Endogeneously arising network allocation rules

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Abstract

In this paper we study endogenously arising network allocation rules. We focus on three allocation rules: the Myerson value, the position value and the component-wise egalitarian solution. For any of these three rules we provide a characterization based on component efficiency and some balanced contribution property. Additionally, we present three mechanisms whose equilibrium payoffs are well defined and coincide with the three rules under consideration if the underlying value function is monotonic. Nonmonotonic value functions are shown to deal with allocation rules applied to monotonic covers. The mechanisms are inspired by the implementation of the Shapley value by Pérez-Castrillo and Wettstein (2001). We conclude with some comments on this implementation of the Shapley value.

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1 Introduction

An important characteristic in many social and economic situations is the organization of communication or cooperation. It is commonly observed that direct relationships exist, but since not everyone interacts directly with anyone else, indirect relationships play an important role as well.

Networks have proven to be an important modeling framework for social and economic situations. Recent years have provided an increasing amount of literature on this subject. Increasingly, the literature moves away from the original setting of Myerson (1977), who considered communication situations consisting of a cooperative game with transferable utilities supplemented with a network, and moves to the setting popularized by Jackson and Wolinsky (1996). This latter setting takes a value function, which assigns a value to any possible network on a fixed set of players, as starting point. It is commonly understood that the latter setting is richer.

Starting with Myerson (1977) the literature on communication situations mainly focused on allocation rules or, more specifically, on the axiomatic foundations of allocation rules for these situations. Myerson (1977) introduced and characterized an allocation rule, which is now called the Myerson value. Later on, alternative characterizations, some valid on restricted sets of communication situations only, were given in Myerson (1980) and Borm et al. (1992).

The main contribution of Meessen (1988) and Borm et al. (1992) was the introduction of an alternative rule for communication situations. This rule is called the position value. Borm et al. (1992) provided a characterization of this rule for communication situations with trees as the underlying graphs only. A general characterization of the position value was recently given by Slikker (2005a). Yet another rule for communication situations has been introduced by Hamiache (1999). Some deficiencies in this paper were recently pointed out and addressed by Bilbao et al. (2006).

Jackson and Wolinsky (1996) focused on the tension between efficiency and stability in networks. Besides this they introduced an allocation rule, a straightforward adaptation of the Myerson value for communication situations, which is simply called the Myerson value as well. Following this work, a vast amount of literature concentrated on the relations between efficiency and stability.

The existence of networks that are stable against changes in links by coalitions in investigated by Jackson and van den Nouweland (2005). They show that the existence of so-called strongly stable networks is equivalent to nonemptiness of the core of an associated cooperative game. Moreover, they show that to investigate the existence of such strongly stable networks one can restrict attention to the value that equally divides the value of a component, i.e., the component-wise egalitarian solution.

For extensive surveys on the subjects described so far, we refer to Slikker and van den Nouweland (2001), who focus on cooperative axiomatic approaches and network formation issues, Dutta and Jackson (2003), containing several state-of-the-art papers, and Jackson (2005b), containing a recent survey focusing on efficiency and stability.

Recently, Jackson (2005a) took a slightly different view in the analysis of network allocation rules by focusing more on the integration between network formation and payoff division rather than assuming some exogenously given network. He introduces and characterizes several new allocation rules and subsequently presents a comparison of these rules. He ends with the question where these allocation rules come from, i.e., he wonders about non-cooperative procedures resulting in the same payoffs as allocation rules.

This last approach has, in the recent literature on cooperative games, been influenced much by the work of Pérez-Castrillo and Wettstein (2001). They presented an implementation of the Shapley value. Follow-ups of this paper considered comparable mechanisms, focusing on efficient outcomes in economic environments (Mutuswami et al. (2004)), the Owen value (Vidal-Puga and Bergantiños (2003)), and networks (Pérez-Castrillo and Wettstein (2005)). In Pérez-Castrillo and Wettstein (2005) the player-based flexible network allocation rule of Jackson (2005a) results. In the concluding section we will come back to the relation with our work.

This paper focuses on three allocation rules in a setting with value functions: the Myerson value, the position value, and the component-wise egalitarian solution. We allow for a solid comparison between the three rules. First, by introducing comparable characterizations, subsequently, by comparable non-cooperative bargaining procedures resulting in the same payoffs as these rules.

The setup of this paper is as follows. Preliminaries on games and networks are treated in Section 2. Section 3 provides three comparable characterizations of the three allocation rules. Subsequently, in Section 4 we provide recursive formulas for the allocation rules. Three comparable mechanisms are presented and analyzed in Section 5. In Section 6 we adjust the mechanisms to deal with nonmonotonic value functions. We conclude in Sections 7 and 8 with some comments on Pérez-Castrillo and Wettstein (2001) and concluding remarks, respectively.

2 Preliminaries

In this section we present notation and definitions.

By $N = \{1, \ldots, n\}$ we denote a set of players. This set will generally be fixed. For convenience we denote $S - i = S \backslash \{i\}$ for any set and any player i. Furthermore, we denote $S - T = S \backslash T$ for any pair of sets S, T.

By g we denote a network, i.e., a set of unordered pairs of players (in N). If $\{i, j\} \in g$ we say that players i and j are adjacent. For notational convenience we sometimes write ij rather than $\{i, j\}$. The set of all unordered pairs within N is denoted by g^N. The set of all networks on N is denoted by $G^N = \{g \mid g \subseteq g^N\}$.

The network that results if link ij is added to g is denoted by $g + ij$. If link ij is deleted from g this is denoted by $g - ij$. Adding or deleting a set of links g' is denoted similarly by $g + g'$ and $g - g'$, respectively.

By $N(g)$ we denote the set of players that are involved in at least one link according to g, i.e., $N(g) = \{i \mid \exists j \in N : ij \in g\}$. We abuse notation and denote the links in g in which player i is involved by g_i. Furthermore, we denote the set of links in g that are a subset of $S \subseteq N$ by $g(S) = \{\{i, j\} \in g \mid i, j \in S\}$. The complete network on S is denoted by $g^S = \{\{i, j\} \mid i, j \in S; i \neq j\}$.

A path in g is a sequence of players $(i_1, \ldots, i_K)$ with $i_k, i_{k+1} \in g$ for all $k \in \{1, \ldots, K-1\}$. A path $(i_1, \ldots, i_K)$ is also called a path between i_1 and i_K. If there exists a path between i_1 and i_K we say that i_1 and i_K are connected. The notion of connectedness induces a partition of the set of players. This partition is denoted by N/g. The elements of this partition are called components. Obviously, two players are in the same component if and only if there is a path between them. Similarly, $S \subseteq N$ is partitioned into components by $g(S)$. The resulting partition is not only denoted by $S/g(S)$ but also by S/g. If there is no ambiguity about g we denote the component that contains player $i \in N$ by C_i.

A value function $v : G^N \to \mathbb{R}$ specifies the total value that is generated by the players in any network structure. We assume that no cooperation results in no value, i.e., $v(\emptyset) = 0$. The set of all value functions (on N) is denoted by V. A value function is called component additive if $v(g) = \sum_{C \in N/g} v(g(C))$, i.e., there are no externalities across components of a network. A value function v is called monotonic if $v(g) \geq v(g')$ for all $g' \subset g$.

Throughout this work we restrict attention to value functions that are component additive. Unless stated explicitly, we restrict ourselves to monotonic value functions as well.

For any value function v on player set N and any network $g \in G^N$ we denote the restriction of v to subsets of g by $v_{|g}$. Furthermore, we slightly abuse notation and denote for any $S \subseteq N$ the restriction of v to networks on S by $v_{|S} = v_{|g(S)}$.

A pair (N, v) with set of players N and value function v is called a network game, a function $\gamma : G^N \times V \to \mathbb{R}^N$ is called an allocation rule, and a pair (g, v) consisting of a network and a value function is simply called a situation.

Unanimity value function u_g is defined by

$$u_g(g') = \begin{cases} 1 & \text{if } g \subseteq g'; \\ 0 & \text{otherwise.} \end{cases}$$

Unanimity value functions are a basis for the set of value functions. A value function can be written as a unique linear combination of unanimity value functions, $v = \sum_{g \subseteq g^N} \alpha_g u_g$. The coefficients α_g, $g \subseteq g^N$ are called the unanimity coefficients of v.

We will describe three allocation rules. The first allocation rule is the Myerson value, a generalization of the Myerson value for communication situations:

$$\mu_i(g, v) := \sum_{A \subseteq g:\ A_i \neq \emptyset} \frac{\alpha_A}{|N(A)|} \quad \text{for all } i \in N. \tag{1}$$

The Myerson value was introduced for communication situations by Myerson (1977). Its extension to value functions as defined here, was introduced by Jackson and Wolinsky (1996). The current definition in terms of unanimity coefficients follows from the description of the Shapley value in terms of unanimity coefficients and the fact that the Myerson value of (g, v) equals the Shapley value of (N, v^g), where $v^g(S) = v(g(S))$ (cf. Slikker (2005b)).

The second allocation rule is the position value. Let v be a value function with unanimity coefficients $(\alpha_A)_{A \subseteq g^N}$. Then the position value $\pi(g, v)$ is defined by

$$\pi_i(g, v) = \sum_{A \subseteq g} \frac{\alpha_A |A_i|}{2|A|} = \sum_{A \subseteq g} \sum_{l \in A_i} \frac{\alpha_A}{2|A|} \quad \text{for all } \ i \in N. \tag{2}$$

This position value is a natural extension of the position value for cooperative games, introduced by Borm et al. (1992), and discussed by Slikker (2005b). Recall that $\Phi_l(g, v_{|g}) = \sum_{A \subseteq g:\ l \in A} \frac{\alpha_A}{|A|}$, where Φ denotes the Shapley value. Hence,

$$\pi_i(g, v) = \sum_{l \in g_i} \frac{1}{2} \Phi_l(g, v_{|g}) \quad \text{for all} \quad i \in N.$$

The third allocation rule is the component-wise egalitarian solution γ^{CE}, for any v, any g, any $C \in N/g$, and any $i \in C$ defined by

$$\gamma_i^{CE}(g, v) = \frac{v(g(C))}{|C|}.$$

3 Three characterizations

In this section we will provide a characterization for each of the allocation rules that were introduced in the previous section.

