}, \ldots, A_0) \in \mathcal{A})\}$ for some compact and non-empty set $\mathcal{A} \subset (\mathbb{R}^{n \times n})^{h+1}$. $\square$

Definition 2.2 The DDI (2.1) is called $\mathcal{D}$-*homogeneous of order* N, $N \in \mathbb{Z}_+$ if for any $r \in \mathbb{R}$ it holds that $F(r\mathbf{x}_{[-h,0]}) = r^N F(\mathbf{x}_{[-h,0]})$ for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$. $\square$

The first property that is considered for the DDI (2.1) is stability. Therefore, let $\mathcal{S}(\mathbf{x}_{[-h,0]})$ denote the set of all solutions to (2.1) that correspond to initial condition $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$.

---

[1]For time-invariant controllers, both delays on the measurement and the actuation link can be lumped [150] into a single delay on the latter link and hence output delays are implicitly taken into account.

Furthermore, let $\Phi := \{\phi_k\}_{k \in \mathbb{Z}_{\geq -h}} \in \mathcal{S}(\mathbf{x}_{[-h,0]})$ denote a solution to the DDI (2.1) such that $\phi_k = x_k$ for all $k \in \mathbb{Z}_{[-h,0]}$ and $\phi_{k+1} \in F(\Phi_{[k-h,k]})$ for all $k \in \mathbb{Z}_+$.

Definition 2.3 *(i)* The origin of (2.1) is called *globally uniformly attractive* if for every $(r, \varepsilon) \in \mathbb{R}^2_{>0}$ there exists a $T(r, \varepsilon) \in \mathbb{Z}_{\geq 1}$ such that if $\|\mathbf{x}_{[-h,0]}\| \leq r$ then $\|\phi_k\| \leq \varepsilon$ for all $(\Phi, k) \in \mathcal{S}(\mathbf{x}_{[-h,0]}) \times \mathbb{Z}_{\geq T(r,\varepsilon)}$; *(ii)* the origin of (2.1) is called *Lyapunov stable (LS)* if for every $\varepsilon \in \mathbb{R}_{>0}$ there exists a $\delta(\varepsilon) \in \mathbb{R}_{>0}$ such that if $\|\mathbf{x}_{[-h,0]}\| \leq \delta(\varepsilon)$ then $\|\phi_k\| \leq \varepsilon$ for all $(\Phi, k) \in \mathcal{S}(\mathbf{x}_{[-h,0]}) \times \mathbb{Z}_+$; and *(iii)* the DDI (2.1) is called *globally asymptotically stable (GAS)* if its origin is globally uniformly attractive and LS. $\qquad\square$

Definition 2.4 *(i)* The DDI (2.1) is called $\mathcal{KL}$-*stable* if there exists a $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $\beta \in \mathcal{KL}$ and $\|\phi_k\| \leq \beta(\|\mathbf{x}_{[-h,0]}\|, k)$ for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $(\Phi, k) \in \mathcal{S}(\mathbf{x}_{[-h,0]}) \times \mathbb{Z}_+$; and *(ii)* the DDI (2.1) is called *globally exponentially stable* (GES) if it is $\mathcal{KL}$-stable with $\beta(r, s) = cr\mu^s$ for some $(c, \mu) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{[0,1)}$. $\qquad\square$

Note that the above notions of stability define global and strong properties, i.e., properties that hold for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $\Phi \in \mathcal{S}(\mathbf{x}_{[-h,0]})$. The following lemma relates DDIs that are GAS to DDIs that are $\mathcal{KL}$-stable.

Lemma 2.1 The following two statements are equivalent:

(i) The DDI (2.1) is GAS and $\delta(\varepsilon)$ in Definition 2.3 can be chosen to satisfy

$$\lim_{\varepsilon \to \infty} \delta(\varepsilon) = \infty;$$

(ii) the DDI (2.1) is $\mathcal{KL}$-stable. $\qquad\square$

The proof of Lemma 2.1 can be obtained *mutatis mutandis* from the proof of Lemma 4.5 in [74], a result for continuous-time systems without delay. An example of a system that is GAS but not $\mathcal{KL}$-stable can be found in the online appendix corresponding to the textbook [119]. The relevance of the result of Lemma 2.1 comes from the fact that $\mathcal{KL}$-stability, as opposed to mere GAS, is a standard assumption in converse Lyapunov theorems, see, e.g., [2,73,107]. Note that, if the map F corresponding to the DDI (2.1) is upper semicontinuous and the set $F(\mathbf{x}_{[-h,0]})$ is compact for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$, then it can be shown, similarly to Proposition 6 in [72], that GAS is equivalent to $\mathcal{KL}$-stability.

2.3 Stability analysis of delay difference inclusions

Next, a variety of stability analysis results, based on Lyapunov theory, for DDIs is presented.

2.3.1 The Krasovskii approach

As pointed out in Section 2.1, a standard approach for studying the stability of discrete-time systems with delay is to augment the state vector with all relevant delayed states/inputs, which yields a standard discrete-time system without delay whose stability can be studied via classical Lyapunov theory. Hence, let $\xi_k := \mathrm{col}(\{x_l\}_{l \in \mathbb{Z}_{[k-h,k]}})$ and consider the difference inclusion

$$\xi_{k+1} \in \bar{F}(\xi_k), \quad k \in \mathbb{Z}_+, \tag{2.2}$$

where the map $\bar{F} : \mathbb{R}^{(h+1)n} \rightrightarrows \mathbb{R}^{(h+1)n}$ is obtained from the map F in (2.1), i.e., $\bar{F}(\xi_0) = \mathrm{col}(\{x_l\}_{l \in \mathbb{Z}_{[-h+1,0]}}, F(\mathbf{x}_{[-h,0]}))$, with $\xi_0 := \mathrm{col}(\{x_l\}_{l \in \mathbb{Z}_{[-h,0]}})$. It should be noted that, by definition, $\bar{F}(\xi_0)$ is non-empty for all $\xi_0 \in \mathbb{R}^{(h+1)n}$ and that $\bar{F}(0) = \{0\}$. In what follows, $\bar{S}(\xi_0)$ is used to denote the set of all solutions to (2.2) from initial condition $\xi_0 \in \mathbb{R}^{(h+1)n}$. Furthermore, $\bar{\Phi} := \{\bar{\phi}_k\}_{k \in \mathbb{Z}_+} \in \bar{S}(\xi_0)$ denotes a solution to (2.2) such that $\bar{\phi}_0 = \xi_0$ and that $\bar{\phi}_{k+1} \in \bar{F}(\bar{\phi}_k)$ for all $k \in \mathbb{Z}_+$.

The following lemma relates stability of the DDI (2.1) to stability of the difference inclusion (2.2). Thus, stability of the set-valued map $F : (\mathbb{R}^n)^{h+1} \rightrightarrows \mathbb{R}^n$ is related to stability of the set-valued map $\bar{F} : \mathbb{R}^{(h+1)n} \rightrightarrows \mathbb{R}^{(h+1)n}$.

Lemma 2.2 The following claims are true:
  (i) The DDI (2.1) is GAS *if and only if* the difference inclusion (2.2) is GAS;
  (ii) the DDI (2.1) is $\mathcal{KL}$-stable *if and only if* the difference inclusion (2.2) is $\mathcal{KL}$-stable;
  (iii) the DDI (2.1) is GES *if and only if* the difference inclusion (2.2) is GES. $\qquad\square$

The proof of Lemma 2.2 can be found in Appendix B.1. In the standard approach, e.g., [22,26,38,42,46,59,61,78,100,132,146], an LF for the augmented system (2.2) is obtained. This LF is then used to conclude that the DDI (2.1) is $\mathcal{KL}$-stable. Lemma 2.2 allows us to formally establish that a LF for the augmented difference inclusion (2.2) implies that the DDI (2.1) is $\mathcal{KL}$-stable. More importantly, we also obtain the converse.

Theorem 2.1 The following three statements are equivalent:
  (i) There exist a $\bar{V} : \mathbb{R}^{(h+1)n} \to \mathbb{R}_+$, some $\bar{\alpha}_1, \bar{\alpha}_2 \in \mathcal{K}_\infty$ and a $\bar{\rho} \in \mathbb{R}_{[0,1)}$ such that

$$\bar{\alpha}_1(\|\xi_0\|) \le \bar{V}(\xi_0) \le \bar{\alpha}_2(\|\xi_0\|), \tag{2.3a}$$

$$\bar{V}(\xi_1) \le \bar{\rho}\bar{V}(\xi_0), \tag{2.3b}$$

for all $\xi_0 \in \mathbb{R}^{(h+1)n}$ and all $\xi_1 \in \bar{F}(\xi_0)$;
  (ii) the difference inclusion (2.2) is $\mathcal{KL}$-stable;
  (iii) the DDI (2.1) is $\mathcal{KL}$-stable. $\qquad\square$

Theorem 2.1 is proven in Appendix B.1. A function $\bar{V}$ that satisfies the hypothesis of statement *(i)* of Theorem 2.1 is called an LKF for the DDI (2.1). Theorem 2.1 recovers typical stability analysis results for DDEs that make use of the Krasovskii approach [131] as a particular case. Moreover, Theorem 2.1 also establishes the converse to these results.

Remark 2.2 For continuous-time systems the Krasovskii approach is based on the interpretation that solutions to the FDE evolve in an infinite-dimensional function space, on which Lyapunov's second method is then applied, see [109]. Hence, the Krasovskii approach relies on a functional that uses trajectory segments and is strictly decreasing along all trajectories of the FDE. Similarly, for the DDI (2.1) the Krasovskii approach is based on the interpretation that solutions to the DDI (2.1) evolve in the $(h + 1)n$-dimensional augmented space, such that $\bar{V}$ uses trajectory segments and is strictly decreasing along all trajectories of the DDI. Therefore, we refer to Theorem 2.1 as an application of the Krasovskii approach. $\quad\square$

Remark 2.3 For systems with external disturbances a result can be obtained that parallels Theorem 2.1 in terms of input-to-state stability (or even integral input-to-state stability). Such a result can be proven based on a parallel of Lemma 2.2 for input-to-state stability and Theorem 1 in [64]. This extension and the extension of other results in this thesis to systems with external disturbances are discussed in more detail in the conclusions of this thesis. $\square$

From Theorem 2.1 we obtain the following two corollaries.

Corollary 2.1 Suppose that the DDI (2.1) is a linear DDE and hence also that the corresponding augmented system (2.2) is a linear difference equation. Then, the following statements are equivalent:

(i) There exist a symmetric matrix $\bar{P} \in \mathbb{R}^{(h+1)n \times (h+1)n}$, some $(c_1, c_2) \in \mathbb{R}_{>0} \times \mathbb{R}_{\geq c_1}$ and a $\bar{\rho} \in \mathbb{R}_{[0,1)}$ such that the quadratic function $\bar{V}(\xi_0) := \xi_0^\top \bar{P} \xi_0$ satisfies

$$c_1 \|\xi_0\|_2^2 \leq \bar{V}(\xi_0) \leq c_2 \|\xi_0\|_2^2, \tag{2.4a}$$

$$\bar{V}(\xi_1) \leq \bar{\rho} \bar{V}(\xi_0), \tag{2.4b}$$

for all $\xi_0 \in \mathbb{R}^{(h+1)n}$ and all $\xi_1 \in \bar{F}(\xi_0)$;

(ii) the linear difference equation (2.2) is GES;

(iii) the linear DDE (2.1) is GES. $\square$

Corollary 2.2 Suppose that the DDI (2.1) is a linear DDI and hence also that the corresponding augmented system (2.2) is a linear difference inclusion. Then the following statements are equivalent:

(i) There exist a matrix $\bar{P} \in \mathbb{R}^{p \times (h+1)n}$, $p \in \mathbb{Z}_{\geq (h+1)n}$ some $(c_1, c_2) \in \mathbb{R}_{>0} \times \mathbb{R}_{\geq c_1}$ and a $\bar{\rho} \in \mathbb{R}_{[0,1)}$ such that the polyhedral function $\bar{V}(\xi_0) := \|\bar{P} \xi_0\|_\infty$ satisfies

$$c_1 \|\xi_0\|_\infty \leq \bar{V}(\xi_0) \leq c_2 \|\xi_0\|_\infty, \tag{2.5a}$$

$$\bar{V}(\xi_1) \leq \bar{\rho} \bar{V}(\xi_0), \tag{2.5b}$$

for all $\xi_0 \in \mathbb{R}^{(h+1)n}$ and all $\xi_1 \in \bar{F}(\xi_0)$;

(ii) the linear difference inclusion (2.2) is GES;

(iii) the linear DDI (2.1) is GES. $\square$

The proof of Corollary 2.1 follows from Corollary 3.1* in [68] and Lemma 2.2. Corollary 2.1 relies on the fact that, if the DDI (2.1) is a linear DDE, then the augmented system (2.2) is a linear difference equation, which admits a quadratic LF if and only if it is GES. Furthermore, the proof of Corollary 2.2 follows from the Corollary in [7], Part III and Lemma 2.2. Note that, the set $\mathcal{A}$ is closed and bounded by assumption but not necessarily convex, which is exactly what is required for the Corollary in [7], Part III. Corollary 2.2 relies on the fact that, if the DDI (2.1) is a linear DDI, then the augmented system (2.2) is a linear difference inclusion, which admits a polyhedral LF if and only if it is GES. A function $\bar{V}(\xi_0) := \xi_0^\top \bar{P} \xi_0$ that satisfies the hypothesis of statement *(i)* of Corollary 2.1 is called a

quadratic LKF. Furthermore, a function $\bar{V}(\xi_0) := \|\bar{P}\xi_0\|_\infty$ that satisfies the hypothesis of statement *(i)* of Corollary 2.2 is called a polyhedral LKF. The above classes of quadratic and polyhedral LKFs are explicitly considered here because both can be constructed efficiently via convex optimization algorithms, see Section 2.6 for further details.

The following example illustrates the results derived above.

Example 2.3 (Part I) Consider the linear DDE

$$x_{k+1} = bx_{k-1} + ax_k, \quad k \in \mathbb{Z}_+, \tag{2.6}$$

where $x_k \in \mathbb{R}$ and $(a, b) \in \mathbb{R}^2$. Note that the linear DDE (2.6) corresponds to the discrete-time model of the water temperature regulation problem in the shower with $T_s = h$ and $c\kappa T_s \in \mathbb{R}_{(0,1)}$, as discussed in Section 1.3.1. Let $\xi_k := \text{col}(x_{k-1}, x_k)$, which yields

$$\xi_{k+1} = \bar{A}\xi_k, \quad k \in \mathbb{Z}_+, \tag{2.7}$$

where $\bar{A} = \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix}$. For all $b \in \mathbb{R}$ with $|b| < 1$ and all $a \in \mathbb{R}$ with $|a| < 1 - b$, the spectral radius of $\bar{A}$ is strictly less than one and hence (2.7) is GES, see, e.g., [68]. Therefore, it follows from Corollary 2.1 that, if $(a, b) \in \mathbb{R}^2$ with $|b| < 1$ and $|a| < 1 - b$, then there exists a symmetric matrix $\bar{P} \in \mathbb{R}^{2 \times 2}$ such that

$$\bar{A}^\top \bar{P} \bar{A} - \bar{\rho}\bar{P} \prec 0, \quad \bar{P} \succ 0, \tag{2.8}$$

for some $\bar{\rho} \in \mathbb{R}_{[0,1)}$. Indeed, Corollary 2.1 implies that if $(a, b) \in \mathbb{R}^2$ with $|b| < 1$ and $|a| < 1 - b$ and hence (2.6) is GES, then (2.6) admits a quadratic LKF, which in turn is equivalent to (2.8). For example, let $a := 1$ and $b := -0.5$. As $\bar{P} := \begin{bmatrix} 0.7 & -0.5 \\ -0.5 & 1.3 \end{bmatrix}$ is a solution to (2.8) for $\bar{\rho} = 0.95$, system (2.7), with $a = 1$ and $b = -0.5$, is GES. Hence, the linear DDE (2.6), with $a = 1$ and $b = -0.5$, is GES. Moreover, the function $\bar{V}(\xi_0) := \xi_0^\top \bar{P}\xi_0$ is a quadratic LF for (2.7) and the same function $\bar{V}(\xi_0) = \bar{V}(\mathbf{x}_{[-1,0]}) = 0.7x_{-1}^2 - x_0 x_{-1} + 1.3x_0^2$ is a quadratic LKF for (2.6). □

To verify stability of the DDI (2.1) via the Krasovskii approach a function has to be constructed that satisfies a set of conditions, e.g., those presented in Theorem 2.1. Unfortunately, this function makes use of trajectory segments of length $h + 1$, and hence its construction becomes increasingly complex when the size of the delay h increases. Therefore, it would be desirable to construct a set of conditions that require the construction of a function that makes use of the system state only, i.e., involving the original DDI directly, rather than the augmented system.

2.3.2 The Razumikhin approach

The Razumikhin approach is a Lyapunov technique for systems with delay that imposes conditions that directly involve the DDI (2.1), as opposed to the augmented system (2.2).

Theorem 2.2 Suppose that there exist a $V : \mathbb{R}^n \to \mathbb{R}_+$ and some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(\|x_0\|) \leq V(x_0) \leq \alpha_2(\|x_0\|), \tag{2.9a}$$

for all $x_0 \in \mathbb{R}^n$ and that there exist a $\rho \in \mathbb{R}_{[0,1)}$ and some $\pi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\pi(r) > r$ for all $r \in \mathbb{R}_{>0}$, $\pi(0) = 0$ and that if

$$\pi(V(x_1)) \geq \max_{i \in \mathbb{Z}_{[-h,0]}} V(x_i) \ \text{ then } \ V(x_1) \leq \rho V(x_0), \tag{2.9b}$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $x_1 \in F(\mathbf{x}_{[-h,0]})$. Then, the DDI (2.1) is $\mathcal{KL}$-stable. $\quad\square$

The proof of the above theorem, which is omitted here for brevity, is similar in nature to the proof of Theorem 6 in [37] and relies on the fact that the conditions in Theorem 2.2 imply that V is decreasing with respect to the maximum over its values for the delayed states. The conditions in Theorem 2.2 are impractical as (2.9b) imposes a condition on $V(x_1)$ if $V(x_1)$ satisfies some other condition. As a consequence, a technique to construct the functions V and π satisfying the hypothesis of Theorem 2.2 is missing. Note that the corresponding Razumikhin theorem for continuous-time systems, e.g., Theorem 4.1 in [51], is more practical because it imposes a condition on *the derivative of $V(x)$* if $V(x)$ satisfies a certain condition. The following result is a particular case of Theorem 2.2 (using $\pi(r) := \rho^{-1}r$), but provides verifiable sufficient conditions for stability of the DDI (2.1).

Theorem 2.3 If there exist a $V : \mathbb{R}^n \to \mathbb{R}_+$, some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a $\rho \in \mathbb{R}_{[0,1)}$ such that

$$\alpha_1(\|x_0\|) \leq V(x_0) \leq \alpha_2(\|x_0\|), \tag{2.10a}$$
$$V(x_1) \leq \rho \max_{i \in \mathbb{Z}_{[-h,0]}} V(x_i), \tag{2.10b}$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $x_1 \in F(\mathbf{x}_{[-h,0]})$, then the DDI (2.1) is $\mathcal{KL}$-stable. $\quad\square$

Theorem 2.3 is proven in Appendix B.1 and, when specialized to DDEs, is similar to Theorems 3.2 and 4.2 in [93]. This, and the same observation for Theorem 2.1 above, indicates that to consider DDIs is a generalization that does not compromise the sharpness of the results when only DDEs are of interest. A function that satisfies the hypothesis of Theorem 2.2 is typically called a *backward* LRF and one that satisfies the hypothesis of Theorem 2.3 a *forward* LRF. For brevity, we will omit the adjective 'forward' in what follows.

Remark 2.4 For continuous-time systems the Razumikhin approach is based on the interpretation that solutions to the FDE evolve in a Euclidean space, on which a variation of Lyapunov's second method is applied, see [109]. Thus, the Razumikhin approach relies on a function that uses the state of the system at a single time instant and is only decreasing if the trajectory of the FDE satisfies a specific condition. Similarly, for the DDI (2.1) the Razumikhin approach is based on the interpretation that solutions to the DDI (2.1) evolve in $\mathbb{R}^n$, such that the function V uses the state of the DDI (2.1) at a single time instant and is only decreasing if the trajectory of the DDI satisfies a specific condition. Therefore, we refer to Theorems 2.2 and 2.3 as applications of the Razumikhin approach. $\quad\square$

An inherent consequence of the interpretations in Remarks 2.2 and 2.4 is that, for both FDEs and DDIs, the Krasovskii approach provides necessary and sufficient conditions for stability

while the Razumikhin approach is relatively simple to apply, e.g., constructing a function on $\mathbb{R}^n$ is simpler than one on $(\mathbb{R}^n)^{h+1}$.

The following corollary follows directly from (B.9).

Corollary 2.3 If there exists a function $V : \mathbb{R}^n \to \mathbb{R}_+$ that satisfies the hypothesis of Theorem 2.3 with $\alpha_1(r) = c_1 r^\lambda$ and $\alpha_2(r) = c_2 r^\lambda$ for some $(c_1, c_2, \lambda) \in \mathbb{R}_{>0} \times \mathbb{R}_{\geq c_1} \times \mathbb{Z}_{>0}$ and some $\rho \in \mathbb{R}_{[0,1)}$, then the DDI (2.1) is GES. $\qquad\square$

Next, we will use Example 2.3 to prove that the converse of Theorems 2.2 and 2.3 are not true in general.

Proposition 2.1 Consider the linear DDE (2.6) and suppose that $b \in \mathbb{R}_{(-1,0)}$ and $a = 1$. Then, the following claims are true:

(i) The linear DDE (2.6) with $b \in \mathbb{R}_{(-1,0)}$ and $a = 1$ is GES;

(ii) the linear DDE (2.6) with $b \in \mathbb{R}_{(-1,0)}$ and $a = 1$ does not admit a backward LRF;

(iii) the linear DDE (2.6) with $b \in \mathbb{R}_{(-1,0)}$ and $a = 1$ does not admit an LRF. $\qquad\square$

Proposition 2.1 is proven in Appendix B.1. Currently, it is unclear how to reformulate the conditions in Theorem 2.2 into an optimization problem that can be used to obtain a backward LRF. The conditions in Theorem 2.3, on the other hand, can be reformulated as a SDP problem whose solution yields an LRF, as it will be shown in Section 2.6. Furthermore, it follows from Proposition 2.1 that both results suffer from a similar conservatism. Therefore, in what follows, we will focus on LRFs and disregard backward LRFs. We refer to [93] for a detailed discussion on LRFs, backward LRFs and their differences.

Most Razumikhin theorems for FDEs provide delay-independent conditions that imply not only GAS or GES but a stronger type of stability that is independent of the delay, see, e.g., [109] (delay-dependent results can be obtained after a coordinate transformation only). Similarly, the Razumikhin conditions of Theorem 2.3 also imply a type of stability that is independent of the delay, see [45, 95] for details. This is, apart from a reduced complexity, another advantage of the Razumikhin approach over the Krasovskii approach.

2.4 Relations between the Krasovskii and Razumikhin approaches

For FDEs, i.e., continuous-time systems with delay, it was shown in [77], Section 4.8, that LRFs form a particular case of LKFs, when only Lyapunov stability (see Definition 2.3) rather than $\mathcal{KL}$-stability is of concern. A similar reasoning as the one used in [77] can be applied to DDIs as well. Suppose that the function V satisfies the hypothesis of Theorem 2.3 *with $\rho = 1$*. Then, it can be easily verified that

$$\bar{V}(\xi_0) := \max_{i \in \mathbb{Z}_{[-h,0]}} V(x_i),$$

satisfies the hypothesis of Theorem 2.1 *with $\bar{\rho} = 1$*. Thus, it follows from (2.3b) that

$$\bar{V}(\bar{\phi}_k) \leq \bar{V}(\xi_0), \quad \forall \xi_0 \in \mathbb{R}^{(h+1)n}, \ \forall(\bar{\Phi}, k) \in \bar{\mathcal{S}}(\mathbf{x}_{[-h,0]}) \times \mathbb{Z}_+.$$

From this fact one can show, using (2.3a), that the DDI (2.1) is LS. However, the same candidate LKF does not satisfy the assumptions of Theorem 2.1 for $\bar{\rho} \in \mathbb{R}_{[0,1)}$, i.e., when

$\mathcal{KL}$-stability is imposed. Interestingly, in [75] an example was provided where the above result was generalized to $\bar{\rho} \in \mathbb{R}_{[0,1)}$ for quadratic functions and continuous-time systems. Next, we show how to construct an LKF from an LRF for $\bar{\rho} \in \mathbb{R}_{[0,1)}$.

Theorem 2.4 Suppose that $V : \mathbb{R}^n \to \mathbb{R}_+$ satisfies the hypothesis of Theorem 2.3. Then,

$$\bar{V}(\xi_0) := \max_{i \in \mathbb{Z}_{[-h,0]}} \rho_{h+1+i} V(x_i), \tag{2.11}$$

where $\rho_i := \frac{\rho+i}{i+1}$, $i \in \mathbb{Z}_{[1,h]}$ and $\rho_{h+1} := 1$, satisfies the hypothesis of Theorem 2.1. □

Theorem 2.4 is proven in Appendix B.1. An alternative construction of an LKF from an LRF is $\bar{V}(\xi_0) := \max_{i \in \mathbb{Z}_{[-h,0]}} \hat{\rho}^{-i} V(x_i)$, where $\hat{\rho} := \rho^{\frac{1}{h+1}}$.

Next, we establish under which additional assumptions the converse is true, i.e., when the existence of an LKF implies the existence of an LRF.

Proposition 2.2 Suppose that the function $\bar{V} : \mathbb{R}^{(h+1)n} \to \mathbb{R}_+$ satisfies the hypothesis of Theorem 2.1. Furthermore, suppose that there exist a function $V : \mathbb{R}^n \to \mathbb{R}_+$, some $\alpha_3, \alpha_4 \in \mathcal{K}_\infty$ and a $\rho \in \mathbb{R}_{[0,1)}$ such that (2.10a) holds, $\alpha_3(r) \leq \alpha_4(r)$ and $\alpha_3(\rho r) \geq \bar{\rho}\alpha_4(r)$ for all $r \in \mathbb{R}_+$ and that

$$\sum_{i=-h}^{0} \alpha_3(V(x_i)) \leq \bar{V}(\xi_0) \leq \sum_{i=-h}^{0} \alpha_4(V(x_i)), \tag{2.12}$$

for all $\xi_0 \in \mathbb{R}^{(h+1)n}$. Then, V satisfies the hypothesis of Theorem 2.3. □

The following corollary is a slight modification of Proposition 2.2.

Corollary 2.4 Suppose that the hypothesis of Proposition 2.2 holds with (2.12) replaced by

$$\max_{i \in \mathbb{Z}_{[-h,0]}} \alpha_3(V(x_i)) \leq \bar{V}(\xi_0) \leq \max_{i \in \mathbb{Z}_{[-h,0]}} \alpha_4(V(x_i)). \tag{2.13}$$

Then, V satisfies the hypothesis of Theorem 2.3. □

Proposition 2.2 and Corollary 2.4 are both proven in Appendix B.1.

The hypothesis and conclusion of Theorem 2.4, Proposition 2.2 and Corollary 2.4 might not seem very intuitive. However, when quadratic or polyhedral functions are considered, these results do provide valuable insights. For example, suppose that $V(x_0) := \|Px_0\|_\infty$ is a polyhedral LRF. Then, it follows from Theorem 2.4 that

$$\bar{V}(\xi_0) := \max_{i \in \mathbb{Z}_{[-h,0]}} \rho_{h+1+i} \|Px_i\|_\infty = \left\| \begin{bmatrix} \rho_1 P & & 0 \\ & \ddots & \\ 0 & & \rho_{h+1}P \end{bmatrix} \xi_0 \right\|_\infty, \tag{2.14}$$

is a polyhedral LKF. Conversely, suppose that the function (2.14) is a polyhedral LKF for some $\bar{\rho} \in \mathbb{R}_{[0,1)}$ such that $\bar{\rho} < \rho_1$. Then, it follows from Corollary 2.4, i.e., by taking $\alpha_3(r) := \rho_1 r$ and $\alpha_4(r) := r$, that $V(x_0) := \|Px_0\|_\infty$ is a polyhedral LRF. In fact, it can

even be shown that the existence of a polyhedral LRF is equivalent to the existence of a polyhedral LKF *that satisfies the hypothesis of Corollary 2.4*.

In contrast, given a quadratic LRF, Theorem 2.4 does not yield a quadratic LKF but rather a more complex LKF, i.e., the maximum over a set of quadratic functions. On the other hand, Proposition 2.2 can provide a quadratic LRF constructed from a quadratic LKF. Indeed, consider the quadratic LKF

$$\bar{V}(\xi_0) := \sum_{i=-h}^{0} x_i^\top P x_i = \xi_0^\top \begin{bmatrix} P & & 0 \\ & \ddots & \\ 0 & & P \end{bmatrix} \xi_0,$$

then it follows from Proposition 2.2 that $V(x_0) := x_0^\top P x_0$ is a quadratic LRF.

Figure 2.2 presents a schematic overview of all results derived in Sections 2.2-2.4 for DDIs that are GAS. Similarly, Figure 2.3 presents a schematic overview of the results derived in Sections 2.2-2.4 for DDIs that are GES. Interestingly, it is unclear if the existence of
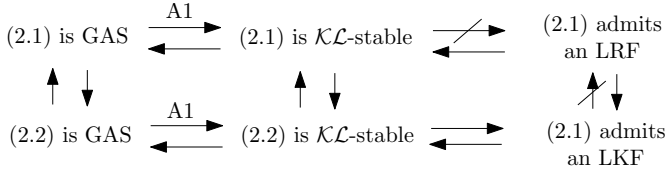
Figure 2.2: A schematic overview of the relations established in the Sections 2.2-2.4 for DDIs that are GAS. $B \rightarrow C$ means that B implies C, $B \nrightarrow C$ means that B does not necessarily imply C and $B \xrightarrow{A1} C$ means that B implies C under the additional assumption that $\delta(\varepsilon)$ in Definition 2.3 can be chosen to satisfy $\lim_{\varepsilon \to \infty} \delta(\varepsilon) = \infty$.
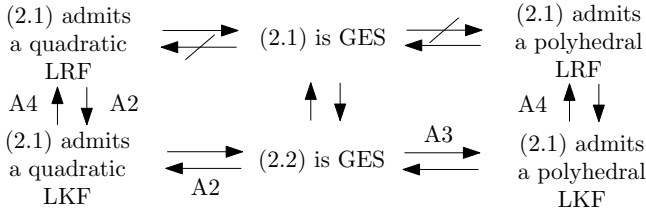
Figure 2.3: A schematic overview of the relations established in the Sections 2.2-2.4 for DDIs that are GES. $B \rightarrow C$ means that B implies C, $B \nrightarrow C$ means that B does not necessarily imply C and $B \xrightarrow{A} C$ means that B implies C under the additional assumption A. The employed assumptions are as follows: (A2) – the DDI (2.1) is a linear DDE; (A3) – the DDI (2.1) is a linear DDI; (A4) – the LKF has certain structural properties.

a quadratic LRF implies the existence of a quadratic LKF. The existence of an LRF and the existence of a polyhedral LRF, on the other hand, have been shown to imply the existence of an LKF and polyhedral LKF, respectively.

2.5 Contractive sets and delay difference inclusions

Invariant and contractive sets are at the basis of virtually every control technique for constrained systems, see, e.g., [14, 17, 54]. In this context, as the sublevel sets of an LF are contractive sets, Lyapunov theory is frequently used to study the existence of such sets. Indeed, as a dynamical system is $\mathcal{KL}$-stable if and only if it admits an LF, it follows that any stable system admits a nontrivial contractive set. Moreover, under suitable assumptions the converse is also true [7, 17, 105].

In what follows, we discuss the existence of contractive and invariant sets for DDIs and derive their relation to LKFs and LRFs. Therefore, consider the following definitions.

Definition 2.5 *(i)* A set $\bar{\mathbb{X}} \subset (\mathbb{R}^n)^{h+1}$ is called λ-*contractive*, $\lambda \in \mathbb{R}_{[0,1)}$ for the DDI (2.1) if $\mathbf{x}_{[-h+1,1]} \in \lambda \bar{\mathbb{X}}$ for all $\mathbf{x}_{[-h,0]} \in \bar{\mathbb{X}}$ and all $x_1 \in F(\mathbf{x}_{[-h,0]})$; and *(ii)* a set that is λ-contractive with $\lambda = 1$ is called *invariant*. □

Definition 2.6 *(i)* A set $\mathbb{X} \subset \mathbb{R}^n$ is called λ-$\mathcal{D}$-*contractive*, $\lambda \in \mathbb{R}_{[0,1)}$ for the DDI (2.1) if $F(\mathbf{x}_{[-h,0]}) \subseteq \lambda \mathbb{X}$ for all $\mathbf{x}_{[-h,0]} \in \mathbb{X}^{h+1}$; and *(ii)* a set that is λ-contractive with $\lambda = 1$ is called $\mathcal{D}$-*invariant*. □

Throughout this thesis, our use of the term invariant is a typographical simplification of the classical term positively invariant; hence, no confusion should arise. Furthermore, like the stability notions in Definitions 2.3 and 2.4, the above definitions define strong properties, i.e., properties that hold for all $\Phi \in \mathcal{S}(\mathbf{x}_{[-h,0]})$ as opposed to for a single solution $\Phi \in \mathcal{S}(\mathbf{x}_{[-h,0]})$. The terminology $\mathcal{D}$-invariance stems from delay-invariance and refers to the fact that the Razumikhin approach (which is closely related to $\mathcal{D}$-invariance) implies a type of delay independent stability, see, e.g., [95] and the discussion at the end of Section 2.3.2.

In the following result we relate the existence of a contractive set to the existence of an LKF. The result can be proven using the sublevel sets of the LKF.

Proposition 2.3 Suppose that the DDI (2.1) is $\mathcal{D}$-homogeneous[2] of order 1. The following two statements are equivalent:

   (i) The DDI (2.1) admits a convex LKF;

   (ii) the DDI (2.1) admits a proper C-set that is λ-contractive, for some $\lambda \in \mathbb{R}_{[0,1)}$. □

The proof of Proposition 2.3 can be obtained, using the augmented system, from the results derived in [16, 17, 105]. It should be noted that the most common LF candidates, such as quadratic and norm-based functions, are inherently convex. Moreover, continuity (which is a consequence of convexity) is a desirable property as continuous LFs guarantee that the corresponding type of stability does not have zero robustness, see, e.g., [88].

Unfortunately, it is unclear what a contractive set $\bar{\mathbb{X}} \subset (\mathbb{R}^n)^{h+1}$ implies for the DDI (2.1) and for the trajectories $\Phi \in \mathcal{S}(\mathbf{x}_{[-h,0]})$ in the original state space $\mathbb{R}^n$, in particular. The above observation indicates a drawback of LKFs. While homogeneous DDIs admit an LKF if and only if the system is $\mathcal{KL}$-stable, the sublevel sets of an LKF do not provide

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[2]For example, linear DDIs are $\mathcal{D}$-homogeneous of order 1.

a contractive set in the original state space, i.e., $\mathbb{R}^n$, but rather a contractive set in the higher dimensional state space corresponding to the augmented system, i.e., $\mathbb{R}^{(h+1)n}$ or equivalently $(\mathbb{R}^n)^{h+1}$. In contrast, an LRF is based on particular Lyapunov conditions that involve the non-augmented system, rather than the augmented one. As such, it is reasonable to expect that such a function is equivalent to the existence of a type of contractive set in $\mathbb{R}^n$. This expectation is confirmed by the following result.

Proposition 2.4 Suppose that the DDI (2.1) is $\mathcal{D}$-homogeneous of order 1. The following two statements are equivalent:

(i) The DDI (2.1) admits a convex LRF;

(ii) the DDI (2.1) admits a proper C-set that is λ-$\mathcal{D}$-contractive, for some $\lambda \in \mathbb{R}_{[0,1)}$. $\square$

Proposition 2.4 is proven in Appendix B.1. Note that the assumptions under which we have proven the equivalence of the statements in Propositions 2.3 and 2.4, i.e., regarding the properties of the contractive sets, the corresponding functions and the system under study, are standard assumptions for the type of results derived in this section, see, e.g., [16, 17, 105]. Furthermore, Proposition 2.4 essentially recovers Proposition 2.3 and similar results in [16, 17, 105] as a particular case, i.e., for $h = 0$.

Let us analyse some of the implications of the results that have been derived so far. Suppose that the DDI (2.1) is $\mathcal{D}$-homogeneous of order 1 and admits a set $\mathbb{V} \subset \mathbb{R}^n$ that is λ-$\mathcal{D}$-contractive. Then, it follows from Proposition 2.4 that the DDI (2.1) admits an LRF. Moreover, it follows from Theorem 2.4 that the DDI (2.1) admits an LKF which in turn, via Proposition 2.3, guarantees the existence of a λ-contractive set for the DDI (2.1). Under some additional assumptions a similar reasoning can be used to obtain a λ-$\mathcal{D}$-contractive set from a λ-contractive set.

2.6 Stabilizing controller synthesis for linear systems

Next, the applicability of the developed results is illustrated via some basic stability analysis and controller synthesis methods. To this end, controlled DDIs are considered, i.e.,

$$x_{k+1} \in f(\mathbf{x}_{[k-h,k]}, \mathbf{u}_{[k-h,k]}), \quad k \in \mathbb{Z}_+, \tag{2.15}$$

where $u_k \in \mathbb{R}^m$ is a control input and $f : (\mathbb{R}^n)^{h+1} \times (\mathbb{R}^m)^{h+1} \rightrightarrows \mathbb{R}^n$.

Throughout this thesis, to obtain problems that can be solved via SDP, we will focus on specific subclasses of the controlled DDI (2.15).

Definition 2.7 *(i)* The controlled DDI (2.15) is called a *linear controlled DDE* if the map $f(\mathbf{x}_{[-h,0]}, \mathbf{u}_{[-h,0]}) = \{\sum_{i=-h}^{0}(A_i x_i + B_i u_i)\}$ for some $(\{A_i, B_i\}_{i \in \mathbb{Z}_{[-h,0]}}) \in (\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m})^{h+1}$; and *(ii)* the controlled DDI (2.15) is called a *linear controlled DDI* if the map $f(\mathbf{x}_{[-h,0]}, \mathbf{u}_{[-h,0]}) = \{\sum_{i=-h}^{0}(A_i x_i + B_i u_i) : (\{A_i, B_i\}_{i \in \mathbb{Z}_{[-h,0]}}) \in \mathcal{AB}\}$ for some compact and non-empty set $\mathcal{AB} \subset (\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m})^{h+1}$. $\square$

Remark 2.5 Linear controlled DDIs form a rather general modeling class that can model, for example, certain types of NCS, e.g., those in Example 2.2 and [46, 150]. $\square$

To obtain stability analysis and stabilizing controller synthesis problems that can be solved via SDP, quadratic LKFs and LRFs are considered in what follows. However, the results derived in the preceding sections are not restricted to a particular type of function. In fact, since the augmented system (2.2) is a standard difference inclusion, synthesis techniques for polyhedral LFs [17, 86], composite LFs [60] and polynomial LFs [112] can be applied directly to obtain an LKF of a corresponding type.

In what follows we will search for a control law of the form

$$u_k = Kx_k, \quad k \in \mathbb{Z}_+, \tag{2.16}$$

where $K \in \mathbb{R}^{m \times n}$. First, a result for stabilizing controller synthesis that is based on the existence of a quadratic LKF is presented. Therefore, let $\tilde{A}_i := A_i G + B_i Y$, $i \in \mathbb{Z}_{[-h,0]}$.

Proposition 2.5 Choose some $\bar{\rho} \in \mathbb{R}_{[0,1)}$. Suppose that the controlled DDI (2.15) is a linear controlled DDI. Furthermore, suppose that there exists a set of matrices $(\bar{P}, G, Y) \in \mathbb{R}^{(h+1)n \times (h+1)n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{m \times n}$ such that $\bar{P} = \bar{P}^\top \succ 0$ and that

$$\begin{bmatrix} \bar{\rho}\bar{P} & * \\ \begin{bmatrix} 0 & G & & 0 \\ & & \ddots & \\ 0 & 0 & & G \\ \tilde{A}_{-h} & \tilde{A}_{-h+1} & \dots & \tilde{A}_0 \end{bmatrix}^\top & \begin{bmatrix} G & & 0 \\ & \ddots & \\ 0 & & G \end{bmatrix} + \begin{bmatrix} G & & 0 \\ & \ddots & \\ 0 & & G \end{bmatrix}^\top - \bar{P} \end{bmatrix} \succeq 0,$$

for all $(\{A_i, B_i\}_{i \in \mathbb{Z}_{[-h,0]}}) \in \mathcal{AB}$. Then, the linear controlled DDI (2.15) in closed loop with the control law (2.16), where $K := YG^{-1}$, is GES. □

Proposition 2.6 is proven in Appendix B.1. Note that the matrix inequality in Proposition 2.5 is linear in the variables $\bar{P}$, G and Y and hence it is a linear matrix inequality (LMI). Therefore, if the set $\mathcal{AB}$ is a matrix polytope, then it suffices to verify the LMI for the vertices of the set $\mathcal{AB}$ and hence the conditions in Proposition 2.5 can be verified by solving an LMI of finite dimensions. Furthermore, if the set $\mathcal{AB}$ consists of a finite number of points, such as it is the case in Example 2.1, then the conditions in Proposition 2.5 can also be verified by solving an LMI of finite dimensions.

Next, a result for stabilizing controller synthesis that is based on the existence of a quadratic LRF is presented.

Proposition 2.6 Choose some $\rho \in \mathbb{R}_{[0,1)}$. Suppose that the controlled DDI (2.15) is a linear controlled DDI. Furthermore, suppose that there exist variables $(\{\delta_i\}_{i \in \mathbb{Z}_{[-h,0]}}, Z, Y) \in \mathbb{R}_{[0,1]}^{h+1} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$ such that $Z = Z^\top \succ 0$, $\sum_{i=-h}^{0} \delta_i \leq 1$ and that

$$\begin{bmatrix} \rho\delta_{-h}Z & & 0 & * \\ & \ddots & & \\ 0 & & \rho\delta_0 Z & * \\ A_{-h}Z + B_{-h}Y & \dots & A_0 Z + B_0 Y & Z \end{bmatrix} \succeq 0, \tag{2.17}$$

for all $(\{A_i, B_i\}_{i \in \mathbb{Z}_{[-h,0]}}) \in \mathcal{AB}$. Then, the linear controlled DDI (2.15) in closed loop with the control law (2.16), where $K := YZ^{-1}$, is GES. $\square$

Proposition 2.6 is proven in Appendix B.1. The matrix inequality (2.17) is bilinear in the scalars δ_i and the matrix Z. The set $\mathbb{R}^{h+1}_{[0,1]}$, where the scalar variables $\{\delta_i\}_{i \in \mathbb{Z}_{[-h,0]}}$ are allowed to take values, can be discretized using a gridding technique. Then, solving (2.17) for each point in the resulting grid amounts to solving an LMI. Thus, a feasible solution to (2.17) can be obtained by solving a sequence of LMIs. Observe that if Z, Y and $\{\delta_i\}_{i \in \mathbb{Z}_{[-h,0]}}$ satisfy (2.17) with $\sum_{i=-h}^{0} \delta_i < 1$, then there exist $\{\hat{\delta}_i\}_{i \in \mathbb{Z}_{[-h,0]}}$ such that $\sum_{i=-h}^{0} \hat{\delta}_i = 1$ and that Z, Y and $\{\hat{\delta}_i\}_{i \in \mathbb{Z}_{[-h,0]}}$ also satisfy (2.17). Therefore, it suffices to consider only those points in the grid such that $\sum_{i=-h}^{0} \delta_i = 1$.

Remark 2.6 Clearly, Propositions 2.5 and 2.6 can also be used for the stability analysis of a linear DDI of the form (2.1), i.e., by setting $B_i = 0$ for all $i \in \mathbb{Z}_{[-h,0]}$ in both results. Note that even in this case the matrix inequality (2.17) remains bilinear as the variables $\{\delta_i\}_{i \in \mathbb{Z}_{[-h,0]}}$ that multiply the matrix Z are related to the maximum in Theorem 2.3 and not due to the fact that controller synthesis is of concern in Proposition 2.6. $\square$

To illustrate the application of Propositions 2.5 and 2.6 we consider Example 2.2 again.

Example 2.2 (Part II) Let us consider again the DC-motor controlled over a communication network that was discussed in Example 2.2, Part I. Note that as the load torque is assumed to be constant, it is possible to remove it via a linear state transformation. Then, it follows from a direct inspection of the eigenvalues of the system that the DC-motor is open-loop stable. However, to guarantee a faster convergence, a control law will be designed that controls the DC-motor over the communication network. Unfortunately, due to the fact that the set $\{\Delta(\tau) : \tau \in \mathbb{R}_{[0,\bar{\tau}]}\}$ is not polytopic, the conditions derived in Propositions 2.5 and 2.6 do not lead to an LMI of finite dimensions when applied to the model derived in Example 2.2, Part I directly. Therefore, a polytopic over-approximation of the uncertain time-varying matrix $\Delta(\tau)$ is computed using the Cayley-Hamilton technique presented in [46]. Numerical results for various values of $\bar{\tau}$ can be found in Appendix A.

The sampling time is chosen to be $T_s = 0.01$s. Firstly, taking $\tau_k \in \mathbb{R}_{[0,0.424T_s]}$, Proposition 2.6 with $\rho = 0.8$ yields[3] the quadratic LRF matrix and corresponding controller matrix

$$P_{\mathrm{LRF}} = \begin{bmatrix} 7.01 & 0.51 \\ 0.51 & 0.04 \end{bmatrix}, \quad K_{\mathrm{LRF}} = \begin{bmatrix} -10.96 & -0.80 \end{bmatrix},$$

along with $\delta_0 = 0.75$ and $\delta_{-1} = 0.25$. However, for $\bar{\tau} > 0.424T_s$ Proposition 2.6 no longer provides a feasible solution. It is worth to point out that for $\delta_0 = \delta_{-1} = 0.5$ and $\bar{\tau} > 0.35T_s$ no stabilizing controller is obtained via Proposition 2.6. This indicates the additional freedom provided by the introduction of $\{\delta_i\}_{i \in \mathbb{Z}_{[-h,0]}}$ as free variables. Secondly,

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[3]Numerical results were obtained using the Multiparametric Toolbox v.2.6.2 and SeDuMi v.1.1.

for $\tau_k \in \mathbb{R}_{[0,0.48T_s]}$ and using Proposition 2.5 with $\bar{\rho} = 0.8$ yields the quadratic LKF matrix and corresponding controller matrix

$$P_{\text{LKF}} = \begin{bmatrix} 51.74 & 6.98 & 17.26 & 2.35 \\ 6.98 & 1.15 & 1.88 & 0.30 \\ 17.22 & 1.88 & 25.17 & 3.35 \\ 2.35 & 0.30 & 3.35 & 0.5249 \end{bmatrix}, \quad K_{\text{LKF}} = \begin{bmatrix} -14.95 & -1.72 \end{bmatrix}.$$

However, for $\bar{\tau} > 0.48T_s$ Proposition 2.5 no longer provides a feasible solution. Therefore, it can be concluded that for this example the Krasovskii approach is able to find a stabilizing controller, i.e., via Proposition 2.5, for a larger range of time-varying delays when compared to the Razumikhin approach, i.e., via Proposition 2.6. □

The observations in Example 2.2, Part II confirm the results that were derived in this chapter. Obviously, this does not discard the Razumikhin approach as a valuable technique as the Razumikhin approach has, among others, a smaller computational complexity when compared to the Krasovskii approach.

2.7 Conclusions

In this chapter DDIs were introduced and explained to be a rich modeling class that can provide models to analyse the properties of most types of sampled-data systems and NCS. Moreover, at the same time it was shown that DDIs allow for the derivation of results that, when specialized to a specific subclass of DDIs, are equivalent to the results that can be derived for that subclass. As such DDIs form a generalization of discrete-time systems with delay that does not compromise the sharpness of the derived results. Then, motivated by the fact that a comprehensive overview of stability analysis methods for discrete-time systems with delay based on Lyapunov theory is missing, the standard stability analysis results for this class of systems were discussed and extended to DDIs. Moreover, for the first time, a converse theorem for the Krasovskii approach was proven. Also, the relation between the Razumikhin and the Krasovskii approach and their relation to certain types of set invariance properties was derived. These results were complemented by the corresponding stability analysis and stabilizing controller synthesis methods for linear DDIs.

An important observation in this chapter is that the Razumikhin approach makes use of conditions that involve the system state only while the Krasovskii approach makes use of conditions involving trajectory segments of a length determined by the size of the delay. As a consequence, only the Razumikhin approach yields computationally attractive conditions that provide information about the system trajectories directly, but the Krasovskii approach provides necessary and sufficient conditions for stability. Hence, the standard stability analysis results that were discussed in this chapter each have their own disadvantages, which may prevent their use in some cases. Motivated by this fact, the following two chapters are aimed at understanding the reason for these disadvantages and, where possible, removing them. Therefore, in Chapter 3 we make use of a relation of DDIs to interconnected systems to obtain a deeper understanding of the Razumikhin approach. Moreover, we also obtain a novel Razumikhin-type stability analysis technique, which is shown to be less conservative than the standard Razumikhin approach.

Chapter 3

The relation of delay difference inclusions to interconnected systems

In this chapter we explore the link between DDIs and interconnected systems to obtain a better understanding of DDIs. More specifically, by considering each delayed state as a subsystem, the behavior of a DDI can be described by an interconnected system with a particular structure. Thus, we show that the Razumikhin approach is a direct application of the small-gain theorem for interconnected systems. Moreover, we obtain an alternative set of Razumikhin-type conditions for stability that make use of dissipativity theory for interconnected systems, i.e., via the selection of storage and supply functions, stability can be established for DDIs. These conditions are shown to provide a trade-off between the conceptual generality of the Krasovskii approach and the computational convenience of the Razumikhin approach. Moreover, for linear DDIs they can be verified by solving an LMI, as opposed to a BMI (see Proposition 2.6) for the Razumikhin approach.

3.1 Introduction

In Chapter 2 the Krasovskii and Razumikhin approaches for DDIs have been discussed. It was shown that the Krasovskii approach provides a set of necessary and sufficient conditions for stability. Unfortunately, these conditions involve trajectory segments and as such do not provide information about the DDI trajectories directly, which causes them to become increasingly complex for large delays. For FDEs these drawbacks were overcome via the Razumikhin approach. However, as it was also indicated in Section 2.1, a direct translation of the Razumikhin approach for FDEs to discrete-time systems with delay results in a set of so-called backward Razumikhin conditions [37], which are typically difficult to verify. Recently, a more practical variant of these conditions was proposed in [93] and extended to systems with disturbances in [92]. Even so, for linear systems with delay these conditions are non-convex and hence not easy to verify. Therefore, in Chapter 2 the aforementioned results were extended to DDIs and, at the cost of some additional conservatism, a BMI was obtained that can be used to verify these conditions. However, even though this BMI has a particular structure with certain computational advantages, finding a solution still requires solving a sequence of LMIs and hence remains computationally demanding.

Therefore, in this chapter the Razumikhin approach in general and the above results

in particular, are evaluated again. We show that by considering each delayed state of a DDI as a subsystem of an interconnected system, the behavior of a DDI can be described by an interconnected system with a particular structure. As a consequence we are able to show that the Razumikhin approach is a direct application of the small-gain theorem for interconnected systems. Moreover, as a by-product an alternative proof for Theorem 2.3 is obtained. This result can be considered a counterpart for discrete-time systems to the results in [133], where a Razumikhin theorem for FDEs was proven using small-gain arguments. Furthermore, this also explains, to some extent, why the stability analysis via the Razumikhin approach even for linear DDIs requires solving a BMI. Indeed, even for linear interconnected systems a method to verify the small-gain theorem by solving a convex optimization problem is missing. Interestingly, for linear interconnected systems with dissipative subsystems, there does exist a computationally tractable set of LMIs that can be solved to verify stability of such systems, see, e.g., [84, 144]. Inspired by these results, we propose a tractable set of Razumikhin-type conditions for stability analysis of DDIs. In particular, via the proper selection of storage and supply functions these conditions can be used to verify stability for DDIs. The proposed conditions are Razumikhin-type conditions as they make use of functions that involve the system state only as opposed to trajectory segments. As a consequence, their verification is typically simpler than verification of the conditions corresponding to the Krasovskii approach. Moreover, for linear systems and quadratic functions both the corresponding stability analysis and controller synthesis problem can be solved via a single LMI that is less conservative than the BMI that was developed in Proposition 2.6 and corresponds to the Razumikhin approach.

3.2 Interconnected systems

In what follows, small-gain and dissipativity theory for interconnected systems are introduced. These stability analysis methods for interconnected systems are then interpreted in the context of DDIs. Essentially, both small-gain and dissipativity theory render the stability analysis of the overall interconnected system tractable by considering smaller subsystems separately, without taking into account the interconnection between these subsystems. Then, a set of coupling conditions, which takes into account the interconnections, is employed to pursue stability analysis of the overall interconnected system in a distributed manner. Therefore, consider a set of $N \in \mathbb{Z}_{\geq 2}$ interconnected systems. The dynamics of the i-th subsystem, $i \in \mathbb{Z}_{[1,N]}$ is given by

$$x_{i,k+1} \in G_i(x_{1,k}, \dots, x_{N,k}), \quad k \in \mathbb{Z}_+, \tag{3.1}$$

where $x_{i,k} \in \mathbb{R}^{n_i}$ and $G_i : \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_N} \rightrightarrows \mathbb{R}^{n_i}$, $i \in \mathbb{Z}_{[1,N]}$ is a mapping with the origin as equilibrium point, i.e., $G_i(0, \dots, 0) := \{0\}$. The interconnected system is described using the state vector $x_k := \mathrm{col}(\{x_{i,k}\}_{i \in \mathbb{Z}_{[1,N]}}) \in \mathbb{R}^n$, which yields

$$x_{k+1} \in G(x_k), \quad k \in \mathbb{Z}_+, \tag{3.2}$$

where $n = \sum_{i=1}^{N} n_i$ and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is obtained from the mappings G_i, i.e., $G(x_0) = \mathrm{col}(\{G_i(x_{1,0}, \dots, x_{N,0})\}_{i \in \mathbb{Z}_{[1,N]}}) \subseteq \mathbb{R}^n$ where $x_0 := \mathrm{col}(\{x_{i,0}\}_{i \in \mathbb{Z}_{[1,N]}}) \in \mathbb{R}^n$.

Next, based on [32, Corollary 5.7], a small-gain theorem for continuous-time systems, a nonlinear small-gain theorem for the interconnected system (3.2) is established.

Theorem 3.1 Suppose that there exist functions $\{W_i, \gamma_{i,j}\}_{(i,j) \in \mathbb{Z}_{[1,N]}^2}$, with $W_i : \mathbb{R}^{n_i} \to \mathbb{R}_+$ and $\gamma_{i,j} \in \mathcal{K}_\infty \cup \{0\}$, and some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that for all $i \in \mathbb{Z}_{[1,N]}$ it holds that

$$\alpha_1(\|x_{i,0}\|) \leq W_i(x_{i,0}) \leq \alpha_2(\|x_{i,0}\|), \tag{3.3a}$$

$$W_i(x_{i,1}) \leq \max_{j \in \mathbb{Z}_{[1,N]}} \gamma_{i,j}(W_j(x_{j,0})), \tag{3.3b}$$

for all $x_0 \in \mathbb{R}^n$ and all $x_{i,1} \in G_i(x_{1,0}, \ldots, x_{N,0})$. Moreover, suppose that for all $y \in \mathbb{R}_+^N \setminus \{0\}$ there exists an $i(y) \in \mathbb{Z}_{[1,N]}$ such that $\max_{j \in \mathbb{Z}_{[1,N]}} \gamma_{i(y),j}([y]_j) < [y]_{i(y)}$. Then, the interconnected system (3.2) is $\mathcal{KL}$-stable. $\qquad\square$

Theorem 3.1 is proven in Appendix B.2 and relies on the fact that if the hypothesis of the theorem holds, then there exist $\bar\alpha_1, \bar\alpha_2 \in \mathcal{K}_\infty$, $\{\sigma_i\}_{i \in \mathbb{Z}_{[1,N]}}$, with $\sigma_i \in \mathcal{K}_\infty$ for all $i \in \mathbb{Z}_{[1,N]}$, and a $\rho \in \mathcal{K}_\infty \cup \{0\}$ such that $\rho(r) < r$ for all $r \in \mathbb{R}_{>0}$ and that

$$\bar\alpha_1(\|x_0\|) \leq W(x_0) \leq \bar\alpha_2(\|x_0\|), \tag{3.4a}$$

$$W(x_1) \leq \rho(W(x_0)), \tag{3.4b}$$

for all $x_0 \in \mathbb{R}^n$ and all $x_1 \in G(x_0)$, where $W(x_0) := \max_{i \in \mathbb{Z}_{[1,N]}} \sigma_i^{-1}(W_i(x_{i,0}))$.

