Synchronous behavior in networks of coupled systems : with applications to neuronal dynamics

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with applications to neuronal dynamics disc

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SYNCHRONOUS BEHAVIOR IN NETWORKS OF COUPLED SYSTEMS

with applications to neuronal dynamics

PROEFSCHRIFT

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Erik Steur

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Contents

Summary

Synchronous behavior in networks of coupled systems

Synchronization in networks of interacting dynamical systems is an interesting phenomenon that arises in nature, science and engineering. Examples include the simultaneous flashing of thousands of fireflies, the synchronous firing of action potentials by groups of neurons, cooperative behavior of robots and synchronization of chaotic systems with applications to secure communication. How is it possible that systems in a network synchronize? A key ingredient is that the systems in the network "communicate" information about their state to the systems they are connected to. This exchange of information ultimately results in synchronization of the systems in the network. The question is how the systems in the network should be connected and respond to the received information to achieve synchronization. In other words, which network structures and what kind of coupling functions lead to synchronization of the systems? In addition, since the exchange of information is likely to take some time, can systems in networks show synchronous behavior in presence of time-delays?

The first part of this thesis focusses on synchronization of identical systems that interact via *diffusive coupling*, that is a coupling defined through the weighted difference of the output signals of the systems. The coupling might contain time-delays. In particular, two types of diffusive time-delay coupling are considered: *coupling type I* is diffusive coupling in which only the transmitted signals contain a time-delay, and *coupling type II* is diffusive coupling in which every signal is time-delayed. It is proven that networks of diffusive time-delay coupled systems that satisfy a strict semipassivity property have solutions that are *ultimately bounded*. This means that the solutions of the interconnected systems always enter some compact set in finite time and remain in that set ever after. Moreover, it is proven that nonlinear minimum-phase strictly semipassive systems that interact via diffusive coupling always synchronize provided the interaction is sufficiently strong. If the coupling functions contain time-delays, then these systems synchronize if, in addition to the sufficiently strong interaction, the product of the time-delay and the coupling

strength is sufficiently small.

Next, the specific role of the topology of the network in relation to synchronization is discussed. First, using symmetries in the network, linear invariant manifolds for networks of the diffusively time-delayed coupled systems are identified. If such a linear invariant manifold is also attracting, then the network possibly shows *partial synchronization*. Partial synchronization is the phenomenon that some, at least two, systems in the network synchronize with each other but not with every system in the network. It is proven that a linear invariant manifold defined by a symmetry in a network of strictly semipassive systems is attracting if the coupling strength is sufficiently large and the product of the coupling strength and the time-delay is sufficiently small. The network shows partial synchronization if the values of the coupling strength and time-delay for which this manifold is attracting differ from those for which all systems in the network synchronize. Next, for systems that interact via *symmetric coupling type II*, it is shown that the values of the coupling strength and time-delay for which *any* network synchronizes can be determined from the structure of that network and the values of the coupling strength and time-delay for which *two* systems synchronize.

In the second part of the thesis the theory presented in the first part is used to explain synchronization in networks of neurons that interact via electrical synapses. In particular, it is proven that four important models for neuronal activity, namely the *Hodgkin-Huxley* model, the *Morris-Lecar* model, the *Hindmarsh-Rose* model and the *FitzHugh-Nagumo* model, all have the semipassivity property. Since electrical synapses can be modeled by diffusive coupling, and all these neuronal models are nonlinear minimum-phase, synchronization in networks of these neurons happens if the interaction is sufficiently strong and the product of the time-delay and the coupling strength is sufficiently small. Numerical simulations with various networks of Hindmarsh-Rose neurons support this result. In addition to the results of numerical simulations, synchronization and partial synchronization is witnessed in an *experimental setup* with type II coupled electronic realizations of Hindmarsh-Rose neurons. These experimental results can be fully explained by the theoretical findings that are presented in the first part of the thesis.

The thesis continues with a study of a network of *pancreatic β-cells*. There is evidence that these β-cells are diffusively coupled and that the synchronous bursting activity of the network is related to the secretion of insulin. However, if the network consists of active (oscillatory) β-cells and inactive (dead) β-cells, it might happen that, due to the interaction between the active and inactive cells, the activity of the network dies out which results in a inhibition of the insulin secretion. This problem is related to Diabetes Mellitus type 1. Whether the activity dies out or not depends on the number of cells that are active relative to the number of inactive cells. A bifurcation analysis gives estimates of the number of active cells relative to the number of inactive cells for which the network remains active.

At last the *controlled* synchronization problem for *all-to-all* coupled strictly semipassive

systems is considered. In particular, a systematic design procedure is presented which gives (nonlinear) coupling functions that guarantee synchronization of the systems. The coupling functions have the form of a definite integral of a scalar weight function on a interval defined by the outputs of the systems. The advantage of these coupling functions over linear diffusive coupling is that they provide high gain only when necessary, i.e. at those parts of the state space of the network where nonlinearities need to be suppressed. Numerical simulations in networks of Hindmarsh-Rose neurons support the theoretical results.

CHAPTER ONE

Introduction

Abstract. In this introductory chapter the synchronization phenomenon is introduced and some historical notes are given. It is shown that synchronization plays an important role in our daily lives, and that there are many important applications of synchronization. In this chapter the motivation for this thesis and the main contributions are presented. In addition, the structure of the thesis is discussed. At the end of this chapter a list of the author's publications is given.

1.1 The synchronization phenomenon and historical notes

Synchronization is everywhere, whether it is the simultaneous flashing of thousands of fireflies that gather in trees along the tidal rivers in Malaysia [141, 27] (see [151] for a nice color picture), or the undesired lateral vibrations of London's Millennium Bridge on its opening day induced by the synchronized feet of pedestrians walking over it [150]. Synchronization is inevitable and plays an important role in our lives. Clusters of synchronized pacemaker neurons regulate our heartbeat [121], synchronized neurons in the olfactory bulb allow us to detect and distinguish between odors [53], and our circadian rhythm is synchronized to (more precisely, entrained to) the 24-hour day-night cycle [40, 167].

Synchronization should be understood as the phenomenon that "things" keep happening simultaneously for an extended period of time [149]. Synchronization is persistent. Two fish that "accidentally" swim in the same direction for some time can not be called synchronized, while a school of fish that moves though the ocean like a single organism can be considered as synchronized. In other words, synchronization is the (stable) time-correlated behavior of two or more processes [23]. Probably one of the clearest examples of synchronization in that sense is the firefly example; all fireflies light up at the same time. Another but probably less clear example is the synchronization of the the orbit of the moon around the earth and its spin. The same side of the moon is always facing earth which is because the moon spins around its axis in the same amount of time it takes the moon to orbit around earth [149]. Also less obvious is the synchronization of the legs of a horse when trotting: the front left leg and the right back leg are in sync but half a period out of sync with the synchronized other pair [37].

To have persistent synchronization of certain systems there should be some kind of interaction between the systems. This interaction can be of the type *master-slave*, where one system influences the other system(s), or there can be *mutual* interaction, where all systems influence each other. A clear example of master-slave synchronization is the synchronization of the circadian rhythm to the 24-hour day-night cycle; a change in our circadian rhythm does not affect the 24-hour day-night cycle. The synchronization of the fireflies is an example of mutual synchronization; there is no single firefly that orchestrates the rhythmic synchronized blinking. Each firefly adjusts its own rhythm of lighting up as a response to the flashes of the others, resulting in a mysterious self-organizing collective behavior. Sometimes one can intuitively explain why systems synchronize but often the mechanism that synchronizes systems is not trivial. A nice non-trivial example is the crowd synchronization on the Millennium Bridge in London on the day it opened. In [150, 45], a theory is presented that explains what happened that day. When a critical number of pedestrians was walking over the bridge, it started to vibrate in lateral direction. As a natural response, to keep their balance, people where stepping to the left or to the right at the same time, counteracting the bridge's lateral movement. The lateral movement of the bridge started the crowd to walk synchronously. As more pedestrians stepped in synchrony, the larger forces acting on the bridge made it vibrate even more, triggering more and more people to synchronize their feet. Eventually a large number of people stepped in synchrony, inducing a movement of the bridge in lateral direction with an amplitude of a couple of centimeters.

The example of the Millennium bridge shows great resemblance to what the Dutch scientist Christiaan Huygens wrote down in his notebook in the seventeenth century [67] (which is probably the first scientific description of the synchronization phenomenon). Huygens observed that two of his famous pendulum clocks that where hanging on a beam supported by two chairs always ended up swinging in opposite direction. This "sympathy", as he called it, was persistent; for any kind of perturbation he applied the clocks ended up in synchrony. Huygens' explanation of this remarkable phenomenon was that the motion of the beam that induced the synchronization of the two clocks [123], just like the motion of the Millennium bridge induced the crowd synchrony. His explanation was remarkably accurate given that differential calculus was still to be invented those days.

About two hundred years later, Lord Rayleigh described in his famous book *"the theory of sound"* the synchronized sound of two organ tubes whose outlets where close to each other [130]. In the beginning of the twentieth century, Balthasar van der Pol and Sir Edward Victor Appleton discussed the synchronization of a triode oscillator to an external input [10, 160]. This result was important as it had applications to radio communication. In the eighties of the twentieth century, in Russia, synchronization in balanced and unbalanced rotors and vibro-exciters was reported [22]. See also [153, 25] and the references therein. These examples of synchronization of (electro-)mechanical systems have important applications in milling processes and electrical generators. In 1990, Pecora and Carroll published their famous paper *"Synchronization in chaotic systems"*, [116], which discussed synchronization of two master-slave coupled chaotic Lorenz systems. Until then it was widely believed that synchronization of chaotic systems was impossible since in a chaotic system small disturbances grow exponentially fast. However, Pecora and Carroll showed that chaotic systems can synchronize. Applications of chaos synchronization are in secure communication; a chaotic master system can mask a message that is recovered by the synchronized slave [39]. Also in 1990, Mirollo and Strogatz published the paper *"Synchronization of pulse-coupled biological oscillators"*, [95], in which a model is presented that explains why, for instance, fireflies synchronize. Motivated by these important works, synchronization became of popular subject of study for physicists, biologists, mathematicians and engineers. See for instance, the special issues [1, 3, 2, 4, 5, 6, 7, 8] and the references therein.

1.2 Applications and controlled synchronization

Synchronization is not only something that just happens, but there are also numerous applications. One application that is already mentioned before is the secure communication via synchronization of chaotic systems [116]. See also [65] and [39]. Another important application is the synchronization of robot manipulators, commonly referred to as *cooperation* or *coordination* [122]. Synchronization of robots can give flexibility and manoeuvrability that can not be achieved by a single manipulator [104]. Examples include tele-operated master-slave systems, multi-actuated positioning systems and medical robotics for minimal invasive surgery.

Interesting applications of synchronization are in the area of automotive engineering. For instance, if vehicles are able to ride in a platoon, i.e. a cluster or string of synchronized vehicles, with relatively short intervehicle distances, a significant reduction of aerodynamics drag is possible, resulting in lower fuel consumption [99]. Another automotive application is the synchronization of windscreen wipers discussed in [79]. To save space and weight, it is suggested to remove the classical bulky rigid mechanical connection between the wipers and drive them instead by independent motors. Synchronization between the wipers is then needed to avoid collisions.

When considered the synchronization of two or more systems, one can distinguish two directions: *synchronization analysis* of interconnected systems with given coupling functions and communication structure, and *design* of coupling functions and network structures that guarantees synchronization of systems. In general, trying to find explanations why synchronization happens is an analysis problem, while for engineering applications of synchronization one often has to find controllers which guarantee that synchronization will be achieved. Designing coupling functions and network structures that lead to synchronization of systems is called *controlled synchronization*. The controlled synchronization of master-slave systems is closely related to *observer design* known from (non)linear control theory [103]. Indeed, using the transmitted signals from the master system, the states of the slave system have to be reconstructed in such a way that they match the states of the master, i.e. there is synchronization of master and slave. The controlled synchronization of master-slave systems can also be considered as a particular case of the (nonlinear) *regulator problem* [66, 114] for which conditions for the solvability exist. Controlled synchronization for mutually coupled systems is discussed in, for instance, [35, 36, 104].

1.3 Motivation, contributions and outline

Consider a network consisting of k systems of the form

$$\dot{x}_i(t) = f(x_i(t), u_i(t)), \tag{1.1a}$$

$$y_i(t) = h(x_i(t)), \quad i = 1, 2, \ldots, k, \tag{1.1b}$$

with state x_i, input u_i and output y_i. The systems are coupled; the inputs of the systems will depend on the outputs of the systems they are connected to. Such couplings are described by the equations

$$u_i(t) = G_i(y_1(t - \tau_{i1}), y_2(t - \tau_{i2}), \ldots, y_k(t - \tau_{ik})), \quad i = 1, 2, \ldots, k, \tag{1.2}$$

with G_i being the coupling function for the i^{th} system. The coupled systems (1.1), (1.2) will be called *synchronized* if their states asymptotically match, i.e. $x_i(t) \to x_j(t)$ as $t \to \infty$ for all i, j. The coupling functions have to satisfy the communication structure of the network; $u_i(t)$ can only be influenced by the output of system $y_j(t - \tau_{ij})$ if system j connects to system i. The constants τ_{ij} represent time-delays. A signal is time-delayed if $\tau_{ij} > 0$ and non-delayed if $\tau_{ij} = 0$. It is relevant to take time-delays into account as the communication between two or more systems can take an amount of time that often can not be neglected. An example is the coupling of two distant neurons; due to the finite propagation speed of the membrane potential through the neuron's axon [73], a neuron "feels" the change of membrane potential of the other neuron it is connected to only after some time has elapsed. It might also be the case that the time-delay is induced by the time that it takes to "compute" the coupling functions. An example of this is when humans are trying to drive their cars at a fixed distance of each other [139]. All drivers compare the distance between their vehicles and the vehicles in front of them and decide whether they should maintain their current speed or have to accelerate or decelerate to keep the desired distance. However, the reaction time[1] of the drivers can not be neglected; experiments and simulator results show that the reaction time varies between 0.6 s and 2 s, [139].

This thesis consists of two parts. The first part presents results on synchronization of diffusively coupled systems. Diffusive coupling is a linear coupling that is proportional to the difference of the (time-delayed) output signals of the interacting systems, cf. [55, 128]. For instance, in a network of two diffusively coupled systems without time-delays the coupling functions are

$$u_1(t) = \sigma a_{12}(y_2(t) - y_1(t)), \tag{1.3a}$$
$$u_2(t) = \sigma a_{21}(y_1(t) - y_2(t)). \tag{1.3b}$$

Here the positive constant σ denotes the coupling strength and nonnegative scalars a_{12} and a_{21} are the weights of the interconnections. The notation σa_{12} and σa_{21} looks a bit cumbersome here; one might have expected simply σ_{12} and σ_{21} instead. The main reason to use this notation is that in this thesis the networks are supposed to be given, i.e. a_{12} and a_{21} are supposed to be fixed and known. Then, for fixed and known values a_{12} and a_{21}, conditions for synchronization will be expressed in terms of the value of the coupling strength σ. In case of time-delayed interaction, two types of diffusive coupling will be considered. In the first type of coupling the time-delay appears only in the "received" signals. For two coupled systems possible coupling functions are

$$u_1(t) = \sigma a_{12}(y_2(t-\tau) - y_1(t)), \tag{1.4a}$$
$$u_2(t) = \sigma a_{21}(y_1(t-\tau) - y_2(t)), \tag{1.4b}$$

where the positive constant τ represents the amount of time-delay. This type of diffusive coupling will in the remainder be referred to as *coupling type I*. Of course, it is also possible that every signal in the coupling functions contains a time-delay. Possible coupling functions that describe this type of interaction in a network of two systems are

$$u_1(t) = \sigma a_{12}(y_2(t-\tau) - y_1(t-\tau)), \tag{1.5a}$$
$$u_2(t) = \sigma a_{21}(y_1(t-\tau) - y_2(t-\tau)). \tag{1.5b}$$

Interaction of this type will be called *coupling type II*. An important difference between coupling type I and coupling type II is that if the systems are synchronized then coupling type II vanishes, i.e. $y_i(t) = y_j(t)$ implies $u_i(t) = u_j(t) = 0$, but coupling type I generally does not vanish[2]. This implies that the solutions of synchronized type II coupled systems are a solution of an uncoupled system whereas the solutions of synchronized type I coupled systems will generally not be a solution of an uncoupled system.

[1]The reaction time consists of the time it takes to receive and process visual information, the time that is needed to make a decision and the time it takes to hit the brakes or the accelerator pedal.

[2]Coupling type I vanishes only if the synchronized systems have τ-periodic or constant steady-state solutions.

Diffusive interaction is an important type of coupling. It is found in, for instance, networks of coupled neurons [21, 34, 38, 74, 80, 89, 162], networks of biological systems [119, 137, 41], coupled mechanical systems [104, 132, 131, 36, 172] and electrical systems [39, 169]. In [124, 128] a framework is introduced to analyze synchronization of systems that interact via non-delayed symmetric diffusive coupling, i.e. (1.3) with $a_{12} = a_{21}$. In this framework, it is assumed that each systems has a property called semipassivity. A semipassive system is a system whose state trajectories remain bounded provided that the supplied energy is bounded[3]. Many physical and biological systems do have such a property, cf. [147]. It is proved in [124, 128] that semipassive systems that interact via symmetric non-delayed diffusive coupling have solutions that are *ultimately bounded*. That is, every solution enters a compact set in finite time and remains there ever after. Moreover, under the assumption that the system is (nonlinear) minimum-phase[4], it proved that there exists a positive constant, say $\bar{\sigma}$, such that the systems synchronize if the coupling strength is larger than or equal to this constant, i.e. $\sigma \geq \bar{\sigma}$.

This thesis extends the ideas presented in [124, 128]. In particular, in *chapter 3*, the semipassivity-based framework for synchronization of diffusively coupled systems is generalized in the sense that

i. the interaction is not assumed to be symmetric, i.e. a_{ij} is not necessarily a_{ji};

ii. the diffusive coupling functions might contain time-delays.

For both coupling type I and coupling type II, it is proven that the solutions of diffusively time-delay coupled strictly semipassive systems are ultimately bounded. Moreover, it is proven that if these systems are also minimum-phase, then the systems synchronize if the coupling is sufficiently strong and, in addition, the product of the coupling strength and the time-delay is sufficiently small. See Figure 1.1. The results presented in this chapter are published in [146].

In *chapter 4* results are presented on *partial synchronization* in networks of diffusively time-delay coupled systems. Partial synchronization, also known as *clustering*, is the phenomenon where some, at least two, systems in the network do synchronize with each other but not with every system in the network. In [129, 125, 126], it is shown that symmetries in networks of systems interacting via non-delayed symmetric diffusive coupling define linear invariant manifolds. Moreover, it is proven that a linear invariant manifold defined by a symmetry in a network of strictly semipassive minimum-phase systems is attracting if the coupling strength is sufficiently large. The network shows partial synchronization if the coupling strength for which this linear invariant manifold is attracting is *lower* than the coupling strength for which all systems in the network synchronize.

[3]A formal definition of a semipassive systems will be presented in section 2.3.

[4]Details are provided in Chapter 3.

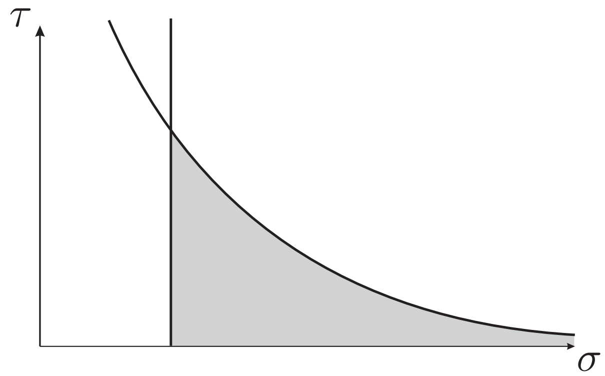


Figure 1.1. Diffusively time-delay coupled strictly semipassive systems synchronize if the coupling strength σ and time-delay τ belong to the shaded area.

Chapter 4 extends the results of [129, 125, 126] to the case of diffusive time-delay interaction which is not assumed to be symmetric. Like in [129, 125, 126], it is shown that symmetries in networks of diffusively time-delay coupled systems define linear invariant manifolds. Such a linear invariant manifold for a network of coupled strictly semipassive minimum-phase systems is attracting if the coupling strength is sufficiently large and the product of the coupling strength and the time-delay is sufficiently small. The network shows partial synchronization if the values of the coupling strength and time-delay for which this manifold is attracting differ from those for which all systems in the network synchronize. Most of the results presented in this chapter are derived for uniform time-delays, i.e. every time-delay has the same value. Section 4.4 presents some results on partial synchronization for systems that interact via coupling type I with non-uniform time-delays.

In *chapter 5* a relation between synchronization of two symmetric type II coupled systems and synchronization in more complex networks of symmetric type II coupled systems is established. In particular, it is shown in this chapter that the knowledge of the values of the coupling strength and time-delay for which two symmetric type II coupled systems synchronize is sufficient to determine those values of the coupling strength and time-delay for which any network of symmetric type II coupled systems synchronizes. For general coupled systems these results that are presented hold *locally*, that is, the systems will synchronize given that they are already sufficiently close. They become *global* if the systems are strictly semipassive and minimum-phase. The results presented in this chapter can be considered as a generalization of the famous Wu-Chua conjecture [170]. This chapter is based on [145].

The second part of this thesis shows how the theory presented in the first part can be applied. Some related results are presented in addition. The focus is on synchronization in networks of neurons. First, in *chapter 6*, it is proven that four of the most popular models for neural activity do have the strict semipassivity property. That is, the Hodgkin-Huxley model, the Morris-Lecar model, the Hindmarsh-Rose model and the FitzHugh-Nagumo model are all strictly semipassive. Moreover, all these models are also minimum-phase. These results are important because they explain, using the theory presented in the first part, the (experimentally) observed synchronous behavior of neurons that interact via so-called *electrical synapses*. Simulations illustrate the theoretical results. The results presented in this chapter are published in [147].

Chapter 7 presents examples of synchronization and partial synchronization in networks of diffusively time-delay coupled Hindmarsh-Rose neurons. The examples that are presented are results of numerical simulations and experiments with setup of type II coupled electronic Hindmarsh-Rose neurons. Some of the results presented in this chapter are published in [101].

Chapter 8 studies synchronization and activation in networks of coupled pancreatic β-cells. These cells play an important role in glucose homeostasis since they release insulin, which is the hormone that is mainly responsible for the blood glucose regulation. The β-cells are known to be diffusively coupled and there is evidence that the synchronized bursting activity is closely related to the insulin secretion. First it is shown that synchronous bursting activity can indeed be expected in a network of properly functioning β-cells. Next networks are considered that consist of cells that are functioning well and cells that are dead. It is shown that all activity of the network stops if the number of dead cells relative to the number of healthy cells exceeds a certain threshold. Analytical estimates of this threshold are derived and numerical simulations verify the results. The results presented in this chapter are published in [12].

In *Chapter 9* is the focus is on the *controlled* synchronization problem. Using the notions of semipassivity, convergent systems[5] and *incremental passivity* [110], a method is described to derive nonlinear integral coupling functions that guarantee synchronization in networks of *all-to-all* coupled systems. The main idea of the approach is to overcome the disadvantages of the conventional linear high gain coupling in practical applications, e.g. when there is a lot of output noise. The proposed method gives coupling gains that are only large in the parts of the state space where the nonlinearities have to be suppressed. The results are illustrated using simulations of a network with two Hindmarsh-Rose neurons. The results presented in this chapter are published in [113][6].

Figure 1.2 shows the structure of the thesis. *Chapter 2* contains some basic definitions and mathematical tools that will be used throughout this thesis. It is strongly advised to read chapter 2 first. Chapters 3, 8 and 9 can be read independently (after reading chapter 2). These chapters are all self-contained with their own introduction and conclusions.

[5] Convergent systems will be defined in section 2.4

[6] The main ideas presented in this chapter are of the first author of [113], A. V. Pavlov. This chapter is included with his permission.