The characterizations are in the spirit of the characterization of the Myerson value for communication situations with component efficiency and balanced contributions that is attributed to Myerson (1980) and the characterization of the position value for communication situations with component efficiency and balanced link contributions of Slikker (2005a).

Consider the following properties for an allocation rule φ defined on a class of situations D, where each element is a network-value function-pair:

Component efficiency (CE): For all $(g, v) \in D$ and all $C \in N/g$,

$$\sum_{i \in C} \varphi_i(g, v) = v(g(C)). \tag{3}$$

Balanced contributions (BC): For all $(g, v) \in D$ and all $i, j \in N$ it holds that

$$\varphi_i(g, v) - \varphi_i(g - g_j, v,) = \varphi_j(g, v) - \varphi_j(g - g_i, v,). \tag{4}$$

Balanced link contributions (BLC): For all $(g, v) \in D$ and all $i, j \in N$,

$$\sum_{l \in g_j} \Big[\varphi_i(g, v) - \varphi_i(g - l, v)\Big] = \sum_{l \in g_i} \Big[\varphi_j(g, v) - \varphi_j(g - l, v)\Big]. \tag{5}$$

Balanced component contributions (BCC): For all $(g, v) \in D$ and all $i, j \in N$,

$$\varphi_i(g, v) - \varphi_i(g - g(C_j), v) = \varphi_j(g, v) - \varphi_j(g - g(C_i), v). \tag{6}$$

The first property is standard and dates back to Myerson (1977). *Balanced contributions* is a straightforward extension of the balanced contributions property for communication situations, cf. Myerson (1980), and deals with the contribution of a player to the payoff of another player, which is measured by the payoff difference a player experiences if all the links of the other player are removed. According to *balanced contributions* such a contribution should be equal to the reverse contribution. *Balanced link contributions* is similar, however, now the total contribution of a player to the payoff of another player is defined as the sum over all links of the first player of the payoff difference the second player experiences if such a link is broken. Finally, *Balanced component contributions* is similar as well, but now the contribution of a player to the payoff of another player is defined as the difference in payoff of the second player if all links in the component of the first player break down.

The following theorem provides characterizations of the Myerson value, the position value, and the component-wise egalitarian solution. Each of the characterizations involve component efficiency and a property that deals with balancedness of certain contributions.

Theorem 3.1 Let D be the set of network-value function-pairs. Then

1. The Myerson value is the unique allocation rule on D that satisfies *component efficiency* and *balanced contributions*.
2. The position value is the unique allocation rule on D that satisfies *component efficiency* and *balanced link contributions*.
3. The component-wise egalitarian solution is the unique allocation rule on D that satisfies *component efficiency* and *balanced component contributions*.

Proof: The first part is a straightforward extension of the result of Myerson (1980). The second part is a straightforward extension of a similar characterization of the position value for communication situations of Slikker (2005a). It remains to show part 3.

First we will show that γ^{CE} satisfies the two properties. Let $(g, v) \in D$ and let $C \in N/g$. Then

$$\sum_{i \in C} \gamma_i^{CE}(g, v) = \sum_{i \in C} \frac{v(g(C))}{|C|} = v(g(C)).$$

Hence, γ^{CE} satisfies CE. To prove *balanced component contributions* let $(g, v) \in D$ and let $i, j \in N$. If there exists $C \in N/g$ with $C = C_i = C_j$ then

$$\gamma_i^{CE}(g, v) - \gamma_i^{CE}(g - g(C), v) = \frac{v(g(C))}{|C|} - 0 = \gamma_j^{CE}(g, v) - \gamma_j^{CE}(g - g(C), v).$$

If $C_i \neq C_j$ then

$$\gamma_i^{CE}(g, v) - \gamma_i^{CE}(g - g(C_j), v) = 0 = \gamma_j^{CE}(g, v) - \gamma_j^{CE}(g - g(C), v).$$

We conclude that γ^{CE} satisfies BCC.

Secondly, suppose φ satisfies CE and BCC. Let $(g, v) \in D$ and let $C \in N/g$. If $|C| = 1$, say $C = \{i\}$, then $\varphi_i(g, v) = v(\emptyset) = \gamma_i^{CE}(g, v)$ by CE of both φ and γ^{CE}. Suppose $|C| > 1$. Fix $i \in C$. Then for all $j \in C$ it holds by BCC and the result for components with 1 player only that

$$\varphi_j(g, v) = \varphi_j(g, v) - \varphi_j(g - g(C)) = \varphi_i(g, v) - \varphi_i(g - g(C)) = \varphi_i(g, v).$$

Combining this with CE we conclude that

$$\varphi_i(g,v) = \frac{v(g(C))}{|C|} = \gamma^{CE}(g,v), \text{ for all } i \in C.$$

This completes the proof. □

4 Recursive formulas

In this section we present recursive formulas for the three allocation rules under consideration. All formulas are in the same spirit as the recursive formula for the Shapley value that is used by Maschler and Owen (1989), Hart and Mas-Colell (1996), and Pérez-Castrillo and Wettstein (2001), which is $\Phi_i(N,v) = \frac{1}{|N|}\big[v(N) - v(N-i) + \sum_{j\in N-i}\Phi_i(N-j, v_{|N-j})\big]$. For each of the rules its recursive formula will be used in the next section to show that a certain mechanism ends in payoffs equal to the allocation rule.

Since the focus of the next section is on the position value, we start with a recursive formula for the position value. Throughout this section we denote the unanimity coefficients of v by $(\alpha_A^v)_{A\subseteq g^N}$.

Theorem 4.1 Let v be a component additive value function and g a network. Then for all $C \in N/g$ and all $i \in C$

$$\pi_i(g,v) = \frac{1}{2|g|}\Big(\sum_{l\in g_i}\big[v(g) - v(g-l)\big] + 2\sum_{l\in g}\pi_i(g-l,v)\Big). \tag{7}$$

Proof: Let $C \in N/g$ and let $i \in C$. Then

$$\begin{aligned}
&\frac{1}{2|g|}\Big(\sum_{l\in g_i}\big[v(g) - v(g-l)\big] + 2\sum_{l\in g}\pi_i(g-l,v)\Big)\\
&=\frac{1}{2|g|}\Big(\sum_{l\in g_i}\sum_{A\subseteq g:\ l\in A}\alpha_A^v + 2\sum_{l\in g}\sum_{A\subseteq g-l}\frac{|A_i|}{2|A|}\alpha_A^v\Big)\\
&=\frac{1}{2|g|}\sum_{A\subseteq g}\Big(|A_i|\alpha_A^v + |g-A|\frac{|A_i|}{|A|}\alpha_A^v\Big)\\
&=\frac{1}{2|g|}\sum_{A\subseteq g}\Big(|A|\frac{|A_i|}{|A|}\alpha_A^v + |g-A|\frac{|A_i|}{|A|}\alpha_A^v\Big)\\
&=\frac{1}{2|g|}\sum_{A\subseteq g}|g|\frac{|A_i|}{|A|}\alpha_A^v\\
&=\sum_{A\subseteq g}\frac{|A_i|}{2|A|}\alpha_A^v\\
&=\pi_i(g,v).
\end{aligned}$$

The first and last equality follow by (2). The other equalities follow by rearranging terms. The second equality uses $|g - A| = \#\{l \in g \mid A \subseteq g - l\}$. This completes the proof. $\square$

This theorem states that the position value of a player in a situation consists of two parts. The first part is the average of his position values in situations with one less link, $\frac{1}{|g|}\sum_{l\in g}\pi_i(g-l,v)$, and the second part is part of the sum of the marginal contributions of its links, $\frac{1}{2|g|}\sum_{l\in g_i}\left[v(g)-v(g-l)\right]$.

Using $\alpha_A^v = \alpha_A^{v_{|g}}$ for all $g \supseteq A$ and (2) we derive that the position value is component decomposable (cf. van den Nouweland (1993)), i.e., for all $C \in N/g$ and all $i \in C$ it holds that $\pi_i(g,v) = \pi_i(g(C), v_{|C})$, and we conclude that for all $C \in N/g$ and all $i \in C$ it holds that $\pi_i(g,v) = \pi_i(g(C),v)$. The following corollary then follows directly from theorem 4.1.

Corollary 4.1 Let v be a component additive value function and g a network. Then for all $C \in N/g$ and all $i \in C$

$$\begin{aligned}\pi_i(g,v) &= \pi_i(g(C),v)\\ &= \frac{1}{2|g(C)|}\Big[\sum_{l\in g_i}\big(v(g(C)) - v(g(C)-l)\big) + 2\sum_{l\in g(C)}\pi_i(g(C)-l,v)\Big].\end{aligned} \tag{8}$$

Subsequently, we derive a recursive formula for the Myerson value similar to the recursive formula for the Shapley value.