The small-gain condition in Theorem 3.1 (the second item of the hypothesis) implies that all compositions of gain functions corresponding to loops in the interconnected system (3.2) are subunitary, i.e., signals are attenuated and not amplified, see [32] for details. Hence, $\gamma_{i,i}(r) < r$ for all $r \in \mathbb{R}_{>0}$ and all $i \in \mathbb{Z}_{[1,N]}$ (i.e., choose $y \in \mathbb{R}^N \setminus \{0\}$ such that $[y]_i := r$ and zero otherwise). Therefore, W_i is an LF for subsystem (3.1), i.e., for $x_{j,0} = 0$ for all $j \neq i$, and the effect of the other subsystems on subsystem (3.1) can be bounded via $\gamma_{i,j}$. As such, the functions W_i and W are called an LF for the i-th subsystem (3.1) and an LF for the overall interconnected system (3.2), respectively. Other small-gain theorems for systems of the form (3.2), which also use the results in [32], can be found in, e.g., [82, 115].

Next, the stability analysis result for interconnected systems of the form (3.2) that stems from the pioneering article [144] is presented. This result has not appeared in the literature before but is, to a large extent, an analogy of the continuous–time result in [84].

Theorem 3.2 Suppose that there exist functions $\{W_i, S_{i,j}\}_{(i,j) \in \mathbb{Z}_{[1,N]}^2}$, with $W_i : \mathbb{R}^{n_i} \to \mathbb{R}_+$ and $S_{i,j} : \mathbb{R}^{n_i} \times \mathbb{R}^{n_j} \to \mathbb{R}$, some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a $\rho \in \mathbb{R}_{[0,1)}$ such that

$$\alpha_1(\|x_{i,0}\|) \leq W_i(x_{i,0}) \leq \alpha_2(\|x_{i,0}\|), \tag{3.5a}$$

$$W_i(x_{i,1}) \leq \rho W_i(x_{i,0}) + \sum_{j=1}^N S_{i,j}(x_{i,0}, x_{j,0}), \tag{3.5b}$$

for all $x_0 \in \mathbb{R}^n$, $x_{i,1} \in G_i(x_{1,0}, \ldots, x_{N,0})$ and all $i \in \mathbb{Z}_{[1,N]}$. Moreover, suppose that there exist $\{\sigma_i\}_{i \in \mathbb{Z}_{[1,N]}}$ such that $\sum_{i=-h}^0 \sum_{j=-h}^0 \sigma_i S_{i,j}(x_{i,0}, x_{j,0}) \leq 0$ for all $x_0 \in \mathbb{R}^n$, where $\sigma_i \in \mathbb{R}_{>0}$ for all $i \in \mathbb{Z}_{[1,N]}$. Then, the interconnected system (3.2) is $\mathcal{KL}$-stable. $\qquad\square$

To prove Theorem 3.2 it suffices to observe that the hypothesis of the theorem implies that the function $W(x_0) := \sum_{i=1}^N \sigma_i W_i(x_{i,0})$ is an LF for the interconnected system (3.2). The remainder of the proof then follows from the proof of Theorem 3.1 in Appendix B.2. Typically, the functions W_i and $S_{i,j}$ are called storage and supply functions, respectively.

With these preliminary results for interconnected systems established we focus again on DDIs and establish a link of DDIs to interconnected systems.

3.3 Delay difference inclusions and the small-gain theorem

The behavior of the DDI (2.1) can be described via an interconnected system of the form (3.2) with a particular structure. To do this, consider each delayed state as a state of one of the subsystems (3.1), which indeed yields an interconnected system of the form (3.2). Figure 3.1 provides a graphical depiction of this reasoning. Hence, conditions similar to
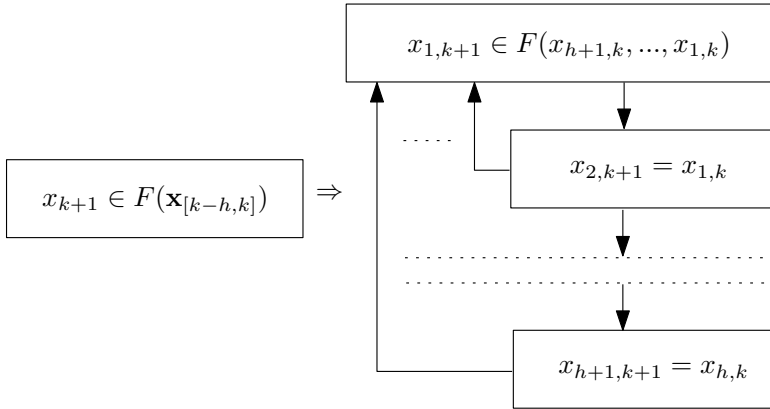
Figure 3.1: A DDI can be expressed as an interconnected system with a particular structure.

those that were used in Theorems 3.1 and 3.2 can be used to establish stability for the DDI (2.1). We formalize this approach in the following result, whose proof is also presented here because it is particularly insightful.

Theorem 3.3 Suppose that there exist functions $\{V, \rho\}$, with $V : \mathbb{R}^n \to \mathbb{R}_+$ and $\rho \in \mathcal{K}_\infty \cup \{0\}$, and some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(\|x_0\|) \leq V(x_0) \leq \alpha_2(\|x_0\|),$$
$$V(x_1) \leq \max_{i \in \mathbb{Z}_{[-h,0]}} \rho(V(x_i)),$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $x_1 \in F(\mathbf{x}_{[-h,0]})$. Moreover, suppose that for all $r \in \mathbb{R}_{>0}$ it holds that $\rho(r) < r$. Then, the DDI (2.1) is $\mathcal{KL}$-stable.

Proof: For all $i \in \mathbb{Z}_{[1,h+1]}$ let $x_{i,k} := x_{k-i+1}$, which yields $N = h + 1$ together with the maps $G_1(x_{1,0}, \ldots, x_{h+1,0}) = F(x_{h+1,0}, \ldots, x_{1,0})$ and $G_i(x_{1,0}, \ldots, x_{h+1,0}) = x_{i-1,0}$ for all $i \in \mathbb{Z}_{[2,h+1]}$. Next, it is shown that V is an LF for the i-th subsystem (3.1) for all $i \in \mathbb{Z}_{[1,h+1]}$. Letting $W_i(x_{i,0}) := V(x_{i,0})$ for all $i \in \mathbb{Z}_{[1,h+1]}$ yields

$$\gamma_{i,j}(r) = \begin{cases} \rho(r), & i = 1, \ j \in \mathbb{Z}_{[1,h+1]}, \\ r, & i \in \mathbb{Z}_{[2,h+1]}, \ j = i - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Note that, by choosing $[y]_i := r$ and zero otherwise, the second item of the hypothesis of Theorem 3.1 implies that $\gamma_{i,i}(r) < r$ for all $r \in \mathbb{R}_{>0}$. As $\rho(r) < r$ for all $r \in \mathbb{R}_{>0}$, $\gamma_{i,j} \in \mathcal{K}_\infty \cup \{0\}$ for all $(i,j) \in \mathbb{Z}^2_{[1,h+1]}$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ it follows that, indeed, V is an LF for the i-th subsystem (3.1) for all $i \in \mathbb{Z}_{[1,h+1]}$. Furthermore, consider any $y \in \mathbb{R}^{h+1}_+ \setminus \{0\}$. If $[y]_1 \geq \max_{i \in \mathbb{Z}_{[2,h+1]}} [y]_i$, then the fact that $\rho(r) < r$ for all $r \in \mathbb{R}_{>0}$ yields that

$$\max_{i \in \mathbb{Z}_{[1,h+1]}} \rho([y]_i) < [y]_1.$$

Moreover, if $[y]_1 < \max_{i \in \mathbb{Z}_{[2,h+1]}} [y]_i$, then there exists an $i(y) \in \mathbb{Z}_{[2,h+1]}$ such that

$$[y]_{i(y)-1} < [y]_{i(y)}.$$

As such the second item and hence the entire hypothesis of Theorem 3.1 is satisfied. Therefore, it follows from Theorem 3.1 that the DDI (2.1) is $\mathcal{KL}$-stable. ∎

Note that Theorem 3.3 has been proven by defining a set of functions and gains that satisfy the hypothesis of Theorem 3.1 and then concluding the result of Theorem 3.1. Furthermore, observe that Theorem 3.3 corresponds to Theorem 2.3 with the generalization that ρ is allowed to be a nonlinear function rather than a constant. Thus it has been shown that *the Razumikhin approach is an exact application of the small-gain theorem* to DDIs. Moreover, an alternative proof for Theorem 2.3 was also obtained.

The above facts and the approach used to prove Theorem 3.3 are a discrete-time counterpart to the results in [133], where a Razumikhin theorem for FDEs was proven using small-gain arguments. From the proof of Theorem 3.1 we obtain the following result.

Proposition 3.1 Suppose that the hypothesis of Theorem 3.3 is satisfied. Then there exist $\bar{\alpha}_1, \bar{\alpha}_2 \in \mathcal{K}_\infty$, $\{\sigma_i\}_{i \in \mathbb{Z}_{[-h,0]}}$, with $\sigma_i \in \mathcal{K}_\infty$ for all $i \in \mathbb{Z}_{[-h,0]}$, and a $\bar{\rho} \in \mathcal{K}_\infty \cup \{0\}$ such that $\bar{\rho}(r) < r$ for all $r \in \mathbb{R}_{>0}$ and that

$$\bar{\alpha}_1(\|\xi_0\|) \leq \bar{V}(\xi_0) \leq \bar{\alpha}_2(\|\xi_0\|), \tag{3.6a}$$
$$\bar{V}(\xi_1) \leq \bar{\rho}(\bar{V}(\xi_0)), \tag{3.6b}$$

for all $\xi_0 \in \mathbb{R}^{(h+1)n}$ and all $\xi_1 \in \bar{F}(\xi_0)$, where $\bar{V}(\xi_0) := \max_{i \in \mathbb{Z}_{[-h,0]}} \sigma_i^{-1}(V(x_i))$. □

The proof of Proposition 3.1 follows directly from the proof of Theorem 3.1 and the definitions used in the proof of Theorem 3.3. An explicit expression for the functions $\{\sigma_i\}_{i \in \mathbb{Z}_{[-h,0]}}$ in Proposition 3.1 can be obtained using the ideas presented in [70]. Thus, an explicit construction of an LKF from an LRF has been obtained. Hence, Proposition 3.1 parallels Theorem 2.4, like Theorem 3.3 parallels Theorem 2.3.

3.3.1 Necessary conditions for the Razumikhin approach

It is well known that the small-gain theorem provides conditions for stability that are relatively simple to verify but conservative. As it has now been established that the Razumikhin approach is an application of the small-gain theorem, this approach is likely to be conservative as well. Of course, this was already established in Section 2.3.2 and illustrated via Example 2.3 and hence does not come as a surprise.

In view of the above and to better understand the conservatism that is associated with the Razumikhin approach, it makes sense to search for further necessary conditions for the Razumikhin approach. Therefore, let $\mathbb{T}_1 := \{0, 1\}$ and let $\mathbb{T}_2 := \{-1, 0, 1\}$ and consider the following family of systems

$$z_{k+1} \in H_\delta(z_k), \quad k \in \mathbb{Z}_+, \tag{3.7}$$

where $z_k \in \mathbb{R}^n$ and $H_\delta(z_0) := F([\delta]_1 z_0, \ldots, [\delta]_{h+1} z_0)$ where $\delta \in \mathbb{T}_1^{h+1}$ or $\delta \in \mathbb{T}_2^{h+1}$. Next, we derive a set of necessary conditions for the Razumikhin approach that makes use of the family of systems (3.7).

Proposition 3.2 Suppose that the hypothesis of Theorem 3.3 is satisfied. Then, the family of systems (3.7) is $\mathcal{KL}$-stable for all $\delta \in \mathbb{T}_1^{h+1}$. □

Under an additional assumption the result of Proposition 3.2 can be sharpened.

Assumption 3.1 The equality $H_\delta(-z_0) = -H_\delta(z_0)$ holds for all $(\delta, z_0) \in \mathbb{T}_2^{h+1} \times \mathbb{R}^n$. □

Proposition 3.3 Suppose that Assumption 3.1 holds and that the hypothesis of Theorem 3.3 is satisfied. Then, the family of systems (3.7) is $\mathcal{KL}$-stable for all $\delta \in \mathbb{T}_2^{h+1}$. □

Propositions 3.2 and 3.3 are proven in Appendix B.2. Assumption 3.1 holds for, among many others, linear and cubic functions. Therefore, for linear DDIs it follows from Proposition 3.3 that the Razumikhin approach can be applied only if

$$\text{sr}\left(\sum_{i=-h}^{0} [\delta]_{i+h+1} A_i\right) < 1, \quad \forall (\delta, \{A_i\}_{i \in \mathbb{Z}_{[-h,0]}}) \in \mathbb{T}_2^{h+1} \times \mathcal{A}.$$

To illustrate the application of the above results, let us reconsider Example 2.3 with $a = 1$ and $b \in \mathbb{R}_{(-1,0)}$.

Example 3.1 (Example 2.3, Part II) Consider the linear DDE

$$x_{k+1} = b x_{k-1} + x_k, \quad k \in \mathbb{Z}_+, \tag{3.8}$$

where $x_k \in \mathbb{R}$ and $b \in \mathbb{R}_{(-1,0)}$. In Proposition 2.1 it was established that the linear DDE (3.8) is GES but does not admit an LRF for any $b \in \mathbb{R}_{(-1,0)}$. Let us try to confirm this result via the necessary conditions that were derived above.

For any $b \in \mathbb{R}_{(-1,0)}$ it is possible to compute the family of systems (3.7), e.g., $H_\delta(z_0) = \{(1-b)z_0\}$ for $\delta = \text{col}(-1, 1)$ while $H_\delta(z_0) = \{bz_0\}$ for $\delta = \text{col}(1, 0)$. Therefore, as $H_\delta(z_0) = \{(1-b)z_0\}$ for $\delta = \text{col}(-1, 1)$, it follows from Proposition 3.3 that for any $b \in \mathbb{R}_{(-1,0)}$ the conditions corresponding to the Razumikhin approach are infeasible while the linear DDE (3.8) is $\mathcal{KL}$-stable, which confirms the result of Proposition 2.1. □

While the above necessary conditions are insightful, it has been established that they are not sufficient, see [45]. Interestingly, in the same article, when considering linear DDEs and polyhedral LRFs only, a necessary condition has been formulated for which no counterexample is available. This may provide a fruitful starting point for a converse Lyapunov theorem for the Razumikhin approach.

3.4 Delay difference inclusions and dissipativity theory

A disadvantage of small-gain theorems is that the conditions contain a multiplication of the functions W_i and $\gamma_{i,j}$ (see (3.3b)), which both need to be determined to establish stability. This and the observation that the Razumikhin approach is an application of the small-gain theorem explains, to some extent, why even for linear DDIs and quadratic functions the conditions corresponding to the Razumikhin approach can be verified via a BMI only, i.e., such as it is the case for Proposition 2.6. Dissipativity theory on the other hand is not limited by such a fundamental nonlinearity, see, e.g., Theorem 3.2. Therefore, we interpret the dissipativity conditions in Theorem 3.2 in the context of DDIs next.

Theorem 3.4 Suppose that there exist functions $\{V_i, S_{i,j}\}_{(i,j) \in \mathbb{Z}^2_{[-h,0]}}$, where $V_i : \mathbb{R}^n \to \mathbb{R}_+$ and $S_{i,j} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a $\rho \in \mathbb{R}_{[0,1)}$ such that

$$\alpha_1(\|x_0\|) \leq V_i(x_0) \leq \alpha_2(\|x_0\|), \tag{3.9a}$$

$$V_i(x_{i+1}) \leq \rho V_i(x_i) + \textstyle\sum_{j=-h}^0 S_{i,j}(x_i, x_j), \tag{3.9b}$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$, $x_1 \in F(\mathbf{x}_{[-h,0]})$ and all $i \in \mathbb{Z}_{[-h,0]}$. Moreover, suppose that there exist $\{\sigma_i\}_{i \in \mathbb{Z}_{[-h,0]}}$ such that $\sum_{i=-h}^0 \sum_{j=-h}^0 \sigma_i S_{i,j}(x_i, x_j) \leq 0$ for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$, where $\sigma_i \in \mathbb{R}_{>0}$ for all $i \in \mathbb{Z}_{[-h,0]}$. Then, the DDI (2.1) is $\mathcal{KL}$-stable. $\qquad\square$

Theorem 3.4 follows directly from Theorem 3.2 and the definitions used in the proof of Theorem 3.3. Similarly, as for the Razumikhin approach, the functions V_i map the state of the DDI (2.1) at a single time instant to the positive numbers and are only decreasing if the trajectory of the DDI satisfies a specific condition. Therefore, we refer to Theorem 3.4 as a Razumikhin-type result. More importantly, the conditions in Theorem 3.4 form a set of distributed stability analysis conditions for DDIs that can be verified efficiently. Further details on the distributed verification of the conditions in Theorem 3.4 can be found in [84].

Remark 3.1 Theorem 3.4 can be simplified by: *(i)* Replacing the second item of the hypothesis by $S_{i,j}(x_i, x_j) + S_{j,i}(x_j, x_i) \leq 0$ for all $(x_i, x_j) \in \mathbb{R}^{n_i} \times \mathbb{R}^{n_j}$, $(i,j) \in \mathbb{Z}^2_{[-h,0]}$; *(ii)* choosing $S_{i,i}$ and $S_{i,j}$ identical to zero for all $i \in \mathbb{Z}_{[-h,0]}$ and each $(i,j) \in \mathbb{Z}^2_{[-h,0]}$ such that x_j does not directly affect x_i via (3.9b), respectively; *(iii)* letting $\sigma_i S_{i,j}(x_i, x_j) := -\sigma_j S_{j,i}(x_j, x_i)$ for all $(i,j) \in \mathbb{Z}^2_{[-h,0]}$; and *(iv)* choosing $\sigma_i := 1$ for all $i \in \mathbb{Z}_{[-h,0]}$. $\qquad\square$

Remark 3.2 Dissipativity theory was used to obtain sufficient conditions for stability of continuous-time systems with delay in [23]. Interestingly, for linear systems these conditions correspond to the Razumikhin approach, cf. [20]. On the other hand, in what follows the conditions in Theorem 3.4 are shown to be different from the Razumikhin-type conditions for discrete-time systems proposed in [93] and Theorems 2.3 and 3.3. $\qquad\square$

It was shown in Sections 2.3.2 and 3.3.1 that the Razumikhin approach can be considered as a particular case of the Krasovskii approach and that as such it is conservative. Similarly,

Theorem 3.4 can be considered as a particular case of the Krasovskii approach, i.e., the function $\bar{V}(\xi_0) := \sum_{i=-h}^{0} \sigma_i V_i(x_i)$ satisfies the hypothesis of Theorem 2.1. Hence, it is probable that the conditions corresponding to Theorem 3.4 are also conservative. Like for the Razumikhin approach, Example 2.3 confirms this expectation.

Proposition 3.4 Consider the linear DDE (2.6) and suppose that $b \in \mathbb{R}_{(-1,0)}$ and $a = 1$. Then, the following claims are true:

(i) The linear DDE (2.6) with $b \in \mathbb{R}_{(-1,0)}$ and $a = 1$ is GES;

(ii) the linear DDE (2.6) with $b \in \mathbb{R}_{(-1,0)}$ and $a = 1$ does not admit a set of functions $\{V_i, S_{i,j}\}_{(i,j) \in \mathbb{Z}^2_{[-1,0]}}$ that satisfies the hypothesis of Theorem 3.4. □

Proposition 3.4 is proven in Appendix B.2. It follows from Proposition 3.4 that Theorem 3.4 indeed provides a set of sufficient, but not necessary, conditions for stability of DDIs.

3.4.1 Implications for set invariance

An interesting feature of the Razumikhin approach is that it yields a $\mathcal{D}$-contractive set as opposed to the standard contractive set obtained via the Krasovskii approach. Next, we investigate what type of contractive sets can be obtained from a set of functions satisfying the hypothesis of Theorem 3.4. Therefore, consider a set of functions $\{V_i, S_{i,j}\}_{(i,j) \in \mathbb{Z}^2_{[-h,0]}}$ that satisfies the hypothesis of Theorem 3.4 and suppose that the functions $\{V_i\}_{i \in \mathbb{Z}_{[-h,0]}}$ are convex. Then, the function $\bar{V}(\xi_0) := \sum_{i=-h}^{0} \sigma_i V_i(x_i)$ is an LKF for the DDI (2.1). Moreover, $\bar{V}$ is convex by construction and hence provides, as established in Proposition 2.3, a standard contractive set for the DDI (2.1). Alternatively, to obtain a relation of the functions corresponding to Theorem 3.4 to $\mathcal{D}$-contractive sets, consider the following result.

Proposition 3.5 Suppose that there exists a set of functions $\{V_i, S_{i,j}\}_{(i,j) \in \mathbb{Z}^2_{[-h,0]}}$ that satisfies the hypothesis of Theorem 3.4. Furthermore, suppose that

$$\rho \sigma_i V_i(x_0) \leq \rho \sigma_0 V_0(x_0) \leq \sigma_i V_i(x_0), \quad \forall x_0 \in \mathbb{R}^n,$$

for all $i \in \mathbb{Z}_{[-h,-1]}$. Then, V_0 is an LRF for the DDI (2.1). □

Proposition 3.5 is proven in Appendix B.2. For example, the inequality in Proposition 3.5 is true when all functions and scalars $\{V_i, \sigma_i\}_{i \in \mathbb{Z}_{[-h,0]}}$ are identical.

Thus, we have proven that a set of functions $\{V_i, S_{i,j}\}_{(i,j) \in \mathbb{Z}^2_{[-h,0]}}$ that satisfies the hypothesis of Proposition 3.5 together with the additional assumption that V_0 is convex implies via Proposition 2.4 that the DDI (2.1) admits a $\mathcal{D}$-contractive set. Hence, it has been shown that, under an additional assumption, the conditions provided in Theorem 3.4 imply the existence of a $\mathcal{D}$-contractive set.

3.5 Stabilizing controller synthesis for linear systems

To compare the Razumikhin-type conditions provided in Theorem 3.4 with the interpretation of the Razumikhin approach for discrete-time systems developed in [93] and Theorem 2.3 in more detail, linear controlled DDIs are considered next. Recall that linear controlled DDIs are DDIs of the form (2.15) with the structure defined in Definition 2.7.

To reformulate the conditions in Theorem 3.4 as an LMI, quadratic storage and supply functions are considered, i.e., such that

$$V_i(x_0) := x_0^\top Z_0^{-1} Z_i Z_0^{-1} x_0, \tag{3.10a}$$

with $(\{Z_i\}_{i \in \mathbb{Z}_{[-h,0]}}) \in (\mathbb{R}^{n \times n})^{h+1}$ and

$$S_{i,j}(x_j, x_j) := \begin{bmatrix} x_i \\ x_j \end{bmatrix}^\top \begin{bmatrix} Z_0^{-1} & 0 \\ 0 & Z_0^{-1} \end{bmatrix} \begin{bmatrix} X_{i,j,1} & X_{i,j,2} \\ X_{i,j,3} & X_{i,j,4} \end{bmatrix} \begin{bmatrix} Z_0^{-1} & 0 \\ 0 & Z_0^{-1} \end{bmatrix} \begin{bmatrix} x_i \\ x_j \end{bmatrix}, \tag{3.10b}$$

with $(\{X_{i,j,l}\}_{(i,j,l) \in \mathbb{Z}_{[-h,0]}^2 \times \mathbb{Z}_{[1,4]}}) \in (\mathbb{R}^{n \times n})^{4(h+1)^2}$. Suppose that $h \in \mathbb{Z}_{\geq 1}$, i.e., the linear controlled DDI (2.15) has a delay term. In this case, for any $(\{A_i, B_i\}_{i \in \mathbb{Z}_{[-h,0]}}) \in \mathcal{AB}$, let $\bar{A}_i \in \mathbb{R}^{n \times (h+1)n}$, $i \in \mathbb{Z}_{[-h,-1]}$ denote matrices with $[\bar{A}_i]_{1:n,(h+1+i)n+1:(h+2+i)n} := I$ and zero elsewhere. Moreover, let $\bar{A}_0 := \begin{bmatrix} A_{-h} & \dots & A_0 \end{bmatrix}$ and let $\bar{B} := \begin{bmatrix} B_{-h} & \dots & B_0 \end{bmatrix}$. Finally, let $\bar{Z}_i, \bar{X}_{i,j} \in \mathbb{R}^{(h+1)n \times (h+1)n}$ denote matrices with

$$[\bar{Z}_i]_{(h+i)n+1:(h+1+i)n,(h+i)n+1:(h+1+i)n} := Z_i,$$
$$[\bar{X}_{i,j}]_{(h+i)n+1:(h+1+i)n,(h+i)n+1:(h+1+i)n} := X_{i,j,1},$$
$$[\bar{X}_{i,j}]_{(h+i)n+1:(h+1+i)n,(h+j)n+1:(h+1+j)n} := X_{i,j,2},$$
$$[\bar{X}_{i,j}]_{(h+j)n+1:(h+1+j)n,(h+i)n+1:(h+1+i)n} := X_{i,j,3},$$
$$[\bar{X}_{i,j}]_{(h+j)n+1:(h+1+j)n,(h+j)n+1:(h+1+j)n} := X_{i,j,4},$$

and zero elsewhere for all $(i,j) \in \mathbb{Z}_{[-h,0]}^2$.

Proposition 3.6 Choose some $\rho \in \mathbb{R}_{[0,1)}$. Suppose that the controlled DDI (2.15) is a linear controlled DDI. If there exist matrices $(\{Y, Z_i, X_{i,j,l}\}_{(i,j,l) \in \mathbb{Z}_{[-h,0]}^2 \times \mathbb{Z}_{[1,4]}}) \in \mathbb{R}^{m \times n} \times (\mathbb{R}^{n \times n})^{h+1} \times (\mathbb{R}^{n \times n})^{4(h+1)^2}$ such that $Z_i = Z_i^\top \succ 0$ for all $i \in \mathbb{Z}_{[-h,0]}$ and

$$\rho \bar{Z}_i + \sum_{l=-h}^0 \bar{X}_{i,l} \succeq \bar{A}_i^\top Z_i \bar{A}_i, \quad \forall i \in \mathbb{Z}_{[-h,-1]}, \tag{3.11a}$$

$$\begin{bmatrix} \rho \bar{Z}_0 + \sum_{l=-h}^0 \bar{X}_{0,l} & * \\ \bar{A}_0 \operatorname{diag}(Z_0, \dots, Z_0) + \bar{B} \operatorname{diag}(Y, \dots, Y) & Z_0 \end{bmatrix} \succeq 0, \tag{3.11b}$$

$$0 \succeq \sum_{i=-h}^0 \sum_{j=-h}^0 \bar{X}_{i,j} \tag{3.11c}$$

for all $(\{A_i, B_i\}_{i \in \mathbb{Z}_{[-h,0]}}) \in \mathcal{AB}$, then the linear controlled DDI (2.15) in closed loop with the control law (2.16), where $K := Y Z_0^{-1}$, is $\mathcal{KL}$-stable. $\square$

Proposition 3.6 is proven in Appendix B.2. Note that, if the set $\mathcal{AB}$ is a matrix polytope or consists of a finite number of points, then the conditions in Proposition 3.6 can be verified by solving an LMI of finite dimensions. Clearly, Proposition 3.6 can also be used for the stability analysis of a linear DDI of the form (2.1), i.e., by setting $\bar{B} = 0$.

Next, we illustrate the advantages of the Razumikhin-type conditions of Theorem 3.4 over the interpretation of the Razumikhin approach developed for discrete-time systems in [93] and extended to DDIs in Theorem 2.3.

Proposition 3.7 Suppose that there exists a set of variables, i.e., $(\{\delta_i\}_{i \in \mathbb{Z}_{[-h,0]}}, Z, Y)$, that satisfies the hypothesis of Proposition 2.6. Then, there exists another set of matrices, i.e., $(\{Y, Z_i, X_{i,j,l}\}_{(i,j,l) \in \mathbb{Z}_{[-h,0]}^2 \times \mathbb{Z}_{[1,4]}})$, that satisfies the hypothesis of Proposition 3.6. $\qquad \square$

Proposition 3.7 is proven in Appendix B.2 and establishes that for linear controlled DDIs Proposition 3.6, which corresponds to the Razumikhin-type conditions proposed in Theorem 3.4, is less conservative than Proposition 2.6, which is a reformulation at the cost of some conservatism of the interpretation of the Razumikhin approach that was developed in [93] and Theorem 2.3. Moreover, the corresponding optimization problem is an LMI as opposed to a BMI and hence the conditions are also computationally more attractive.

The following example illustrates the application of Proposition 3.6 and shows that the hypothesis of Proposition 2.6 is strictly stronger than the hypothesis of Proposition 3.6.

Example 3.2 (Part I) Consider the linear DDE

$$x_{k+1} = \begin{bmatrix} 0 & 0 \\ 0 & 0.75 \end{bmatrix} x_{k-1} + \begin{bmatrix} 0.75 & 0 \\ 0 & 0 \end{bmatrix} x_k, \quad k \in \mathbb{Z}_+, \tag{3.12}$$

where $x_k \in \mathbb{R}^2$. The properties of the linear DDE (3.12) are studied in what follows. $\quad \square$

Proposition 3.8 The following claims are true:

(i) The linear DDE (3.12) admits a set of matrices that satisfies the hypothesis of Proposition 3.6 and hence, (3.12) is $\mathcal{KL}$-stable;

(ii) the linear DDE (3.12) does not admit a set of variables that satisfies the hypothesis of Proposition 2.6. $\qquad \square$

Proposition 3.8 is proven in Appendix B.2. Note that the function $V(x_0) := \|x_0\|_\infty$ satisfies the hypothesis of Theorem 2.3. Therefore, Proposition 3.8 should not be used to draw any conclusions about Theorems 2.3 and 3.4 in their full generality, but applies to the related computational procedures only. Indeed, Propositions 3.7 and 3.8 indicate that the conditions in Proposition 3.6 are less conservative than those in Proposition 2.6.

Next, Example 2.2 is revisited to illustrate that the controller synthesis conditions corresponding to the approach proposed in Theorem 3.4 are more general than the controller synthesis conditions corresponding to the Razumikhin approach while computationally more attractive than those corresponding to the Krasovskii approach.

Example 3.3 (Example 2.2, Part III) As before we try to stabilize the DC-motor over the communication network by designing a controller, this time via Proposition 3.6. Again, the numerical values for the polytopic over-approximation of the uncertain time-varying matrix $\Delta(\tau)$ can be found in Appendix A. Taking $\tau_k \in \mathbb{R}_{[0,0.44T_s]}$, Proposition 3.6 with $\rho = 0.8$ yields the storage function matrices and corresponding controller matrix

$$Z_0 = \begin{bmatrix} 0.39 & -4.97 \\ -4.97 & 68.14 \end{bmatrix}, \quad Z_1 = \begin{bmatrix} 0.05 & -0.48 \\ -0.48 & 6.26 \end{bmatrix}, \quad K_{\text{DISSIP}} = \begin{bmatrix} -23.97 & -1.74 \end{bmatrix}.$$

Table 3.1: The MAD and the type and dimension of the corresponding controller synthesis problem for each method.

approach	controller synthesis result	MAD	solution method	dimension
Razumikhin	Proposition 2.6	4.24ms	BMI	(26×26)
Razumikhin-type	Proposition 3.6	4.40ms	LMI	(32×32)
Krasovskii	Proposition 2.5	4.80ms	LMI	(36×36)

However, for $\bar{\tau} > 0.44T_s$ Proposition 3.6 no longer provides a feasible solution. In Table 3.1 the above results and those that were obtained in Example 2.2, Part II are summarized. In particular, the table shows which theoretical framework was used and what the corresponding controller synthesis result is. Furthermore, the maximal admissible delay (MAD), i.e., $\bar{\tau}$, for which a feasible solution to the optimization problem was found and the type of optimization problem are also shown. Finally, the dimension of the corresponding controller synthesis LMI or BMI is shown. The latter provides an indication of the complexity of the optimization problem.

The results in Table 3.1 confirm the observations in this chapter and Chapter 2, i.e., the Razumikhin approach has the lowest computational complexity (ignoring the fact that the corresponding optimization problem is a BMI) while the Krasovskii approach provides the most general results. Furthermore, the approach corresponding to Theorem 3.4 provides a trade-off between these two properties. □

3.6 Conclusions

In this chapter we explored the link between DDIs and interconnected systems. In particular, it was shown that the behavior of a DDI can be described by an interconnected system with a particular structure. Thus, we were able to prove that the Razumikhin approach is a direct application of the small-gain theorem for interconnected systems, which explains, to some extent, the conservatism that is typically associated with the Razumikhin approach. Furthermore, it also allowed us to derive a Razumikhin-type stability analysis result based on a stability analysis technique for interconnected systems with dissipative subsystems. The corresponding conditions were shown to be computationally more attractive and less conservative than existing conditions that are based on the more standard interpretation of the Razumikhin approach for discrete-time systems.

The results in this chapter have lead to a deeper understanding of the Razumikhin approach and even resulted in a novel Razumikhin-type stability analysis technique. Unfortunately, the standard Razumikhin approach as discussed in Section 2.3.2 and the novel Razumikhin-type conditions that were developed in Section 3.4 remain sufficient only and are not necessary for stability. Therefore, in the next chapter we propose a relaxation of the Razumikhin approach that leads to necessary and sufficient conditions for stability.

Chapter 4

Simple, necessary and sufficient conditions for stability

The Razumikhin-type stability analysis methods for DDIs that were discussed in the previous two chapters provide sufficient, *but not necessary* conditions for stability. Nevertheless, the Razumikhin approach is of interest because it makes use of conditions that involve the system state, as opposed to trajectory segments. As a consequence, the corresponding function provides information about the trajectories of the DDI directly and the corresponding computations can be executed in the underlying low-dimensional state space of the DDI dynamics. Therefore, we propose a relaxation of the Razumikhin approach in this chapter which leads to necessary and sufficient conditions for stability of DDIs. For linear DDEs the stability analysis problem that corresponds to these relaxed Razumikhin conditions can be solved via SDP. Furthermore, for positive linear DDEs, these novel conditions are equivalent with the standard Razumikhin approach, which implies that this approach is non-conservative for positive linear DDEs. The implications of the proposed conditions for the construction of invariant sets are also briefly discussed.

4.1   Introduction

In Chapter 2 we showed that, like for continuous-time systems with delay, the Krasovskii approach provides necessary and sufficient conditions for stability. Unfortunately, these conditions involve trajectory segments and as such do not provide information about the DDI trajectories directly, which causes them to become increasingly complex for large delays. The Razumikhin approach, on the other hand, relies on a Lyapunov-like function defined in the original, non-augmented state space. As such, the LRF provides information about the trajectories of the DDI directly and the corresponding computations can be executed in the underlying low-dimensional state space of the DDI dynamics. A direct translation of the Razumikhin approach for continuous-time systems with time-delay to DDEs yields a set of so-called backward Razumikhin conditions [37], which are typically difficult to verify. A more practical variant of these conditions was first proposed in [93]. Even so, for quadratic functions and linear DDEs the conditions obtained therein are nonlinear and non-convex and hence, difficult to verify. In Section 2.6, at the cost of some additional conservatism, synthesis of quadratic LRFs for linear DDIs was reduced to a bilinear matrix inequality

(BMI), which is still non-convex, but less difficult to solve. In Chapter 3 an alternative, less conservative, set of Razumikhin-type conditions for DDIs, which can be verified by solving a single LMI for linear DDEs and quadratic functions, was obtained via dissipativity theory. However, even though these conditions, as well as all of the other, above-mentioned Razumikhin conditions, are relatively simple to verify, they remain conservative. This was proven in Propositions 2.1 and 3.4 via Example 2.3, which consists of a scalar linear DDE that is stable but for which the aforementioned Razumikhin-type conditions are infeasible.

Motivated by the above facts, we propose a modification of the Razumikhin approach in this chapter and prove that this relaxation yields necessary and sufficient conditions for stability of DDIs. More specifically, the candidate LRF is required to be less than the maximum over the function values for a number of delayed states. When this number is chosen equal to the size of the delay, LRFs as introduced in [93] are recovered. However, to attain necessity, typically, a larger number of delayed states has to be considered. For exponentially stable DDIs, an estimate is constructed for the lower bound on the value of the number of states for which necessity is obtained. Furthermore, for linear DDEs and quadratic functions the developed conditions are shown to be equivalent with an LMI. Interestingly, for positive linear DDEs we prove that the newly proposed conditions are equivalent with the Razumikhin approach. This establishes the non-conservatism of the Razumikhin approach and, hence, the dominance of the Razumikhin approach over the Krasovskii approach for such systems, in the sense that both approaches are non-conservative but only the Razumikhin approach yields relatively simple conditions for stability that provide information about the system trajectories directly. The implications of the relaxed Razumikhin-type conditions for the construction of invariant sets are also briefly discussed.

4.2   Non-conservative Razumikhin-type conditions for stability

As it was also indicated above, the Krasovskii approach provides necessary and sufficient conditions for stability of the DDI (2.1). Unfortunately, these conditions can be difficult to verify in practice. The Razumikhin approach provides relatively simple conditions for stability of the DDI (2.1), which, however, are conservative (see Proposition 2.1). This motivates us to introduce a relaxation of the Razumikhin approach in what follows.

4.2.1 $\mathcal{KL}$-stability

In this section we present a relaxation of Theorem 2.3. More precisely, we will show that by imposing (2.10b) with respect to the maximum over the function values for the $M \in \mathbb{Z}_{\geq h}$ most recent states, necessity is obtained for M large enough. Clearly, for $M > h$ a sequence of states $\mathbf{x}_{[-M,0]} \in (\mathbb{R}^n)^{M+1}$ is generated consistently by the dynamics (2.1).

Definition 4.1 A sequence of states $\mathbf{x}_{[-M,0]} \in (\mathbb{R}^n)^{M+1}$, $M \in \mathbb{Z}_{\geq h}$ is called *a solution to the DDI* (2.1) *of length* $M + 1$ if, for each $M \in \mathbb{Z}_{\geq h+1}$, it holds that $x_{i+1} \in F(\mathbf{x}_{[-h+i,i]})$ for all $i \in \mathbb{Z}_{[-M+h,-1]}$. Obviously, any $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ is called a solution to the DDI (2.1) of length $h + 1$. $\qquad\square$

Note that the space of solutions to the DDI (2.1) of length $M + 1$ corresponds, after a shift in time, to the space $\{\Phi_{[-h,M-h]} : \mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}, \ \Phi \in \mathcal{S}(\mathbf{x}_{[-h,0]})\}$.

For our first main result we will use the following assumption.

Assumption 4.1 The map $F : (\mathbb{R}^n)^{h+1} \rightrightarrows \mathbb{R}^n$ that generates the dynamics (2.1) is $\mathcal{K}$-continuous with respect to zero. $\qquad\square$

$\mathcal{K}$-continuity is a generalization of Hölder continuity and, hence, every DDI that is, e.g., Lipschitz continuous is also $\mathcal{K}$-continuous, see [40]. $\mathcal{K}$-continuity with respect to zero is a weaker version of $\mathcal{K}$-continuity that requires the property to hold for any vector $x_0 \in \mathbb{R}^n$ with respect to 0. Note that a system that is $\mathcal{K}$-continuous with respect to zero need not even be continuous, as such Assumption 4.1 is not very restrictive. More importantly, even under Assumption 4.1, the conditions corresponding to the Razumikhin approach in general and Theorem 2.3 in particular remain sufficient only and are not necessary. Indeed, the scalar linear DDE (2.6) satisfies Assumption 4.1 and hence the conclusion of Proposition 2.1 remains true. Therefore, consider the following relaxation of the Razumikhin approach.

Theorem 4.1 Suppose that Assumption 4.1 holds. The following statements are equivalent:

(i) There exist a function $V : \mathbb{R}^n \to \mathbb{R}_+$, some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, a $\rho \in \mathbb{R}_{[0,1)}$ and, for each compact set $\mathbb{X} \subset (\mathbb{R}^n)^{h+1}$, a finite $M(\mathbb{X}) \in \mathbb{Z}_{\geq h}$ such that

$$\alpha_1(\|x_0\|) \leq V(x_0) \leq \alpha_2(\|x_0\|), \quad \forall x_0 \in \mathbb{R}^n, \tag{4.1a}$$

$$V(x_1) \leq \rho \max_{i \in \mathbb{Z}_{[-M,0]}} V(x_i), \tag{4.1b}$$

for all $\mathbf{x}_{[-M,0]} \in (\mathbb{R}^n)^{M+1}$ and all $x_1 \in F(\mathbf{x}_{[-h,0]})$ such that $\mathbf{x}_{[-M,0]}$ is a solution to the DDI (2.1) of length $M + 1$ and satisfies $\mathbf{x}_{[-M,-M+h]} \in \mathbb{X}$;

(ii) the DDI (2.1) is $\mathcal{KL}$-stable. $\qquad\square$

The proof of Theorem 4.1 is presented in Appendix B.3 and relies on the fact that for any stable system, all trajectories that start in a proper C-set return to this set after some time. As a consequence, one can use an LKF with a particular structure without introducing any conservatism if the typical decrease condition is imposed over a non-unitary horizon. Hence, the result follows from the reasoning that was used in Corollary 2.4. We emphasize here that the set $\mathbb{X}$ is not necessarily invariant. Furthermore, it is important to observe that given an $M(\mathbb{X}) \in \mathbb{Z}_{\geq h}$ the same value is necessary for any subset of $\mathbb{X}$.

Remark 4.1 Theorem 4.1 recovers the interpretation of the Razumikhin approach that was presented in [93] and Theorem 2.3 for $M = h$. Moreover, the sublevel sets of the function V corresponding to Theorem 4.1 provide information about the evolution of the trajectories of the DDI (2.1) in the original state space $\mathbb{R}^n$, as opposed to $(\mathbb{R}^n)^{h+1}$ for the Krasovskii approach. As such the computations corresponding to Theorem 4.1 can be executed with respect to the original state space, which yields a computational advantage. $\qquad\square$

Unfortunately, it remains unclear if there exists a single M for which the conditions in Theorem 4.1 become necessary and sufficient. Moreover, if such an M exists, it would be interesting to provide an estimate on the value for which necessity is attained. In what follows, it is shown that both issues can be resolved for DDIs that are GES.

4.2.2   Global exponential stability

For DDIs that are GES a somewhat stronger version of Assumption 4.1 is required.

Assumption 4.2 The map $F : (\mathbb{R}^n)^{h+1} \rightrightarrows \mathbb{R}^n$ that generates the dynamics (2.1) is Lipschitz continuous with respect to zero. □

Lipschitz continuity with respect to zero is a weaker version of Lipschitz continuity that requires the standard property to hold for any vector $x_0 \in \mathbb{R}^n$ with respect to 0. Note that a system that is Lipschitz continuous with respect to zero need not even be continuous, as such Assumption 4.2 is not very restrictive. More importantly, Assumption 4.2 does not change the fact that the conditions corresponding to the Razumikhin approach for GES, i.e., those provided in Corollary 2.3, remain sufficient only and are not necessary. Indeed, Example 2.3 satisfies Assumption 4.2 and hence the conclusion of Proposition 2.1 remains true. Therefore, consider the following relaxation of Corollary 2.3.

Theorem 4.2 Suppose that Assumption 4.2 holds. The following statements are equivalent:
  (i) There exist a function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$, some $(c_1, c_2, \lambda) \in \mathbb{R}_{>0} \times \mathbb{R}_{\geq c_1} \times \mathbb{Z}_{\geq 1}$, a $\rho \in \mathbb{R}_{[0,1)}$ and a finite $M \in \mathbb{Z}_{\geq h}$ such that

$$c_1 \|x_0\|^\lambda \leq V(x_0) \leq c_2 \|x_0\|^\lambda, \quad \forall x_0 \in \mathbb{R}^n, \tag{4.2a}$$

$$V(x_1) \leq \rho \max_{i \in \mathbb{Z}_{[-M,0]}} V(x_i), \tag{4.2b}$$

for all $\mathbf{x}_{[-M,0]} \in (\mathbb{R}^n)^{M+1}$ and all $x_1 \in F(\mathbf{x}_{[-h,0]})$ such that $\mathbf{x}_{[-M,0]}$ is a solution to the DDI (2.1) of length $M+1$;
 (ii) the DDI (2.1) is GES with constants $(c, \mu) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{[0,1)}$.

Moreover, (4.2b) can be satisfied for any $M \in \mathbb{Z}_{\geq h}$ such that $M > \log_\mu(\frac{1}{c}) + h - 1$. □

The proof of Theorem 4.2 is presented in Appendix B.3. Like Theorem 4.1, Theorem 4.2 provides conditions for GES that provide information about the evolution of the trajectories of the DDI (2.1) in $\mathbb{R}^n$ directly. Furthermore, Theorem 4.2 recovers the interpretation of the Razumikhin approach that was presented in Corollary 2.3 for $M = h$.

Remark 4.2 Increasing $M \in \mathbb{Z}_{\geq h}$ reduces the conservatism of the conditions in Theorem 4.2. An estimate of the smallest M for which necessity of the conditions in Theorem 4.2 is attained was also provided in Theorem 4.2. Therein, the constants μ and c correspond to the GES property of the DDI (2.1). □

To illustrate the non-conservatism of the developed conditions let us revisit Example 2.3 with $a = 1$ and $b = -0.5$. Recall that this example was also used to show that the standard Razumikhin approach and the Razumikhin-type conditions in Chapter 3 provide merely sufficient and not necessary conditions for stability of DDIs, see Propositions 2.1 and 3.4 for details.

Example 4.1 (Example 2.3, Part III) Consider the scalar linear DDE

$$x_{k+1} = -0.5x_{k-1} + x_k, \quad k \in \mathbb{Z}_+. \tag{4.3}$$

It can be concluded from standard Lyapunov arguments [65] that the linear DDE (4.3) is GES with $\mu = 0.7071$ and $c = 3.71$, i.e., by constructing a quadratic LKF for this system and using the facts that $\mu := \bar{\rho}^{\frac{1}{2}}$ and $c := (\frac{\bar{c}_2}{\bar{c}_1})^{\frac{1}{2}}$.

Furthermore, a direct calculation verifies that the function $V(x_0) := |x_0|$ satisfies the hypothesis of Theorem 4.2 with $c_1 = 1$, $c_2 = 1$, $\lambda = 1$, $\rho = 0.5$ and $M = 3$, which confirms, via Theorem 4.2, that the scalar linear DDE (4.3) is GES. Interestingly, the estimate of the lower bound on M indicated in Theorem 4.2 yields that for any integer $M > 3.78$ there exists a function V that satisfies the hypothesis of the theorem. This shows that the bound indicated in Theorem 4.2 is not necessarily tight. $\qquad \square$

4.2.3 Stability analysis for linear systems

For linear DDEs and quadratic functions V it was shown in Proposition 2.6, at the cost of some additional conservatism, that the conditions in Theorem 2.3 can be verified by solving a BMI. Next, we prove that for linear DDEs and quadratic functions the conditions in Theorem 4.2 can be verified by solving a single LMI. To prove this, consider the following definitions, i.e., let

$$\bar{A} := \begin{bmatrix} 0 & I_n & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & I_n \\ A_{-h} & A_{-h+1} & \cdots & A_0 \end{bmatrix},$$

and consider a set of matrices $\{A_{i,j} \in \mathbb{R}^{n \times n}\}_{(i,j) \in \mathbb{Z}_{[-h,0]} \times \mathbb{Z}_{[M-h,M]}}$ such that

$$\bar{A}^M = \begin{bmatrix} A_{-h,M-h} & \cdots & A_{0,M-h} \\ \vdots & \ddots & \vdots \\ A_{-h,M} & \cdots & A_{0,M} \end{bmatrix},$$

with $M \in \mathbb{Z}_{\geq h}$. Thus, we obtain an alternative description of a linear DDE of the form (2.1), i.e.,

$$x_{k+1} = \sum_{i=-h}^{0} A_{i,M} x_{k-M+h+i}, \quad k \in \mathbb{Z}_+.$$

Proposition 4.1 Suppose that the DDI (2.1) is a linear DDE. Consider any $\rho \in \mathbb{R}_{(0,1)}$. The following two statements are equivalent:

(i) There exist a matrix $P \in \mathbb{R}^{n \times n}$ and a finite $M \in \mathbb{Z}_{\geq h}$ such that $P = P^\top \succ 0$ and

$$\begin{bmatrix} \rho^{\frac{1}{h+1}} P & & 0 & * \\ & \ddots & & \vdots \\ 0 & & \rho^{\frac{1}{h+1}} P & * \\ P A_{-h,M} & \cdots & P A_{0,M} & P \end{bmatrix} \succeq 0; \tag{4.4}$$

(ii) the linear DDE (2.1) is GES with constants $(c, \mu) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{[0,1)}$.

Above, (4.4) can be satisfied for any $M \in \mathbb{Z}_{\geq h}$ such that $M \geq \log_\mu(\frac{1}{c}(\rho \frac{1}{h+1})^{\frac{1}{2}}) + h - 1.\square$

Proposition 4.1 is proven in Appendix B.3 and uses the fact that one can use an LKF with a particular structure without introducing any conservatism if the typical decrease condition is imposed over a non-unitary horizon. Indeed, given $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ the matrices $\{A_{i,M}\}_{i \in \mathbb{Z}_{[-h,0]}}$ can be used to obtain the state ϕ_{M+1-h}. Hence, Proposition 4.1 follows from the reasoning that was used in Proposition 2.2. Note that the matrix inequality (4.4) is linear in P if the matrices $\{A_{i,M}\}_{i \in \mathbb{Z}_{[-h,0]}}$ are known and hence, for fixed values of M, it is an LMI. The matrices $\{A_{i,M}\}_{i \in \mathbb{Z}_{[-h,0]}}$ can be obtained by computing $\bar{A}^M$, which can be computed efficiently in a distributed fashion. Therefore, Proposition 4.1 provides a tractable, *necessary and sufficient* stability analysis test for linear DDEs.

Remark 4.3 The LMI (4.4) is similar to the BMI (2.17) with $\delta_i = \frac{1}{h+1}$ for all $i \in \mathbb{Z}_{[-h,0]}$, which can be used to verify the conditions in Theorem 2.3. It follows from Proposition 4.1 that, in contrast to the situation for Proposition 2.6, the aforementioned simple choice for δ_i does not introduce any conservatism when M is chosen large enough. $\square$

Next, we illustrate the results that were derived so far via an example.

Example 4.2 Consider the scalar linear DDE

$$x_{k+1} = -0.11 x_{k-6} + 1.1 x_k, \quad k \in \mathbb{Z}_+. \tag{4.5}$$

The DDE (4.5) is GES with constants $\mu = 0.95$ and $c = 29.6$. Therefore, it follows from Theorem 4.2 that (4.2b) can be satisfied for all $M > 78.6$, which makes the DDE (4.5) a challenging example for the method that is proposed in this chapter. Nevertheless, using $M := 79$ and $\rho := 0.99$ it is possible to establish that the scalar linear DDE (4.5) is GES. To this end we computed

$$\bar{A}^{79} = \begin{bmatrix} 0.26 & -0.04 & -0.04 & -0.03 & -0.03 & -0.03 & -0.03 \\ 0.27 & -0.04 & -0.04 & -0.04 & -0.03 & -0.03 & -0.03 \\ 0.28 & -0.04 & -0.04 & -0.04 & -0.04 & -0.03 & -0.03 \\ 0.30 & -0.04 & -0.04 & -0.04 & -0.04 & -0.04 & -0.03 \\ 0.31 & -0.04 & -0.04 & -0.04 & -0.04 & -0.04 & -0.04 \\ 0.32 & -0.05 & -0.04 & -0.04 & -0.04 & -0.04 & -0.04 \\ 0.34 & -0.05 & -0.05 & -0.04 & -0.04 & -0.04 & -0.04 \end{bmatrix},$$

and solved an LMI of dimension 9×9 with 1 optimization variable (i.e., a combination of the LMIs (4.4) and $P \succ 0$). Alternatively, using the Krasovskii approach, in the form of Proposition 2.5, it is possible to establish that (4.5) is GES by solving an LMI of dimension 21×21 with 49 optimization variables (i.e., a combination of an LMI that guarantees the decrease condition and one that guarantees the positive definiteness of the LKF). The above

results illustrate the differences in computational complexity of the approach proposed in this chapter and the Krasovskii approach. Interestingly, Proposition 3.3 indicates that the Razumikhin conditions of [93] and Theorem 2.3 are infeasible for this example. □

Currently, it remains unclear how to formulate an algorithm for stabilizing controller synthesis that can be solved via convex optimization algorithms.

4.2.4 *Positive linear delay difference equations*

It was shown in [49] that GES for positive linear DDEs is equivalent to a set of simple algebraic conditions, which have no clear relation to the Krasovskii and Razumikhin approaches. Motivated by this observation we proceed with establishing a tight lower bound on M for positive linear DDEs, which form an important class of systems that can model [49], e.g., biological systems and economic systems.

Definition 4.2 The DDI (2.1) is called a *positive linear DDE* if it is a linear DDE and the set of matrices $(A_{-h}, \ldots, A_0) \in (\mathbb{R}_+^{n \times n})^{h+1}$. □

Recall that $\mathbb{R}_+^{n \times n}$ is the set of all $n \times n$ matrices with nonnegative elements. For positive linear DDEs only initial conditions in $(\mathbb{R}_+^n)^{h+1}$ are of interest.

Proposition 4.2 Suppose that the DDE (2.1) is a positive linear DDE. If the positive linear DDE (2.1) is GES, then the hypothesis of the first statement of Theorem 4.2 can be satisfied with $M = h$. □

Proposition 4.2 is proven in Appendix B.3. As the first statement of Theorem 4.2 with $M = h$ corresponds to the interpretation of the Razumikhin approach that was proposed in [93] and Theorem 2.3, it follows from Proposition 4.2 that for positive linear DDEs the Razumikhin approach provides a set of necessary and sufficient conditions for GES. This establishes the dominance of the Razumikhin approach over the Krasovskii approach for positive linear DDEs, in the sense that both approaches are non-conservative but only the Razumikhin approach yields relatively simple conditions for stability that provide information about the system trajectories directly.

Remark 4.4 Crucial to Proposition 4.2 is that the augmented system constructed from the positive linear DDE admits a diagonal polyhedral LF if and only if the positive linear DDE is GES. Hence, a result similar to Proposition 4.2 is plausible for linear DDEs for which the corresponding augmented system satisfies one of the conditions derived in [71, Section 2.7], which also imply the existence of a diagonal polyhedral LF. □

4.3 Implications for set invariance

The sublevel sets of a function that corresponds to the Krasovskii approach are invariant. However, the conditions corresponding to the Krasovskii approach make use of the augmented system. As a consequence, the corresponding computations have to be executed

with respect to the augmented state space $(\mathbb{R}^n)^{h+1}$, which is not tractable when the computation of the maximal invariant set is of interest. On the other hand, the conditions corresponding to the Razumikhin approach apply to the original state space $\mathbb{R}^n$ directly and hence lead to computational procedures of much lower complexity. Unfortunately, the Razumikhin approach is conservative and, hence, unable to characterize the maximal invariant set. Similarly, the recent extension of periodic invariance, termed cyclic invariance [95], is less conservative than the standard Razumikhin approach, but remains conservative.