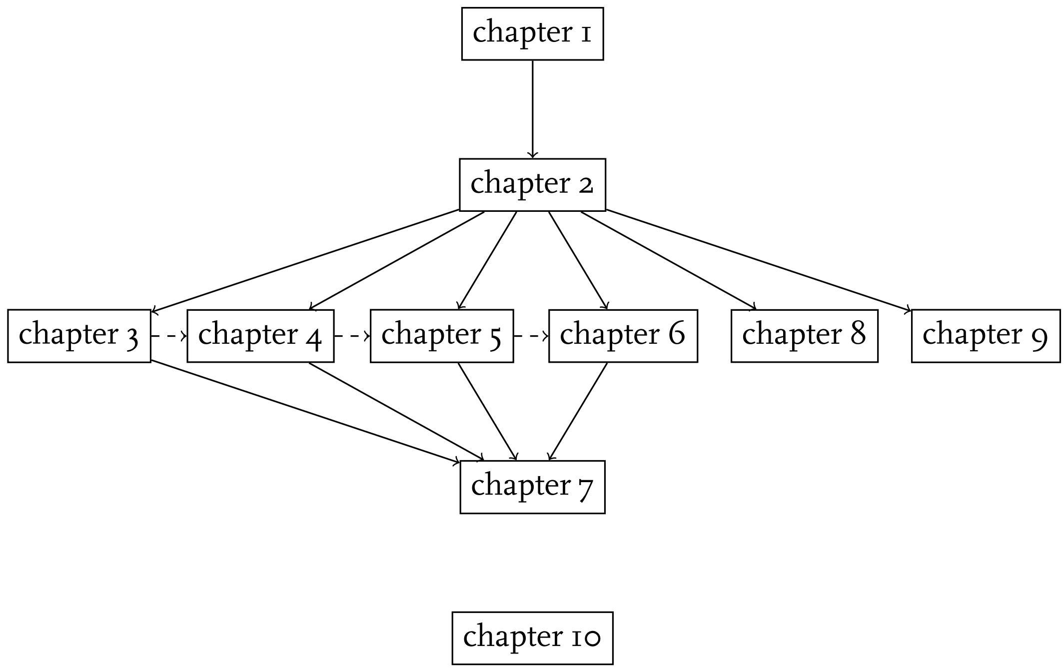


Figure 1.2. Structure of the thesis.

It is recommended to read chapter 3 before reading chapters 4, 5 and 6. Chapter 7, in which simulation results and experimental results are presented, should be read only after reading chapters 3, 4, 5 and 6.

Chapter 10 summarizes the most important conclusions of all chapters. In addition, some recommendations for future research are given. Not shown in Figure 1.2 are the appendices. *Appendix A* provides the proofs of the technical results. *Appendix B* presents a parameter estimation procedure for a Hindmarsh-Rose neuron. These results are published in [148] and generalized in [158, 155]. The machinery that is used is published in [157]. In [156] a more general procedure for the estimation of parameters of such models is presented.

1.4 List of Publications

Refereed journal publications

- E. Steur and H. Nijmeijer, “Synchronization in networks of diffusively time-delay coupled (semi-)passive systems,” *IEEE trans. Circ. Syst. I*, vol. 58, no. 6, pp. 1358—1371, 2011. (Chapter 3)
- E. Steur, I. Tyukin, and H. Nijmeijer, “Semi-passivity and synchronization of diffusively coupled neuronal oscillators,” *Physica D*, vol. 238, no. 21, pp. 2119–2128, 2009. (Chapter 6)
- P. J. Neefs, E. Steur, and H. Nijmeijer, “Network complexity and synchronous behavior: An experimental approach,” *Int. J. Neural Systems*, vol. 20, no. 3, pp. 233–247, 2010. (Chapter 7)
- J. G. Barajas Ramírez, E. Steur, R. Femat, and H. Nijmeijer, “Synchronization and activation in a model of a network of β-cells,” *Automatica*, vol. 47, no. 6, pp. 1243–1248, 2011. (Chapter 8)
- I. Tyukin, E. Steur, H. Nijmeijer, D. Fairhurst, I. Song, A. Semyanov, and C. v. Leeuwen, “State and parameter estimation for canonic models of neural oscillators,” *Int. J. Neural Syst.*, vol. 20, no. 3, pp. 193–207, 2010.
- I. Tyukin, E. Steur, H. Nijmeijer, and C. v. Leeuwen, “Non-uniform small-gain theorems for systems with unstable invariant sets,” *SIAM J. Opt. Contr.*, vol. 47, no. 2, pp. 849–882, 2008.

Submitted journal publications

- E. Steur, W. Michiels, H. J. C. Huijberts, and H. Nijmeijer, “Networks of diffusively time-delay coupled systems: Synchronization and its relation to the network topology.” (Chapter 5)
- I. Tyukin, E. Steur, H. Nijmeijer, and C. v. Leeuwen, “Adaptive observers and parametric identification for systems without a canonical adaptive observer form.”

Journal publications in preparation

- E. Steur and H. Nijmeijer, “Partial synchronization in networks of diffusively time-delay coupled systems.” (Chapter 4)
- A. Gorban, I. Tyukin, E. Steur and H. Nijmeijer, “Positive invariance lemmas for control problems with convergence to Lyapunov-unstable sets.”

Book chapters

- E. Steur, L. Kodde and H. Nijmeijer, "Synchronization of diffusively coupled electronic Hindmarsh-Rose oscillators," in *Dynamics and Control of Hybrid Mecahnical Systems*, ser. World Scientific series on Nonlinear Science, Series B, vol. 14, pp. 195–208, G. Leonov, H. Nijmeijer, A. Pogromsky, and A. Fradkov, Eds. World Scientific, 2010.

Refereed proceedings

- E. Steur, I. Tyukin, H. Nijmeijer, and C. v. Leeuwen, "Reconstructing dynamics of spiking neurons from input-output measurements in vitro," in *Proceedings of the 3rd IEEE Conference on Physics and Control, Potsdam, Germany*, 2007. (Appendix B)
- E. Steur, L. Kodde and H. Nijmeijer, "Synchronization of diffusively coupled electronic Hindmarsh-Rose oscillators," in *Sixth European Nonlinear Dynamics Conference (ENOC2008), Saint Petersburg, Russia*, 2008.
- I. Tyukin, E. Steur, H. Nijmeijer and C. v. Leeuwen, "State and Parameter Estimation for Systems in Non-canonical Adaptive Observer Form," in *17th IFAC World Congress on Automation Control, Seoul, Korea*, 2008.
- I. Tyukin, E. Steur, H. Nijmeijer and C. v. Leeuwen, "Non-uniform small-gain theorems for systems with unstable invariant sets," in *47th IEEE Conference on Decision and Control, Cancun, Mexico*, 2008.
- E. Steur, I. Tyukin and H. Nijmeijer, "Semi-passivity and synchronization of neuronal oscillators," in *IFAC CHAOS 2009, London, UK*, 2009.
- A. V. Pavlov, E. Steur and N. v. d. Wouw, "Controlled synchronization via nonlinear integral coupling," in *joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference, Shanghai, China*, 2009. (Chapter 9)

CHAPTER TWO

Preliminaries

Abstract. In this chapter the notation and (mathematical) concepts that will be used throughout the thesis are introduced. In section 2.1 the notation is introduced. Section 2.2 discusses stability concepts for ordinary differential equations. The notions of (semi)passivity and convergent systems are presented in sections 2.3 and 2.4, respectively. Section 2.5 deals with retarded function differential equations and stability of retarded functional differential equations. Finally, in section 2.6 some basic graph theoretical results are discussed.

2.1 Notation

The symbol $\mathbb{R}$ stands for the real numbers $(-\infty, \infty)$, $\mathbb{R}_{>0}$ ($\mathbb{R}_{\geq 0}$) denotes the set of positive (non-negative) real numbers and $\mathbb{R}^n$ denotes the n-fold cartesian product $\mathbb{R} \times \ldots \times \mathbb{R}$. The symbol $\mathbb{C}$ stands for the complex numbers, $\mathbb{C}_{>0}$ ($\mathbb{C}_{\geq 0}$) denotes the set of complex numbers with positive (non-negative) real part. The set of integers is denoted by $\mathbb{Z}$, and $\mathbb{N}$ is the set of positive integers. The Euclidian norm in $\mathbb{R}^n$ is denoted by $|\cdot|$, $|x|^2 := x^\top x$, where $x^\top$ denotes the transpose of x. Let $\epsilon \in \mathbb{R}_{>0}$, then $|x|_\epsilon$ stands for the following:

$$|x|_\epsilon = \begin{cases} |x| - \epsilon, & \text{if } |x| > \epsilon, \\ 0, & \text{otherwise.} \end{cases}$$

The induced norm of a matrix $A \in \mathbb{R}^{n\times n}$, denoted by $\|A\|$, is defined as $\|A\| = \max_{x\in\mathbb{R}^n, |x|=1} |Ax|$. The $n \times n$ identity matrix is denoted by I_n. Simply I is written if no confusion can arise. The notation $\operatorname{col}(x_1, \ldots, x_n)$ denotes the column vector with entries $x_1, \ldots, x_n$. Here x_i might be scalars or column vectors. The symbol $\otimes$ denotes the Kronecker product of two matrices, i.e. let $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{p\times l}$, then the matrix

$A \otimes B \in \mathbb{R}^{np \times ml}$ is given as

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1m}B \\ a_{21}B & a_{22}B & \dots & a_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \dots & a_{nm}B \end{pmatrix},$$

where a_{ij} denotes the ij^{th} entry of the matrix A. The spectrum, determinant and trace of a matrix A are denoted by $\operatorname{spec}(A)$, $\det(A)$ and $\operatorname{trace}(A)$, respectively.

Let $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{Y} \subset \mathbb{R}^m$. The space of continuous functions from $\mathcal{X}$ to $\mathcal{Y}$ that are (at least) $r \geq 0$ times continuously differentiable is denoted by $\mathcal{C}^r(\mathcal{X}, \mathcal{Y})$, $\mathcal{C}(\mathcal{X}, \mathcal{Y})$ is simply the space of continuous functions from $\mathcal{X}$ to $\mathcal{Y}$. If the derivatives of a function of all orders ($r = \infty$) exist the function is called *smooth* and if the derivatives up to a sufficiently high order exist the function is called *sufficiently smooth*. Let $\mathcal{L}_\infty(\mathcal{X}, \mathcal{Y})$ be the space of essentially bounded functions that map elements of $\mathcal{X}$ into elements of $\mathcal{Y}$, i.e. $\mathcal{L}_\infty(\mathcal{X}, \mathcal{Y})$ is the space of all measurable functions $f : \mathcal{X} \to \mathcal{Y}$ for which $\operatorname{ess\,sup} |f| < \infty$. A function $V : \mathcal{D} \to \mathbb{R}_{\geq 0}$, $\mathcal{D} \subset \mathbb{R}^n$ contains 0, is called positive (semi)definite, denoted by $V(\cdot) > 0$ ($V(\cdot) \geq 0$), if $V(0) = 0$ and $V(x) > 0$ ($V(x) \geq 0$) for all $x \in \mathcal{D} \setminus \{0\}$. It is radially unbounded if $\mathcal{D} = \mathbb{R}^n$ and $|x| \to \infty$ implies $V(x) \to \infty$. If the quadratic form $x^\top P x$ with a symmetric matrix $P = P^\top$ is positive (semi)definite, then the matrix P is positive (semi)definite, denoted by $P > 0$ ($P \geq 0$).

2.2 Stability concepts for ordinary differential equations

Consider a system of ordinary differential equations,

$$\dot{x}(t) = f(t, x(t)), \tag{2.1}$$

with state $x \in \mathbb{R}^n$ and $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ being piecewise continuous in t and locally Lipschitz continuous in x for all $t \geq t_0$. The dot notation, " $\cdot$ ", stands, as usual, for the derivative with respect to t. A solution of (2.1) on $[t_0, t_0 + T]$ is a function $x(t)$ that satisfies (2.1) on $[t_0, t_0 + T]$ almost everywhere. A solution of (2.1) though (t_0, x_0), denoted by $x(t; t_0, x_0)$, is a solution of (2.1) for which $x(t_0) = x_0$. The assumptions on f guarantee existence and uniqueness of solutions.

Definition 2.1 (Lyapunov stability [114, 135]). Suppose that $f(t, 0) = 0$ for all $t \geq t_0$ and let $x(t_0) = x_0$. Then the trivial solution $x \equiv 0$ is

i. *stable* (in the sense of Lyapunov) if for any number $\varepsilon > 0$ and any $t_0 \in \mathbb{R}$, there is $\delta = \delta(\varepsilon, t_0) > 0$ such that $|x_0| < \delta$ implies $|x(t; t_0, x_0)| < \varepsilon$ for all $t \geq t_0$;

ii. *uniformly stable* (in the sense of Lyapunov) if it is stable and the number δ can be chosen independently of t_0;

iii. *asymptotically stable* (in the sense of Lyapunov) if it is stable and there is a number $\bar{\delta} = \bar{\delta}(t_0) > 0$ such that $|x_0| < \bar{\delta}$ implies $|x(t; t_0, x_0)| \to 0$ as $t \to \infty$;

iv. *uniformly asymptotically stable* (in the sense of Lyapunov) if it is uniformly stable and there is a number $\bar{\delta} > 0$ such that for any $\varepsilon > 0$ there is a $T = T(\varepsilon) > 0$ such that $|x_0| < \bar{\delta}$ implies $|x(t; t_0, x_0)| < \varepsilon$ for all $t \geq t_0 + T$;

v. *exponentially stable* (in the sense of Lyapunov) if there are constants $m, \alpha > 0$ such that $|x(t; t_0, x_0)| \leq me^{-\alpha(t-t_0)} |x_0|$ for all $t \geq t_0$.

△

Remark 2.1. All definitions for Lyapunov stability are given *locally*. They become *global* if the definitions hold for all $x_0 \in \mathbb{R}^n$. △

The stability of an equilibrium of an *ordinary differential equation* can be ensured by constructing a (suitable) *Lyapunov function*.

Theorem 2.1 (Lyapunov's second method [72]). *Consider* (2.1) *and suppose that* $f(t, 0) = 0$ *for all* $t \geq t_0$. *Let* $u, v, w : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *be continuous nondecreasing functions,* $u(s)$ *and* $v(s)$ *are positive for* $s > 0$, *and* $u(0) = v(0) = 0$. *Suppose that there exists a positive definite function* $V \in \mathcal{C}^1(\mathbb{R} \times \mathcal{D}, \mathbb{R}_{\geq 0})$ *such that*

$$u(|x|) \leq V(t, x) \leq v(|x|).$$

and

$$\dot{V}(t, x) = \frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} f(t, x) \leq -w(|x|)$$

for all $x \in \mathcal{D}$ *and* $t \geq t_0$, *then the origin of* (2.1) *is* uniformly stable. *The origin of* (2.1) *is* asymptotically uniformly stable *if it is uniformly stable and* $w > 0$ *for all* $x \in \mathcal{D} \backslash \{0\}$. *The origin of* (2.1) *is* globally uniformly (asymptotically) stable *if it is uniformly (asymptotically) stable with* $\mathcal{D} = \mathbb{R}^n$ *and* $u \to \infty$ *as* $|x| \to \infty$.

Stability (in the sense of Lyapunov) of (2.1) can also be defined with respect to sets. First invariance and attractivity of a set with respect to (2.1) are defined.

Definition 2.2 (Invariance of sets [78]). Let $\mathcal{A}$ be a nonempty set and $x(t; t_0, x_0)$ a solution of (2.1) through (t_0, x_0). Then $\mathcal{A}$ is called

i. *invariant* under (2.1) if $x_0 \in \mathcal{A}$ implies $x(t; t_0, x_0) \in \mathcal{A}$ for all $t \in \mathbb{R}$;

ii. *positive invariant* under (2.1) if $x_0 \in \mathcal{A}$ implies $x(t; t_0, x_0) \in \mathcal{A}$ for all $t \geq t_0$.

$\triangle$

Definition 2.3 (Attractivity of sets [157, 94]). Let $\mathcal{A}$ be a nonempty set and $x(t; t_0, x_0)$ a solution of (2.1) through (t_0, x_0). Then $\mathcal{A}$ is called an *attracting set* if

i. $\mathcal{A}$ is positively invariant under the dynamics (2.1), and

ii. there exists a set $\mathcal{T} \subset \mathbb{R}^n$ of strictly positive measure such that $\lim_{t\to\infty} \operatorname{dist}(x(t; t_0, x_0), \mathcal{A}) = 0$ for all $x_0 \in \mathcal{T}$, with $\operatorname{dist}(x, \mathcal{A}) := \inf_{x^* \in \mathcal{A}} |x - x^*|$.

The set $\mathcal{A}$ is a *globally attracting set* if $\mathcal{T} = \mathbb{R}^n$. $\triangle$

Stability of sets is defined as follows:

Definition 2.4 (Stability of sets [176]). Let $\mathcal{A} \subset \mathbb{R}^n$ be compact and positively invariant under (2.1). The set $\mathcal{A}$ is

i. *stable* (in the sense of Lyapunov) with respect to (2.1) if for any $\varepsilon > 0$ there is a $\delta > 0$ such that $\operatorname{dist}(x_0, \mathcal{A}) < \delta$ implies $\operatorname{dist}(x(t; t_0, x_0), \mathcal{A}) < \varepsilon$ for all $t \geq t_0$;

ii. *asymptotically stable* (in the sense of Lyapunov) with respect to (2.1) if it is a stable and attracting set;

iii. *uniformly asymptotically stable* (in the sense of Lyapunov) with respect to (2.1) if it is asymptotically stable and there is a number $\bar{\delta} > 0$ such that for any $\varepsilon > 0$ there is a $T = T(\varepsilon) > 0$ such that $\operatorname{dist}(x_0, \mathcal{A}) < \bar{\delta}$ implies $\operatorname{dist}(x(t; t_0, x_0), \mathcal{A}) < \varepsilon$ for all $t \geq t_0 + T$.

$\triangle$

Remark 2.2. The definitions for Lyapunov stability of sets are given *locally*. They become *global* if the definitions hold for all $x_0 \in \mathbb{R}^n$. $\triangle$

The stability (in the sense of Lyapunov) of (2.1) can also be defined with respect to a solution of (2.1). Section 2.4 of this chapter provides conditions for a solutions of (2.1) to be stable.

Definition 2.5 (Lyapunov stability of a solution [114]). Let $\bar{x}(t)$ be a solution of (2.1) defined for $t \in (t^*, \infty)$. The solution $\bar{x}(t)$ is called

i. *stable* (in the sense of Lyapunov) if for any $t_0 \in (t^*, \infty)$ and number $\varepsilon > 0$ there is $\delta = \delta(\varepsilon, t_0) > 0$ such that $|x_0 - \bar{x}(t_0)| < \delta$ implies $|x(t; t_0, x_0) - \bar{x}(t)| < \varepsilon$ for all $t \geq t_0$;

ii. *uniformly stable* (in the sense of Lyapunov) if it is stable and the number δ can be chosen independently of t_0;

iii. *asymptotically stable* (in the sense of Lyapunov) if it is stable and there is a number $\bar{\delta} = \bar{\delta}(t_0) > 0$ such that $|x_0 - \bar{x}(t_0)| < \bar{\delta}$ implies $|x(t; t_0, x_0) - \bar{x}(t)| \to 0$ as $t \to \infty$;

iv. *uniformly asymptotically stable* (in the sense of Lyapunov) if it is uniformly stable and there is a number $\bar{\delta} > 0$ such that for any $\varepsilon > 0$ there is a $T = T(\varepsilon) > 0$ such that $|x_0 - \bar{x}(t_0)| < \bar{\delta}$ implies $|x(t; t_0, x_0) - \bar{x}(t)| < \varepsilon$ for all $t \geq t_0 + T$;

v. *exponentially stable* (in the sense of Lyapunov) if there are constants $m, \alpha > 0$ such that $|x(t; t_0, x_0) - \bar{x}(t)| \leq me^{-\alpha(t-t_0)} |x_0 - \bar{x}(t_0)|$ for all $t \geq t_0$.

△

Finally some notions of boundedness of the system (2.1) are given.

Definition 2.6 (Lagrange stability and $\mathcal{L}$-dissipativity, [127]). The system (2.1) is called

i. *Lagrange stable* if every solution is bounded in forward time;

ii. *$\mathcal{L}$-dissipative* if the system is Lagrange stable and there exists a constant $c > 0$ such that $\limsup_{t\to\infty} |x(t)| \leq c$ for every initial condition $x_0 \in \mathbb{R}^n$.

△

Remark 2.3. The solutions of a $\mathcal{L}$-dissipative system are *ultimately bounded*, that is, all solutions enter independent of the initial conditions a compact set in finite time. △

2.3 Passive systems and semipassive systems

This section deals with systems having inputs and outputs. The theory of dissipative systems provides a nice and intuitive framework to analyze (and design) such *open* systems. With the introduction of storage functions and supply rates by J.C. Willems in 1972, [165, 166], the connection between physical energy-related phenomena and the mathematical input-output description of a system was established. A dissipative system is a system for which the supply (or energy) at the current time does not exceed the initial supply *plus* the supplied energy. Roughly speaking, a dissipative system is a system that does not generate energy and dissipates the energy supplied by its surrounding. Passive systems are dissipative systems with a particular supply rate, namely a supply rate being the bilinear product of the input(s) and output(s). Semipassive systems are systems that behave as passive systems except that these systems do generate a finite amount of energy itself. Formally passivity and semipassivity are defined as follows:

Definition 2.7 (Passivity and semipassivity, [165, 124]). Consider a system

$$\dot{x}(t) = f(x(t), u(t)), \quad y(t) = h(x(t)), \tag{2.2}$$

where state $x \in \mathbb{R}^n$, output $y \in \mathbb{R}^m$, input $u \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R}^m)$, sufficiently smooth functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^m$. Suppose that there exists a nonnegative storage function $V \in \mathcal{C}^r(\mathbb{R}^n, \mathbb{R}_{\geq 0})$, $r \geq 0$, $V(0) = 0$, such that the following dissipation inequality

$$V(x(t)) - V(x(t_0)) \leq \int_{t_0}^{\infty} y^\top(s)u(s) - H(x(s))\mathrm{d}s, \tag{2.3}$$

holds where $H \in \mathcal{C}(\mathbb{R}^n, \mathbb{R})$. The system (2.2) is called

i. $\mathcal{C}^r$-passive if there exists a $\mathcal{C}^r$ storage function V and a function H such that (2.3) holds with $H(\cdot) \geq 0$;

ii. strictly $\mathcal{C}^r$-passive if there exists a $\mathcal{C}^r$ storage function V and a function H such that (2.3) holds with $H(\cdot) > 0$.

iii. $\mathcal{C}^r$-semipassive if there exists a $\mathcal{C}^r$ storage function V and a function H such that (2.3) holds with $H(\cdot) \geq 0$ outside a ball $\mathcal{B} = \mathcal{B}(0, R) \subset \mathbb{R}^n$ with radius R centered around 0, i.e.

$$\exists R > 0, \ |x| \geq R \Rightarrow H(x) \geq \varrho\left(|x|\right),$$

with some nonnegative continuous function $\varrho(|x|)$ defined for all $|x| \geq R$;

iv. strictly $\mathcal{C}^r$-semipassive if there exists a $\mathcal{C}^r$ storage function V and a function H such that (2.3) holds with $H(\cdot) > 0$ outside a ball $\mathcal{B} = \mathcal{B}(0, R) \subset \mathbb{R}^n$.

△

Remark 2.4. If the storage function $V \in \mathcal{C}^r(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ with $r \geq 1$, inequality (2.3) can be replaced by

$$\dot{V}(x(t)) \leq y^\top(t)u(t) - H(x(t)).$$

△

Passive systems systems have, from a control theoretical point of view, some interesting properties. For instance, a “free” $\mathcal{C}^1$-passive system, that is a $\mathcal{C}^1$-passive system with $u \equiv 0$, or a $\mathcal{C}^1$-passive system with a feedback $u = -\gamma(y)$ satisfying $y^\top\gamma(y) \geq 0$ for all y, is, under some detectability assumptions, stable in the sense of Lyapunov. Moreover, if the storage function is positive definite, then the zero-dynamics of a strictly $\mathcal{C}^1$-passive system (2.2), i.e. the dynamics (2.2) with the constraint $y \equiv 0$, are asymptotically stable. A nonlinear system with asymptotically stable zero dynamics is also called *nonlinear*

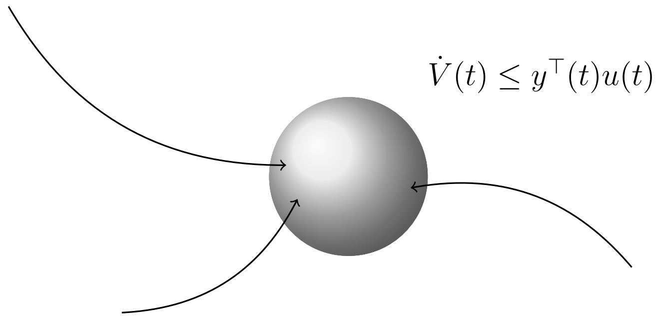


Figure 2.1. Semipassivity: a systems behaving as a passive systems outside some ball in its state-space. For any smooth passive feedback $u(t) = \gamma(y(t))$ such that $-y^\top(t)\gamma(y(t)) \leq 0$, the solutions of a strictly semipassive system enter a compact set in finite time [127].

minimum-phase. See [136, 26, 159, 59] for (many) more details and interesting properties of dissipative and passive systems.