Theorem 4.2 Let v be a component additive value function and g a network. Then for all $C \in N/g$ and all $i \in C$

$$\begin{aligned}\mu_i(g,v) &= \mu_i(g(C),v)\\ &= \frac{1}{|C|}\Big[v(g) - v(g-g_i) + \sum_{j\in C-i}\mu_i(g-g_j,v)\Big].\end{aligned} \tag{9}$$

Proof: Let $C \in N/g$ and let $i \in C$. Then

$$\begin{aligned}&\frac{1}{|C|}\Big[v(g) - v(g-g_i) + \sum_{j\in C-i}\mu_i(g-g_j,v)\Big]\\ &= \frac{1}{|C|}\Big[\sum_{A\subseteq g: A_i\neq\emptyset}\alpha_A^v + \sum_{j\in C-i}\ \sum_{A\subseteq g-g_j: A_i\neq\emptyset}\frac{\alpha_A^v}{|N(A)|}\Big]\\ &= \frac{1}{|C|}\Big[\sum_{A\subseteq g: A_i\neq\emptyset}\frac{|N(A)|}{|N(A)|}\alpha_A^v + \sum_{A\subseteq g: A_i\neq\emptyset}|C-N(A)|\frac{\alpha_A^v}{|N(A)|}\Big]\\ &= \frac{1}{|C|}\sum_{A\subseteq g: A_i\neq\emptyset}\frac{|C|}{|N(A)|}\alpha_A^v\\ &= \sum_{A\subseteq g: A_i\neq\emptyset}\frac{1}{|N(A)|}\alpha_A^v\\ &= \mu_i(g,v) = \mu_i(g(C),v).\end{aligned}$$

The first and fifth equality follow by (1), the last by definition, and the others by rearranging terms. The second equality uses $|C - N(A)| = \#\{j \mid A \subseteq g - g_j\}$. The last equality follows along the lines of reasoning right in front of corollary 4.1. □

This theorem states that the Myerson value of a player in a situation consists of two parts. The first part is the average of his Myerson values in situations with one player deleting all its links, $\frac{1}{|C|}\sum_{j\in C-i}\mu_i(g-g_j,v)$, and the second part is part of the marginal contributions of all its links, $\frac{1}{|C|}\big[v(g(C)) - v(g(C)-g_i)\big]$.

For the sake of completeness and for further reference we provide a similar recursive formula for the component-wise egalitarian solution as well.

Theorem 4.3 Let v be a component additive value function and g a network. Then for all $C \in N/g$ and all $i \in C$

$$\gamma_i^{CE}(g,v) = \frac{1}{|C|}\Big[v(g(C)) - v(\emptyset) + \sum_{j\in C-i}\gamma_i^{CE}(g(C)-g(C),v)\Big] \tag{10}$$

Proof: Let $C \in N/g$ and let $i \in C$. Then

$$\begin{aligned}
&\frac{1}{|C|}\Big[v(g(C)) - v(\emptyset) + \sum_{j\in C-i}\gamma_i^{CE}(g(C)-g(C),v)\Big]\\
&\quad = \frac{1}{|C|}\Big[v(g(C)) - 0\Big) + \sum_{j\in C-i} 0\Big]\\
&\quad = \gamma_i^{CE}(g,v).
\end{aligned}$$

This completes the proof. □

5 Three mechanisms

In the previous sections we have provided similar characterizations and similar recursive formulas for the position value, the Myerson value, and the component-wise egalitarian solution. In this section we will study three mechanisms. Each of the mechanisms implements one of the allocation rules in subgame perfect Nash equilibrium (SPNE).

5.1 The position value

In this section we introduce and analyze a mechanism that implements the position value.

We will formally describe the mechanism. The mechanism is specified recursively and componentwise. Let C be a component of g in some situation (g, v) and let $g(C)$ be the links in component C. If $|g(C)| = 0$ then the unique player i in this component receives his stand-alone value 0. Now, suppose $|g(C)| = m \geq 1$ and that the mechanism has been specified for components with at most $m-1$ links. The mechanism is played in several rounds. Each round consists of five steps (t=1 through t=5) and after a round the game ends or new rounds start for all components remaining. We will describe such a round for a component C with m links:[1]

t=1 Each player $i \in C$ makes to any player $k \neq i$, a bid $b_k^i \in \mathbb{R}$. For each $i \in C$ determine the net bid

$$B^i = \sum_{k \neq i} b_k^i - \sum_{k \neq i} b_i^k.$$

Let $\alpha = \text{argmax}_i(B^i)$, with α chosen according to an arbitrary process among the maximizing indices in case of a nonunique maximizer.

t=2 Every player $k \neq \alpha$ divides the bid in his direction from player α over the links of player α, i.e., he specifies $b_k^{\alpha,l} \in \mathbb{R}$ for all $l \in g_\alpha$ under the condition that $\sum_{l \in g_\alpha} b_k^{\alpha,l} = b_k^\alpha$. We remark that $b_k^{\alpha,l}$ is allowed to be negative.

t=3 Player α chooses a link $l_\alpha \in g_\alpha$ and pays b_k^{α,l_α} to every $k \neq \alpha$.

t=4 Player α proposes payoffs y_k to players $k \neq \alpha$.

t=5 Players other than α sequentially either accept or reject the proposed offers. If the proposal is rejected by at least one of the players then the players of C proceed to play the next round where the set of links within C is $g(C) - l_\alpha$. Note that this next round consists of several separate (sub-)mechanisms in case $|C/(g(C) - l_\alpha)| > 1$. If the proposal is accepted, each $k \neq \alpha$ receives y_k and player α receives $v(g(C)) - \sum_{k \neq \alpha} y_k$. We remark that these payoffs come on top of the bids that were paid at $t = 3$ and, perhaps, payoffs from previous rounds. So, if the offers are accepted the payoff to player $k \neq \alpha$ *in this round* equals $y_k + b_k^{\alpha,l_\alpha}$, while for player α this payoff equals $v(g(C)) - \sum_{k \neq \alpha}(y_k + b_k^{\alpha,l_\alpha})$.

We remark that at $t = 5$ three different payoffs for each player come to the fore. First, the payoffs that are in case of acceptance equal to the payoffs that are actually paid at $t = 5$ and in case of rejection equal to the continuation payoff, i.e., payoffs received in subsequent rounds. Secondly, these first payoffs with on top of these the payoffs at $t = 3$ in the round under consideration. Finally, these second payoffs with on top of these the payoffs from previous rounds. Note that this requires rejection in these previous rounds. Furthermore, note that if we talk about maximizing profit at $t = 1$, $t = 2$, or $t = 3$ it does not matter whether we talk about the second or third specification (difference between the two is already 'sunk'). Finally, at $t = 4$ or $t = 5$ all three specifications can be used (again because the difference is already 'sunk'). Oftentimes the analysis is independent of the specification we are looking at. For notational convenience, we will in general not specify which specification we are looking at, but concentrate on payoff comparisons for some fixed specification.

[1] If no ambiguity can arise we will, throughout the remainder of this paper, write simply $k \neq i$ rather than $k \in C - i$.

The mechanism described above applied to situation (g, v) and component $C \in N/g$ will be denoted by $\Gamma(g, v, C)$.

Theorem 5.1 For all situations (g, v) and all $C \in N/g$ there exists an SPNE in $\Gamma(g, v, C)$ such that the payoffs to the players correspond to the position value.

Proof: The proof will be by induction on the number of links within a component. Obviously, the statement in the theorem holds for all situations and all its components with no link.

Now, let $m \geq 1$ and assume that the statement in the theorem is true for all situations and for all components with at most $m - 1$ links. Let (g, v) be a situation with $C \in N/g$ and $|g(C)| = m$.

Consider the following strategies:

t=1 Each player $i \in C$ makes to any player $k \neq i$ bid

$$b_k^i = \sum_{l \in g_i} [\pi_k(g, v) - \pi_k(g - l, v)].$$

t=2 Each player $k \neq \alpha$ chooses for all $l \in g_\alpha$:[2]

$$b_k^{\alpha,l} = \pi_k(g, v) - \pi_k(g - l, v) + \frac{b_k^\alpha - \sum_{l' \in g_\alpha} [\pi_k(g, v) - \pi_k(g - l', v)]}{|g_\alpha|} \tag{11}$$

(Note that $\sum_{l \in g_\alpha} b_k^{\alpha,l} = b_k^\alpha$).

t=3 Player α makes a subgame perfect choice, i.e., he chooses a link l that maximizes his payoff when taking into account the strategies at $t = 4$ and $t = 5$.[3]

t=4 Player α proposes to every player $k \neq \alpha$ payoff $y_k^\alpha = \pi_k(g - l_\alpha, v)$.

t=5 Player k accepts if $y_k^\alpha \geq \pi_k(g - l_\alpha, v)$ and rejects the offer otherwise.

First, we argue that these strategies have the position value as their final payoff. Note that following the choices at $t = 1$, the last part of the right-hand-side of equation (11) equals zero. Using this, we find that player $k \neq \alpha$ will receive

$$\begin{aligned} y_k^\alpha + b_k^{\alpha,l_\alpha} &= \pi_k(g - l_\alpha, v) + [\pi_k(g, v) - \pi_k(g - l_\alpha, v)] \\ &= \pi_k(g, v). \end{aligned}$$

Player α will receive $v(g(C)) - \sum_{k \neq \alpha} [y_k^\alpha + b_k^{\alpha,l_\alpha}]$. Using the expression above and component efficiency of the position value it follows that player α receives $v(g(C)) - \sum_{k \neq \alpha} \pi_k(g, v) = \pi_\alpha(g, v)$.

[2]The last part of this expression is 0 in the node that follows the choices as just specified for $t = 1$. However, this part needs to be included since we cannot restrict attention to the equilibrium path to make sure that a strategy is a subgame perfect Nash equilibrium.

[3]Recall our remarks on different specifications of the payoffs of the players.

Note that the payoff determination is independent of the identity of the proposer α and the identity of the link l_α. Furthermore, we note that the strategies above result in net bids equal to zero for all $i \in N$ since

$$\begin{aligned} B^i &= \sum_{k \neq i} b^i_k - \sum_{k \neq i} b^k_i. \\ &= \sum_{k \neq i} \Big[\sum_{l \in g_i} [\pi_k(g,v) - \pi_k(g-l,v)] - \sum_{l \in g_k} [\pi_i(g,v) - \pi_i(g-l,v)] \Big]. \end{aligned}$$

By the balanced link contributions property, we know that

$$\sum_{l \in g_k} \Big[\pi_i(g,v) - \pi_i(g-l,v) \Big] = \sum_{l \in g_i} \Big[\pi_k(g,v) - \pi_k(g-l,v) \Big].$$

We conclude that $B^i = 0$.