In what follows, we investigate what type of invariant set can be obtained from the conditions that were proposed in this chapter. To this end observe that an alternative set of necessary and sufficient conditions for $\mathcal{KL}$-stability (that better suits the construction of an invariant set) is the existence of a function V that satisfies (4.1a) and, for each compact set $\mathbb{X}$, of a finite $M(\mathbb{X}) \in \mathbb{Z}_{\geq h}$ such that

$$V(x_1) \leq \rho \max_{i \in \mathbb{Z}_{[-M, -M+h]}} V(x_i). \tag{4.6}$$

for all $\mathbf{x}_{[-M,0]} \in (\mathbb{R}^n)^{M+1}$ that are solutions to the DDI (2.1) of length $M+1$ and satisfy $\mathbf{x}_{[-M,-M+h]} \in \mathbb{X}$. This fact follows directly from the part of the proof of Theorem 4.1 that shows that the second statement implies the first, and from the observation that (4.6) implies (4.1b). Therefore, let us consider a function V that satisfies the hypothesis of Theorem 4.1 with (4.1b) replaced by (4.6). Define

$$X_M(\Phi) := \left\{ \{\phi_{-h+i}\} \times \ldots \times \{\phi_i\} : i \in \mathbb{Z}_{[0,M]} \right\},$$

where $\Phi \in \mathcal{S}(\mathbf{x}_{[-h,0]})$ for some $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$. Note that $X_M(\Phi) \subset (\mathbb{R}^n)^{h+1}$. Now, let $\mathbb{V}_1 := \{x \in \mathbb{R}^n : V(x) \leq 1\}$ and define

$$\mathbb{W} := \{X_M(\Phi) : \Phi \in \mathcal{S}(\mathbf{x}_{[-h,0]}), \mathbf{x}_{[-h,0]} \in \mathbb{V}_1^{h+1}\}, \tag{4.7}$$

such that $\mathbb{W} \subset (\mathbb{R}^n)^{h+1}$. Then, the inequality (4.6) yields that for all $\mathbf{x}_{[-h,0]} \in \mathbb{V}_1^{h+1}$ and all $\Phi \in \mathcal{S}(\mathbf{x}_{[-h,0]})$ it also holds that $\Phi_{[M-h+1,M+2]} \in \mathbb{V}_1^{h+1}$, which implies that the set $\mathbb{W}$ is invariant.

Note that the set $\mathbb{W}$ can be constructed directly from all solutions to the DDI (2.1) of length $M+1$ that satisfy $\mathbf{x}_{[-h,0]} \in \mathbb{V}^{h+1}$. Hence, given a function V and a constant M that satisfy the hypothesis of Theorem 4.1 with (4.1b) replaced by (4.6), one can obtain an invariant set by computing the system solutions and taking their union (in a proper manner).

Remark 4.5 If the sublevel sets of the function V are polytopes and the DDI (2.1) is a linear DDI, then it suffices to consider in the definition of $\mathbb{W}$ only the vertices of the set $\mathbb{V}^{h+1}$ and to use the implicit form of the convex hull of the resulting set. $\qquad\square$

To illustrate the above procedures, let us revisit Example 4.1.

Example 4.1 (Example 2.3, Part IV) Consider the function $V(x_0) := |x_0|$ and the constant $M := 3$ that were shown to satisfy the hypothesis of Theorem 4.2 in Example 4.1, Part I. Note that they also satisfy the hypothesis of Theorem 4.2 with (4.2b) replaced by

(4.6). Observe that $\mathbb{V}_1 = \mathbb{R}_{[-1,1]}$. As the sublevel sets of the function $V(x_0) := |x_0|$ are polytopic, it follows from Remark 4.5 that it suffices to consider $\mathbf{x}_{[-1,0]} \in \{-1, 1\} \cup \{1, -1\} \cup \{1, 1\} \cup \{-1, -1\}$ only. Hence, the invariant set $\mathbb{W}$ can be constructed from the convex hull of 4 trajectories of length $M + 1 = 4$. The results are illustrated in Figure 4.1. Note that invariance is guaranteed for trajectories of length $M + 1$, but in some cases shorter
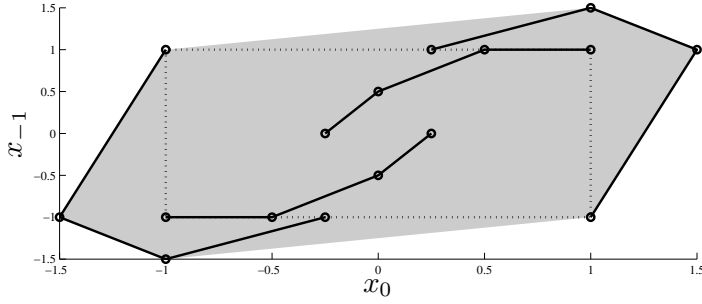

Figure 4.1: The invariant set $\mathbb{W}$ (grey), the set $\mathbb{V}_1^2 = \mathbb{R}_{[-1,1]}^2$ ($\cdots$) and the trajectories from the four vertices of $\mathbb{V}_1^2$ ($- \circ -$).

trajectories may suffice, e.g., as it is the case in this example for trajectories of length 3. $\quad\square$

The above example illustrates that the proposed concept allows to construct invariant sets for DDEs efficiently, which is a fact that we will explore in more detail in the next chapter.

4.4 Conclusions

In this chapter we proposed a relaxation of the Razumikhin approach for DDIs and we proved that the corresponding relaxed conditions are necessary and sufficient for stability. Moreover, we also showed that the relaxed Razumikhin conditions still yield computational procedures for the construction of invariant sets that can be executed in the original, non-augmented state space of the DDI dynamics. Throughout this chapter, the benefits of these novel conditions for stability analysis of linear DDEs and positive linear DDEs were indicated. For positive linear DDEs we proved that the newly proposed conditions are equivalent with the Razumikhin approach. This establishes the non-conservatism and, hence, dominance of the Razumikhin approach over the Krasovskii approach for such systems.

It should be clear that the method that is proposed in this chapter has several important advantages for the stability analysis of DDIs when compared to the techniques that were discussed in Chapters 2 and 3. Unfortunately, it remains unclear how to use this method for stabilizing controller synthesis. In the next chapter we study the stability analysis of constrained DDIs. To this end we further develop the implications for set invariance of the conditions that were derived in this chapter. In particular, we will propose an invariance concept that combines the conceptual generality of the Krasovskii approach with the computational advantages of the Razumikhin approach. Thus, a computationally efficient method for the construction of invariant sets for DDIs will be obtained that is able to characterize the maximal invariant set.

Chapter 5

Stability analysis of constrained delay difference inclusions

In this chapter we consider the stability analysis of constrained delay difference inclusions. To this end, the construction of invariant sets for DDIs is studied. Existing methods for the construction of invariant sets suffer either from computational intractability or come with considerable conservatism, with respect to their ability to provide a nontrivial invariant set as well as their ability to characterize the maximal invariant set. Therefore, we apply the concept of invariant families of sets to DDIs in this chapter. This notion enjoys computational practicability and at the same time is non-conservative, both in terms of the type of sets it produces and its ability to characterize the maximal invariant set. Moreover, this technique also provides a tractable stability analysis tool for DDIs. Throughout this chapter, we analyse the properties of invariant families of sets and we illustrate their application via several examples. A variety of methods that can be used for the construction of invariant families of sets via convex optimization algorithms is also provided.

5.1 Introduction

The stability analysis and stabilization of dynamical systems that are subject to constraints, as it is often the case in practice, forms a challenging topic in the field of control theory. An essential tool for almost every control approach for constrained systems, such as, e.g., model predictive control (MPC) [119] and Lyapunov-based control [3], is the construction of an invariant set. In this context, Lyapunov theory is frequently used to study the existence and construction of such sets [14,17,86]. However, often finding just any invariant set is not sufficient, rather one would like to find the largest possible invariant set or maximal invariant set. To this end, iterative algorithms for the approximate construction of the maximal invariant can be used, see, e.g., [6,17,118]. Unfortunately, the corresponding computational procedures are of considerable complexity.

This computational drawback will prove to be crucial for the construction of invariant sets for systems with delay. As explained in Section 2.3, an augmented system without delay can be constructed from a discrete-time system with delay. To this augmented system the classical methods for the construction of invariant sets, such as the ones that were mentioned above, can be applied. Then, the invariant set for the augmented system provides an

invariant set for the system with delay. Obviously, this approach, which has been applied in, e.g., [97, 101, 110], is in some sense an application of the Krasovskii approach, see Section 2.5. However, the dimension of the augmented system increases with the size of the delay so that the corresponding computational effort renders the Krasovskii-based methods impracticable. Hence, a second category of synthesis algorithms, essentially based on the Razumikhin approach, has been considered. Therein, a set in a lower-dimensional state space is obtained satisfying particular conditions, i.e., $\mathcal{D}$-invariance, such that the Cartesian product of this set provides a standard invariant set for the system with delay [30, 55, 95, 96]. However, also when set invariance is of concern, the Razumikhin approach is conservative, i.e., there exist systems for which, in contrast to the Krasovskii approach, the Razumikhin approach does not yield an invariant set (excluding the trivial invariant set $\{0\}$). Moreover, in most cases a set obtained via this approach is not able to characterize the maximal invariant set. Furthermore, the recent notion of cyclic invariance [95], which is of a similar computational complexity but less conservative than the Razumikhin approach, remains conservative and, hence, suffers from similar drawbacks. As such, it would be desirable to obtain a method that is based on necessary and sufficient conditions for stability and at the same time enjoys computational practicability.

Therefore, we apply the notion of invariant families of sets [5] in combination with the concepts of vector LFs [83, 141] and set dynamics [4] to DDIs. This approach is essentially an application of the results in Chapter 4 for set invariance. More specifically, a relatively simple comparison system is constructed from a given family of sets and the corresponding set dynamics. Then, by constructing a standard invariant set for the relatively low-dimensional comparison system, an invariant family of sets can be obtained for the DDI, which is similar to the approach proposed in [15]. We show that the proposed framework combines the computational convenience of the Razumikhin approach with the conceptual generality of the Krasovskii approach, i.e., it can characterize the maximal invariant set at a relatively low computational cost. Moreover, this technique also provides a tractable stability analysis tool for DDIs. Throughout this chapter, we analyse the properties of invariant families of sets and we illustrate their conceptual generality and computational advantages via several examples. To obtain tractable synthesis methods, two different parameterized families of sets are proposed, which lead to synthesis problems that can be solved via convex optimization algorithms.

5.2 Invariant sets for delay difference inclusions

In this chapter the DDI (2.1) is considered together with a set of state constraints, i.e.,

$$\mathbf{x}_{[k-h,k]} \in \mathbb{C}, \quad \forall k \in \mathbb{Z}_+.$$

The standing assumptions, throughout this chapter, are:

Assumption 5.1 $\mathbb{C} \subseteq (\mathbb{R}^n)^{h+1}$ is a proper C-set. □

Assumption 5.2 The map $F : (\mathbb{R}^n)^{h+1} \rightrightarrows \mathbb{R}^n$ that generates the dynamics (2.1) is upper semicontinuous and the set $F(\mathbf{x}_{[-h,0]})$ is compact for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$. □

To take into account the constraints, consider the following addition to Definition 2.5.

Definition 5.1 *(i)* A set $\bar{\mathbb{X}} \subseteq (\mathbb{R}^n)^{h+1}$ is called an *invariant set for the DDI* (2.1) *and the constraint set* $\mathbb{C}$ if $\bar{\mathbb{X}} \subseteq \mathbb{C}$ and $\mathbf{x}_{[-h+1,1]} \in \bar{\mathbb{X}}$ for all $\mathbf{x}_{[-h,0]} \in \bar{\mathbb{X}}$ and all $x_1 \in F(\mathbf{x}_{[-h,0]})$; and *(ii)* the set $\bar{\mathbb{X}}_{\mathrm{MAX}}$ is called *the maximal invariant set for* (2.1) *and* $\mathbb{C}$ if $\bar{\mathbb{X}}_{\mathrm{MAX}}$ is an invariant set for (2.1) and $\mathbb{C}$ and $\bar{\mathbb{X}}_{\mathrm{MAX}}$ contains all other invariant sets. $\qquad\square$

Throughout this chapter when a set is called invariant it is (unless specified otherwise) an invariant set for the DDI (2.1) and the constraint set $\mathbb{C}$.

When combined with a stability analysis, e.g., using the results in Chapters 2 - 4, an invariant set *in the presence of constraints* provides information about the set of initial conditions for which the corresponding trajectories converge to the origin and constraint satisfaction is guaranteed at all times. Such an invariant set can be constructed based on the Krasovskii approach. Indeed, suppose that there exists a function $\bar{V}$ that satisfies the hypothesis of the first statement of Theorem 2.1 with $\bar{\rho} = 1$. Then, any sublevel set of $\bar{V}$ that is a subset of $\mathbb{C}$ is an invariant set, i.e., let

$$\bar{\mathbb{V}}_\gamma := \{\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1} \; : \; \bar{V}(\mathbf{x}_{[-h,0]}) \leq \gamma\},$$

for some $\gamma \in \mathbb{R}_{\geq 0}$ such that $\bar{\mathbb{V}}_\gamma \subseteq \mathbb{C}$. Then, it follows from (2.3b) that $\mathbf{x}_{[-h+1,1]} \in \bar{\mathbb{V}}_\gamma$ for all $\mathbf{x}_{[-h,0]} \in \bar{\mathbb{V}}_\gamma$ and all $x_1 \in F(\mathbf{x}_{[-h,0]})$. Hence, $\bar{\mathbb{V}}_\gamma$ is an invariant set for the DDI (2.1) and the constraint set $\mathbb{C}$. Alternatively, an invariant set *in the presence of constraints* can also be constructed based on the Razumikhin approach. Indeed, suppose that there exists a function V that satisfies the hypothesis of Theorem 2.3 with $\rho = 1$. Then, the set $\mathbb{V}_\gamma^{h+1}$ obtained from the $h + 1$-times Cartesian product of a sublevel set of V, i.e.,

$$\mathbb{V}_\gamma := \{x \in \mathbb{R}^n : V(x) \leq \gamma\},$$

for some $\gamma \in \mathbb{R}_{\geq 0}$ satisfying $\mathbb{V}_\gamma^{h+1} \subseteq \mathbb{C}$, is an invariant set for the DDI (2.1) and the constraint set $\mathbb{C}$ as (2.10b) ensures that $F(\mathbf{x}_{[-h,0]}) \subseteq \mathbb{V}_\gamma$ for all $\mathbf{x}_{[-h,0]} \in \mathbb{V}_\gamma^{h+1}$.

In the above context the Krasovskii approach leads to necessary and sufficient conditions for the stability and invariance analyses of the DDI (2.1). However, it also requires the corresponding computations to be carried out with respect to the augmented state space $(\mathbb{R}^n)^{h+1}$ and, hence, it might fail to be practicable for systems with large delays. On the other hand, the Razumikhin approach enjoys computational practicability since the underlying computations can be executed with respect to the non-augmented state space $\mathbb{R}^n$. Unfortunately, the Razumikhin approach provides only sufficient conditions and in most cases is not able to characterize the maximal invariant set. Therefore, the main aim of this chapter is to develop an invariance notion that preserves the conceptual generality of the Krasovskii approach while, at the same time, has a computational complexity comparable to the Razumikhin approach. Moreover, at the very least this notion should be such that an invariant set can be computed for linear DDEs via convex optimization algorithms.

5.3 Invariant families of sets

A combination of conceptual generality and computational convenience for the stability analysis of DDIs was already attained in Chapter 4 via the introduction of the variable M. Now consider the following definition, which is an extension of the results in Section 4.3.

Definition 5.2 A family of ($h + 1$-tuples of) sets $\mathcal{X}$, $\mathcal{X} \subset (\mathrm{Com}(\mathbb{R}^n))^{h+1}$ is called *an invariant family of sets for the DDI* (2.1) *and the constraint set* $\mathbb{C}$ if for all $(X_1, \ldots, X_{h+1}) \in \mathcal{X}$ it holds that

$$X_1 \times X_2 \times \ldots \times X_{h+1} \subseteq \mathbb{C},$$

and there exists a $(X_1', X_2', \ldots, X_{h+1}') \in \mathcal{X}$ such that

$$F(X_1, \ldots, X_{h+1}) \subseteq X_{h+1}' \text{ and } X_{i+1} \subseteq X_i', \quad \forall i \in \mathbb{Z}_{[1,h]},$$

where $F(X_1, \ldots, X_{h+1}) := \{F(\mathbf{x}_{[-h,0]}) : x_{-h} \in X_1, \ldots, x_0 \in X_{h+1}\}$. $\qquad\square$

In what follows, whenever a family of sets $\mathcal{X}$ is called an invariant family it is (unless specified otherwise) an invariant family for the DDI (2.1) and the constraint set $\mathbb{C}$.

Remark 5.1 In view of the relation of DDIs to interconnected systems that was developed in Chapter 3, it follows that Definition 5.2 is compatible with the practical set invariance notions for interconnected systems that were developed in [116, 117]. $\qquad\square$

It is important to note that an *invariant family of sets* is not necessarily composed of invariant sets. An invariant family of sets $\mathcal{X}$, cf. Definition 5.2, is a set of $h + 1$-tuples of sets each of which is a subset of $\mathbb{R}^n$ and whose Cartesian product is not necessarily an invariant set. Nevertheless, the notion of an invariant family of sets is closely related to the classical set invariance notion. The following four propositions, which are all proven in Appendix B.4, serve to clarify this relation. In particular, the first two results establish analogues of classical set invariance properties, which rely on the following definitions, i.e.,

$$\mathcal{X} \cup \mathcal{Y} := \{(Z_1, \ldots, Z_{h+1}) : (Z_1, \ldots, Z_{h+1}) \in \mathcal{X} \text{ or } (Z_1, \ldots, Z_{h+1}) \in \mathcal{Y}\},$$
$$\lambda \mathcal{X} + (1 - \lambda)\mathcal{Y} := \{(\lambda X_1 \oplus (1 - \lambda)Y_1, \ldots, \lambda X_{h+1} \oplus (1 - \lambda)Y_{h+1}) :$$
$$(X_1, \ldots, X_{h+1}) \in \mathcal{X}, (Y_1, \ldots, Y_{h+1}) \in \mathcal{Y}\}, \quad \lambda \in \mathbb{R}_{[0,1]},$$
$$\mathrm{co}(\mathcal{X}, \mathcal{Y}) := \bigcup_{\lambda \in \mathbb{R}_{[0,1]}} \lambda \mathcal{X} + (1 - \lambda)\mathcal{Y}.$$

Proposition 5.1 Let $\mathcal{X}$ and $\mathcal{Y}$ be two non-empty invariant families of sets. Then, $\mathcal{X} \cup \mathcal{Y}$ is an invariant family of sets. $\qquad\square$

Proposition 5.2 Suppose that the DDI (2.1) is a linear DDI. Let $\mathcal{X}$ and $\mathcal{Y}$ be two non-empty invariant families of sets. Then, the following claims hold:
  (i) $\lambda \mathcal{X} + (1 - \lambda)\mathcal{Y}$ is an invariant family of sets for all $\lambda \in \mathbb{R}_{[0,1]}$;
  (ii) $\mathrm{co}(\mathcal{X}, \mathcal{Y})$ is an invariant family of sets. $\qquad\square$

Next, we establish a relation of invariant families of sets to standard invariant sets for DDIs.

Proposition 5.3 Let $\mathcal{X}$ denote an invariant family of sets. Then, $\bigcup_{(X_1, \ldots, X_{h+1}) \in \mathcal{X}} X_1 \times \ldots \times X_{h+1} \subset (\mathbb{R}^n)^{h+1}$ is an invariant set. $\qquad\square$

Furthermore, it is possible to construct an invariant family of sets from an invariant set.

Proposition 5.4 Consider a set $\mathbb{X} \subseteq \mathbb{C}$ that is invariant for the DDI (2.1). Then, there exists an invariant family of sets $\mathcal{X}$ such that $\bigcup_{(X_1,\ldots,X_{h+1}) \in \mathcal{X}} X_1 \times \ldots \times X_{h+1} = \mathbb{X}$. $\qquad\square$

Note that the proof of Proposition 5.4 provides a procedure to construct an invariant family of sets from an invariant set, i.e., for any invariant set, e.g., $\bar{\mathbb{V}}_\gamma \subseteq \mathbb{C}$, the family of sets

$$\mathcal{X} := \left\{ \left(\bigcup_{\Phi \in \mathcal{S}(\mathbf{x}_{[-h,0]})} \{\phi_{k-h}\}, \ldots, \bigcup_{\Phi \in \mathcal{S}(\mathbf{x}_{[-h,0]})} \{\phi_k\} \right) : (\mathbf{x}_{[-h,0]}, k) \in \bar{\mathbb{V}}_\gamma \times \mathbb{Z}_+ \right\},$$

is, by construction, an invariant family of sets. Hence, the concept of an invariant family of sets is as general as the standard invariance notion from Definition 5.1.

An important consequence of Proposition 5.4 is that it is possible to characterize the maximal invariant set via an invariant family of sets. Interestingly, there may be more invariant families of sets that yield, via Proposition 5.3, the maximal invariant set. On the other hand, as the union of two invariant families of sets is also invariant, it is possible to define the maximal invariant family of sets. The corresponding notion is not discussed here as it lies beyond the intended scope of this chapter.

The concept of an invariant family of sets is more general than the type of sets that can be obtained via the Razumikhin approach. Indeed, consider an invariant set, e.g., $\mathbb{V}_\gamma^{h+1} \subseteq \mathbb{X}$, constructed via the Razumikhin approach. Then, $F(\mathbb{V}_\gamma, \ldots, \mathbb{V}_\gamma) \subseteq \mathbb{V}_\gamma$. Hence, the family of sets $\mathcal{X} := \{(\mathbb{V}_\gamma, \ldots, \mathbb{V}_\gamma)\}$ is an invariant family of sets. A similar observation applies to the recent concept of cyclic invariance [95]. The following example, however, shows that there exist DDIs for which it is not possible to find a nontrivial invariant set via the Razumikhin approach even though an invariant family of sets can be constructed for that DDI. This example corresponds to Examples 2.3, 3.1 and 4.1 with $a = 1$ and $b = -0.5$.

Example 5.1 (Example 2.3, Part V) Consider the scalar linear DDE

$$x_{k+1} = -0.5x_{k-1} + x_k, \quad k \in \mathbb{Z}_+, \tag{5.1}$$

with the constraints $\mathbf{x}_{[k-1,k]} \in \mathbb{C} := \mathbb{R}_{[-1,1]} \times \mathbb{R}_{[-1,1]}$ for all $k \in \mathbb{Z}_+$. In Proposition 2.1 it was established that the linear DDE (5.1) does not admit a function V that satisfies the hypothesis of Theorem 2.3 and hence it follows from Proposition 2.4 that (5.1) does not admit a nontrivial invariant set constructed via the Razumikhin approach. The only trivial invariant set is $\{0\} \times \{0\}$.

Consider the following family of pairs of singleton sets

$$\mathcal{X} := \{(\{\eta\}, \{\eta\}), (\{\eta\}, \{0.5\eta\}), (\{\eta\}, \{0\})\}, (\{0\}, \{\eta\}) : \eta \in \mathbb{R}_{[-1,1]}\}.$$

Note that this family of sets has some similarities to the one that was used in the proof of Proposition 5.4 and discussed above. By definition of $\mathcal{X}$, for all $(X_1, X_2) \in \mathcal{X}$ it holds that $X_1 \times X_2 \subseteq \mathbb{C}$. Moreover, a direct calculation verifies that for all $(X_1, X_2) \in \mathcal{X}$ there exists an $X_3 \in \text{Com}(\mathbb{R})$ such that $(X_2, X_3) \in \mathcal{X}$ and $-\frac{1}{2}X_1 \oplus X_2 \subseteq X_3$. Therefore, $\mathcal{X}$ is

an invariant family of sets for the linear DDE (5.1) and the constraint set $\mathbb{C}$. Similarly, the family of sets

$$\mathcal{Y} := \{(\{\eta\}, \{-0.5\eta\}), (\{0.5\eta\}, \{\eta\}), (\{\eta\}, \{0.75\eta\}), (\{\eta\}, \{0.33\eta\}) : \eta \in \mathbb{R}_{[-1,1]}\},$$

is also an invariant family of sets. Moreover, by Proposition 5.2 and Proposition 5.3, $\mathrm{co}(\mathcal{X}, \mathcal{Y})$ is an invariant family of sets and $\mathbb{X} := \bigcup_{(X_1, X_2) \in \mathrm{co}(\mathcal{X}, \mathcal{Y})} X_1 \times X_2$ is an invariant set. Interestingly, this set is equal to the maximal invariant set $\mathbb{X}_{\mathrm{MAX}}$ for the linear DDE (5.1) and the constraint set $\mathbb{C}$, which can be computed via the Krasovskii approach. In Figure 5.1 the set $\mathbb{C}$, the invariant family of sets $\mathcal{X}$, the invariant set constructed from the
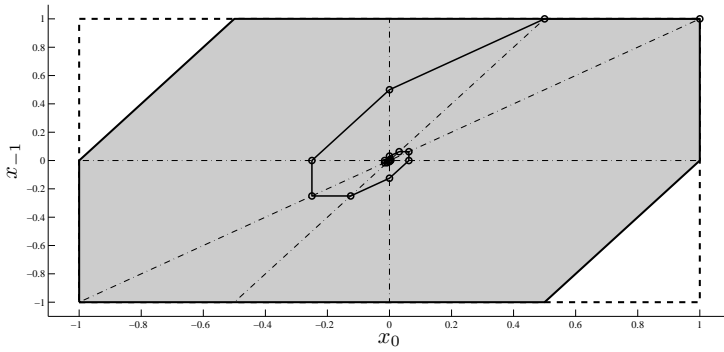

Figure 5.1: The constraint set $\mathbb{C}$ (- - -), the invariant family of sets $\mathcal{X}$ $(- \cdot -)$, the invariant set constructed from the family $\mathrm{co}(\mathcal{X}, \mathcal{Y})$ (——), the maximal invariant set $\mathbb{X}_{\mathrm{MAX}}$ (in grey) and a trajectory $(- \circ -)$ for the linear DDE (5.1) and the constraint set $\mathbb{C}$.

family $\mathrm{co}(\mathcal{X}, \mathcal{Y})$, the maximal invariant set $\mathbb{X}_{\mathrm{MAX}}$ and a trajectory for (5.1) are shown. $\qquad \square$

As proven above and illustrated in Example 5.1, the concept of an invariant family of sets is as general as the classical invariance notion associated with the Krasovskii approach. Moreover, note that with respect to the discussion in Section 4.3 the family of sets

$$\mathcal{X} := \left\{ \left(\bigcup_{\Phi \in \mathcal{S}(\mathbf{x}_{[-h,0]})} \{\phi_{i-h}\}, \ldots, \bigcup_{\Phi \in \mathcal{S}(\mathbf{x}_{[-h,0]})} \{\phi_i\} \right) : \mathbf{x}_{[-h,0]} \in \mathbb{V}_1^{h+1}, \ i \in \mathbb{Z}_{[0,M]} \right\},$$

is an invariant family of sets. Therefore, the discussion in Section 4.3 indicates that every linear DDI that is stable admits a nontrivial invariant family of sets that can be described by a finite number of $h+1$-tuples of sets and parameters, as it is also the case in Example 5.1.

In what follows, dynamics of vector LFs [141] are employed in conjunction with set dynamics [4] to demonstrate that an invariant family of sets can be constructed via an invariant set for a relatively low-dimensional comparison system. Therefore, let the dynamics of the $h+1$-th set be given by the map $g_{h+1} : (\mathrm{Com}(\mathbb{R}^n))^{h+1} \rightrightarrows \mathrm{Com}(\mathbb{R}^n)$ with

$$g_{h+1}(X_1, \ldots, X_{h+1}) := F(X_1, \ldots, X_{h+1}),$$

where the range of g_{h+1} is $\mathrm{Com}(\mathbb{R}^n)$ due to Assumption 5.2. Furthermore, let the dynamics of the j-th set, $j \in \mathbb{Z}_{[1,h]}$ be given by the map $g_j : (\mathrm{Com}(\mathbb{R}^n))^{h+1} \rightrightarrows \mathrm{Com}(\mathbb{R}^n)$ with

$$g_j(X_1, \ldots, X_{h+1}) := X_{j+1}, \quad \forall j \in \mathbb{Z}_{[1,h]}.$$

Then, consider a set of functions $W_j : \mathrm{Com}(\mathbb{R}^n) \to \mathbb{R}_+$, $j \in \mathbb{Z}_{[1,h+1]}$ such that, for all $X_1 \in \mathrm{Com}(\mathbb{R}^n)$ and all $j \in \mathbb{Z}_{[1,h+1]}$ it holds that

$$\alpha_1(\|X_1\|) \le W_j(X_1) \le \alpha_2(\|X_1\|), \tag{5.2a}$$

for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$. Above, $\|X_1\| := \max\{\|x_0\| : x_0 \in X_1\}$. The above definitions implicitly define a set of functions $f_j : \mathbb{R}_+^{h+1} \to \mathbb{R}_+$, $j \in \mathbb{Z}_{[1,h+1]}$ such that, for all $(X_1, \ldots, X_{h+1}) \in (\mathrm{Com}(\mathbb{R}^n))^{h+1}$ it holds that

$$W_j(g_j(X_1, \ldots, X_{h+1})) = f_j(W_1(X_1), \ldots, W_{h+1}(X_{h+1})), \tag{5.2b}$$

for all $j \in \mathbb{Z}_{[1,h+1]}$. The equations (5.2b), $j \in \mathbb{Z}_{[1,h+1]}$ form a dynamical system that provides information about the evolution of the $h+1$-tuples of sets in $(\mathrm{Com}(\mathbb{R}^n))^{h+1}$ "filtered" through the functions W_j, $j \in \mathbb{Z}_{[1,h+1]}$. Hence, define $W := \mathrm{col}(\{W_i\}_{i \in \mathbb{Z}_{[1,h+1]}})$ and $f(W) := \mathrm{col}(\{f_i(W)\}_{i \in \mathbb{Z}_{[1,h+1]}})$ to obtain the dynamics of the comparison system

$$W_{k+1} = f(W_k), \quad k \in \mathbb{Z}_+. \tag{5.3}$$

Observe that (5.3) is, by construction, a positive dynamical system. Finally, define the projection of the constraints $\mathbb{C}$ onto the state space of the comparison system (5.3), i.e.,

$$\bar{\mathbb{T}} := \{\mathrm{col}(\{W_i(X_i)\}_{i \in \mathbb{Z}_{[1,h+1]}}) \in \mathbb{R}_+^{h+1} : (X_1, \ldots, X_{h+1}) \in (\mathrm{Com}(\mathbb{R}^n))^{h+1}$$
$$\text{with } X_1 \times \ldots \times X_{h+1} \subseteq \mathbb{C}\}.$$

Theorem 5.1 Consider the family of sets $(\mathrm{Com}(\mathbb{R}^n))^{h+1}$. Suppose that there exist $h+1$ functions $W_j : \mathrm{Com}(\mathbb{R}^n) \to \mathbb{R}_+$, $j \in \mathbb{Z}_{[1,h+1]}$ that satisfy (5.2a) and (5.2b), for some $f_j : \mathbb{R}_+^{h+1} \to \mathbb{R}_+$. Then, the following claims hold:

(i) If $\mathbb{T} \ne \emptyset$, $\mathbb{T} \subseteq \bar{\mathbb{T}}$ is an invariant set for (5.3), then the family of sets $\mathcal{X}_{\mathbb{T}} := \{(X_1, \ldots, X_{h+1}) \subseteq (\mathrm{Com}(\mathbb{R}^n))^{h+1} : \mathrm{col}(\{W_i(X_i)\}_{i \in \mathbb{Z}_{[1,h+1]}}) \in \mathbb{T}\}$ is invariant;

(ii) if the comparison system (5.3) is $\mathcal{KL}$-stable, then the DDI (2.1) is $\mathcal{KL}$-stable. $\qquad\square$

Theorem 5.1 is proven in Appendix B.4. Using the above notions, the construction of an invariant family of sets for the DDI (2.1) requires the construction of a comparison system and the search for an invariant set for that comparison system in $\mathbb{R}_+^{h+1}$. Consequently, the complexity of the construction of an invariant set for the DDI (2.1) taking into account the constraint set $\mathbb{C}$ is reduced while the conceptual generality of the Krasovskii approach is preserved. Thus, for systems with large delays, constructing an invariant family of sets remains practicable while the Krasovskii approach becomes computationally prohibitive.

Remark 5.2 The above results can be extended directly to allow for a local and relaxed version of Theorem 5.1. Indeed, the analysis above and assertions of Theorem 5.1 remain valid if the family of sets $(\text{Com}(\mathbb{R}^n))^{h+1}$ is replaced by any non-empty family of sets, say $\mathcal{X} \subseteq (\text{Com}(\mathbb{R}^n))^{h+1}$ that is invariant with respect to the system dynamics (2.1), i.e., such that, for all $(X_1, \ldots, X_{h+1}) \in \mathcal{X}$ it holds that $(X'_1, \ldots, X'_{h+1}) \in \mathcal{X}$ where $X'_j := g_j(X_1, \ldots, X_{h+1})$, $j \in \mathbb{Z}_{[1,h+1]}$. This relevant but direct extension offers no challenge since it requires merely notational changes and, hence, it is omitted here. □

5.4 Parameterized families of sets

To obtain more specific and explicit results, *only linear DDIs are considered in the remainder of this chapter*. Recall that linear DDIs are DDIs of the form (2.1) with the structure defined in Definition 2.1. Furthermore, to facilitate the construction of an invariant family of sets, we consider the parameterized family of sets

$$\mathcal{X}(S_1, \ldots, S_{h+1}, \Theta) := \{([\theta]_1 S_1, \ldots, [\theta]_{h+1} S_{h+1}) : \theta \in \Theta\}, \tag{5.4}$$

with $\theta \in \mathbb{R}_+^{h+1}$ and where $\Theta \subseteq \mathbb{R}_+^{h+1}$ and $S_i \subseteq \mathbb{R}^n$, $i \in \mathbb{Z}_{[1,h+1]}$ are fixed. In what follows, the third standing assumption is:

Assumption 5.3 For all $i \in \mathbb{Z}_{[1,h+1]}$ the set $S_i \subseteq \mathbb{R}^n$ is a proper C-set. □

If certain conditions on θ can be imposed, the parametrization (5.4) provides an invariant family of sets. These conditions are derived next.

Proposition 5.5 Suppose that the DDI (2.1) is a linear DDI. Consider the parameterized family of sets (5.4) with $\Theta \neq \emptyset$, $\Theta \subseteq \mathbb{R}_+^{h+1}$. If, for all $\theta \in \Theta$, it holds that

$$[\theta]_1 S_1 \times [\theta]_2 S_2 \times \ldots \times [\theta]_{h+1} S_{h+1} \subseteq \mathbb{C},$$

and there exists a $\theta' \in \Theta$ such that $[\theta]_{j+1} S_{j+1} \subseteq [\theta']_j S_j$ for all $j \in \mathbb{Z}_{[1,h]}$ and that

$$A_{-h}[\theta]_1 S_1 \oplus \ldots \oplus A_0[\theta]_{h+1} S_{h+1} \subseteq [\theta']_{h+1} S_{h+1},$$

for all $(A_{-h}, \ldots, A_0) \in \mathcal{A}$, then $\mathcal{X}(S_1, \ldots, S_{h+1}, \Theta)$ is an invariant family of sets. □

Proposition 5.5 is proven in Appendix B.4. If the parameterized family of sets (5.4) is invariant, the properties established in Propositions 5.1 and 5.2 can be sharpened.

Proposition 5.6 Suppose that the DDI (2.1) is a linear DDI. Let $\mathcal{X}(S_1, \ldots, S_{h+1}, \Theta_i)$, $i \in \mathbb{Z}_{[1,2]}$ denote two invariant parameterized families of sets with $\Theta_i \neq \emptyset$, $i \in \mathbb{Z}_{[1,2]}$ and $(\Theta_1, \Theta_2) \subseteq (\mathbb{R}_+^{h+1})^2$. Then, the following claims hold:

(i) $\mathcal{X}(S_1, \ldots, S_{h+1}, \Theta_1 \cup \Theta_2)$ is an invariant parameterized family of sets;

(ii) $\mathcal{X}(S_1, \ldots, S_{h+1}, \lambda \Theta_1 \oplus (1 - \lambda)\Theta_2)$ is invariant for all $\lambda \in \mathbb{R}_{[0,1]}$;

(iii) $\mathcal{X}(S_1, \ldots, S_{h+1}, \text{co}(\Theta_1 \cup \Theta_2))$ is an invariant parameterized family of sets. □

Proposition 5.6 is proven in Appendix B.4. Note that Proposition 5.6 provides the necessary tools to characterize the maximal invariant parameterized family of sets for fixed shape sets $S_1, \ldots, S_{h+1}$. Namely, this family is induced by the maximal (with respect to set inclusion) set Θ satisfying conditions postulated in Proposition 5.5.

Given the parametrization (5.4) the comparison system (5.3) reduces to the θ-dynamics generated by the function $f_{h+1} : \mathbb{R}_+^{h+1} \to \mathbb{R}_+$ satisfying

$$f_{h+1}(\theta) := \max_{(A_{-h}, \ldots, A_0) \in \mathcal{A}} \min_{\eta} \{ \eta \in \mathbb{R}_+ : \bigoplus_{j=1}^{h+1} A_{-h-1+j} [\theta]_j S_j \subseteq \eta S_{h+1} \},$$

and the functions $f_j : \mathbb{R}_+^{h+1} \to \mathbb{R}_+$, satisfying

$$f_j(\theta) := \min_{\eta} \{ \eta \in \mathbb{R}_+ : [\theta]_{j+1} S_{j+1} \subseteq \eta S_j \}, \quad \forall j \in \mathbb{Z}_{[1,h]}.$$

Letting $f : \mathbb{R}_+^{h+1} \to \mathbb{R}_+^{h+1}$ with $f(\theta) := \mathrm{col}(\{f_i(\theta)\}_{i \in \mathbb{Z}_{[1,h+1]}})$ yields the θ-dynamics

$$\theta_{k+1} = f(\theta_k), \quad k \in \mathbb{Z}_+. \tag{5.5}$$

In this case, the constraints on θ are given by

$$\theta \in \bar{\Theta} := \{ \theta \in \mathbb{R}_+^{h+1} : [\theta]_1 S_1 \times \ldots \times [\theta]_{h+1} S_{h+1} \subseteq \mathbb{C} \},$$

which is the equivalent of $\bar{\mathbb{T}}$ for parameterized families of sets. Note that it follows from Assumptions 5.1 and 5.3 that $\mathrm{int}(\bar{\Theta}) \neq \emptyset$.

The following result is a relevant analogue of Theorem 5.1.

Theorem 5.2 Suppose that the DDI (2.1) is a linear DDI and consider the parameterized family of sets (5.4). Then, the following claims hold:

   (i) The functions f_i, $i \in \mathbb{Z}_{[1,h+1]}$ are sublinear functions;

  (ii) if the set $\Theta \neq \emptyset$ and $\Theta \subseteq \bar{\Theta}$ is an invariant set for (5.5), then $\mathcal{X}(S_1, \ldots, S_{h+1}, \Theta)$ is an invariant family of sets;

 (iii) if (5.5) is $\mathcal{KL}$-stable, then the linear DDI (2.1) is $\mathcal{KL}$-stable.   □

Theorem 5.2 is proven in Appendix B.4. To simplify the computations required to obtain an invariant set for the sublinear dynamics (5.5), a linear upper bound on f can be employed. To this end, consider a matrix $M \in \mathbb{R}_+^{h+1 \times h+1}$ and let, for all $j \in \mathbb{Z}_{[1,h+1]}$

$$[M]_{h+1,j} := \max_{(A_{-h}, \ldots, A_0) \in \mathcal{A}} \min_{\eta \in \mathbb{R}_+} \{ \eta : A_{-h-1+j} S_j \subseteq \eta S_{h+1} \}. \tag{5.6a}$$

Thus, for all $\theta \in \mathbb{R}_+^{h+1}$, it holds that $f_{h+1}(\theta) \leq \sum_{j=1}^{h+1} [M]_{h+1,j} [\theta]_j$. Furthermore, let

$$[M]_{j,j+1} := \min_{\eta \in \mathbb{R}_+} \{ \eta : S_{j+1} \subseteq \eta S_j \}, \tag{5.6b}$$

for all $j \in \mathbb{Z}_{[1,h]}$ and let the elements of M be equal to zero elsewhere. Thus, linear θ-dynamics that provide an upper bound for the sublinear dynamics (5.5) are obtained, i.e.,

$$\theta_{k+1} = M\theta_k, \quad k \in \mathbb{Z}_+. \tag{5.7}$$

As (5.5) is a positive system and (5.7) upper bounds (5.5), it follows that Theorem 5.2 also applies to (5.7).

The following result relates the scalars specified in (5.6) to the stability of (5.7).

Lemma 5.1 The matrix $M \in \mathbb{R}_+^{h+1 \times h+1}$ satisfies $\mathrm{sr}(M) < 1$ if and only if

$$\sum_{j=1}^{h} \prod_{i=j}^{h} [M]_{i,i+1} [M]_{h+1,j} + [M]_{h+1,h+1} < 1. \tag{5.8}$$

□

Lemma 5.1 is proven in Appendix B.4. Obviously, using the linear dynamics (5.7) rather than the sublinear dynamics (5.5) simplifies the computations required to obtain an invariant family of sets. Indeed, as M is a positive matrix, the comparison system (5.7) is $\mathcal{KL}$-stable if and only if [11] there exists a vector $p \in \mathbb{R}_{>0}^{h+1}$ such that $[Mp]_i < [p]_i$ for all $i \in \mathbb{Z}_{[1,h+1]}$. Moreover, the fact that $[Mp]_i < [p]_i$ for all $i \in \mathbb{Z}_{[1,h+1]}$ also implies that the set $\Theta = \{\theta \in \mathbb{R}_+^{h+1} : [\theta]_i \leq [p]_i, \ \forall i \in \mathbb{Z}_{[1,h+1]}\}$ is an invariant set for the comparison system (5.7). Note that the vector $p \in \mathbb{R}_{>0}^{h+1}$ can be obtained by solving a simple linear program and, hence, an invariant set for (5.7) is easy to obtain.

Furthermore, the maximal invariant set Θ_{MAX} for the comparison system (5.7) subject to the constraints $\bar{\Theta}$ is [17] the Hausdorff limit of the sequence of sets $\{\Theta_k\}_{k \in \mathbb{Z}_+}$, where $\Theta_0 := \bar{\Theta}$ and

$$\Theta_{k+1} := \{\theta \in \mathbb{R}_+^{h+1} : M\theta \in \Theta_k\} \cap \Theta_0, \tag{5.9}$$

for all $k \in \mathbb{Z}_+$, which is a set sequence that can be computed efficiently. More importantly, if $\bar{\Theta} \neq \emptyset$ and (5.8) holds, then Θ_∞ is a C-set with non-empty interior that can be computed in finite time, see [116] for details. Thus, given a family of sets, a procedure to construct an invariant family of sets for the linear DDI (2.1) via the construction of a standard invariant set in the relatively low-dimensional space $\mathbb{R}_+^{h+1}$ has been obtained.

Any invariant set $\mathbb{V}_\gamma^{h+1}$ constructed via the Razumikhin approach also induces the invariant parameterized family of sets $\mathcal{X}(\mathbb{V}_\gamma, \ldots, \mathbb{V}_\gamma, \mathbb{R}_{[0,1]}^N)$. However, the following example demonstrates that even the parameterized family of sets (5.4) in combination with the linear θ-dynamics (5.7) leads to invariant sets that provide a better approximation of the maximal invariant set than those constructed via the Razumikhin approach.

Example 5.2 Consider the scalar linear DDE

$$x_{k+1} = 0.25x_{k-1} + 0.25x_k, \quad k \in \mathbb{Z}_+, \tag{5.10}$$

and the constraints $\mathbf{x}_{[k-1,k]} \in \mathbb{C} := \{\mathbf{x}_{[-1,0]} \in (\mathbb{R})^2 : |x_{-1}| + |x_0| \leq 1\}$ for all $k \in \mathbb{Z}_+$. Let $S_1 := S_2 := \mathbb{R}_{[-1,1]}$. Then, the linear dynamics (5.7) are given by

$$\theta_{k+1} = \begin{bmatrix} 0 & 1 \\ 0.25 & 0.25 \end{bmatrix} \theta_k, \quad k \in \mathbb{Z}_+. \tag{5.11}$$

while the constraints on θ are $\bar{\Theta} = \{\theta \in \mathbb{R}_+^2 : [\theta]_1 + [\theta]_2 \leq 1\}$. Executing the recursion (5.9) yields the maximal invariant set for the comparison system (5.11) and the constraints $\bar{\Theta}$, i.e., $\Theta_{\mathrm{MAX}} = \Theta_\infty = \{\theta \in \mathbb{R}_+^2 : [\theta]_1 + [\theta]_2 \leq 1, \ 0.25[\theta]_1 + 1.25[\theta]_2 \leq 1\}$. Hence, it follows from Theorem 5.2 claim *(ii)* that the family $\mathcal{X}(S_1, S_2, \Theta_\infty)$ is an invariant parameterized family of sets for the linear DDE (5.10) and the constraint set $\mathbb{C}$.
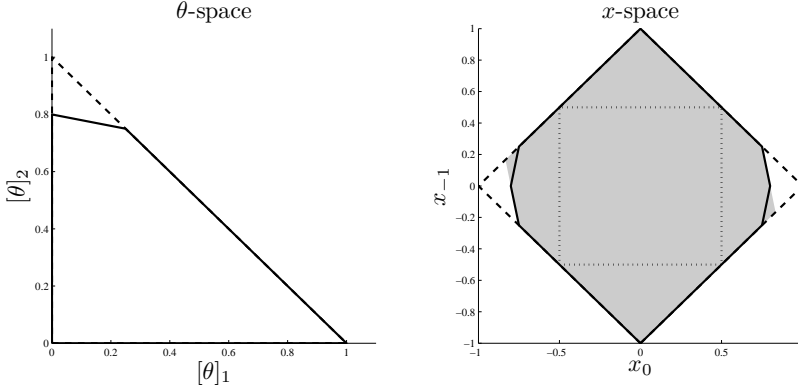
Figure 5.2: Left: The constraint set $\bar{\Theta}$ (- - -) and the maximal invariant set Θ_∞ (——). Right: The constraint set $\mathbb{C}$ (- - -), the invariant set obtained from $\mathcal{X}(S_1, S_2, \Theta_\infty)$ (——), $\mathbb{V}_{0.5}^2$ ($\cdots$) and the maximal invariant set (in grey) for (5.10) and $\mathbb{C}$.

Alternatively, the function $V(x_0) := |x_0|$ together with Theorem 2.3 (with $\rho = 1$) can be used to obtain the invariant set $\mathbb{V}_{0.5}^2 = \mathbb{R}_{[-0.5, 0.5]}^2$ associated with the Razumikhin approach. Note that the invariant set constructed from the invariant family of sets $\mathcal{X}(S_1, S_2, \Theta_\infty)$ is about twice as large as $\mathbb{V}_{0.5}^2$ and, in fact, is almost equal to the maximal invariant set for the scalar linear DDE (5.10) and the constraints $\mathbb{C}$ (which can only be obtained via the Krasovskii approach). The results are shown in Figure 5.2. $\square$

So far, the construction of an invariant family of sets, as outlined above, relies on the construction of an invariant set for the linear dynamics (5.7) rather than the sublinear dynamics (5.5). The use of the linear dynamics (5.7) can introduce considerable conservatism as illustrated by the following example.

Example 5.3 (Example 3.2, Part II) Consider the linear DDE

$$x_{k+1} = \begin{bmatrix} 0.75 & 0 \\ 0 & 0 \end{bmatrix} x_{k-1} + \begin{bmatrix} 0 & 0 \\ 0 & 0.75 \end{bmatrix} x_k, \quad k \in \mathbb{Z}_+, \tag{5.12}$$

where $\mathbf{x}_{[k-1,k]} \in \mathbb{C} := (\mathbb{R}_{[-1,1]}^2)^2$ for all $k \in \mathbb{Z}_+$. Let $S_1 := S_2 := \mathbb{R}_{[-1,1]}^2$ so that $\bar{\Theta} = \mathbb{R}_{[0,1]}^2$. Indeed, for all $\theta \in \bar{\Theta}$, it holds that $[\theta]_1 S_1 \times [\theta]_2 S_2 \subseteq \mathbb{C}$. Furthermore, the parameterized family of sets $\mathcal{X}(S_1, S_2, \bar{\Theta})$ yields the sublinear dynamics

$$\theta_{k+1} = \begin{bmatrix} [\theta_k]_2 \\ 0.75 \max\{[\theta_k]_1, [\theta_k]_2\} \end{bmatrix}, \quad k \in \mathbb{Z}_+. \tag{5.13}$$

The set $\Theta := \bar{\Theta}$ is an invariant set for the comparison system (5.13) and the constraint set $\bar{\Theta}$. Hence, it follows from Theorem 5.2 that $\mathcal{X}(S_1, S_2, \Theta)$ is an invariant parameterized family of sets for the linear DDE (5.12) and the constraint set $\mathbb{C}$.

However, linear dynamics that upper bound (5.13) are

$$\theta_{k+1} = \begin{bmatrix} 0 & 1 \\ 0.75 & 0.75 \end{bmatrix} \theta_k, \quad k \in \mathbb{Z}_+. \tag{5.14}$$

By Lemma 5.1 the system (5.14) is not $\mathcal{KL}$-stable while by direct inspection Θ is not an invariant set for (5.14). Hence, the linear dynamics (5.14) can not be employed to verify that $\mathcal{X}(S_1, S_2, \Theta)$ is an invariant family of sets for (5.12) and $\mathbb{C}$ (a fact which is established by utilizing the sublinear dynamics (5.13)).

Furthermore, it is interesting to observe that analyzing the stability of a DDI via the sublinear comparison system (5.5) can sometimes provide a positive answer when analyzing stability via Proposition 2.6, which corresponds to the Razumikhin approach, provides a negative answer (i.e., the BMI (2.17) does not admit a feasible solution). This indicates an advantage of Theorem 5.2 over the stability analysis test that was developed in Proposition 2.6, which corresponds to the Razumikhin approach. □

Example 5.3 illustrates that it might be highly beneficial to employ the sublinear dynamics (5.5) instead of the linear dynamics (5.7). Hence, computationally tractable synthesis algorithms enabling the use of these dynamics are discussed next.

5.5 Computation of invariant parameterized families of sets

To enable the construction of invariant parameterized families of sets via convex optimization algorithms, particular types of shape sets are used in what follows. In particular, both polyhedral and ellipsoidal shape sets are considered.

5.5.1 Parameterized families of polyhedral sets

Whenever parameterized families of *polyhedral* sets are considered, to facilitate the computational procedures, *the results are restricted to linear DDEs*, i.e., systems of the form (2.1) with a property as specified in Definition 2.1.

Ideally, the synthesis of invariant families of sets should be performed in such a way that the shape sets $S_1, \ldots, S_{h+1}$ and the dynamics of the associated comparison system (generated by functions $f_1, \ldots, f_{h+1}$) together with the corresponding invariant set Θ are computed jointly. Unfortunately, at present this task is not computationally practicable. Therefore, to enhance computational tractability, a two stage design procedure is proposed in what follows. The first stage detects a suitable collection of shape sets $S_1, \ldots, S_{h+1}$ while the second stage yields the dynamics of the corresponding comparison system and the associated invariant set Θ. Hence, the first step to obtain an invariant family of sets, is to construct a set of proper C-polytopic shape sets $S_i \subseteq \mathbb{R}^n$, $i \in \mathbb{Z}_{[1,h+1]}$ specified[1] as

$$S_i := \{x_0 \in \mathbb{R}^n : [C_i]_{:,j}^\top x_0 \leq 1, \ \forall j \in \mathbb{Z}_{[1,p_i]}\}, \tag{5.15}$$

where, for all $i \in \mathbb{Z}_{[1,h+1]}$, $C_i \in \mathbb{R}^{n \times p_i}$ and $p_i \in \mathbb{Z}_{\geq n+1}$. Two cases are considered, namely the cases when a single and multiple shape sets are utilized for the construction of the invariant families of sets, i.e., $\mathcal{X}(S, \ldots, S, \Theta)$ and $\mathcal{X}(S_1, \ldots, S_{h+1}, \Theta)$, respectively.

---

[1]The matrices $C_i \in \mathbb{R}^{n \times p_i}$ induce irreducible representations of the proper C-polytopic sets S_i, i.e., S_i is the intersection of p_i half-spaces.

Single shape set case

For the construction of an invariant parameterized family of sets with a single shape set S, we require, in view of Definition 5.2, that the set S is a proper C-polytopic set which satisfies for all $i \in \mathbb{Z}_{[-h,0]}$

$$A_i S \subseteq \lambda_i S \text{ with } \lambda_i \in \mathbb{R}_{\geq \mathrm{sr}(A_i)}, \tag{5.16}$$

and that $S^{h+1} \subseteq \mathbb{C}$. The detection and computation of such a set S can be achieved via the following algorithm.

Algorithm 5.1 Set $k = 0$ and choose for each $i \in \mathbb{Z}_{[-h,0]}$ some $\lambda_i \in \mathbb{R}_{>\mathrm{sr}(A_i)}$.

Step 1: Choose a proper C-polytopic set $Z_0 \subset \mathbb{R}^n$ such that $Z_0^{h+1} \subseteq \mathbb{C}$ and obtain its irreducible representation

$$Z_0 := \{x_0 \in \mathbb{R}^n : [D_0]_{:,j}^\top x_0 \leq 1, \ \forall j \in \mathbb{Z}_{[1,p_0]}\},$$

where $D_0 \in \mathbb{R}^{n \times p_0}$, $p_0 \in \mathbb{Z}_{\geq n+1}$.

Step 2: Construct the set

$$Z_{k+1} := \{x_0 \in \mathbb{R}^n : [D_k]_{:,j}^\top A_i x_0 \leq \lambda_i, \ \forall (i,j) \in \mathbb{Z}_{[-h,0]} \times \mathbb{Z}_{[1,p_k]}\} \cap Z_0, \tag{5.17}$$

and obtain $D_{k+1} \in \mathbb{R}^{n \times p_{k+1}}$ and $p_{k+1} \in \mathbb{Z}_{\geq 2}$ yielding the irreducible representation of Z_{k+1} such that

$$Z_{k+1} = \{x_0 \in \mathbb{R}^n : [D_{k+1}]_{:,j}^\top x_0 \leq 1, \ \forall j \in \mathbb{Z}_{[1,p_{k+1}]}\}.$$

Step 3: If $Z_{k+1} \neq Z_k$, set $k = k + 1$ and repeat Step 2, otherwise set $S = Z_{k+1}$ and terminate the algorithm. □

It is important to observe that Algorithm 5.1 can be implemented directly by utilizing the recursion (5.17). In principle, any proper C-polytopic set Z_0 such that $Z_0^{h+1} \subseteq \mathbb{C}$ can be employed. However, it is natural to anticipate that sets with a larger volume will lead to better results in general.

Essentially, Algorithm 5.1 is a modification of the recursion (5.9). Therefore, it generates a sequence of non-empty, monotonically non-increasing with respect to set inclusion, proper C-polytopic sets $\{Z_k\}_{k \in \mathbb{Z}_+}$ whose Hausdorff limit is guaranteed to be at least a C-set (possibly a trivial C-set $\{0\}$). If Algorithm 5.1 terminates in finite time then its Hausdorff limit is the set S which is guaranteed to be a proper C-polytopic set. Moreover, the above and the fact that $Z_0^{h+1} \subseteq \mathbb{C}$ guarantees that $S^{h+1} \subseteq \mathbb{C}$. Furthermore, under certain reasonable assumptions the set S constructed via Algorithm 5.1 will ultimately lead to the successful construction of an invariant family of sets.

Proposition 5.7 Suppose that the set S obtained by Algorithm 5.1 is a nontrivial C-set and that $\sum_{i=-h}^{0} \lambda_i < 1$. Then, there exists a C-set $\Theta \subseteq \bar{\Theta}$ in $\mathbb{R}_+^{h+1}$ with non-empty interior such that $\mathcal{X}(S, \ldots, S, \Theta)$ is an invariant family of sets. □

Proposition 5.7 is proven in Appendix B.4. Next, the multiple shape sets case is considered.