As follows from its definition, a semipassive system behaves, roughly speaking, as a passive system outside some ball in the system's state-space. See Figure 2.1. A nice property of semipassive systems (that will be heavily exploited in this thesis) is that a system of semipassive systems for which $y^\top u \leq 0$ is Lagrange stable. The closed-loop system is even $\mathcal{L}$-dissipative if the systems are strictly semipassive. See [127] for details.

Many (physical) systems are semipassive. In the next example it is shown that the well-known Lorenz (chaotic) oscillator is a strictly semipassive system.

Example 2.1 ([124]). Consider the Lorenz equations [86] with input u,

$$\dot{x}_1 = \sigma(x_2 - x_1) + u, \tag{2.4a}$$

$$\dot{x}_2 = rx_1 - x_2 - x_1x_3, \tag{2.4b}$$

$$\dot{x}_3 = -bx_3 + x_1x_2, \tag{2.4c}$$

where $\sigma, r, b > 0$ are constant parameters. The Lorenz system is strictly semipassive with respect to output $y = x_1$ and input u with the positive definite storage function $V = \frac{1}{2}\left(x_1^2 + x_2^2 + (x_3 - \sigma - r)^2\right)$. Indeed, a straightforward computation shows that $\dot{V} \leq yu - H(x)$ with $H(x) = \sigma x_1^2 + x_2^2 + b\left(x_3 - \frac{\sigma+r}{2}\right)^2 - b\frac{(\sigma+r)^2}{4}$ being positive outside the ball $\mathcal{B}$ centered around $(0, 0, \sigma + r)$ with radius R, $R^2 = (\sigma + r)^2\left(\frac{1}{4} + \frac{b}{4}\max\left(\frac{1}{\sigma}, 1\right)\right)$. △

2.4 Convergent systems

In this section the notion of convergent systems is introduced. Convergent systems are nonlinear systems with inputs that have some interesting properties. The most important property of a convergent system is that the solutions of such system "forget" their initial conditions such that, after some transient time, the solutions only depend on the input signal that excites the system. Note that this property is natural for asymptotically stable linear systems, but nonlinear systems do not have this property in general. A convergent system is formally defined as follows:

Definition 2.8 (Convergent systems, [42, 114]). Consider the system

$$\dot{x}(t) = f(x(t), w(t)), \tag{2.5}$$

with state $x \in \mathbb{R}^n$, external signal $w(t) \in \mathbb{PC}(\mathbb{R}, \mathcal{W})$, that is, $w(t)$ is piecewise continuous in t and takes values from a compact set $\mathcal{W} \subset \mathbb{R}^m$, and the function $f \in \mathcal{C}(\mathbb{R}^n \times \mathbb{PC}(\mathbb{R}, \mathcal{W}), \mathbb{R}^n)$ is locally Lipschitz and $\mathcal{C}^1$ in x. The system (2.5) is called

i. *convergent* if

   (a) for any continuous input $w(t) \in \mathbb{PC}(\mathbb{R}, \mathcal{W})$ all solutions $x(t)$ are defined and bounded for all $t \in [t_0, \infty)$ and all initial conditions $x_0 = x(t_0) \in \mathbb{R}^n$;

   (b) for any input $w(t) \in \mathbb{PC}(\mathbb{R}, \mathcal{W})$ there exists a unique globally asymptotically stable solution $x_w(t)$ on the interval $t \in (-\infty, +\infty)$, i.e. for all initial conditions the following holds:

$$\lim_{t \to \infty} |x(t) - x_w(t)| = 0;$$

ii. *uniformly convergent* if the system is convergent and the solution $x_w(t)$ is globally uniformly asymptotically stable;

iii. *exponentially convergent* if the system is convergent and the solution $x_w(t)$ is globally exponentially stable.

$\triangle$

As follows from its definition, a convergent system has a unique limit solution that is determined by the input signal, and every solution converges to it independent of the choice of initial conditions. As mentioned in [114], the notion of convergence is has the advantage over other existing formulations of this property (such as *incremental stability* [9], *contraction theory* [85] and *incremental ISS* [9]) that it is coordinate independent and does not require an operator description of the system.

A sufficient condition for the system (2.5) to be an exponentially convergent system is given in the following lemma.

Lemma 2.2 (Demidovich Lemma, [42, 114]). *Consider the system* (2.5). *If there exists a matrix $P \in \mathbb{R}^{n\times n}$, $P = P^\top > 0$, such that all eigenvalues $\lambda_i(Q)$ of the symmetric matrix*

$$Q(x,w) = \frac{1}{2}\left(P\left(\frac{\partial f}{\partial x}(x,w)\right) + \left(\frac{\partial f}{\partial x}(x,w)\right)^\top P\right)$$

are negative and separated away from zero, i.e. there is a $\delta > 0$ such that

$$\lambda_i(Q(x,w)) \leq -\delta < 0,$$

for all $i = 1,\dots,n$ and all $x \in \mathbb{R}^n$, $w \in \mathcal{W}$, then the system (2.5) *is exponentially convergent.*

2.5 Retarded functional differential equations

In this thesis the systems in the network will be described by ordinary differential equations. Since the systems interact via time-delayed diffusive coupling, the closed-loop dynamics are given by a set of *delay differential equations*. The specific type of delay differential equations that will be encountered are *retarded functional differential equations*. In this section some basic theory about solutions and stability of solutions of retarded functional differential equations is being introduced.

The following has been adopted from [56]. Let $\tau \geq 0$ be a real number and let $\mathbf{C} = \mathcal{C}([-\tau,0],\mathbb{R}^n)$. The norm of an element ϕ of $\mathbf{C}$ is $|\phi| = \sup_{-\tau\leq\theta\leq 0}|\phi(\theta)|$. Even though $|\cdot|$ also defines a norm in $\mathbb{R}^n$, no confusion should arise. If $t_0 \in \mathbb{R}$, $T \geq 0$ and $x \in \mathcal{C}([t_0-\tau, t_0+T])$, for any $t \in [t_0, t_0+T]$, $x_t(\theta) \in \mathbf{C}$ is defined as $x_t(\theta) = x(t+\theta)$, $-\tau \leq \theta \leq 0$. Let $\Omega \subset \mathbb{R}\times\mathbf{C}$, $f : \Omega \to \mathbb{R}^n$ is a given functional and " $\cdot$ " represents the right-hand derivative with respect to time[1], then the relation

$$\dot{x}(t) = f(t, x_t), \tag{2.6}$$

is called a *retarded functional differential equation* (on Ω), denoted by RFDE(f). A function x is a solution of (2.6) on the interval $[t_0-\tau, t_0+T)$ if there are $t_0 \in \mathbb{R}$ and $T > 0$ such that $x \in \mathcal{C}([t_0-\tau, t_0+T), \mathbb{R}^n)$, $(t, x_t) \in \Omega$ and $x(t)$ satisfies (2.6) for all $t \in [t_0-\tau, t_0+T)$. For given $t_0 \in \mathbb{R}$ and $\phi \in \mathbf{C}$, $x(t; t_0, \phi)$ denotes a solution of (2.6) through (t_o, ϕ). That is, $x(t; t_0, \phi)$ is a solution of (2.6) for which $x_{t_0} = \phi$. In the remainder it will be assumed that the function f is *completely continuous*, that is, $f : \Omega \to \mathbb{R}^n$ is continuous and takes closed bounded sets of Ω into bounded subsets of $\mathbb{R}^n$. In addition, it will be assumed that f is Lipschitz in ϕ in each compact set in Ω and has bounded continuous first order derivatives with respect to ϕ. These assumptions on f guarantee existence and uniqueness of an *absolutely continuous* solution $x(t; t_0, \phi)$.

[1]The right-hand derivative of the function $x(t)$ is $\dot{x}(t) = \lim_{h\to 0^+} \frac{x(t+h)-x(t)}{h}$.

2.5.1 Stability theory for RFDE

The notions of (Lyapunov) stability for *ordinary differential equations* presented in section 2.2 naturally extend to notions of (Lyapunov) stability for *retarded functional differential equations.*

Definition 2.9 (Stability of RFDE(f), [56, 54]). Consider the RFDE(f),

$$\dot{x}(t) = f(t, x_t), \tag{2.7}$$

and suppose that $f(t, 0) = 0$ for all $t \in \mathbb{R}$. Then the solution $x \equiv 0$ is

i. *stable* if for any $t_0 \in \mathbb{R}$ and number $\varepsilon > 0$ there is $\delta = \delta(t_0, \varepsilon) > 0$ such that $|\phi| < \delta$ implies $|x_t(t_0, \phi)| < \varepsilon$ for all $t \geq t_0$;

ii. *uniformly stable* if it is stable and the number δ can be chosen independently of t_0;

iii. *asymptotically stable* if it is stable and there exists a number $\bar{\delta} = \bar{\delta}(t_0) > 0$ such that $|\phi| \leq \bar{\delta}$ implies $x(t; t_0, \phi) \to 0$ as $t \to \infty$;

iv. *uniformly asymptotically stable* if it is uniformly stable and there exists a number $\bar{\delta} = \bar{\delta}(t_0) > 0$ such that for every $\varepsilon > 0$ there is a $T' = T'(\varepsilon) > 0$ such that $|\phi| \leq \bar{\delta}$ implies $|x_t(t_0, \phi)| < \varepsilon$ for all $t \geq t_0 + T'$.

v. *exponentially stable* if there are constants $m, \alpha > 0$ such that $|x(t; t_0, \phi)| \leq me^{-\alpha(t-t_0)} |\phi|$ for all $t \geq t_0$.

$\triangle$

Like the stability of (an equilibrium) of an *ordinary differential equation* can be ensured by constructing a (suitable) *Lyapunov function*, the stability of (an equilibrium) of a *retarded functional differential equation* can be ensured by constructing a (suitable) *Lyapunov functional.* If $V : \mathbb{R} \times \mathbf{C} \to \mathbb{R}$ is continuous and $x(t; t_0, \phi)$ is a solution of (2.6) through (t_0, ϕ), then

$$\dot{V}(t, \phi) := \limsup_{h \to 0^+} \tfrac{1}{h}[V(t + h, x_{t+h}(t, \phi)) - V(t, \phi)]. \tag{2.8}$$

That is, $\dot{V}(t, \phi)$ is the upper right-hand derivative of $V(t, \phi)$ along the solution $x(t; t_0, \phi)$.

Theorem 2.3 (Method of Lyapunov functionals, [56] §5.2, Theorem 2.1). *Consider the RFDE(f) and suppose $f : \mathbb{R} \times \mathbf{C} \to \mathbb{R}^n$ is completely continuous and $u, v, w : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ are continuous nondecreasing functions, $u(s)$ and $v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$. If there is a continuous functional $V : \mathbb{R} \times \mathbf{C} \to \mathbb{R}$ such that*

$$u(|\phi(0)|) \leq V(t, \phi) \leq v(|\phi|),$$

$$\dot{V}(t,\phi) \le -w(|\phi(0)|),$$

then the solution $x \equiv 0$ of (2.7) is uniformly stable. If $u(s) \to \infty$ if $s \to \infty$ the solutions of (2.7) are uniformly bounded. If $w(s) > 0$ for $s > 0$ the solution $x \equiv 0$ is uniformly asymptotically stable.

Remark 2.5. Theorem 2.3 is sometimes referred to as the Lyapunov-Krasovskii theorem since N. N. Krasovskii proved asymptotic stability in Theorem 2.3. △

The following example shows how Theorem 2.3 can be applied to assess the stability of the zero solutions of a simple linear scalar system.

Example 2.2. From [56]. Consider the scalar system

$$\dot{x}(t) = -ax(t) + bx(t-\tau), \tag{2.9}$$

with constants $a > 0$ and b and finite time-delay τ. Take $V(\phi) = \frac{1}{2}\phi^2(0) + \frac{a}{2}\int_{-\tau}^{0} \phi^2(\theta)\mathrm{d}\theta$, then $\dot{V}(\phi) = -\frac{a}{2}\phi^2(0) - b\phi(0)\phi(-\tau) - \frac{a}{2}\phi^2(-\tau)$. It is easy to see that $\dot{V}(\phi)$ is negative definite if $|b| < a$. Hence the zero solution of (2.9) is uniformly asymptotically stable for any $|b| < a$. △

Remark 2.6. The condition $|b| < a$ is sufficient but certainly not necessary for the global stability of the zero solution of (2.9). Indeed, for the linear system (2.9), the exact region of stability is obtained for those parameters a, b, τ for which the roots of the characteristic equation $\lambda + a - be^{-\lambda\tau} = 0$ have strictly negative real part. The upper bound for the region of stability is given parametrically by the equation $a = b\cos(\zeta\tau)$, $b\sin(\zeta\tau) = -\zeta$ where $0 < \zeta < \frac{\pi}{\tau}$. The region $|b| < a$ is exactly the region where the zero solution of (2.9) is uniformly asymptotically stable for any $\tau > 0$. See [56], §5.2. △

To apply Theorem 2.3, a functional has to be defined which has negative definite derivatives along the solutions of RFDE(f). In this sense Theorem 2.3 can be seen as the natural extension of Lyapunov's second method for ODE's. However, often it is preferable to determine the stability of a system using functions rather than functionals as functions are, in general, easier to apply. Moreover, it is often intuitive to assess the stability of a system by defining functions like a distance function or a energy function and the rate of change of such function. In the following theorem sufficient conditions for stability of the RFDE(f) are given using functions instead of functionals. If $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is continuous and $x(t; t_0, \phi)$ is a solution of (2.6) through (t_0, ϕ), then

$$\dot{V}(t,\phi(0)) := \limsup_{h\to 0^+} \tfrac{1}{h}[V(t+h, x(t+h; t, \phi)) - V(t, \phi(0))]. \tag{2.10}$$

Theorem 2.4 (Lyapunov-Razumikhin theorem, [56], §5.4, Theorem 4.1 and Theorem 4.2). *Consider the RFDE(f) and suppose that $f : \mathbb{R} \times \mathbf{C} \to \mathbb{R}^n$ is completely continuous. Suppose $u, v, w : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ are continuous nondecreasing functions, $u(s)$ and $v(s)$ are positive* *for* $s > 0$, $u(0) = v(0) = 0$, *and* v *strictly increasing. If there is a continuous function* $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *such that*

$$u(|x|) \leq V(t,x) \leq v(|x|), \quad t \in \mathbb{R}, x \in \mathbb{R}^n.$$

and

$$\dot{V}(t,\phi(0)) \leq -w(|\phi(0)|) \quad \textit{if} \quad V(t+\theta,\phi(\theta)) \leq V(t,\phi(0)),$$

for $\theta \in [-\tau, 0]$, *then the solution* $x \equiv 0$ *of the RFDE*(f) *is* uniformly stable. *If, in addition,* $w(s) > 0$ *for* $s > 0$ *and there is a continuous nondecreasing function* $p(s) > s$ *for* $s > 0$ *such that*

$$\dot{V}(t,\phi(0)) \leq -w(|\phi(0)|) \quad \textit{if} \quad V(t+\theta,\phi(\theta)) < p(V(t,\phi(0))),$$

for $\theta \in [-\tau, 0]$, *then the solution* $x \equiv 0$ *is* uniformly asymptotically stable. *If* $u(s) \to \infty$ *if* $s \to \infty$ *the solution* $x \equiv 0$ *is a global attractor for the RFDE*(f).

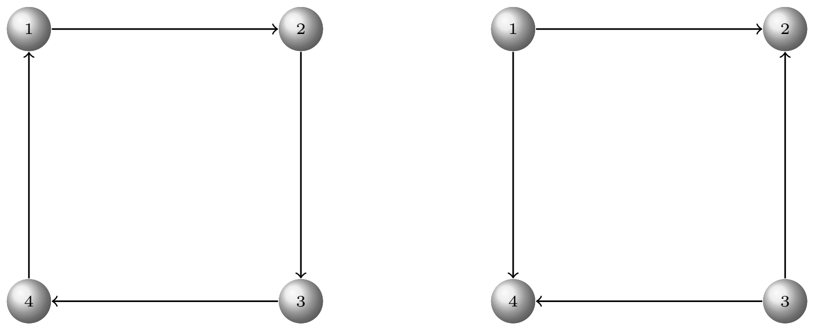
The next example shows how Theorem 2.4 can be applied to determine stability of the zero solution of the simple linear scalar system (2.9).

Example 2.3. From [56]. Consider again (2.9) and let $V(x(t)) = \frac{1}{2}x^2(t)$. Then $\dot{V}(x(t)) = -ax^2(t) + bx(t)x(t-\tau) \leq -ax^2(t) + bx^2(t)$ if $|x(t)| \geq |x(t-\tau)|$, hence the solution $x \equiv 0$ is uniformly stable if $|b| \leq a$. Take $p(s) = c^2 s$ with some constant $c > 1$, then $\dot{V}(x(t)) \leq -(a - bc)x^2(t)$ if $p(V(x(t))) > V(x(t-\tau))$. Hence if $|b| < a$ there is a $c > 1$ such that $\dot{V}(x(t)) \leq 0$ (whenever $p(V(x(t))) > V(x(t-\tau))$). This implies that the zero solution of (2.9) is uniformly asymptotically stable for $|b| < a$. In fact, $x \equiv 0$ is a global attractor for (2.9). △

2.6 Elementary Graph theory

In this section some basic terminology from graph theory is presented. The notation and terminology has been adopted from [24, 44]. A *graph* is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with $\mathcal{V} = \mathcal{V}(\mathcal{G})$ denoting the set of nodes of the graph $\mathcal{G}$ and $\mathcal{E} = \mathcal{E}(\mathcal{G})$ are its edges. It will always be assumed that $\mathcal{V}$ and $\mathcal{E}$ are finite and, for unambiguous notation, $\mathcal{V} \cap \mathcal{E} = \emptyset$. Let x and y be two nodes, then $\{x, y\} \in \mathcal{E}$ denotes the directed edge *from* node x *to* node y. A network is called *undirected* if $\{x, y\} \in \mathcal{E}$ for each $\{y, x\} \in \mathcal{E}$. Otherwise the graph is a *directed* graph, or *digraph* for short. The set of all directed edges *to* a node x is denoted as $\mathcal{E}_x$. A graph contains a *self-loop* if there is a node x with an edge $\{x, x\}$. If a graph does not contain self-loops the graph is called *simple*.

Example 2.4. The graphs depicted in Figure 2.2 are both simple digraphs. Consider the graph depicted in Figure 2.2(a), then $\mathcal{V}(\mathcal{G}_1) = \{1, 2, 3, 4\}$ and $\mathcal{E}(\mathcal{G}_2) = \{\{4, 1\}, \{1, 2\}, \{2, 3\}, \{3, 4\}\}$. For the graph depicted in Figure 2.2(b) it follows that $\mathcal{V}(\mathcal{G}_2) = \{1, 2, 3, 4\}$ and $\mathcal{E}(\mathcal{G}_2) = \{\{1, 4\}, \{1, 2\}, \{3, 2\}, \{3, 4\}\}$. △



(a) (b)

Figure 2.2. Two simple digraphs: (a) graph $\mathcal{G}_1$, (b) graph $\mathcal{G}_2$.

A graph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ is a *subgraph* of $\mathcal{G}$, denoted by $\mathcal{G}' \subset \mathcal{G}$, if $\mathcal{V}' \subset \mathcal{V}$ *and* $\mathcal{E}' \subset \mathcal{E}$. A path is a nonempty graph $\mathcal{P}(\mathcal{V}', \mathcal{E}') \subset \mathcal{G}(\mathcal{V}, \mathcal{E})$ of the form $\mathcal{V}' = \{x_0, x_1, \ldots, x_k\}$, $\mathcal{E}' = \{\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{k-1}, x_k\}\}$ with all x_i distinct.) A digraph $\mathcal{G}$ is called *strongly connected* if every two nodes are joined by some path. A digraph $\mathcal{G}$ is *connected* if every two nodes are joined by some path not taken the direction of the interconnection into account. A maximal (strongly) connected subgraph of $\mathcal{G}$ is a *(strongly) connected component* of $\mathcal{G}$.

Example 2.5. The graph $\mathcal{G}_1$ depicted in Figure 2.2(a) is strongly connected while the graph $\mathcal{G}_2$ in Figure 2.2(b) is *not* strongly connected. $\triangle$

If two nodes have a directed edge in common they are called *adjacent*. Suppose that the network consists of k nodes, then the *adjacency matrix* $A \in \mathbb{R}^{k \times k}$ is defined as $A = \left(a_{ij}\right)$ where

$$a_{ij} = \begin{cases} 1, & \text{if } \{j, i\} \in \mathcal{E}(\mathcal{G}), \\ 0, & \text{otherwise.} \end{cases}$$

The *degree matrix* $D \in \mathbb{R}^{k \times k}$ is a diagonal matrix with the degrees $d_i = \sum_j a_{ij}$ as entries on its diagonal. The matrix $L := D - A$ is the *Laplacian matrix* (or Kirchhoff matrix) of the graph $\mathcal{G}$.

Example 2.6. The adjacency matrices of the graphs depicted in Figure 2.2(a) and Figure 2.2(b) are, respectively,

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

$\triangle$

Of course, it is also possible to consider *weighted graphs*. Let w_{ij} be the weight on edge

$\{j,i\}$, then the *weighted adjacency matrix* $A_w := \left(a_{w,ij}\right)$ with

$$a_{w,ij} = \begin{cases} w_{ij}, & \text{if } \{j,i\} \in \mathcal{E}(\mathcal{G}), \\ 0, & \text{otherwise.} \end{cases}$$

Then the *weighted degree matrix* D_w is defined as the diagonal matrix with entries $\sum_j w_{ij}$ and the *weighted Laplacian matrix* $L_w = D_w - A_w$. In this thesis the weighted versions of the adjacency matrix, the degree matrix and the Laplacian matrix will be denoted, for notational simplicity, as A, D and L, respectively. Also, rather than weighted adjacency matrix, weighted degree matrix and weighted Laplacian matrix, simply adjacency matrix, degree matrix and Laplacian matrix will be written.

This thesis considers dynamical systems on a simple strongly connected graph. The following proposition gives a useful properties of the Laplacian matrix of a simple strongly connected graph.

Proposition 2.5 (Properties of the Laplacian matrix). *Let $\mathcal{G}$ be a simple strongly connected digraph. Then the Laplacian matrix L is irreducible*[2] *and* $\operatorname{spec}(L) \subset \mathbb{C}_{>0} \cup 0$.

Proof of proposition 2.5. The adjacency matrix A is irreducible if and only if its associated digraph is strongly connected. Then L is irreducible. In addition L is singular because all rowsums are zero. Since L is strongly connected the number of strongly connected components is 1 ($\mathcal{G}$ itself), and because the (algebraic) multiplicity of the zero eigenvalue is equal to the number of strongly connected components, cf. [24], it follows that the zero eigenvalue is simple. Gerschgorin's theorem about the location of the eigenvalues of L in the complex plane, cf. [64], and the fact that the zero eigenvalue is simple implies $\operatorname{spec}(L) \subset \mathbb{C}_{>0} \cup 0$. □

[2]The matrix $L \in \mathbb{R}^k \times k$, $k > 1$, is reducible if there is a permutation matrix $P \in \mathbb{R}^k \times k$ and an integer $1 \leq m \leq k-1$ such that

$$P^\top L P = \begin{pmatrix} L_1 & L_2 \\ 0 & L_3 \end{pmatrix},$$

where $L_1 \in \mathbb{R}^{m\times m}$,$L_2 \in \mathbb{R}^{m\times(k-m)}$ and $L_3 \in \mathbb{R}^{(k-m)\times(k-m)}$, cf. [64]. The matrix L is irreducible if L is not reducible [64].

Part I

Synchronization of diffusively coupled semipassive systems

CHAPTER THREE

Synchronization of semipassive systems

Abstract. In this chapter results are presented on synchronization of strictly semipassive systems. First it will be proven that the solutions of strictly semipassive systems that interact via time-delayed diffusive coupling are ultimately bounded. Then, assuming that the systems are identical and satisfy an internal stability property, it is proven that these systems synchronize whenever the coupling is sufficiently strong and the product of the coupling strength and time-delay is sufficiently small. The results presented in this chapter are published in [146].