It remains to check that the strategies constitute an SPNE. It is obvious that the strategies at $t = 5$ are best responses. To check that the offers at $t = 4$ are subgame perfect, note that in case of rejection proposer α obtains $\pi_\alpha(g - l_\alpha, v)$. In case all others accept, player α can, taking into account the choices at $t = 5$ obtain at most $v(g(C)) - \sum_{k \neq \alpha} \pi_k(g - l_\alpha, v)$. Since v is monotonic we have that $v(g(C)) \geq v(g(C) - l_\alpha)$. Consequently, $v(g(C)) - \sum_{k \neq \alpha} \pi_k(g - l_\alpha, v) \geq \pi_\alpha(g - l_\alpha, v)$. Hence, player α maximizes his payoff by making the offers as described in the strategy.

The strategies at $t = 3$ are subgame perfect by definition.

To check that the strategies at $t = 2$ form a best response, note that for any $k \neq \alpha$ the current choices at $t = 2$ imply that the payoff of k is independent of the choice of player α at $t = 3$, namely $b^{\alpha,l}_k + y_k = \pi_k(g,v) + \frac{b^\alpha_k - \sum_{l' \in g_\alpha} [\pi_k(g,v) - \pi_k(g-l',v)]}{|g_\alpha|}$. Consequently, the same holds for player α, i.e., he can choose an arbitrary link at $t = 3$. For a fixed link $l \in g_\alpha$ consider a change of δ^k_l of player k in $b^{\alpha,l}_k$, i.e., change it to $b^{\alpha,l}_k + \delta^k_l$. Assuming that l is chosen at $t = 3$ it follows that the payoff of player k changes by δ^k_l, whereas the payoff to player α changes by $-\delta^k_l$. Any change in the strategy of player k results in a negative δ^k_l for at least one link l. Hence, player α will choose at $t = 3$ a link that increases his payoff (recall that currently his payoff is independent of his choice) and, consequently, will decrease the payoff of player k.

Finally, consider the strategies at $t = 1$. Suppose player i changes his bids such that he will be the proposer with certainty. Then he has increased his total bids and thereby, he has decreased his eventual payoff (given his strategies at future periods and our analysis of these periods, which holds both on and off the equilibrium path!). If he changes his bids such that, with certainty, he will not be the proposer, another player will propose, meaning that his payoff will not change. Finally, suppose that player i has changed his bids, and that he remains in $\text{argmax}_j(B^j)$, while simultaneously $|\text{argmax}_j(B^j)| \geq 2$. Then there exist $j, k \in C$ such that b^i_j has been increased, while b^i_k has been decreased. To make sure that the net bid of player i is at least as much as the (new) net bid of player k (which has been increased), the total bid of player i must have been increased by at least the decrease in b^i_k. So, if player i is chosen with a positive probability he will increase his total bid and, hence, decrease his eventual payoff. If he is not chosen with a positive probability his payoff will remain unchanged. □

The following theorem shows that the payoff to the players coincide with the position value according to any SPNE. Though random devices may be used in the selection of the proposer and the selection of the link we stress that any SPNE results in the position value with certainty, and not in expectation only.

Theorem 5.2 For all situations (g, v), all $C \in N/g$, and each SPNE in $\Gamma(g, v, C)$ the payoffs to the players correspond to the position value.

Proof: The proof will be by induction on the number of links within a component. Obviously, the statement in the theorem holds for all situations for all its components with no link.

Now, let $m \geq 1$ and assume that the statement in the theorem is true for all situations and for all its components with at most $m-1$ links. Let (g, v) be a situation with $C \in N/g$ such that $|g(C)| = m$.

We will prove the induction step by proving several claims:

Claim 1 In any SPNE, at $t = 5$ all players who are not the proposer accept the proposal if $y_k^\alpha > \pi_k(g - l_\alpha, v)$ for all $k \neq \alpha$. Moreover, if $y_k^\alpha < \pi_k(g - l_\alpha, v)$ for some $k \neq \alpha$ then the proposal is rejected.

By the induction hypothesis we have that in case of rejection the payoffs to players coincide with $\pi(g - l_\alpha, v)$. Hence, if all players but the last one have accepted the offer then the last player in $C - \alpha$, say k, will accept any offer higher than $\pi_k(g - l_\alpha, v)$ and reject any offer lower than $\pi_k(g - l_\alpha, v)$. Anticipating this decision, the player right in front of the last player accepts if he and the last player both receive more than their respective position values in $g - l_\alpha$. He rejects if he receives less, while the last player would receive more, and he is indifferent about his choice if the last player (k) receives less than $\pi_k(g - l_\alpha, v)$, irrespective of his own payoff. Working our way back to the first player who has to decide at $t = 5$ in a similar manner, proves the claim.

Claim 2 If $v(g(C)) > v(g(C) - l_\alpha)$ then all SPNE of a game that starts at $t = 4$ satisfy the following specifications:[4] at $t = 4$ player α proposes $y_k^\alpha = \pi_k(g - l_\alpha, v)$ to all $k \neq \alpha$ and at $t = 5$ every player $k \neq \alpha$ accepts offer y_k^α if $y_j^\alpha = \pi_j(g - l_\alpha, v)$ for all $j \neq \alpha$.
If $v(g(C)) = v(g(C) - l_\alpha)$ then any SPNE that does not satisfy the specifications in the first part of this claim satisfies the following specifications: at $t = 4$ player α proposes, among others, $y_k^\alpha \leq \pi_k(g - l_\alpha, v)$ to some player $k \neq \alpha$ and at $t = 5$ this offer is rejected by some player.
In all SPNE of this subgame the payoffs to the players are given by $v(g(C)) - v(g(C) - l_\alpha) - \sum_{k \neq \alpha} b_k^{\alpha, l_\alpha}$ for player α and $\pi(g - l_\alpha) + b_k^{\alpha, l_\alpha}$ for all $k \neq \alpha$.

First, consider the case in which $v(g(C)) > v(g(C) - l_\alpha)$. Then, if the offer is rejected by one of the players, player α will receive $\pi_\alpha(g - l_\alpha, v)$. Define $\Delta = v(g(C)) - v(g(C) - l_\alpha) > 0$. Then player α could have offered $\pi_k(g - l_\alpha, v) + \frac{\Delta}{|C|}$ to all $k \neq \alpha$. This offer will be accepted and it

[4] We will not give a complete description of the SPNE.

will improve the payoff of player α by $\Delta - (|C|-1)\frac{\Delta}{|C|} = \frac{\Delta}{|C|}$. So, any SPNE requires an offer at $t=4$ that is accepted at $t=5$. By Claim 1 this implies $y_k^\alpha \geq \pi_k(g-l_\alpha, v)$ for all $k \neq \alpha$. If $\epsilon_k := y_k^\alpha - \pi_k(g-l_\alpha, v) > 0$ for some $k \neq \alpha$ then player α can clearly improve his payoff by offering any $j \neq \alpha$ payoff $\pi_j(g-l_\alpha, v) + \frac{\epsilon_k}{|C|}$, which is accepted by all players and increases the payoff of player α by $\frac{\epsilon_k}{|C|}$. Hence, $y_k^\alpha = \pi_k(g-l_\alpha, v)$ for all $k \neq \alpha$. Since any SPNE requires an offer at $t=4$ that is accepted at $t=5$ we conclude that at $t=5$ every player $k \neq \alpha$ accepts any offer y_k^α if $y_j^\alpha = \pi_j(g-l_\alpha, v)$ for all $j \neq \alpha$.

If $v(g(C)) = v(g(C)-l_\alpha)$ we may have two types of SPNE, namely with an accepted or rejected set of offers. The proposer has to offer at least $\pi_j(g-l_\alpha, v)$ for all $j \neq \alpha$ to have the offers accepted (cf Claim 1). Hence, the total offer is at least $\sum_{j\neq\alpha}\pi_j(g-l_\alpha) = v(g(C)) - \pi_\alpha(g-l_\alpha, v)$. As in the previous case of the proof of this claim, every accepted offer in equilibrium should be equal to $\pi_j(g-l_\alpha, v)$ for all $j \neq \alpha$. Furthermore, by Claim 1 we know that an offer at $t=4$ can only be rejected at $t=5$ if $y_k^\alpha \leq \pi_k(g-l_\alpha, v)$ for some $k \neq \alpha$. Note that both types of SPNE result in the same payoffs.

The last part of the claim now follows immediately.