Multiple shape sets case

In the case of multiple shape sets $S_1, S_2, \ldots, S_{h+1}$, we require, in view of Definition 5.2, that the sets S_i are proper C-polytopic sets which satisfy, for all $i \in \mathbb{Z}_{[1,h+1]}$

$$A_{-h-1+i} S_i \subseteq \lambda_{-h-1+i} S_i \text{ with } \lambda_{-h-1+i} \in \mathbb{R}_{\geq \text{sr}(A_{-h-1+i})}, \tag{5.18}$$

and that $S_1 \times \ldots \times S_{h+1} \subseteq \mathbb{C}$. Similarly to the case of a single shape set, the detection and computation of such sets $S_1, \ldots, S_{h+1}$ can be achieved via the following algorithm.

Algorithm 5.2 For all $i \in \mathbb{Z}_{[1,h+1]}$, choose $\lambda_{-h-1+i} \in \mathbb{R}_{>\text{sr}(A_{-h-1+i})}$ and set $i^* = 1$.
  Step 1: Choose $h+1$ proper C-sets $Z_{0,i} \subset \mathbb{R}^n$, $i \in \mathbb{Z}_{[1,h+1]}$ such that $Z_{0,1} \times \ldots \times Z_{0,h+1} \subseteq \mathbb{C}$ and obtain their irreducible representations

$$Z_{0,i} := \{x_0 \in \mathbb{R}^n : [D_{0,i}]_{:,j}^\top x_0 \leq 1, \ \forall j \in \mathbb{Z}_{[1,p_{0,i}]}\},$$

where $D_{0,i} \in \mathbb{R}^{n \times p_{0,i}}$ and $p_{0,i} \in \mathbb{Z}_{\geq n+1}$ for all $i \in \mathbb{Z}_{[1,h+1]}$.
  Step 2: Set $k = 0$.
  Step 2.1: Construct the set

$$Z_{k+1,i^*} := \{x_0 \in \mathbb{R}^n : [D_{k,i^*}]_{:,j}^\top A_{-h-1+i^*} x_0 \leq \lambda_{-h-1+i^*}, \ \forall j \in \mathbb{Z}_{[1,p_{k,i^*}]}\} \cap Z_{0,i^*}, \tag{5.19}$$

and obtain $D_{k+1,i^*} \in \mathbb{R}^{n \times p_{k+1,i^*}}$ and $p_{k+1,i^*} \in \mathbb{Z}_{\geq n+1}$ yielding the irreducible representation of Z_{k+1,i^*} so that

$$Z_{k+1,i^*} = \{x_0 \in \mathbb{R}^n : [D_{k+1,i^*}]_{:,j}^\top x_0 \leq 1, \ \forall j \in \mathbb{Z}_{[1,p_{k+1,i^*}]}\}.$$

  Step 2.2: If $Z_{k+1,i^*} \neq Z_{k,i^*}$, set $k = k+1$ and repeat Step 2.1.
  Step 3: Set $S_{i^*} = Z_{k+1,i^*}$ and, if $i^* \neq h+1$, set $i^* = i^* + 1$ and repeat Step 2. Otherwise, terminate the algorithm. □

Similarly to Algorithm 5.1, Algorithm 5.2 can be implemented directly by using the recursion (5.19) while any $h+1$ proper C-sets $Z_{0,i}$, $i \in \mathbb{Z}_{[1,h+1]}$ such that $Z_{0,1} \times \ldots \times Z_{0,h+1} \subseteq \mathbb{C}$ can be employed. As before, it is natural to anticipate that sets with a larger volume will lead to better results in general. Furthermore, under certain reasonable assumptions Algorithm 5.2 will lead to the construction of useful shape sets S_i, $i \in \mathbb{Z}_{[1,h+1]}$.

Proposition 5.8 Consider Algorithm 5.2 and suppose that $\max\{\lambda_i : i \in \mathbb{Z}_{[-h,0]}\} < 1$. Then, Algorithm 5.2 terminates in finite time and the resulting sets S_{i^*}, $i^* \in \mathbb{Z}_{[1,h+1]}$ are proper C-polytopic sets. □

Proposition 5.8 is proven in Appendix B.4. Suppose that the hypothesis of Proposition 5.8 is satisfied and consider the linear θ-dynamics (5.7) defined by the sets S_i, $i \in \mathbb{Z}_{[1,h+1]}$ constructed via Algorithm 5.2. If (5.8) holds then Θ, constructed via (5.9), is a C-set with non-empty interior and hence $\mathcal{X}(S_1, \ldots, S_{h+1}, \Theta)$ is a nontrivial family of sets. The above indicates that, if the linear θ-dynamics induced by the sets constructed via Algorithm 5.2 are stable, then these sets can be used to construct an invariant family of sets.

Sublinear and linear θ-dynamics and computation of Θ

The second stage in designing an invariant family of sets is the detection of the sublinear dynamics of the associated comparison system (generated by functions $f_1, \ldots, f_{h+1}$) together with the corresponding invariant set Θ. Therefore, given the proper C-polytopic shape sets S_i, $i \in \mathbb{Z}_{[1,h+1]}$ and the corresponding representation (5.15), the sublinear and linear θ-dynamics (5.5) and (5.7) are obtained in what follows.

A direct use of the polytopic structure of the sets S_i, $i \in \mathbb{Z}_{[1,h+1]}$ in conjunction with the algebra of support functions [126] yields that, for all $\theta \in \mathbb{R}_+^{h+1}$

$$f_{h+1}(\theta) = \min_\eta \{\eta \in \mathbb{R}_+ : \operatorname{supp}(\bigoplus_{j=1}^{h+1} A_{-h-1+j}[\theta]_j S_j, [C_{h+1}]_{:,i})$$
$$\leq \operatorname{supp}(\eta S_{h+1}, [C_{h+1}]_{:,i}), \ \forall i \in \mathbb{Z}_{[1,p_{h+1}]}\},$$

or equivalently, due to the additivity and homogeneity of the support function in its first argument [126]

$$f_{h+1}(\theta) = \min_\eta \{\eta \in \mathbb{R}_+ : [\theta]_1 \operatorname{supp}(A_{-h}S_1, [C_{h+1}]_{:,i}) + \ldots$$
$$+ [\theta]_{h+1} \operatorname{supp}(A_0 S_{h+1}, [C_{h+1}]_{:,i}) \leq \eta, \ \forall i \in \mathbb{Z}_{[1,p_{h+1}]}\}.$$

By the same token, for all $\theta \in \mathbb{R}_+^{h+1}$ and all $j \in \mathbb{Z}_{[1,h]}$, it holds that

$$f_j(\theta) = \min_\eta \{\eta \in \mathbb{R}_+ : [\theta]_{j+1} \operatorname{supp}(S_{j+1}, [C_j]_{:,i}) \leq \eta, \ \forall i \in \mathbb{Z}_{[1,p_j]}\}$$
$$= [M]_{j,j+1}[\theta]_{j+1} \text{ where } [M]_{j,j+1} := \max_{i \in \mathbb{Z}_{[1,p_j]}} \operatorname{supp}(S_{j+1}, [C_j]_{:,i}).$$

The above support functions can be evaluated directly by a straightforward use of linear programming. For example, for all $j \in \mathbb{Z}_{[1,h+1]}$ and all $i \in \mathbb{Z}_{[1,p_j]}$, it holds that

$$\operatorname{supp}(S_{j+1}, [C_j]_{:,i}) = \max\{[C_j]_{:,i}^\top x_0 : x_0 \in S_{j+1}\}.$$

Hence, the functions $f_j : \mathbb{R}_+^{h+1} \to \mathbb{R}_+$, $j \in \mathbb{Z}_{[1,h]}$ are, in fact, linear while the function $f_{h+1} : \mathbb{R}_+^{h+1} \to \mathbb{R}_+$ is sublinear and can be obtained by solving the following parametric linear programming problem.

Algorithm 5.3 For every $\theta \in \mathbb{R}_+^{h+1}$ solve

$$\min \eta,$$

subject to $\eta \in \mathbb{R}_+$ and, for all $i \in \mathbb{Z}_{[1,p_{h+1}]}$

$$[\theta]_1 \operatorname{supp}(A_{-h}S_1, [C_{h+1}]_{:,i}) + \ldots + [\theta]_{h+1} \operatorname{supp}(A_0 S_{h+1}, [C_{h+1}]_{:,i}) \leq \eta.$$

$\square$

As the function f_{h+1} is the value function of the parametric linear program defined in Algorithm 5.3, it follows that f_{h+1} is a continuous piecewise affine function. This fact together

with claim *(i)* of Theorem 5.2 implies that this function is a continuous piecewise linear function. Therefore, the dynamics (5.5) are continuous piecewise linear and their form is

$$\theta_{k+1} = \{F_i \theta_k : \theta_k \in \mathbb{S}_i, \ i \in \mathbb{Z}_{[1,L_f]}\}, \quad k \in \mathbb{Z}_+, \tag{5.20}$$

where $F_i \in \mathbb{R}_+^{h+1 \times h+1}$ for all $i \in \mathbb{Z}_{[1,L_f]}$ and $L_f \in \mathbb{Z}_{\geq 1}$, while the sets $\mathbb{S}_i \subseteq \mathbb{R}_+^{h+1}$, $i \in \mathbb{Z}_{[1,L_f]}$ have non-overlapping interiors (relative interiors) and satisfy $\bigcup_{i=1}^{L_f} \mathbb{S}_i = \mathbb{R}_+^{h+1}$.

Now, the maximal invariant set for (5.20) and the constraint set $\bar{\Theta}$ is the Hausdorff limit of the set sequence $\{\Theta_k\}_{k \in \mathbb{Z}_+}$ where $\Theta_0 := \bar{\Theta}$ and, for all $k \in \mathbb{Z}_+$

$$\Theta_{k+1} := \bigcup_{i=1}^{L_f} \{\theta \in \mathbb{S}_i : F_i \theta \in \Theta_k\} \cap \Theta_0. \tag{5.21}$$

The sets Θ_k, $k \in \mathbb{Z}_+$ can be computed by the standard computational geometry methods, and can be shown to be C-polytopic sets in $\mathbb{R}_+^{h+1}$. Clearly, the set Θ_∞ can be obtained by recursively computing sets Θ_k until $\Theta_{k^*+1} = \Theta_{k^*}$ for some $k^* \in \mathbb{Z}_+$ in which case $\Theta_\infty = \Theta_{k^*}$. Hence, a computationally tractable approach to obtain the sublinear dynamics (5.5) and the corresponding maximal invariant set Θ_∞ has been identified.

Furthermore, an explicit expression for the linear dynamics (5.7) is induced by the matrix M whose entries are given, for all $j \in \mathbb{Z}_{[1,h+1]}$, by

$$[M]_{h+1,j} = \max_{i \in \mathbb{Z}_{[1,p_{h+1}]}} \text{supp}(A_{-h-1+j} S_j, [C_{h+1}]_{:,i})$$
$$= \max_{i \in \mathbb{Z}_{[1,p_{h+1}]}} \text{supp}(S_j, A_{-h-1+j}^\top [C_{h+1}]_{:,i}),$$

and, for all $j \in \mathbb{Z}_{[1,h]}$, by

$$[M]_{j,j+1} = \max_{i \in \mathbb{Z}_{[1,p_j]}} \text{supp}(S_{j+1}, [C_j]_{:,i}),$$

and $[M]_{i,j} = 0$ otherwise.

The application of Algorithms 5.1 and 5.2 in conjunction with the solution to Algorithm 5.3, is illustrated by revisiting Example 5.3.

Example 5.3 (Example 3.2, Part III) Consider the parameters that were used in Example 5.3. Put $Z_0 := \mathbb{R}_{[-1,1]}^2$ so that $Z_0 \times Z_0 \subset \mathbb{C}$. Then, executing Algorithm 5.1 yields $Z_1 = \mathbb{R}_{[-1,1]}^2 = Z_0$, which terminates the algorithm with the result $S = \mathbb{R}_{[-1,1]}^2$. Thus, solving Algorithm 5.3 yields that for all $\theta \in \mathbb{R}_+^2$

$$f_1(\theta) = [\theta]_2, \qquad\qquad f_2(\theta) = \begin{cases} \frac{3}{4}[\theta]_1, & [\theta]_1 \geq [\theta]_2 \\ \frac{3}{4}[\theta]_2, & [\theta]_2 > [\theta]_1 \end{cases}.$$

Moreover, observing that $\bar{\Theta} = \mathbb{R}_{[0,1]}^2$ and solving (5.21) yields the maximal invariant set $\Theta_{\text{MAX}} = \Theta_\infty = \mathbb{R}_{[0,1]}^2$. Hence, it can be concluded that $\mathcal{X}(S, S, \Theta_\infty)$ is an invariant family of sets for the scalar linear DDE (5.12) and the constraint set $\mathbb{C}$. Note that, the above results correspond to the results that were found in Example 5.3, Part I. $\square$

Similar to the above example and via a proper choice of initial conditions, the results that were obtained in Example 5.2 can also be reproduced using Algorithms 5.1-5.3.

The following example illustrates the computational advantages of the results that were developed in this chapter.

Example 5.4 Consider the linear DDE

$$x_{k+1} = A_{-2}x_{k-2} + A_0 x_k, \tag{5.22}$$

where $k \in \mathbb{Z}_+$ and

$$A_0 := \begin{bmatrix} -0.5 & 0.05 & 0.1 & 0 \\ -0.2 & 0.1 & 0.2 & 0.2 \\ 0 & 0 & 0.3 & 0.05 \\ -0.4 & 0 & 0.1 & 0.15 \end{bmatrix}, \quad A_{-2} := \begin{bmatrix} -0.1 & 0.05 & 0.1 & 0.1 \\ -0.2 & 0.1 & 0.1 & 0.2 \\ 0 & -0.2 & 0.3 & 0.05 \\ -0.2 & -0.5 & 0.2 & 0.4 \end{bmatrix}.$$

The constraints, associated with the DDE (5.22), are specified via $\mathbb{C} := (\mathbb{R}^4_{[-2,2]})^3$.

For the linear DDE (5.22) Algorithm 5.2 can be used, with the initial condition $Z_{0,i} := \mathbb{R}^4_{[-2,2]}$ for all $i \in \mathbb{Z}_{[1,3]}$, to obtain the shape sets shown in Figure 5.3. Then, Algorithm 5.3
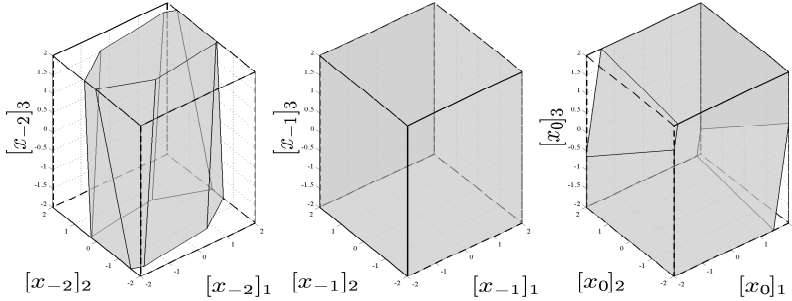
Figure 5.3: A projection on $[x_i]_4 = 0$, $i \in \mathbb{Z}_{[-2,0]}$ of the shape sets S_1, S_2 and S_3 (grey) and the constraint set $\mathbb{R}^4_{[-2,2]}$ (- - -).

can be used to obtain the sublinear dynamics (5.20) together with the regions shown in Figure 5.4. For this example, the dynamics (5.20) consist of 3 regions for which the corresponding matrices are

$$F_1 = \begin{bmatrix} 0 & 2.5 & 0 \\ 0 & 0 & 1 \\ 0.18 & 0 & 0.48 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 & 2.5 & 0 \\ 0 & 0 & 1 \\ 0.13 & 0 & 0.52 \end{bmatrix}, \quad F_3 = \begin{bmatrix} 0 & 2.5 & 0 \\ 0 & 0 & 1 \\ 0.27 & 0 & 0.27 \end{bmatrix}.$$

Finally, the maximal invariant set for the dynamics (5.20) is also shown in Figure 5.4 and can be obtained via the recursion (5.21).

Figure 5.4: Left: The regions $\mathbb{S}_i$, $i \in \mathbb{Z}_{[1,3]}$. Right: The constraints $\bar{\Theta}$ (- - -) and the maximal invariant set Θ_∞ (in grey).
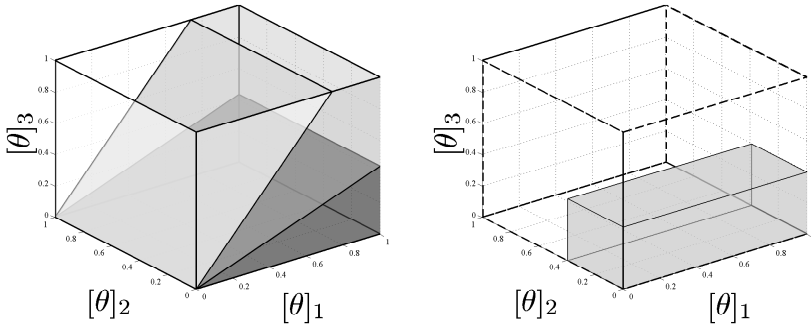
Note that the invariant family of polyhedral sets $\mathcal{X}(S_1, S_2, S_3, \Theta_\infty)$ was obtained via the computation of an invariant set in $\mathbb{R}^3$. To obtain an invariant set for the linear DDE (5.22) via the Krasovskii approach, the computation of an invariant set in $\mathbb{R}^{12}$ is required, which indicates the computational advantage of utilizing invariant families of sets. □

The following example illustrates the advantage of using multiple shape sets as opposed to a single shape set, which corresponds to using Algorithm 5.2 as opposed to Algorithm 5.1.

Example 5.5 Consider the linear DDE

$$x_{k+1} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.05 \end{bmatrix} x_{k-1} + \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} x_k, \quad k \in \mathbb{Z}_+, \tag{5.23}$$

where the constraints are specified by $\mathbb{C} := (\mathbb{R}_{[-1,1]} \times \mathbb{R}_{[-10,10]}) \times \mathbb{R}^2_{[-1,1]}$. Let $S_1 := \mathbb{R}_{[-1,1]} \times \mathbb{R}_{[-10,10]}$ and let $S_2 := \mathbb{R}^2_{[-1,1]}$. Then, the dynamics (5.7) takes form

$$\theta_{k+1} = \begin{bmatrix} 0 & 1 \\ 0.5 & 0.5 \end{bmatrix} \theta_k, \quad k \in \mathbb{Z}_+, \tag{5.24}$$

while the constraints on θ are $\theta \in \bar{\Theta} = \mathbb{R}^2_{[0,1]}$. The set $\Theta_\infty = \bar{\Theta}$ is the maximal invariant set for the linear dynamics (5.24) and the constraint set $\bar{\Theta}$. Hence, the family $\mathcal{X}(S_1, S_2, \Theta_\infty)$ is an invariant parameterized family of sets for the DDE (5.23) and the constraint set $\mathbb{C}$.

On the other hand, the largest invariant family of sets parameterized via a single shape set for the DDE (5.23) is the family $\mathcal{X}(S_2, S_2, \Theta_\infty)$. Since $S_2 \subset S_1$ it is obvious that this family induces an invariant set that is a subset of the set induced by $\mathcal{X}(S_1, S_2, \Theta_\infty)$. □

5.5.2 *Parameterized families of ellipsoidal sets*

In the last part of this chapter, parameterized families of ellipsoidal sets are considered. The advantage of this class of families of sets is that the shape sets and a corresponding controller

can be found at the same time. Furthermore, the computational procedures remain tractable even when linear DDIs (as opposed to linear DDEs) are considered. However, the exact set dynamics (5.5) can be found via multiparametric SDP only, which is a type of optimization problem that currently allows for an approximate solution only [9].

In what follows, linear controlled DDIs (as opposed to linear DDEs, which were considered in Section 5.5.1) are considered, i.e., DDIs of the form (2.15) with the structure defined in Definition 2.7. Furthermore, the linear controlled DDI is subject to the following state and input constraints, i.e., for all $k \in \mathbb{Z}_+$ it holds that

$$\mathbf{x}_{[k-h,k]} \in \mathbb{C}_x \quad \text{and} \quad u_k \in \mathbb{C}_u, \tag{5.25}$$

for some $\mathbb{C}_x \subseteq (\mathbb{R}^n)^{h+1}$ and $\mathbb{C}_u \subseteq \mathbb{R}^m$. To satisfy Assumption 5.1, the sets $\mathbb{C}_x \subseteq (\mathbb{R}^n)^{h+1}$ and $\mathbb{C}_u \subseteq \mathbb{R}^m$ are assumed to be proper C-sets. Furthermore, the state-feedback control law (2.16), for some $K \in \mathbb{R}^{m \times n}$, is used. In this case, for a given feedback matrix K the constraints (5.25) result in the constraint set $\mathbb{C} = \{\mathbf{x}_{[-h,0]} \in \mathbb{C}_x : Kx_i \in \mathbb{C}_u, \forall i \in \mathbb{Z}_{[-h,0]}\}$ for the closed-loop system, which satisfies Assumption 5.1.

A suitable collection of ellipsoidal shape sets $(S, \ldots, S)$ together with a controller can be obtained via Proposition 2.6. Indeed, as the set $\mathbb{V}_\gamma := \{x_0 \in \mathbb{R}^n : x_0^\top P x_0 \leq \gamma\}$ with $P := Z^{-1}$ is the sublevel set of an LRF it is, for γ small enough, $\mathcal{D}$-invariant (in fact, this set is even $\mathcal{D}$-contractive). Therefore, the parameterized family of sets $\mathcal{X}(S, \ldots, S, \Theta)$, with $S := \{x_0 \in \mathbb{R}^n : x_0^\top P x_0 \leq 1\}$ and $\Theta := \mathbb{R}^{h+1}_{[0,\gamma]}$, where $\gamma := \max_{\eta \in \mathbb{R}_+}\{\eta : (\eta S)^{h+1} \subseteq \mathbb{C}\}$, is an invariant family of sets. Furthermore, for an ellipsoidal shape set of the form $S = \{x_0 \in \mathbb{R}^n : x_0^\top P x_0 \leq 1\}$, for some $P \in \mathbb{R}^{n \times n}$, the sublinear θ-dynamics (5.5) can be obtained by solving the optimization problem $\min_{\eta \in \mathbb{R}_+} \eta$ subject to

$$\begin{bmatrix} \eta^2 P & \ldots & 0 & P(A_{-h} + B_{-h}K)^\top[\theta]_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & \eta^2 P & P(A_0 + B_0 K)^\top[\theta]_{h+1} \\ * & \ldots & * & P \end{bmatrix} \succeq 0,$$

for all $(\{A_i, B_i\}_{i \in \mathbb{Z}_{[-h,0]}}) \in \mathcal{AB}$. Unfortunately, this problem can be solved via parametric SDP [9] over the parameter vector $\theta \in \bar{\Theta}$ only, which is a type of optimization problem that currently allows for an approximate solution only [9].

On the other hand, the linear dynamics (5.7) can be obtained by solving an LMI of finite dimensions. Indeed, as $(A_{-h-1+j} + B_{-h-1+j}K)S \subseteq [M]_{h+1,j}S$ if and only if

$$[M]^2_{h+1,j}P \succeq (A_{-h-1+j} + B_{-h-1+j}K)^\top P(A_{-h-1+j} + B_{-h-1+j}K),$$

part of the linear dynamics (5.7) can be obtained by solving the optimization problem

$$\min_{M \in \mathbb{R}^{h+1 \times h+1}_+} \sum_{j=1}^{h+1} [M]^2_{h+1,j} \tag{5.26}$$

subject to, for all $j \in \mathbb{Z}_{[1,h+1]}$

$$[M]^2_{h+1,j}P \succeq (A_{-h-1+j} + B_{-h-1+j}K)^\top P(A_{-h-1+j} + B_{-h-1+j}K),$$

for all $(\{A_i, B_i\}_{i \in \mathbb{Z}_{[-h,0]}}) \in \mathcal{AB}$. Furthermore, the remainder of the dynamics is given by $[M]_{j,j+1} = 1$ for all $j \in \mathbb{Z}_{[1,h]}$ and zero elsewhere. The above techniques can be implemented via SDP and hence they yield directly an invariant parameterized family of ellipsoidal sets and corresponding control law for the linear controlled DDI (2.15).

These results are illustrated in the following example.

Example 5.6 (Example 2.2, Part IV) Let us consider again the DC-motor controlled over the communication network that was also considered in Example 2.2, Parts I-III. Consider the case where the DC-motor is subject to the following constraints: The armature voltage needs to be positive and smaller than 15 Volts such that $e_a(t) \in \mathbb{R}_{[0,15]}$ for all $t \in \mathbb{R}$. Furthermore, the load imposes the constraint $\dot{\omega}(t) \in \mathbb{R}_{[-1250,1250]}$ on the angular velocity of the motor. The nominal operating point for the motor is $x_{\text{nom}} = \text{col}(1.413, 400)$ together with the load torque $T_l = 1.9 \cdot 10^{-3}$, which yields the nominal armature voltage $e_a = 12$.

As indicated in Section 5.5.2 it is possible to use a solution to Proposition 2.6 to obtain a suitable collection of shape sets and corresponding control law. Let us assume that $\tau_k \in \mathbb{R}_{[0,0.348T_s]}$ and compute the polytopic over-approximation of the uncertain time-varying matrix $\Delta(\tau)$. Like in Parts II and III of this example, the numerical values for this over-approximation can be found in Appendix A. Then, the quadratic LRF matrix and corresponding controller matrix

$$P_{\text{LRF}} := \begin{bmatrix} 3.11 & 0.26 \\ 0.26 & 0.02 \end{bmatrix}, \quad K_{\text{LRF}} := \begin{bmatrix} -11.81 & -0.95 \end{bmatrix},$$

form a feasible solution to Proposition 5.6 for $\rho = 0.7$ with $\delta_0 = 0.8$ and $\delta_{-1} = 0.2$. Given the matrix K_{LRF} it is possible to transform the equilibrium point to zero via a linear state transformation and to obtain the constraint set $\mathbb{C}$, i.e., $\mathbb{C} = \{\mathbf{x}_{[-1,0]} \in (\mathbb{R}^2)^2 : K_{\text{LRF}}x_i \in \mathbb{R}_{[-12,3]}, \ i \in \mathbb{Z}_{[-1,0]}, \ ([x_0]_2 - [x_{-1}]_2) \in \mathbb{R}_{[-12.5,12.5]}\}$. Furthermore, the collection of shape sets (S, S), where $S := \{x_0 \in \mathbb{R}^2 : x_0^\top P_{\text{LRF}} x_0 \leq 1\}$, yields via the LMI (5.26) the linear θ-dynamics (5.7), i.e.,

$$\theta_{k+1} = \begin{bmatrix} 0 & 1 \\ 0.25 & 0.74 \end{bmatrix} \theta_k, \quad k \in \mathbb{Z}_+. \tag{5.27}$$

The constraints on θ are $\bar{\Theta} = \{\theta \in \mathbb{R}_+^2 : 3.85[\theta]_1 + 3.85[\theta]_2 \leq 1, \ 5.08[\theta]_1 \leq 1, \ 5.08[\theta]_2 \leq 1\}$, which follow from the constraint set $\mathbb{C}$ and the collection of shape sets (S, S). In this case, the maximal invariant set Θ_∞ can be computed via the recursion (5.9), which yields the results shown in Figure 5.5. Also shown in Figure 5.5 is the shape set S.

Hence, for any initial condition $\mathbf{x}_{[-1,0]} \in \mathcal{X}(S, S, \Theta_\infty) + \{x_{\text{nom}}, x_{\text{nom}}\}$ the DC-motor in closed loop with the control law $u_k = K_{\text{LRF}} x_k$ controlled over the communication network satisfies the constraints for all time $k \in \mathbb{Z}_+$. $\qquad\square$
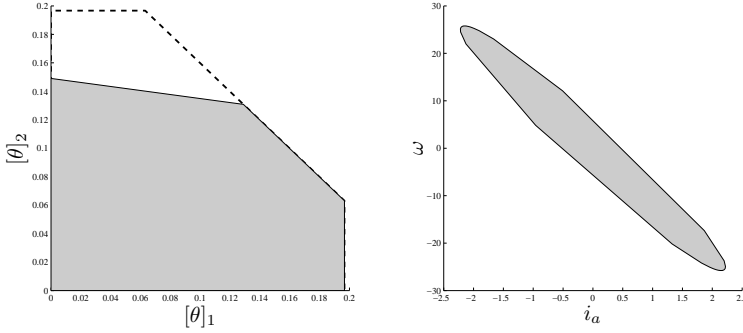
Figure 5.5: Left: The constraint set $\bar{\Theta}$ (- - -) and the maximal invariant set Θ_∞ (in grey). Right: The shape set S.

5.6 Conclusions

Motivated by the observation that a method for the construction of invariant sets for DDIs that combines computational tractability and conceptual generality is missing, this problem was considered in the present chapter. More specifically, the ideas that were outlined in Chapter 4 in general and Section 4.3 in particular were explored in the context of set invariance. It was shown that this concept, termed the invariant family of sets, enjoys computational practicability and at the same time is non-conservative. Furthermore, the properties and usage of this concept were analyzed formally and illustrated via several examples. Thus, a set of algorithms was derived that yields an approximation of the maximal invariant set for DDIs at a relatively low computational cost.

The results that were discussed in Chapters 2 - 5 provide a comprehensive collection of stability analysis methods for DDIs as well as constrained DDIs. Indeed, a stability analysis, e.g., using the results in Chapters 2 - 4, combined with an approximation of the maximal invariant set, constructed via the results discussed in this chapter, provides information about the set of initial conditions for which the corresponding trajectories converge to the origin and constraint satisfaction is guaranteed at all times. However, for stabilizing controller synthesis the situation is more complicated. Indeed, the results in Chapters 2 and 3 can be used to design a controller for a controlled DDI with constraints while an approximation of the maximal invariant set can be computed for the resulting closed-loop system via the techniques that were discussed in this chapter. Unfortunately, this approach does not necessarily lead to the largest possible region for which constraint satisfaction can be guaranteed because the control law and invariant set are designed separately rather than at the same time. Motivated by this fact, two more advanced control schemes for constrained DDIs are presented in the next chapter.

Chapter 6

Stabilization of constrained delay difference inclusions

In this chapter the stabilization of linear controlled DDIs with constraints is considered. The results are restricted to linear DDIs because this class of systems is able to model a wide variety of relevant processes, including many types of sampled-data systems and NCS, and allows the derivation of control problems that can be solved via convex optimization algorithms. In particular, two advanced control schemes are proposed that make use of online optimization. The first approach employs an LRF for the unconstrained system that is obtained via Proposition 2.6. In the presence of constraints, this LRF is valid only locally. Therefore, a receding horizon optimization problem is proposed based on the local LRF that takes constraints into account and which can be solved via SDP. The second approach makes use of a quadratic state-dependent LRF to reduce the computational complexity that is typically associated with the stabilization of linear controlled DDIs. Moreover, by allowing the shape matrix of the LRF to be dependent on all relevant delayed states the conservatism of the Razumikhin approach is avoided. Thus, a non-conservative control scheme is obtained which takes constraints into account and requires solving online a relatively low-dimensional SDP problem.

6.1 Introduction

Linear controlled DDIs have the ability to model a wide variety of relevant processes including many types of sampled-data systems [41] and NCS [46, 150, 153]. Therefore, the stabilization of linear controlled DDIs, possibly subject to constraints, is a frequently studied problem. Within the context of Lyapunov theory, the most common approach to solve this problem is to use the Krasovskii approach. For example, state-feedback controllers were obtained for uncertain systems with delay in [26, 132, 146], for uncertain systems with time-varying delay in [22, 38] and for uncertain singular systems with time-varying delay in [100]. Furthermore, control strategies that can handle constraints were developed for uncertain systems with delay in [63, 78, 97] and for uncertain systems with time-varying delay in [61]. In one of these references, i.e., [78], an MPC scheme for systems without delay is presented and it is pointed out that this scheme can be easily extended to handle systems with delay. Similarly, other MPC schemes for systems without delay, such as the

ones discussed in [119], can be used for the stabilization of constrained linear DDIs as well. Unfortunately, the computational complexity of all of the above-mentioned approaches increases significantly with the size of the delay, which are therefore not tractable for systems with large delays. As the conditions corresponding to the Razumikhin approach are typically less complex than those corresponding to the Krasovskii approach, the Razumikhin approach has the potential to overcome this problem. Motivated by this advantage, the Razumikhin approach was used in [91] to obtain a static state-feedback controller for systems with delay. Unfortunately, these results apply to stable systems with input delay only and as such their application is limited.

The above discussion indicates that a comprehensive framework that can deal with large delays and constraints is missing. Therefore, in this chapter, two controller synthesis solutions for linear controlled DDIs that are subject to constraints are developed within the Razumikhin framework. For the first method a LRF and corresponding static state-feedback control law are obtained for the unconstrained system via Proposition 2.6. In the presence of constraints the so-obtained LRF and corresponding controller remain valid only locally. Therefore, a receding horizon control algorithm, which relaxes the LRF conditions of the unconstrained case, is proposed, along with a closed-loop stability analysis. Furthermore, it is shown that by exploiting properties of the Minkowski addition of polytopes and the structure of the developed control law, the online component of the control scheme merely requires solving a relatively low-dimensional SDP problem.

Unfortunately, as the Razumikhin approach provides conservative conditions for stability, any method based on this approach is inherently conservative. Motivated by this conservatism, the second control scheme that is proposed in this chapter adds an additional degree of freedom to the Razumikhin approach while preserving its computational advantages. In particular, the corresponding function is restricted to be quadratic but with a shape matrix that is dependent on all relevant delayed states, which leads to a set of necessary and sufficient Razumikhin-type conditions for stability of linear DDIs. This concept is then used to design a control scheme for linear controlled DDIs that are subject to constraints. Again, Minkowski set addition properties are used to obtain a computationally efficient algorithm. As the proposed technique does not, as opposed to the first method, require the offline computation of a locally stabilizing controller, the overall synthesis method remains computationally tractable even for linear controlled DDIs with large delays and constraints.

6.2 Controller synthesis for constrained systems

In this chapter linear controlled DDIs are considered, i.e., DDIs of the form (2.15) with the structure defined in Definition 2.7. It is assumed that the linear controlled DDI is subject to a set of state and input constraints, i.e., for all $k \in \mathbb{Z}_+$ it holds that

$$\mathbf{x}_{[k-h,k]} \in \mathbb{C}_x, \quad u_k \in \mathbb{C}_u, \tag{6.1}$$

for some $\mathbb{C}_x \subset (\mathbb{R}^n)^{h+1}$ and $\mathbb{C}_u \subset \mathbb{R}^m$.

For the stabilization of the linear controlled DDI (2.15) a control law $\pi : \mathbb{C}_x \rightrightarrows \mathbb{C}_u$ will be used. The system in closed loop with this control law yields the DDI

$$x_{k+1} \in F_\pi(\mathbf{x}_{[k-h,k]}), \quad k \in \mathbb{Z}_+, \tag{6.2}$$

where

$$F_\pi(\mathbf{x}_{[-h,0]}) := \{ f(\mathbf{x}_{[-h,0]}, \mathbf{u}_{[-h,0]}) : u_0 \in \pi(\mathbf{x}_{[-h,0]}) \},$$

and $\mathbf{u}_{[-h,-1]}$ is assumed to be known. For notational convenience, it is assumed that the initial input sequence $\mathbf{u}_{[-h,-1]} \in \mathbb{C}_u^h$ is dependent on the initial state sequence $\mathbf{x}_{[-h,0]} \in \mathbb{C}_x$ only. As a consequence, it follows that $\mathbf{u}_{[k-h,k-1]}$ is dependent on $\mathbf{x}_{[k-h,k]}$ only. Therefore, the dependence of F_π on $\mathbf{u}_{[k-h,k-1]}$ can be omitted, as done in (6.2). Like for the DDI (2.1), $\mathcal{S}(\mathbf{x}_{[-h,0]})$ denotes the set of all solutions to (6.2) from $\mathbf{x}_{[-h,0]} \in \mathbb{C}_x$ while $\Phi \in \mathcal{S}(\mathbf{x}_{[-h,0]})$ denotes a specific solution to (6.2).

To facilitate the stabilization of the controlled DDI (2.15) when it is subject to the constraints (6.1), consider the following definition.

Definition 6.1 Suppose that the function $V : \mathbb{R}^n \to \mathbb{R}_+$ satisfies (2.10a). Moreover, suppose that there exists a control law $\pi : \mathbb{C}_x \rightrightarrows \mathbb{C}_u$ such that, for all $\mathbf{x}_{[-h,0]} \in \mathbb{C}_x$ and all $x_1 \in F_\pi(\mathbf{x}_{[-h,0]})$, the closed-loop system (6.2) is such that $\mathbf{x}_{[-h+1,1]} \in \mathbb{C}_x$ and that (2.10b) holds. Then, V is called a *control Lyapunov-Razumikhin function with respect to $\mathbb{C}_x$ and $\mathbb{C}_u$* (or, shortly, cLRF($\mathbb{C}_x$,$\mathbb{C}_u$)) for the linear controlled DDI (2.15). $\qquad\square$

Given a cLRF($\mathbb{C}_x$,$\mathbb{C}_u$) the stability of the closed-loop system (6.2) follows from Theorem 2.3. Moreover, as the set $\mathbb{C}_x$ is invariant, it can also be concluded that the constraints (6.1) are satisfied for any initial condition in $\mathbb{C}_x$ (and corresponding $\mathbf{u}_{[-h,-1]} \in \mathbb{C}_u^h$).

In what follows the following definition will prove to be useful.

Definition 6.2 The set $\mathbb{C}_x \subseteq (\mathbb{R}^n)^{h+1}$ is called *constrained control invariant with respect to $\mathbb{C}_u$* (or, shortly, CCI($\mathbb{C}_u$)) for the controlled DDI (2.15) if there exists a control law $\pi : \mathbb{C}_x \rightrightarrows \mathbb{C}_u$ such that $\mathbf{x}_{[-h+1,1]} \in \mathbb{C}_x$ for all $\mathbf{x}_{[-h,0]} \in \mathbb{C}_x$ and all $x_1 \in F_\pi(\mathbf{x}_{[-h,0]})$. $\qquad\square$

In the remainder of this chapter, two methods are proposed to find a control law π for the linear controlled DDI (2.15) that renders the closed-loop system (6.2) GAS or $\mathcal{KL}$-stable and that guarantees that the constraints (6.1) are satisfied.

6.3 Stabilization via local Lyapunov-Razumikhin functions

The first method makes use of a cLRF($(\mathbb{R}^n)^{h+1}$,$\mathbb{R}^m$), i.e., a cLRF for the unconstrained system, to stabilize the linear controlled DDI (2.15) in the presence of the constraints (6.1). To this end, the following assumptions will be used.

Assumption 6.1 The set $\mathbb{C}_x \subseteq (\mathbb{R}^n)^{h+1}$ is CCI($\mathbb{C}_u$) for the controlled DDI (2.15). $\qquad\square$

Assumption 6.2 The linear controlled DDI (2.15) admits a cLRF($(\mathbb{R}^n)^{h+1}$,$\mathbb{R}^m$) with a corresponding control law that is continuous and satisfies $\pi(0) = 0$. $\qquad\square$

It should be noted that the assumption that $\mathbb{C}_x$ is CCI($\mathbb{C}_u$) for the controlled DDI (2.15) does not provide a controller such that the closed-loop system (6.2) GAS and that the constraints (6.1) are satisfied. Assumption 6.1 merely implies that there exists a control law π such that

all solutions that start in $\mathbb{C}_x$ remain in $\mathbb{C}_x$ and that the constraints (6.1) are satisfied for all time. If $\mathbb{C}_x$ is not CCI($\mathbb{C}_u$), the results apply for any subset of $\mathbb{C}_x$ that is CCI($\mathbb{C}_u$). Furthermore, note that a function V and a corresponding control law π that satisfy Assumption 6.2 can be obtained via Proposition 2.6 for linear controlled DDIs.

Now consider the following algorithm.

Algorithm 6.1 At time $k \in \mathbb{Z}_+$, let $\mathbf{x}_{[k-h,k]}$ and $\mathbf{u}_{[k-h,k-1]}$ be known[1] and solve

$$\min_{(u_k, \lambda_k)} \lambda_k \tag{6.3}$$

subject to

$$u_k \in \mathbb{C}_u, \quad \mathbf{x}_{[k-h+1,k+1]} \in \mathbb{C}_x, \quad \lambda_k \in \mathbb{R}_+, \tag{6.4a}$$

$$V(x_{k+1}) \leq \max_{i \in \mathbb{Z}_{[-h,0]}} \rho V(x_{k+i}) + \lambda_k, \tag{6.4b}$$

for all $x_{k+1} \in f(\mathbf{x}_{[k-h,k]}, \mathbf{u}_{[k-h,k]})$. $\square$

The role of the variable λ_k, $k \in \mathbb{Z}_+$ is to introduce additional flexibility in (2.10b) and to enhance feasibility in the presence of constraints [85], as it will be made clear further in this section. Note that Algorithm 6.1 defines the set-valued control law

$$\pi(\mathbf{x}_{[k-h,k]}) := \{u_k \in \mathbb{R}^m : \exists \lambda_k \in \mathbb{R}_+ \text{ s.t. (6.3) holds}\}, \quad k \in \mathbb{Z}_+. \tag{6.5}$$

In what follows, it will be established under which assumptions Algorithm 6.1 is recursively feasible in $\mathbb{C}_x$ and sufficient conditions for the stability of the closed-loop system (6.2) obtained from the control law (6.5) will be presented.

Let λ_k^* denote the optimum in Algorithm 6.1 at time $k \in \mathbb{Z}_+$.

Theorem 6.1 Suppose that the controlled DDI (2.15) is a linear controlled DDI. Moreover, suppose that $0 \in \text{int}(\mathbb{C}_x)$, $0 \in \text{int}(\mathbb{C}_u)$ and that Assumptions 6.1 and 6.2 hold. Then, the following claims hold:

  (i) There exists a neighborhood of the origin $\mathcal{N}$ such that V is a cLRF($\mathcal{N}^{h+1}$,$\mathbb{C}_u$);

  (ii) Algorithm 6.1 is feasible for all $\mathbf{x}_{[-h,0]} \in \mathbb{C}_x$ and remains feasible for all $k \in \mathbb{Z}_+$. Moreover, the constraints (6.1) are satisfied, i.e., $\Phi_{[k-h,k]} \in \mathbb{C}_x$ and $u_k \in \mathbb{C}_u$, for all $\mathbf{x}_{[-h,0]} \in \mathbb{C}_x$ and all $(\Phi, k) \in \mathcal{S}(\mathbf{x}_{[-h,0]}) \times \mathbb{Z}_+$;

  (iii) if $\lim_{k \to \infty} \lambda_k^* = 0$ for all $\mathbf{x}_{[-h,0]} \in \mathbb{C}_x$, then the closed-loop system (6.2) obtained from the control law (6.5) is GAS. $\square$

Theorem 6.1 is proven in Appendix B.5.

---

[1]This assumption merely implies that the controller is able to measure the current state and to store relevant past states and control actions.

Remark 6.1 By augmenting Algorithm 6.1 with the constraint

$$0 \leq \lambda_k \leq \rho \max_{i \in \mathbb{Z}_{[1,M]}} \lambda^*_{k-i}, \quad \forall k \in \mathbb{Z}_{\geq M}, \tag{6.6}$$

for some $M \in \mathbb{Z}_{\geq 1}$, the property $\lim_{k \to \infty} \lambda^*_k = 0$ can be guaranteed. The constraint (6.6) is non-conservative in the sense that a non-monotone evolution of λ^*_k is allowed, while $\lambda^*_k \to 0$ as $k \to \infty$. □

When (6.6) is added to Algorithm 6.1 claim *(ii)* of Theorem 6.2 remains inherently valid only locally. Claims *(ii)* and *(iii)* of Theorem 6.2 can then be reformulated as typically done in sub-optimal MPC [87, 128], i.e., as a result of the type "feasibility implies stability". However, it would be desirable to identify sufficient conditions under which Algorithm 6.1 yields a stabilizing control law, without explicitly restricting the evolution of $\{\lambda_k\}_{k \in \mathbb{Z}_+}$, and thus, removing the recursive feasibility guarantee that Algorithm 6.1 has.

To this end, we will firstly establish that the set of feasible choices for $\{\lambda_k\}_{k \in \mathbb{Z}_+}$ can be upper bounded by a function of the state trajectory. Therefore, consider a cLRF($\mathcal{N}^{h+1}$,$\mathbb{C}_u$) together with the corresponding control law π (which exist due to Assumption 6.2). Let $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\rho \in \mathbb{R}_{[0,1)}$ denote the corresponding functions and constant. Now consider the following assumption.

Assumption 6.3 The linear controlled DDI (2.15) admits a cLRF($\mathbb{C}_x$,$\mathbb{C}_u$) denoted by V_1 with corresponding control law π_1, satisfying (2.10a) with $\alpha_3, \alpha_4 \in \mathcal{K}_\infty$ and (2.10b) with $\rho_1 \in \mathbb{R}_{[0,1)}$. □

The control law π_1 in closed loop with the linear controlled DDI (2.15) yields a system of the form (6.2), denoted by F_{π_1}.

Assumption 6.4 There exists a $\sigma \in \mathcal{K}_\infty$ such that, for all $\mathbf{x}_{[-h,0]} \in \mathbb{C}_x$ and all $(x_1, \bar{x}_1) \in F_\pi(\mathbf{x}_{[-h,0]}) \times F_{\pi_1}(\mathbf{x}_{[-h,0]})$, it holds that $V(x_1) \leq \sigma(V_1(\bar{x}_1))$. □

Lemma 6.1 Suppose that Assumptions 6.2-6.4 hold. Then, there exists a $\lambda : \mathbb{C}_x \to \mathbb{R}_+$ that is bounded on bounded sets, $\lambda(\mathbf{0}_{[-h,0]}) = 0$ and such that

$$V(x_1) \leq \max_{i \in \mathbb{Z}_{[-h,0]}} \rho V(x_i) + \lambda(\mathbf{x}_{[-h,0]}), \tag{6.7}$$

for all $\mathbf{x}_{[-h,0]} \in \mathbb{C}_x$ and all $x_1 \in F_\pi(\mathbf{x}_{[-h,0]})$. □

Notice that the result of Lemma 6.1, which is proven in Appendix B.5, establishes that if the linear controlled DDI (2.15) admits a cLRF($\mathbb{C}_x$,$\mathbb{C}_u$), i.e., V_1, then any candidate cLRF($(\mathbb{R}^n)^{h+1}$,$\mathbb{R}^m$), e.g., V, can be employed to "approximate" the evolution of V_1 via a suitable sequence of variables $\{\lambda_k\}_{k \in \mathbb{Z}_+}$.

The following result makes use of Lemma 6.1 to obtain sufficient conditions under which Algorithm 6.1 yields a stabilizing control law and hence under which the closed-loop system (6.2) obtained from (6.5) is GAS and satisfies the constraints (6.1).

Theorem 6.2 Suppose that Assumptions 6.2-6.4 hold and that $\rho \in \mathbb{R}_{\geq \rho_1}$. Furthermore, suppose that the function

$$\beta(r, s) := \alpha_1^{-1} \left((\rho \alpha_2(r) - \rho \alpha_1(r) + \sigma(\rho_1 \alpha_4(r)))^s \right), \qquad (6.8)$$

is such that $\beta \in \mathcal{KL}$. Then, the closed-loop system (6.2) obtained from the control law (6.5) is $\mathcal{KL}$-stable and for all $\mathbf{x}_{[-h,0]} \in \mathbb{C}_x$ the constraints (6.1) are satisfied. $\qquad \square$

Theorem 6.2 is proven in Appendix B.5. An inherent consequence of Theorem 6.2 is that $\lim_{k \to \infty} \lambda_k^* = 0$, which is in accordance with the hypothesis of Theorem 6.1-*(iii)*.

Remark 6.2 An alternative[2] set of sufficient conditions for stability of the closed-loop system (6.2) corresponding to Algorithm 6.1 can be obtained using the recent article [47] on recursive feasibility and stability of MPC. The results in [47] also rely on the construction of a $\mathcal{KL}$-bound on the closed-loop trajectories from an unknown control LF for the constrained system. The application of these results to the setting of Algorithm 6.1 indicates that the conditions proposed within Theorem 6.2 are less conservative. $\qquad \square$

6.3.1 Large delays

Control schemes for linear controlled DDIs, such as the ones discussed in [22, 26, 38, 61, 63, 78, 91, 100, 132, 146] and the ones corresponding to Propositions 2.5, 2.6 and 3.6, typically require the solution to a set of LMIs that need to be verified for the vertices of the matrix polytope $\mathcal{AB}$. However, for linear controlled DDIs with large delays and even more so for linear controlled DDIs with large delays that arise from systems with parametric uncertainty, i.e., for matrix polytopes of the form

$$\mathcal{AB} := \mathcal{A}_{-h} \times \mathcal{B}_{-h} \times \ldots \times \mathcal{A}_0 \times \mathcal{B}_0,$$

where $\mathcal{A}_i \subset \mathbb{R}^{n \times n}$ and $\mathcal{B}_i \subset \mathbb{R}^{n \times m}$, $i \in \mathbb{Z}_{[-h,0]}$ are compact sets, the number of vertices can become extremely large. As a consequence, the computational complexity of the control problem becomes an issue. In particular for optimization based controllers, which includes MPC schemes in general and Algorithm 6.1 in particular, this is not acceptable. To make the complexity of the computation of the control update less dependent on h, let

$$\hat{f}(\mathbf{x}_{[-h,0]}, \mathbf{u}_{[-h,0]}) := \left\{ B_0 u_0 + v : B_0 \in \mathcal{B}_0, \ v \in \mathcal{V}(\mathbf{x}_{[-h,0]}, \mathbf{u}_{[-h,-1]}) \right\}, \qquad (6.9)$$

where

$$\mathcal{V}(\mathbf{x}_{[-h,0]}, \mathbf{u}_{[-h,-1]}) := \mathcal{A}_0 x_0 \oplus \left(\bigoplus_{i=-h}^{-1} (\mathcal{A}_i x_i \oplus \mathcal{B}_i u_i) \right).$$

It should be mentioned here that if the matrix polytope $\mathcal{AB}$ does not have the particular structure indicated above, the results that follow require minor modifications only.

---

[2]Classical stabilization conditions employed in MPC [119], which rely on a sufficiently large prediction horizon N, are not suitable for linear DDIs, as increasing N leads to an exponential increase of the complexity of the corresponding optimization problem.

Lemma 6.2 Consider the linear controlled DDI (2.15) as defined in Definition 2.7 and let $\hat{f}$ be defined as in (6.9). Then,

$$\hat{f}(\mathbf{x}_{[-h,0]}, \mathbf{u}_{[-h,0]}) = f(\mathbf{x}_{[-h,0]}, \mathbf{u}_{[-h,0]}),$$

for all $(\mathbf{x}_{[-h,0]}, \mathbf{u}_{[-h,0]}) \in \mathbb{C}_x \times \mathbb{C}_u^{h+1}$. □

Lemma 6.2 follows straightforwardly from the properties [126] that the Minkowski addition is commutative and associative.

In Algorithm 6.1, at time $k \in \mathbb{Z}_+$, $\mathbf{x}_{[k-h,k]}$ and $\mathbf{u}_{[k-h,k-1]}$ are known before computation of the control update and hence, the set $\mathcal{V}$ can be computed at each time instant. The computation of $\mathcal{V}$ can be performed efficiently using, for example, the tools for performing the Minkowski addition of polytopes available within the Multi Parametric Toolbox for Matlab. Moreover, recent research [8] has led to much faster algorithms for performing Minkowski additions of polytopes. As the number of vertices spanning a polytope resulting from a Minkowski addition is likely to be much smaller than all possible combinations of vertices spanning the original polytopes, the control of a system of the form (6.9) is far simpler than the control of the original system. However, determining an exact upper bound on the number of vertices spanning a polytope resulting from a Minkowski addition is a nontrivial problem that has attracted much interest. The interested reader is referred to [125, 143] and the references therein for further reading.

Remark 6.3 The structure that is exploited in Lemma 6.2 to obtain a reduction in complexity of the control algorithm is particular to Algorithm 6.1. Other optimization based control schemes, such as the ones in [63, 78], can not benefit from this structure to obtain a similar reduction in complexity. □

6.3.2 Semidefinite programming implementation of Algorithm 6.1

Next, it is shown how Algorithm 6.1 can be solved via an SDP problem. To this end, suppose[3] that the set $\mathcal{V}(\mathbf{x}_{[k-h,k]}, \mathbf{u}_{[k-h,k-1]})$ is known at each time $k \in \mathbb{Z}_+$. Furthermore, suppose that the matrix $P \in \mathbb{R}^{n \times n}$ defines a quadratic cLRF$((\mathbb{R}^n)^{h+1}, \mathbb{R}^m)$ that satisfies Assumption 6.2, e.g., which can be obtained via Proposition 2.6.

Proposition 6.1 At time $k \in \mathbb{Z}_+$, consider the following optimization problem

$$\min_{(u_k, \lambda_k)} \lambda_k \tag{6.10}$$

subject to

$$\mathbf{x}_{[k-h+1,k+1]} \in \mathbb{C}_x, \quad u_k \in \mathbb{C}_u, \quad \lambda_k \in \mathbb{R}_+, \tag{6.11a}$$

$$\begin{bmatrix} \rho \max_{i \in \mathbb{Z}_{[-h,0]}} x_{k+i}^\top P x_{k+i} + \lambda_k & * \\ P x_{k+1} & P \end{bmatrix} \succ 0, \tag{6.11b}$$

---

[3]$\mathcal{V}$ can be computed at each time $k \in \mathbb{Z}_+$, as explained in Section 6.3.1.

for all $(B_0, v) \in \mathcal{B}_0 \times \mathcal{V}(\mathbf{x}_{[k-h,k]}, \mathbf{u}_{[k-h,k-1]})$, where $x_{k+1} := B_0 u_k + v$. Then, any solution (λ_k, u_k) to (6.10) satisfies the inequalities (6.4) with $V(x_0) := x_0^\top P x_0$. $\qquad \square$

Proposition 6.1 follows from the Schur complement and Lemma 6.2. If $\mathbb{C}_x$ and $\mathbb{C}_u$ are polytopes or ellipsoids and the set $\mathcal{AB}$ is a matrix polytope (or consists of a finite number of matrices), then finding a solution that satisfies (6.11) amounts to solving an LMI of finite dimensions. Thus, a solution to Algorithm 6.1 can be obtained via SDP.

The following example illustrates the application of Algorithm 6.1.

Example 6.1 Consider the linear controlled DDI (2.15) with $h = 3$ and let

$$\mathcal{A}_0 := \mathrm{co}\left(\begin{bmatrix} 0.5 & 1.3 \\ -1.1 & 1 \end{bmatrix}, \begin{bmatrix} 0.5 & 1 \\ -1.1 & 1 \end{bmatrix}, \begin{bmatrix} 0.5 & 1 \\ -1.1 & 0.8 \end{bmatrix}, \begin{bmatrix} 0.8 & 1 \\ -1.1 & 0.8 \end{bmatrix}\right), \quad \mathcal{B}_0 := \left\{\begin{bmatrix} 1 \\ 0.5 \end{bmatrix}\right\},$$

$\mathcal{A}_i := 0.05\mathcal{A}_0$ and $\mathcal{B}_i := \{0_{n \times m}\}$, $i \in \mathbb{Z}_{[-3,-1]}$. Furthermore, the constraints (6.1) are given by the sets $\mathbb{C}_x := (\mathbb{R}_{[-0.9,0.9]} \times \mathbb{R}_{[-1.6,1.6]})^4$ and $\mathbb{C}_u := \mathbb{R}_{[-0.8,0.8]}$.