3.1 Introduction

Diffusive coupling arises naturally in various areas varying from physiology [137, 41] and neuroscience [34, 38, 74, 80, 162] to electrical systems [169, 161] and mechanical engineering [132, 131, 36, 172]. The study of synchronization of these diffusively coupled systems has received a lot of attention from the scientific community. Most works do discuss synchronization in networks of diffusively coupled systems of a specific type, e.g. networks of Lorenz systems. Some general theory about synchronization in networks of diffusively coupled systems without time-delays is presented in [55]. It is shown that the systems synchronize if the solutions of the interconnected systems are bounded and the coupling is sufficiently strong. In [124, 128] a general framework to analyze synchronization of systems that interact via symmetric non-delayed diffusive coupling is proposed. Sufficient conditions are presented for the solutions of the systems to be ultimately bounded, and this boundedness result is independent of the topology of the network. If the systems satisfy an additional internal stability property, then the systems are guaranteed to synchronize for sufficiently strong coupling. The main result of [128] is given in the following theorem.

Theorem 3.1 ([128]). *Consider k systems on a simple strongly connected graph:*

$$\dot{z}_i(t) = q(z_i(t), y_i(t)), \tag{3.1a}$$
$$\dot{y}_i(t) = a(y_i(t), z_i(t)) + Bu_i(t), \quad i \in \mathcal{I} := \{1, 2, \ldots, k\}, \tag{3.1b}$$

with $z_i \in \mathbb{R}^p$, $y_i \in \mathbb{R}^m$, $u_i \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R}^m)$, sufficiently smooth functions $a : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^p$ and $q : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^m$, and the matrix $B \in \mathbb{R}^{m\times m}$ is (similar to) a positive definite matrix. The systems (3.1) *are coupled via*

$$u_i(t) = \sigma \sum_{j\in\mathcal{E}_i} a_{ij}(y_j(t) - y_i(t)), \tag{3.2}$$

with coupling strength $\sigma > 0$ and interconnection weights $a_{ij} = a_{ji} \geq 0$ for all $i, j \in \mathcal{I}$[1]*. Suppose that*

(H3.1) each system (3.1) *is strictly $\mathcal{C}^1$-semipassive with a radially unbounded storage function;*

(H3.2) there exists a positive definite function $V_0 \in \mathcal{C}^2(\mathbb{R}^p, \mathbb{R}_{\geq 0})$ and a positive constant α such that

$$\nabla V_0^\top(z_i - z_j)(q(z_i, y) - q(z_j, y)) \leq -\alpha \left|z_i - z_j\right|^2,$$

for all $z_i, z_j \in \mathbb{R}^p$ and all $y \in \mathbb{R}^m$.

Then the solutions of the closed-loop system (3.1), (3.2) *are ultimately bounded and there exists a constant $\sigma^* > 0$ such that if $\sigma\lambda_2(L) \geq \sigma^*$, $\lambda_2(L)$ denotes the smallest nonzero eigenvalue of the symmetric Laplacian matrix, there is a globally asymptotically stable subset of the diagonal set*

$$\left\{\operatorname{col}(z_1, \ldots, z_k, y_1, \ldots, y_k) \in \mathbb{R}^{k(p+m)} | y_i = y_j \text{ and } z_i = z_j \text{ for all } i, j \in \mathcal{I}\right\}.$$

An important observation is that assumptions (H3.1) and (H3.2) in Theorem 3.1 do *not* depend on the network. This implies that systems that satisfy assumptions (H3.1) and (H3.2) *can* synchronize and this result is independent of the specific network topology. Theorem 3.1 states that the systems *will* synchronize if $\sigma\lambda_2(L)$ is sufficiently large. Here the network topology plays a role since $\lambda_2(L)$ denotes the smallest nonzero eigenvalue of the Laplacian of the network. The smallest nonzero eigenvalue of the Laplacian matrix is also known as the *algebraic connectivity* of the network, [47].

In the results presented above the coupling functions do not contain time-delays. However, it is relevant and important to consider time-delayed interaction. Indeed, in every

[1]The notation σa_{ij} for the "effective" coupling strength (between systems i and j) looks a bit cumbersome at first sight. The reason to use this notation is that for a fixed network topology, i.e. a_{ij} are fixed, sufficient conditions for synchronization can be expressed in terms of the value of the coupling strength σ (and, in case of time-delayed coupling, the value of the time-delay).

physical setup or biological system, it is likely that the transmission of a signal takes some (small) amount of time. In the last decade, a quite some attention is given to synchronization of diffusively coupled systems with time-delayed interaction. For instance, in [81, 173], synchronization of master-slave coupled nonlinear systems in Lur'e form is investigated. In [107], the circle criterion for time-delay systems is used to analyze synchronization in a network of mutually coupled Lur'e systems. In [51, 171, 106, 31], synchronization in networks of time-delay coupled systems is investigated and sufficient conditions for synchronization are presented in terms of *Linear Matrix Inequalities* (LMIs). A drawback of the LMI approach is that the conditions that follow are not very transparent and, moreover, such conditions tend to be (very) conservative. Conditions for *local* stability of the synchronized state in networks of nonlinear bidirectionally diffusively coupled systems are presented in [174]. In [93] an approach is presented where the *local* stability of the synchronized equilibria in directed networks is investigated. In [32, 33] it is shown that passive and weakly minimum-phase relative degree one systems on a balanced graph synchronize in presence of time-delays, see also [108] for results of synchronization of agents on undirected graphs. Synchronization of Lagrangian systems is discussed in [35, 36]. In [163, 35] contraction analysis [85] is used to analyze synchronization in networks with time-delayed interaction. Hereby it is assumed that the coupling is either uniform or symmetric. In addition, to use contraction theory, one has to find a domain that is positively invariant under the given network dynamics [134], something that is also nontrivial, especially when time-delays are involved.

In this chapter synchronization of diffusively time-delay coupled semipassive systems is discussed in the spirit of [128]. Sufficient conditions are given for the solutions of the coupled systems to be bounded and, in addition, sufficient condition are presented for synchronization of these systems. The results generalize the main result of [128], i.e. Theorem 3.1, in the sense that the coupling is not assumed to be symmetric and might contain time-delays. (See Theorem 3.10, Theorem 3.12 and Corollary 3.11 in this chapter.) The systems will interact via two different types of time-delay diffusive coupling, namely *coupling type I*

$$u_i(t) = \sigma \sum_{j\in\mathcal{E}_i} a_{ij} \left(y_j(t-\tau_{ij}) - y_i(t)\right), \tag{3.3}$$

or *coupling type II*

$$u_i(t) = \sigma \sum_{j\in\mathcal{E}_i} a_{ij} \left(y_j(t-\tau_{ij}) - y_i(t-\tau_{ji})\right). \tag{3.4}$$

In case of coupling type I the transmitted signal (the output of node j) is delayed by a factor τ_{ij} and "compared" with the current output of node i. This type of coupling arises naturally for interconnected systems since the transmission of signals can be expected to take some time. Coupling type II differs from coupling type I as the "reference" signals $y_i(t)$ in case of coupling type II also contain a time-delay. Such a coupling might arise, for instance, when the systems are synchronized by a centralized control law. It is important to realize that coupling type II vanishes if the systems are synchronized while coupling type I generally does not vanish. This means that the dynamics of synchronized type II coupled systems are the same as the dynamics of an uncoupled system, while the dynamics of the synchronized type I coupled systems differ from that of an uncoupled system.

The remainder of this chapter is organized as follows. In section 3.2 it will be proven that the solutions of interconnected semipassive systems are ultimately bounded. Some results related to synchronization of diffusively time-delay coupled passive systems are presented as well. Section 3.3 gives sufficient conditions for synchronization of semipassive systems in the spirit of Theorem 3.1. In section 3.4, it will be shown that strictly semipassive minimum-phase systems synchronize for suitable values of the coupling strength and the time-delay. Section 3.5 discusses the results presented in this chapter.

3.2 Interconnected semipassive systems

Consider k systems on a simple strongly connected graph:

$$\dot{x}_i(t) = f_i(x_i(t)) + g_i(x_i(t))u_i(t), \tag{3.5a}$$

$$y_i(t) = h_i(x_i(t)), \quad i \in \mathcal{I} := \{1, 2, \ldots, k\}, \tag{3.5b}$$

with state $x_i \in \mathbb{R}^n$, input $u_i \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R}^m)$, output $y_i \in \mathbb{R}^m$, sufficiently smooth functions $f_i : \mathbb{R}^n \to \mathbb{R}^n$, $g_i : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $h_i : \mathbb{R}^n \to \mathbb{R}^m$. First, to avoid any possible confusion about when systems can be called synchronized, the formal definition of the notion of synchronization that will be used throughout this thesis is given.

Definition 3.1 (Synchronization). Consider k systems (3.5) on a simple strongly connected graph. Let the systems be coupled via (3.3) or (3.4). The interconnected systems are said to *locally synchronize* if, for all continuous initial history ϕ_i, ϕ_j, there is $\delta > 0$ such that $|\phi_i - \phi_j| < \delta$ implies $|x_i(t; t_0, \phi_i) - x_j(t; t_0, \phi_j)| \to 0$ as $t \to \infty$ for every $i, j \in \{1, 2, \ldots, k\}$. When $\delta = \infty$ the systems are said to *globally synchronize* (or simply *synchronize* for short). △

Definition 3.1 states that the coupled systems locally synchronize if the initial history is sufficient close and the corresponding solutions converge to each other as time goes to infinity. For global synchronization the the initial history of the systems is not required to be close initially.

The main goal of the remainder of this section is to give sufficient conditions for the coupled systems to have bounded solutions. As shown in [127, 147], it is not trivial that the solutions of interconnected systems are bounded. See also chapter 6, section 6.4, in which it is shown that the solutions of two diffusively coupled linear systems $\dot{x}_i(t) =$

$Ax_i(t) + Bu_i(t)$, with A a Hurwitz matrix, grow unbounded if the coupling strength exceeds some threshold value. The following (technical) lemma will be used to prove the main results of this section.

Lemma 3.2. *Let L be the Laplacian matrix of a simple strongly connected graph. Then there exists a vector ν with strictly positive entries such that $\nu^\top L = 0$, i.e. the positive vector ν is the left eigenvector of L corresponding to the simple zero eigenvector.*

Proof of Lemma 3.2. The claim follows from Proposition 2.5 and the Perron-Frobenius theorem for irreducible non-negative matrices, cf. [64]. □

3.2.1 Semipassive systems interacting via coupling type I

Consider k systems on a simple strongly connected graph which interact via coupling type I,

$$u_i(t) = \sigma \sum_{j \in \mathcal{E}_i} a_{ij} \left(y_j(t - \tau_{ij}) - y_i(t)\right). \tag{3.3}$$

Theorem 3.3. *Consider k (not necessarily identical) systems* (3.5) *on a simple strongly connected graph that interact via coupling type I* (3.3). *Suppose that each system is strictly C^1-semipassive with a radially unbounded storage function. Then the solutions of the closed-loop system* (3.5), (3.3) *are ultimately bounded.*

Proof of Theorem 3.3. Since the graph is a simple and strongly connected graph there exists, by Lemma 3.2, a vector ν with strictly positive entries such that $\nu^\top L = 0$. Let ν_i be the i^{th} entry of ν. By assumption, each system is strictly semipassive with a radially unbounded storage function $V_i(x_i)$. Define the functional

$$W(x_t(\theta)) = \sum_{i=1}^{k} \nu_i \left(V_i(x_i(t)) + \tfrac{\sigma}{2} \sum_{j \in \mathcal{E}_i} a_{ij} \int_{-\tau_{ij}}^{0} |y_j(t+s)|^2 \,\mathrm{d}s \right), \tag{3.6}$$

with $\max_{i,j \in \mathcal{I}}(\tau_{ij})\theta \leq 0$ Clearly the functional W is positive definite. Then, by assumption,

$$\dot{W}(x_t(\theta)) \leq \sum_{i=1}^{k} \nu_i \left(y_i^\top(t) u_i(t) - H_i(x_i(t)) + \tfrac{\sigma}{2} \sum_{j \in \mathcal{E}_i} a_{ij} \left(|y_j(t)|^2 - |y_j(t - \tau_{ij})|^2 \right) \right). \tag{3.7}$$

Consider the term

$$\sum_{i=1}^{k} \nu_i y_i^\top(t) u_i(t) = \sigma \left(\sum_{i=1}^{k} \nu_i \left(-d_i \,|y_i(t)|^2 + \sum_{j \in \mathcal{E}_i} a_{ij} y_i^\top(t) y_j(t - \tau_{ij}) \right) \right), \tag{3.8}$$

and note that $\nu^\top L = \nu^\top (D - A) = 0$ implies that $\nu^\top D\xi = \nu^\top A\xi$ for any vector $\xi \in \mathbb{R}^k$. Then, using the inequality $y_i^\top(t)y_j(t-\tau_{ij}) \leq \frac{1}{2}|y_i(t)|^2 + \frac{1}{2}|y_j(t-\tau_{ij})|^2$ and equations (3.7) and (3.8), it is easy to verify that

$$\dot{W}(x_t(\theta)) \leq -\sum_{i=1}^{k} \nu_i H_i(x_i(t)). \tag{3.9}$$

The result follows now from the arguments presented in [127]. □

Remark 3.1. The result stated in Theorem 3.3 is independent of the values of the time-delays. △

Remark 3.2. If the systems are semipassive (thus not strictly semipassive) the closed-loop system (3.5),(3.3) will be Lagrange stable. See [127] for the details. △

Corollary 3.4. *Consider a network of k identical systems* (3.5) *on a simple strongly connected graph that interact via coupling type I. Suppose that the conditions stated in Theorem 3.3 hold. Then the solutions of the interconnected systems converge to the set* $\cup_{i=1}^{k} \{x_i \in \mathbb{R}^n | V(x_i) \leq c^*\}$ *where* $c^* = \sup_{H(\xi)=0} V(\xi)$.

Proof of Corollary 3.4. See appendix A.1.1. □

Theorem 3.3 is important as it provides conditions for the solutions of the interconnected systems to be bounded independent of the specific network topology. Corollary 3.4 gives an even stronger result as it implies that the solutions of identical systems converge to a set which is completely determined by the semipassivity property of the systems.

Using the strong connection between passive systems and semipassive systems, see section 2.3, the following result is immediate (hence it will be presented without proof).

Corollary 3.5. *Consider k not necessarily identical systems* (3.5) *on a simple strongly connected graph. Let the systems interact via coupling type I and suppose that the systems* (3.5) *with* $f_i(0) = 0$ *are*

i. strictly $\mathcal{C}^1$-passive with a radially unbounded storage function, then the systems synchronize in the sense of Definition 3.1;

ii. strictly $\mathcal{C}^1$-output passive[2] *with a radially unbounded storage function, then the systems synchronize in the sense that $h(x_i(t;t_0,\phi_i)) \to h(x_j(t;t_0,\phi_j))$ as $t \to \infty$ for any continuous initial history ϕ_i, ϕ_j for all $i, j \in \mathcal{I}$. If, in addition, the systems are* zero-state detectible[3]*, then the systems synchronize in the sense of Definition 3.1.*

[2] A system is strictly $\mathcal{C}^1$-output passive if the system is $\mathcal{C}^1$-passive and the function $H(x) - \epsilon|h(x(t))|^2 \geq 0$ for some $\epsilon > 0$.

[3] The system (3.5) is zero-state detectable if, for any trajectory such that $u_i(t) = 0$, $y_i(t) = 0$ implies $x_i(t) = 0$. See [59].

Remark 3.3. Suppose that the systems (3.5) with $f_i(0) = 0$ on a simple strongly connected graph are $\mathcal{C}^1$-passive with a radially unbounded positive definite storage function V_i. Examples of such systems are systems of the form $\dot{x}_i(t) = Ax_i(t) + Bu_i(t)$, $y_i(t) = C^\top x_i(t)$, with a storage function $V_i(x_i) = x_i^\top(t)Px_i(t)$ for some positive definite P such that $PB = C^\top$. Then the interconnected systems (3.5), (3.3) synchronize in the sense that $\left|y_i(t) - y_j(t - \tau_{ij}^*)\right| \to 0$ as $t \to \infty$ for all initial history for all $i, j \in \mathcal{I}$, where τ_{ij}^* denotes the minimum of the sum of delays over the paths from i to j. This assertion follows directly from the proof of Theorem 3.3 using a LaSalle type of argument and the fact that $\sum_{i=1}^k \sum_{j\in\mathcal{E}_i} \nu_i a_{ij} y_i^\top(t) y_j(t-\tau_{ij}) = \sum_{i=1}^k \sum_{j\in\mathcal{E}_i} \nu_i a_{ij} \left(\frac{1}{2}\left|y_i(t)\right|^2 + \frac{1}{2}\left|y_j(t-\tau_{ij})\right|^2\right)$ only if $\left|y_i(t) - y_j(t - \tau_{ij}^*)\right|$ for all $i, j \in \mathcal{I}$. Often this implies that $\left|y_i(t) - y_j(t)\right| \to 0$ as $t \to \infty$ for all initial history for all $i, j \in \mathcal{I}$, e.g. if the solutions of the interconnected systems all converge to some constant function. This is for instance the case when the systems are of the form $\dot{x}_i(t) = u_i(t)$ with output $y_i(t) = x_i(t)$. See also [32, 33] for examples. However, for a network with identical harmonic oscillators, it is not necessarily true that $\left|y_i(t) - y_j(t - \tau_{ij}^*)\right| \to 0$ as $t \to \infty$ implies that $\left|y_i(t) - y_j(t)\right| \to 0$ as $t \to \infty$ for any initial history. Indeed, for a network consisting of three identical harmonic oscillators with frequency ω that are coupled in a directed ring and all time-delays are $\tau_{ij} = \omega/3$, there exist initial history such that $y_1(t) = y_2(t - \omega/3) = y_3(t - 2\omega/3) = y_1(t - \omega) = y_1(t)$ (as $t \to \infty$). △

3.2.2 Semipassive systems interacting via coupling type II

As interconnected strictly semipassive systems on a simple strongly connected graph that interact via coupling type I have ultimately bounded solutions, one might expect that the solutions of those systems interacting via coupling type II,

$$u_i(t) = \sigma \sum_{j\in\mathcal{E}_i} a_{ij}\left(y_j(t-\tau_{ij}) - y_i(t-\tau_{ji})\right), \tag{3.4}$$

are being ultimately bounded as well.

Theorem 3.6. *Consider k (not necessarily identical) systems* (3.5) *on a simple strongly connected graph that interact via coupling type II* (3.4). *Suppose that each system is strictly $\mathcal{C}^1$-semipassive with a radially unbounded storage function and the functions $H_i(x_i)$ are such that there exist $R_i > 0$ such that $|x_i| > R_i$ implies $H_i(x_i) - 2\sigma d_i \left|y_i\right|^2 > 0$ with $d_i = \sum_{j\in\mathcal{E}_i} a_{ij}$. Then the solutions of the closed-loop system* (3.5), (3.4) *are ultimately bounded.*

Proof of Theorem 3.6. See appendix A.1.2 □

Remark 3.4. Of course, it can be that the conditions of Theorem 3.6 are satisfied only for values σ on some interval $\mathcal{I}_\sigma$. Then the solutions of the interconnected systems are ultimately bounded for $\sigma \in \mathcal{I}_\sigma$. △

It is easy to derive the counterparts of Corollaries 3.4 and 3.5 for coupling type II. (Hence these results are presented without proofs.)

Corollary 3.7. *Consider a network of k identical systems* (3.5) *on a simple strongly connected graph. Let the systems interact via coupling type II and suppose that the conditions stated in Theorem 3.6 hold. Then the solutions of the interconnected systems converge to the set* $\cup_{i=1}^{k}\{x_i \in \mathbb{R}^n | V(x_i) \leq c^*(\sigma)\}$ *where* $c^*(\sigma) = \sup_{H(\xi)-2\sigma\max_i(d_i)|h(\xi)|^2=0} V(\xi)$.

Corollary 3.8. *Consider k not necessarily identical systems* (3.5) *on a simple strongly connected graph. Let the systems interact via coupling type II and suppose that the systems* (3.5) *with* $f_i(0) = 0$ *are*

i. *strictly $\mathcal{C}^1$-passive with a radially unbounded storage function and the functions $H_i(x_i)$ are such that $H_i(x_i) - 2\sigma d_i |y_i|^2 > 0$, then the systems synchronize in the sense of Definition 3.1;*

ii. *strictly $\mathcal{C}^1$-passive with a radially unbounded storage function and the functions $H_i(x_i)$ are such that $H_i(x_i) - 2\sigma d_i |y_i|^2 > 0$, then the systems synchronize in the sense that $h(x_i(t; t_0, \phi_i)) \to h(x_j(t; t_0, \phi_j))$ as $t \to \infty$ for any continuous initial history ϕ_i, ϕ_j for all $i, j \in \mathcal{I}$. If, in addition, the systems are* zero state detectible*, then the systems synchronize in the sense of Definition 3.1.*

Remark 3.5. Note that in case *ii* in Corollary 3.5 the systems are required to satisfied a stronger condition than being strictly output passive. Indeed, the systems have to be strictly output passive with the constant $\epsilon \geq 2\sigma d_i$ (instead of $\epsilon > 0$). △

3.3 Synchronization of semipassive systems

This section considers *identical* systems (3.5) that can be transformed into the normal form

$$\dot{z}_i(t) = q(z_i(t), y_i(t)), \tag{3.10a}$$
$$\dot{y}_i(t) = a(y_i(t), z_i(t)) + B(y_i(t), z_i(t))u_i(t), \quad i \in \mathcal{I}, \tag{3.10b}$$

with $y_i \in \mathbb{R}^m$, $z_i \in \mathbb{R}^p$, $p = n - m$, sufficiently smooth Lipschitz continuous functions $q : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^p$, $a : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^m$, and $B(\cdot, \cdot)$ being nonsingular. See for instance [30] for necessary and sufficient conditions for the existence of such coordinate transformation. For simplicity, however, it will be assumed that $B(\cdot, \cdot) = I$, hence

$$\dot{z}_i(t) = q(z_i(t), y_i(t)), \tag{3.11a}$$
$$\dot{y}_i(t) = a(y_i(t), z_i(t)) + u_i(t). \tag{3.11b}$$

In this section sufficient conditions are presented for synchronization of strictly semipassive systems (3.11) that interact via either coupling type I or coupling type II. For both types of coupling, conditions are presented such that the synchronization manifold, i.e. the linear manifold

$$\mathcal{M} = \left\{ \mathrm{col}\,(z_1, \ldots, z_k, y_1, \ldots, y_k) \in \mathbb{R}^{k(p+m)} | z_i = z_j \text{ and } y_i = y_j \text{ for all } i, j \in \mathcal{I} \right\},$$

is invariant under the given dynamics. Sufficient conditions for synchronization of those systems will be presented in term of the coupling strength and the time-delays.

Remark 3.6. The systems (3.11) have relative degree one. In [30] it is shown that, under mild regularity conditions, any passive system with a positive definite storage function has to be a relative degree one system. As in the remainder of this section it will be assumed that the systems are strictly semipassive, and semipassive systems are basically passive systems outside some ball in the state space, it is natural to consider relative degree one systems. △

3.3.1 Semipassive systems interacting via coupling type I

Consider the systems (3.11) on a simple strongly connected graph. Let the systems be coupled via

$$u_i(t) = \sigma \sum_{j \in \mathcal{E}_i} a_{ij} \left(y_j(t - \tau_{ij}) - y_i(t) \right). \tag{3.3}$$

First conditions will be derived that guarantee that $\mathcal{M}$ is invariant under the given closed-loop dynamics (3.11), (3.3).

Proposition 3.9. *The linear manifold $\mathcal{M}$ is invariant under the dynamics* (3.11), (3.3) *on a simple strongly connected graph if (at least) one of the following conditions is satisfied:*

i. *the time-delays $\tau_{ij} = 0$ for all $i, j \in \mathcal{I}$;*

ii. *the time-delays $\tau_{ij} = \tau$ with $\tau \in \mathbb{R}_{>0}$, and $\sum_{j \in \mathcal{E}_i} a_{ij} = \text{constant}$ for all $i, j \in \mathcal{I}$;*

iii. *there are continuous initial history ϕ_i on $\mathcal{M}$ such that all $y_i(t; t_0, \phi_i)$[4] on $\mathcal{M}$ are T-periodic with period time $T = \min_{i,j \in \mathcal{I}}(\tau_{ij})$;*

iv. *there are continuous initial history ϕ_i on $\mathcal{M}$ such that all $y_i(t; t_0, \phi_i)$ on $\mathcal{M}$ are constant, i.e. $y_i(t; t_0, \phi_i) = \text{constant}$ on $\mathcal{M}$ for every $i \in \mathcal{I}$.*

[4]The notation $y_i(t; t_0, \phi_i)$ is used to denote the y_i-components of the solution of (3.11) that coincides with ϕ_i on $t \in [-\tau, 0]$.