Claim 3 In any SPNE of a game that starts at $t=2$ the strategy of player $k \neq \alpha$ at $t=2$ results in the same payoff for player k independent of the choice of player α at $t=3$. Hence, for all $l \in g_\alpha$:

$$b_k^{\alpha,l} = \pi_k(g,v) - \pi_k(g-l,v) + \frac{b_k^\alpha - \sum_{l'\in g_\alpha}[\pi_k(g,v) - \pi_k(g-l',v)]}{|g_\alpha|}.$$

First we will show that player α is indifferent about his choice at $t=3$. Subsequently, we will show that the same holds for all $k \neq \alpha$. Let $\mathcal{L} \subseteq g_\alpha$ be the set of links that maximize the payoff of player α and let l_α be the choice of player α. Suppose $\mathcal{L} \subset g_\alpha$. Then for each $l \in \mathcal{L}$ player α receives more than on average over all possible choices in g_α. Since we know by claim 2 that the sum of the payoffs to the players is constant there exists $j_l \in C\backslash\{\alpha\}$ who receives less if l is chosen than he receives on average over all possible choices in g_α. Let Δ be the payoff decrease for player α if he could only choose a link from the set $g_\alpha\backslash\mathcal{L}$, which is nonempty by assumption. Denote the choice of player j_{l_α} at $t=2$ by $b_{j_{l_\alpha}}^{\alpha,l}$, $l \in g_\alpha$. Consider the deviation to $d_{j_{l_\alpha}}^{\alpha,l}$, $l \in g_\alpha$ defined by

$$d_{j_{l_\alpha}}^{\alpha,l} = \begin{cases} b_{j_{l_\alpha}}^{\alpha,l} + \frac{\Delta}{4|g_\alpha|} & \text{if } l = l_\alpha; \\ b_{j_{l_\alpha}}^{\alpha,l} + 2\frac{\Delta}{4|g_\alpha|} & \text{if } l \in \mathcal{L}\backslash\{l_\alpha\}; \\ b_{j_{l_\alpha}}^{\alpha,l} - \frac{(2|\mathcal{L}|-1)\frac{\Delta}{4|g_\alpha|}}{|g_\alpha\backslash\mathcal{L}|} & \text{if } l \in g_\alpha\backslash\mathcal{L}. \end{cases}$$

So, player j_{l_α} increases the bid of player α on l_α for j_{l_α} by $\frac{\Delta}{4|g_\alpha|}(< \frac{1}{2}\Delta)$, the bid on any $l \in \mathcal{L}\backslash\{l_\alpha\}$ by $2\frac{\Delta}{4|g_\alpha|}$, and he decreases the amount on $l \in g_\alpha\backslash\mathcal{L}$ by $\frac{(2|\mathcal{L}|-1)\frac{\Delta}{4|g_\alpha|}}{|g_\alpha\backslash\mathcal{L}|}(< \frac{1}{2}\Delta)$. Note that the sum of these deviations equals 0. After this deviation l_α is the unique link that maximizes the payoff of α at $t=3$. Furthermore, this deviation increases the payoff of player j_{l_α} by $\frac{\Delta}{4|g_\alpha|}$.

We conclude that $\mathcal{L} = g_\alpha$. Let l_α be the link selected by α. We will argue that player $j \neq \alpha$ does not prefer any other link. Suppose that player j would have preferred that α chose l_j

since that would have increased his payoff by Δ. Consider the following deviation of player j at $t = 2$,

$$d_j^{\alpha,l} = \begin{cases} b_j^{\alpha,l} + \frac{\Delta}{10} & \text{if } l = l_\alpha; \\ b_j^{\alpha,l} - \frac{\Delta}{10} & \text{if } l = l_j; \\ b_j^{\alpha,l} & \text{if } l \in g_\alpha \backslash \{l_\alpha, l_j\}. \end{cases}$$

Player α, who was indifferent between all links in g_α before the deviation will after the deviation choose l_j. Hence, the (expected) payoff of player j increases by $\Delta - \frac{\Delta}{10} = 9\frac{\Delta}{10}$. This contradicts that we are considering an SPNE. So, no player $j \neq \alpha$ prefers the choice of another link. Additionally, since α is indifferent between all $l \in \mathcal{L} = g_\alpha$ and, by Claim 2, the sum of the payoffs to the players is constant ($v(g(C))$) we conclude that no player prefers l_α to any link in $g_\alpha \backslash \{l_\alpha\}$ since that would require a reverse preference of another player.

We conclude that in any SPNE strategies at $t = 2$ are such that all players are indifferent about the choice of player α at $t = 3$. Then, for any $k \neq \alpha$ we can combine $\sum_{l \in g_\alpha} b_k^{\alpha,l} = b_k^\alpha$ with the payoff player k receives at $t = 5$, i.e., $\pi_k(g - l, v)$ if l is chosen by α at $t = 3$, to conclude that

$$b_k^{\alpha,l} = \pi_k(g, v) - \pi_k(g - l, v) + \frac{b_k^\alpha - \sum_{l' \in g_\alpha} [\pi_k(g, v) - \pi_k(g - l', v)]}{|g_\alpha|}.$$

Claim 4 In any SPNE, each player is indifferent about the selection of the proposer among $\text{argmax}\{B^i : i \in C\}$.

Suppose that Claim 4 does not hold true. Consider an SPNE such that $|\text{argmax}\{B^i : i \in C\}| \geq 2$ and there is a player (say k^*) who prefers one proposer (say $i^* \in \text{argmax}\{B^i : i \in C\}$) to some other proposer (say $j^* \in \text{argmax}\{B^i : i \in C\}$). By Claim 2 we know that the total payoffs to the players in C are the same and hence, we conclude that there is another player (say m^*) who has a reverse preference. Since i^* and j^* being proposer result in different payoff vectors, we have that for any proposer $r^* \in C \backslash \{i^*, j^*\}$ there exists a proposer (say t^*, equal to i^* or j^*) such that r^* and t^* being proposer result in different payoff vectors. Hence, using Claim 2 (the sum of the payoffs to the players is constant) we know that there is a player who prefers t^* to r^*. We conclude that for *any* proposer in $\text{argmax}\{B^i : i \in C\}$ we can identify a player who prefers another player to be the proposer.

Let $\alpha \in \text{argmax}\{B^r : r \in C\}$ be a player who will be proposer with a positive probability p (possibly equal to 1). Then there exists $i \in C$ who prefers $j \in \text{argmax}\{B^r : r \in C\}$ to α and who weakly prefers j to any proposer in $\text{argmax}\{B^r : r \in C\}$. Let Δ be the difference in payoff for player i if j is the proposer rather than α. We now distinguish three cases. Note that $j \neq \alpha$.

Case 1: $i \neq j$ and $i \neq \alpha$. Increase b_α^i and decrease b_j^i (both by the same arbitrary amount). Then j becomes the proposer with certainty and player i improves his (expected) payoff by at least $p\Delta$.

Case 2: $i = j$ (and $i \neq \alpha$): Increase b_α^i by $\frac{1}{2}p\Delta > 0$. Then $i(= j)$ becomes the proposer with certainty. The gain (for player i) by going from proposer α (who proposed with probability p) to proposer i would be $p\Delta$. Moving from other proposers to i increases the possible gain for player i. Bidding a higher bid decreases his payoff by at most $\frac{1}{2}p\Delta$. Hence, the payoff of player i will increase by at least $\frac{1}{2}p\Delta$.

Case 3: $i = \alpha$ (and $i \neq j$): Decrease b^i_j (by, e.g., $\frac{1}{2}p\Delta > 0$). Then j becomes the proposer with certainty and player i improves his (expected) payoff by at least $p\Delta$.

Claim 5 In any SPNE, $B^i = 0$ for all $i \in C$.

Using Claims 2 and 3 we conclude that if i will be proposer, then his payoff will be

$$\pi_i(g,v) + \frac{\sum_{k\neq i}[-b^i_k + \sum_{l\in g_i}(\pi_k(g,v) - \pi_k(g-l,v))]}{|g_i|}.$$

Hence, his payoff is linear in $\sum_{k\neq i} b^i_k$, assuming that player i remains the proposer. By construction of the net bid we know that $\sum_{j\in C} B^j = 0$. Suppose there exists $k \in C$ with $B^k < 0$. Let i be a player with maximum net bid. Note that Claim 4 implies that the payoff to player i is independent of the choice of the proposer among argmax$\{B^j : j \in C\}$. Let $\mathcal{C}$ be the set of players with maximal B^j. Let r be a player with the highest net bid smaller than B^i. Let $\Delta = \min\{B^i - B^r, |B^k|)\}$. Obviously, $\Delta > 0$. Consider the following deviation of player i:

$$d^i_j = \begin{cases} b^i_j - \frac{\Delta}{2} & \text{if } j = k; \\ b^i_j + \frac{\Delta}{2|\mathcal{C}|-1} & \text{if } j \in \mathcal{C}\backslash\{i\}; \\ b^i_j & \text{if } j \in C\backslash(\mathcal{C}\cup\{k\}). \end{cases}$$

Then the net bid of player k increases by $\frac{\Delta}{2}$ (but remains negative), whereas the net bid of any player $j \in C\backslash(\mathcal{C}\cup\{k\})$ remains unchanged. The net bid of any player $j \in \mathcal{C}\backslash\{i\}$ decreases by $\frac{\Delta}{2|\mathcal{C}|-1}$. Finally, the net bid of player i decreases by $\frac{\Delta}{2} - (|C|-1)\frac{\Delta}{2|\mathcal{C}|-1} = \frac{\frac{1}{2}\Delta}{2|\mathcal{C}|-1}$. We conclude that player i will be the proposer for sure. Since his total bid decreased by $\frac{\frac{1}{2}\Delta}{2|\mathcal{C}|-1} = \frac{\Delta}{4|\mathcal{C}|-2}$, he will increase his payoff by $\frac{\frac{\Delta}{4|\mathcal{C}|-2}}{|g_i|}$.

Claim 6 In any SPNE, the payoff of each of the players coincides with his position value.

Consider an SPNE. Let $i \in C$. If player i is the proposer and he chooses link $l \in g_i$, then his payoff equals $x^{i,l}_i = \pi_i(g-l,v) + v(g(C)) - v(g(C)-l) - \sum_{l\in g_i}\sum_{j\neq i} b^j_{i,l}$. If $j \neq i$ is the proposer and he proposes $l \in g_j$ then player i receives $x^{j,l}_i = \pi_i(g-l,v) + b^{j,l}_i$. Hence, the sum of the payoffs of player i over all proposer-link combinations is given by

$$\begin{aligned}
\sum_{j\in C}\sum_{l\in g_j} x^{j,l}_i &= \sum_{l\in g_i}[\pi_i(g-l,v) + v(g(C)) - v(g(C)-l) - \sum_{l\in g_i}\sum_{j\neq i} b^j_{i,l}] \\
&\quad + \sum_{j\neq i}\sum_{l\in g_j}[\pi_i(g-l,v) + b^{j,l}_i] \\
&= \sum_{l\in g_i}(v(g(C)) - v(g(C)-l)) + \sum_{j\in C}\sum_{l\in g_j}[\pi_i(g-l,v)] - B^i \\
&= \sum_{l\in g_i}(v(g(C)) - v(g(C)-l)) + \sum_{j\in C}\sum_{l\in g_j}[\pi_i(g-l,v)] \\
&= 2|L|\pi_i(g,v),
\end{aligned}$$

where the third equality follows by Claim 5 and last equality by theorem 4.1. Since player i is indifferent about the proposers (Claims 4 and 5) and about the link chosen by the proposer (Claim 3) we have that all $x_i^{j,l}$ coincide and, hence, that $x_i^{j,l} = \pi_i(g, v)$ for all $j \in C$ and all $l \in g_j$. □

5.2 The Myerson value

In this section we introduce and analyze a mechanism that implements the Myerson value.