To stabilize the system under study via Algorithm 6.1 a cLRF$((\mathbb{R}^n)^{h+1}, \mathbb{R}^m)$ is obtained via Proposition 2.6. To this end, the set in which $\{\delta_i\}_{i \in \mathbb{Z}_{[-3,0]}}$ can take values is split in 16^4 points, which results in 793 points satisfying $\sum_{i=-3}^{0} \delta_i = 1$. Thus, Proposition 2.6 with $\rho = 0.9$ yields the function $V(x_0) := x_0^\top P x_0$ and the control law $u_k = K x_k$, where

$$P = \begin{bmatrix} 1 & -0.1993 \\ -0.1993 & 0.2109 \end{bmatrix}, \quad K = \begin{bmatrix} -0.8068 & -1.0909 \end{bmatrix}.$$

Now it is possible to compute the sets $\mathbb{C}_x$, $\mathbb{C}_\pi$ and $\mathcal{N}$, as defined in the proof of Theorem 6.2. A projection of the aforementioned sets onto the current state is shown in Figure 6.1. Furthermore, using Algorithm 6.1 to stabilize the system under study for the initial
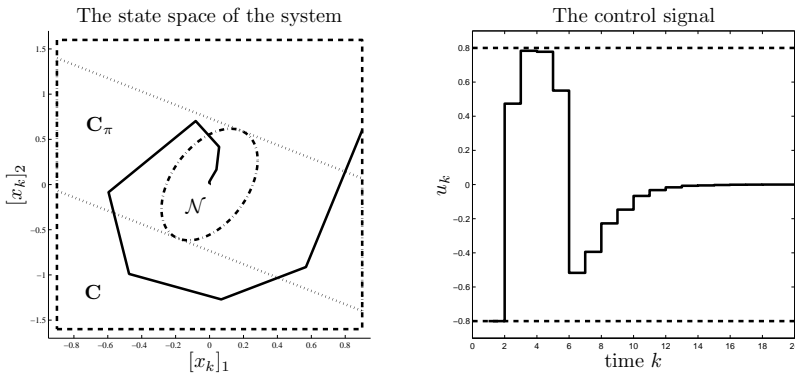

Figure 6.1: Left: A projection of the state constraints $\mathbb{C}_x$ (- - -) and the set $\mathbb{C}_\pi$ (· · · · ·) onto the current state, the neighborhood $\mathcal{N}$ (- · - · -) and the state trajectory (——). Right: The control signal (——) and the set $\mathbb{C}_u$ (- - -).

condition $x_i = [0.9\ 0.6]^\top$ for all $i \in \mathbb{Z}_{[-3,0]}$ and $u_i = 0$ for all $i \in \mathbb{Z}_{[-3,-1]}$ yields the trajectory and corresponding control signal also shown in Figure 6.1. Shown in the Figure 6.2 are the value of $V(x_k)$ and λ_k^* as a function of time. Moreover, also shown in Figure 6.2 are
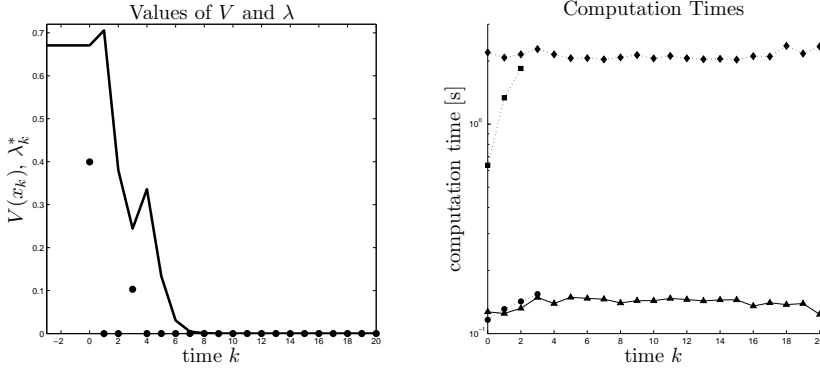

Figure 6.2: Left: The values of the cLRF (——) and λ_k (•) as a function of time. Right: The computation time of the control update in logarithmic scale, i.e., Algorithm 6.1 with (▲) and without (♦) Minkowski addition. Also shown are the minimization of V (•) and the MPC-like control approach (■).

the computation times[4] at each time instant. Note that the computation time corresponding to Algorithm 6.1 is a factor 15 smaller with Minkowski addition than without Minkowski addition, which indicates the advantage of performing the Minkowski addition online.

Next, several other control approaches are used to stabilize the system under study. *Firstly*, as $\mathbf{x}_{[-h,0]} \notin \mathbb{C}_\pi$, $Kx_0 \notin \mathbb{C}_u$ and hence the control law corresponding to the cLRF$((\mathbb{R}^n)^{h+1}, \mathbb{R}^m)$ is not feasible. *Secondly*, the control scheme proposed in [78] leads to an infeasible problem at time $k = 0$. *Thirdly*, for the considered initial condition a minimization of the quadratic cLRF candidate V, i.e., $\min_{u_k \in \mathbb{C}_u} x_{k+1}^\top P x_{k+1}$, with Minkowski addition leads to an infeasible problem at time $k = 4$. This is indicated in the plot of Figure 6.2, i.e., the time needed for solving a specific problem is shown until the first sampling instant when the problem becomes infeasible. Notice that this strategy is a direct application of the standard control LF approach [3] to time-delay systems and corresponds to Algorithm 6.1 without the constraint $\lambda_k \geq 0$. Hence, infeasibility of this approach demonstrates that the 'maximum decrease' approach of the standard control LF design is not always the feasible choice in the presence of constraints, which demonstrates the effectiveness of the control design method developed in this chapter (see also [85]). *Fourthly*, minimization of the cost function $\sum_{i=1}^N x_{k+i}^\top Q_i x_{k+i} + u_{k+i}^\top R_i u_{k+i}$, for some $N \in \mathbb{Z}_{\geq 1}$, $Q_i \in \mathbb{R}^{2 \times 2}$ and $R_i \in \mathbb{R}$, $i \in \mathbb{Z}_{[1,N]}$ is pursued. This method corresponds to a typical terminal cost MPC design approach, where one selects, e.g., based on the results in [121], a suitable cost func-

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[4]Simulations were performed using Matlab 7.5.0 on an Intel Q9400, 2.66GHz desktop PC.

tion and pursues minimization of the cost, while taking N sufficiently large. For example, choosing $N := 2$, $Q_1 := I_2$, $Q_2 := P$, $R_1 := 0.01$ and $R_2 := 0.01$ leads to an infeasible problem at time $k = 3$. Again, this can also be observed in Figure 6.2. Furthermore, choosing a larger horizon or adding a terminal constraint set, which guarantees recursive feasibility [119], either leads to an infeasible problem at time $k = 0$ or to an intractable optimization problem, i.e., the solver[5] does not return a solution, not even after a very large period of time. This indicates the conservativeness of the terminal constraint set method and, for that matter, of any other MPC design that requires a sufficiently large N for recursive feasibility, see, e.g., [119]. Unfortunately, other methods that take constraints into account such as the ones proposed in [41, 91] can be used to obtain a stabilizing controller for linear controlled DDIs without state delays only. $\qquad\square$

Recently, the applicability of Algorithm 6.1 in an automotive setting was studied in [21]. Therein, the control of a vehicle drivetrain over a control area network, which can be considered as an NCS setup, was studied and it was found that the application of Algorithm 6.1 dramatically improves the results when compared to the current control law.

While Example 6.1 and the article [21] illustrate that Algorithm 6.1 provides a control scheme that has certain advantages over other existing control schemes, its application is limited to linear controlled DDIs that admit a cLRF$((\mathbb{R}^n)^{h+1}, \mathbb{R}^m)$. This implies that Algorithm 6.1 inherits the conservatism that is associated with the Razumikhin approach, which motivates us to propose a second control scheme in what follows.

6.4 Stabilization via state-dependent Lyapunov-Razumikhin functions

Motivated by the above facts, an additional degree of freedom is added to the Razumikhin approach in what follows, which reduces the conservatism associated with this approach while preserving its computational advantages. In particular, the corresponding function is restricted to be quadratic but with a shape matrix that is dependent on all relevant delayed states, which leads to a set of necessary and sufficient Razumikhin-type conditions for stability of linear DDIs.

To simplify the presentation of the following results, autonomous DDIs are considered, i.e., DDIs of the form (2.1) as opposed to (2.15). Thereafter, a control scheme for linear controlled DDIs of the form (2.15) is derived based on the techniques that were developed for autonomous DDIs. Therefore, consider the following result.

Theorem 6.3 Suppose that the DDI (2.1) is a linear DDI and hence that the augmented system (2.2) is a linear difference inclusion. Then, the following statements are equivalent:

(i) There exist a function $\bar{P} : \mathbb{R}^{(h+1)n} \to \mathbb{R}^{(h+1)n \times (h+1)n}$, some $(\bar{c}_1, \bar{c}_2) \in \mathbb{R}_{>0} \times \mathbb{R}_{\geq \bar{c}_1}$ and a $\bar{\rho} \in \mathbb{R}_{[0,1)}$ such that $\bar{V}(\xi_0, \bar{P}) := \xi_0^\top \bar{P}(\xi_0)\xi_0$ satisfies

$$\bar{c}_1 I_{(h+1)n} \preceq \bar{P}(\xi_0) \preceq \bar{c}_2 I_{(h+1)n}, \tag{6.12a}$$

$$\bar{V}(\bar{\phi}_{k+1}, \bar{P}) \leq \bar{\rho}\bar{V}(\bar{\phi}_k, \bar{P}), \tag{6.12b}$$

---

[5]Simulations were performed using SeDuMi, LMILab and SDPT3. Similar results were obtained for all solvers. The results shown in Figures 6.1 and 6.2 were obtained using SeDuMi.

for all $\xi_0 \in \mathbb{R}^{(h+1)n}$ and all $(\bar{\Phi}, k) \in \bar{\mathcal{S}}(\xi_0) \times \mathbb{Z}_+$;

(ii) the linear difference inclusion (2.2) is GES;

(iii) the linear DDI (2.1) is GES. □

The proof of Theorem 6.3 is an adaptation of its continuous-time counterpart in [140, Section 5.4.3] and can be found in Appendix B.5. Theorem 1 in [29] provides a similar set of necessary and sufficient conditions for stability albeit with a quadratic LF that is based on the uncertain system parameters rather than the system state. When Theorem 6.3 is interpreted for the linear DDI (2.1), it provides a set of necessary and sufficient conditions for GES which correspond to the Krasovskii approach. Therefore, we refer to a function $\bar{V}$ that satisfies the hypothesis of Theorem 6.3 as a quadratic state-dependent LKF. Note that an explicit expression for the function $\bar{P}$ is provided in the proof of Theorem 6.3.

Next, using Theorem 6.3 and based on the Razumikhin approach, necessary and sufficient conditions for stability of the linear DDI (2.1) are obtained.

Theorem 6.4 Suppose that the DDI (2.1) is a linear DDI and hence that the augmented system (2.2) is a linear difference inclusion. Then, the following statements are equivalent:

(i) There exist a function $P : (\mathbb{R}^n)^{h+1} \to \mathbb{R}^{n \times n}$, some $(c_1, c_2) \in \mathbb{R}_{>0} \times \mathbb{R}_{\geq \bar{c}_1}$ and a $\rho \in \mathbb{R}_{[0,1)}$ such that $V(x_0, P) := x_0^\top P(\mathbf{x}_{[-h,0]})x_0$ satisfies

$$c_1 I_n \preceq P(\mathbf{x}_{[-h,0]}) \preceq c_2 I_n, \tag{6.13a}$$

$$V(\phi_{k+1}, P) \leq \rho \max_{i \in \mathbb{Z}_{[-h,0]}} V(\phi_{k+i}, P), \tag{6.13b}$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $(\Phi, k) \in \mathcal{S}(\mathbf{x}_{[-h,0]}) \times \mathbb{Z}_+$;

(ii) the linear difference inclusion (2.2) is GES;

(iii) the linear DDI (2.1) is GES. □

Theorem 6.4 is proven in Appendix B.5 and establishes that, for linear DDIs, the Razumikhin approach is not conservative when the function is restricted to be quadratic but with a shape matrix that is dependent on all relevant delayed states. Thus, necessary and sufficient conditions for stability of the linear DDI (2.1) have been obtained that are based on the Razumikhin approach. A function V that satisfies the hypothesis of Theorem 6.4 is called a quadratic state-dependent LRF. Figure 6.3 provides a schematic overview of the above results and summarizes some of the results in Section 2.3.

6.4.1 Quadratic state-dependent control Lyapunov-Razumikhin functions

Clearly, finding the function P in Theorem 6.4 is not necessarily much simpler than finding the function $\bar{V}$ corresponding to Theorem 2.1. However, in what follows we will use an online optimization algorithm that approximates the function P along a trajectory of the system and thus we obtain a computationally tractable control scheme for the linear controlled DDI (2.15) and the constraints (6.1). To this end, consider the following definition.
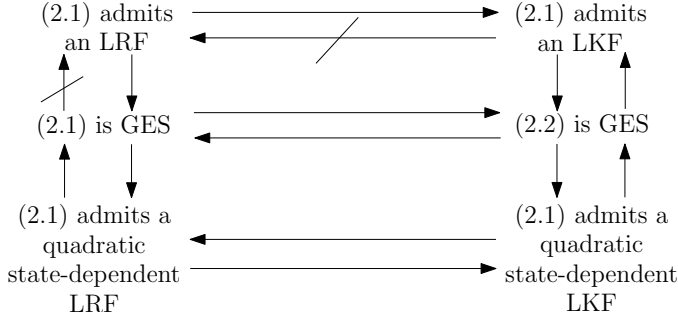
Figure 6.3: The existence of LKFs and LRFs when related to the stability of the DDI (2.1) under the assumption that (2.1) is a linear DDI. $A \to B$ means that A implies B and $A \nrightarrow B$ means that A does not necessarily imply B.

Definition 6.3 Suppose that the function $P : (\mathbb{R}^n)^{h+1} \to \mathbb{R}^{n \times n}$ satisfies (6.13a). Moreover, suppose that there exists a control law $\pi : \mathbb{C}_x \rightrightarrows \mathbb{C}_u$ such that, for all $\mathbf{x}_{[-h,0]} \in \mathbb{C}_x$ and all $x_1 \in F_\pi(\mathbf{x}_{[-h,0]})$, the closed-loop system (6.2) is such that $\mathbf{x}_{[-h+1,1]} \in \mathbb{C}_x$ and that the function $V(x_0, P) := x_0^\top P(\mathbf{x}_{[-h,0]})x_0$ satisfies (6.13b). Then, V is called a *quadratic state-dependent cLRF($\mathbb{C}_x$,$\mathbb{C}_u$)* for the linear controlled DDI (2.15). □

Given a quadratic state-dependent cLRF($\mathbb{C}_x$,$\mathbb{C}_u$) the stability of the closed-loop system (6.2) follows from Theorem 6.4. Moreover, as the set $\mathbb{C}_x$ is invariant, it can also be concluded that the constraints (6.1) are satisfied for any initial condition in $\mathbb{C}_x$ (and corresponding $\mathbf{u}_{[-h,-1]} \in \mathbb{C}_u^h$). The following algorithm is based on Definition 6.3 and aims to facilitate the control of linear controlled DDIs via a quadratic state-dependent cLRF. However, rather than finding an explicit expression for the function P we aim to approximate, at each time $k \in \mathbb{Z}_+$, the matrix $P(\Phi_{[k-h,k]}) \in \mathbb{R}^{n \times n}$ with the matrix $P_k \in \mathbb{R}^{n \times n}$. Therefore, let $P_k := c_2 I_n$ for all $k \in \mathbb{Z}_{[-h,0]}$.

Algorithm 6.2 At time $k \in \mathbb{Z}_+$, suppose that $\{P_{k+j}\}_{j \in \mathbb{Z}_{[-h,0]}}$, $\mathbf{x}_{[k-h,k]}$ and $\mathbf{u}_{[k-h,k-1]}$ are known[6] and find a $(u_k, P_{k+1}) \in \mathbb{R}^m \times \mathbb{R}^{n \times n}$ that satisfy

$$u_k \in \mathbb{C}_u, \quad \mathbf{x}_{[k-h+1,k+1]} \in \mathbb{C}_x, \tag{6.14a}$$

$$c_1 I_n \preceq P_{k+1} \preceq c_2 I_n, \tag{6.14b}$$

$$V(x_{k+1}, P_{k+1}) \leq \max_{i \in \mathbb{Z}_{[-h,0]}} \rho V(x_{k+i}, P_{k+i}), \tag{6.14c}$$

for all $x_{k+1} \in f(\mathbf{x}_{[k-h,k]}, \mathbf{u}_{[k-h,k]})$, where $V(x_0, P_0) := x_0^\top P_0 x_0$. □

Note that Algorithm 6.2 defines the set-valued control law

$$\pi(\mathbf{x}_{[k-h,k]}) := \{u_k \in \mathbb{R}^m : \exists P_{k+1} \in \mathbb{R}^{n \times n}, \text{ s.t. (6.14) holds}\}, \quad k \in \mathbb{Z}_+. \tag{6.15}$$

---

[6]This assumption merely implies that the controller is able to measure the current state and to store relevant past states, control actions and the corresponding matrices P_k.

In what follows, it will be established under which assumptions Algorithm 6.2 is recursively feasible in $\mathbb{C}_x$ and sufficient conditions for the stability of the closed-loop system (6.2) obtained from the control law (6.15) will be presented.

Proposition 6.2 Consider the closed-loop system (6.2) obtained from (6.15). Suppose that Algorithm 6.2 is recursively feasible for all $\mathbf{x}_{[-h,0]} \in \mathbb{C}_x$ and all $\Phi \in \mathcal{S}(\mathbf{x}_{[-h,0]})$. Then, the closed-loop system (6.2) obtained from (6.15) is GES. Moreover, the constraints (6.1) are satisfied at all times $k \in \mathbb{Z}_+$, i.e., $\Phi_{[k-h,k]} \in \mathbb{C}_x$ and $u_k \in \mathbb{C}_u$ for all $\mathbf{x}_{[-h,0]} \in \mathbb{C}_x$ and all $(\Phi, k) \in \mathcal{S}(\mathbf{x}_{[-h,0]}) \times \mathbb{Z}_+$. $\qquad\square$

Proposition 6.2 is proven in Appendix B.5 and of the type 'feasibility implies stability'. Hence, solving Algorithm 6.2 online for some initial conditions (assuming that it remains feasible) one does not obtain a quadratic state-dependent cLRF but merely a function that satisfies (6.14) for the corresponding closed-loop trajectory, see [89] for more details. In view of these observations we introduce the following definition.

Definition 6.4 Let $\Phi \in \mathcal{S}(\mathbf{x}_{[-h,0]})$ denote a trajectory of the closed-loop system (6.2) obtained from (6.15) for which Algorithm 6.2 is recursively feasible and let $V(\phi_k, P_k)$ denote the corresponding function. Then, $\{V(\phi_k, P_k)\}_{k \in \mathbb{Z}_{\geq -h}}$ is called a *quadratic trajectory-dependent cLRF* for the linear DDI (2.15). $\qquad\square$

It was established in Theorem 6.4 that any linear DDI that is GES admits a quadratic state-dependent LRF, which indicates the generality of Algorithm 6.2. However, choosing the right variables out of the feasible set such that Algorithm 6.2 becomes recursively feasible is not straightforward. To facilitate this choice, consider adding the cost function

$$J(u_k, P_{k+1}) := \max_{i \in \mathbb{Z}_{[-h,0]}} \rho V(x_{k+i}, P_{k+i}) - V(x_{k+1}, P_{k+1}), \qquad (6.16)$$

to Algorithm 6.2. Then, minimization of (6.16) under the conditions (6.14) implies a maximization of $V(x_{k+1}, P_{k+1})$ and also guarantees that J is lower bounded by zero. In other words, minimizing J achieves the least decrease in the value of $V(x_{k+1}, P_{k+1})$. The example presented at the end of this chapter confirms that adding (6.16) to Algorithm 6.2 improves recursive feasibility.

Similarly as discussed in Section 6.3.1, the computational requirements for the control scheme corresponding to Algorithm 6.2 can be reduced via an online Minkowski addition of sets, i.e., based on Lemma 6.2. Thus, it is possible to make the complexity of the computation of the control update less dependent on h. In this case, Algorithm 6.2 has an important advantage over Algorithm 6.1. Indeed, as Algorithm 6.2 does not, in contrast to Algorithm 6.1, require the construction of an LRF for the unconstrained system, a control scheme is obtained that is not limited by the conservatism of the Razumikhin approach and is computationally tractable independently of the size of the delay.

6.4.2 Semidefinite programming implementation of Algorithm 6.2

Next, it is shown how Algorithm 6.2 can be solved via an SDP problem. To this end, suppose[7] that the set $\mathcal{V}(\mathbf{x}_{[k-h,k]}, \mathbf{u}_{[k-h,k-1]})$ is known at each time $k \in \mathbb{Z}_+$. Moreover, suppose that $\gamma \in \mathbb{R}_{>0}$ and $\Gamma \in \mathbb{R}_{\geq\gamma}$ are fixed and let $P_k := \Gamma I_n$ for all $k \in \mathbb{Z}_{[-h,0]}$.

Proposition 6.3 At time $k \in \mathbb{Z}_+$, find a $(u_k, Z_{k+1}) \in \mathbb{R}^m \times \mathbb{R}^{n \times n}$ such that

$$\mathbf{x}_{[k-h+1,k+1]} \in \mathbb{C}_x, \quad u_k \in \mathbb{C}_u, \tag{6.17a}$$

$$\Gamma^{-1} I_n \preceq Z_{k+1} \preceq \gamma^{-1} I_n, \tag{6.17b}$$

$$\begin{bmatrix} \rho \max_{i \in \mathbb{Z}_{[-h,0]}} x_{k+i}^\top P_{k+i} x_{k+i} & * \\ x_{k+1} & Z_{k+1} \end{bmatrix} \succeq 0, \tag{6.17c}$$

for all $(B_0, v) \in \mathcal{B}_0 \times \mathcal{V}(\mathbf{x}_{[k-h,k]}, \mathbf{u}_{[k-h,k-1]})$, where $x_{k+1} := B_0 u_k + v$. Then, any solution (u_k, Z_{k+1}) to (6.17) satisfies (6.14) with $P_{k+1} = Z_{k+1}^{-1}$, $c_1 = \gamma$ and $c_2 = \Gamma$. $\quad\square$

Proposition 6.3 is proven in Appendix B.5. If $\mathbb{C}_x$ and $\mathbb{C}_u$ are polytopes or ellipsoids and the set $\mathcal{AB}$ is a matrix polytope (or consists of a finite number of matrices), then finding a solution that satisfies (6.17) amounts to solving an LMI of finite dimensions. Thus, a solution to Algorithm 6.2 can be obtained via SDP. The resulting control law (6.15) is stabilizing if the corresponding optimization problem is recursively feasible.

As it was observed above, to facilitate the right choice of variables a cost function can be employed. For example, using Proposition 6.3 and some nontrivial facts about positive semidefinite matrices, see, e.g., [12], it can be shown that a solution to Algorithm 6.2 that minimizes the cost (6.16) is obtained by solving the following SDP problem, i.e.,

$$\min_{(Z_{k+1}, u_k, \varepsilon_k)} \varepsilon_k, \tag{6.18}$$

subject to (6.17), $\varepsilon_k \geq 0$ and

$$\begin{bmatrix} \rho \max_{i \in \mathbb{Z}_{[-h,0]}} x_{k+i}^\top P_{k+i} x_{k+i} & * \\ x_{k+1} & Z_{k+1} \end{bmatrix} \preceq \varepsilon_k I_{n+1}.$$

The optimization algorithm (6.18) provides a stabilizing control law for the linear controlled DDI (2.15) taking into account the constraints (6.1). The application of this control law is illustrated in the following example.

Example 6.2 Consider the linear controlled DDI (2.15) with $h = 3$ and let

$$\mathcal{A}_0 := \mathrm{co}\left(\begin{bmatrix} 1.5 & 0.1 \\ 0.1 & 0.3 \end{bmatrix}, \begin{bmatrix} 1.5 & -0.1 \\ 0.1 & 0.3 \end{bmatrix}, \begin{bmatrix} 1.5 & 0.1 \\ -0.1 & 0.3 \end{bmatrix}, \begin{bmatrix} 1.5 & -0.1 \\ -0.1 & 0.3 \end{bmatrix}\right),$$

$$\mathcal{B}_0 := \mathrm{co}\left(\begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \begin{bmatrix} 1 \\ 0.25 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix}\right),$$

---

[7]$\mathcal{V}$ can be computed at each time $k \in \mathbb{Z}_+$, as explained in Section 6.3.1.

$\mathcal{A}_i := 0.1 \cdot \mathcal{A}_0$ and $\mathcal{B}_i := 0.1 \cdot \mathcal{B}_0$, $i \in \mathbb{Z}_{[-3,-1]}$ such that $\mathcal{AB} := \mathcal{A}_{-h} \times \ldots \times \mathcal{B}_0$. Furthermore, consider the constraints $\mathbb{C}_x := (\mathbb{R}_{[-0.8,0.8]} \times \mathbb{R}_{[-0.6,0.6]})^4$ and $\mathbb{C}_u := \mathbb{R}_{[-0.6,0.6]}$.

To stabilize the system under study, Algorithm 6.2 is used together with the cost (6.16). A solution to Algorithm 6.2 is obtained by solving at each time $k \in \mathbb{Z}_+$ the optimization problem (6.18) with the initialization $\rho = 0.95$, $\gamma = 0.5$, $\Gamma = 5$ and $\{P_k = \Gamma I_n\}_{k \in \mathbb{Z}_{[-3,0]}}$. For a large variety of initial conditions, the control algorithm was able to stabilize the linear controlled DDI (2.15). Figure 6.4 shows the state trajectories and input values as a function of time for the initial conditions $x_i = [-0.5 \; 0.6]^\top$ for all $i \in \mathbb{Z}_{[-3,0]}$ and $u_i = 0$ for all $i \in \mathbb{Z}_{[-3,-1]}$. Observe that the constraints are satisfied nontrivially at all times.
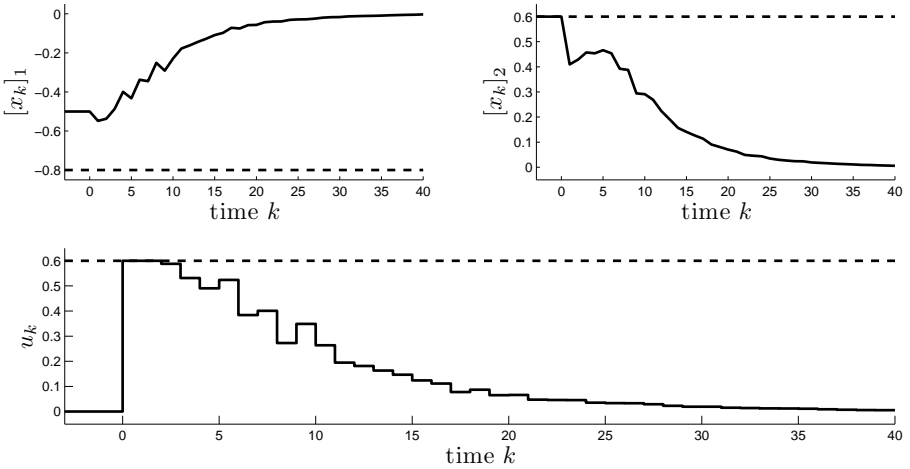

Figure 6.4: Left: A projection of the state constraints $\mathbb{C}_x$ (- - -) onto the current state and the state trajectory (——). Right: u_k as a function of time (——) and the set $\mathbb{C}_u$.

To illustrate the computational advantages of Algorithm 6.2, the dimension of the LMI that needs to be solved to stabilize the linear controlled DDI (2.15) is compared for a selection of control solutions, i.e., the offline synthesis methods presented in [26, 146], the online optimization-based method presented in [78] and for the optimization problem corresponding to Algorithms 6.1 and 6.2. The results are shown in Table 6.1, therein, to obtain a fair comparison, the constraints were not taken into account. Furthermore, as the bottleneck for Algorithm 6.1 is finding an offline solution to an LMI of large dimensions (as opposed to the online component), these dimensions are shown in Table 6.1. Observe that the control schemes that are based on the Razumikhin approach, i.e., Algorithms 6.1 and 6.2, have a smaller complexity than their counterparts based on the Krasovskii approach.

The values in Table 6.1 clearly show the need for a technique whose computational complexity is independent of the size of the delay, which justifies the approach presented in Algorithm 6.2. Note that, the comparison in Table 6.1 merely indicates the importance of the complexity issue for the stabilization of linear controlled DDIs and should not be used

Table 6.1: Dimension of the LMI for various control approaches.

method	type	dimension
Theorem 3 in [146]	offline	$(589 \cdot 10^4 \times 589 \cdot 10^4)$
Theorem 1 in [78]	online	$(164 \cdot 10^4 \times 164 \cdot 10^4)$
Theorem 5 in [26]	offline	$(144 \cdot 10^4 \times 144 \cdot 10^4)$
Algorithm 6.1	offline	$(655 \cdot 10^3 \times 655 \cdot 10^3)$
Algorithm 6.2 without Minkowski addition	online	$(197 \cdot 10^3 \times 197 \cdot 10^3)$
Algorithm 6.2 with Minkowski addition	online	(100×100)

to draw any further conclusions regarding the various control schemes. □

6.5 Conclusions

In this chapter we proposed two frameworks for stabilizing controller synthesis which deal with both constraints and large delays via the Razumikhin approach. Firstly, based on a local LRF, obtained for the system in the absence of constraints, a stabilizing receding horizon control scheme was developed that can handle constraints. It was then demonstrated that by exploiting properties of the Minkowski addition of polytopes and the structure of the developed control law, an efficient implementation of the online component of the control scheme can be attained even for large delays. Then, a second approach that makes use of a quadratic state-dependent cLRF to reduce the computational complexity that is typically associated with the stabilization of linear controlled DDIs was developed. Moreover, by allowing the shape matrix of the cLRF to be dependent on all relevant delayed states the corresponding conditions were shown to be necessary and sufficient for stability. Therefore, this control scheme remains computationally tractable for large delays and is not hampered by the conservatism that is typically associated with the Razumikhin approach.

In view of the close relation of DDIs to interconnected systems, it is reasonable to expect that the results that were derived in this thesis can be extended to interconnected systems with delay. In the following chapter we establish such an extension for the results that were presented in Chapter 2. The so-obtained results are illustrated via a practical case study, i.e., a power system that is controlled only locally and over a communication network, which gives rise to local delays in the power plants.

Chapter 7

Interconnected systems with delay

In this chapter we consider the stability analysis of large-scale interconnected systems with delay. We consider both interconnection delays, which arise in the paths connecting the subsystems, and local delays, which arise in the dynamics of the subsystems. Using small-gain arguments it is proven that interconnection delays do not affect the stability of an interconnected system if a delay-independent small-gain condition holds. Furthermore, also using small-gain arguments, stability for interconnected systems with local delay is established via the Razumikhin as well as the Krasovskii approach. By combining the above results, we obtain a scalable stability analysis framework for interconnected systems with both interconnection and local delays. The applicability of this stability analysis framework is illustrated via a classical power systems example wherein the power plants are controlled only locally via a communication network, which gives rise to local delays.

7.1   Introduction

Large-scale interconnections of dynamical systems, such as power systems, chemical processes, biological systems and urban water supply networks, form an important topic in the field of control systems, see, e.g., [103, 141] and the references therein. The stability analysis of such systems is generally hampered by the large size and complexity of the overall system. Therefore, to render the stability analysis of the overall interconnected system tractable, smaller subsystems are typically considered separately, without taking into account the interconnections between the subsystems. Then, a set of coupling conditions, which take into account the interconnections, is employed to pursue stability analysis of the overall interconnected system in a distributed manner. To this end, two fundamental approaches, which rely on the concept of vector LFs [83], were proposed, i.e., small–gain theory, see, e.g., [32, 70], and dissipativity theory [84, 144].

In practice, interconnections of dynamical systems, such as, for example, power systems, often show a geographical separation of the subsystems. Hence, the propagation of signals takes place over large distances which can induce interconnection delays. Furthermore, due to inherent delays in the dynamical processes, local delays can also arise in the subsystems. Indeed, for example, in power systems interconnection delays are introduced by water flowing through rivers that connect hydro-thermal power plants [142] while local delays can be introduced by human operators in the local control loops. Therefore, several

small-gain theorems that make use of the Krasovskii as well as the Razumikhin approach have appeared recently for interconnected systems with delay. For example, based on the Krasovskii approach, the stability analysis of interconnected systems with both interconnection and local delays was performed in [31, 62]. Furthermore, an alternative set of sufficient conditions for stability, this time based on the Razumikhin approach and small-gain arguments, was presented in [31]. Alternatively, the relation of the Razumikhin approach to the small-gain theorem established in [133] was used in [136] to formulate a small-gain theorem for interconnected systems with both interconnection and local delays. A different approach was taken in [137], where a small-gain theorem for interconnected systems with both interconnection and local delays was established using standard small-gain arguments, but without using Lyapunov theory. However, none of the above results applies to interconnections of discrete-time systems with delay. Moreover, interconnection delays and local delays have thus far mostly been considered at the same time, while in [62] it was shown that considering them separately can be advantageous.

Therefore, in this chapter we study the stability of interconnections of *discrete-time* systems with delay. Moreover, interconnection delays and local delays are considered separately. Based on the relation of DDIs to interconnected systems that was established in Chapter 5 and small-gain arguments, we prove that interconnection delays do not affect the stability of an interconnected system if a delay-independent small-gain condition holds. While such a small-gain condition might seem to be a strong requirement, it is not uncommon in, e.g., cooperative control [39]. Furthermore, under a similar small-gain condition, it is shown that interconnected systems with local delay admit an LRF for the overall interconnected system if each subsystem admits an LRF. Similarly, it is shown that the interconnected system admits an LKF if each subsystem admits an LKF and a small-gain condition is satisfied. A combination of the results for interconnected systems with interconnection delay and local delay, respectively, provides a scalable framework for the stability analysis of general interconnected systems with delay. The applicability of this stability analysis framework is illustrated via a classical power systems example, i.e., the CIGRÉ 7-machine power system. In this example, the power plants are controlled only locally via a communication network, which gives rise to local delays.

7.2 Interconnected systems with interconnection delay

First, interconnection delays are considered only. Then, in the second part of this chapter, local delays are treated. Therefore, consider the subsystems (3.1) but now with interconnection delays, i.e., for all $i \in \mathbb{Z}_{[1,N]}$

$$x_{i,k+1} \in G_i(x_{1,k-h_{i,1}}, \ldots, x_{N,k-h_{i,N}}), \quad k \in \mathbb{Z}_+, \tag{7.1}$$

with G_i as defined in (3.1) and where $h_{i,j} \in \mathbb{Z}_+$, $(i,j) \in \mathbb{Z}_{[1,N]}^2$ is the interconnection delay from subsystem j to subsystem i. For now, *it is assumed that $h_{i,i} = 0$ for all $i \in \mathbb{Z}_{[1,N]}$*, i.e., the subsystems are not affected by local delays. To describe the complete interconnected system with interconnection delay the usual definition $x_0 := \mathrm{col}(\{x_{l,0}\}_{l \in \mathbb{Z}_{[1,N]}}) \in \mathbb{R}^n$ is used, which yields a DDI of the form (2.1) with $n = \sum_{i=1}^{N} n_i$, $h = \max_{(i,j) \in \mathbb{Z}_{[1,N]}^2} h_{i,j}$ and where $F : (\mathbb{R}^n)^{h+1} \rightrightarrows \mathbb{R}^n$ is obtained from the maps G_i and the delays $h_{i,j}$, $(i,j) \in \mathbb{Z}_{[1,N]}^2$.

The stability analysis of the DDI (2.1) obtained from (7.1) using the techniques pre-

sented in, e.g., Chapter 2 is hampered by the size and complexity of the overall system. On the other hand, as illustrated in Figure 7.1, using the transformation that was established in Chapter 3, the DDI (2.1) obtained from (7.1) can be transformed into an augmented interconnected system. For the so-obtained interconnected system, Theorem 3.1 provides a
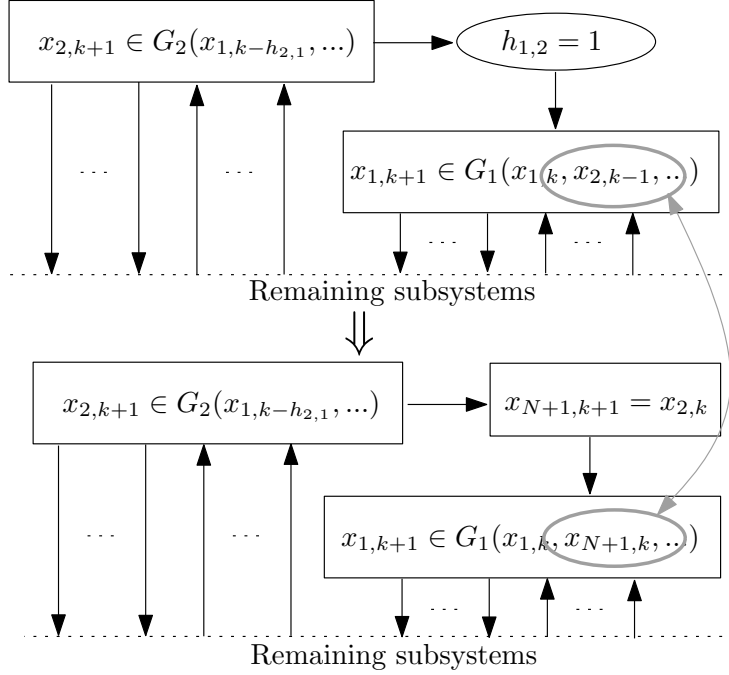

Figure 7.1: A graphical depiction of the transformation of an interconnected system with interconnection delay into an augmented interconnected system.

scalable stability analysis method. The following result makes use of this approach and reaches a surprising conclusion.

Theorem 7.1 Suppose that the subsystems without interconnection delay (3.1) satisfy the hypothesis of Theorem 3.1. Then, the DDI (2.1) obtained from the subsystems with interconnection delay (7.1) is $\mathcal{KL}$-stable. $\qquad\square$

Theorem 7.1 is proven in Appendix B.6 and establishes that, if the delay-independent small-gain condition in the hypothesis of Theorem 3.1 holds, *finite interconnection delays can not cause in*stability *of an interconnected system*. Hence, the stability analysis of interconnected systems with interconnection delay can be reduced, via Theorem 7.1, to the stability analysis of standard interconnected systems. This result stands in sharp contrast with the typical destabilizing effect that time delays have. Note that, while such a small-gain condition might seem to be a strong requirement, it is not uncommon in, e.g., cooperative control [39]

and allows to significantly simplify the analysis of an otherwise very complicated problem. The above discussion indicates an advantage of considering interconnection delays and local delays separately as opposed to considering both types of delay at once, as done in other works, see, e.g., [31, 136].

Remark 7.1 For continuous-time systems a similar relation was also observed in [62, 114, 123]. However, the derivations for such systems rely on different arguments. Indeed, the transformation applied to prove Theorem 7.1 does not apply to the continuous-time case. □

Remark 7.2 It should be noted that Theorem 7.1 does not assume any knowledge about the interconnection delays. If the interconnection delays are assumed to be known, potentially less conservative delay-dependent small-gain conditions can be derived, see, e.g., [62] for the continuous-time case. □

Now that interconnection delays have been treated, local delays are considered.

7.3 Interconnected systems with local delay

When a subsystem is subject to delays in the local dynamical process, an interconnected system with local delay is obtained. Therefore, such systems are considered in what follows. In particular, the small-gain theory presented in Theorem 3.1 is combined with the Krasovskii and Razumikhin approaches to establish stability for interconnected systems with local delay. Therefore, consider an interconnection of $N \in \mathbb{Z}_{\geq 2}$ subsystems affected by local delays. The dynamics of the i-th subsystem, $i \in \mathbb{Z}_{[1,N]}$ is described by

$$x_{i,k+1} \in F_i(\mathbf{x}_{i,[k-\hat{h},k]}, x_{1,k}, \ldots, x_{N,k}), \quad k \in \mathbb{Z}_+, \tag{7.2}$$

where for each $i \in \mathbb{Z}_{[1,N]}$, $F_i : (\mathbb{R}^{n_i})^{\hat{h}+1} \times \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_N} \rightrightarrows \mathbb{R}^{n_i}$, the notation $\mathbf{x}_{i,[-\hat{h},0]} := \{x_{i,l}\}_{l \in \mathbb{Z}_{[-\hat{h},0]}}$ and $\hat{h} \in \mathbb{Z}_+$ is the maximal delay affecting (7.2). Note that, with a slight abuse of notation, to simplify the exposition, $x_{i,k}$ appears twice as an argument of F_i. Moreover, it can be assumed without any restriction to the generality of the results that all subsystems (7.2) share the same maximal delay. To describe the complete interconnected system obtained from the subsystems with local delay (7.2) use the standard definition $x_0 := \mathrm{col}(\{x_{l,0}\}_{l \in \mathbb{Z}_{[1,N]}}) \in \mathbb{R}^n$, which yields a DDI of the form (2.1) with $h = \hat{h}$, $n = \sum_{i=1}^N n_i$ and where F is obtained from F_i, $i \in \mathbb{Z}_{[1,N]}$, i.e., $F(\mathbf{x}_{[-h,0]}) = \mathrm{col}(\{F_i(\mathbf{x}_{i,[-\hat{h},0]}, x_{1,0}, \ldots, x_{N,0})\}_{i \in \mathbb{Z}_{[1,N]}})$.

The following result, which can be used for the stability analysis of an interconnected system of the form (2.1) obtained from (7.2), combines the Razumikhin approach with small-gain theory, i.e., Theorems 3.3 and 3.1.

Theorem 7.2 Suppose that there exist functions $\{V_i, \gamma_{i,j}\}_{(i,j)\in\mathbb{Z}^2_{[1,N]}}$, with $V_i : \mathbb{R}^{n_i} \to \mathbb{R}_+$ and $\gamma_{i,j} \in \mathcal{K}_\infty \cup \{0\}$, and some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that for all $i \in \mathbb{Z}_{[1,N]}$ it holds that

$$\alpha_1(\|x_{i,0}\|) \leq V_i(x_{i,0}) \leq \alpha_2(\|x_{i,0}\|), \tag{7.3a}$$

$$V_i(x_{i,1}) \leq \max\{\max_{j' \in \mathbb{Z}_{[-h,0]}} \gamma_{i,i}(V_i(x_{i,j'})), \max_{j \in \mathbb{Z}_{[1,N]}, j \neq i} \gamma_{i,j}(V_j(x_{j,0}))\}, \tag{7.3b}$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $x_{i,1} \in F_i(\mathbf{x}_{i,[-h,0]}, x_{1,0}, \ldots, x_{N,0})$. Suppose that for all $y \in \mathbb{R}_+^N \setminus \{0\}$ there exists an $i(y) \in \mathbb{Z}_{[1,N]}$ such that $\max_{j \in \mathbb{Z}_{[1,N]}} \gamma_{i(y),j}([y]_j) < [y]_{i(y)}$. Then, the interconnected system with local delay (2.1) obtained from (7.2) is $\mathcal{KL}$-stable. $\square$

Theorem 7.2 is proven in Appendix B.6 and uses the fact that, if all subsystems with local delay (7.2) admit an LRF-like function, i.e., V_i, and the small-gain condition in the second item of the hypothesis of Theorem 7.2 holds, then the interconnected system (2.1) obtained from (7.2) admits an LRF. Hence, if the hypothesis of Theorem 7.2 is satisfied, then it follows from Theorem 3.3 that the DDI (2.1) obtained from (7.2) is $\mathcal{KL}$-stable.

The following result parallels Theorem 7.2 and combines the Krasovskii approach with small-gain theory, i.e., Theorems 2.1 and 3.1.

Theorem 7.3 Suppose that there exist $\{\bar{V}_i, \bar{\gamma}_{i,j}\}_{(i,j) \in \mathbb{Z}_{[1,N]}^2}$, with $\bar{V}_i : (\mathbb{R}^{n_i})^{h+1} \to \mathbb{R}_+$ and $\bar{\gamma}_{i,j} \in \mathcal{K}_\infty \cup \{0\}$, and some $\bar{\alpha}_1, \bar{\alpha}_2 \in \mathcal{K}_\infty$ such that for all $i \in \mathbb{Z}_{[1,N]}$ it holds that

$$\bar{\alpha}_1(\|\mathbf{x}_{i,[-h,0]}\|) \leq \bar{V}_i(\mathbf{x}_{i,[-h,0]}) \leq \bar{\alpha}_2(\|\mathbf{x}_{i,[-h,0]}\|), \tag{7.4a}$$

$$\bar{V}_i(\mathbf{x}_{i,[-h+1,1]}) \leq \max_{j \in \mathbb{Z}_{[1,N]}} \bar{\gamma}_{i,j}(\bar{V}_j(\mathbf{x}_{j,[-h,0]})), \tag{7.4b}$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $x_{i,1} \in F_i(\mathbf{x}_{i,[-h,0]}, x_{1,0}, \ldots, x_{N,0})$. Suppose that for all $y \in \mathbb{R}_+^N \setminus \{0\}$ there exists an $i(y) \in \mathbb{Z}_{[1,N]}$ such that $\max_{j \in \mathbb{Z}_{[1,N]}} \bar{\gamma}_{i(y),j}([y]_j) < [y]_{i(y)}$. Then, the interconnected system with local delay (2.1) obtained from (7.2) is $\mathcal{KL}$-stable. $\square$

The proof of Theorem 7.3 is similar to the proof of Theorem 7.2 and is omitted here for brevity. Theorem 7.3 uses that, if all subsystems with local delay (7.2) admit an LKF-like function, i.e., $\bar{V}_i$, and the small-gain condition in the second item of the hypothesis of Theorem 7.3 holds, then the interconnected system (2.1) obtained from (7.2) admits an LKF with a nonlinear function $\rho \in \mathcal{K}_\infty \cup \{0\}$ such that $\rho(r) < r$ for all $r \in \mathbb{R}_{>0}$, as opposed to a constant. Still, if the hypothesis of Theorem 7.2 is satisfied, then it follows from a reasoning similar to the proof of Theorem 2.1 that the DDI (2.1) obtained from (7.2) is $\mathcal{KL}$-stable.

If the graph corresponding to the subsystems with local delay (7.2) is strongly connected, see [32] for details, then the following corollaries follow directly from Theorem 5.2 in [32] and the reasoning used to prove Theorems 7.2 and 7.3.

Corollary 7.1 Suppose that there exist functions $\{V_i, \gamma_{i,j}\}_{(i,j) \in \mathbb{Z}_{[1,N]}^2}$, with $V_i : \mathbb{R}^{n_i} \to \mathbb{R}_+$ and $\gamma_{i,j} \in \mathcal{K}_\infty \cup \{0\}$, and some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that for all $i \in \mathbb{Z}_{[1,N]}$ it holds that

$$\alpha_1(\|x_{i,0}\|) \leq V_i(x_{i,0}) \leq \alpha_2(\|x_{i,0}\|), \tag{7.5a}$$

$$V_i(x_{i,1}) \leq \max_{j' \in \mathbb{Z}_{[-h,0]}} \gamma_{i,i}(V_i(x_{i,j'})) + \sum_{j \in \mathbb{Z}_{[1,N]}, j \neq i} \gamma_{i,j}(V_j(x_{j,0})), \tag{7.5b}$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $x_{i,1} \in F_i(\mathbf{x}_{i,[-h,0]}, x_{1,0}, \ldots, x_{N,0})$. Suppose that for all $y \in \mathbb{R}_+^N \setminus \{0\}$ there exists an $i(y) \in \mathbb{Z}_{[1,N]}$ such that $\sum_{j \in \mathbb{Z}_{[1,N]}} \gamma_{i(y),j}([y]_j) < [y]_{i(y)}$. Then, the interconnected system with local delay (2.1) obtained from (7.2) is $\mathcal{KL}$-stable. $\square$

Corollary 7.2 Suppose that there exist $\{\bar{V}_i, \bar{\gamma}_{i,j}\}_{(i,j)\in\mathbb{Z}_{[1,N]}^2}$, with $\bar{V}_i : (\mathbb{R}^{n_i})^{h+1} \to \mathbb{R}_+$ and $\bar{\gamma}_{i,j} \in \mathcal{K}_\infty \cup \{0\}$, and some $\bar{\alpha}_1, \bar{\alpha}_2 \in \mathcal{K}_\infty$ such that for all $i \in \mathbb{Z}_{[1,N]}$ it holds that

$$\bar{\alpha}_1(\|\mathbf{x}_{i,[-h,0]}\|) \leq \bar{V}_i(\mathbf{x}_{i,[-h,0]}) \leq \bar{\alpha}_2(\|\mathbf{x}_{i,[-h,0]}\|), \tag{7.6a}$$

$$\bar{V}_i(\mathbf{x}_{i,[-h+1,1]}) \leq \textstyle\sum_{j\in\mathbb{Z}_{[1,N]}} \bar{\gamma}_{i,j}(\bar{V}_j(\mathbf{x}_{j,[-h,0]})), \tag{7.6b}$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $x_{i,1} \in F_i(\mathbf{x}_{i,[-h,0]}, x_{1,0}, \ldots, x_{N,0})$. Suppose that for all $y \in \mathbb{R}_+^N \setminus \{0\}$ there exists an $i(y) \in \mathbb{Z}_{[1,N]}$ such that $\max_{j\in\mathbb{Z}_{[1,N]}} \bar{\gamma}_{i(y),j}([y]_j) < [y]_{i(y)}$. Then, the interconnected system with local delay (2.1) obtained from (7.2) is $\mathcal{KL}$-stable. $\square$

The advantages of standard Krasovskii and Razumikhin theorems when compared to each other, see the various discussions throughout Chapter 2, also apply to Theorems 7.2 and 7.3 and Corollaries 7.1 and 7.2, i.e., computational simplicity is traded for conceptual generality. In fact, three options can be distinguished for interconnected systems with local delay, i.e., (i) consider the interconnected system as a single system with delay and apply the Krasovskii approach; (ii) consider the interconnected system as an interconnection of several subsystems with delay, apply the Krasovskii approach locally and then use small-gain arguments (see, e.g., Theorem 7.3); (iii) consider, based on the transformation developed in Chapter 3, the interconnected system with delay as the interconnection of an augmented, potentially very large, set of subsystems and apply small-gain arguments directly (see, e.g., Theorem 7.2 and the observations in Chapter 3). Due to the large size and complexity of interconnected systems (i) is generally not a tractable approach. Both approaches (ii) and (iii) lead to a tractable stability analysis framework for large-scale systems. Furthermore, while the conditions related to (iii) are more conservative, these conditions are simpler to verify compared to the conditions related to (ii).

Remark 7.3 Theorems 7.2 and 7.3 and Corollaries 7.1 and 7.2 are discrete-time counterparts of Theorem 3.4 and Theorem 3.7 in [31]. However, the reasoning required to prove the results for the discrete-time case differs significantly with respect to the continuous-time case, mainly due to the different conditions involved in the Razumikhin approach. As such, the aforementioned results provide a valuable addition to the results presented in [31]. $\square$

Remark 7.4 Suppose that, for the subsystems with local delay (7.2) for all $i \in \mathcal{I} \subset \mathbb{Z}_{[1,N]}$, the functions V_i satisfy the hypothesis of Theorem 7.2. Furthermore, suppose for all $i \in \mathbb{Z}_{[1,N]} \setminus \mathcal{I}$, the functions $\bar{V}_i$ satisfy the hypothesis of Theorem 7.3. Then, for the subsystems (7.2) for all $i \in \mathcal{I}$ functions $\bar{V}_i$ that satisfy the hypothesis of Theorem 7.3 can be constructed from V_i via Proposition 3.1. Thus, the stability analysis of the interconnected system (2.1) obtained from (7.2) can be performed via Theorem 7.3. $\square$

In Theorem 7.1 small-gain arguments are used to establish *robustness of stability with respect to interconnection delays* for an interconnected system. On the other hand, in Theorem 7.2 and Theorem 7.3 small-gain arguments are used to establish *robustness of stability*

with respect to inputs from the other subsystems for a subsystem with local delay. Hence, while all proofs are based on small-gain arguments, the reasoning used to prove Theorem 7.1 is different from the reasoning used to prove Theorem 7.2 and Theorem 7.3. The fact that the reasoning of Theorem 7.1 also applies in the context of Theorem 7.2 and Theorem 7.3 is exploited in the next section.

7.4 General interconnected systems with delay

If an interconnected system consists of one or more subsystems with delays in the dynamical process and the subsystems are located in different geographical places, a general interconnected system with delay is obtained. For the stability analysis of such systems, a combination of Theorem 7.1 with Theorem 7.2 or Theorem 7.3, respectively, is required. In what follows such results are derived. Therefore, consider an interconnection of $N \in \mathbb{Z}_{\geq 2}$ subsystems with local delay described by (7.2) that are subject to interconnection delays. Then, the dynamics of the i-th subsystem, $i \in \mathbb{Z}_{[1,N]}$ is given by

$$x_{i,k+1} \in F_i(\mathbf{x}_{i,[k-\hat{h}]}, x_{1,k-h_{i,1}}, \dots, x_{N,k-h_{i,N}}), \quad k \in \mathbb{Z}_+, \tag{7.7}$$

with $x_{i,k} \in \mathbb{R}^{n_i}$ and F_i, $i \in \mathbb{Z}_{[1,N]}$ as defined in (7.2). Above, $h_{i,j} \in \mathbb{Z}_+$, $(i,j) \in \mathbb{Z}_{[1,N]}^2$ is the interconnection delay from the j-th to the i-th subsystem. It is assumed that $h_{i,i} = 0$ for all $i \in \mathbb{Z}_{[1,N]}$. To describe the complete interconnected system, let $x_0 :=$ $\mathrm{col}(\{x_{l,0}\}_{l \in \mathbb{Z}_{[1,N]}}) \in \mathbb{R}^n$, which yields a system of the form (2.1) with $n = \sum_{i=1}^N n_i$, $h = \max\{\hat{h}, \max_{(i,j) \in \mathbb{Z}_{[1,N]}^2} h_{i,j}\}$ and where F is obtained from the functions F_i and the delays $h_{i,j}$, $(i,j) \in \mathbb{Z}_{[1,N]}^2$. The following corollary, which is based on the Razumikhin approach, can be obtained directly from Theorems 7.1 and 7.2 and Corollary 7.1.

Corollary 7.3 Suppose that the subsystems with local delay (7.2) satisfy the hypothesis of Theorem 7.2 or Corollary 7.1. Then, the interconnected system (2.1) obtained from the subsystems with both local and interconnection delays (7.7) is $\mathcal{KL}$-stable. □

Moreover, a similar result can be obtained, based on the Krasovskii approach, from Theorems 7.1 and 7.3 and Corollary 7.2.

Corollary 7.4 Suppose that the subsystems with local delay (7.2) satisfy the hypothesis of Theorem 7.3 or Corollary 7.2. Then, the interconnected system (2.1) obtained from the subsystems with both local and interconnection delays (7.7) is $\mathcal{KL}$-stable. □

The above general results provide a framework for the stability analysis of interconnected systems with delay. Moreover, the results for interconnected systems with interconnection delay only or with local delay only are recovered as a particular case, i.e., for $\hat{h} = 0$ and for $h_{i,j} = 0$, $(i,j) \in \mathbb{Z}_{[1,N]}^2$, respectively. Furthermore, note that Corollaries 7.3 and 7.4 reduce the stability analysis of interconnected systems with both interconnection and local delays to the stability analysis of interconnected systems with local delay only via a delay-independent small-gain condition. As the stability analysis of interconnected systems with local delay only is in general less complex, Corollaries 7.3 and 7.4 provide a simpler

tool to analyse the stability for interconnected systems with delay, when compared to the continuous-time results in, e.g., [31, 136]. Moreover, Corollary 7.3 provides a counterpart for discrete-time systems to the results presented in [62].

7.5 Case study: Power systems with delay

To illustrate the applicability of the results that were derived in this chapter, the stability analysis of power systems is considered. Typically, two different control layers can be distinguished in power systems. Firstly, an upper market-based layer, which operates on a time scale of several minutes up to several hours, which is employed for the scheduling of power generation, see, e.g., [66, 127] and the references therein. The market-based layer handles predictable variations in the supply and demand of energy and takes care of constraints such as tie line constraints and generator capacities. As the scheduling algorithms in the upper control layer are not time critical they can make use of the exchange of information between nodes and, hence, are often centralized or iteration-based. Secondly, the lower control layer, which operates on a time scale of seconds, handles the continuous balancing of the supply and demand of energy. Because of the fast time scale on which these control schemes need to act, they consist of (almost) decentralized controllers that require very little communication. This control layer is referred to as automatic generation control (AGC), see, e.g., [81, 124]. In general, AGC does not take constraints into account as its design is based on the assumption that constraints are taken care of via constraint margins in the upper control layer.

This case study focuses on the AGC control layer and considers the situation where the power plants in the power system are controlled only locally via a communication network. This setup gives rise to a power system with local delay, which is a situation that has been considered in, e.g., [13, 148]. Unfortunately, the aforementioned references make use of centralized stability analysis techniques and as such are not suitable for the AGC control layer. On the other hand, the small-gain theorems that were derived in this chapter provide a decentralized stability analysis method that can take delays into account. Moreover, the stability of the power system can even be guaranteed in case of a tie line or power plant failure. As such, the method is very well suited for the stability analysis of large-scale power systems with delay. To illustrate the application of the results, a benchmark power systems example is considered, i.e., the CIGRÉ 7-machine power system [56, 111].