Proof of Proposition 3.9. Since the systems (3.11) are assumed to be identical it follows that $\mathcal{M}$ is invariant under (3.11), (3.3) if

$$u_i(t) = u_j(t), \tag{3.12}$$

for all $i, j \in \mathcal{I}$. Case *i* and *iv* are obvious. For case *ii* it is assumed that all $\tau_{ij} = \tau$ such that, on $\mathcal{M}$, $0 = u_i(t) - u_j(t) = \sigma(d_i - d_j)(y(t) - y(t-\tau))$. Thus (3.12) holds if $d_i = \sum_{j\in\mathcal{E}_i} a_{ij} = \text{constant}$ for all $i \in \mathcal{I}$. For case *iii* it follows that any $u_i(t) = \sigma \sum_{j\in\mathcal{E}_i} a_{ij}(y_j(t-\tau_{ij}) - y_i(t)) = \sigma \sum_{j\in\mathcal{E}_i} a_{ij}(y_j(t-\tau_{ij}) - y_i(t-T^*))$ with T^* being some integer multiple of T. Then all solutions on $\mathcal{M}$ being T-periodic implies each $u_i(t) \equiv 0$ on $\mathcal{M}$. □

Remark 3.7. The conditions for the existence of the synchronized state presented in Proposition 3.9 are sufficient but not necessary as will be shown in Example 4.3 in chapter 4. △

Only the cases *i* and *ii* will be discussed since cases *iii* and *iv* of Proposition 3.9 are rather restrictive and, in general, difficult to verify *a priori*. First it will be assumed that the couplings functions are such that the conditions in case *ii* are satisfied. In particular, it will be assumed the the systems (3.11) interact via coupling type I with

(H3.3) $\tau_{ij} = \tau$ and $\sum_{j\in\mathcal{E}_i} a_{ij} = 1$ for every $i, j \in \mathcal{I}$.

Here, for notational convenience and without loss of generality, $\sum_{j\in\mathcal{E}_i} a_{ij}$ is chosen to be equal to one for every $i \in \mathcal{I}$.

Theorem 3.10. *Consider k identical systems* (3.11) *which interact via coupling type I on a simple strongly connected graph. Suppose that (H3.3) holds and, in addition, assume that*

(H3.1) each system (3.11) *is strictly C^1-semipassive with a radially unbounded positive definite storage function $V(\cdot)$;*

(H3.2) there exists a positive definite function $V_0 \in \mathcal{C}^2(\mathbb{R}^p, \mathbb{R}_{>0})$ such that for all $z_i, z_j \in \mathbb{R}^p$ and all $y^* \in \mathbb{R}^m$ there is a constant $\alpha_0 \in \mathbb{R}_{>0}$ such that

$$(\nabla V_0(z_i - z_j))^\top (q(z_i, y^*) - q(z_j, y^*)) \le -\alpha_0 |z_i - z_j|^2 . \tag{3.13}$$

Then there exists constants $\bar{\sigma}$ and $\bar{\gamma}$ such that if $\sigma \ge \bar{\sigma}$ and $\sigma\tau \le \bar{\gamma}$ the set $\mathcal{M}$ contains a globally asymptotically stable subset.

Proof of Theorem 3.10. See appendix A.1.3. □

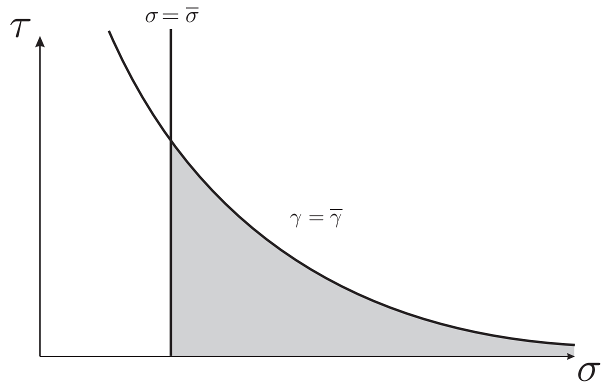


Figure 3.1. Strictly semipassive minimum phase systems that interact via coupling type I synchronize whenever $(\sigma, \tau) \in \mathcal{S}$ (shaded area).

The result of Theorem 3.10 boils down to the following. Given k strictly $\mathcal{C}^1$-semipassive identical systems with minimum phase[5] internal dynamics, then under restriction that the coupling is sufficiently strong and at the same time the time-delay is sufficiently small, the systems will synchronize. In other words, there exists always a region $\mathcal{S} = \{\sigma, \tau \in \mathbb{R}_{\geq 0} | \sigma \geq \bar\sigma \text{ and } \sigma\tau \leq \bar\gamma\}$, indicated in gray in Figure 3.1, such that if $(\sigma, \tau) \in \mathcal{S}$, then the systems synchronize.

Remark 3.8. In Theorem 3.10, assumption (H3.1) implies, by Theorem 3.3 and Corollary 3.4, that the solutions of the network are being ultimately bounded. This boundedness property plays an important role in the proof of Theorem 3.10. An advantage of the semipassivity approach that is used here is that proving that a *single* system is strictly semipassive guarantees boundedness of solutions of the *whole* network. However, these conditions for boundedness are only sufficient and Theorem 3.10 remains true without assumption (H3.1) if it can be proved that the solutions of the interconnected systems are ultimately bounded. △

Remark 3.9. For simplicity it is stated that assumption (H3.2) holds on the whole state space. Of course, since ultimate boundedness is already guaranteed by assumption (H3.1) (and Theorem 3.3), assumption (H3.2) has to hold only on the compact set to which the solutions converge in finite time. △

Consider now case i of Proposition 3.9, i.e. the k identical systems (3.11) on a simple strongly connected graph interact via coupling of the form

$$u_i(t) = \sigma \sum_{j \in \mathcal{E}_i} a_{ij}(y_j(t) - y_i(t)), \tag{3.14}$$

[5] a system is minimum-phase if it has stable zero dynamics, i.e. the internal tracking dynamics are stable, cf. [135]. Assumption (H3.2) implies that the systems are minimum-phase.

where a_{ij} is not assumed to be equal to a_{ji}. Since this coupling vanishes on the synchronization manifold, assumption (H3.3) of Theorem 3.10 can be omitted, and the following result can easily be derived.

Corollary 3.11. *Consider k identical systems* (3.11) *on a simple strongly connected graph that interact via non-delayed diffusive coupling* (3.14). *Suppose that assumptions (H3.1) and (H3.2) hold. Then there exists a constant $\bar{\sigma}$ such that if $\sigma \geq \bar{\sigma}$ the set $\mathcal{M}$ contains a globally asymptotically stable subset.*

Proof of Corollary 3.11. See appendix A.1.4. ☐

Obviously Corollary 3.11 generalizes the main result of [128], i.e. Theorem 3.1, in the sense that the assumption that the coupling is symmetric ($a_{ij} = a_{ji}$) is not needed anymore.

3.3.2 Semipassive systems interacting via coupling type II

Consider the systems (3.11) on a simple strongly connected graph and let the systems be coupled via

$$u_i(t) = \sigma \sum_{j \in \mathcal{E}_i} a_{ij} \left(y_j(t - \tau_{ij}) - y_i(t - \tau_{ji}) \right). \tag{3.4}$$

To guarantee invariance of the synchronization manifold $\mathcal{M}$ under the dynamics (3.11), (3.4) it will be assumed that

(H3.4) $\tau_{ij} = \tau_{ji}$ for all $i, j \in \mathcal{I}$.

Clearly, if assumption (H3.4) is satisfied, all couplings function (3.4) vanish if the systems are synchronized. Hence the linear manifold $\mathcal{M}$ is positively invariant under the given dynamics. Sufficient conditions for the stability of the synchronization manifold are presented in the following theorem.

Theorem 3.12. *Consider k identical systems* (3.11) *on a simple strongly connected graph that interact via coupling type II. Suppose that (H3.4) holds and assume, in addition, that*

(H3.5) each system is strictly $\mathcal{C}^1$-semipassive with a radially unbounded storage function and the functions $H(x_i)$ are such that there exist $R > 0$ such that $|x_i| > R$ implies $H(x_i) - 2\sigma d \left|y_i\right|^2 > 0$ with $d = \max_{i \in \mathcal{I}} \sum_{j \in \mathcal{E}_i} a_{ij}$;

(H3.2) there exists a positive definite function $V_0 \in \mathcal{C}^2(\mathbb{R}^p, \mathbb{R}_{>0})$ such that for all $z_i, z_j \in \mathbb{R}^p$ and all $y^* \in \mathbb{R}^m$ there is a constant $\alpha_0 \in \mathbb{R}_{>0}$ such that

$$(\nabla V_0(z_i - z_j))^\top (q(z_i, y^*) - q(z_j, y^*)) \leq -\alpha_0 \left|z_i - z_j\right|^2. \tag{3.13}$$

Then there exists constants $\bar{\sigma}$ and $\bar{\gamma}$ such that if $\sigma \geq \bar{\sigma}$ and $\sigma\tau^* \leq \bar{\gamma}$, $\tau^* = \max_{i,j\in\mathcal{I}} \tau_{ij}$, the set $\mathcal{M}$ contains a globally asymptotically stable subset.

Proof of Theorem 3.12. See appendix A.1.5. ☐

Remark 3.10. As remarked before for systems that interact via coupling type I (Remark 3.8), assumption (H3.5) in Theorem 3.12 is sufficient to have boundedness of solutions of type II coupled systems. Suppose that assumption (H3.5) is not satisfied but it can be proven that the solutions of the closed-loop system are ultimately bounded, then Theorem 3.12 can be applied without (H3.5). △

Remark 3.11. It might be that assumption (H3.5) is only satisfied for values σ on some interval $\mathcal{I}_\sigma$. Then the systems (3.11) coupled via (3.4) synchronize if $(\sigma, \tau^*) \in \{\sigma \in \mathcal{I}_\sigma, \tau^* \in \mathbb{R}_{\geq 0} | \sigma \geq \bar{\sigma} \text{ and } \sigma\tau^* \leq \bar{\gamma}\}$. Note that it is then implicitly assumed that $\bar{\sigma} \in \mathcal{I}_\sigma$. △

3.4 Convergent systems

In Theorems 3.1, 3.10, 3.12 and Corollary 3.11 it has been assumed that

(H3.2) there exists a positive definite function $V_0 \in \mathcal{C}^2(\mathbb{R}^p, \mathbb{R}_{>0})$ such that for all $z_i, z_j \in \mathbb{R}^p$ and all $y^* \in \mathbb{R}^m$ there is a constant $\alpha_0 \in \mathbb{R}_{>0}$ such that

$$(\nabla V_0(z_i - z_j))^\top (q(z_i, y^*) - q(z_j, y^*)) \leq -\alpha_0 |z_i - z_j|^2 . \tag{3.13}$$

In this section it is shown how this function V_0 can be constructed using the concept of *convergent systems* (which is introduced in chapter 2, section 2.4).

Proposition 3.13. *If the subsystem* (3.11a) *satisfies the Demidovich condition (Lemma 2.2) with a positive definite matrix P, then assumption (H3.2) holds with*

$$V_0(z_i - z_j) = (z_i - z_j)^\top P (z_i - z_j). \tag{3.15}$$

Proof of Proposition 3.13. (The proof can also be found in [111].) The derivative of (3.15) along two trajectories of (3.11a) with the constraint $y_i(t) = y_j(t) = y^*(t)$ is given by

$$\begin{aligned} \dot{V}_0\Big|_{y_i=y_j=y^*}(z_i - z_j) &= (\nabla V_0(z_i - z_j))^\top (q(z_i, y^*) - q(z_j, y^*)) \\ &= (z_i - z_j)^\top P (q(z_i, y^*) - q(z_j, y^*)). \end{aligned} \tag{3.16}$$

Denote

$$\Phi(\zeta) := (z_i - z_j)^\top P q(z_j + \zeta(z_i - z_j), y^*), \tag{3.17}$$

with constant $\zeta \in [0,1]$. Note that (3.16) can be written as $\Phi(1) - \Phi(0)$. By the mean value theorem and $q \in \mathcal{C}^1$

$$\Phi(1) - \Phi(0) = \frac{\mathrm{d}\Phi}{\mathrm{d}\zeta}(\zeta^*), \tag{3.18}$$

for some $\zeta^* \in [0,1]$. Hence

$$\begin{aligned} \dot{V}_0\Big|_{y_i=y_j=y^*}(z_i - z_j) &= (z_i - z_j)^\top P \frac{\mathrm{d}q}{\mathrm{d}z}(z^*, y^*)(z_i - z_j) \\ &= (z_i - z_j)^\top Q(z^*, y^*)(z_i - z_j), \end{aligned} \tag{3.19}$$

with $z^* = z_j + \zeta^*(z_i - z_j)$ and the matrix $Q(\cdot,\cdot)$ as defined in Lemma 2.2. Since $Q(\cdot,\cdot)$ is uniformly negative definite by assumption it can be concluded that

$$\dot{V}_0\Big|_{y_i=y_j=y^*}(z_i - z_j) \leq -\alpha_0 \left|z_i - z_j\right|^2, \tag{3.20}$$

with constant $\alpha_0 = \frac{\delta}{\lambda_{\max}(P)}$, δ as defined in Lemma 2.2 and $\lambda_{\max}(P)$ being the largest eigenvalue of the positive definite matrix P. □

In the remainder of this thesis it will often be assumed that a system satisfies assumption (H3.2). For notational convenience, it will simply be stated that such a system satisfies the Demidovich condition (with a positive definite matrix P). Of course, this assumption can replaced by the more general assumption (H3.2).

3.5 Discussion

In this chapter synchronization of semipassive systems that interact via time-delay diffusive coupling has been discussed. First it is proven that the solutions of strictly semipassive systems that interact via coupling type I or type II are ultimately bounded. Next it is proven that coupled identical strictly semipassive minimum-phase systems always synchronize provided that the coupling is sufficiently strong and the product of the coupling and the time-delay is sufficiently small. Finally the notion of convergent systems is used to present a condition (that is relatively easy to verify) to ensure that the systems are minimum-phase. An important observation is that (as follows from the proofs of the theorems) the main result of [128], i.e. Theorem 3.1, is a special case of Theorem 3.10 and Theorem 3.12. See also Corollary 3.11. Thus the results presented in this chapter generalize Theorem 3.1.

It has to be noted that the results presented in this chapter are mainly on the existence level, i.e. for strictly semipassive minimum-phase systems there *exist* constants $\bar{\sigma}$ and $\bar{\gamma}$ such that if $\sigma \geq \bar{\sigma}$ and $\sigma\tau \leq \bar{\gamma}$ the systems synchronize. One has to realize that the estimates of the threshold values $\bar{\sigma}$ and $\bar{\gamma}$ which are provided in the proofs of the theorems can be quite conservative. Moreover, it is important to notice that these estimates depend on the particular dynamical systems in the network, e.g. the threshold values for a network of Lorenz systems can be expected to differ from the threshold values that can be found for a network of Hindmarsh-Rose oscillators.

The results presented in this chapter hold for networks with a fixed topology and constant time-delays. It would, from a practical point of view, be very interesting to extend the results for networks with time-varying topologies and time-varying delays. See, for instance, [108, 96] for some results for consensus of multi agent systems. It has to be noted that for networks with a fixed topology but varying delays, the part of the proofs of Theorem 3.10 and Theorem 3.12 concerning the convergence of the solutions to each other is still valid. This is because the proof of convergence uses the Lyapunov-Razumikhin theorem which remains true for bounded time-varying delays. Hence to extend the results presented here in that direction it has (only) to be proven that the solutions are ultimately bounded.

CHAPTER FOUR

Synchronization and network topology Part I: Partial synchronization

Abstract. This chapter presents results on partial synchronization of identical strictly semipassive systems. Linear invariant manifolds corresponding to the regimes of partial synchronization are identified by symmetries in the network. First, the results on partial synchronization of strictly semipassive systems that interact via symmetric non-delayed diffusive coupling presented in [129, 125, 126] are extended to the asymmetric coupling case. Then results are presented for partial synchronization of systems that interact via coupling type I or coupling type II. In both cases, it is assumed that the time-delays are uniform. Finally, some results are presented for partial synchronization of semipassive systems that interact via a specific type of coupling type I with non-uniform time-delays.

4.1 Introduction

In this chapter *partial synchronization* of k *identical* systems that interact via diffusive time-delay coupling is discussed. Partial synchronization, also known as *clustering*, is the phenomenon where some systems in the network do synchronize while others do not. This implies that, if there is partial synchronization in a network, there should exist linear manifolds of the type $\left\{\operatorname{col}(x_1, \ldots, x_k) \in \mathbb{R}^{kn} | x_i = x_j \text{ for some } i, j \in \mathcal{I}\right\}$ that invariant under the closed-loop dynamics and attracting for values of the coupling strength and time-delay other than those for which the full synchronization manifold $\left\{\operatorname{col}(x_1, \ldots, x_k) \in \mathbb{R}^{kn} | x_i = x_j \text{ for all } i, j \in \mathcal{I}\right\}$ is attracting. As the systems in the network are assumed to be identical, the existence of such linear invariant manifolds is likely to follow from the specific topology of the network [18, 20, 129, 125, 126]. In particular, in [18, 20] it is shown how the structure of the network can be chosen such that partial synchronization occurs. In [129, 125, 126], a systematic approach is presented to identify the linear invariant manifolds using *local* and *global* symmetries in the closed-loop dynamical system. Local symmetries are the symmetries that are present in the system itself. For example, the Lorenz system [86] is a system [129] with a local symmetry. Indeed, the Lorenz equations

$$\dot{x}_1 = \sigma(x_2 - x_1), \tag{4.1a}$$

$$\dot{x}_2 = rx_1 - x_2 - x_1x_3, \tag{4.1b}$$

$$\dot{x}_3 = -bx_3 + x_1x_2, \tag{4.1c}$$

are invariant under the change of coordinates $x \mapsto y$ with $y_1 = -x_1$, $y_2 = -x_2$ and $y_3 = x_3$. Global symmetries are the symmetries that are present in the network. In this chapter the results of [129, 125, 126] will be extended to the case with asymmetric time-delayed interaction. Only global symmetries will be considered as the focus in this chapter is on the topology of the the network. However, it is important to realize that local symmetries also might influence the synchronization in a network. See [129] for an example.

In case of a global symmetry, i.e. if the network contains a certain symmetry, the symmetry must be present in the adjacency matrix A (and thus also in the Laplacian matrix L). Hence a rearrangement of (some of) the entries of A leaves the network unchanged. Mathematically the rearrangement of the entries of A is described by the (pre)multiplication of A with a permutation matrix. Recall that a permutation matrix is a matrix with exactly one entry equal to one in each row and each column and zeros everywhere else. Permutation matrices are orthogonal and form a group (with identity element I) under multiplication. In [129, 125, 126] networks are considered that consist of identical systems that interact via non-delayed diffusive coupling,

$$u_i(t) = \sigma \sum_{j \in \mathcal{E}_i} a_{ij}(y_j(t) - y_i(t)), \tag{4.2}$$

with $a_{ij} = a_{ji}$. In particular, in [129] it is proven that if there is a permutation matrix Π that commutes with the Laplacian matrix L, i.e. $\Pi L = L\Pi$, then the set $\ker(I_{kn} - \Pi \otimes I_n)$ defines a linear invariant manifold for the network. In addition, sufficient conditions are presented for this manifold to be globally attracting for the closed-loop dynamics. In [125, 126], the assumption that Π and L commute is relaxed; it is proven that the result of [129] remains true if there exists a solution X of the matrix equation $(I-\Pi)L = X(I-\Pi)$. (Obviously, if Π and L commute, $L = X$.)

Partial synchronization will be formally defined as follows:

Definition 4.1 (Partial synchronization). Consider k systems on a simple strongly connected graph:

$$\dot{x}_i(t) = f_i(x_i(t)) + g_i(x_i(t))u_i(t), \tag{4.3a}$$

$$y_i(t) = h_i(x_i(t)), \quad i \in \mathcal{I} := \{1, 2, \ldots, k\}, \tag{4.3b}$$

with state $x_i \in \mathbb{R}^n$, input $u_i \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R}^m)$, output $y_i \in \mathbb{R}^m$, sufficiently smooth functions $f_i : \mathbb{R}^n \to \mathbb{R}^n$, $g_i : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ and $h_i : \mathbb{R}^n \to \mathbb{R}^m$. Let the systems interact via *coupling type I*

$$u_i(t) = \sigma \sum_{j \in \mathcal{E}_i} a_{ij} \left(y_j(t - \tau_{ij}) - y_i(t) \right), \tag{4.4}$$

or *coupling type II*

$$u_i(t) = \sigma \sum_{j \in \mathcal{E}_i} a_{ij} \left(y_j(t - \tau_{ij}) - y_i(t - \tau_{ji}) \right). \tag{4.5}$$

The interconnected systems are said to *locally partially synchronize* if, for all continuous initial history ϕ_i, ϕ_j, there is $\delta > 0$ such that $|\phi_i - \phi_j| < \delta$ implies $|x_i(t; t_0, \phi_i) - x_j(t; t_0, \phi_j)| \to 0$ as $t \to \infty$ for at least two but not all $i, j \in \mathcal{I}$. When $\delta = \infty$ the systems are said to *globally partially synchronize* (or simply *partially synchronize* for short). △

The remainder of this chapter is organized as follows. Section 4.2 gives sufficient conditions for partial synchronization in networks coupled without delays. The result generalizes the main results presented in [129, 125, 126] in the sense that the non-delayed interaction is not assumed to be symmetric anymore. In section 4.3 sufficient conditions are presented for partial synchronization of systems that interact via time-delay diffusive coupling of type I or type II. In this section it is assumed that the time-delays are uniform, i.e. the value of the time-delay on each interconnection is the same. In section 4.4 some results are presented for systems that interact via coupling type I with non-uniform time-delays. Section 4.5 concludes the chapter.

4.2 Non-delayed interaction

Consider k systems (in the normal form introduced in chapter 3, section 3.3) on a simple strongly connected graph,

$$\dot{z}_i(t) = q(z_i(t), y_i(t)), \tag{4.6a}$$

$$\dot{y}_i(t) = a(y_i(t), z_i(t)) + u_i(t), \quad i \in \mathcal{I}, \tag{4.6b}$$

with $z_i \in \mathbb{R}^p$, $y_i \in \mathbb{R}^m$, input $u_i \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}^m) \cap \mathcal{L}_\infty$, and sufficiently smooth functions $q : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^p$, $a : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^m$. Let the systems (4.6) interact via non-delayed diffusive coupling

$$u_i(t) = \sigma \sum_{j \in \mathcal{E}_i} a_{ij} (y_j(t) - y_i(t)). \tag{4.7}$$

It will *not* be assumed that the coupling (4.7) is symmetric, i.e. a_{ij} is not necessarily equal to a_{ji}. The following lemma shows that a symmetry in the network defines a linear invariant manifold for the closed-loop dynamics.

Lemma 4.1. *Let $\Pi \in \mathbb{R}^{k \times k}$ be a permutation matrix and the matrix X be the solution of the matrix equation $(I - \Pi)L = X(I - \Pi)$. Then the set $\ker(I_{kn} - \Pi \otimes I_n)$ defines a linear invariant manifold for the system* (4.6), (4.7).