We will formally describe the mechanism. The mechanism is specified recursively and componentwise. Let C be a component of g in some situation (g, v) and let $g(C)$ be the links in component C. If $|g(C)| = 0$ then the unique player i in this component receives his stand-alone value 0. Now, suppose $|g(C)| = m \geq 1$ and that the mechanism has been specified for components with at most $m - 1$ links. The mechanism is played in several rounds. Each round consists of three steps (t=1 through t=3) and after a round the game ends or new rounds start for all components remaining. We will describe such a round for a component C with m links:

t=1 Each player $i \in C$ makes to any player $k \neq i$, a bid $b_k^i \in \mathbb{R}$. For each $i \in C$ determine the net bid

$$B^i = \sum_{k \neq i} b_k^i - \sum_{k \neq i} b_i^k.$$

Let $\alpha = \text{argmax}_i(B^i)$, with α chosen according to an arbitrary mechanism among the maximizing indices in case of a nonunique maximizer. Player α pays b_k^α to every $k \neq \alpha$.

t=2 Player α proposes payoffs y_k to players $k \neq \alpha$.

t=3 Players other than α sequentially either accept or reject the proposal. If the offer is rejected by at least one of the players then the players of C proceed to play the next round where the set of links within C is $g(C) - g_\alpha$. Note that this next round might consist of several separate (sub-)mechanisms. If the offer is accepted, each $k \neq \alpha$ receives y_k and player α receives $v(g(C)) - \sum_{k \neq \alpha} y_k$. We remark that these payoffs come on top of the bids that were paid at $t = 3$ and, perhaps, payoffs from previous rounds. So, if the offers are accepted the payoff to player $k \neq \alpha$ *in this round* equals $y_k + b_k^\alpha$, while for player α this payoff equals $v(g(C)) - \sum_{k \neq \alpha}(y_k + b_k^\alpha)$.

Similar to the mechanism for the position value several different (relevant) payoffs come to the fore. As for the position value we will in general not specify which specification we are looking at, but concentrate on payoff comparisons for some fixed specification.

This mechanism applied to situation (g, v) and component $C \in N/g$ will be denoted by $\Gamma^M(g, v, C)$.

Since the Myerson value of (g, v) coincides with the Shapley value of (N, v^g) the proof of the following theorems follows by Pérez-Castrillo and Wettstein (2001). A sketch of a slightly different proof, similar to the proof that the mechanism in the previous subsection implements the position value, can be found in the appendix. We refer to section 8 for some comments on Pérez-Castrillo and Wettstein (2001).

Theorem 5.3 For all situations (g, v) and all $C \in N/g$ there exists an SPNE in $\Gamma^M(g, v, C)$ such that the payoffs to the players correspond to the Myerson value.

Theorem 5.4 For all situations (g, v), all $C \in N/g$, and each SPNE in $\Gamma^M(g, v, C)$ the payoffs to the players correspond to the Myerson value.

5.3 The component-wise egalitarian solution

In this subsection we introduce and analyze a mechanism that implements the component-wise egalitarian solution.

We will formally describe the mechanism. The mechanism is specified recursively and componentwise. Let C be a component of g in some situation (g, v) and let $g(C)$ be the links in component C. If $|g(C)| = 0$ then the unique player i in this component receives his stand-alone value 0. Now, suppose $|g(C)| = m \geq 1$ and that the mechanism has been specified for components with at most $m - 1$ links. The mechanism is played in several rounds. Each round consists of three steps (t=1 through t=3) and after a round the game ends or new rounds start for all components remaining. We will describe such a round for a component C with m links:

t=1 Each player $i \in C$ makes to any player $k \neq i$, a bid $b^i_k \in \mathbb{R}$. For each $i \in C$ determine the net bid

$$B^i = \sum_{k \neq i} b^i_k - \sum_{k \neq i} b^k_i.$$

Let $\alpha = \text{argmax}_i(B^i)$, with α chosen according to an arbitrary mechanism among the maximizing indices in case of a nonunique maximizer. Player α pays b^α_k to every $k \neq \alpha$.

t=2 Player α proposes payoffs y_k to players $k \neq \alpha$.

t=3 Players other than α sequentially either accept or reject the proposal. If the offer is rejected by at least one of the players then the players of C proceed to play the next round where the set of links within C is $g(C) - g(C_\alpha) = \emptyset$. If the offer is accepted, each $k \neq \alpha$ receives y_k and player α receives $v(g(C)) - \sum_{k \neq \alpha} y_k$. We remark that these payoffs come on top of the bids that were paid at $t = 3$ and, perhaps, payoffs from previous rounds. So, if the offers are accepted the payoff to player $k \neq \alpha$ *in this round* equals $y_k + b^{\alpha,l_\alpha}_k$, while for player α this payoff equals $v(g(C)) - \sum_{k \neq \alpha}(y_k + b^{\alpha,l_\alpha}_k)$.

Similar to the mechanisms in the previous subsections several different (relevant) payoffs come to the fore. Once again, we will in general not specify which specification we are looking at, but concentrate on payoff comparisons for some fixed specification.

This mechanism applied to situation (g, v) and component $C \in N/g$ will be denoted by $\Gamma^{CE}(g, v, C)$.

A sketch of the proofs, similar to the proofs for the theorems in the previous subsections, can be found in the appendix.

Theorem 5.5 For all situations (g, v) and all $C \in N/g$ there exists an SPNE in $\Gamma(g, v, C)$ such that the payoffs to the players correspond to the component-wise egalitarian solution.

Theorem 5.6 For all situations (g, v), all $C \in N/g$, and each SPNE in $\Gamma^{CE}(g, v, C)$ the payoffs to the players correspond to the component-wise egalitarian solution.

6 Nonmonotonic value functions

So far, we have restricted attention to monotonic value functions. In such a setting players jointly have an incentive to stick to the starting network since subnetworks do not have a higher value. In each of the three mechanisms, independent of the starting network, an equilibrium exists that does not break down any cooperation.

In case we drop our monotonicity assumption, the issue of network formation is added to our setting. This is similar to the extension of Pérez-Castrillo and Wettstein (2001) in their work on the Shapley value when they move from the restricted setting of zero-monotonic transferable utility games to general transferable utility games. They altered their basic mechanism slightly and showed that this resulted in an efficient partition as cooperation structure and the Shapley value of the superadditive cover as payoffs. In their approach in the one-but-last stage the proposer proposes a coalition and a payoff vector for this coalition. If in the last stage all players in this coalition accept, the next round is played with the remaining players (the players not in this coalition). If a player refuses the next round is played with all players accept the proposer.

We take a somewhat different approach, which has similar results. In the one-but last stage, the proposer does not only choose a payoff to all other players in his component C but also a network $g' \subseteq g(C)$. This network need not be connected. In the last round then, all players in C have to accept or reject, as before.

For any of the three mechanisms we described in the previous section, this adaptation results in the same allocation rule as before, but now applied to the monotonic cover.

6.1 The position value

In this subsection we consider the position value. Consider the mechanism of subsection 5.1. Consider the following mechanism that coincides with the mechanism of subsection 5.1 except for the fourth and fifth step. These steps are changed as follows:

t=$\overline{4}$ Player α proposes $g' \subseteq g(C)$ and payoffs y_k to players $k \neq \alpha$.

t=$\overline{5}$ As original step 5, except for player α receiving $v(g') - \sum_{k \neq \alpha} y_k$ rather than $v(g(C)) - \sum_{k \neq \alpha} y_k$.

Denote this mechanism applied to situation (g, v) and component $C \in N/g$ by $\overline{\Gamma}(g, v, C)$. Furthermore, denote the monotonic of v by $\overline{v}$, i.e., $\overline{v}(g) = \max_{g' \subseteq g} v(g')$. Along the lines of the proofs in section 5.1 one can prove the following theorems.

Theorem 6.1 For all situations (g, v) and all $C \in N/g$ there exists an SPNE in $\overline{\Gamma}(g, v, C)$ such that the payoffs to the players correspond to the position value of $(g, \overline{v})$.

Theorem 6.2 For all situations (g, v), all $C \in N/g$, and each SPNE in $\overline{\Gamma}(g, v, C)$ the payoffs to the players correspond to the position value of $(g, \overline{v})$.

Though not explicitly mentioned in the theorems, all SPNE result in networks that maximize total profit, taking into account that starting from a certain network, links can only be broken and not restored. So, notice that though the payoffs that result in $\overline{\Gamma}(g, v, C)$ correspond to the position value of $(g, \overline{v})$, the resulting network will be some $\overline{g} \subseteq g$ with $v(\overline{g}) = \max_{g' \subseteq g} v(g')$.

6.2 The Myerson value and the component-wise egalitarian solution

Similar to the previous subsection we extend in this section the results of sections 5.2 and 5.3 to nonmonotonic value functions.

First, for any (g, v) and component $C \in N/g$ let $\overline{\Gamma}^M(g, v, C)$ correspond to $\Gamma^M(g, v, C)$ except for steps 2 and 3:

t=$\overline{2}$ Player α proposes $g' \subseteq g(C)$ and payoffs y_k to player $k \neq \alpha$.

t=$\overline{3}$ As original step 3, except for player α receiving $v(g') - \sum_{k \neq \alpha} y_k$ rather than $v(g(C)) - \sum_{k \neq \alpha} y_k$.