7.5.1 Modeling power systems with delay

The CIGRÉ 7-machine power system [56, 111] is a power system that consists of an interconnection of 7 generator buses and 10 load buses. The schematic layout of the CIGRÉ 7-machine power system is depicted in Figure 7.2. For simplicity, *the load disturbances are not considered* in this case study. The analysis of the influence of disturbances has been considered in the conference paper related to this case study, see Section 1.5. In what follows we first introduce a model for the power plants in the power system and then we include the effects of controlling the power plants locally over a communication network.

Typically, the upper control layer determines slowly varying generation profiles for each power plant while for the lower control layer, relatively small changes with respect to these profiles are considered. In this setting, an accurate model for the i-th generator dynamics,
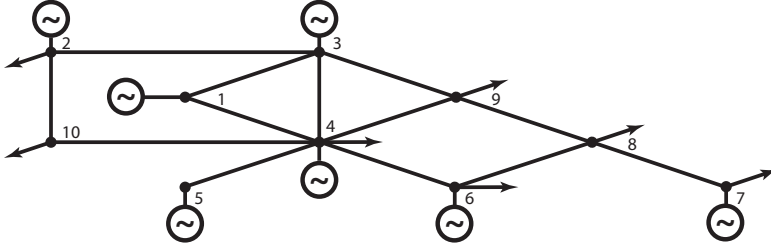
Figure 7.2: The CIGRÉ 7-machine power system.

$i \in \mathbb{Z}_{[1,7]}$ is given by [56, 81] the linearized equations

$$
\begin{aligned}
\dot{\delta}_i(t) &= \omega_i(t) \\
\dot{\omega}_i(t) &= \frac{1}{H_i}\left(P_{\text{T}_i}(t) - D_i\omega_i(t) - \sum_{j=1}^{7}[\Upsilon]_{i,j}\delta_j(t)\right) \\
\dot{P}_{\text{T}_i}(t) &= \frac{1}{\tau_{\text{T}_i}}(P_{\text{G}_i}(t) - P_{\text{T}_i}(t)) \\
\dot{P}_{\text{G}_i}(t) &= \frac{1}{\tau_{\text{G}_i}}(P_{\text{ref}_i}(t) - P_{\text{G}_i}(t) - \frac{1}{r_i}\omega_i(t))
\end{aligned}
\quad , \quad t \in \mathbb{R}_+,
\tag{7.8}
$$

where δ_i [rad] is the rotor phase angle, ω_i [rad/s] the rotor frequency, P_{T_i} [MW] the turbine state and P_{G_i} [MW] the governor state of the i-th generator, respectively. Furthermore, P_{ref_i} [MW] denotes the relative control input of generator i. The parameters corresponding to each of the 7 power plants are provided in Table 7.1. The interconnections in the power

Table 7.1: System parameters for the CIGRÉ 7-machine power system

		$i = 1$	2	3	4	5	6	7
inertia	H_i	100	30.3	35.8	28.6	26	34.8	26.4
damping coefficient	D_i	0.8	0.85	0.8	0.8	0.9	0.7	0.8
governor time constant	τ_{G_i}	0.2	0.15	0.2	0.2	0.25	0.2	0.2
turbine time constant	τ_{T_i}	0.5	0.4	0.5	0.5	0.4	0.5	0.5
regulation constant	r_i	$\frac{1}{20}$	$\frac{1}{23}$	$\frac{1}{19}$	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{18}$	$\frac{1}{20}$

system are described by a weighted adjacency matrix $\Upsilon \in \mathbb{R}^{7\times7}$, where the elements of the matrix Υ have the unit $[\Omega^{-1}]$ and define the virtual inductive reactance between bus i and bus j. The virtual inductive reactance is obtained via an elimination of the buses that do not contain generators but only external loads (nodes 8, 9 and 10), i.e., such that $\Upsilon := (B_{11} - B_{12}B_{22}^{-1}B_{21}) \in \mathbb{R}^{7\times7}$ where $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} := A - \text{diag}(A1_7)$. The matrix $A \in \mathbb{R}^{10\times10}$ contains the real inductive reactances of the CIGRÉ 7-machine power system and its values are provided in Table 7.2.

In this case study the power plants are controlled by local control centers that communicate with the power plants over communication networks. As it is also the case in Example 2.2, the communication network introduces uncertain time-varying input and output

Table 7.2: Inductive reactance matrix for the CIGRÉ 7-machine power system

$[A]_{1,3} = 1/24.5$	$[A]_{1,4} = 1/24.5$	$[A]_{2,3} = 1/62.6$	$[A]_{2,10} = 1/32.3,$
$[A]_{3,4} = 1/40$	$[A]_{3,9} = 1/28$	$[A]_{4,5} = 1/10$	$[A]_{4,6} = 1/10,$
$[A]_{4,10} = 1/33$	$[A]_{6,8} = 1/31.8$	$[A]_{7,8} = 1/39.5$	$[A]_{8,9} = 1/97,$
$[A]_{4,9} = 1/97$	$[A]_{j,i} = [A]_{i,j}$	$\forall\,(i,j) \in \mathbb{Z}_{[1,7]}^2$ such that $j > i$	

delays, such that the control signal P_{ref_i} satisfies

$$P_{\mathrm{ref}_i}(t) = u_{k,i}, \quad \forall t \in \mathbb{R}_{[t_k+\tau_{k,i},\,t_{k+1}+\tau_{k+1,i})},$$

where $t_k = kT_s$, $k \in \mathbb{Z}_+$ is the sampling instant, $T_s = 1$ [s] denotes the sampling period and $u_{k,i} \in \mathbb{R}$ is the control action for the i-th power plant generated at time $t = t_k$. Moreover, $\tau_{k,i} \in \mathbb{R}_{[0,\bar{\tau}]}$ denotes the sum of the input and output delay[1] at time $k \in \mathbb{Z}_+$ and $\bar{\tau} = 0.3$ [s] is the maximal delay induced by the communication networks. A discrete-time model of the power plants and the communication networks is given by

$$x_{i,k+1} = A_i x_{i,k} + (B_i - \Delta_i(\tau_{i,k}))\, u_{i,k} + \Delta_i(\tau_{i,k}) u_{i,k-1} + \sum_{j=1}^{N} A_{i,j} x_{j,k}, \qquad (7.9)$$

with $k \in \mathbb{Z}_+$ and where $\tau_{i,k} \in \mathbb{R}_{[0,\bar{\tau}]}$ and $x_{i,k} := \mathrm{col}(\delta_{i,k}, \omega_{i,k}, P_{\mathrm{T}_i,k}, P_{\mathrm{G}_i,k})$. The matrices corresponding to the discrete-time approximation of the CIGRÉ 7-machine power system (7.9) are given by the expressions $A_i := e^{\bar{A}_i T_s}$, $B_i := \int_0^{T_s} e^{\bar{A}_i(T_s-\theta)} \bar{B}_i d\theta$, $A_{i,j} := \int_0^{T_s} e^{\bar{A}_i(T_s-\theta)} \bar{C}_{i,j} d\theta$ and $\Delta_i(\tau_{i,k}) := \int_0^{\tau_{i,k}} e^{\bar{A}_i(T_s-\theta)} \bar{B}_i d\theta$. In these expressions the matrices $\bar{A}_i$, $\bar{B}_i$ and $\bar{C}_{i,j}$ are obtained directly from the continuous-time model (7.8) using $C_{i,i} := 0_{n_i \times n_i}$.

7.5.2 Stability analysis

In what follows we will construct a set of controllers that stabilize the CIGRÉ 7-machine power system over the communication network that was described above. To this end we will use the model of the power plants (7.9) in combination with Corollary 7.1.

Therefore, we first use Proposition 2.6 to construct a suitable LRF V_i and corresponding control law $u_{i,k} = K_i x_{i,k}$ for each power plant ignoring the interconnections. Note that the model (7.9) gives rise to an interconnected system with local delays of the form (7.2), i.e., the inclusion is obtained because $\tau_{i,k}$ can take any value in the interval $\mathbb{R}_{[0,\bar{\tau}]}$. However, as the set $\{\Delta_i(\tau_i) : \tau_i \in \mathbb{R}_{[0,\bar{\tau}]}\}$ is not polytopic, the conditions derived in Proposition 2.6 do not lead to an LMI of finite dimensions. Therefore, a polytopic over-approximation of the uncertain time-varying matrix $\Delta(\tau_i)$ is computed using the Cayley-Hamilton technique presented in [46], see Example 2.2, Part II for further details. Thus, using $\rho = 0.7$ for all

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[1]For time-invariant controllers, both delays on the measurement and the actuation link can be lumped [150] into a single delay on the latter link and hence output delays are implicitly taken into account.

power plants we obtain the quadratic LRF matrices and corresponding controller matrices

$$P_1 = \begin{bmatrix} 132.52 & 384.63 & 1.75 & 0.67 \\ 384.63 & 1314.67 & 6.20 & 2.42 \\ 1.75 & 6.20 & 0.03 & 0.01 \\ 0.67 & 2.42 & 0.01 & 0.01 \end{bmatrix}, \quad K_1 = \begin{bmatrix} -22.30 & -85.47 & -0.41 & -0.16 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 150.61 & 190.68 & 2.51 & 0.94 \\ 190.68 & 246.31 & 3.24 & 1.21 \\ 2.51 & 3.24 & 0.04 & 0.02 \\ 0.94 & 1.21 & 0.02 & 0.01 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -17.99 & -22.87 & -0.30 & -0.11 \end{bmatrix},$$

$$\vdots$$

$$P_7 = \begin{bmatrix} 40.65 & 51.09 & 0.96 & 0.39 \\ 51.09 & 64.92 & 1.22 & 0.49 \\ 0.96 & 1.22 & 0.02 & 0.01 \\ 0.39 & 0.49 & 0.01 & 0.01 \end{bmatrix}, \quad K_7 = \begin{bmatrix} -15.81 & -20.09 & -0.38 & -0.15 \end{bmatrix},$$

Next we construct the functions $\gamma_{i,j} \in \mathcal{K}_\infty \cup \{0\}$. To this end note that the functions $V_i(x_{i,0}) := (x_{i,0}^\top P_i x_{i,0})^{\frac{1}{2}}$ satisfy the triangle inequality and, hence, are such that

$$V_i(x_{i,k+1}) = V_i((A_i + (B_i - \Delta_i(\tau_{i,k}))K_i)x_{i,k} + \Delta_i(\tau_{i,k})K_i x_{i,k-1} + \sum_{j=1}^N A_{i,j}x_{j,k})$$
$$\leq V_i((A_i + (B_i - \Delta_i(\tau_{i,k}))K_i)x_{i,k} + \Delta_i(\tau_{i,k})K_i x_{i,k-1}) + \sum_{j=1}^N V_i(A_{i,j}x_{j,k})$$
$$\leq \max_{j' \in \mathbb{Z}_{[-1,0]}} \rho^{\frac{1}{2}} V_i(x_{i,k+j'}) + \sum_{j=1,j\neq i}^N V_i(A_{i,j}x_{j,k}),$$

for all $\tau_{i,k} \in \mathbb{R}_{[0,\bar{\tau}]}$. Hence, it can be concluded that $\gamma_{i,i}(r) := \rho^{\frac{1}{2}}r$ for all $i \in \mathbb{Z}_{[1,N]}$. Furthermore, solving the the LMI

$$A_{i,j}^\top P_i A_{i,j} \preceq \rho_{i,j} P_j, \tag{7.10}$$

for all $(i,j) \in \mathbb{Z}_{[1,N]}^2$ such that $i \neq j$, yields $\gamma_{i,j}(r) = \rho_{i,j}^{\frac{1}{2}}r$. The so-obtained functions $\{V_i, \gamma_{i,j}\}_{(i,j)\in\mathbb{Z}_{[1,7]}^2}$ satisfy, by construction, (7.5a) and (7.5b) for the power system (7.9) in closed loop with the control laws $u_{i,k} = K_i x_{i,k}$ for all $\tau_{i,k} \in \mathbb{R}_{[0,\bar{\tau}]}$, $i \in \mathbb{Z}_{[1,7]}$.

Let $\Gamma \in \mathbb{R}^{N \times N}$ denote a matrix with $[\Gamma]_{i,i} := \rho^{\frac{1}{2}}$ for all $i \in \mathbb{Z}_{[1,7]}$ and $[\Gamma]_{i,j} := \rho_{i,j}^{\frac{1}{2}}$ otherwise. Thus, we obtain

$$\Gamma = 10^{-1} \begin{bmatrix} 8.37 & 0 & 0.29 & 0.22 & 0 & 0 & 0 \\ 0 & 8.37 & 0.15 & 0.11 & 0 & 0 & 0 \\ 0.03 & 0.03 & 8.37 & 0.11 & 0 & 0.02 & 0.01 \\ 0 & 0.02 & 0.15 & 8.37 & 0.25 & 0.50 & 0 \\ 0 & 0 & 0 & 0.55 & 8.37 & 0 & 0 \\ 0 & 0 & 0.02 & 0.37 & 0 & 8.37 & 0.05 \\ 0 & 0 & 0.01 & 0 & 0 & 0.07 & 8.37 \end{bmatrix}.$$

Now observe that Theorem 2.1.11 in [11] yields that the second item of the hypothesis of Corollary 7.1 is satisfied if $\mathrm{sr}(\Gamma) < 1$. Therefore, as $\mathrm{sr}(\Gamma) = 0.8963$, it follows from Corollary 7.1 that the CIGRÉ 7-machine power system controlled over local communication networks via the static state-feedback laws $u_{i,k} = K_i x_{i,k}$, is $\mathcal{KL}$-stable. A simulation of the generator dynamics from a set of random initial conditions is depicted in Figure 7.3. It should be emphasized that the control laws $u_{i,k} = K_i x_{i,k}$ do not contain an integral
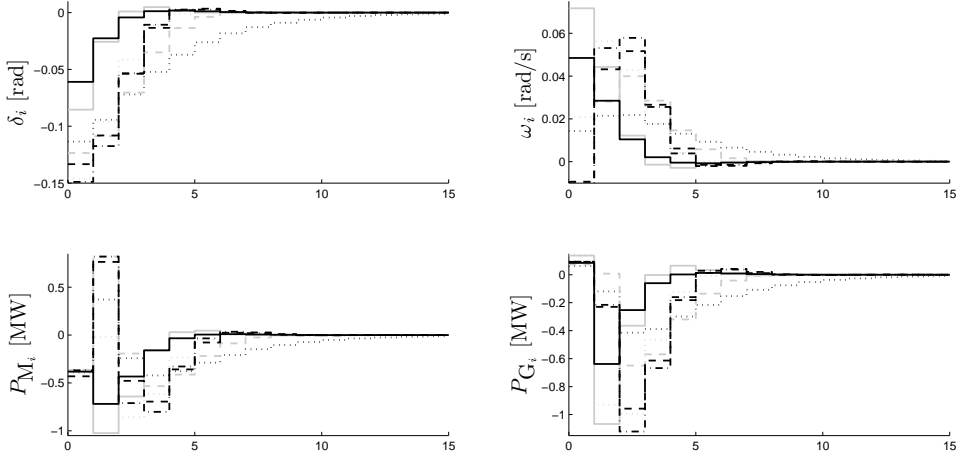

Figure 7.3: A simulation of the closed-loop CIGRÉ 7-machine power system.

action and hence, in the presence of persistent disturbances, there will be a steady-state error. However a controller with integral action can be obtained by designing a controller for a new model that consists of an integrator in series with the model (7.9).

Remark 7.5 Above, the stability of a power system with local delay has been established by solving a decentralized LRF synthesis problem, solving a set of decoupled LMIs for neighboring systems and computing the eigenvalues of a matrix in $\mathbb{R}^{N \times N}$, where N denotes the number of systems in the interconnected system. As efficient numerical algorithms exist for the computation of the spectral radius of a large matrix, it follows that Corollary 7.1 provides *a tractable stability analysis method* for power systems with delay. □

Remark 7.6 A tie line failure between node i and j corresponds to setting $[A]_{i,j} = 0$. The effect of this failure on the gains $\gamma_{i,j}$ can be computed by repeating the procedure outlined above. More interestingly, a tie line failure between node i and j corresponds to reducing $\gamma_{i,j}$ (potentially to zero). In this case the hypothesis of Corollary 7.1 still holds and hence the power system remains stable. Furthermore, when a power plant is decoupled from the power system due to a failure, a similar reasoning applies. □

7.6 Conclusions

In this chapter the stability analysis results of Chapter 2 were extended to large-scale inter-connected systems with both interconnection and local delays. To obtain a tractable stability analysis method for such systems the Razumikhin as well as the Krasovskii approach were combined with small-gain arguments. Thus, it was demonstrated that interconnection delays do not affect the stability of an interconnected system if a delay-independent small-gain condition holds. Moreover, scalable conditions for the stability analysis of interconnected systems with delay were also obtained. The applicability of these results was illustrated via a classical power systems example wherein the power plants are controlled only locally via a communication network, which gives rise to local delays.

Essentially, the present chapter made use of the relation of DDIs to interconnected systems, to extend the results in Chapter 2 to interconnected systems with delay. Clearly, the same reasoning can be applied to the results in Chapters 3 - 5. For brevity these results are not considered in this thesis.

Chapter 8

Conclusions and recommendations

A summary of the main contributions and a collection of possible, relevant directions for extensions and future research conclude this thesis.

8.1 Discussion of the results in this thesis

In this thesis, we introduced delay difference inclusions (DDIs) as a modeling class for discrete-time systems with delay. The generality and capabilities of this modeling class were highlighted. In particular, it was shown that DDIs can provide models to analyse the properties of most types of sampled-data systems and networked control systems, see Section 2.2. Moreover, in many cases the results that were derived for DDIs were shown to recover the corresponding results for delay difference equations as a particular case.

For this general class of systems we have supplied stability analysis and controller synthesis tools that can take constraints into account and provide a suitable trade-off between computational tractability and conceptual generality. More specifically, we have discussed the following subjects.

8.1.1 Stability analysis of delay difference inclusions

For continuous-time systems with delay two extensions of Lyapunov theory exist, i.e., the Krasovskii and Razumikhin approaches. In Chapter 2 the corresponding counterparts for these two approaches were derived for discrete-time systems. Similarly as for the stability analysis of continuous-time systems, the Krasovskii approach relies on a function that maps trajectory segments of a length determined by the size of the delay to the positive numbers and is strictly decreasing along all trajectories of the system. The Razumikhin approach on the other hand relies on a function that maps the state of the system at a single time instant to the positive numbers and is only decreasing if the trajectory of the DDI satisfies a specific condition. As a consequence, the Krasovskii approach provides necessary and sufficient conditions for stability analysis of DDIs while the Razumikhin approach is relatively simple to apply but conservative. Furthermore, it is only via the Razumikhin approach that one obtains information about system trajectories directly, as opposed to information about trajectory segments, such that the corresponding computations can be executed in the underlying low-dimensional state space of the DDI dynamics. Interestingly, the Razumikhin approach was proven to be an application of the small-gain theorem in Section 3.3, which

explains the conservatism that is associated with this approach to some extent. Then, also in Chapter 3, we derived, inspired by results for interconnected systems with dissipative subsystems, an alternative set of Razumikhin-type conditions for stability analysis of DDIs. These conditions provide a trade-off between the conceptual generality associated with the Krasovskii approach and the computational tractability of the Razumikhin approach. Moreover, as these conditions are of Razumikhin-type, they provide information about system trajectories directly. Unfortunately, even these novel Razumikhin-type conditions remain conservative. Therefore, we proposed a relaxation of the Razumikhin approach in Chapter 4, which leads to necessary and sufficient conditions for stability of DDIs. Thus, necessary and sufficient conditions for stability were obtained that provide information about system trajectories directly. Interestingly, it was shown that for positive linear delay difference *equations* the standard Razumikhin approach is non-conservative and hence dominant over the Krasovskii approach, in the sense that both approaches are non-conservative but only the Razumikhin approach provides relatively simple conditions for stability.

In summary, it was shown that the Razumikhin approach provides a simple but conservative test for stability. At the cost of a slight increase in complexity this conservatism can be reduced via an alternative set of Razumikhin-type conditions that was developed in Chapter 3. Furthermore, the most complex but necessary and sufficient conditions for stability of DDIs are provided by the Krasovskii approach. Perhaps the most attractive stability analysis method is provided by the Razumikhin-type conditions that were developed in Chapter 4. Indeed, they form a set of necessary and sufficient conditions for stability that provide information about system trajectories directly, such that the corresponding computations can be executed in the underlying low-dimensional state space of the DDI dynamics. It should be noted that for positive linear delay difference *equations* it suffices to consider the standard Razumikhin approach only, as the corresponding conditions are necessary for this class of systems. A graphical summary of the relations of the Krasovskii and Razumikhin approaches to the set of all DDIs that are stable is shown in Figure 8.1.
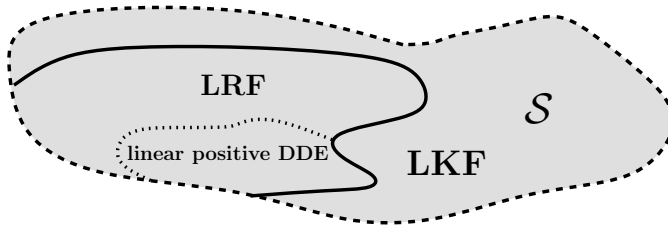

Figure 8.1: The set $\mathcal{S}$ (grey) consists of all DDIs that are $\mathcal{KL}$-stable and is equivalent to the set of all DDIs which admit an LKF (- - -). Furthermore, the set of all DDIs which admit an LRF (——) forms a strict subset of $\mathcal{S}$ while the set of all $\mathcal{KL}$-stable linear positive delay difference equations ($\cdots$) is a strict subset of the set of all DDIs which admit an LRF.

8.1.2 *Stability analysis of constrained delay difference inclusions*

When the DDI under study is subject to constraints, its stability has to be analyzed via one of the methods that were discussed in Section 8.1.1 in combination with an invariant set,

preferably the maximal invariant set. In this context, it was shown in Section 2.5 that the Krasovskii approach corresponds to a standard invariant set for the DDI while the Razumikhin approach gives rise to a set with particular invariance properties, called $\mathcal{D}$-invariance. However, it was also observed that both techniques are not perfectly suited for the construction of invariant sets for DDIs. The Krasovskii approach leads to algorithms that are not computationally tractable, in particular when finding the maximal invariant set is desirable, while the Razumikhin approach is, due to its conservatism, not always able to provide a suitable invariant set. Therefore, we introduced the concept of invariant families of sets in Chapter 5. It was shown that this concept is able to characterize the maximal invariant set and has the potential to lead to simple algorithms for its construction. Based on a parametrization of the space of admissible families of sets, a wide variety of algorithms for the construction of invariant families of sets was proposed. Even though the parametrization of the space of admissible families of sets introduces some conservatism, the resulting synthesis algorithms were proven to be more general than those corresponding to the Razumikhin approach. Moreover, the computational complexity of the corresponding synthesis algorithms remains tractable even for relatively large systems with large delays, for which the Krasovskii approach fails to provide tractable algorithms.

8.1.3 Stability analysis of interconnected delay difference inclusions

In Chapter 7 we considered the stability analysis of interconnected systems with delay. Both interconnection delays, which arise in the paths connecting the subsystems, and local delays, which arise in the dynamics of the subsystems, were considered. Using small-gain arguments it was shown that interconnection delays do not affect the stability of an interconnected system if a delay-independent small-gain condition holds. This result stands in sharp contrast with the typical destabilizing effect that time delays have. Furthermore, also using small-gain arguments, it what shown that stability for interconnected systems with local delay can be established via both the Razumikhin and Krasovskii approaches. A combination of the above results led to a scalable stability analysis framework for general interconnected systems with delay. The advantages of the Razumikhin and Krasovskii approaches for DDIs when compared to each other carry over to the corresponding results for interconnected systems. We illustrated the applicability of the derived stability analysis framework for interconnected systems with delay via a classical power systems example wherein the power plants are controlled only locally via a communication network, which gives rise to local delays.

8.1.4 Stabilization of delay difference inclusions

In the absence of constraints, linear DDIs can be stabilized via the techniques that were developed in Chapters 2 and 3. More specifically, in Sections 2.6 and 3.5 it was shown how to design a feedback control law via the Krasovskii approach, the Razumikhin approach and the Razumikhin-type conditions that were developed in Chapter 3. Unfortunately, it currently remains unclear how to formulate an algorithm for stabilizing controller synthesis that makes use of the non-conservative Razumikhin-type conditions in Chapter 4. In the presence of constraints, the aforementioned controllers for the unconstrained system can be used in combination with an invariant set, e.g., an approximation of the maximal invariant set for the resulting closed-loop system, which can be computed via the techniques that

were discussed in Chapter 5. However, this approach does not necessarily lead to the largest possible region for which constraint satisfaction can be guaranteed because the control law and invariant set are designed separately rather than at the same time.

Motivated by this fact, two advanced control schemes for constrained DDIs were presented in Chapter 6. Both control schemes make use of the concept of model predictive control and, hence, rely on online optimization. The first control scheme uses a Lyapunov-Razumikhin function for the unconstrained system. Then, a relaxation of the standard conditions related to the Razumikhin approach, allows to handle constraints in an efficient manner. The second control scheme makes use of a variation of the Razumikhin approach that removes the conservatism that is usually associated with this approach. Moreover, an efficient implementation for DDIs with large delays was obtained via Minkowskii addition properties. As the second control scheme does not, as opposed to the first control scheme, require the construction of a local controller via the Razumikhin approach, it is non-conservative and remains tractable even for DDIs with very large delays.

8.2 Extensions of the results in this thesis

Some of the results in this thesis can be extended to a more general class of systems or modified such that they are less conservative. Such, relatively straightforward, extensions have not been included in this thesis for brevity only and are discussed in what follows.

8.2.1 Delay difference inclusions with external disturbances

In this thesis, to simplify the presentation and to ensure a uniform set of results, we considered DDIs without external disturbances only. However, the vast majority of the results in this thesis can be extended to DDIs with external disturbances with only minor modifications. In this case, conditions that guarantee (integral) input-to-state stability or a corresponding notion for linear DDIs, i.e., ℓ_2-disturbance rejection, can be obtained. In fact, many of the articles underlying this thesis deal with DDIs with external disturbances, see Section 1.5 for the appropriate references. The extension of the results on invariant families of sets to DDIs with external disturbances is particularly interesting. For example, in this case the concept of the minimal robust invariant set and the minimal robust invariant family of sets gives rise to many open questions. Note that such an extension was already considered for the related concept of practical invariance for interconnected systems in [117] which, therefore, provides the necessary tools for this extension.

8.2.2 Global asymptotic stability of a set

The theorems in this thesis that contain (necessary and) sufficient conditions for stability, e.g., such as the ones presented in Section 2.3, can be extended to global asymptotic stability of a set as opposed to a single equilibrium point, i.e., in our case the origin. For difference inclusions without delay this extension was already considered in [73], which therefore provides all the tools that are necessary for the extension of the results in this thesis to global asymptotic stability of a set. For example, the extension of Theorem 2.1 requires to replace the norms in (2.3a) with the distance to the set of interest while the inequality (2.3b) should be required to hold outside the set only. The extension of other results in this thesis is more involved but also possible with only minor modifications.

Global asymptotic stability of a set is useful for DDIs with external disturbances that

take values in a compact set as it allows to study stability without explicitly considering the disturbance, i.e., using $\tilde{F}(\mathbf{x}_{[-h,0]}) := \{F(\mathbf{x}_{[-h,0]}, w) : w \in \mathbb{W}\}$ where $w \in \mathbb{W}$ is an external disturbance and $\mathbb{W} \subset \mathbb{R}^m$ is some compact set. More importantly, this extension allows to study the stability of DDIs with time-varying delays with a bounded rate of variation by including the value of the delay at the previous time instant in the state vector and thus obtaining a standard DDI, i.e., as indicated in Remark 2.1 and considered in, e.g., [153]. Interestingly, the set that is globally asymptotically stable does not need to be bounded. Therefore, the notion of global asymptotic stability of a set also allows to study time-varying DDIs by modeling them as a time-invariant DDI. Indeed, consider the time-varying DDI $x_{k+1} \in F(k, \mathbf{x}_{[k-h,k]})$. The stability of this DDI can be established by considering stability of the time-invariant DDI $\tilde{x}_{k+1} \in \tilde{F}(\tilde{\mathbf{x}}_{[k-h,k]})$ with respect to the unbounded set $\{0\}^{h+1} \times \mathbb{Z}_+$, where $\tilde{x}_k := \mathrm{col}(x_k, k)$ and $\tilde{F}(\tilde{\mathbf{x}}_{[-h,0]}) := \mathrm{col}(F(\mathbf{x}_{[-h,0]}), k+1)$.

8.2.3 Interconnected delay difference inclusions

In Chapter 3 it was shown that by considering each delayed state as a subsystem, the behavior of a DDI can be described by an interconnected system with a particular structure. We have used this relation in Chapter 7 to obtain a stability analysis framework for interconnected systems with delay via the small-gain theorem. Similarly, it is possible to extend the Razumikhin-type conditions in Chapter 3 to interconnected systems with delay using results for interconnected systems with dissipative subsystems. Thus, an alternative Razumikhin-type stability analysis method for interconnected systems with delay can be obtained that parallels the results in Chapter 7. Furthermore, as the concepts of practical invariance for interconnected systems [116] and invariant families of sets are closely related, it is possible to extend the results in Chapter 5 to interconnected systems with delay as well. Thus, a tractable tool for the construction of invariant sets for interconnected systems with delay can be obtained. Again, this combination of methods essentially relies on the fact that DDIs can be transformed into an interconnected system. Proceeding along this line of reasoning, results for interconnected systems, such as distributed control schemes [139], may be translated to DDIs or even interconnected DDIs to obtain new theoretical tools. Conversely, insights for DDIs and other classes of discrete-time systems with delay may prove to be insightful when interpreted in the context of interconnected systems.

8.2.4 Parameter dependent Lyapunov functions

All controller synthesis results in this thesis, e.g., such as Proposition 2.5, are based on quadratic functions. To reduce the conservatism of the corresponding stability analysis conditions it is possible to replace this function with a parameter dependent function, which is a technique that has been applied in, e.g., [26, 59]. This modification requires almost no changes to the proofs of the results and is therefore immediate.

8.2.5 Parameterized families of sets

While the concept of an invariant family of sets is able to characterize the maximal invariant set, the parametrization that is proposed in Section 5.4 does not have this ability, e.g., as it is illustrated in Example 5.2. Therefore, a different parametrization of families of sets is needed that is able to characterize the maximal invariant set and leads to tractable algorithms for the computation of an invariant family of sets. In this context, an interesting suggestion

is presented in Section 4.3. The results therein suggest that an invariant family of sets can be obtained from the union of a number of system trajectories of finite length.

8.3 Future work related to the results in this thesis

Even though an attempt has been made to present a comprehensive set of stability analysis and controller synthesis results for DDIs, some questions remain unanswered. Moreover, some important issues have not been treated in this thesis at all. In what follows, we discuss the most important topics for future research related to DDIs.

8.3.1 Stabilization of constrained delay difference inclusions

While the stabilization of constrained delay difference inclusions is discussed in this thesis, quite some open issues remain. For example, it is unclear how to verify the guarantee of recursive feasibility that was presented in Theorem 6.2. Furthermore, it is unclear under which conditions Algorithm 6.2 is recursively feasible. It currently even remains an open question if, given a set of initial conditions for which recursive feasibility is guaranteed, the same holds for initial conditions that are a convex combination of those with a guarantee of recursive feasibility. While it is reasonable to expect that this is true, it is not straightforward to prove this property. Moreover, we do not claim that the algorithms that were presented in Chapter 6 are optimal in any sense. They merely try to address the issue of constraints handling while guaranteeing a limited computational complexity.

8.3.2 Sampled-data systems with delay

Currently, of all the results that are presented in this thesis, only the first control scheme proposed in Chapter 6 has been used in an application, i.e., the automotive setup discussed in [21]. In order to make all results in this thesis readily applicable to practical engineering problems, a framework is needed that rigorously extends the results for sampled-data systems that were derived in [107, 108] to sampled-data systems with delay. Indeed, these results are needed to guarantee that the properties of a discrete-time model with delay can be carried over to the original process. As such a framework requires an accurate discretization of the original continuous-time model with delay for various values of the sampling time, a study on the discretization of continuous-time systems with delay should accompany the aforementioned results for sampled-data systems with delay. Note that some preliminary results in this direction can be found in [113, 138].

8.3.3 Stochastic delay difference inclusions

For communication networks it is reasonable to assume that some stochastic information about the delay or the variation of the delay is known. If this information is ignored, the resulting stability analysis results will be conservative. As such to obtain more accurate stability analysis and controller synthesis results for networked control systems, stochastic DDIs should be considered. It would be interesting to see which results in this thesis can be extended to stochastic DDIs and which results require certain modifications.

8.3.4 Further suggestions

This thesis essentially provides an exploratory study of the stability analysis and control of discrete-time systems with delay, which is a topic that has been scarcely studied so far. In contrast, the stability analysis and control of continuous-time systems with delay has been

an active topic of research for many years. Therefore, insights that are quite common for continuous-time systems with delay are often not that well understood in the discrete-time case. For example, the interpretation of Lyapunov's second method for time-delay systems due to Barnea, see, e.g., [52], has thus far not been interpreted for discrete-time systems with delay. Also, the relation of the Razumikhin approach to the S-procedure, which is well understood for FDEs [109], remains unclear. Many more of these interesting possibilities can be found in the excellent monographs [48, 51, 52, 77, 79, 104, 109].

8.4   Final thoughts

The results in this thesis should be considered in the context of sampled-data and networked control systems. In particular, the considerations that are presented in this thesis have the potential to provide researchers in the aforementioned fields with sufficient information to make well-funded design choices. For example, while the controller structure that is used throughout this thesis might not be the most convenient in a specific networked control systems setting, the relations of the different approaches when compared to each other remain valid for different controller structures as well. As such, when computational complexity is the most important issue it is advisable to use the Razumikhin approach. On the other hand, if one is more interested in obtaining optimal control performance it would be better to use the Krasovskii approach. Therefore, while the results in this thesis will probably not answer the specific question you may have, they should provide the necessary information to choose the right solution concept for your problem.

Appendix A

Numerical values for Example 2.2

The matrices A_d and B_d can be computed via Matlab, which yields

$$A_d = e^{A_c T_s} = \begin{bmatrix} 0.7586 & -0.0008 \\ 2.9984 & 0.9876 \end{bmatrix}, \quad B_d = \int_0^{T_s} e^{A_c(T_s - \theta)} d\theta B_c = \begin{bmatrix} 0.0514 \\ 0.0924 \end{bmatrix}.$$

The Cayley-Hamilton method presented in [46] yields a set of matrices $\{\hat{\Delta}_i\}_{i \in \mathbb{Z}_{[1,4]}}$, with $\hat{\Delta}_i \in \mathbb{R}^{n \times m}$ for all $i \in \mathbb{Z}_{[1,4]}$, which can be used to form a polytopic over-approximation of $\Delta(\tau)$ for all $\tau \in \mathbb{R}_{[0,\bar{\tau}]}$, i.e., such that $\{\Delta(\tau) : \tau \in \mathbb{R}_{[0,\bar{\tau}]}\} \subseteq \text{co}(\{\hat{\Delta}_i\}_{i \in \mathbb{Z}_{[1,4]}})$. In what follows, the matrices $\hat{\Delta}_i$ that span the aforementioned polytopic over-approximation are computed for several values of $\bar{\tau}$. These values for $\bar{\tau}$ correspond to the MAD for which the stability of the networked control system described in Example 2.2 can be established via Propositions 2.5, 2.6 and 3.6. Furthermore, the matrices that were used to compute the invariant set in Example 5.6 are also presented. For $\bar{\tau} := 0.348 T_s$ the method yields

$$\hat{\Delta}_1 = \begin{bmatrix} 0.0163 \\ 0.0519 \end{bmatrix}, \quad \hat{\Delta}_2 = \begin{bmatrix} -0.0041 \\ 0.0519 \end{bmatrix}, \quad \hat{\Delta}_3 = \begin{bmatrix} 0.0204 \\ 0 \end{bmatrix}, \quad \hat{\Delta}_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Note that these are the generators of the polytope denoted by equation (9) in [46]. For $\bar{\tau} := 0.424 T_s$ the method yields

$$\hat{\Delta}_1 = \begin{bmatrix} 0.0201 \\ 0.0605 \end{bmatrix}, \quad \hat{\Delta}_2 = \begin{bmatrix} -0.0048 \\ 0.0605 \end{bmatrix}, \quad \hat{\Delta}_3 = \begin{bmatrix} 0.0249 \\ 0 \end{bmatrix}, \quad \hat{\Delta}_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Letting $\bar{\tau} := 0.44 T_s$ yields

$$\hat{\Delta}_1 = \begin{bmatrix} 0.0209 \\ 0.0622 \end{bmatrix}, \quad \hat{\Delta}_2 = \begin{bmatrix} -0.0050 \\ 0.0622 \end{bmatrix}, \quad \hat{\Delta}_3 = \begin{bmatrix} 0.0258 \\ 0 \end{bmatrix}, \quad \hat{\Delta}_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Furthermore, letting $\bar{\tau} := 0.48 T_s$ yields

$$\hat{\Delta}_1 = \begin{bmatrix} 0.0229 \\ 0.0663 \end{bmatrix}, \quad \hat{\Delta}_2 = \begin{bmatrix} -0.0053 \\ 0.0663 \end{bmatrix}, \quad \hat{\Delta}_3 = \begin{bmatrix} 0.0282 \\ 0 \end{bmatrix}, \quad \hat{\Delta}_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Appendix B

Proofs of nontrivial results

B.1 Proofs of Chapter 2

Proof of Lemma 2.2

Before proceeding with the proof of the first claim of Lemma 2.2, some preliminary results are derived. On a finite dimensional vector space $\mathbb{R}^n$ all norms are equivalent [80], i.e., for any two norms $\|\cdot\|_{p_1}$ and $\|\cdot\|_{p_2}$, there exist constants $(\underline{c}, \bar{c}) \in \mathbb{R}_{>0} \times \mathbb{R}_{\geq \underline{c}}$ such that $\underline{c}\|x\|_{p_1} \leq \|x\|_{p_2} \leq \bar{c}\|x\|_{p_1}$ for all $x \in \mathbb{R}^n$. Consider any $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and let $\xi_0 := \mathrm{col}(\{x_l\}_{l \in \mathbb{Z}_{[-h,0]}})$. Then, for any norm $\|\cdot\|_p$, $p \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$ there exist constants $(c_1, c_2) \in \mathbb{R}^2_{>0}$ such that

$$\|\mathbf{x}_{[-h,0]}\|_p = \left\| \begin{bmatrix} \|x_{-h}\|_p \\ \vdots \\ \|x_0\|_p \end{bmatrix} \right\|_\infty \geq \left\| c_1 \begin{bmatrix} \|x_{-h}\|_\infty \\ \vdots \\ \|x_0\|_\infty \end{bmatrix} \right\|_\infty = c_1\|\xi_0\|_\infty \geq c_1 c_2 \|\xi_0\|_p. \quad \text{(B.1)}$$

Similarly, there exist constants $(c_3, c_4) \in \mathbb{R}^2_{>0}$ such that

$$\|\mathbf{x}_{[-h,0]}\|_p = \left\| \begin{bmatrix} \|x_{-h}\|_p \\ \vdots \\ \|x_0\|_p \end{bmatrix} \right\|_\infty \leq \left\| c_3 \begin{bmatrix} \|x_{-h}\|_\infty \\ \vdots \\ \|x_0\|_\infty \end{bmatrix} \right\|_\infty = c_3\|\xi_0\|_\infty \leq c_3 c_4 \|\xi_0\|_p. \quad \text{(B.2)}$$

Furthermore, the definition of the p-norm, $p \in \mathbb{Z}_{\geq 1}$ yields

$$\|\xi_0\|_p^p = \sum_{i=1}^{(h+1)n} |[\xi_0]_i|^p = \sum_{i=-h}^{0} \sum_{j=1}^{n} |[x_i]_j|^p \geq \sum_{j=1}^{n} |[x_0]_j|^p = \|x_0\|_p^p. \quad \text{(B.3)}$$

From (B.3) and the fact that $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $f(r) := r^{\frac{1}{p}}$ and $p \in \mathbb{Z}_{>0}$ is strictly increasing, it follows that $\|x_0\|_p \leq \|\xi_0\|_p$ for all $p \in \mathbb{Z}_{>0}$. It is straightforward to see from the definition of the infinity norm that $\|x_0\|_\infty \leq \|\xi_0\|_\infty$ holds as well. In what follows, let $\bar{\Phi} \in \bar{\mathcal{S}}(\xi_0)$ correspond to $\Phi \in \mathcal{S}(\mathbf{x}_{[-h,0]})$.

Proof of claim (i): Suppose that the DDI (2.1) is GAS. Using (B.1), (B.2) and the fact that the origin of the DDI (2.1) is globally uniformly attractive by assumption, we obtain

Appendix B. Proofs of nontrivial results

that for each $(\bar{r}, \bar{\varepsilon}) \in \mathbb{R}_{>0}^2$ there exists a $\bar{T}(\bar{r}, \bar{\varepsilon}) := T(c_5\bar{r}, \frac{1}{c_6}\bar{\varepsilon}) + h \in \mathbb{Z}_{\geq 1}$ such that if $\|\mathbf{x}_{[-h,0]}\| \leq c_5\|\xi_0\| \leq c_5\bar{r}$ then

$$\|\bar{\phi}_k\| \leq c_6\|\Phi_{[k-h,k]}\| \leq \bar{\varepsilon},$$

for all $(\bar{\Phi}, k) \in \bar{\mathcal{S}}(\xi_0) \times \mathbb{Z}_{\geq \bar{T}(\bar{r}, \bar{\varepsilon})}$ and some $(c_5, c_6) \in \mathbb{R}_{>0}^2$. Hence, the origin of (2.2) is globally uniformly attractive. Furthermore, as the DDI (2.1) is LS, it follows from (B.1) that for all $\varepsilon \in \mathbb{R}_{>0}$ there exist $(\delta, c_7) \in \mathbb{R}_{>0}^2$, with $\delta \leq \varepsilon$, such that if $\|\mathbf{x}_{[-h,0]}\| \leq \delta$ then

$$\|\bar{\phi}_k\| \leq c_7\|\Phi_{[k-h,k]}\| \leq c_7\varepsilon,$$

for all $(\Phi, k) \in \mathcal{S}(\mathbf{x}_{[-h,0]}) \times \mathbb{Z}_+$. Moreover, (B.2) implies that there exists a $c_8 \in \mathbb{R}_{>0}$ such that $\|\mathbf{x}_{[-h,0]}\| \leq c_8\|\xi_0\|$. Therefore, we conclude that for every $\bar{\varepsilon} := c_7\varepsilon \in \mathbb{R}_{>0}$ there exists a $\bar{\delta} := \frac{1}{c_8}\delta \in \mathbb{R}_{>0}$ such that if $\|\xi_0\| \leq \bar{\delta}$ and hence, $\|\mathbf{x}_{[-h,0]}\| \leq \delta$, then

$$\|\bar{\phi}_k\| \leq c_7\|\Phi_{[k-h,k]}\| \leq c_7\varepsilon = \bar{\varepsilon},$$

for all $(\bar{\Phi}, k) \in \bar{\mathcal{S}}(\xi_0) \times \mathbb{Z}_+$. Hence, (2.2) is LS and therefore, (2.2) is GAS.

Conversely, suppose that difference inclusion (2.2) is GAS. Using (B.1) and the fact that the origin of the difference inclusion (2.2) is globally uniformly attractive by assumption, we obtain that for each $(r, \varepsilon) \in \mathbb{R}_{>0}^2$ there exists a $T(r, \varepsilon) := \bar{T}(c_9r, \varepsilon) \in \mathbb{Z}_{\geq 1}$ such that if $\|\xi_0\| \leq c_9\|\mathbf{x}_{[-h,0]}\| \leq c_9r$ then

$$\|\phi_k\| \leq \|\bar{\phi}_k\| \leq \varepsilon,$$

for all $(\Phi, k) \in \mathcal{S}(\mathbf{x}_{[-h,0]}) \times \mathbb{Z}_{\geq T(r, \varepsilon)}$ and some $c_9 \in \mathbb{R}_{>0}$. Hence, the origin of (2.1) is globally uniformly attractive. Furthermore, using (B.3) and as (2.2) is LS it follows that for all $\bar{\varepsilon} \in \mathbb{R}_{>0}$ there exists a $\bar{\delta} \in \mathbb{R}_{>0}$ such that if $\|\xi_0\| \leq \bar{\delta}$ then

$$\|\phi_k\| \leq \|\bar{\phi}_k\| \leq \bar{\varepsilon},$$

for all $(\bar{\Phi}, k) \in \bar{\mathcal{S}}(\xi_0) \times \mathbb{Z}_+$. Moreover, it follows from (B.1) that there exists a $c_{10} \in \mathbb{R}_{>0}$ such that $\|\xi_0\| \leq c_{10}\|\mathbf{x}_{[-h,0]}\|$. Therefore, we conclude that for every $\varepsilon := \bar{\varepsilon} \in \mathbb{R}_{>0}$ there exists a $\delta := \frac{1}{c_{10}}\bar{\delta} \in \mathbb{R}_{>0}$ such that if $\|\mathbf{x}_{[-h,0]}\| \leq \delta$ and hence, $\|\xi_0\| \leq \bar{\delta}$, then

$$\|\phi_k\| \leq \|\bar{\phi}_k\| \leq \bar{\varepsilon} = \varepsilon,$$

for all $(\Phi, k) \in \mathcal{S}(\mathbf{x}_{[-h,0]}) \times \mathbb{Z}_+$. Thus, it was shown that (2.1) is LS and hence, that (2.1) is GAS, which proves claim *(i)*.

Proof of claim (ii): Suppose that the DDI (2.1) is $\mathcal{KL}$-stable. Then, it follows from Lemma 2.1 that the DDI (2.1) is GAS and that for $\delta(\varepsilon)$, $\lim_{\varepsilon \to \infty} \delta(\varepsilon) = \infty$ is an admissible choice. Hence, as $\bar{\delta} := \frac{1}{c_6}\delta$ and $\bar{\varepsilon} := c_5\varepsilon$ with $(c_5, c_6) \in \mathbb{R}_{>0}^2$ it follows that $\lim_{\bar{\varepsilon} \to \infty} \bar{\delta}(\bar{\varepsilon}) = \infty$ is an admissible choice as well. Thus, using Lemma 2.1 again, it follows that system (2.2) is $\mathcal{KL}$-stable.

Conversely, suppose that system (2.2) is $\mathcal{KL}$-stable. Then, Lemma 2.1 yields that system (2.2) is GAS and that $\lim_{\bar{\varepsilon} \to \infty} \bar{\delta}(\bar{\varepsilon}) = \infty$ is an admissible choice. Hence, as $\delta := \frac{1}{c_7}\bar{\delta}$ with $c_7 \in \mathbb{R}_{>0}$ it follows that $\lim_{\varepsilon \to \infty} \delta(\varepsilon) = \infty$ is an admissible choice as well. Thus, using Lemma 2.1 again, it follows that the DDI (2.1) is $\mathcal{KL}$-stable.

Proof of claim (iii): Suppose that the DDI (2.1) is GES. Then, there exist some $(c, \mu) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{[0,1)}$ such that

$$\|\phi_k\| \leq c\|\mathbf{x}_{[-h,0]}\|\mu^k,$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $(\Phi, k) \in \mathcal{S}(\mathbf{x}_{[-h,0]}) \times \mathbb{Z}_+$. Hence, it follows from (B.1) and (B.2) that there exist $(c_1, c_2) \in \mathbb{R}_{>0}^2$ such that

$$\|\bar{\phi}_k\| \leq c_1\|\Phi_{[k-h,k]}\| \leq c_1 c_2 c\|\xi_0\|\mu^{k-h},$$

for all $\xi_0 \in \mathbb{R}^{(h+1)n}$ and all $(\bar{\Phi}, k) \in \bar{\mathcal{S}}(\xi) \times \mathbb{Z}_+$. As $\bar{c} := cc_1 c_2 \mu^{-h} \in \mathbb{R}_{\geq 1}$ and $\bar{\mu} := \mu \in \mathbb{R}_{[0,1)}$ it follows that the augmented system (2.2) is GES.

Conversely, suppose that the augmented system (2.2) is GES. Then, there exist some $(\bar{c}, \bar{\mu}) \in \mathbb{R}_{\geq 1} \times \in \mathbb{R}_{[0,1)}$ such that

$$\|\bar{\phi}_k\| \leq \bar{c}\|\xi_0\|\bar{\mu}^k,$$

for all $\xi_0 \in \mathbb{R}^{(h+1)n}$ and all $(\bar{\Phi}, k) \in \bar{\mathcal{S}}(\xi) \times \mathbb{Z}_+$. It then follows from (B.3) and (B.1) that there exists a $c_3 \in \mathbb{R}_{>0}$ such that

$$\|\phi_k\| \leq \|\bar{\phi}_k\| \leq c_3 c\|\mathbf{x}_{[-h,0]}\|\bar{\mu}^k,$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $(\Phi, k) \in \mathcal{S}(\mathbf{x}_{[-h,0]}) \times \mathbb{Z}_+$. As $c := c_3 \bar{c} \in \mathbb{R}_{\geq 1}$ and $\mu := \bar{\mu} \in \mathbb{R}_{[0,1)}$ it follows that the DDI (2.1) is GES, which completes the proof. ∎

Proof of Theorem 2.1
The equivalence of statements *(i)* and *(ii)* was proven in [73], Theorem 2.7, under the additional assumptions that the map $\bar{F}$ is upper semicontinuous and compact and that the function V is smooth. However, these assumptions were only used to prove certain robustness properties and can therefore be omitted. Alternatively, this equivalence can be shown following *mutatis mutandis* the reasoning used in the proof of Lemma 4 in [107], which is a result for difference equations that does not use any regularity assumptions. Furthermore, the equivalence of statements *(ii)* and *(iii)* follows from Lemma 2.2. ∎

Proof of Theorem 2.3
Suppose that $\rho \neq 0$. Let $\hat{\rho} := \rho^{\frac{1}{h+1}} \in \mathbb{R}_{(0,1)}$ and let

$$i_k^*(\Phi) := \arg\max_{i \in \mathbb{Z}_{[-h,0]}} \hat{\rho}^{-(k+i)} V(\phi_{k+i}),$$
$$U_k(\Phi) := \max_{i \in \mathbb{Z}_{[-h,0]}} \hat{\rho}^{-(k+i)} V(\phi_{k+i}), \tag{B.4}$$

where $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and $(\Phi, k) \in \mathcal{S}(\mathbf{x}_{[-h,0]}) \times \mathbb{Z}_+$. Next, it will be proven that

$$U_{k+1}(\Phi) \leq U_k(\Phi), \tag{B.5}$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $(\Phi, k) \in \mathcal{S}(\mathbf{x}_{[-h,0]}) \times \mathbb{Z}_+$. Therefore, suppose that $i_{k+1}^*(\Phi) = 0$ for some $\mathbf{x}_{[-h,0]}$ and some (Φ, k). Then, (2.10b) yields

$$U_{k+1}(\Phi) = \hat{\rho}^{-(k+1)} V(\phi_{k+1}) \leq \hat{\rho}^{-(k+1)} \max_{i \in \mathbb{Z}_{[-h,0]}} \hat{\rho}^{(h+1)} V(\phi_{k+i})$$
$$\leq \max_{i \in \mathbb{Z}_{[-h,0]}} \hat{\rho}^{-(k+i)} V(\phi_{k+i}) = U_k(\Phi). \tag{B.6}$$

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Furthermore, if $i^*_{k+1}(\Phi) \in \mathbb{Z}_{[-h,-1]}$, then

$$U_{k+1}(\Phi) = \max_{i \in \mathbb{Z}_{[-h,-1]}} \hat{\rho}^{-(k+i+1)} V(\phi_{k+i+1}) = \max_{i \in \mathbb{Z}_{[-h+1,0]}} \hat{\rho}^{-(k+i)} V(\phi_{k+i})$$
$$\leq U_k(\Phi). \tag{B.7}$$

Note that, (B.6) and (B.7) imply that (B.5) holds. Then, applying (B.5) recursively yields

$$U_k(\Phi) \leq U_0(\Phi) \leq \max_{i \in \mathbb{Z}_{[-h,0]}} V(x_i). \tag{B.8}$$

Next, combining (B.4) and (B.8) yields

$$V(\phi_k) \leq \hat{\rho}^k U_k(\Phi) \leq \hat{\rho}^k \max_{i \in \mathbb{Z}_{[-h,0]}} V(x_i),$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $(\Phi, k) \in \mathcal{S}(\mathbf{x}_{[-h,0]}) \times \mathbb{Z}_+$. Applying (2.10a) and using the fact that $\max_{i \in \mathbb{Z}_{[-h,0]}} \alpha_2(\|x_i\|) = \alpha_2(\|\mathbf{x}_{[-h,0]}\|)$ yields

$$\|\phi_k\| \leq \alpha_1^{-1}(\hat{\rho}^k \alpha_2(\|\mathbf{x}_{[-h,0]}\|)), \tag{B.9}$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $(\Phi, k) \in \mathcal{S}(\mathbf{x}_{[-h,0]}) \times \mathbb{Z}_+$. As the function $\beta(r,s) := \alpha_1^{-1}(\hat{\rho}^s \alpha_2(r))$ is such that $\beta \in \mathcal{KL}$, it follows that (2.1) is $\mathcal{KL}$-stable.

Suppose that $\rho = 0$. Then, it follows from (2.10b) and (2.10a) that $\|x_1\| = 0$ for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $x_1 \in F(\mathbf{x}_{[-h,0]})$. Hence, $\|\phi_k\| \leq \|\mathbf{x}_{[-h,0]}\| \frac{1}{2}^k$ for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $(\Phi, k) \in \mathcal{S}(\mathbf{x}_{[-h,0]}) \times \mathbb{Z}_+$. Observing that the function $\beta(r,s) := r\frac{1}{2}^s$ is such that $\beta \in \mathcal{KL}$, completes the proof. ∎

Proof of Proposition 2.1
It was shown in Example 2.3 that the augmented system (2.7) with $a, b \in \mathbb{R}$ such that $|b| < 1$ and $|a| < 1 - b$ is GES. Hence, it follows from Lemma 2.2 that the DDE (2.6) with $b \in \mathbb{R}_{(-1,0)}$ and $a = 1$ is also GES.

The proof of claim *(ii)* proceeds by contradiction. Therefore, suppose ad absurdum that there exists a backward LRF $V : \mathbb{R} \to \mathbb{R}_+$ for the DDE (2.6) with $b \in \mathbb{R}_{(-1,0)}$ and $a = 1$. Let $x_0 = 1$, $x_{-1} = 0$ and let $\pi : \mathbb{R}_+ \to \mathbb{R}_+$ be any function such that $\pi(r) > r$ for all $r \in \mathbb{R}_{>0}$ and $\pi(0) = 0$. Then, (2.6) yields that $x_1 = 1$. As

$$\pi(V(x_1)) = \pi(V(1)) \geq V(1) = \max_{i \in \mathbb{Z}_{[-1,0]}} V(x_i),$$

it follows from (2.9b) that

$$V(x_1) = V(1) \leq \rho V(1) = \rho V(x_0).$$

Obviously, as $\rho \in \mathbb{R}_{[0,1)}$ a contradiction has been reached and hence, V is not a backward LRF for the DDE (2.6). As the functions V and π and the constant ρ were chosen, with the restriction that $\pi(r) > r$ for all $r \in \mathbb{R}_{>0}$ and that $\rho \in \mathbb{R}_{[0,1)}$, arbitrarily, it follows that the second claim has been established.