Proof of Lemma 4.1. The following new variables are introduced for notational convenience: $\Xi := I_{kn} - \Pi \otimes I_n$, $\xi(t) := \operatorname{col}(z_1(t), y_1(t), \ldots, z_k(t), y_k(t))$, $F(\xi(t)) := \operatorname{col}(q(z_1(t), y_1(t)), a(y_1(t), z_1(t)), \ldots, q(z_k(t), y_k(t)), a(y_k(t), z_k(t)))$ and

$$B := \begin{pmatrix} 0 & 0 \\ 0 & I_m \end{pmatrix}. \tag{4.8}$$

Then, using the new notation, the closed-loop system (4.6), (4.7) can be written as

$$\dot{\xi}(t) = F(\xi(t)) - \sigma(L \otimes B)\xi(t). \tag{4.9}$$

Let $\xi^*(t)$ be such that $\Xi\xi^*(t) = 0$, hence $\Xi\dot{\xi}^*(t) = 0$. This implies that

$$\Xi F(\xi^*(t)) - \sigma\Xi(L \otimes B)\xi^*(t) = 0. \tag{4.10}$$

Since $(\Pi \otimes I_n)F(\xi(t)) = F((\Pi \otimes I_n)\xi(t))$ because Π is a permutation matrix, it follows from the assumptions that

$$0 = F(\xi^*(t)) - F((\Pi \otimes I_n)\xi^*(t)) - (X \otimes B)(I_{kn} - \Pi \otimes I_n)\xi^*(t), \tag{4.11}$$

hence $\ker(I_{kn} - \Pi \otimes I_n)$ defines a linear invariant manifold for the system (4.6), (4.7). □

Using Lemma 4.1 and the machinery presented in the previous chapter, cf. Corollary 3.11, the following theorem is a straightforward extension of the main result of [125].

Theorem 4.2. *Consider k systems* (4.6) *on a simple strongly connected graph that are coupled via non-delayed diffusive coupling* (4.7). *Let $\Pi \in \mathbb{R}^{k \times k}$ be a permutation matrix and X be the solution of the matrix equation $(I - \Pi)L = X(I_k - \Pi)$. Suppose that*

(H4.1) each system (4.6) *is strictly $\mathcal{C}^1$-semipassive with a radially unbounded storage function;*

(H4.2) each subsystem (4.6a) *satisfies the Demidovich condition;*

(H4.3) there is a constant $\lambda' \in \mathbb{R}_{>0}$ such that for all $\zeta \in \mathbb{R}^k$

$$\tfrac{1}{2}\zeta^\top(I_k - \Pi)^\top(X + X^\top)(I_k - \Pi)\zeta \geq \lambda'\zeta^\top(I_k - \Pi)^\top(I_k - \Pi)\zeta.$$

Then there exists a positive constant σ^* such that if $\sigma \geq \sigma^*$, then a subset of the set $\ker(I_{kn} - \Pi \otimes I_n)$ is globally asymptotically stable.

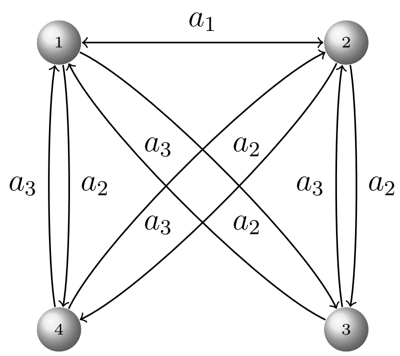


Figure 4.1. The network of Example 4.1.

Proof of Theorem 4.2. See appendix A.2.1. □

Remark 4.1. The existence of a solution X of the matrix equation $(I-\Pi)L = X(I-\Pi)$ and assumption (H4.3) can be easily verified using the singular value decomposition of $(I-\Pi)$, cf. [125, 126]. △

Theorem 4.2 provides sufficient conditions for subset of the set $\ker(I_{kn} - \Pi \otimes I_n)$ to be globally asymptotically stable. The network shows partial synchronization if the values of the coupling strength for which this subset is globally asymptotically stable do not coincide with the values of the coupling strength for which the network fully synchronizes. Let σ^* be as in Theorem 4.2, i.e. σ^* is such that a subset of the set $\ker(I_{kn} - \Pi \otimes I_n)$ is globally asymptotically stable if $\sigma \geq \sigma^*$. Note that assumptions (H4.1) and (H4.2) imply, by Corollary 3.11, that *all* systems synchronize whenever the coupling is sufficiently strong, say $\sigma \geq \bar{\sigma}$. Then partial synchronization of the systems can only be expected if $\sigma^* < \bar{\sigma}$.

The following two examples show how Theorem 4.2 can be applied.

Example 4.1. Consider the network depicted in Figure 4.1. The corresponding Laplacian matrix is given as

$$L = \begin{pmatrix} a_1 + 2a_3 & -a_1 & -a_3 & -a_3 \\ -a_1 & a_1 + 2a_3 & -a_3 & -a_3 \\ -a_2 & -a_2 & 2a_2 & 0 \\ -a_2 & -a_2 & 0 & 2a_2 \end{pmatrix}.$$

Consider the permutation matrix

$$\Pi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

It can easily be verified that this matrix commutes with L, i.e. $\Pi L = L\Pi$. Hence $X = L$ is a solution of the matrix equation $(I-\Pi)L = X(I-\Pi)$. A straightforward computation

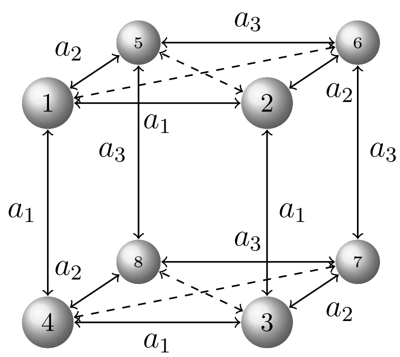


Figure 4.2. The network of Example 4.2. The interconnections represented by the dashed lines have weights a_4.

shows that (H4.3) holds with $\lambda' = 2a_2 > 0$. Suppose that assumptions (H4.1) and (H4.2) are satisfied. Then Theorem 4.2 implies that the set $\ker(I_{4n} - \Pi \otimes I_n)$ contains a globally attracting subset for sufficiently large σ which implies that systems 3 and 4 synchronize for sufficiently large σ. There is partial synchronization if the network does not fully synchronize for the values of σ for which systems 3 and 4 synchronize. △

Example 4.2 ([125]). Consider the network depicted in Figure 4.2. The Laplacian matrix of this network is

$$L = \begin{pmatrix} d_1 & -a_1 & 0 & -a_1 & -a_2 & -a_4 & 0 & 0 \\ -a_1 & d_1 & -a_1 & 0 & -a_4 & -a_2 & 0 & 0 \\ 0 & -a_1 & d_1 & -a_1 & 0 & 0 & -a_2 & -a_4 \\ -a_1 & 0 & -a_1 & d_1 & 0 & 0 & -a_4 & -a_2 \\ -a_2 & -a_4 & 0 & 0 & d_5 & -a_3 & 0 & -a_3 \\ -a_4 & -a_2 & 0 & 0 & -a_3 & d_5 & -a_3 & 0 \\ 0 & 0 & -a_2 & -a_4 & 0 & -a_3 & d_5 & -a_3 \\ 0 & 0 & -a_4 & -a_2 & -a_3 & 0 & -a_3 & d_5 \end{pmatrix},$$

with $d_1 = 2a_1 + a_2 + a_4$ and $d_5 = 2a_3 + a_2 + a_4$. The permutation matrix

$$\Pi = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

does *not* commute with L but there exists a solution X of the matrix equation $(I - \Pi)L = X(I - \Pi)$. This can be verified (numerically) using the singular value decomposition. For instance, let $a_1 = a_3 = 1$ and $a_2 = a_4 = \frac{1}{2}$. Consider the singular value decomposition of

$I - \Pi$, $I - \Pi = U\Sigma V^\top$, with unitary matrices U and V and Σ is a diagonal matrix with the singular vales of $I - \Pi$ as entries, cf. [64]. Note that $\dim\ker(I - \Pi) = 2$. Assume that Σ has the form

$$\Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix},$$

with $\Sigma_1 \in \mathbb{R}^{6\times 6}$ being a diagonal matrix with the nonzero singular values of $I - \Pi$ as entries. It is easy to verify that the matrix

$$X = \frac{1}{4}\begin{pmatrix} 11 & -5 & -1 & -5 & 0 & 0 & 0 & 0 \\ -5 & 11 & -5 & -1 & 0 & -2 & 0 & 2 \\ -1 & -5 & 11 & -5 & 0 & 0 & 0 & 0 \\ -5 & -1 & -5 & 11 & 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 11 & -5 & -1 & -5 \\ 0 & -2 & 0 & 2 & -5 & 11 & -5 & -1 \\ 0 & 0 & 0 & 0 & -1 & -5 & 11 & -5 \\ 0 & 2 & 0 & -2 & -5 & -1 & -5 & 11 \end{pmatrix}$$

solves $(I - \Pi)L = X(I - \Pi)$. Let

$$\tfrac{1}{2}U^\top(X + X^\top)U = \begin{pmatrix} X_1 & X_2^\top \\ X_2 & X_3 \end{pmatrix},$$

then λ' can be chosen as the smallest eigenvalue of the matrix X_1. For this example $\lambda' = 2 > 0$. Then Theorem 4.2 implies that if assumptions (H4.1) and (H4.2) are satisfied, the set $\ker(I_{8n} - \Pi \otimes I_n)$ contains a globally attracting subset for sufficiently large σ. △

4.3 Delayed interaction with uniform time-delays

In this section the result presented in the previous section will be extended to the case of systems that interact via diffusive time-delay coupling. First sufficient conditions for partial synchronization in networks of systems with coupling type I are given, then similar results are presented for systems interacting via coupling type II. For both cases it will be assumed that the time-delays are all the same, i.e. $\tau_{ij} = \tau$ for every $i, j \in \mathcal{I}$.

4.3.1 Coupling type I

Consider a network with systems (4.6) on a simple strongly connected graph that interact via coupling type I with uniform time-delays, that is, coupling of the form

$$u_i(t) = \sigma \sum_{j\in\mathcal{E}_i} a_{ij}(y_j(t-\tau) - y_i(t)). \tag{4.12}$$

Like in Chapter 3, it will be assumed that

(H4.4) $\sum_j a_{ij} = 1$ for all $i \in \mathcal{I}$.

Like in the previous section, before sufficient conditions for the convergence to a linear invariant manifold are given, it is first shown that such invariant manifolds exist under appropriate conditions.

Lemma 4.3. *Let $\Pi \in \mathbb{R}^{k\times k}$ be a permutation matrix and the matrix X be the solution of the matrix equation $(I-\Pi)A = X(I-\Pi)$. Suppose that (H4.4) holds, then the set* $\ker(I_{kn} - \Pi \otimes I_n)$ *defines a linear invariant manifold for the system* (4.6), (4.12).

Proof of Lemma 4.3. The proof is almost the same as the proof of Lemma 4.1. Using the notation as in Lemma 4.1, the closed-loop system (4.6), (4.12) can be written as

$$\dot{\xi}(t) = F(\xi(t)) - \sigma(I\otimes B)\xi(t) + \sigma(A\otimes B)\xi(t-\tau). \tag{4.13}$$

Let $\xi^*(t+\theta)$, $\theta \in [-\tau, 0]$, be such that $\Xi\xi^*(t+\theta) = 0$. Hence $\Xi\dot{\xi}^*(t+\theta) = 0$ such that

$$\Xi F(\xi^*(t)) - \sigma\Xi[(I\otimes B)\xi^*(t) - (A\otimes B)\xi^*(t-\tau)] = 0. \tag{4.14}$$

Since $(\Pi\otimes I_n)F(\xi(t)) = F((\Pi\otimes I_n)\xi(t))$, it follows from the assumptions that

$$0 = F(\xi^*(t)) - F((\Pi\otimes I_n)\xi^*(t)) - \sigma[(I\otimes B)\Xi\xi^*(t) - (X\otimes B)\Xi\xi^*(t-\tau)]. \tag{4.15}$$

Thus $\ker(I_{kn} - \Pi\otimes I_n)$ defines a linear invariant manifold for the system (4.6), (4.12). $\square$

Conditions for the convergence of the solutions of the closed-loop system to (a subset of) the set $\ker(I - \Pi\otimes I_n)$ are presented in the following theorem.

Theorem 4.4. *Consider k systems* (4.6) *on a simple strongly connected graph that interact via coupling* (4.12). *Let $\Pi \in \mathbb{R}^{k\times k}$ be a permutation matrix and the matrix X be the solution of the matrix equation $(I-\Pi)A = X(I-\Pi)$. Suppose that (H4.4) holds and assume, in addition, that*

(H4.1) each system (4.6) *is strictly C^1-semipassive with a radially unbounded storage function;*

(H4.2) each subsystem (4.6a) *satisfies the Demidovich condition;*

(H4.3) there is a constant $\lambda' \in \mathbb{R}_{>0}$ such that for all $\zeta \in \mathbb{R}^k$

$$\tfrac{1}{2}\zeta^\top(I_k - \Pi)^\top(X + X^\top)(I_k-\Pi)\zeta \geq \lambda'\zeta^\top(I_k-\Pi)^\top(I_k-\Pi)\zeta.$$

Then there exist positive constants σ^* and γ^* such that if $\sigma \geq \sigma^*$ and $\sigma\tau \leq \gamma^*$, then the set $\ker(I - \Pi\otimes I_n)$ *contains a globally asymptotically stable subset.*

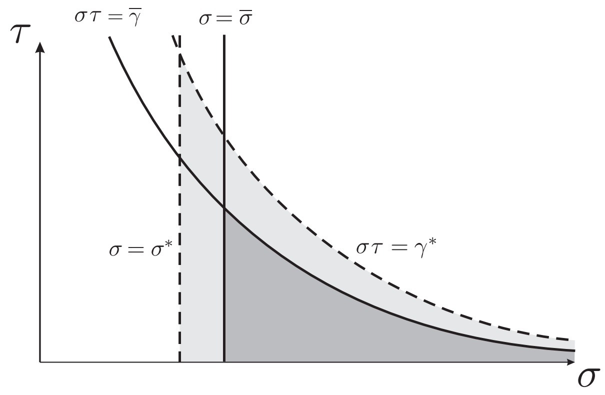


Figure 4.3. Values of σ and τ for which a subset of the set $\ker(I_{kn} - \Pi \otimes I_n)$ is globally asymptotically stable (light shaded area) and the values of σ and τ for which there is full synchronization.

Proof of Theorem 4.4. See appendix A.2.2 □

Like Theorem 4.2, Theorem 4.4 gives sufficient conditions for a subset of the set $\ker(I_{kn} - \Pi \otimes I_n)$ to be globally asymptotically stable. It follows that partial synchronization occurs if the values for the coupling strength and the time-delay for which a subset of the set $\ker(I_{kn} - \Pi \otimes I_n)$ is globally asymptotically stable do not coincide with the values of the coupling strength and time-delay for which there is full synchronization. Theorem 3.10 implies that, if the conditions of Theorem 4.4 are satisfied, there are positive constants $\bar{\sigma}$ and $\bar{\gamma}$ such that if $\sigma \geq \bar{\sigma}$ and $\sigma\tau \leq \bar{\gamma}$ the coupled systems (4.6), (4.12) all synchronize. Thus partial synchronization can only occur if $\sigma^* < \bar{\sigma}$, $\gamma^* > \bar{\gamma}$, or both $\sigma^* < \bar{\sigma}$ and $\gamma^* > \bar{\gamma}$. See Figure 4.3.

Corollary 4.5. *Consider k systems* (4.6) *on a simple strongly connected graph that interact via coupling* (4.12) *with $a_{ij} = a_{ji}$ for every $i, j \in \mathcal{I}$. Suppose that (H4.4), (H4.1) and (H4.2) hold. Let $\Pi \in \mathbb{R}^{k\times k}$ be a permutation matrix that commutes with A, i.e. $\Pi A = A\Pi$, and let $\lambda_i(A)$ be the eigenvalues of A ordered as $0 \leq |\lambda_1(A)| \leq |\lambda_2(A)| \leq \ldots \leq |\lambda_{k-1}(A)| \leq |\lambda_k(A)| = 1$. Define $\underline{\lambda}(L) = \underline{\lambda}(I - A) := \max(\lambda_2(L), \lambda^*(L))$, $\lambda_2(L)$ is the smallest nonzero eigenvalue of $L = I - A$ and $\lambda^*(L)$ denotes the smallest nonzero eigenvalue of L with corresponding eigenvector in* $\operatorname{range}(I - \Pi)$*, and $\overline{\lambda}(A) := \min(|\lambda_{k-1}(A)|, |\lambda'(A)|)$, $|\lambda'(A)|$ is the largest $|\lambda_i(A)|$ with corresponding eigenvector in* $\operatorname{range}(I - \Pi)$. *Then there exist positive constants $\sigma^\star$ and $\gamma^\star$ such that the set $\ker(I_{kn} - \Pi \otimes I_n)$ contains a globally asymptotically stable subset if*

$$\sigma\underline{\lambda}(L) \geq \sigma^\star \text{ and } \sigma\tau\overline{\lambda}(A) \leq \gamma^\star.$$

Proof of Corollary 4.5. The proof follows immediately from the proof of Theorem 4.4 using the facts that

i. $A = A^\top$ since $a_{ij} = a_{ji}$ for any $i, j \in \mathcal{I}$;

ii. Π and A commute implies $X = A$ is a solution of the matrix equation $(I - \Pi)A = X(I - \Pi)$;

iii. $\zeta^\top (I_k - \Pi)^\top A(I_k - \Pi)\zeta \geq \underline{\lambda}(L)\zeta^\top (I_k - \Pi)^\top (I_k - \Pi)\zeta$ for any $\xi \in \mathbb{R}^k$;

iv. $\zeta^\top (I_k - \Pi)^\top A(I_k - \Pi)\zeta \leq \overline{\lambda}(A)\zeta^\top (I_k - \Pi)^\top (I_k - \Pi)\zeta$ for any $\xi \in \mathbb{R}^k$.

□

Corollary 4.5 implies that a network of systems that interact via symmetric coupling type I might show partial synchronization only if $\underline{\lambda}(L) > \lambda_2(L)$ and/or $\overline{\lambda}(A) < \lambda'(A)$. Indeed, as follows from the proof of Theorem 3.10, if $\underline{\lambda}(L) = \lambda_2(L)$ and $\overline{\lambda}(A) = \lambda'(A)$, the values of σ and τ for which a subset of the set $\ker(I_{kn} - \Pi \otimes I_n)$ is globally asymptotically stable coincide with the values of σ and τ for which all systems synchronize.

4.3.2 Coupling type II

Consider again the systems (4.6) on a simple strongly connected graph but let the systems now interact via coupling type II with uniform time-delays, i.e. let the system be coupled via

$$u_i(t) = \sigma \sum_j a_{ij}(y_j(t-\tau) - y_i(t-\tau)). \tag{4.16}$$

The results on partial synchronization for systems interacting via coupling type I can easily be extended to the case where the systems interact with coupling type II. The following lemma gives conditions for the existence of a linear invariant manifold.

Lemma 4.6. *Let $\Pi \in \mathbb{R}^{k\times k}$ be a permutation matrix and the matrix X be the solution of the matrix equation $(I - \Pi)L = X(I - \Pi)$. Then the set $\ker(I_{kn} - \Pi \otimes I_n)$ defines a linear invariant manifold for the system* (4.6), (4.16).

Proof of Lemma 4.6. The proof follows with minor modifications from the proof of Lemma 4.1. □

Using Lemma 4.6, hence the existence of a linear invariant manifold, the following theorem gives sufficient conditions for the linear invariant manifold to be attracting for the dynamics (4.6), (4.16).

Theorem 4.7. *Consider k systems* (4.6) *on a simple strongly connected graph that interact via coupling* (4.16). *Let $\Pi \in \mathbb{R}^{k\times k}$ be a permutation matrix and the matrix X be the solution of the matrix equation $(I - \Pi)L = X(I - \Pi)$. Suppose that*

(H4.5) *each system is strictly $\mathcal{C}^1$-semipassive with a radially unbounded storage function and the functions $H(x_i)$ are such that there exist $R > 0$ such that $|x_i| > R$ implies $H(x_i) - 2\sigma d\,|y_i|^2 > 0$ with $d = \max_{i\in\mathcal{I}} \sum_{j\in\mathcal{E}_i} a_{ij}$;*

(H4.2) *each subsystem* (4.6a) *satisfies the Demidovich condition;*

(H4.3) *there is a constant $\lambda' \in \mathbb{R}_{>0}$ such that for all $\zeta \in \mathbb{R}^k$*

$$\tfrac{1}{2}\zeta^\top(I_k - \Pi)^\top(X + X^\top)(I_k - \Pi)\zeta \geq \lambda'\zeta^\top(I_k - \Pi)^\top(I_k - \Pi)\zeta.$$

Then there exist positive constants σ^* and γ^* such that if $\sigma \geq \sigma^*$ and $\sigma\tau \leq \gamma^*$, then the set $\ker(I - \Pi \otimes I_n)$ contains a globally asymptotically stable subset.

Proof of Theorem 4.7. Assumption (H4.5) implies, by Theorem 3.6 and Corollary 3.7, that the solutions of the closed-loop system are ultimately bounded, and the bounds are independent of the network topology. Lemma 4.6 implies that the set $\ker(I_{kn} - \Pi \otimes I_n)$ defines a linear invariant manifold for the closed-loop system (4.6), (4.16). The proof can now easily be constructed from arguments used in the proofs of Theorem 3.12 and Theorem 4.4. ☐

The next corollary is the counterpart of Corollary 4.5 for systems that interact via coupling type II. (Hence it will be presented without proof.)

Corollary 4.8. *Consider k systems* (4.6) *on a simple strongly connected graph that interact via coupling* (4.16) *with $a_{ij} = a_{ji}$ for any $i, j \in \mathcal{I}$. Suppose that (H4.5) and (H4.2) hold. Let $\Pi \in \mathbb{R}^{k\times k}$ be a permutation matrix that commutes with L, i.e. $\Pi L = L\Pi$, and let $\lambda_i(L)$ be the eigenvalues of L ordered as $0 = \lambda_1(L) < \lambda_2(L) \leq \ldots \leq \lambda_k(L)$. Define $\underline{\lambda}(L) := \max(\lambda_2(L), \lambda^*(L))$, $\lambda^*(L)$ is the smallest eigenvalue of L with corresponding eigenvector in $\operatorname{range}(I - \Pi)$, and $\overline{\lambda}(L) = \min(\lambda_k(L), \lambda'(L))$, $\lambda'(L)$ is the largest eigenvalue of L with corresponding eigenvector in $\operatorname{range}(I - \Pi)$. Then there exist positive constants $\sigma^\star$ and $\gamma^\star$ such that the set $\ker(I_{kn} - \Pi \otimes I_n)$ contains a globally asymptotically stable subset if*

$$\sigma\underline{\lambda}(L) \geq \sigma^\star \;\; \textit{and} \;\; \sigma\tau\overline{\lambda}(L) \leq \gamma^\star.$$

Corollary 4.8 implies that partial synchronization in a network of systems that interact via symmetric coupling type II can only happen if

i. $\underline{\lambda}(L) > \lambda_2(L)$, $\lambda_2(L)$ is the smallest nonzero eigenvalue of L, and/or

ii. $\overline{\lambda}(L) < \lambda_k(L)$, $\lambda_k(L)$ is the largest eigenvalue of L.

4.4 Delayed interaction with non-uniform time-delays

In the previous section it is assumed that the time-delays are uniform, i.e. all the time-delays in the network are identical. Suppose now that the time-delays are non-uniform, but there is a certain symmetry in the network with respect to the interactions *and* time-delays. That is, some *simultaneous* rearrangements of the entries of the adjacency matrix A and the matrix (τ_{ij}), i.e.

$$\left(\tau_{ij}\right) = \begin{pmatrix} 0 & \tau_{12} & \cdots & \tau_{1k} \\ \tau_{21} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \tau_{(k-1)k} \\ \tau_{k1} & \cdots & \tau_{k(k-1)} & 0 \end{pmatrix}, \tag{4.17}$$

leaves the network unchanged. For example, if Π is a permutation matrix that commutes with A, $\Pi A = A\Pi$, then Π should also commute with the matrix (τ_{ij}). Note that for a network with uniform time-delays, a symmetry in the network with respect to the interactions is always a symmetry with respect to the time-delays. Is it possible to have partial synchronization in networks with non-uniform time-delays? In this section some results are presented for systems that interact via coupling type I with non-uniform time-delays.

To avoid (very) complicated notation only a special type of coupling type I will be considered. Let $\mathcal{P}_{\Pi(k)}$ denote the family of $k \times k$ dimensional symmetric permutation matrices with zero trace and let the systems be coupled via

$$u(t) = -\sigma I y(t) + \sigma \sum_{\ell} \eta_\ell (\Pi_\ell \otimes I) y(t - \tau_\ell), \tag{4.18}$$

where $\Pi_\ell \in \mathcal{P}_{\Pi(k)}$ and constants $\eta_\ell \geq 0$ are such that $\sum_\ell \eta_\ell = 1$. Note that the zero trace assumption implies that there is no self-interaction. In addition, due to the specific form of coupling (4.18), a symmetry in the adjacency matrix is also a symmetry in the matrix (τ_{ij}).