Similarly, for any (g, v) and component $C \in N/g$ let $\overline{\Gamma}^{CE}(g, v, C)$ correspond to $\Gamma^{CE}(g, v, C)$ except for steps 2 and 3, which are altered similar as for $\overline{\Gamma}^M(g, v, C)$ above.

Along the lines of the results in sections 5.2 and 5.3 the following theorems are easily shown.

Theorem 6.3 For all situations (g, v) and all $C \in N/g$ there exists an SPNE in $\overline{\Gamma}^M(g, v, C)$ such that the payoffs to the players correspond to the Myerson value of $(g, \overline{v})$.

Theorem 6.4 For all situations (g, v), all $C \in N/g$, and each SPNE in $\overline{\Gamma}^M(g, v, C)$ the payoffs to the players correspond to the Myerson value of $(g, \overline{v})$.

Theorem 6.5 For all situations (g, v) and all $C \in N/g$ there exists an SPNE in $\overline{\Gamma}^{CE}(g, v, C)$ such that the payoffs to the players correspond to the component-wise egalitarian solution of $(g, \overline{v})$.

Theorem 6.6 For all situations (g, v), all $C \in N/g$, and each SPNE in $\overline{\Gamma}^{CE}(g, v, C)$ the payoffs to the players correspond to the component-wise egalitarian solution of $(g, \overline{v})$.

As for the nonmonotonic result for the position value, in both models all SPNE result in networks that maximize total profit, taking into account that starting from a certain network, links can only be broken and not restored.

7 Some comments on Pérez-Castrillo and Wettstein (2001)

In this section we present some comments on Pérez-Castrillo and Wettstein (2001). For a description and analysis of this model we refer to Pérez-Castrillo and Wettstein (2001).

First, it seems that their model is not specified completely. In the introduction of their model at page 281 they note that "...the element of randomness is inconsequential to our proofs. Our results still hold if ties in net bids are broken deterministically..." In the proof of their Claim (c) on page 284, they conclude that a player has the same probability of becoming the proposer if the set of players with maximal net bid is not changed. Hence, the mechanism to choose the proposer should depend on the set of players with maximal net bid only. Altering their Claims (c) and (d) by making them similar to Claims 4 and 5 in the proof of theorem 5.2 would prove that their results are truly independent of the mechanism to select the proposer among the players with maximal net bid (i.e., first prove that the payoff of any player is independent of the specific choice between the players with maximal net bid and, secondly, prove that all net bids should be equal to zero).

Secondly, at page 283, Pérez-Castrillo and Wettstein (2001) claim that "If player i increases his net bid $\sum_{j\neq i} b_j^i$, then that player will be chosen as proposer with certainty." Now, suppose player 1 changes his bids to players 2 and 3 by -2 and 3, respectively. Then the net bid of player 1 increases by 1, whereas the net bid of player 2 increases by 2. Hence, player 2 would become the proposer (which would leave the payoff of player 1 unchanged). The conclusion that their strategy under consideration constitutes an SPNE, however, still holds.

Finally, in Claim (b) at page 284 they claim that "If $v(N) > v(N\backslash\{\alpha\}) + v(\{\alpha\})$, the only SPNE of the game that starts at $t = 2$ is the following: At $t = 2$, player α offers $y_i^\alpha = \Phi_i(N\backslash\{\alpha\})$ to all $i \neq \alpha$; at $t = 3$, every player $i \neq \alpha$ accepts any offer $y_i^\alpha \geq \Phi(N\backslash\{\alpha\})$ and rejects the offer otherwise." Though they claim that the equilibrium is unique, they can only justify a conclusion that the equilibrium path is unique. This is illustrated in the following example. We stress that this example deals with the mechanism defined in Pérez-Castrillo and Wettstein (2001).

Example 7.1 Let $N = \{1, 2, 3\}$ and $v = u_N$. Suppose that at $t = 1$ all bids have been chosen equal to $\frac{1}{3}$ and that player 1 has been selected as the proposer. Then the following strategies constitute an SPNE of the game that starts at $t = 2$: At $t = 2$, player 1 chooses $y_2^1 = y_3^1 = 0$. At $t = 3$, the last player (say k) rejects if $y_k < 0$ and accepts otherwise, while the one-but-last player rejects if $\min\{y_2^1, y_3^1\} < 0$ and accepts otherwise. It follows straightforwardly that these strategies constitute an SPNE. ◇

8 Concluding remarks

The previous sections provide a solid basis to provide a comparison between our three allocation rules. We will summarize and discuss the main insights. Furthermore, we will compare the mechanisms with similar mechanisms in the literature.

First, the results in section 5 make clear that the Myerson value, the position value and the component-wise egalitarian solution have a different focus. The Myerson value can be seen as focusing on the role of a player, the position value on the role of the links, and the component-wise egalitarian solution, as the name suggests, on the role of the component. For each of these rules, a characterization is provided with a balancedness property that takes this focus into account, together with component efficiency, which is standard. For a more detailed discussion on the difference between balanced contributions and balanced link contributions we refer to Slikker (2005a).

The same difference in focus can be seen in the mechanisms that were shown to have these allocation rules as their payoffs. A survey on similarities and (sometimes only subtle) differences between the mechanisms can be found in table 1.

stage	description of stage		
1.a.	Bid to be proposer		
1.b.	μ: Winner pays his bids	π: Each non winner splits bid of winner to him over links of winner; Winner chooses one of his links and pays associated bids	γ^{CE}: Winner pays his bids
2.	Winner proposes payoffs		
3.a	Sequential acceptance/rejection by non winners.		
3.b.	If accepted by all then pay proposed payoffs. Otherwise: μ: Remove all links of winner and restart	π: Remove link chosen by winner and restart	γ^{CE}: Disconnect component of winner and restart

Table 1: Survey of mechanisms

In the literature several similar mechanisms have appeared. None of them deal with the position value or the component-wise egalitarian solution. Taking this into account, together with the fact that the Myerson value is the Shapley value of some game associated with the underlying situation, it is not surprising that our mechanisms for the Myerson value come closest to the mechanisms in the literature. In fact the mechanism for monotonic value functions can be seen as a straightforward adaptation of the model of Pérez-Castrillo and Wettstein (2001) to a setting with networks and value functions. In a separate paper, Pérez-Castrillo and Wettstein (2005) describe a mechanism that ends in the Myerson value of the monotonic cover of the value function.[5] Their results differ in three ways from the current paper. First, they consider the full graph only, whereas we take an arbitrary graph as our starting point. Secondly, in step 2 they let the proposer choose a coalition, a network on the players in this coalition, and payoffs for players in this coalition. Moreover, if all players in this coalition accept in the final round then the remaining players restart. In our model no coalition is specified, but a graph, which might consist of several components. After acceptance, no new round starts (for the component under consideration). Finally, they consider a slightly different bidding mechanism. Though Pérez-Castrillo and Wettstein (2005) only consider starting from the full graph, their mechanism can be applied starting from an arbitrary graph and can be shown to always end up in the Myerson value of the monotonic cover.

This last difference provides the basis for an adjusted model to implement the Shapley value of the monotonic cover for arbitrary cooperative games with transferable utilities. Rather than picking a coalition and payoffs for players in this coalition only as in Pérez-Castrillo and Wettstein (2001), a partition and a payoff vector for all players should be picked. Similar proof techniques result in similar results as Pérez-Castrillo and Wettstein (2001) derive for their mechanism.

Finally, we remark that much of our analysis could have been carried out for a fourth rule as well, namely the rule that equally divides the value of a network among the players involved in at least one link. Whereas the position value focuses on a link, the Myerson value on a player, and the component-wise egalitarian solution on a component, this rule focuses on all cooperating players. Though this rule is not component efficient, it is efficient and satisfies a balancedness property, in which the contribution of a player to another player is measured by the payoff difference the second player experiences if *all* cooperation breaks down. Mechanisms can be described similarly as well. We concentrated on the Myerson value, the position value and the component-wise egalitarian solution only, because in our opinion they are more appealing then this egalitarian type of rule.

Appendix

In this appendix we provide proofs of several theorems.

Proof of theorem 5.3

Proof: The proof will be by induction on the number of links within a component. Obviously,

[5] They conclude that their mechanism ends up with the *player-based flexible allocation* and an efficient graph. However, this rule corresponds to the Myerson value of the monotonic cover of the value function in conjunction with the complete graph.

the statement in the theorem holds for all situations for all its components with no link.

Now, let $m \geq 1$ and assume that the statement in the theorem is true for all situations and for all components with at most $m-1$ links. Let (g, v) be a situation with $C \in N/g$ and $|g(C)| = m$.

Consider the following strategies:

t=1 Each player $i \in C$ makes to any player $k \neq i$ bid

$$b_k^i = \mu_k(g, v) - \mu_k(g - g_i, v).$$

t=2 Player α proposes to every player $k \neq \alpha$ payoff $y_k^\alpha = \mu_k(g - g_\alpha, v)$.

t=3 Player k accepts if $y_k^\alpha \geq \mu_k(g - g_\alpha, v)$ and rejects the offer otherwise.

First, we argue that these strategies have the Myerson value as their final payoff. Player $k \neq \alpha$ will receive

$$\begin{aligned} y_k^\alpha + b_k^\alpha =& \mu_k(g - g_\alpha, v) + [\mu_k(g, v) - \mu_k(g - g_\alpha, v)] \\ =& \mu_k(g, v). \end{aligned}$$

Player α will receive $v(g(C)) - \sum_{k \neq \alpha}[y_k^\alpha + b_k^\alpha]$. Using the expression above and component efficiency of the Myerson value it follows that player α receives $v(g(C)) - \sum_{k \neq \alpha} \mu_k(g, v) = \mu_\alpha(g, v)$.