The same contradiction argument and initial conditions as the ones used in the proof of claim *(ii)* can be used to establish claim *(iii)*. ∎

Proof of Theorem 2.4
First, it is established that

$$\rho < \rho_1 < \ldots < \rho_h < \rho_{h+1} = 1. \tag{B.10}$$

As $\rho < 1$ it holds that $\rho < 1 = (i+1)^2 - (i+2)i$, which is equivalent to

$$(i+2)(\rho+i) < (i+1)(\rho+(i+1)).$$

Therefore, it follows that $\rho_i < \rho_{i+1}$, for all $i \in \mathbb{Z}_{[1,h]}$. Obviously, $\rho_i < \frac{1+i}{i+1} = 1$, which establishes that (B.10) holds. Next, let $\pi_i := \frac{\rho_{i-1}}{\rho_i}$, $i \in \mathbb{Z}_{[1,h+1]}$, and let $\rho_0 := \rho$. Then, as $\rho_{i-1} < \rho_i$ it follows that $\pi_i < \frac{\rho_i}{\rho_i} = 1$. Letting $\pi := \max_{i \in \mathbb{Z}_{[1,h+1]}} \pi_i$, yields $\pi \in \mathbb{R}_{[0,1)}$.

Next, consider any $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$. Then, (2.10b) yields that

$$
\begin{aligned}
\bar{V}(\xi_1) &= \max\{\rho_{h+1}V(x_1), \max_{i \in \mathbb{Z}_{[-h+1,0]}} \rho_{h+i}V(x_i)\} \\
&\leq \max\{\max_{i \in \mathbb{Z}_{[-h,0]}} \rho V(x_i), \max_{i \in \mathbb{Z}_{[-h+1,0]}} \rho_{h+i}V(x_i)\} \\
&= \max\{\rho V(x_{-h}), \max_{i \in \mathbb{Z}_{[-h+1,0]}} \rho_{h+i}V(x_i)\} \\
&= \max_{i \in \mathbb{Z}_{[-h,0]}} \pi_{h+i+1}\rho_{h+i+1}V(x_i) \leq \pi \bar{V}(\xi_0),
\end{aligned}
$$

for all $\xi_0 \in \mathbb{R}^{(h+1)n}$ and all $\xi_1 \in \bar{F}(\xi_0)$. Let $\bar{\rho} := \pi$, $\bar{\alpha}_1(r) := \rho_1 \alpha_1(r)$ and $\bar{\alpha}_2(r) := \alpha_2(r)$. As $\bar{\alpha}_1, \bar{\alpha}_2 \in \mathcal{K}_\infty$ and $\bar{\rho} \in \mathbb{R}_{[0,1)}$, it follows that $\bar{V}$ satisfies the hypothesis of Theorem 2.1, which completes the proof. ∎

Proof of Proposition 2.2
Applying (2.12) in (2.3b) yields

$$\alpha_3(V(x_1)) - \bar{\rho}\alpha_4(V(x_{-h})) + \sum_{i=-h+1}^{0}(\alpha_3(V(x_i)) - \bar{\rho}\alpha_4(V(x_i))) \leq 0, \tag{B.11}$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $x_1 \in F(\mathbf{x}_{[-h,0]})$. The inequality $\alpha_3(r) \geq \alpha_3(\rho r) \geq \bar{\rho}\alpha_4(r)$ for all $r \in \mathbb{R}_+$, which holds by assumption, yields that

$$\sum_{i=-h+1}^{0}(\alpha_3(V(x_i)) - \bar{\rho}\alpha_4(V(x_i))) \geq 0. \tag{B.12}$$

The inequality (B.12) in combination with $V(x_{-h}) \leq \max_{i \in \mathbb{Z}_{[-h,0]}} V(x_i)$ yields that (B.11) is a sufficient condition for

$$\alpha_3(V(x_1)) - \bar{\rho}\alpha_4(\max_{i \in \mathbb{Z}_{[-h,0]}} V(x_i)) \leq 0, \tag{B.13}$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $x_1 \in F(\mathbf{x}_{[-h,0]})$. Then, using that there exists a $\rho \in \mathbb{R}_{[0,1)}$ such that $\rho r \geq \alpha_3^{-1}(\bar{\rho}\alpha_4(r))$ for all $r \in \mathbb{R}_+$ yields

$$V(x_1) - \rho \max_{i \in \mathbb{Z}_{[-h,0]}} V(x_i) \leq 0,$$

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for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $x_1 \in F(\mathbf{x}_{[-h,0]})$. Hence, the hypothesis of Theorem 2.3 is satisfied and the proof is complete. ∎

Proof of Corollary 2.4
Using the bounds (2.13) in (2.3b) yields that

$$\max_{i \in \mathbb{Z}_{[-h+1,1]}} \alpha_3(V(x_i)) - \bar{\rho} \max_{j \in \mathbb{Z}_{[-h,0]}} \alpha_4(V(x_j)) \leq 0, \qquad \text{(B.14)}$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $x_1 \in F(\mathbf{x}_{[-h,0]})$. As $\max\{r_1, r_2\} \geq r_2$ for any $(r_1, r_2) \in \mathbb{R}_+^2$, (B.14) is a sufficient condition for

$$\alpha_3(V(x_1)) - \bar{\rho}\alpha_4(\max_{i \in \mathbb{Z}_{[-h,0]}} V(x_i)) \leq 0,$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $x_1 \in F(\mathbf{x}_{[-h,0]})$. Thus, (B.13) is recovered and the result follows from the remainder of the proof of Proposition 2.2. ∎

Proof of Proposition 2.4
First, the implication *(ii)* ⇒ *(i)* is proven. Let $\mathbb{V}$ denote a λ-$\mathcal{D}$-contractive set for the DDI (2.1). Consider the gauge function [126] of $\mathbb{V}$, i.e.,

$$V(x_0) := \inf\{\mu \in \mathbb{R}_+ : x_0 \in \mu\mathbb{V}\}. \qquad \text{(B.15)}$$

Then, it follows from the fact that $\mathbb{V}$ is a proper C-set that the function V is sublinear and hence convex [126]. Furthermore, letting $a_1 := \max_{x_0 \in \mathbb{V}} \|x_0\| > 0$ and $a_2 := \min_{x_0 \in \partial\mathbb{V}} \|x_0\| > 0$ yields

$$a_1^{-1}\|x_0\| \leq V(x_0) \leq a_2^{-1}\|x_0\|, \quad \forall x_0 \in \mathbb{R}^n.$$

Observe that this maximum and minimum are well-defined and positive because $\mathbb{V}$ is a proper C-set. Next, consider any $\nu \in \mathbb{R}_{>0}$ and let $\mathbf{x}_{[-h,0]} \in (\nu\mathbb{V})^{h+1}$. Then, $\nu^{-1}\mathbf{x}_{[-h,0]} \in \mathbb{V}^{h+1}$ and therefore $F(\nu^{-1}\mathbf{x}_{[-h,0]}) \subseteq \lambda\mathbb{V}$. As the DDI (2.1) is $\mathcal{D}$-homogeneous of order 1 it follows that $F(\mathbf{x}_{[-h,0]}) = \nu F(\nu^{-1}\mathbf{x}_{[-h,0]}) \subseteq \nu\lambda\mathbb{V}$. Thus, it it has been established that if $\mathbb{V}$ is λ-$\mathcal{D}$-contractive then $\nu\mathbb{V}$ is λ-$\mathcal{D}$-contractive as well. As $\nu\mathbb{V}$ is λ-$\mathcal{D}$-contractive for all $\nu \in \mathbb{R}_{>0}$, it follows that for any $\mu \in \mathbb{R}_+$ such that $x_i \in \mu\mathbb{V}$ for all $i \in \mathbb{Z}_{[-h,0]}$ it also holds that $F(\mathbf{x}_{[-h,0]}) \subseteq \mu\lambda\mathbb{V}$. Hence

$$\lambda \max_{i \in \mathbb{Z}_{[-h,0]}} V(x_i) = \lambda \max_{i \in \mathbb{Z}_{[-h,0]}} \inf\{\mu \in \mathbb{R}_+ : x_i \in \mu\mathbb{V}\}$$
$$\geq \inf\{\mu \in \mathbb{R}_+ : x_1 \in \mu\mathbb{V}\}$$
$$= V(x_1),$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $x_1 \in F(\mathbf{x}_{[-h,0]})$. Therefore, the function (B.15) satisfies the hypothesis of Theorem 2.3 with $\alpha_1(r) = a_1^{-1}r$, $\alpha_2(r) = a_2^{-1}r$, $\rho = \lambda$.

Next, the implication *(i)* ⇒ *(ii)* is proven. Consider a sublevel set of V, i.e., $\mathbb{V}_1 := \{x_0 \in \mathbb{R}^n : V(x_0) \leq 1\}$. As the function V is convex (and hence continuous), the set $\mathbb{V}_1$ is convex and closed [126]. Moreover, boundedness follows from the lower bound on the function V. Furthermore, for any $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ if $\max_{i \in \mathbb{Z}_{[-h,0]}} V(x_i) \leq 1$ then it follows from

(2.10b) that $V(x_1) \leq \rho$ for all $x_1 \in F(\mathbf{x}_{[-h,0]})$. Hence, there exists a $\lambda \in \mathbb{R}_{[0,1)}$ such that $F(\mathbf{x}_{[-h,0]}) \subseteq \lambda \mathbb{V}_1$ for all $\mathbf{x}_{[-h,0]} \in \mathbb{V}_1^{h+1}$, which implies that $\mathbb{V}_1$ is λ-$\mathcal{D}$-contractive for the DDI (2.1). ∎

Proof of Proposition 2.5

Using a reasoning that is similar to the reasoning that is used in the proof of Theorem 3 in [34] and substituting $Y = KG$ yields that $\bar{P} \succ 0$ and that

$$
\begin{bmatrix} 0_{n\times hn} & (A_{-h} + B_{-h}K)^\top \\ & (A_{-h+1} + B_{-h+1}K)^\top \\ I_{hn} & \vdots \\ & (A_0 + B_0K)^\top \end{bmatrix} \bar{P} \begin{bmatrix} 0_{n\times hn} & (A_{-h} + B_{-h}K)^\top \\ & (A_{-h+1} + B_{-h+1}K)^\top \\ I_{hn} & \vdots \\ & (A_0 + B_0K)^\top \end{bmatrix}^\top - \bar{\rho}\bar{P} \preceq 0,
$$

for all $(\{A_i, B_i\}_{i\in\mathbb{Z}_{[-h,0]}}) \in \mathcal{AB}$. Therefore, the function $\bar{V}(\xi_0) := \xi_0^\top \bar{P}\xi_0$ satisfies (2.4b) for the linear controlled DDI (2.15) in closed loop with the control law (2.16). Moreover, this function also satisfies (2.4a) with $\bar{c}_1 = \lambda_{\min}(\bar{P}) \in \mathbb{R}_{>0}$ and $\bar{c}_2 = \lambda_{\max}(\bar{P}) \in \mathbb{R}_{\geq c_1}$. Hence, the claim follows from Corollary 2.1. It should be mentioned here that only the implication *(iii)* $\Rightarrow$ *(i)* actually makes use of the assumption that the closed-loop system is a linear DDE (as opposed to a linear DDI). Hence, the implication *(i)* $\Rightarrow$ *(iii)* can indeed be used to obtain the desired result. ∎

Proof of Proposition 2.6

A congruence transformation with a matrix that has Z^{-1} on its diagonal and zero elsewhere, substituting $Z^{-1} := P$, $K = YZ^{-1}$ and applying the Schur complement to (2.17) yields that $P \succ 0$ and that

$$
\begin{bmatrix} \rho\delta_{-h}P & & 0 \\ & \ddots & \\ 0 & & \rho\delta_0 P \end{bmatrix} - \begin{bmatrix} (A_{-h} + B_{-h}K)^\top \\ \vdots \\ (A_0 + B_0K)^\top \end{bmatrix} P \begin{bmatrix} (A_{-h} + B_{-h}K)^\top \\ \vdots \\ (A_0 + B_0K)^\top \end{bmatrix}^\top \succeq 0,
$$

for all $(\{A_i, B_i\}_{i\in\mathbb{Z}_{[-h,0]}}) \in \mathcal{AB}$. Therefore, the function $V(x_0) := x_0^\top P x_0$ satisfies

$$
V\left(\sum_{i=-h}^0 (A_i + B_iK)x_i\right) - \rho \sum_{i=-h}^0 \delta_i V(x_i) \leq 0, \tag{B.16}
$$

for all $(\{A_i, B_i\}_{i\in\mathbb{Z}_{[-h,0]}}) \in \mathcal{AB}$ and all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$. As

$$
\sum_{i=-h}^0 \delta_i V(x_i) \leq \max_{i\in\mathbb{Z}_{[-h,0]}} V(x_i),
$$

it follows that the function $V(x_0) := x_0^\top P x_0$ satisfies (2.10b) for the linear controlled DDI (2.15) in closed loop with the control law (2.16). Moreover, V also satisfies (2.10a) with $\alpha_1(r) = \lambda_{\min}(P)r^2$ and $\alpha_2(r) = \lambda_{\max}(P)r^2$. Hence, Corollary 2.3 yields that the linear controlled DDI (2.15) in closed loop with the control law (2.16) is GES. ∎

B.2   Proofs of Chapter 3

Proof of Theorem 3.1

Before proceeding with the proof of Theorem 3.1 a lemma that is used in the proof of Theorem 3.1 is presented first.

Lemma B.1 Let $\rho \in \mathcal{K}$ and such that $\rho(r) < r$ for all $r \in \mathbb{R}_{>0}$. Then, $\beta(r,s) := \rho^s(r)$ is such that $\beta \in \mathcal{KL}$. $\qquad \square$

Lemma B.1 was proven in [90]. Next, Theorem 3.1 is proven.

Proof of Theorem 3.1: It follows from the second item of the hypothesis of Theorem 3.1 that the functions $\gamma_{i,j}$ satisfy the hypothesis of claim *(iii)* of Theorem 5.2 in [32]. Therefore, there exist $\{\sigma_i\}_{i \in \mathbb{Z}_{[1,N]}}$ with $\sigma_i \in \mathcal{K}_\infty$ for all $i \in \mathbb{Z}_{[1,N]}$ such that

$$\max_{j \in \mathbb{Z}_{[1,N]}} \gamma_{i,j} \circ \sigma_j(r) < \sigma_i(r), \tag{B.17}$$

for all $r \in \mathbb{R}_{>0}$ and all $i \in \mathbb{Z}_{[1,N]}$. Let

$$\gamma(r) := \max_{i \in \mathbb{Z}_{[1,N]}} \max_{j \in \mathbb{Z}_{[1,N]}} \sigma_i^{-1} \circ \gamma_{i,j} \circ \sigma_j(r).$$

Then, it follows from (B.17) that $\gamma(r) < r$ for all $r \in \mathbb{R}_{>0}$. Furthermore, $\gamma \in \mathcal{K}_\infty \cup \{0\}$. Now, consider the function $W(x_0) := \max_{i \in \mathbb{Z}_{[1,N]}} \sigma_i^{-1}(W_i(x_{i,0}))$ and consider any $x_0 \in \mathbb{R}^n$. Then, it follows from (3.3b) that

$$W(x_1) = \max_{i \in \mathbb{Z}_{[1,N]}} \sigma_i^{-1}(W_i(x_{i,1})) \le \max_{i \in \mathbb{Z}_{[1,N]}} \sigma_i^{-1} \circ \max_{j \in \mathbb{Z}_{[1,N]}} \gamma_{i,j}(W_j(x_{j,0}))$$
$$\le \max_{i \in \mathbb{Z}_{[1,N]}} \max_{j \in \mathbb{Z}_{[1,N]}} \sigma_i^{-1} \circ \gamma_{i,j} \circ \sigma_j \circ \sigma_j^{-1}(W_j(x_{j,0}))$$
$$\le \gamma \circ \max_{i' \in \mathbb{Z}_{[1,N]}} \sigma_{i'}^{-1}(W_{i'}(x_{i',0})) \le \gamma(W(x_0)).$$

for all $x_{i,1} \in G_i(x_{1,0}, \dots, x_{N,0})$ and hence all $x_1 \in G(x_0)$, which implies that (3.4b) holds with $\rho(r) = \gamma(r)$. Furthermore, similar to the derivations at the beginning of Appendix B.1, the equivalence of norms [80] yields that there exist some $(c_1, c_2) \in \mathbb{R}^2_{>0}$ such that

$$c_1 \max_{i \in \mathbb{Z}_{[1,N]}} \|x_{i,0}\| \le \|x_0\| \le c_2 \max_{i \in \mathbb{Z}_{[1,N]}} \|x_{i,0}\|, \quad \forall x_0 \in \mathbb{R}^n.$$

Hence, it follows from (3.3a) that for all $x_0 \in \mathbb{R}^n$ it holds that

$$\min_{i \in \mathbb{Z}_{[1,N]}} \sigma_i^{-1} \circ \alpha_1(c_2^{-1}\|x_0\|) \le W(x_0) \le \max_{i \in \mathbb{Z}_{[1,N]}} \sigma_i^{-1} \circ \alpha_2(c_1^{-1}\|x_0\|),$$

which implies that the function W satisfies (3.4a).

Now, using (3.4), it is possible to show that the interconnected system (3.2) is $\mathcal{KL}$-stable. Indeed, applying the inequality (3.4b) recursively and using the bounds (3.4a) yields that

$$\|\tilde{\phi}_k\| \le \alpha_1^{-1} \circ \gamma^k \circ \alpha_2(\|x_0\|), \quad \forall x_0 \in \mathbb{R}^n, \ \forall(\tilde{\Phi}, k) \in \tilde{\mathcal{S}}(x_0) \times \mathbb{Z}_+.$$

Above, $\tilde{\mathcal{S}}(x_0)$ is the space of all solutions to (3.2) while $\tilde{\Phi} := \{\tilde{\phi}_k\}_{k \in \mathbb{Z}_+} \in \tilde{\mathcal{S}}(x_0)$ denotes a solution to (3.2) from initial condition $x_0 \in \mathbb{R}^n$. Letting $\beta(r,s) := \alpha_1^{-1} \circ \gamma^s \circ \alpha_2(r)$ it follows, from the fact that $\gamma(r) < r$ for all $r \in \mathbb{R}_{>0}$ and Lemma B.1, that $\beta \in \mathcal{KL}$.

Therefore, the interconnected system (3.2) is $\mathcal{KL}$-stable, which completes the proof. ∎

Proof of Proposition 3.2
Consider any $\delta \in \mathbb{T}_1^{h+1}$. As the first item of the hypothesis of Theorem 3.3 holds for any $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$, it holds for $\tilde{\mathbf{x}}_{[-h,0]} := \{[\delta]_i z_0\}_{i \in \mathbb{Z}_{[1,h+1]}}$ for any $z_0 \in \mathbb{R}^n$. Hence, the Lyapunov decrease condition in Theorem 3.3 yields that

$$V(z_1) = V(\tilde{x}_1) \leq \max_{i \in \mathbb{Z}_{[1,h+1]}} \rho(V([\delta]_i z_0)) \leq \rho(V(z_0)),$$

for all $z_0 \in \mathbb{R}^n$ and all $z_1 \in H_\delta(z_0)$ and each corresponding $\tilde{x}_1 \in F(\tilde{\mathbf{x}}_{[-h,0]})$. The remainder of the proof then follows from standard Lyapunov arguments such as the ones used in the proof of Theorem 3.1. ∎

Proof of Proposition 3.3
Consider any $\delta \in \mathbb{T}_2^{h+1}$. As the first item of the hypothesis of Theorem 3.3 holds for any $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$, it also holds for $\tilde{\mathbf{x}}_{[-h,0]} := \{[\delta]_i z_0\}_{i \in \mathbb{Z}_{[1,h+1]}}$ for any $z_0 \in \mathbb{R}^n$. Therefore, the Lyapunov decrease condition in Theorem 3.3 and Assumption 3.1 yield that

$$V(z_1) \leq \max\{\rho(V(z_0)), \rho(V(-z_0))\},$$
$$V(-z_1) \leq \max\{\rho(V(z_0)), \rho(V(-z_0))\},$$

for all $z_0 \in \mathbb{R}^n$ and all $z_1 \in H_\delta(z_0)$. It follows from the above two inequalities that the function $\tilde{V}(z_0) := \max\{V(z_0), V(-z_0)\}$ satisfies $\tilde{V}(z_1) \leq \rho(\tilde{V}(z_0))$ for all $z_0 \in \mathbb{R}^n$ and all $z_1 \in H_\delta(z_0)$. The remainder of the proof can then be obtained using standard Lyapunov arguments such as the ones used in the proof of Theorem 3.1. ∎

Proof of Proposition 3.4
Claim *(i)* was proven in Proposition 2.1. To prove claim *(ii)* suppose, ad absurdum, that the set of functions $\{V_i, S_{i,j}\}_{(i,j) \in \mathbb{Z}_{[-1,0]}^2}$ satisfies the hypothesis of Theorem 3.4 for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty, \rho \in \mathbb{R}_{[0,1)}$ and some $\{\sigma_i\}_{i \in \mathbb{Z}_{[-h,0]}}$, with $\sigma_i \in \mathbb{R}_{>0}$ for all $i \in \mathbb{Z}_{[-h,0]}$. Let $x_0 = 1$ and $x_{-1} = 0$. Then, (2.6) yields that $x_1 = 1$. Furthermore, (3.9b) yields that

$$V_0(1) = V_0(x_1) \leq \rho V_0(x_0) + \sum_{j=-1}^0 S_{0,j}(x_0, x_j)$$
$$= \rho V_0(1) + S_{0,-1}(1,0) + S_{0,0}(1,1),$$
$$V_{-1}(1) = V_{-1}(x_0) \leq \rho V_{-1}(x_{-1}) + \sum_{j=-1}^0 S_{-1,j}(x_{-1}, x_j)$$
$$= S_{-1,0}(0,1) + S_{-1,-1}(0,0).$$

The above two inequalities and (3.9a) imply that $\sum_{i=-1}^0 \sum_{j=-1}^0 \sigma_i S_{i,j}(x_i, x_j) > 0$, which contradicts the second item of the hypothesis of Theorem 3.4 and, hence, proves the claim. ∎

Proof of Proposition 3.5
The inequality (3.9b) implies that for all $i \in \mathbb{Z}_{[-h,-1]}$ it holds that

$$-\sum_{j=-h}^0 S_{i,j}(x_i, x_j) \leq \rho V_i(x_i) - V_i(x_{i+1}), \tag{B.18}$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$. When combined (B.18), (3.9b) and the second item of the hypothesis of Theorem 3.4 yield that

$$V_0(x_1) \leq \rho V_0(x_0) + \frac{\sigma_j}{\sigma_0} \sum_{j=-h}^{-1}(\rho V_j(x_j) - V_j(x_{j+1})),$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $x_1 \in F(\mathbf{x}_{[-h,0]})$. Therefore, it follows from the hypothesis of Proposition 3.5 that

$$\begin{aligned}
V_0(x_1) &\leq \rho V_0(x_0) + \sum_{j=-h}^{-1} \frac{\sigma_j}{\sigma_0}(\rho V_j(x_j) - V_j(x_{j+1})) \\
&\leq \rho V_0(x_0) + \sum_{j=-h}^{-1}(\rho V_0(x_j) - \rho V_0(x_{j+1})) \\
&\leq \rho V_0(x_0) + \max\{0, \max_{j \in \mathbb{Z}_{[-h,-1]}} \sum_{i=j}^{-1}(\rho V_0(x_i) - \rho V_0(x_{i+1}))\} \\
&= \rho V_0(x_0) + \max\{0, \max_{j \in \mathbb{Z}_{[-h,-1]}} \rho V_0(x_j) - \rho V_0(x_0)\} \\
&= \max_{i \in \mathbb{Z}_{[-h,0]}} V_0(x_i),
\end{aligned}$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $x_1 \in F(\mathbf{x}_{[-h,0]})$, which corresponds to (2.10b). Furthermore, observing that (2.10a) follows from (3.9a) implies that the function V_0 satisfies the hypothesis of Theorem 2.3. ∎

Proof of Proposition 3.6

Firstly, the fact that the matrices Z_i, $i \in \mathbb{Z}_{[-h,0]}$ are symmetric and positive definite implies that $Z_0^{-1} Z_i Z_0^{-1} \succ 0$ for all $i \in \mathbb{Z}_{[-h,0]}$. Therefore, and as the matrices Z_i are real-valued, it follows that the functions $\alpha_1(r) := \min_{i \in \mathbb{Z}_{[-h,0]}} \lambda_{\min}(Z_0^{-1} Z_i Z_0^{-1})r^2$ and $\alpha_2(r) := \max_{i \in \mathbb{Z}_{[-h,0]}} \lambda_{\max}(Z_0^{-1} Z_i Z_0^{-1})r^2$ satisfy $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$. Therefore, the functions (3.10a) satisfy (3.9a) with the above α_1 and α_2 for all $i \in \mathbb{Z}_{[-h,0]}$. Secondly, applying the Schur complement to (3.11b) and applying a congruence transformation with $\mathrm{diag}(Z_0^{-1}, \ldots, Z_0^{-1})$ yields

$$\begin{aligned}
\mathrm{diag}(Z_0^{-1}, \ldots, &Z_0^{-1})(\rho \bar{Z}_0 + \sum_{l=-h}^{0} \bar{X}_{0,l}) \mathrm{diag}(Z_0^{-1}, \ldots, Z_0^{-1}) \\
&\succeq (\bar{A}_0 + \bar{B} \mathrm{diag}(K, \ldots, K))^\top Z_0^{-1} (\bar{A}_0 + \bar{B} \mathrm{diag}(K, \ldots, K)), \qquad \text{(B.19)}
\end{aligned}$$

for all $(\{A_i, B_i\}_{i \in \mathbb{Z}_{[-h,0]}}) \in \mathcal{AB}$. Thus, by pre-multiplying (B.19) with $\xi_0^\top$ and post-multiplying by ξ_0, it follows that the functions (3.10) satisfy (3.9b) with $i = 0$ for the linear controlled DDI (2.15) in closed loop with the control law (2.16). Similarly, applying a congruence transformation to (3.11a), pre-multiplying with $\xi_0^\top$ and post-multiplying by ξ_0 yields that the functions (3.10) satisfy (3.9b) for all $i \in \mathbb{Z}_{[-h,-1]}$. Thirdly, applying a congruence transformation to (3.11c), pre-multiplying with $\xi_0^\top$ and post-multiplying by ξ_0 yields that the functions (3.10b) satisfy the second item of the hypothesis of Theorem 3.4 with $\sigma_i = 1$ for all $i \in \mathbb{Z}_{[-h,0]}$. Therefore, the hypothesis of Theorem 3.4 is satisfied with the functions (3.10a) and (3.10b) and it follows from Theorem 3.4 that the linear controlled DDI (2.15) in closed loop with the control law (2.16) is $\mathcal{KL}$-stable. ∎

Proof of Proposition 3.7

The result is proven for $h = 2$. The generalization to $h \in \mathbb{Z}_{\geq 3}$ is omitted for brevity.

Suppose that there exist variables $(\{\delta_i\}_{i\in\mathbb{Z}_{[-h,0]}}, Z, Y)$ that satisfy the hypothesis of Proposition 2.6 for some $\hat{\rho} \in \mathbb{R}_{[0,1)}$. Then, the hypothesis of Proposition 2.6 implies that the function $V(x_0) := x_0^\top Z^{-1} x_0$ satisfies (B.16). Now consider, for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$, the following definitions

$$
\begin{aligned}
V_0(x_0) &:= \rho^{-2} x_0^\top Z^{-1} x_0, \\
V_{-1}(x_{-1}) &:= \rho^{-1}(\delta_{-2} + \delta_{-1}) x_{-1}^\top Z^{-1} x_{-1}, \\
V_{-2}(x_{-2}) &:= \delta_{-2} x_{-2}^\top Z^{-1} x_{-2}, \\
S_{0,-1}(x_0, x_{-1}) &:= -S_{-1,0}(x_{-1}, x_0) \\
&:= \mathrm{col}(x_0, x_{-1})^\top \mathrm{diag}(-\tfrac{\delta_{-1}+\delta_{-2}}{\rho} Z^{-1}, \delta_{-1} Z^{-1}) \mathrm{col}(x_0, x_{-1}), \\
S_{0,-2}(x_0, x_{-2}) &:= -S_{-2,0}(x_{-2}, x_0) \\
&:= \mathrm{col}(x_0, x_{-2})^\top \mathrm{diag}(0, \rho\delta_{-2} Z^{-1}) \mathrm{col}(x_0, x_{-2}), \\
S_{-1,-2}(x_{-1}, x_{-2}) &:= -S_{-2,-1}(x_{-2}, x_{-1}) \\
&:= \mathrm{col}(x_{-1}, x_{-2})^\top \mathrm{diag}(-\delta_{-2} Z^{-1}, 0) \mathrm{col}(x_{-1}, x_{-2}),
\end{aligned}
$$

which define, via (3.10), the matrices $(\{Y, Z_i, X_{i,j,l}\}_{(i,j,l)\in(\mathbb{Z}_{[-h,0]})^2\times\mathbb{Z}_{[1,4]}})$. Then, it follows that, by definition, $Z_i = Z_i^\top \succ 0$ for all $i \in \mathbb{Z}_{[-2,0]}$. Furthermore, it follows that, also by definition, (3.11a) with $\rho := \hat{\rho}^{\frac{1}{h+1}}$ and (3.11c) hold. Finally, it follows from (2.17) and (B.16) that (3.11b) holds with $\rho := \hat{\rho}^{\frac{1}{h+1}}$, which completes the proof. ∎

Proof of Proposition 3.8
Claim *(i)* follows directly by solving the LMI (3.11) which yields a feasible solution for $\rho = 0.99$. Therefore, it follows from Proposition 3.6 and Theorem 3.4 that the linear DDE (3.12) is $\mathcal{KL}$-stable.

Claim *(ii)* is proven by contradiction. Therefore, suppose that there exists a set of variables $(\{\delta_i\}_{i\in\mathbb{Z}_{[-1,0]}}, Z, Y)$ that satisfies the hypothesis of Proposition 2.6. Then, applying the Schur complement to (2.17) yields that

$$
\begin{bmatrix} \rho\delta_{-1} Z & 0 \\ 0 & \rho\delta_0 Z \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0.75 \\ 0.75 & 0 \\ 0 & 0 \end{bmatrix} Z \begin{bmatrix} 0 & 0 \\ 0 & 0.75 \\ 0.75 & 0 \\ 0 & 0 \end{bmatrix}^\top \succeq 0, \tag{B.20}
$$

Pre-multiplying (B.20) with $\xi_0 := \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$ and post-multiplying with $\xi_0^\top$ yields that $\delta_0 > 0.56$. Similarly, pre-multiplying (B.20) with $\xi_0 := \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$ and post-multiplying with $\xi_0^\top$ yields that $\delta_{-1} > 0.56$. The above implies that $\delta_0 + \delta_{-1} > 1.12$, which is a contradiction and, hence, completes the proof. ∎

B.3   Proofs of Chapter 4

Proof of Theorem 4.1

Let us prove that *(i)* $\Rightarrow$ *(ii)*. Consider any compact set $\mathbb{X} \subset (\mathbb{R}^n)^{h+1}$. Then, applying (4.1b) recursively and using (4.1a) yields that

$$\|\phi_{k'(M+1)-h}\| \leq \alpha_1^{-1}(V(\phi_{k'(M+1)-h})) \leq \max_{i\in\mathbb{Z}_{[-h,M-h]}} \alpha_1^{-1}(\rho^{k'}\alpha_2(\|\phi_i\|)),$$

for all $\mathbf{x}_{[-h,0]} \in \mathbb{X}$ and all $(\Phi, k') \in \mathcal{S}(\mathbf{x}_{[-h,0]}) \times \mathbb{Z}_+$. Next, observe that Assumption 4.1 implies that there exists an $\bar{\alpha} \in \mathcal{K}$ such that $\|\phi_{k''}\| \leq \bar{\alpha}(\|\mathbf{x}_{[-h,0]}\|)$ for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $(\Phi, k'') \in \mathcal{S}(\mathbf{x}_{[-h,0]}) \times \mathbb{Z}_{[0,M-h]}$. Recalling the definition of the norm of a sequence, it holds that

$$\|\phi_{k'(M+1)-h}\| \leq \alpha_1^{-1}(\rho^{k'}\alpha_2 \circ \bar{\alpha}(\|\mathbf{x}_{[-h,0]}\|)),$$

which, together with the fact that $\rho^{\frac{h-M}{M+1}}\rho^{\frac{k''}{M+1}} \geq 1$ for all $k'' \in \mathbb{Z}_{[0,M-h]}$, yields

$$\begin{aligned}
\|\phi_k\| &= \|\phi_{k'(M+1)-h+k''}\| \\
&\leq \max_{i\in\mathbb{Z}_{[-h,0]}} \alpha_1^{-1}(\rho^{\frac{k'(M+1)}{M+1}}\alpha_2 \circ \bar{\alpha}(\|\phi_{i+k''}\|)) \\
&\leq \alpha_1^{-1}(\rho^{\frac{h-M}{M+1}}\rho^{\frac{k''}{M+1}}\rho^{\frac{k'(M+1)}{M+1}}\alpha_2 \circ \bar{\alpha} \circ \bar{\alpha}(\|\mathbf{x}_{[-h,0]}\|)),
\end{aligned}$$

for all $\mathbf{x}_{[-h,0]} \in \mathbb{X}$ and all $(\Phi, k) \in \mathcal{S}(\mathbf{x}_{[-h,0]}) \times \mathbb{Z}_+$. As $\bar{\rho} := \rho^{\frac{1}{M+1}} \in \mathbb{R}_{[0,1)}$ it follows that the function $\beta_{\mathbb{X}}(r, s) := \alpha_1^{-1}(\rho^{\frac{2h-M}{M+1}}\bar{\rho}^s\alpha_2 \circ \bar{\alpha} \circ \bar{\alpha}(r))$ is such that $\beta_{\mathbb{X}} \in \mathcal{KL}$. Next, let $\tilde{\beta}(r, s) := \max_{i\in\mathbb{Z}_{[1,\lceil r+1\rceil]}} \beta_{i(\mathbb{B}^n)^{h+1}}(r, s)$. Then, $\tilde{\beta}$ satisfies all properties of a $\mathcal{KL}$-function except continuity with respect to r. But Lemma 3 in [134] yields that there exists a function $\bar{\beta} \in \mathcal{KL}$ such that $\bar{\beta}(r, s) \leq \tilde{\beta}(r, s)$ for all $(r, s) \in \mathbb{R}_+ \times \mathbb{Z}_+$. Hence, $\|\phi_k\| \leq \bar{\beta}(\|\mathbf{x}_{[-h,0]}\|, k)$ for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $(\Phi, k) \in \mathcal{S}(\mathbf{x}_{[-h,0]}) \times \mathbb{Z}_+$, which yields the desired claim.

To prove that *(ii)* $\Rightarrow$ *(i)*, suppose that the DDI (2.1) is $\mathcal{KL}$-stable. Consider any $\rho \in \mathbb{R}_{(0,1)}$. Then, for every $\mathbb{X} \subset (\mathbb{R}^n)^{h+1}$ there exists a finite $M(\mathbb{X}) \in \mathbb{Z}_+$ such that $\beta(r, M - h + 1) \leq \rho r$ for all $r \in \mathbb{R}_{[0,\bar{r}]}$ where $\bar{r} = \max\{\|\mathbf{x}_{[-h,0]}\| : \mathbf{x}_{[-h,0]} \in \mathbb{X}\}$. Therefore, it follows from the $\mathcal{KL}$-stability property of the DDI (2.1) that $\|x_1\| \leq \rho\|\mathbf{x}_{[-M,-M+h]}\|$ for all $\mathbf{x}_{[-M,0]} \in (\mathbb{R}^n)^{M+1}$ and all $x_1 \in F(\mathbf{x}_{[-h,0]})$ such that $\mathbf{x}_{[-M,0]}$ is a solution to the DDI (2.1) of length $M + 1$ and satisfies $\mathbf{x}_{[-M,-M+h]} \in \mathbb{X}$. Let $V(x) := \|x\|$. Then, (4.1a) holds with $\alpha_1(r) = r, \alpha_2(r) = r$. Moreover, the definition of the norm of a sequence yields

$$V(x_1) \leq \rho\|\mathbf{x}_{[-M,-M+h]}\| \leq \rho \max_{i\in\mathbb{Z}_{[-M,0]}} V(x_i),$$

for all $\mathbf{x}_{[-M,0]} \in (\mathbb{R}^n)^{M+1}$ and all $x_1 \in F(\mathbf{x}_{[-h,0]})$ such that $\mathbf{x}_{[-M,0]}$ is a solution to the DDI (2.1) of length $M + 1$ and satisfies $\mathbf{x}_{[-M,-M+h]} \in \mathbb{X}$. Hence, (4.1b) holds, which completes the proof of Theorem 4.1. ∎

Proof of Theorem 4.2

The implication *(i)* $\Rightarrow$ *(ii)* follows from the same reasoning as the one used in the proof of Theorem 4.1 by choosing $\mathbb{X} := (\mathbb{R}^n)^{h+1}$ and using the fact that $\bar{\alpha}(r) := \bar{c}r, \bar{c} \in \mathbb{R}_{>0}$

due to Assumption 4.2. To prove the implication *(ii)* $\Rightarrow$ *(i)*, suppose that the DDI (2.1) is GES. Then, for $M := \log_\mu(\frac{1}{c}) + h - 1$ it holds that $c\mu^{M-h+1} = 1$. Therefore, for any $M > \log_\mu(\frac{1}{c}) + h - 1$ there exists a $\rho \in \mathbb{R}_{[0,1)}$ such that $\|\phi_{M-h+1}\| \leq \rho\|\mathbf{x}_{[-h,0]}\|$ for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $\Phi \in \mathcal{S}(\mathbf{x}_{[-h,0]})$. Hence, $\|x_1\| \leq \rho\|\mathbf{x}_{[-M,-M+h]}\|$ for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $\Phi \in \mathcal{S}(\mathbf{x}_{[-h,0]})$ such that $\mathbf{x}_{[-M,0]}$ is a solution to the DDI (2.1) of length $M+1$. Let $V(x) := \|x\|$. Then, V satisfies (4.2a) with $c_1 = 1$, $c_2 = 1$ and $\lambda = 1$. Furthermore, it also holds that

$$V(x_1) \leq \rho\|\mathbf{x}_{[-M,-M+h]}\| \leq \rho \max_{i \in \mathbb{Z}_{[-M,0]}} V(x_i),$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $\Phi \in \mathcal{S}(\mathbf{x}_{[-h,0]})$ such that $\mathbf{x}_{[-M,0]}$ is a solution to the DDI (2.1) of length $M+1$ and, hence, (4.2b) holds. $\blacksquare$

Proof of Proposition 4.1
To prove the implication *(i)* $\Rightarrow$ *(ii)*, suppose that the linear DDE (2.1) admits a matrix $P \in \mathbb{R}^{n \times n}$ and a finite $M \in \mathbb{Z}_{\geq 1}$ such that $P = P^\top \succ 0$ and that (4.4) holds. Then, as P is real-valued and symmetric, $P \succ 0$ implies that the function $V(x_0) := x_0^\top P x_0$ satisfies (4.2a) with $c_1 = \lambda_{\min}(P)$, $c_2 = \lambda_{\max}(P)$ and $\lambda = 2$. Furthermore, applying the Schur complement to (4.4) yields

$$\rho \frac{1}{h+1} \begin{bmatrix} P & & 0 \\ & \ddots & \\ 0 & & P \end{bmatrix} \succeq \begin{bmatrix} A_{0,M}^\top \\ \vdots \\ A_{-h,M}^\top \end{bmatrix} P \begin{bmatrix} A_{0,M} & \ldots & A_{-h,M} \end{bmatrix}.$$

Pre- and post-multiplying the above inequality with $y := \mathrm{col}(x_{-M+h}, \ldots, x_{-M})^\top$ and $y^\top$, respectively, yields that the function $V(x_0) := x_0^\top P x_0$ satisfies

$$V(F(\mathbf{x}_{[-h,0]})) = V(\textstyle\sum_{i=-h}^0 A_{i,M} x_{-M+h+i}) \leq \rho \frac{1}{h+1} \textstyle\sum_{i=-h}^0 V(x_{-M+h+i})$$
$$\leq \rho \max_{i \in \mathbb{Z}_{[-M,0]}} V(x_i),$$

for all $\mathbf{x}_{[-M,0]} \in (\mathbb{R}^n)^{M+1}$ that are a solution to the linear DDE (2.1) of length $M+1$, and hence satisfies (4.2b). Thus, Theorem 4.2 yields that the linear DDE (2.1) is GES.

Conversely, to prove that *(ii)* $\Rightarrow$ *(i)*, suppose that the linear DDE (2.1) is GES. Let $P := I_n$, which implies that $P = P^\top \succ 0$ holds by construction. Furthermore, as the linear DDE (2.1) is GES there exists an $M \in \mathbb{Z}_{\geq 1}$ such that $c\mu^{M-h+1} \leq (\rho\frac{1}{h+1})^{\frac{1}{2}}$. In this case, c and μ correspond to the GES property with respect to the 2-norm. Hence, the GES property of the linear DDE (2.1) implies that $\|\sum_{i=-h}^0 A_{i,M} x_{-M+h+i}\|_2^2 \leq \rho\frac{1}{h+1}\|\mathbf{x}_{[-M,-M+h]}\|_2^2$ for all $\mathbf{x}_{[-M,0]} \in (\mathbb{R}^n)^{M+1}$ that are a solution to the linear DDE (2.1) of length $M+1$. In view of the fact that $\|x_0\|_2^2 = x_0^\top I_n x_0$ we obtain that

$$(\textstyle\sum_{i=-h}^0 A_{i,M} x_{-M+h+i})^\top I_n (\textstyle\sum_{i=-h}^0 A_{i,M} x_{-M+h+i}) \leq \rho\frac{1}{h+1} \max_{i \in \mathbb{Z}_{[-M,-M+h]}} x_i^\top I_n x_i$$
$$\leq \rho\frac{1}{h+1} \textstyle\sum_{i=-M}^{-M+h} x_i^\top I_n x_i,$$

for all $\mathbf{x}_{[-M,0]} \in (\mathbb{R}^n)^{M+1}$ that are a solution to the linear DDE (2.1) of length $M+1$. The

above inequality is equivalent to (4.4) with $P = I_n$ via the Schur complement. ∎

Proof of Proposition 4.2

Consider the positive linear DDE (2.1) and let $\xi_k := \text{col}(\{x_l\}_{l \in \mathbb{Z}_{[k-h,k]}})$, which yields the augmented positive linear system

$$\xi_{k+1} = \bar{A}\xi_k, \quad k \in \mathbb{Z}_+, \tag{B.21}$$

where $\bar{A} \in \mathbb{R}^{(h+1)n \times (h+1)n}$. As the positive linear DDE (2.1) is GES it follows from Lemma 2.2 that the augmented positive linear system (B.21) is GES as well. Therefore, $\bar{A}$ is Schur stable and hence it follows from Theorem 2.1.11 in [11] that there exist a vector $p \in \mathbb{R}_{>0}^{(h+1)n}$ and a constant $\rho \in \mathbb{R}_{[0,1)}$ such that

$$[\bar{A}p]_i \leq \rho[p]_i, \quad \forall i \in \mathbb{Z}_{[1,(h+1)n]}. \tag{B.22}$$

Let $\bar{P} \in \mathbb{R}^{(h+1)n \times (h+1)n}$ denote a matrix with $[\bar{P}]_{i,i} := \frac{1}{[p]_i}$ for all $i \in \mathbb{Z}_{[1,(h+1)n]}$ and zero elsewhere and let $\bar{Q} := \bar{P}\bar{A}\bar{P}^{-1}$. Clearly, $\bar{P}$ and $\bar{Q}$ are well-defined because all elements of p are strictly positive. Furthermore, by construction of $\bar{Q}$ it holds that $\bar{P}\bar{A} = \bar{Q}\bar{P}$. Moreover, (B.22) implies that

$$\|\bar{Q}1_n\|_\infty = \|\bar{P}\bar{A}\bar{P}^{-1}1_n\|_\infty = \|\bar{P}\bar{A}p\|_\infty \leq \|\rho\bar{P}p\|_\infty = \rho.$$

Therefore, the function $\bar{V}(\xi_0) := \|\bar{P}\xi_0\|_\infty$ is [86] a diagonal polyhedral LF for the positive linear system (B.21). Let $P_i \in \mathbb{R}^{n \times n}$, $i \in \mathbb{Z}_{[-h,0]}$ denote diagonal matrices such that $\text{diag}(\{P_i\}_{i \in \mathbb{Z}_{[-h,0]}}) = \bar{P}$. Then, as $\bar{V}(\xi_0) := \|\bar{P}\xi_0\|_\infty$ is a diagonal polyhedral LF it also holds that

$$\left\| \begin{bmatrix} P_{-h}x_{-h+1} \\ \vdots \\ P_0(\sum_{i=-h}^0 A_i x_i) \end{bmatrix} \right\|_\infty \leq \rho \left\| \begin{bmatrix} P_{-h}x_{-h} \\ \vdots \\ P_0 x_0 \end{bmatrix} \right\|_\infty, \tag{B.23}$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$. As $\bar{A}$ has matrices I_n on its superdiagonal, (B.22) implies that the diagonal matrices $(P_{-h}, \dots, P_0)$ are such that $[P_{i-1}]_{j,j} \leq \rho[P_i]_{j,j}$ for all $(i,j) \in \mathbb{Z}_{[-h+1,0]} \times \mathbb{Z}_{[1,n]}$. Therefore, it also holds that $\|P_{i-1}x_0\|_\infty \leq \|P_i x_0\|_\infty$ for all $x_0 \in \mathbb{R}^n$ and all $i \in \mathbb{Z}_{[-h+1,0]}$. Combining the above facts with (B.23) yields that

$$\|P_0(\sum_{i=-h}^0 A_i x_i)\|_\infty \leq \rho \max_{i \in \mathbb{Z}_{[-h,0]}} \|P_0 x_i\|_\infty,$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$. Therefore, the function $V(x_0) := \|P_0 x_0\|_\infty$ satisfies (4.2b) with $M = h$. Furthermore, the function V satisfies (4.2a) with [86] some $(c_1, c_2) \in \mathbb{R}_{>0} \times \mathbb{R}_{\geq c_1}$ and $\lambda = 1$, which completes the proof. ∎

B.4 Proofs of Chapter 5

Proof of Proposition 5.1
Take any $(Z_1, \ldots, Z_{h+1}) \in \mathcal{X} \cup \mathcal{Y}$. If $(Z_1, \ldots, Z_{h+1}) \in \mathcal{X}$ then $Z_1 \times \ldots \times Z_{h+1} \subseteq \mathbb{C}$. Moreover, there exists a $(Z'_1, \ldots, Z'_{h+1}) \in \mathcal{X} \subseteq \mathcal{X} \cup \mathcal{Y}$ such that $F(Z_1, \ldots, Z_{h+1}) \subseteq Z'_{h+1}$ and $Z_{i+1} \subseteq Z'_i$ for all $i \in \mathbb{Z}_{[1,h]}$. Similarly, if $(Z_1, \ldots, Z_{h+1}) \in \mathcal{Y}$ then $Z_1 \times \ldots \times Z_{h+1} \subseteq \mathbb{C}$. Moreover, there exists a $(Z'_1, \ldots, Z'_{h+1}) \in \mathcal{Y} \subseteq \mathcal{X} \cup \mathcal{Y}$ such that $F(Z_1, \ldots, Z_{h+1}) \subseteq Z'_{h+1}$ and that $Z_{i+1} \subseteq Z'_i$ for all $i \in \mathbb{Z}_{[1,h]}$, which establishes the claim. ∎

Proof of Proposition 5.2
To prove claim *(i)*, consider any $\lambda \in \mathbb{R}_{[0,1]}$ and any $(Z_1, \ldots, Z_{h+1}) \in \lambda \mathcal{X} + (1-\lambda)\mathcal{Y}$. Then, by definition, there exist $(X_1, \ldots, X_{h+1}) \in \mathcal{X}$ and $(Y_1, \ldots, Y_{h+1}) \in \mathcal{Y}$ such that $(Z_1, \ldots, Z_{h+1}) = (\lambda X_1 \oplus (1-\lambda)Y_1, \ldots, \lambda X_{h+1} \oplus (1-\lambda)Y_{h+1})$. As $X_1 \times \ldots \times X_{h+1} \subseteq \mathbb{C}$ and $Y_1 \times \ldots \times Y_{h+1} \subseteq \mathbb{C}$ it also holds [126] that

$$Z_1 \times \ldots \times Z_{h+1} = \lambda \left(X_1 \times \ldots \times X_{h+1} \right) \oplus (1-\lambda) \left(Y_1 \times \ldots \times Y_{h+1} \right)$$
$$\subseteq \lambda \mathbb{C} \oplus (1-\lambda)\mathbb{C} = \mathbb{C}.$$

Furthermore, there exist $(X'_1, \ldots, X'_{h+1}) \in \mathcal{X}$ and $(Y'_1, \ldots, Y'_{h+1}) \in \mathcal{Y}$ such that $X_{i+1} \subseteq X'_i$ for all $i \in \mathbb{Z}_{[1,h]}$, $\bigoplus_{j=1}^{h+1} A_{-h-1+j} X_j \subseteq X'_{h+1}$ for all $(A_{-h}, \ldots, A_0) \in \mathcal{A}$ and similarly for $(Y'_1, \ldots, Y'_{h+1})$. Then, the $h+1$-tuple of sets $(Z'_1, \ldots, Z'_{h+1})$ with $Z'_i := \lambda X'_i \oplus (1-\lambda)Y'_i \in \mathrm{Com}(\mathbb{R}^n)$ satisfies, by definition, $(Z'_1, \ldots, Z'_{h+1}) \in \lambda \mathcal{X} + (1-\lambda)\mathcal{Y}$. Furthermore, $Z_{i+1} = \lambda X_{i+1} \oplus (1-\lambda)Y_{i+1} \subseteq \lambda X'_i \oplus (1-\lambda)Y'_i = Z'_i$ for all $i \in \mathbb{Z}_{[1,h]}$. Moreover, it follows from the commutativity and associativity [126] of the Minkowski addition that

$$\bigoplus_{j=1}^{h+1} A_{-h-1+j} Z_j = \lambda (\bigoplus_{j=1}^{h+1} A_{-h-1+j} X_j) \oplus (1-\lambda)(\bigoplus_{j=1}^{h+1} A_{-h-1+j} Y_j)$$
$$\subseteq \lambda X'_{h+1} \oplus (1-\lambda)Y'_{h+1} = Z'_{h+1},$$

for all $(A_{-h}, \ldots, A_0) \in \mathcal{A}$, which establishes claim *(i)*. Claim *(ii)* follows from claim *(i)* and Proposition 5.1. ∎

Proof of Proposition 5.3
As $X_1 \times \ldots \times X_{h+1} \subseteq \mathbb{C}$ for all $(X_1, \ldots, X_{h+1}) \in \mathcal{X}$ it follows that $\bigcup_{(X_1, \ldots, X_{h+1}) \in \mathcal{X}} X_1 \times \ldots \times X_{h+1} \subseteq \mathbb{C}$. Furthermore, consider any $\mathbf{x}_{[-h,0]} \in \bigcup_{(X_1, \ldots, X_{h+1}) \in \mathcal{X}} X_1 \times \ldots \times X_{h+1}$, then there exists some $(Z_1, \ldots, Z_{h+1}) \in \mathcal{X}$ such that $\mathbf{x}_{[-h,0]} \in Z_1 \times \ldots \times Z_{h+1}$. Hence, as $\mathcal{X}$ is an invariant family of sets, it follows that there exists some $(Z'_1, \ldots, Z'_{h+1}) \in \mathcal{X}$ such that $\mathbf{x}_{[-h+1,1]} \in Z'_1 \times \ldots \times Z'_{h+1}$ for all $x_1 \in F(\mathbf{x}_{[-h,0]})$. As $(Z'_1, \ldots Z'_{h+1}) \in \mathcal{X}$ it follows that $\mathbf{x}_{[-h+1,1]} \in \bigcup_{(X_1, \ldots, X_{h+1}) \in \mathcal{X}} X_1 \times \ldots \times X_{h+1}$ for all $x_1 \in F(\mathbf{x}_{[-h,0]})$, which establishes that the set $\bigcup_{(X_1, \ldots, X_{h+1}) \in \mathcal{X}} X_1 \times \ldots \times X_{h+1}$ is invariant. ∎

Proof of Proposition 5.4
As $\mathbb{X} \subseteq \mathbb{C}$ and $\mathbb{X}$ is invariant it follows that the family of sets

$$\mathcal{X} := \left\{ (\bigcup_{\Phi \in \mathcal{S}(\mathbf{x}_{[-h,0]})} \{\phi_{k-h}\}, \ldots, \bigcup_{\Phi \in \mathcal{S}(\mathbf{x}_{[-h,0]})} \{\phi_k\}) : (\mathbf{x}_{[-h,0]}, k) \in \mathbb{X} \times \mathbb{Z}_+ \right\},$$

is such that $X_1 \times \ldots \times X_{h+1} \subseteq \mathbb{C}$ for all $(X_1, \ldots, X_{h+1}) \in \mathcal{X}$. Furthermore, by construction of $\mathcal{X}$ it follows that for all $(X_1, \ldots, X_{h+1}) \in \mathcal{X}$ there exists a $(X_1', \ldots, X_{h+1}') \in \mathcal{X}$ such that $F(X_1, \ldots, X_{h+1}) \subseteq X_{h+1}'$ and that $X_{i+1} \subseteq X_i'$ for all $i \in \mathbb{Z}_{[1,h]}$. Hence, the family of sets $\mathcal{X}$ is invariant. Furthermore, $\bigcup_{(X_1, \ldots, X_{h+1}) \in \mathcal{X}} X_1 \times \ldots \times X_{h+1} = \mathbb{X}$ also holds by definition of $\mathcal{X}$, which completes the proof. ∎

Proof of Theorem 5.1

To prove claim *(i)*, take any $(X_1, \ldots, X_{h+1}) \in \mathcal{X}_{\mathbb{T}}$ (note that $\mathbb{T} \neq \emptyset$ and, hence, $\mathcal{X}_{\mathbb{T}} \neq \emptyset$). Since $\mathbb{T} \subseteq \bar{\mathbb{T}}$ it follows that $X_1 \times \ldots \times X_{h+1} \subseteq \mathbb{C}$. Furthermore, as $\mathbb{T}$ is an invariant set for (5.3) it follows that the $h+1$-tuple $(X_1', \ldots, X_{h+1}')$ where $X_i' := g_i(X_1, \ldots, X_{h+1}) \in \mathrm{Com}(\mathbb{R}^n)$ for all $i \in \mathbb{Z}_{[1,h+1]}$ is such that

$$\mathrm{col}(\{W_i(X_i')\}_{i \in \mathbb{Z}_{[1,h+1]}}) = \mathrm{col}(\{f_i(W_1(X_1), \ldots, W_{h+1}(X_{h+1}))\}_{i \in \mathbb{Z}_{[1,h+1]}}) \in \mathbb{T}.$$

Therefore, we have that $(X_1', \ldots, X_{h+1}') \in \mathcal{X}_{\mathbb{T}}$ which implies, together with the definition of g_i, that $\mathcal{X}_{\mathbb{T}}$ is an invariant family of sets.