Lemma 4.9. *For any matrix $\Pi \in \mathcal{P}_{\Pi(k)}$, the set $\ker(I_{kn} - \Pi \otimes I_n)$ defines a linear invariant manifold for the closed-loop dynamics* (4.6), (4.18).

Proof of Lemma 4.9. It will be proven that all matrices in $\mathcal{P}_{\Pi(k)}$ commute. Then the proof of invariance follows from the proof of Lemma 4.3 with $\xi^*(t+\theta)$, $\theta = [-\tau^*, 0]$ and $\tau^* = \max_\ell(\tau_\ell)$, such that $(I_{kn} - \Pi \otimes I_n)\xi^*(\theta) = 0$.

Take arbitrary matrices $\Pi_i, \Pi_j \in \mathcal{P}_{\Pi(k)}$. Then, using Corollary 4.5.18(b) of [64][1] and each matrix $\Pi_i, \Pi_j \in \mathcal{P}_{\Pi(k)}$ being symmetric and orthogonal, it follows that Π_i and Π_j are

[1]Corollary 4.5.18(b) states that there exists a unitary matrix U such that $UAU^\top$ and $UBU^\top$ are both diagonal, A and B are both symmetric, if and only if $A\bar{B}$ is normal, with $\bar{B}$ the component-wise conjugate of B.

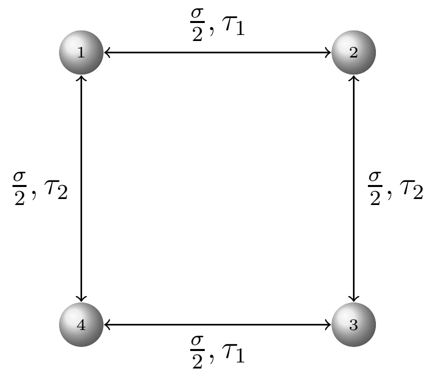


Figure 4.4. Four systems in a ring.

simultaneously diagonalizable. Thus Π_i and Π_j commute. Since Π_i and Π_j are chosen arbitrary from $\mathcal{P}_{\Pi(k)}$ it can be concluded that all matrices in $\mathcal{P}_{\Pi(k)}$ commute. $\square$

Sufficient conditions for partial synchronization of the systems (4.6) coupled via (4.18) are given in the next theorem.

Theorem 4.10. *Consider k systems* (4.6) *on a simple strongly connected graph. Let the systems be coupled via* (4.18) *and suppose that*

(H4.1) each system (4.6) *is strictly C^1-semipassive with a radially unbounded storage function;*

(H4.2) each subsystem (4.6a) *satisfies the Demidovich condition.*

Then for any $\Pi_\ell \in \mathcal{P}_{\Pi(k)}$, if $\eta_\ell > 0$, there exist positive constants σ' and γ' such that if $\sigma \geq \sigma'$ and $\sigma\tau_\ell \leq \gamma'$ there exists a globally attractive subset of the set $\ker(I_{kn} - \Pi \otimes I_n)$.

Proof of Theorem 4.10. See appendix A.2.3. $\square$

To show how Theorem 4.10 should be applied, a simple example with four systems coupled in a ring will be presented.

Example 4.3. Consider four systems coupled in a ring as depicted in Figure 4.4. Note that for this network the coupling (4.18) can be written as

$$u(t) = -\sigma y(t) + \tfrac{\sigma}{2}(\Pi_1 \otimes I)y(t-\tau_1) + \tfrac{\sigma}{2}(\Pi_2 \otimes I)y(t-\tau_2), \tag{4.19}$$

with

$$\Pi_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } \Pi_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Theorem 4.10 implies that, given that (H4.1) and (H4.2) hold, the set $\ker(I_{kn} - \Pi_\ell \otimes I_n)$ is invariant under the closed-loop dynamics and there exist constant σ' and γ' such that if $\sigma \geq \sigma'$ and $\sigma\tau_\ell \leq \gamma'$ a subset of the set $\ker(I_{kn} - \Pi \otimes I_n)$ is globally attracting. Hence, if $\sigma \geq \sigma'$ and $\sigma\tau_1 \leq \gamma'$, systems 1 and 2 synchronize, and systems 3 and 4 synchronize. This result is independent of the value of τ_2. Of course, if $\sigma \geq \sigma'$ and $\sigma\tau_2 \leq \gamma'$, systems 1 and 4 synchronize, and systems 2 and 3 synchronize. If $\sigma \geq \sigma'$, $\sigma\tau_1 \leq \gamma'$ and $\sigma\tau_2 \leq \gamma'$, a subset of the set $\ker(I - \Pi_1 \otimes I) \cup \ker(I - \Pi_2 \otimes I)$ is globally attracting, i.e. all four systems in the network synchronize. △

Remark 4.2. It can be concluded from Example 4.3 that assumption (H3.3) in section 3.3, i.e. $\tau_{ij} = \tau$ and $\sum_{j \in \mathcal{E}_i} a_{ij} = 1$ for every $i, j \in \mathcal{I}$, is sufficient but not necessary for the set $\mathcal{M} = \{\mathrm{col}\,(z_1, \ldots, z_k, y_1, \ldots, y_k) \in \mathbb{R}^{k(m+p)} | z_i = z_j \text{ and } y_i = y_j \text{ for all } i, j \in \mathcal{I}\}$ to be invariant under the closed-loop dynamics. △

4.5 Discussion

In this chapter results have been presented on partial synchronization of strictly semipassive systems. Sufficient conditions for the existence of linear invariant manifolds for networks of diffusively (time-delay) coupled systems are derived. In addition conditions are presented for these linear invariant manifolds manifolds to be stable and attracting. In particular, in networks of strictly semipassive minimum phase systems that interact via diffusive time-delay coupling, a linear invariant manifolds manifold defined by a symmetry is proven to be stable and attracting if the coupling strength is sufficiently large and the product of the coupling strength and time-delay is sufficiently small. The network shows partial synchronization if the values of the coupling strength and time-delay for which this manifold is attracting differ from those for which all systems in the network synchronize. In such networks a decrease of coupling strength or an increase of the time-delay results in loss of full synchronization but some systems in the network will still synchronize, i.e. there is partial synchronization. See Figure 4.3.

Most of the results presented in this chapter are derived for diffusive coupling with uniform time-delay. Some results are presented for partial synchronization of strictly semipassive systems that interact via a special type of coupling type I with non-uniform time-delays. Although the type of coupling functions is rather restricted, an example with four systems coupled in a ring with two different time-delays, Example 4.3, shows interesting results. First of all it has to be noted that the conditions for the linear invariant manifold defined by a symmetry to be stable and attracting in this example depend on the coupling strength and only *one* of the two delays. Moreover, this example shows that assumption (H3.3), i.e. $\sum_j a_{ij} = 1$ for all i and all time-delays are the same, is sufficient but not necessary to guarantee invariance of the full synchronization manifold under the closed-loop dynamics. For future research it would be very interesting to extend the results presented in this chapter to the case of non-uniform time-delays general coupling type I and, of course, coupling type II. However, it has to be noted that the notation will quickly become complicated.

CHAPTER FIVE

Synchronization and network topology Part II: Scaling laws in networks

Abstract. This chapter considers systems that interact via symmetric coupling type II. It will be shown how the knowledge that two coupled systems synchronize for certain values of the coupling strength and time-delay can be used to predict synchronization in any network with more than two systems. Both local and global results are presented. It will be shown that these results are closely related to the *Wu-Chua conjecture* (which is derived for non-delayed interaction). The results that are presented in this chapter are submitted for publication, [145].

5.1 Introduction

In this chapter systems the focus is, like in chapter 3, again on full synchronization. The problem that will be discussed is how the topology of the particular network does affect the synchronization of its nodes. Suppose that it is known that the systems in some network synchronize for a certain set of parameters, i.e. the coupling strength and time-delay. Can this information be used to predict synchronization in some other network? Of course, it can be extremely useful to know the relation between the topology of a network and the synchronization of its nodes. Think for instance about robots that have to carry out some task in synchrony. What kind of communication structure is needed to synchronize anyways, and can there still be synchrony if a communication link is suddenly missing?

For networks of systems that interact via non-delayed diffusive coupling there are quite some results that relate synchronization and the topology of the network. Probably the best-known result for diffusively coupled systems without time-delays is a conjecture proposed in [170].

Conjecture 5.1 (Wu-Chua, [170]). *Consider two arrays of diffusively coupled systems of sizes $k_1 \geq 2$ and $k_2 \geq 2$, respectively,*

$$\dot{x}(t) = \begin{pmatrix} f(t, x_1(t)) \\ \vdots \\ f(t, x_{k_1}(t)) \end{pmatrix} - \sigma_1 (L_1 \otimes D) x(t), \tag{5.1}$$

$$\dot{y}(t) = \begin{pmatrix} f(t, y_1(t)) \\ \vdots \\ f(t, y_{k_2}(t)) \end{pmatrix} - \sigma_2 (L_2 \otimes D) y(t), \tag{5.2}$$

where $x \in \mathbb{R}^{k_1 n}$, $y \in \mathbb{R}^{k_2 n}$, $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, symmetric Laplacian matrices $L_1 \in \mathbb{R}^{k_1 \times k_1}$, $L_2 \in \mathbb{R}^{k_2 \times k_2}$, matrix $D \in \mathbb{R}^{n \times n}$ and σ_1 and σ_2 are positive constants. Suppose that

$$\sigma_1 \lambda_2(L_1) = \sigma_2 \lambda_2(L_2), \tag{5.3}$$

where $\lambda_2(L)$ denotes the smallest nonzero eigenvalue of the symmetric Laplacian matrix L, then the systems in array (5.1) *synchronize if and only if the systems in array* (5.2) *synchronize.*

Conjecture 5.1, commonly referred to as the Wu-Chua conjecture, is shown to be wrong in general [115]. In particular, the Wu-Chua conjecture fails if the coupled systems lose synchrony when the coupling strength is increased. However, for a large class of systems, i.e. those system that remain synchronized for increasing coupling strength, the conjecture seems to hold and is useful to predict synchronization for any network. Note that systems that are strictly semipassive and have internal convergent dynamics have the convenient property that synchronization is maintained for increasing coupling strength, cf. [128].

In [117], the *Master Stability function* (MSF) is introduced to investigate synchronization of diffusively coupled systems. The main idea of the MSF approach is that, if the systems are synchronized, local perturbations transversal to the synchronization manifold vanish as time increases. Suppose that the network consists of identical systems of the form

$$\dot{x}_i(t) = f(x_i(t)) + Bu_i(t), \tag{5.4a}$$
$$y_i(t) = Cx_i(t), \tag{5.4b}$$

where $y_i(t)$ and $u_i(t)$ have the same dimension. If the variational system

$$\dot{\xi}(t) = [Df(s(t)) - \sigma \lambda_j BC] \xi(t), \tag{5.5}$$

with $Df(s(t))$ being the Jacobian of f evaluated along the solution of a free system $\dot{s}(t) = f(s(t))$, is locally stable for every λ_j, then the diffusively coupled systems locally synchronize. Here λ_j, $j = 2, \ldots, k$, are the nonzero eigenvalues of the Laplacian matrix L. It is not assumed that L is symmetric, hence λ_j might be complex. Typically the stability of (5.5) is evaluated by computing its Lyapunov exponents, cf. [168]. The systems locally synchronize, if for every λ_j, the Lyapunov exponents of (5.5) are all negative. It follows that if the Lyapunov exponents of the system

$$\dot{\xi}(t) = [Df(s(t)) - \sigma(a + b\sqrt{-1})BC]\xi(t), \tag{5.6}$$

with real constants a and b, are negative for any $(a, b) \in \mathcal{S}$, where $\mathcal{S}$ is a nonempty subset of $\mathbb{R} \times \mathbb{R}$, then the diffusively coupled systems (5.4) locally synchronize if every $\lambda_j = a_j + b_j\sqrt{-1}$ is such that $(a_j, b_j) \in \mathcal{S}$. Thus the MSF approach relates synchronization and network topology implicitly.

In [19, 16] a graph theoretical approach is presented which relates the network topology and synchronization. In this *Connection Graph Stability* (CGS) method the coupling strength required to synchronize two diffusively symmetrically coupled systems is assumed to be known. Then the coupling strength that ensures synchronization in a larger network can be computed using this information and the sum of path lengths on the edges of the graph. An advantage of this approach is that it can also be used for time-varying couplings [17]. In [14] the CGS method is generalized for asymmetric diffusive interaction.

There are not so many results that show the connection between synchronization and the topology of the network for diffusively time-delay coupled systems. In [43] conditions for synchronization of type II coupled systems are presented using a MSF approach. Hence, synchronization and network topology are implicitly related. In [174] some results for synchronization of type II coupled systems are presented for particular types of network topologies (all-to-all, nearest neighbor, star, small-world).

This chapter presents results regarding synchronization of systems on a simple strongly connected graph that interact via symmetric coupling type II. It will be shown that if the (local) synchronization diagram[1] for two mutually coupled systems is known, then this information is sufficient to predict *local* synchronization in any network of these type II coupled systems. In particular, it will be shown that taking the intersections of scaled copies of the (local) synchronization diagram of two coupled systems gives the local synchronization diagram of any network of coupled systems. The reason to consider only coupling type II is that this type of coupling is non-invasive, i.e. the coupling terms vanish if the systems are synchronized. Note that coupling type I does not have this (nice) property. Because of the non-invasive character of coupling type II, a natural decomposition can be made between the structure of the network and the dynamics of its nodes. The symmetry assumption guarantees that the factors with which the synchronization diagram of two systems are scaled are real valued.

The remainder of this chapter is organized as follows. Section 5.2 presents the main result. In this section it will be proven that the local synchronization diagram of any net-

[1]The local synchronization diagram denotes the set of parameters σ and τ for the the systems locally synchronize.

work can be constructed from the synchronization diagram of two systems. In section 5.3 it will be assumed that the synchronization diagram of two systems has a particular shape. It will be shown that, if this is the case, it is much easier to construct the synchronization diagram for any network from the synchronization diagram of the two coupled systems. In addition, these results will be related to the Wu-Chua conjecture. In section 5.4, a global version of the result of section 5.2 is presented for networks consisting of systems that are strictly semipassive and minimum phase. Finally, section 5.5 concludes the chapter.

5.2 A local analysis

Consider k *identical* systems on a simple strongly connected graph

$$\dot{x}_i(t) = f(x_i(t)) + Bu_i(t), \tag{5.7a}$$

$$y_i(t) = Cx_i(t), \quad i \in \mathcal{I} := \{1, 2 \ldots, k\}, \tag{5.7b}$$

with state $x_i \in \mathbb{R}^n$, input $u_i \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R}^m)$, output $y_i \in \mathbb{R}^m$, sufficiently smooth functions $f : \mathbb{R}^n \to \mathbb{R}^n$, and matrices $B, C^\top \in \mathbb{R}^{n \times m}$. Let the systems interact via the symmetric version of coupling type II with uniform time-delays, i.e. the systems are coupled via

$$u_i(t) = \sigma \sum_{j \in \mathcal{E}_i} a_{ij}(y_j(t-\tau) - y_i(t-\tau)), \tag{5.8}$$

with coupling strength $\sigma > 0$, time-delay τ, and interconnection weights $a_{ij} = a_{ji}$. Note that $a_{ij} = a_{ji}$ implies that the adjacency matrix is symmetric, hence the Laplacian matrix L is symmetric. Thus the eigenvalues of L are real. Then, by Proposition 2.5 in section 2.6 of chapter 2, L has eigenvalues that can be ordered as $0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k$. Throughout this section it will be assumed that

(H5.1) the solutions of the coupled systems (5.7), (5.8) are bounded.

See the chapter 3 for sufficient conditions for boundedness of solutions of coupled systems. In addition, it will be assumed that

(H5.2) a "free" system (5.7), i.e. the system

$$\dot{s}(t) = f(s(t)), \quad s_0 = s(t_0), \tag{5.9}$$

has an attractor $\mathcal{A}$ with basin of attraction $\mathbb{B}(\mathcal{A})$[2].

[2] $\mathcal{A}$ is called an attractor if it is a closed, invariant and Lyapunov stable set. The basin of attraction $\mathbb{B}(\mathcal{A})$ is the union of all solutions of (5.9) that converge to $\mathcal{A}$.

5.2.1 $k = 2$ coupled systems

First, conditions are presented for the local synchronization of two coupled systems. Consider $k = 2$ systems (5.7) that are coupled via (5.8) with $a_{12} = a_{21} = 1$, i.e.

$$\dot{x}_1(t) = f(x_1(t)) + \sigma BC(x_2(t-\tau) - x_1(t-\tau)), \tag{5.10a}$$
$$\dot{x}_2(t) = f(x_2(t)) + \sigma BC(x_1(t-\tau) - x_2(t-\tau)). \tag{5.10b}$$

Theorem 5.2. *Consider the systems* (5.10) *and assume (H5.1) and (H5.2). Suppose, in addition, that*

(H5.3) there is a nonempty set $\mathcal{S}^* \subset \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0}$ such that for all $(\sigma, \tau) \in \mathcal{S}^*$, the origin of the linear system

$$\dot{\eta}(t) = Df(s(t))\eta(t) - 2\sigma BC\eta(t-\tau), \tag{5.11}$$

with state $\eta \in \mathbb{R}^n$ and $Df(s(t))$ being the Jacobian of f evaluated along a solution $s(t)$ of (5.9) *through $s_0 \in \mathbb{B}(\mathcal{A})$, is uniformly locally asymptotically stable.*

Then the systems (5.10) *with sufficiently close history $\phi_1, \phi_2 \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ in $\mathbb{B}(\mathcal{A})$ locally synchronize if $(\sigma, \tau) \in \mathcal{S}^*$.*

Proof of Theorem 5.2. It is proven first that assumptions (H5.2) and (H5.3) imply that the systems

$$\dot{s}(t) = f(s(t)), \tag{5.12a}$$
$$\dot{\xi}(t) = f(\xi(t)) + 2\sigma BC(s(t-\tau) - \xi(t-\tau)). \tag{5.12b}$$

initialized in $\mathbb{B}(\mathcal{A})$ locally synchronize when $(\sigma, \tau) \in \mathcal{S}^*$. Define $\zeta(t) = s(t) - \xi(t)$, then

$$\dot{s}(t) = f(s(t)), \tag{5.13a}$$
$$\dot{\xi}(t) = f(\xi(t)) + 2\sigma BC\zeta(t-\tau), \tag{5.13b}$$
$$\begin{aligned}\dot{\zeta}(t) &= f(s(t)) - f((s-\zeta)(t)) - 2\sigma BC\zeta(t-\tau) \\ &= f((\zeta+\xi)(t)) - f(\xi(t)) - 2\sigma BC\zeta(t-\tau).\end{aligned} \tag{5.13c}$$

Linearize around $\zeta \equiv 0$ to obtain

$$\begin{aligned}\dot{\tilde{\zeta}}(t) &= Df(s(t))\tilde{\zeta}(t) - 2\sigma BC\tilde{\zeta}(t-\tau) \\ &= Df(\xi(t))\tilde{\zeta}(t) - 2\sigma BC\tilde{\zeta}(t-\tau)\end{aligned} \tag{5.14}$$

and

$$\dot{\xi}(t) = f(\xi(t)) + 2\sigma BC\tilde{\zeta}(t-\tau) \tag{5.15}$$

Assumption (H5.3) implies that the origin of system (5.14) driven by the “free” system (5.12a) is locally uniformly asymptotically stable for $(\sigma, \tau) \in \mathcal{S}^*$. Because the solutions of (5.12a) are bounded by (H5.2) hence the solutions of (5.12) are bounded, the systems (5.12) locally synchronize. It follows that assumptions (H5.2) and (H5.3) also imply that the origin of (5.14) driven by (5.15) is uniformly asymptotically stable for $(\sigma, \tau) \in \mathcal{S}^*$.

It will now shown that two mutually coupled systems have local error dynamics of the form (5.14), (5.15). Consider two mutually coupled systems (5.10) and define $e(t) = \frac{1}{2}(x_2(t) - x_1(t))$. Then

$$\begin{aligned} \dot{x}_1(t) &= f(x_1(t)) + 2\sigma BCe(t-\tau), && (5.16a)\\ \dot{e}(t) &= -\tfrac{1}{2}f(x_1(t)) + \tfrac{1}{2}f((x_1+2e)(t)) - 2\sigma BCe(t-\tau), && (5.16b) \end{aligned}$$

and linearizing around $e \equiv 0$ gives

$$\begin{aligned} \dot{x}_1(t) &= f(x_1(t)) + 2\sigma BC\tilde{e}(t-\tau), && (5.17a)\\ \dot{\tilde{e}}(t) &= Df(x_1(t))\tilde{e}(t) - 2\sigma BC\tilde{e}(t-\tau). && (5.17b) \end{aligned}$$

Since the systems (5.14), (5.15) and (5.17) share the same dynamics, it is concluded that the set $\{\tilde{e} = 0\}$ is locally uniformly asymptotically stable if $(\sigma, \tau) \in \mathcal{S}^*$. This immediately implies local synchronization of the systems (5.10). ☐

Remark 5.1. If the coupled systems do not have bounded solutions and the free system does not have a local attractor, asymptotic stability of the origin the variational system (5.11) driven by a free system does not necessarily imply synchronization. See [88] for a counterexample for non-delayed diffusively coupled systems.

5.2.2 $k > 2$ coupled systems

The main result of this chapter is given in the following theorem.

Theorem 5.3. *Consider k systems* (5.7) *on a simple strongly connected graph that are coupled via* (5.8). *Assume the conditions of Theorem 5.2 hold, i.e. (H5.1), (H5.2) and (H5.3) are satisfied, and let λ_j, $j = 2, \dots, k$, be the nonzero eigenvalues of L. Then k coupled systems* (5.7), (5.8) *with continuous initial history sufficiently close in* $\mathbb{B}(\mathcal{A})$ *locally synchronize if* $(\sigma, \tau) \in \mathcal{S} := \cap_{j=2}^{k} \mathcal{S}_j$, *where*

$$\mathcal{S}_j := \{(\sigma, \tau) | (\lambda_j \sigma/2, \tau) \in \mathcal{S}^*\}.$$

Proof of Theorem 5.3. Introduce the auxiliary system

$$\dot{\xi}(t) = f(\xi(t)) - \sigma\lambda_2 BC(\xi(t-\tau) - \tfrac{1}{k}(\mathbf{1}^\top \otimes I)x(t-\tau)), \qquad (5.18)$$

with $\xi \in \mathbb{R}^n$, $x(t) := \mathrm{col}\,(x_1(t), \dots, x_k(t))$, $\mathbf{1} := \mathrm{col}\,(1, \dots, 1)$. Let the initial condition for (5.18) be in $\mathbb{B}(\mathcal{A})$. Define $e_i(t) = x_i(t) - \xi(t)$, then

$$\begin{aligned} \dot{\xi}(t) &= f(\xi(t)) + \tfrac{1}{k}\sigma\lambda_2 BC(\mathbf{1}^\top \otimes I)e(t-\tau), && (5.19a)\\ \dot{e}(t) &= F(e(t), \xi(t)) - \sigma(L \otimes BC)e(t-\tau) - \tfrac{1}{k}\sigma\lambda_2(\mathbf{1}\mathbf{1}^\top \otimes BC)e(t-\tau), && (5.19b) \end{aligned}$$

with $e(t) := \mathrm{col}\,(e_1(t), \ldots, e_k(t))$ and

$$F(e(t), \xi(t)) := \mathrm{col}\,(f((e_1 + \xi)(t)) - f(\xi(t), \ldots, f((e_k + \xi)(t)) - f(\xi(t)) .$$

Because L is symmetric, there is an orthonormal matrix $U \in \mathbb{R}^{k\times k}$ such that

$$U^{-1}LU = \begin{pmatrix} 0 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_k \end{pmatrix} =: \Lambda. \tag{5.20}$$

The eigenspace corresponding to the simple zero eigenvalue of L is $\mathrm{span}\{\mathbf{1}\}$, hence the first column of U is $\frac{\sqrt{k}}{k}\mathbf{1}$. Because the matrix U is orthonormal with the first column equal to $\frac{\sqrt{k}}{k}\mathbf{1}$, for any vector $c\mathbf{1}$ for some constant c, $cU^\top \mathbf{1} = \begin{pmatrix} c\sqrt{k} & 0 & \ldots & 0 \end{pmatrix}^\top$. Define new coordinates $\bar{e}(t) = \frac{\sqrt{k}}{k}(U^{-1} \otimes I)e(t)$, then

$$\dot{\xi}(t) = f(\xi(t)) + \sigma\lambda_2 BC(\gamma^\top \otimes I)\bar{e}(t - \tau), \tag{5.21a}$$
$$\dot{\bar{e}}(t) = \bar{F}(\bar{e}(t), \xi(t)) - \sigma(\Lambda \otimes BC)\bar{e}(t - \tau) - \tfrac{1}{k}\sigma\lambda_2(\gamma\gamma^\top \otimes BC)\bar{e}(t - \tau), \tag{5.21b}$$

with $\gamma = \mathrm{col}\,(1, 0, \ldots, 0)$ and $\bar{F}(\bar{e}(t), \xi(t)) = \frac{\sqrt{k}}{k}(U^{-1} \otimes I)F(\sqrt{k}(U \otimes I)\bar{e}(t), \xi(t))$. Note that, because U is orthonormal with its first column equal to $\frac{\sqrt{k}}{k}\mathbf{1}$, $\bar{e}_j(t) \to 0$ as $t \to \infty$ for all $j \in \{2, \ldots, k\}$ implies synchronization of the coupled systems. Linearize around $e \equiv 0$ to obtain

$$\dot{\xi}(t) = f(\xi(t)) + \sigma\lambda_2 BC\tilde{e}_1(t - \tau) \tag{5.22a}$$
$$\dot{\tilde{e}}_1(t) = Df(\xi(t))\tilde{e}_1(t) - \sigma\lambda_2 BC\tilde{e}_1(t - \tau), \tag{5.22b}$$
$$\dot{\tilde{e}}_j(t) = Df(\xi(t))\tilde{e}_j(t) - \sigma\lambda_j BC\tilde{e}_j(t - \tau), \quad j = 2, \ldots, k. \tag{5.22c}$$

It will now shown that the conditions in the theorem imply that $\tilde{e}_j(t) \to 0$ as $t \to \infty$ which in turn implies local synchronization of the systems (5.7), (5.8).