Note that the payoff determination does not need the identity of the proposer α. Furthermore, we note that the strategies above result in net bids equal to zero for all $i \in N$ since

$$\begin{aligned} B^i &= \sum_{k \neq i} b_k^i - \sum_{k \neq i} b_i^k. \\ &= \sum_{k \neq i} \Big[[\mu_k(g, v) - \mu_k(g - g_i, v)] - [\mu_i(g, v) - \mu_i(g - g_k, v)] \Big] \\ &= 0, \end{aligned}$$

where the last equality follows from the balanced contributions property.

It remains to check that the strategies constitute an SPNE. It is obvious that the strategies at $t = 3$ are best responses. To check that the offers at $t = 2$ are subgame perfect, note that in case of rejection proposer α obtains $\mu_\alpha(g - g_\alpha, v)$. In case all others accept, player α can, taking into account the choices at $t = 3$ obtain at most $v(g(C)) - \sum_{k \neq \alpha} \mu_k(g - g_\alpha, v)$. Since v is monotonic we have that $v(g(C)) \geq v(g(C) - g_\alpha)$. Consequently, $v(g(C)) - \sum_{k \neq \alpha} \mu_k(g - g_\alpha, v) \geq \mu_\alpha(g - g_\alpha, v)$. Hence, player α maximizes his payoff by making the offers as described in the strategy.

Finally, consider the strategies at $t = 1$. Suppose player i changes his bids such that he will be the proposer with certainty. Then he has increased his total bids and thereby, he has decreased his eventual payoff (given his strategies at future periods and our analysis of these periods, which holds both on and off the equilibrium path!). If he changes his bids such that, with certainty, he will not be the proposer, another player will propose, meaning that his payoff will not change. Finally, suppose that player i has changed his bids, and that he remains in $\text{argmax}_j(B^j)$, while simultaneously $|\text{argmax}_j(B^j)| \geq 2$. Then there exist $j, k \in C$ such that b_j^i has been increased, while b_k^i has been decreased. To make sure that the net bid of player i is at least as much as the (new) net bid of player k (which has been increased), the total bid of player i must have been increased by at least the decrease in b_k^i. So, if player i is chosen with a positive probability he will increase his total bid and, hence, decrease his eventual payoff. If he is not chosen with a positive probability his payoff will remain unchanged. □

Proof of theorem 5.4

Proof: The proof will be by induction on the number of links within a component. Obviously, the statement in the theorem holds for all situations and all its components with no link.

Now, let $m \geq 1$ and assume that the statement in the theorem is true for all situations for all its components with at most $m-1$ links. Let (g, v) be a situation with $C \in N/g$ such that $|g(C)| = m$.

We will prove the induction step by proving several claims:

Claim 1 In any SPNE, at $t = 3$ all players who are not the proposer accept the proposal if $y_k^\alpha > \mu_k(g - g_\alpha, v)$ for all $k \neq \alpha$. Moreover, if $y_k^\alpha < \mu_k(g - g_\alpha, v)$ for some $k \neq \alpha$ then the proposal is rejected.

Claim 2 If $v(g(C)) > v(g(C) - g_\alpha)$ then all SPNE of a game that starts at $t = 2$ satisfy the following specifications:[6] at $t = 2$ player α proposes $y_k^\alpha = \mu_k(g - g_\alpha, v)$ to all $k \neq \alpha$ and at $t = 3$ every player $k \neq \alpha$ accepts offer y_k^α if $y_j^\alpha = \mu_j(g - g_\alpha, v)$ for all $j \neq \alpha$.
If $v(g(C)) = v(g(C) - g_\alpha)$ then any SPNE that does not satisfy the specifications in the first part of this claim satisfies the following specifications: at $t = 2$ player α proposes, among others, $y_k^\alpha \leq \mu_k(g - g_\alpha, v)$ to some player $k \neq \alpha$ and at $t = 3$ this offer is rejected by some player.
In all SPNE of this subgame the payoffs to the players are given by $v(g(C)) - v(g(C) - g_\alpha) - \sum_{k \neq \alpha} b_k^\alpha$ for player α and $\mu(g - g_\alpha) + b_k^\alpha$ for all $k \neq \alpha$.

Claim 3 In any SPNE, each player is indifferent about the selection of the proposer among $\text{argmax}\{B^i : i \in C\}$.

Claim 4 In any SPNE, $B^i = 0$ for all $i \in C$.

Claim 5 In any SPNE, the payoff of each of the players coincides with his Myerson value.

The proofs of the claims are along the lines of similar claims for the position value and have therefore been omitted. □

[6] We will not give a complete description of the SPNE.

Proof of theorem 5.5

Proof: The proof will be by induction on the number of links within a component. Obviously, the statement in the theorem holds for all situations and all its components with no link.

Now, let $m \geq 1$ and assume that the statement in the theorem is true for all situations and for all components with at most $m-1$ links. Let (g, v) be a situation with $C \in N/g$ and $|g(C)| = m$.

Consider the following strategies:

t=1 Each player $i \in C$ makes to any player $k \neq i$ bid

$$b_k^i = \gamma_k^{CE}(g, v).$$

t=2 Player α proposes to every player $k \neq \alpha$ payoff $y_k^\alpha = 0$.

t=3 Player k accepts if $y_k^\alpha \geq 0$ and rejects the offer otherwise.

First, we argue that these strategies have the component-wise egalitarian solution as their final payoff. Player $k \neq \alpha$ will receive

$$y_k^\alpha + b_k^\alpha = \gamma_k^{CE}(g, v).$$

Player α will receive $v(g(C)) - \sum_{k \neq \alpha}[y_k^\alpha + b_k^\alpha]$. Using the expression above and component efficiency of the position value it follows that player α receives $v(g(C)) - \sum_{k \neq \alpha} \gamma_k^{CE}(g, v) = \gamma_\alpha^{CE}(g, v)$.

Note that the payoff determination does not need the identity of the proposer α. Furthermore, we note that the strategies above result in net bids equal to zero for all $i \in N$ since

$$\begin{aligned} B^i &= \sum_{k \neq i} b_k^i - \sum_{k \neq i} b_i^k. \\ &= \sum_{k \neq i} \Big[[\gamma_k^{CE}(g, v) - \gamma_k^{CE}(g - g(C_i), v)] - [\gamma_i^{CE}(g, v) - \gamma_i^{CE}(g - g(C_k), v)] \Big] \\ &= 0, \end{aligned}$$

where the last equality follows from the balanced component contributions property.

It remains to check that the strategies constitute an SPNE. It is obvious that the strategies at $t = 3$ are best responses. To check that the offers at $t = 2$ are subgame perfect, note that in case of rejection proposer α obtains 0. In case all others accept, player α can, taking into account the choices at $t = 3$ obtain at most $v(g(C)) - \sum_{k \neq \alpha} \gamma_k^{CE}(g - g(C_\alpha), v) = v(g(C)) - 0 = v(g(C))$. Since v is monotonic we have that $v(g(C)) \geq 0$. Hence, player α maximizes his payoff by making the offers as described in the strategy.

Finally, consider the strategies at $t = 1$. Suppose player i changes his bids such that he will be the proposer with certainty. Then he has increased his total bids and thereby, he has decreased his eventual payoff (given his strategies at future periods and our analysis of these periods, which holds both on and off the equilibrium path!). If he changes his bids such that, with certainty, he will not be the proposer, another player will propose, meaning that his payoff will not change. Finally, suppose that player i has changed his bids, and that he remains in $\text{argmax}_j(B^j)$, while simultaneously $|\text{argmax}_j(B^j)| \geq 2$. Then there exist $j, k \in C$ such that b^i_j has been increased, while b^i_k has been decreased. To make sure that the net bid of player i is at least as much as the (new) net bid of player k (which has been increased), the total bid of player i must have been increased by at least the decrease in b^i_k. So, if player i is chosen with a positive probability he will increase his total bid and, hence, decrease his eventual payoff. If he is not chosen with a positive probability his payoff will remain unchanged. □

Proof of theorem 5.6

Proof: The proof will be by induction on the number of links within a component. Obviously, the statement in the theorem holds for all situations for all its components with no link.

Now, let $m \geq 1$ and assume that the statement in the theorem is true for all situations and for all its components with at most $m-1$ links. Let (g, v) be a situation with $C \in N/g$ such that $|g(C)| = m$.

We will prove the induction step by proving several claims:

Claim 1 In any SPNE, at $t = 3$ all players who are not the proposer accept the proposal if $y^\alpha_k > 0$ for all $k \neq \alpha$. Moreover, if $y^\alpha_k < 0$ for some $k \neq \alpha$ then the proposal is rejected.

Claim 2 If $v(g(C)) > 0$ then all SPNE of a game that starts at $t = 2$ satisfy the following specifications:[7] at $t = 2$ player α proposes $y^\alpha_k = 0$ to all $k \neq \alpha$ and at $t = 3$ every player $k \neq \alpha$ accepts offer y^α_k if $y^\alpha_j = 0$ for all $j \neq \alpha$.
If $v(g(C)) = 0$ then any SPNE that does not satisfy the specifications in the first part of this claim satisfies the following specifications: at $t = 2$ player α proposes, among others, $y^\alpha_k \leq 0$ to some player $k \neq \alpha$ and at $t = 3$ this offer is rejected by some player.
In all SPNE of this subgame the payoffs to the players are given by $v(g(C)) - \sum_{k \neq \alpha} b^\alpha_k$ for player α and b^α_k for all $k \neq \alpha$.

Claim 3 In any SPNE, each player is indifferent about the selection of the proposer among $\text{argmax}\{B^i : i \in C\}$.

Claim 4 In any SPNE, $B^i = 0$ for all $i \in C$.

Claim 5 In any SPNE, the payoff of each of the players coincides with his component-wise egalitarian solution.

The proofs of the claims are along the lines of similar claims for the position value and have therefore been omitted. □

[7] We will not give a complete description of the SPNE.

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