Outline of the proof of claim *(ii)*: By construction, a direct extension of the standard stability arguments for vector LFs [141] yields that the set dynamics

$$\mathrm{col}(\{X_i'\}_{i \in \mathbb{Z}_{[1,h+1]}}) = \mathrm{col}(\{g_i(X_1, \ldots, X_{h+1})\}_{i \in \mathbb{Z}_{[1,h+1]}}),$$

is $\mathcal{KL}$-stable if (5.3) is $\mathcal{KL}$-stable. But, also by construction, for all $\mathbf{x}_{[-h,0]} \in X_1 \times \ldots \times X_{h+1}$ it holds that $F(\mathbf{x}_{[-h,0]}) \subseteq g_{h+1}(X_1, \ldots, X_{h+1})$ and that $x_i \in g_{h+i}(X_1, \ldots, X_{h+1})$ for all $i \in \mathbb{Z}_{[-h+1,0]}$. Hence, the solutions of the DDI (2.1) are included in the solutions of the set dynamics $X_1' \times \ldots \times X_{h+1}' = g_1(X_1, \ldots, X_{h+1}) \times \ldots \times g_{h+1}(X_1, \ldots, X_{h+1})$. Consequently, $\mathcal{KL}$-stability of the set dynamics, given by the comparison system, implies the $\mathcal{KL}$-stability of the DDI (2.1). Hence, the DDI (2.1) is $\mathcal{KL}$-stable if (5.3) is $\mathcal{KL}$-stable. ∎

Proof of Proposition 5.5

Choose any $(X_1, \ldots, X_{h+1}) \in \mathcal{X}(S_1, \ldots, S_{h+1}, \Theta)$. Then, there exists a $\theta \in \Theta$ such that, for all $i \in \mathbb{Z}_{[1,h+1]}$, $X_i = [\theta]_i S_i$. Hence, $X_1 \times \ldots \times X_{h+1} \subseteq \mathbb{C}$. Furthermore, there exist $h+1$ sets $X_i' := [\theta']_i S_i$, $i \in \mathbb{Z}_{[1,h+1]}$ for some $\theta' \in \Theta$ such that $\bigoplus_{j=1}^{h+1} A_{-h-1+j} X_j \subseteq X_{h+1}'$ for all $(A_{-h}, \ldots, A_0) \in \mathcal{A}$ and that $X_{i+1} \subseteq X_i'$ for all $i \in \mathbb{Z}_{[1,h]}$. As $\theta' \in \Theta$ it follows that $(X_1', \ldots, X_{h+1}') \in \mathcal{X}(S_1, \ldots, S_{h+1}, \Theta)$, which implies that the family of sets $\mathcal{X}(S_1, \ldots, S_{h+1}, \Theta)$ is invariant. ∎

Proof of Proposition 5.6

To prove claim *(i)*, take any $(X_1, \ldots, X_{h+1}) \in \mathcal{X}(S_1, \ldots, S_{h+1}, \Theta_1 \cup \Theta_2)$. Then, there exists a $\theta \in \Theta_1 \cup \Theta_2$ such that, for all $j \in \mathbb{Z}_{[1,h+1]}$, $X_j = [\theta]_j S_j$. If $\theta \in \Theta_1$ then $X_1 \times \ldots \times X_{h+1} \subseteq \mathbb{C}$ and there exists a $\theta' \in \Theta_1 \subseteq \Theta_1 \cup \Theta_2$ such that the sets $X_j' := [\theta']_j S_j$, $j \in \mathbb{Z}_{[1,h+1]}$ satisfy $\bigoplus_{j=1}^{h+1} A_{-h-1+j} X_j \subseteq X_{h+1}'$ for all $(A_{-h}, \ldots, A_0) \in \mathcal{A}$ and that $X_{i+1} \subseteq X_i'$ for all $i \in \mathbb{Z}_{[1,h]}$. If $\theta \in \Theta_2$ then the same reasoning applies.

To prove claim *(ii)*, consider an arbitrary $\lambda \in \mathbb{R}_{[0,1]}$ and take any $(X_1, \ldots, X_{h+1}) \in \mathcal{X}(S_1, \ldots, S_{h+1}, \lambda \Theta_1 \oplus (1-\lambda)\Theta_2)$. Then, there exists a $\theta_1 \in \Theta_1$ and $\theta_2 \in \Theta_2$ such that, for all $i \in \mathbb{Z}_{[1,h+1]}$, $X_i = (\lambda[\theta_1]_i + (1-\lambda)[\theta_2]_i)S_i$. But, for all $i \in \mathbb{Z}_{[1,h+1]}$,

$(\lambda[\theta_1]_i + (1 - \lambda)[\theta_2]_i)S = \lambda[\theta_1]_iS_i \oplus (1 - \lambda)[\theta_2]_iS_i$ and hence, for all $i \in \mathbb{Z}_{[1,h+1]}$, $X_i = \lambda[\theta_1]_iS_i \oplus (1 - \lambda)[\theta_2]_iS_i$. Therefore, the commutativity and associativity of the Minkowski addition imply that $X_1 \times \ldots \times X_{h+1} \subseteq \mathbb{C}$. Furthermore, there exist $\theta'_1 \in \Theta_1$ and $\theta'_2 \in \Theta_2$ such that $\bigoplus_{j=1}^{h+1} A_{-h-1+j}[\theta_1]_jS_j \subseteq [\theta'_1]_{h+1}S_{h+1}$ for all $(A_{-h}, \ldots, A_0) \in \mathcal{A}$, $[\theta_1]_{i+1}S_{i+1} \subseteq [\theta'_1]_iS_i$ for all $i \in \mathbb{Z}_{[1,h]}$ and similarly for θ'_2. Therefore, it also holds that

$$
\begin{aligned}
\bigoplus_{j=1}^{h+1} A_{-h-1+j}X_j &= \bigoplus_{j=1}^{h+1} A_{-h-1+j}(\lambda[\theta_1]_j + (1 - \lambda)[\theta_2]_j)S_j \\
&\subseteq \lambda[\theta'_1]_{h+1}S_{h+1} \oplus (1 - \lambda)[\theta'_2]_{h+1}S_{h+1} = X'_{h+1},
\end{aligned}
$$

for all $(A_{-h}, \ldots, A_0) \in \mathcal{A}$ and that $X_{i+1} = \lambda[\theta_1]_{i+1}S_{i+1} \oplus (1 - \lambda)[\theta_2]_{i+1}S_{i+1} \subseteq X'_i$ for all $i \in \mathbb{Z}_{[1,h]}$, which verifies the claim. Claim *(iii)* follows from a direct combination of the arguments used in the proofs of claims *(i)* and *(ii)*. ∎

Proof of Theorem 5.2

To prove claim *(i)* it suffices to prove that f_{h+1} is sublinear as the functions f_i, $i \in \mathbb{Z}_{[1,h]}$ are simplifications of f_{h+1}. Therefore, consider any $\theta \in \mathbb{R}_+^{h+1}$ and any $\alpha \in \mathbb{R}_+$. Then, it follows from the associativity of the Minkowski addition [126] that

$$
\begin{aligned}
f_{h+1}(\alpha\theta) &= \max_{(A_{-h},\ldots,A_0)\in\mathcal{A}} \min_\eta\{\eta \in \mathbb{R}_+ : \alpha(\bigoplus_{j=1}^{h+1} A_{-h-1+j}[\theta]_jS_j) \subseteq \eta S_i\} \\
&= \alpha f_{h+1}(\theta).
\end{aligned}
$$

Furthermore, for any $X_i \subseteq \mathbb{R}^n$, $i \in \mathbb{Z}_{[1,3]}$ and any $(\eta_1, \eta_2) \in \mathbb{R}_{>0}^2$, it holds that $X_1 \oplus X_2 \subseteq (\eta_1 + \eta_2)X_3$ if $X_1 \subseteq \eta_1 X_3$ and $X_2 \subseteq \eta_2 X_3$. Therefore, for any $(\theta_1, \theta_2) \in (\mathbb{R}_+^{h+1})^2$ it follows from the commutativity of the Minkowski addition [126] that

$$
\begin{aligned}
f_{h+1}(\theta_1 + \theta_2) &= \max_{(A_{-h},\ldots,A_0)\in\mathcal{A}} \min_\eta\{\eta \in \mathbb{R}_+ : \bigoplus_{j=1}^{h+1} A_{-h-1+j}[\theta_1]_jS_j \oplus \\
&\qquad\qquad \bigoplus_{j=1}^{h+1} A_{-h-1+j}[\theta_2]_jS_j \subseteq \eta S_{h+1}\} \\
&\leq \max_{(A_{-h},\ldots,A_0)\in\mathcal{A}} \min_\eta\{\eta \in \mathbb{R}_+ : \bigoplus_{j=1}^{h+1} A_{-h-1+j}[\theta_1]_jS_j \subseteq \eta S_{h+1}\} \\
&\quad + \max_{(A_{-h},\ldots,A_0)\in\mathcal{A}} \min_\eta\{\eta \in \mathbb{R}_+ : \bigoplus_{j=1}^{h+1} A_{-h-1+j}[\theta_2]_jS_j \subseteq \eta S_{h+1}\} \\
&= f_{h+1}(\theta_1) + f_{h+1}(\theta_2),
\end{aligned}
$$

which completes the proof of claim *(i)*.

To prove claim *(ii)*, take any $\theta \in \Theta$ (note that $\Theta \neq \emptyset$). Since $\Theta \subseteq \bar{\Theta}$ it follows that $[\theta]_1S_1 \times \ldots \times [\theta]_{h+1}S_{h+1} \subseteq \mathbb{C}$. Furthermore, as Θ is an invariant set for (5.5) it follows that $\theta' = f(\theta) \in \Theta$ and, by definition, that $\bigoplus_{j=1}^{h+1} A_{-h-1+j}[\theta]_jS_j \subseteq f_{h+1}(\theta)S_{h+1}$ for all $(A_{-h}, \ldots, A_0) \in \mathcal{A}$ and that $[\theta]_{i+1}S_{i+1} \subseteq f_i(\theta)S_i$ for all $i \in \mathbb{Z}_{[1,h]}$.

To prove the last claim, let $V_j(X_j) = V_j([\theta]_jS_j) := [\theta]_j$ for all $j \in \mathbb{Z}_{[1,h+1]}$. The functions V_j satisfy (5.2a) with $\alpha_1(r) = \min_{i\in\mathbb{Z}_{[1,h+1]}} (\max_{x_0 \in S_i} \|x_0\|)^{-1} r$ and $\alpha_2(r) = \max_{i\in\mathbb{Z}_{[1,h+1]}} (\min_{x_0 \in \partial S_i} \|x_0\|)^{-1} r$. Furthermore, the functions V_j satisfy (5.2b) with f_i as defined in (5.5). Although "=" is possibly replaced by "≤" in (5.2b), this does not induce any changes to our arguments and results. Hence, a direct utilization of Theorem 5.1, claim

(ii) yields that the linear DDI (2.1) is $\mathcal{KL}$-stable if (5.5) is $\mathcal{KL}$-stable. ∎

Proof of Lemma 5.1

First, apply a similarity transformation to M, i.e., $\tilde{M} := T^{-1}MT$, such that

$$
\tilde{M} := \begin{bmatrix} 0 & & 1 & & 0 \\ \vdots & & & \ddots & \\ 0 & & 0 & & 1 \\ \prod_{i=1}^{h}[M]_{i,i+1}[M]_{h+1,1} & \prod_{i=2}^{h}[M]_{i,i+1}[M]_{h+1,2} & \cdots & [M]_{h+1,h+1} \end{bmatrix},
$$

where $T := \operatorname{diag}(1, [M]_{1,2}^{-1}, \ldots, (\prod_{i=1}^{h}[M]_{i,i+1})^{-1})$. Next, note that $\operatorname{sr}(M) < 1$ if and only if $\operatorname{sr}(\tilde{M}) < 1$. Therefore, it suffices to show the equivalence for the matrix $\tilde{M}$.

Theorem 2.1.11 in [11] implies that $\operatorname{sr}(\tilde{M}) < 1$ if and only if for all $x \in \mathbb{R}_+^{h+1} \setminus \{0\}$ there exists an $i(x) \in \mathbb{Z}_{[1,h+1]}$ such that $[\tilde{M}x]_{i(x)} < [x]_{i(x)}$. Hence, as $1_{h+1} \in \mathbb{R}_+^{h+1} \setminus \{0\}$ and $[\tilde{M}1_{h+1}]_i = 1$ for all $i \in \mathbb{Z}_{[1,h]}$, $\operatorname{sr}(\tilde{M}) < 1$ yields that

$$
1 > [\tilde{M}1_{h+1}]_{h+1} = [M]_{h+1,h+1} + \sum_{j=1}^{h}\prod_{i=j}^{h}[M]_{i,i+1}[M]_{h+1,j}.
$$

Conversely, consider any $x \in \mathbb{R}_+^{h+1} \setminus \{0\}$. If $[x]_i \le [x]_{i+1}$ for all $i \in \mathbb{Z}_{[1,h]}$, then

$$
\begin{aligned}
[\tilde{M}x]_{h+1} &= \sum_{j=1}^{h}\prod_{i=j}^{h}[M]_{i,i+1}[M]_{h+1,j}[x]_j + [M]_{h+1,h+1}[x]_{h+1} \\
&\le (\sum_{j=1}^{h}\prod_{i=j}^{h}[M]_{i,i+1}[M]_{h+1,j} + [M]_{h+1,h+1})[x]_{h+1} < [x]_{h+1}.
\end{aligned}
$$

Therefore, it follows that either $[x]_{i+1} < [x]_i$ for some $i \in \mathbb{Z}_{[1,h]}$ or $[\tilde{M}x]_{h+1} < [x]_{h+1}$, which implies [11, Theorem 2.1.11] that $\operatorname{sr}(\tilde{M}) < 1$. ∎

Proof of Proposition 5.7

It follows, by design of Algorithm 5.1 and the matrix M, that M satisfies $[M]_{h+1,j} \le \lambda_j$ for all $j \in \mathbb{Z}_{[1,h+1]}$, $[M]_{j,j+1} = 1$ for all $j \in \mathbb{Z}_{[1,h]}$ and that $[M]_{i,j} = 0$ otherwise. Hence, the hypothesis of Lemma 5.1 is satisfied. Furthermore, as the set S is a nontrivial C-set (i.e., a C-set not equal to $\{0\}$), the constraint set $\bar{\Theta}$ is a C-set in $\mathbb{R}_+^{h+1}$ with non-empty interior. As a consequence of the above two facts, the set recursion (5.9) yields a C-set in $\mathbb{R}_+^{h+1}$ with non-empty interior. Claim *(ii)* of Theorem 5.2 then completes the proof. ∎

Proof of Proposition 5.8

Step 2.1 of Algorithm 5.2 generates, for any $i^* \in \mathbb{Z}_{[1,h+1]}$, a sequence of non-empty, monotonically non-increasing, proper C-polytopic sets $\{Z_{k,i^*}\}_{k \in \mathbb{Z}_+}$ whose Hausdorff limit is guaranteed to be at least a C-set (possibly a trivial C-set $\{0\}$). Moreover, if $\max\{\lambda_i : i \in \mathbb{Z}_{[-h,0]}\} < 1$, and hence $\max\{\operatorname{sr}(A_i) : i \in \mathbb{Z}_{[-h,0]}\} < 1$, then Algorithm 5.2 is guaranteed to terminate in finite time. Furthermore, the condition that $\max\{\operatorname{sr}(A_i) : i \in \mathbb{Z}_{[-h,0]}\} < 1$ (together with selection of $\lambda_i \in \mathbb{R}_{>\operatorname{sr}(A_i)}$, $i \in \mathbb{Z}_{[-h,0]}$ such that $\max\{\lambda_i : i \in \mathbb{Z}_{[-h,0]}\} < 1$) ensures that the sets S_{i^*}, $i^* \in \mathbb{Z}_{[1,h+1]}$ which are the Hausdorff limits of the sequences $\{Z_{k,i^*}\}_{k \in \mathbb{Z}_+}$, $i^* \in \mathbb{Z}_{[1,h+1]}$ are proper C-polytopic sets. ∎

B.5 Proofs of Chapter 6

Proof of Theorem 6.1

The following lemma is required to prove Theorem 6.1.

Lemma B.2 Let $\rho \in \mathbb{R}_{[0,1)}$ and consider a sequence $\{\lambda_i\}_{i \in \mathbb{Z}_+}$ with $\lambda_i \in \mathbb{R}_+$ and bounded for all $i \in \mathbb{Z}_+$. If $\lim_{i \to \infty} \lambda_i = 0$, then $\lim_{k \to \infty} \sum_{i=0}^{k} \rho^{k-i} \lambda_i = 0$.

Proof: An arbitrary sequence $\{s_k\}_{k \in \mathbb{Z}_+}$ of real numbers is convergent [130] and has the limit s if, for all $\varepsilon \in \mathbb{R}_{>0}$ there exists $N(\varepsilon) \in \mathbb{Z}_+$ such that $|s_k - s| < \varepsilon$ for all $k \in \mathbb{Z}_{\geq N(\varepsilon)}$. Therefore, fix any $\varepsilon \in \mathbb{R}_{>0}$. Observe that the boundedness of each $\lambda_i \geq 0$ implies the existence of an $M \in \mathbb{R}_+$ such that $\lambda_i \leq M$ for all $i \in \mathbb{Z}_+$. Moreover, $\lim_{i \to \infty} \lambda_i = 0$ implies that for all ε there exists an $i^*(\varepsilon) \in \mathbb{Z}_+$ such that $\lambda_i < m := \frac{\varepsilon}{2}(1 - \rho)$ for all $i > i^*(\varepsilon)$. Consider now the sequence $s_k := \sum_{i=0}^{k} \rho^{k-i} \lambda_i$ and the desired limit $s = 0$. Then, for all $\varepsilon \in \mathbb{R}_{>0}$ and for any $k > i^*(\varepsilon)$ it holds that

$$|s_k - 0| = \left| \sum_{i=0}^{k} \rho^{k-i} \lambda_i \right| < \left| \sum_{i=0}^{i^*(\varepsilon)} \rho^{k-i} M + \sum_{i=i^*(\varepsilon)+1}^{k} \rho^{k-i} m \right|.$$

As $(\rho^{-1} - 1) \sum_{i=0}^{i^*(\varepsilon)} \rho^{-i} = \rho^{-i^*(\varepsilon)-1} - 1$, we further obtain that

$$|s_k - 0| < \left| \frac{\rho^{k-i^*(\varepsilon)} - \rho^{k+1}}{1 - \rho} M + \frac{1 - \rho^{k-i^*(\varepsilon)}}{1 - \rho} m \right|.$$

Choosing $N(\varepsilon)$ large enough, such that $\frac{\rho^{N(\varepsilon)-i^*(\varepsilon)} - \rho^{N(\varepsilon)+1}}{1-\rho} M \leq \frac{\varepsilon}{2}$, yields that the series s_k converges to the desired limit 0, which completes the proof. ∎

Next, we proceed with the proof of Theorem 6.1.

Proof of Theorem 6.1: Consider the functions V and π that correspond to Assumption 6.2. As the control law π is continuous, satisfies $\pi(0) = 0$ and as by assumption $0 \in \text{int}(\mathbb{C}_x)$ and $0 \in \text{int}(\mathbb{C}_u)$, there exists an $\varepsilon \in \mathbb{R}_{>0}$ such that for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ satisfying $\|\mathbf{x}_{[-h,0]}\| \leq \varepsilon$ it holds that $\mathbf{x}_{[-h,0]} \in \mathbb{C}_x$ and $\pi(\mathbf{x}_{[-h,0]}) \subseteq \mathbb{C}_u$. Thus, letting $\mathbb{C}_\pi := \{\mathbf{x}_{[-h,0]} \in \mathbb{C}_x : \pi(\mathbf{x}_{[-h,0]}) \subseteq \mathbb{C}_u\}$ it follows that $0 \in \text{int}(\mathbb{C}_\pi)$. Next, from (2.10a) it follows that there exists a $\gamma \in \mathbb{R}_{>0}$ such that $\mathbb{V}_\gamma := \{x_0 \in \mathbb{R}^n : V(x_0) \leq \gamma\}$ satisfies $\mathbb{V}_\gamma^{h+1} \subseteq \mathbb{C}_\pi$. Thus, letting $\mathcal{N} := \mathbb{V}_\gamma$ it follows that V is a cLRF($\mathcal{N}^{h+1}, \mathbb{C}_u$) with corresponding control law π, which asserts claim *(i)*.

Consider any $k \in \mathbb{Z}_+$. By Assumption 6.1 the set $\mathbb{C}_x$ is CCI($\mathbb{C}_u$) for the linear controlled DDI (2.15). Therefore, the constraints in (6.4a) are feasible for all $\mathbf{x}_{[k-h,k]} \in \mathbb{C}_x$. Moreover, by setting

$$\lambda_k := \sup_{\substack{\mathbf{x}_{[-h,0]} \in \mathbb{C}_x, \mathbf{u}_{[-h,0]} \in \mathbb{C}_u^{h+1} \\ x_1 \in f(\mathbf{x}_{[-h,0]}, \mathbf{u}_{[-h,0]})}} \left(V(x_1) - \max_{i \in \mathbb{Z}_{[-h,0]}} \rho V(x_i) \right)$$

in (6.4b), yields that (6.4b) is feasible for all $\mathbf{x}_{[k-h,k]} \in \mathbb{C}_x$. Note that the supremum exists due to boundedness of $\mathbb{C}_x$, $\mathbb{C}_u$, the assumption that the controlled DDI (2.15) is a linear controlled DDI, (2.10a) and continuity of the class $\mathcal{K}_\infty$ functions in (2.10a). Therefore,

Appendix B. Proofs of nontrivial results

Algorithm 6.1 is feasible for all $\mathbf{x}_{[-h,0]} \in \mathbb{C}_x$ and remains feasible for all $k \in \mathbb{Z}_+$. Furthermore, the second part of the claim follows from (6.4a) and hence, claim *(ii)* is proven.

To prove claim *(iii)*, consider any $k \in \mathbb{Z}_+$. Let $\hat{\rho} := \rho^{\frac{1}{h+1}}$ and, for any trajectory of (6.2) obtained from the control law (6.5), i.e., $\Phi \in \mathcal{S}(\mathbf{x}_{[-h,0]})$, let

$$i_k^*(\Phi) := \arg\max_{i \in \mathbb{Z}_{[-h,0]}} \hat{\rho}^{-(k+i)} V(\phi_{k+i}),$$
$$U_k(\Phi) := \max_{i \in \mathbb{Z}_{[-h,0]}} \hat{\rho}^{-(k+i)} V(\phi_{k+i}).$$

Next, it will be shown that $U_{k+1}(\Phi) \leq U_k(\Phi) + \hat{\rho}^{-(k+1)}\lambda_k^*$ for all $\mathbf{x}_{[-h,0]} \in \mathbb{C}_x$ and all $(\Phi, k) \in \mathcal{S}(\mathbf{x}_{[-h,0]}) \times \mathbb{Z}_+$. If $i_{k+1}^* = 0$, then (6.4b) yields that

$$U_{k+1}(\Phi) = \hat{\rho}^{-(k+1)} V(\phi_{k+1}) \leq \hat{\rho}^{-(k+1)} \left(\max_{i \in \mathbb{Z}_{[-h,0]}} \hat{\rho}^{h+1} V(\phi_{k+i}) + \lambda_k^* \right)$$
$$\leq U_k(\Phi) + \hat{\rho}^{-(k+1)} \lambda_k^*. \tag{B.24}$$

Furthermore, if $i_{k+1}^* \in \mathbb{Z}_{[-h,-1]}$ it holds, by definition of U_k, that

$$U_{k+1}(\Phi) = \max_{i \in \mathbb{Z}_{[-h,-1]}} \hat{\rho}^{-(k+i+1)} V(\phi_{k+i+1})$$
$$\leq \max_{i \in \mathbb{Z}_{[-h,0]}} \hat{\rho}^{-(k+i)} V(\phi_{k+i}) = U_k(\Phi). \tag{B.25}$$

Together (B.24) and (B.25) yield that $U_{k+1}(\Phi) \leq U_k(\Phi) + \hat{\rho}^{-(k+1)}\lambda_k^*$ for all $\mathbf{x}_{[-h,0]} \in \mathbb{C}_x$ and all $(\Phi, k) \in \mathcal{S}(\mathbf{x}_{[-h,0]}) \times \mathbb{Z}_+$. Furthermore, from claim *(ii)* it follows that Algorithm 6.1 is feasible for all $\mathbf{x}_{[-h,0]} \in \mathbb{C}_x$ and all $k \in \mathbb{Z}_+$ and hence, it is *recursively feasible*. As such, the inequality $U_{k+1}(\Phi) \leq U_k(\Phi) + \hat{\rho}^{-(k+1)}\lambda_k^*$ can be applied recursively, which yields

$$U_k(\Phi) \leq \max_{i \in \mathbb{Z}_{[-h,0]}} \hat{\rho}^{-h} V(x_i) + \sum_{i=0}^{k-1} \hat{\rho}^{-(i+1)} \lambda_i^*, \tag{B.26}$$

for all $k \in \mathbb{Z}_+$. Combining the definition of U_k and (B.26) yields

$$V(\phi_k) \leq \hat{\rho}^k U_k(\Phi) \leq \hat{\rho}^k \max_{i \in \mathbb{Z}_{[-h,0]}} \hat{\rho}^{-h} V(x_i) + \hat{\rho}^{-1} \sum_{i=0}^{k-1} \hat{\rho}^{k-i} \lambda_i^*, \tag{B.27}$$

which holds for all $\mathbf{x}_{[-h,0]} \in \mathbb{C}_x$ and all $(\Phi, k) \in \mathcal{S}(\mathbf{x}_{[-h,0]}) \times \mathbb{Z}_+$. Next, (2.10a), (B.27) and the inequality $\alpha_1^{-1}(r+s) \leq \alpha_1^{-1}(2\max\{r,s\}) \leq \alpha_1^{-1}(2r) + \alpha_1^{-1}(2s)$ yield

$$\|\phi_k\| \leq \alpha_1^{-1}(2\hat{\rho}^k \max_{i \in \mathbb{Z}_{[-h,0]}} \hat{\rho}^{-h} \alpha_2(x_i)) + \alpha_1^{-1}(2\hat{\rho}^{-1} \sum_{i=0}^{k-1} \hat{\rho}^{k-i} \lambda_i^*).$$

Moreover, it follows from the proof of claim *(ii)* that λ_k is bounded for all $k \in \mathbb{Z}_+$. Therefore, as $\hat{\rho} \in \mathbb{R}_{[0,1)}$ and $\lim_{k \to \infty} \lambda_k^* = 0$, Lemma B.2 yields that $\lim_{k \to \infty} \|\phi_k\| = 0$ for all $\mathbf{x}_{[-h,0]} \in \mathbb{C}_x$ and all $\Phi \in \mathcal{S}(\mathbf{x}_{[-h,0]})$. Now we observe that as the controlled DDI (2.15) is linear by assumption and the control law (6.5) is upper semicontinuous by construction, the above equation implies that the closed-loop system (6.2) obtained from the control law (6.5) is globally uniformly attractive (we make use of the fact that global attractivity is equivalent to global uniform attractivity for upper semicontinuous systems).

Global uniform attractivity of the origin for the closed-loop system (6.2) further implies that for all $\mathbf{x}_{[-h,0]} \in \mathbb{C}_x$ and all $\Phi \in \mathcal{S}(\mathbf{x}_{[-h,0]})$ there exists some finite $k^*(\mathbf{x}_{[-h,0]}) \in \mathbb{Z}_+$ such that $\Phi_{[k^*-h,k^*]} \in \mathcal{N}^{h+1}$. As V is a cLRF($\mathcal{N}^{h+1}, \mathbb{C}_u$), it follows from (2.10b) that,

for all $\Phi_{[k^*-h,k^*]} \in \mathcal{N}^{h+1}$, there exists a feasible solution to Algorithm 6.1 with $\lambda_{k^*} = 0$. Hence, it follows by optimality that solving Algorithm 6.1 yields $\lambda_k^* = 0$ for all $k \in \mathbb{Z}_{\geq k^*}$. Thus, it follows from Theorem 2.3 that the origin of (6.2) is LS, which completes the proof of claim *(iii)* and, hence, the theorem. ∎

Proof of Lemma 6.1

Let

$$\lambda(\mathbf{x}_{[-h,0]}) := \max\{0, \sigma(\max_{i \in \mathbb{Z}_{[-h,0]}} \rho_1 V_1(x_i)) - \max_{i' \in \mathbb{Z}_{[-h,0]}} \rho V(x_{i'})\}.$$

Using (2.10a) for both V_1 and V yields that

$$\lambda(\mathbf{x}_{[-h,0]}) \leq \max\left\{0, \sigma(\rho_1 \alpha_4(\|\mathbf{x}_{[-h,0]}\|)) - \rho \alpha_1(\|\mathbf{x}_{[-h,0]}\|)\right\}, \tag{B.28}$$

and hence that $\lambda(\mathbf{0}_{[-h,0]}) = 0$ and $\lambda(\mathbf{x}_{[-h,0]})$ is bounded on bounded sets. Moreover, using (2.10b) for V_1 yields

$$V(x_1) \leq \sigma(V_1(\bar{x}_1)) \leq \sigma\left(\max_{i \in \mathbb{Z}_{[-h,0]}} \rho_1 V_1(x_i)\right) \leq \max_{i \in \mathbb{Z}_{[-h,0]}} \rho V(x_i) + \lambda(\mathbf{x}_{[-h,0]}),$$

for all $\mathbf{x}_{[-h,0]} \in \mathbb{C}_x$ and all $(x_1, \bar{x}_1) \in F_\pi(\mathbf{x}_{[-h,0]}) \times F_{\pi_1}(\mathbf{x}_{[-h,0]})$. ∎

Proof of Theorem 6.2

Consider any $k \in \mathbb{Z}_+$ and $\mathbf{x}_{[k-h,k]} \in \mathbb{C}_x$. By optimality it follows from Lemma 6.1 that λ_k^* satisfies the upperbound in (B.28). Using (6.4b), the above observation and the bounds in (2.10a) for V yield

$$V(x_{k+1}) \leq \max_{i \in \mathbb{Z}_{[-h,0]}} \rho V(x_{k+i}) + \lambda_k^*$$
$$\leq \rho \alpha_2(\|\mathbf{x}_{[k-h,k]}\|) - \rho \alpha_1(\|\mathbf{x}_{[k-h,k]}\|) + \sigma(\rho_1 \alpha_4(\|\mathbf{x}_{[k-h,k]}\|)),$$

for all $x_{k+1} \in F_\pi(\mathbf{x}_{[k-h,k]})$. As Algorithm 6.1 is recursively feasible by Theorem 6.1-(ii), inequality (6.4b) can be applied recursively and as such, the above inequality can also be applied recursively. This together with (6.8) yields $\|\phi_k\| \leq \beta(\|\mathbf{x}_{[-h,0]}\|, k)$ for all $\mathbf{x}_{[-h,0]} \in \mathbb{C}_x$ and all $(\Phi, k) \in \mathcal{S}(\mathbf{x}_{[-h,0]}) \times \mathbb{Z}_+$. Hence, as $\beta \in \mathcal{KL}$ it follows that the closed-loop system (6.2) obtained from the control law (6.5) is $\mathcal{KL}$-stable. Furthermore, it follows from (6.4a) that for all $\mathbf{x}_{[-h,0]} \in \mathbb{C}_x$ the constraints (6.1) are satisfied. ∎

Proof of Theorem 6.3

The equivalence of the statements *(ii)* and *(iii)* follows directly from Lemma 2.2. Furthermore, the proof that *(i)* $\Rightarrow$ *(ii)* follows from standard Lyapunov arguments and is omitted.

Next, it is proven that *(ii)* $\Rightarrow$ *(i)*. To this end, consider the function

$$V_1(\xi_0) := \sup_{(\bar{\Phi}, k) \in \bar{\mathcal{S}}(\xi_0) \times \mathbb{Z}_+} \|\bar{\phi}_k\|_2^2 \mu^{-k}. \tag{B.29}$$

As the augmented system (2.2) is GES, there exist two constants $(c, \mu) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{[0,1)}$ such that $\|\bar{\phi}_k\|_2^2 \leq c\|\xi_0\|_2^2 \mu^k$ for all $\xi_0 \in \mathbb{R}^{(h+1)n}$ and all $(\bar{\Phi}, k) \in \bar{\mathcal{S}}(\xi_0) \times \mathbb{Z}_+$. Therefore

$$V_1(\xi_0) \geq \sup_{\bar{\Phi} \in \bar{\mathcal{S}}(\xi_0)} \|\bar{\phi}_0\|_2^2 = \|\xi_0\|_2^2, \tag{B.30a}$$

$$V_1(\xi_0) \leq \sup_{k \in \mathbb{Z}_+} c\|\xi_0\|_2^2 \mu^k \mu^{-k} = c\|\xi_0\|_2^2. \tag{B.30b}$$

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Furthermore, for any $\bar{\Phi} \in \bar{\mathcal{S}}(\xi_0)$ it holds that

$$
\begin{aligned}
V_1(\bar{\phi}_1) &= \sup_{(\bar{\Phi},k) \in \bar{\mathcal{S}}(\bar{\phi}_1) \times \mathbb{Z}_+} \|\bar{\phi}_k\|_2^2 \mu^{-k} \\
&\leq \sup_{(\bar{\Phi},k) \in \bar{\mathcal{S}}(\xi_0) \times \mathbb{Z}_{\geq 1}} \|\bar{\phi}_k\|_2^2 \mu^{-k+1} \leq \mu V_1(\xi_0).
\end{aligned}
\tag{B.31}
$$

As the DDI (2.1) is a linear DDI by assumption, it follows that the difference inclusion (2.2) is linear also and defined by a matrix polytope $\bar{\mathcal{A}}$. Moreover, as $\mathcal{A}$ is compact, the set $\bar{\mathcal{A}}$ is compact as well. Hence, it is possible to rewrite (B.29) into

$$
V_1(\xi_0) = \sup_{k \in \mathbb{Z}_+,\ \{\bar{A}_i\}_{i \in \mathbb{Z}_{[0,k]}} \in \bar{\mathcal{A}}^{k+1}} \| \textstyle\prod_{i=0}^k \bar{A}_i \xi_0 \|_2^2 \mu^{-k}.
$$

As the augmented system (2.2) is GES, there exists a bounded $k \in \mathbb{Z}_+$ for which the supremum in (B.29) is attained, i.e., denoted by k^*. Therefore, due to (B.30b) and the compactness of $\bar{\mathcal{A}}$, the supremum in (B.29) is a maximum. Hence, for each $\xi_0 \in \mathbb{R}^{(h+1)n}$ it follows that $V_1(\xi_0) = V(\xi_0, \bar{P})$ with $\bar{P}(\xi_0) := (\prod_{i=0}^{k'} \bar{A}'_i)^\top (\prod_{i=0}^{k'} \bar{A}'_i) \mu^{-k'}$ and where

$$
(k', \{\bar{A}'_i\}_{i \in \mathbb{Z}_{[0,k']}}) := \arg\max_{k \in \mathbb{Z}_+,\ \{\bar{A}_i\}_{i \in \mathbb{Z}_{[0,k]}} \in \bar{\mathcal{A}}^{k+1}} \| \textstyle\prod_{i=0}^k \bar{A}_i \xi_0 \|_2^2 \mu^{-k}.
$$

The above derivations imply that the hypothesis of statement *(i)* is satisfied with $\bar{c}_1 = 1$, $\bar{c}_2 = c \in \mathbb{R}_{\geq 1}$ and $\bar{\rho} = \mu \in \mathbb{R}_{[0,1)}$, which completes the proof. ∎

Proof of Theorem 6.4

The equivalence of the statements *(ii)* and *(iii)* follows directly from Lemma 2.2. Furthermore, the proof that *(i)* $\Rightarrow$ *(ii)* follows from standard Lyapunov arguments and is omitted.

Next, it is proven that *(ii)* $\Rightarrow$ *(i)*. As the augmented difference inclusion (2.2) is GES it admits a state-dependent function $\bar{V}$ that satisfies the conditions in statement *(i)* of Theorem 6.3. The remainder of the proof relies on the construction of a state-dependent function V that equals $\bar{V}$ except for some particular sequences of states. Therefore, consider any $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$, let $\xi_0 := \mathrm{col}(\{x_i\}_{i \in \mathbb{Z}_{[-h,0]}})$ and let $\bar{\Phi} \in \bar{\mathcal{S}}(\xi_0)$ correspond to $\Phi \in \mathcal{S}(\mathbf{x}_{[-h,0]})$. Furthermore, recall that (B.1) implies that $\|\xi_0\|_2^2 \leq c_3 \|\mathbf{x}_{[-h,0]}\|_2^2$ for some $c_3 \in \mathbb{R}_{>0}$.

Using all of the above we will define the function $P : (\mathbb{R}^n)^{h+1} \to \mathbb{R}^{n \times n}$ which gives rise to the state-dependent LRF $V(x_0, P)$. Therefore, for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ such that $\|\mathbf{x}_{[-h,0]}\|_2^2 \neq 0$ and all corresponding $\xi_0 := \mathrm{col}(\{x_i\}_{i \in \mathbb{Z}_{[-h,0]}})$, define

$$
P(\mathbf{x}_{[-h,0]}) := \begin{cases} \dfrac{\bar{V}(\xi_0, \bar{P})}{\|x_0\|_2^2} I_n, & x_0 \in \mathbb{R}^n \setminus \mathcal{N}(\mathbf{x}_{[-h,0]}) \\[2mm] \bar{c}_2 c_3 I_n, & x_0 \in \mathcal{N}(\mathbf{x}_{[-h,0]}) \end{cases},
$$

and, for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ such that $\|\mathbf{x}_{[-h,0]}\|_2^2 = 0$, define $P(\mathbf{x}_{[-h,0]}) := \bar{c}_2 c_3 I_n$. Above, $\mathcal{N}(\mathbf{x}_{[-h,0]}) := \{x_0 \in \mathbb{R}^n : \|x_0\|_2^2 < \frac{1}{\bar{c}_2 c_3} \|\xi_0\|_2^2\}$.

Consider any $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$. If $\|\mathbf{x}_{[-h,0]}\|_2^2 \neq 0$ and $x_0 \in \mathbb{R}^n \setminus \mathcal{N}(\mathbf{x}_{[-h,0]})$, then (6.12a) and the definition of $\mathcal{N}$ yield

$$
\bar{c}_1 I_n \preceq \frac{\bar{c}_1 \|\xi_0\|_2^2}{\|x_0\|_2^2} I_n \preceq P(\mathbf{x}_{[-h,0]}) \preceq \frac{\bar{c}_2 \|\xi_0\|_2^2}{\|x_0\|_2^2} I_n \preceq (\bar{c}_2)^2 c_3 I_n.
$$

Moreover, if $x_0 \in \mathcal{N}(\mathbf{x}_{[-h,0]})$, then $\bar{c}_2 c_3 I_n \preceq P(\mathbf{x}_{[-h,0]}) \preceq \bar{c}_2 c_3 I_n$. Therefore, the function P is well-defined for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and satisfies (6.13a) with $c_1 = \min\{\bar{c}_1, \bar{c}_2 c_3\} \in \mathbb{R}_{>0}$ and $c_2 = \max\{(\bar{c}_2)^2 c_3, \bar{c}_2 c_3\} \in \mathbb{R}_{\geq c_1}$.

Next, consider the state-dependent function $V(x_0, P) := x_0^\top P(\mathbf{x}_{[-h,0]}) x_0$ defined by the function P as given above. In what follows, it will be shown, using three different cases, that the function V satisfies (6.13b). Firstly, consider any $k \in \mathbb{Z}_{\geq h}$ and suppose that $\phi_{k+i^*} \notin \mathcal{N}(\bar{\phi}_{k+i^*})$ for some $i^* \in \mathbb{Z}_{[-h,0]}$. Then,

$$V(\phi_{k+1}, P) \leq \bar{V}(\bar{\phi}_{k+1}, \bar{P}) \leq \bar{\rho}\bar{V}(\bar{\phi}_{k+i^*}, \bar{P}) = \bar{\rho}V(\phi_{k+i^*}, P)$$
$$\leq \bar{\rho} \max_{i \in \mathbb{Z}_{[-h,0]}} V(\phi_{k+i}, P),$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $\Phi \in \mathcal{S}(\mathbf{x}_{[-h,0]})$. Alternatively, consider any $k \in \mathbb{Z}_{\geq h}$ and suppose that $\phi_{k+i} \in \mathcal{N}(\bar{\phi}_{k+i})$ for all $i \in \mathbb{Z}_{[-h,0]}$. Then,

$$V(\phi_{k+1}, P) \leq \bar{V}(\bar{\phi}_{k+1}, \bar{P}) \leq \bar{\rho}\bar{V}(\bar{\phi}_k, \bar{P}) \leq \bar{\rho}\bar{c}_2 \|\bar{\phi}_k\|_2^2$$
$$\leq \bar{\rho}\bar{c}_2 c_3 \max_{i \in \mathbb{Z}_{[-h,0]}} \|\phi_{k+i}\|_2^2 = \bar{\rho} \max_{i \in \mathbb{Z}_{[-h,0]}} V(\phi_{k+i}, P),$$

for all $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $\Phi \in \mathcal{S}(\mathbf{x}_{[-h,0]})$. Thirdly, let $k \in \mathbb{Z}_{[0,h-1]}$. Then, for $k = 0$, (6.13b) follows from the second case considered above. Moreover, for each $k \in \mathbb{Z}_{[1,h-1]}$, (6.13b) follows from the second case if $\phi_{k+i} \in \mathcal{N}(\bar{\phi}_{k+i})$ for all $i \in \mathbb{Z}_{[-k+1,0]}$ and (6.13b) follows from the first case otherwise. Hence, V satisfies (6.13b) with $\rho = \bar{\rho} \in \mathbb{R}_{[0,1)}$. ∎

Proof of Proposition 6.2

Let $\hat{\rho} := \rho^{\frac{1}{h+1}}$. As, for all $\mathbf{x}_{[-h,0]} \in \mathbb{C}_x$ and all $\Phi \in \mathcal{S}(\mathbf{x}_{[-h,0]})$, Algorithm 6.2 is recursively feasible by assumption, it follows from (6.14a) that $\Phi_{[k-h,k]} \in \mathbb{C}_x$ and that $u_k \in \mathbb{C}_u$ for all $\mathbf{x}_{[-h,0]} \in \mathbb{C}_x$ and all $(\Phi, k) \in \mathcal{S}(\mathbf{x}_{[-h,0]}) \times \mathbb{Z}_+$. Moreover, (6.14c) can be applied recursively, which yields

$$V(\phi_k, P_k) \leq \hat{\rho}^k \max_{i \in \mathbb{Z}_{[-h,0]}} V(x_i, P_i).$$

Thus, using that $P_k := c_2 I_n$ for all $k \in \mathbb{Z}_{[-h,0]}$, it follows from (6.14b) that

$$\|\phi_k\|_2^2 \leq \hat{\rho}^k \frac{c_2}{c_1} \|\mathbf{x}_{[-h,0]}\|_2^2,$$

for all $\mathbf{x}_{[-h,0]} \in \mathbb{C}_x$ and all $(\Phi, k) \in \mathcal{S}(\mathbf{x}_{[-h,0]}) \times \mathbb{Z}_+$. Therefore, the closed-loop system (6.2) obtained from the control law (6.15) is GES with $\mu = \hat{\rho}^{\frac{1}{2}} \in \mathbb{R}_{[0,1)}$ and $c = (\frac{c_2}{c_1})^{\frac{1}{2}} \in \mathbb{R}_{\geq 1}$, which completes the proof. ∎

Proof of Proposition 6.3

It follows from (6.17a) and Lemma 6.2 that (6.14a) holds. Moreover, (6.17b) yields that $\gamma I_n \preceq Z_{k+1}^{-1} = P_{k+1} \preceq \Gamma I_n$, which implies that (6.14b) holds with $c_1 = \gamma$ and $c_2 = \Gamma$. Applying the Schur complement to (6.17c) yields

$$(Bu_k + v)^\top Z_{k+1}^{-1}(Bu_k + v) - \rho \max_{i \in \mathbb{Z}_{[-h,0]}} x_{k+i}^\top P_{k+i} x_{k+i} \leq 0,$$

which in view of Lemma 6.2 implies that (6.14c) holds with $P_{k+1} = Z_{k+1}^{-1}$. ∎

B.6   Proofs of Chapter 7

Proof of Theorem 7.1

The proof consists of three parts. In the first part the interconnected system with interconnection delay is transformed into an augmented interconnected system. Therefore, consider the following procedure. Define, for all $i \in \mathbb{Z}_{[1,N]}$, the subsystem dynamics $\hat{G}_i(x_{1,0}, \ldots, x_{N+\bar{h},0}) := G_i(x_{1,0}, \ldots, x_{N,0})$ and let $\bar{h} := \sum_{i=1}^{N} \sum_{j=1}^{N} h_{i,j}$. Furthermore, let $I := 1$ and $J := 2$. If $h_{I,J} \geq 1$ let

$$\hat{G}_{N+1}(x_{1,0}, \ldots, x_{N+\bar{h},0}) := x_J.$$

Note that, the above definition corresponds to the case where $x_{N+1,k} := x_{J,k-1}$ for all $k \in \mathbb{Z}_+$. Furthermore, if $h_{I,J} \geq 2$ let

$$\hat{G}_{N+l+1}(x_{1,0}, \ldots, x_{N+\bar{h},0}) := x_{N+l,0},$$

for all $l \in \mathbb{Z}_{[1,h_{I,J}-1]}$. Repeat this procedure for all $(I, J) \in \mathbb{Z}_{[1,N]}^2$ such that the interconnection delays in (7.1) are replaced by new subsystems. Thus, an augmented interconnection of $N + \bar{h}$ subsystems is obtained, i.e.,

$$x_{i,k+1} = \hat{G}_i(x_{1,k}, \ldots, x_{N+\bar{h},k}), \quad k \in \mathbb{Z}_+, \tag{B.32}$$

with $i \in \mathbb{Z}_{[1,N+\bar{h}]}$. Let $x_0 := \mathrm{col}(\{x_{l,0}\}_{l \in \mathbb{Z}_{[1,N+\bar{h}]}})$, which yields

$$x_{k+1} = \hat{G}(x_k), \quad k \in \mathbb{Z}_+, \tag{B.33}$$

where $\hat{G}(x_0) = \mathrm{col}(\{\hat{G}_i(x_{1,0}, \ldots, x_{N+\bar{h},0})\}_{i \in \mathbb{Z}_{[1,N+\bar{h}]}})$.

In the second part of the proof it is shown that the functions W_i satisfy (3.3) for all $i \in \mathbb{Z}_{[1,N+\bar{h}]}$. Moreover, it is also shown that, if the subsystems without interconnection delay (3.1) admit a set of functions and gains that satisfy the second item of the hypothesis of Theorem 3.1, then the subsystems (B.32) corresponding to the augmented interconnected system (B.33) also admit such a set of functions and gains. It follows from the hypothesis of Theorem 3.1 that the functions W_i satisfy (3.3) for the subsystems (B.32) *for all $i \in \mathbb{Z}_{[1,N]}$*. Furthermore, if $h_{1,2} \geq 1$, then it follows, by definition of $x_{N+1,k}$, that

$$W_{N+1}(\hat{G}_{N+1}(x_{1,0}, \ldots, x_{N+\bar{h},0})) = W_2(x_{2,0}).$$

Hence, let $W_{N+1}(x_{N+1,0}) := W_2(x_{N+1,0})$ for all $x_{N+1,0} \in \mathbb{R}^{n_{N+1}} = \mathbb{R}^{n_2}$. If $h_{1,2} \geq 2$, then it follows, by definition of $x_{N+l+1,k}$, that

$$W_{N+l+1}(\hat{G}_{N+l+1}(x_{1,0}, \ldots, x_{N+\bar{h},0})) = W_{N+l}(x_{N+l,0}),$$

for all $l \in \mathbb{Z}_{[1,h_{I,J}-1]}$. Hence, define the function $W_{N+l+1}(x_{N+l+1,0}) := W_2(x_{N+l+1,0})$ for all $x_{N+l+1,0} \in \mathbb{R}^{n_{N+l+1}} = \mathbb{R}^{n_2}$. Thus, it follows that W_i satisfies (3.3) with respect to the subsystems (B.32) for all $i \in \mathbb{Z}_{[N+1,N+\bar{h}]}$. Next, the corresponding gain functions $\gamma_{i,j}$ are defined recursively and it is shown that they satisfy the third item of the hypothesis of Theorem 3.1. Therefore, let $\gamma_{i,j}^0(r) := \gamma_{i,j}(r), r \in \mathbb{R}_+$ for all $(i, j) \in \mathbb{Z}_{[1,N]}^2$. Furthermore,

let $(I, J) \in \mathbb{Z}^2_{[1,N+l]}$ correspond to the interconnection with delay between subsystem J and I for which the new state $x_{N+l+1,0}$ was introduced. Then, for all $r \in \mathbb{R}_+$ define

$$\gamma^{l+1}_{i,j}(r) := \begin{cases} 0, & i = I,\, j = J \text{ or } i \neq I,\, j = N+l+1 \\ & \text{or } i = N+l+1,\, j \neq J \\ \gamma^l_{I,J}(r), & i = I,\, j = N+l+1 \\ r, & i = N+l+1,\, j = J \\ \gamma^l_{i,j}(r), & \text{otherwise}, \end{cases}$$

for all $l \in \mathbb{Z}_{[0,\bar{h}-1]}$ and all $(i,j) \in \mathbb{Z}^2_{[1,N+l+1]}$. In what follows, we prove, by induction, that for all $l \in \mathbb{Z}_{[1,\bar{h}]}$ and all $y \in \mathbb{R}^{N+l}_+ \setminus \{0\}$, there exists an $i(y) \in \mathbb{Z}_{[1,N+l]}$ such that

$$\max_{j \in \mathbb{Z}_{[1,N+l]}} \gamma^l_{i(y),j}([y]_j) < [y]_{i(y)}. \tag{B.34}$$

Therefore, choose $l = 0$ and let $y := \operatorname{col}(\bar{y}, \tilde{y})$ for any $\bar{y} \in \mathbb{R}^N_+$ and $\tilde{y} \in \mathbb{R}_+$ such that $y \neq 0$. If $\tilde{y} \leq [\bar{y}]_J$, then it follows from the second item of the hypothesis of Theorem 3.1 that (B.34) with $l = 1$ holds for $i(y) = i(\bar{y})$. Conversely, if $\tilde{y} > [\bar{y}]_J$, then

$$\max_{j \in \mathbb{Z}_{[1,N+1]}} \gamma^1_{N+1,j}([y]_j) = [\bar{y}]_J < \tilde{y} = [y]_{N+1}.$$

Thus, it has been established that (B.34) with $l = 1$ holds either with $i(y) = N+1$ or with $i(y) = i(\bar{y})$. Next, consider any $\ell \in \mathbb{Z}_{[0,\bar{h}-1]}$ and suppose that (B.34) with $l = \ell$ holds, i.e., for all $\bar{y} \in \mathbb{R}^{N+\ell}_+$ there exists some $i(\bar{y})$ such that (B.34) holds. Let $y := \operatorname{col}(\bar{y}, \tilde{y})$ for any $\bar{y} \in \mathbb{R}^{N+\ell}_+$ and $\tilde{y} \in \mathbb{R}_+$ such that $y \neq 0$. If $\tilde{y} \leq [\bar{y}]_J$, then it follows from (B.34) with $l = \ell$ that (B.34) with $l = \ell + 1$ also holds for $i(y) = i(\bar{y})$. Conversely, if $\tilde{y} > [\bar{y}]_J$, then

$$\max_{j \in \mathbb{Z}_{[1,N+\ell+1]}} \gamma^{\ell+1}_{N+\ell+1,j}([y]_j) = [\bar{y}]_J < \tilde{y} = [y]_{N+\ell+1}.$$

Hence, (B.34) with $l = \ell + 1$ holds either with $i(y) = N + \ell + 1$ or with $i(y) = i(\bar{y})$. Thus, it has been established, by induction, that (B.34) holds for any $l \in \mathbb{Z}_{[0,\bar{h}]}$. Therefore, the functions and gains $(W_i, \gamma_{i,j})$ corresponding to the subsystems (B.32) satisfy the second item of the hypothesis of Theorem 3.1 and, hence, it follows from Theorem 3.1 that the augmented interconnected system (B.33) is $\mathcal{KL}$-stable.

What remains to be shown is that the interconnected system with interconnection delay, i.e., the DDI (2.1) obtained from (7.1), is $\mathcal{KL}$-stable if the augmented interconnected system (B.33) obtained from (B.32) is $\mathcal{KL}$-stable. As the proof of this claim is similar to the proof of Lemma 2.2, it is omitted here. ∎

Proof of Theorem 7.2
The proof of Theorem 7.2 is similar to the proof of Theorem 3.1. It follows from the second item of the hypothesis of the theorem that the functions $\gamma_{i,j}$ satisfy the hypothesis corresponding to claim *(iii)* of Theorem 5.2 in [32]. Therefore, there exist $\sigma_i \in \mathcal{K}_\infty$, $i \in \mathbb{Z}_{[1,N]}$ such that (B.17) holds for all $r \in \mathbb{R}_{>0}$ and all $i \in \mathbb{Z}_{[1,N]}$. Let

$$\gamma(r) := \max_{i \in \mathbb{Z}_{[1,N]}} \max_{j \in \mathbb{Z}_{[1,N]}} \sigma^{-1}_i \circ \gamma_{i,j} \circ \sigma_j(r), \quad r \in \mathbb{R}_+.$$

Appendix B. Proofs of nontrivial results

Then, it follows from (B.17) that $\gamma(r) < r$ for all $r \in \mathbb{R}_{>0}$. Furthermore, $\gamma \in \mathcal{K}_\infty \cup \{0\}$. Now, define $V(x_0) := \max_{i \in \mathbb{Z}_{[1,N]}} \sigma_i^{-1}(V_i(x_{i,0}))$ and choose any $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$. Then, it follows from (7.3b) that

$$
\begin{aligned}
V(x_1) &= \max_{i \in \mathbb{Z}_{[1,N]}} \sigma_i^{-1}(V_i(x_{i,1})) \\
&\leq \max_{i \in \mathbb{Z}_{[1,N]}} \sigma_i^{-1} \circ \max\{\max_{j' \in \mathbb{Z}_{[-h,0]}} \gamma_{i,i}(V_i(x_{i,j'})), \max_{j \in \mathbb{Z}_{[1,N]}, j \neq i} \gamma_{i,j} \circ V_j(x_{j,0})\} \\
&\leq \max_{j' \in \mathbb{Z}_{[-h,0]}} \max_{i \in \mathbb{Z}_{[1,N]}} \max_{j \in \mathbb{Z}_{[1,N]}} \sigma_i^{-1} \circ \gamma_{i,j} \circ \sigma_j \circ \sigma_j^{-1}(V_j(x_{j,j'})) \\
&\leq \max_{j' \in \mathbb{Z}_{[-h,0]}} \gamma \circ \max_{i' \in \mathbb{Z}_{[1,N]}} \sigma_{i'}^{-1}(V_{i'}(x_{i',j'})) \\
&= \max_{j' \in \mathbb{Z}_{[-h,0]}} \gamma(V(x_{j'})),
\end{aligned}
$$

for all $x_{i,1} \in F_i(\mathbf{x}_{i,[-h,0]}, x_{1,0}, \ldots, x_{N,0})$ and hence all $x_1 \in F(\mathbf{x}_{[-h,0]})$, which implies that the Lyapunov decrease condition in Theorem 3.3 holds with $\rho(r) := \gamma(r)$, $r \in \mathbb{R}_+$. Moreover, it follows from (B.17) that the second item of the hypothesis of Theorem 3.3 holds as well. Furthermore, it follows from (B.1)-(B.3) that there exist some $(c_1, c_2) \in \mathbb{R}_{>0}^2$ such that $c_1 \max_{i \in \mathbb{Z}_{[1,N]}} \|x_{i,0}\| \leq \|x_0\| \leq c_2 \max_{i \in \mathbb{Z}_{[1,N]}} \|x_{i,0}\|$ for all $x_0 \in \mathbb{R}^n$. Hence, the $\mathcal{K}_\infty$ bounds for the functions V_i, $i \in \mathbb{Z}_{[1,N]}$ yield

$$
\min_{i \in \mathbb{Z}_{[1,N]}} \sigma_i^{-1} \circ \alpha_1(\tfrac{1}{c_2}\|x_0\|) \leq V(x_0) \leq \max_{i \in \mathbb{Z}_{[1,N]}} \sigma_i^{-1} \circ \alpha_2(\tfrac{1}{c_1}\|x_0\|).
$$

Therefore, V satisfies the hypothesis of Theorem 3.3 and it follows from Theorem 3.3 that the interconnected system with local delay (2.1) obtained from (7.2) is $\mathcal{KL}$-stable. ∎

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> "*Happiness is only real when shared*"
>
> Chris McCandless

As I believe this quote to be true, it is clear that I am very grateful to those with whom I have shared the past four happy years of my life.

Rob Gielen,
Eindhoven, December 6, 2012.

Curriculum Vitae

Rob H. Gielen was born in Nijmegen, The Netherlands on June 9, 1984. He received his M.Sc. degree (cum laude) in Electrical Engineering from the Eindhoven University of Technology, Eindhoven, The Netherlands in 2009. Thereafter, he started working towards a Ph.D. degree in the Control Systems group of the Electrical Engineering Faculty at the same institution. In 2011 he was awarded a Fulbright Fellowship to visit the Electrical and Computer Engineering Department of the University of California in Santa Barbara where he spent four months in the end of 2011.

His research interests include the stability analysis and control of discrete-time systems with delay and he has studied the application of these techniques for networked control systems, automotive systems and electrical power systems. Recently, he also started working on the theory of large-scale interconnected systems.