Consider the master system

$$\dot{s}(t) = f(s(t)), \tag{5.23}$$

with $s(t_0) \in \mathbb{B}(\mathcal{A})$, that drives $k - 1$ slaves

$$\dot{q}_j(t) = f(q_j(t)) + \sigma\lambda_j BC(s(t - \tau) - q_j(t - \tau)), \quad j = 2, \ldots, k. \tag{5.24}$$

Define $\bar{q}_j(t) = q_j(t) - s(t)$ and linearize around $\bar{q} := \mathrm{col}\,(\bar{q}_2, \ldots, \bar{q}_k) \equiv 0$. Following the proof of Theorem 5.2, one concludes that if the origin of

$$\dot{\tilde{q}}_j(t) = Df(s(t))\tilde{q}_j(t) - \sigma\lambda_j BC\tilde{q}_j(t - \tau), \tag{5.25}$$

is uniformly asymptotically stable for $(\sigma, \tau) \in \mathcal{S}_j$, then the j^{th} slave system (5.24) locally synchronizes with the master (5.23). Then all slaves locally synchronize with the master if $(\sigma,\tau) \in \cap_{j=2}^{k}\mathcal{S}_j$. The arguments used in the proof of Theorem 5.2 also imply that if $(\sigma,\tau) \in \cap_{j=2}^{k}\mathcal{S}_j$, then the set $\{\mathrm{col}\,(\tilde{q}_2,\dots,\tilde{q}_k) = 0\}$ is locally uniformly asymptotically stable for the systems

$$\dot{q}_2(t) = f(q_2(t)) + \sigma\lambda_2 BC\tilde{q}_2(t-\tau), \tag{5.26a}$$

$$\dot{\tilde{q}}_j(t) = Df(q_2(t))\tilde{q}_j(t) - \sigma\lambda_j BC\tilde{q}_j(t-\tau), \quad j = 2,\dots,k. \tag{5.26b}$$

Since the dynamics of (5.26b), (5.26a) and (5.22) are the same, it is concluded that $\tilde{e}_i(t) \to 0$ as $t \to \infty$ for every $i \in \mathcal{I}$ if $(\sigma,\tau) \in \cap_{j=2}^{k}\mathcal{S}_j$. As noted before, $\tilde{e}_j \to 0$ as $t \to \infty$ for every $j \in \{2,\dots,k\}$ implies that $x_i(t) - x_\ell(t) \to 0$ as $t \to \infty$ for every $i,\ell \in \mathcal{I}$, $i \neq \ell$. □

Remark 5.2. The conditions in Theorem 5.3 guarantee that the origin of (5.22b) (driven by (5.22a)) is locally uniformly asymptotically stable. Note that $e_1(t) \to 0$ as $t \to \infty$ implies that $x_i(t) - s(t) \to 0$ for $t \to \infty$ for every $i \in \mathcal{I}$. Although the origin of every system (5.22c) being uniformly asymptotically stable already implies local synchronization of the systems, it is important that the origin of (5.22b) is uniformly stable to guarantee that the linearization around $e \equiv 0$ remains valid. Usually the auxiliary system $\dot{\xi}(t) = f(\xi(t))$ is chosen instead of (5.18). Then the linearized e_1-dynamics would have been $\dot{\tilde{e}}_1(t) = Df(\xi(t))\tilde{e}_1(t)$. Stability of these dynamics cannot be guaranteed if the system $\dot{\xi}(t) = f(\xi(t))$ has for example a chaotic attractor.

A graphical illustration of Theorem 5.3 is given in the next example.

Example 5.1. Consider a network consisting of four systems. Suppose that the nonzero eigenvalues of the Laplacian matrix L are $\lambda_2 < 2$, $\lambda_3 = 2$ and $\lambda_4 > 2$ and let (H5.3) be satisfied with the region $\mathcal{S}^*$ as depicted in Figure 5.1(a). The regions $\mathcal{S}_j$ are determined by scaling $\mathcal{S}^*$ with a factor $2/\lambda_j$ over the σ-axis. Since $\lambda_3 = 2$ it follows that $\mathcal{S}_3 = \mathcal{S}^*$. The scaled copies $\mathcal{S}_2$ and $\mathcal{S}_4$ are, together with $\mathcal{S}^* = \mathcal{S}_3$, shown in Figure 5.1(b). The region $\mathcal{S}_2$ is at the right of $\mathcal{S}^*$ since $\lambda_2 < 2$, and, because $\lambda_4 > 2$, the region $\mathcal{S}_4$ is on the left of $\mathcal{S}^*$. Then $\mathcal{S}$ is determined by taking the intersections of $\mathcal{S}_j$ as shown in Figure 5.1(c). △

5.3 Unimodal functions and the Wu-Chua conjecture

In this section, a region $\mathcal{S}^*$ with a specific shape will be considered. More precise, it will be assumed that $\mathcal{S}^*$ is bounded by the line $\tau = 0$ and a unimodal continuous function $\partial\tau(\sigma)$ which is defined for $\sigma \in [\sigma_{\min},\sigma_{\max}]$, where $\sigma_{\min}$ and $\sigma_{\max}$ denote the minimal and maximal coupling strength for which the two systems synchronize with $\tau \equiv 0$, respectively. Recall that the function $\partial\tau(\sigma)$ is a unimodal function if there is some $\hat{\sigma}$ such that $\partial\tau(\bar{\sigma})$ is monotonically increasing for $\bar{\sigma} \in [\sigma_{\min},\hat{\sigma}]$, and $\partial\tau(\underline{\sigma})$ is monotonically decreasing for $\underline{\sigma} \in [\hat{\sigma},\sigma_{\max}]$. Hence $\partial\tau(\hat{\sigma}) = \hat{\tau}$ is the maximum value of $\partial\tau(\sigma)$ and there are no other local maxima.

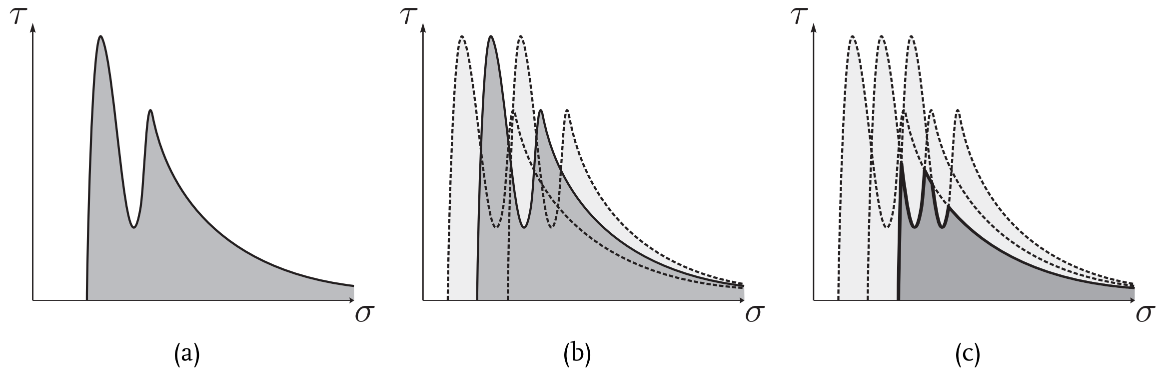


Figure 5.1. A graphical representation of Theorem 5.3 for $k = 4$ systems: (a) the region $\mathcal{S}^* = \mathcal{S}_3$ (shaded area), (b) scaled copies $\mathcal{S}_2$ and $\mathcal{S}_4$(light shaded) and $\mathcal{S}_3$ (dark shaded), and (c) the region $\mathcal{S}$ (dark shaded area) determined by the intersections of scaled copies of $\mathcal{S}^*$.

Let $\overline{\partial\tau}(\cdot)$ be the monotonically increasing part of $\partial\tau(\cdot)$ and denote by $\underline{\partial\tau}(\cdot)$ the monotonically decreasing part of $\partial\tau(\cdot)$. See Figure 5.2(a). In addition, let $\overline{\partial\sigma}(\cdot)$ and $\underline{\partial\sigma}(\cdot)$ be such that $\overline{\partial\tau}(\overline{\partial\sigma}(\tau)) = \tau$ and $\underline{\partial\tau}(\underline{\partial\sigma}(\tau)) = \tau$ for any $\tau \in [0, \hat{\tau}]$, respectively.

Corollary 5.4. *Consider k systems (5.7) on a simple strongly connected graph. Let the systems (5.7) be coupled via (5.8) and λ_j, $j = 2, \ldots, k$, be the positive eigenvalues of the Laplacian matrix L with ordering $\lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_k$. Suppose that the solutions of the closed-loop system are bounded and assume that (H5.2) and (H5.3) are satisfied with $\mathcal{S}^*$ bounded by the line $\tau \equiv 0$ and a unimodal function $\partial\tau(\sigma)$. Then the systems locally synchronize when $(\sigma, \tau) \in \mathcal{S}_2 \cap \mathcal{S}_k$.*

Proof of Corollary 5.4. Theorem 5.3 implies that the systems locally synchronize whenever $(\sigma, \tau) \in \cap_{j=2}^{k} \mathcal{S}_j$. It will be assumed that $\mathcal{S}_2 \cap \mathcal{S}_k \neq \emptyset$. (Otherwise there is nothing to prove.) Then $\mathcal{S}^*$ being bounded by a unimodal function $\partial\tau(\sigma)$ and the line $\tau \equiv 0$ implies that each $\mathcal{S}_j$ is bounded by a unimodal function $\partial\tau(2\sigma/\lambda_j)$ and the line $\tau \equiv 0$. Let $\tau_{\max} = \max\{\tau \in (0, \hat{\tau}] | \exists\sigma \text{ s.t. } (\sigma, \tau) \in \cap_{j=2}^{k} \mathcal{S}_j\}$. Such $\tau_{\max}$ clearly exists since $\partial\tau$ is unimodal. For any fixed $\tau^* \in [0, \tau_{\max}]$, $(\sigma, \tau^*) \in \cap_{j=2}^{k} \mathcal{S}_j$ implies $\sigma \in [2\overline{\partial\sigma}(\tau^*)/\lambda_2, 2\underline{\partial\sigma}(\tau^*)/\lambda_k]$ due to monotonicity of $\overline{\partial\sigma}(\cdot)$ and $\underline{\partial\sigma}(\cdot)$. Hence, if $\mathcal{S}^*$ is bounded by a unimodal function $\partial\tau(\sigma)$ and the line $\tau \equiv 0$, $\cap_{j=2}^{k} \mathcal{S}_j = \mathcal{S}_2 \cap \mathcal{S}_k$. ☐

As illustrated in Figure 5.2(b), if the conditions stated in Corollary 5.4 hold, the region in the (σ, τ) space for which the systems locally synchronize is completely determined by the intersection of $\mathcal{S}_2$ and $\mathcal{S}_k$.

Corollary 5.4 can be reformulated as follows.

Corollary 5.4′. *Consider k systems (5.7) on a simple strongly connected graph. Let the systems (5.7) be coupled via (5.8) and λ_j, $j = 2, \ldots, k$, be the positive eigenvalues of the Laplacian*

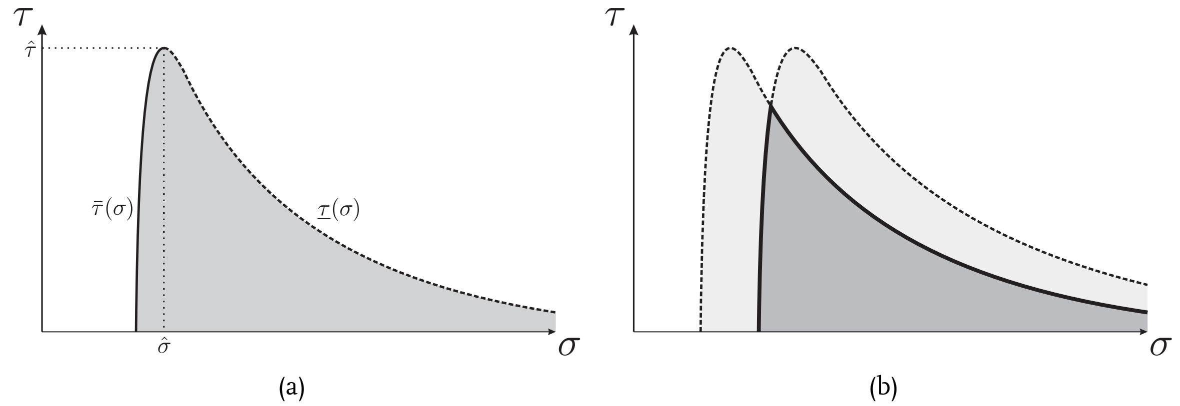


Figure 5.2. Unimodal curves: (a) $\mathcal{S}^*$ is bounded by $\tau \equiv 0$ and a unimodal function $\tau(\sigma)$, and (b) the region of synchronization in the parameter space is determined by the intersections of $\mathcal{S}_2$ and $\mathcal{S}_k$.

matrix L with ordering $\lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_k$. Suppose that the solutions of the closed-loop system are bounded and assume that (H5.2) and (H5.3) are satisfied with $\mathcal{S}^*$ bounded by the line $\tau \equiv 0$ and a unimodal function $\partial\tau(\sigma)$. Then the systems locally synchronize for any $\tau \in [0, \tau_{\max}]$ if

$$\frac{2\overline{\partial\sigma}(\tau)}{\lambda_2} \leq \sigma \leq \frac{2\underline{\partial\sigma}(\tau)}{\lambda_k},$$

where $\tau_{\max}$ is such that

$$\frac{\overline{\partial\sigma}(\tau_{\max})}{\lambda_2} = \frac{\underline{\partial\sigma}(\tau_{\max})}{\lambda_k}.$$

The result stated in Corollary 5.4′ shows a close relation to the Wu-Chua conjecture, i.e. Conjecture 5.1, which is derived for non-delayed coupling. Indeed, for non-delayed interaction, $\tau = 0$, it follows that $\sigma\lambda_2 \geq 2\overline{\partial\sigma}(0) = 2\sigma_{\min}$ and $\sigma\lambda_k \leq 2\underline{\partial\sigma}(0) = 2\sigma_{\max}$. Assuming $\sigma_{\max} = \infty$, i.e. the systems doe not lose synchronize for increasing coupling strength, it follows that the k systems synchronize for $\sigma\lambda_2 \geq 2\sigma_{\min}$. This is exactly the Wu-Chua conjecture since the two systems are assumed to synchronize for any coupling strength not smaller than $\sigma_{\min}$. (Note that the smallest nonzero eigenvalue of the Laplacian matrix for two coupled systems equals two.)

5.4 Global results

In chapter 3, section 3.3, it is proven that strictly semipassive minimum phase systems that interact via coupling type II synchronize whenever the coupling is sufficiently strong and the product of the time-delay with the coupling strength is sufficiently small. In this section a result will be presented that states that taking intersections of scaled copies of the synchronization diagram of two coupled systems is a sufficient condition for *global* synchronization of k type II coupled strictly semipassive minimum phase systems.

Consider the systems (5.7) in the normal form presented in chapter 3, section 3.3, i.e.

$$\dot{z}_i(t) = q(z_i(t), y_i(t)), \tag{5.27a}$$

$$\dot{y}_i(t) = a(y_i(t), z_i(t)) + u_i(t). \tag{5.27b}$$

where $y_i \in \mathbb{R}^m$, $z_i \in \mathbb{R}^p$, $p = n - m$, sufficiently smooth Lipschitz continuous functions $q : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^p$, $a : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^m$. Let the systems (5.27) interact via symmetric coupling type II,

$$u_i(t) = \sigma \sum_{j \in \mathcal{E}_i} a_{ij}(y_j(t-\tau) - y_i(t-\tau)), \tag{5.8}$$

with coupling strength $\sigma > 0$, time-delay τ, and interconnection weights $a_{ij} = a_{ji}$.

It follows from Theorem 3.12 that if

(H5.4) each system (5.27) is strictly $\mathcal{C}^1$-semipassive with a radially unbounded storage function and the functions $H(x_i)$ are such that there exist $R > 0$ such that $|x_i| > R$ implies $H(x_i) - 2\sigma d\,|y_i|^2 > 0$ with $d = \max_{i\in\mathcal{I}} \sum_{j\in\mathcal{E}_i} a_{ij}$;

(H5.5) each subsystem (5.27a) satisfies the Demidovich condition,

then there exist constants $\bar{\sigma}$ and $\bar{\gamma}$ such that the two systems (5.27) coupled via (5.8) globally synchronize whenever $(\sigma, \tau) \in \bar{\mathcal{S}}$ with

$$\bar{\mathcal{S}} = \{(\sigma, \tau) \in \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} | \sigma \geq \bar{\sigma} \text{ and } \sigma\tau \leq \bar{\gamma}\}. \tag{5.28}$$

Clearly, $\bar{\mathcal{S}}$ is bounded by the line $\tau \equiv 0$ and a unimodal function. Hence, by Corollary 5.4, a network consisting of k systems (5.27) coupled via (5.8) shows *local* synchronization if $(\sigma, \tau) \in \bar{\mathcal{S}}_2 \cap \bar{\mathcal{S}}_k$ with $\bar{\mathcal{S}}_j := \left\{(\sigma, \tau) |\, (2\sigma/\lambda_j, \tau) \in \bar{\mathcal{S}}\right\}$. It turns out that the condition that $(\sigma, \tau) \in \bar{\mathcal{S}}_2 \cap \bar{\mathcal{S}}_k$ is sufficient to guarantee *global synchronization* of systems (5.27) coupled via (5.8).

Theorem 5.5. *Consider k systems* (5.27) *on a simple strongly connected graph. Let the systems* (5.27) *be coupled via* (5.8) *and λ_j, $j = 2, \ldots, k$, be the positive eigenvalues of the Laplacian matrix L with ordering $\lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_k$. Suppose that (H5.4) and (H5.5) are satisfied such that two systems globally synchronize if $(\sigma, \tau) \in \bar{\mathcal{S}}$ with $\bar{\mathcal{S}}$ as in* (5.28). *Then the k systems globally synchronize if $(\sigma, \tau) \in \bar{\mathcal{S}}_2 \cap \bar{\mathcal{S}}_k$ with $\bar{\mathcal{S}}_j := \left\{(\sigma, \tau) |\, (2\sigma/\lambda_j, \tau) \in \bar{\mathcal{S}}\right\}$.*

Proof of Theorem 5.5. See Appendix A.3.1. □

5.5 Discussion

In this chapter it is shown that if the values of σ and τ are known for which two systems that interact via symmetric coupling type II (locally) synchronize, then this knowledge is sufficient to determine those values σ and τ for which the systems in a network with more than two systems *locally* synchronize. In particular, the local synchronization diagram for synchronization of k systems is constructed by taking the intersections of scaled copies of the synchronization diagram for synchronization of two systems. The scaling factors are the nonzero eigenvalues of the Laplacian matrix L, and it is shown that, in general, all eigenvalues of L have to be taken into account. If the synchronization diagram of two coupled systems has a particular shape, i.e. it is bounded by the line $\tau \equiv 0$ and a unimodal function, then only the largest and the smallest nonzero eigenvalues of L determine the synchronization diagram for local synchronization of k systems. This result is closely related to the Wu-Chua conjecture. It is also shown that, for the class of systems that is considered in the previous chapters, taking intersections of the scaled synchronization diagram of two systems gives conditions for *global* synchronization.

Using the theory presented in this chapter, one can conclude that there is an important difference between synchronization of systems with delayed interaction and synchronization of systems with non-delayed interaction. In case of non-delayed interaction, synchronization is achieved if the smallest nonzero eigenvalue of the Laplacian matrix is sufficiently large. (Note that the smallest nonzero eigenvalue of the Laplacian of a graph is related to the connectedness of the graph, cf. [47].) This explains directly why it is relatively easy to have synchronization in small-world network while it is hard to synchronize a large network of nearest-neighbor coupled systems. (For a small-world network the value of the smallest nonzero eigenvalue of its Laplacian is large compared to the value of the smallest nonzero eigenvalue of the Laplacian of a nearest-neighbor network.) However, for delayed interaction every eigenvalue of the Laplacian matrix is important as it determines an upper bound on the coupling strength for a fixed time-delay. Since the largest eigenvalue of a Laplacian matrix of a small-world network is relatively large, it is still hard to have synchronization in such network when the coupling contains time-delays.

The results presented in this chapter hold for systems that interact via symmetric coupling type II. The symmetry assumption is convenient since it guarantees the eigenvalues of the Laplacian to be real. However, the symmetry assumption is quite restrictive and it would be interesting to investigate the relation of the network topology and synchronization of asymmetrically coupled systems. The non-invasive character of coupling type II allows to derive conditions that do only depend on the structure of the network. For coupling type I this is certainly less trivial as the coupling structure and coupling strength do influence the dynamics of the systems even if they are synchronized. Of course, it is interesting to have results for coupling type I like those presented in this chapter as well.

Part II

Networks of neurons and related results

CHAPTER SIX

Every neuron is semipassive

Abstract. In this chapter it will be proven that four important models in computational neuroscience have something in common; they are all strictly semipassive. Using the theory presented in the first part of this thesis, it is shown that neurons that interact via so-called electrical synapses will synchronize for sufficiently strong coupling (and sufficiently small time-delays). In addition, two examples are presented that show that diffusive interaction can cause instabilities and/or induces unexpected oscillations. The results presented in this chapter are published in [147].

6.1 Introduction

Single neurons are important functional units for the computational properties of the brain [73, 28]. The most important biophysical variable in neural computation is the neuron's membrane potential, which can rapidly change and controls a vast number of ionic channels. Roughly speaking, there are three distinct cases of activity of a neuron. First, the membrane potential can be constant over time. Second, the neuron can fire *action potentials* at a constant rate. An action potential is an electrical impulse that is characterized by a rapid increase of the neuron's membrane potential followed by a sudden drop to its initial level. The activity where the neuron fires action potentials at a constant rate is called *spiking*. Third, the neuron might produce bursts of action potentials. That is, the neuron fires a couple of action potentials, remains silent for a while, and fires again a couple of action potentials. This type of activity is called *bursting*. These different types of activity are shown in Figure 6.1. As also can be seen in Figure 6.1, the type of activity of a neuron can be "controlled" by applying a certain input current. This input current might be an external clamping current, see [73] for details, or it might be induced by activity of other neurons.

Throughout the years many models are developed that are capable mimicking this kind of behavior. See, for instance, [70] for a review. Letting the membrane potential be the

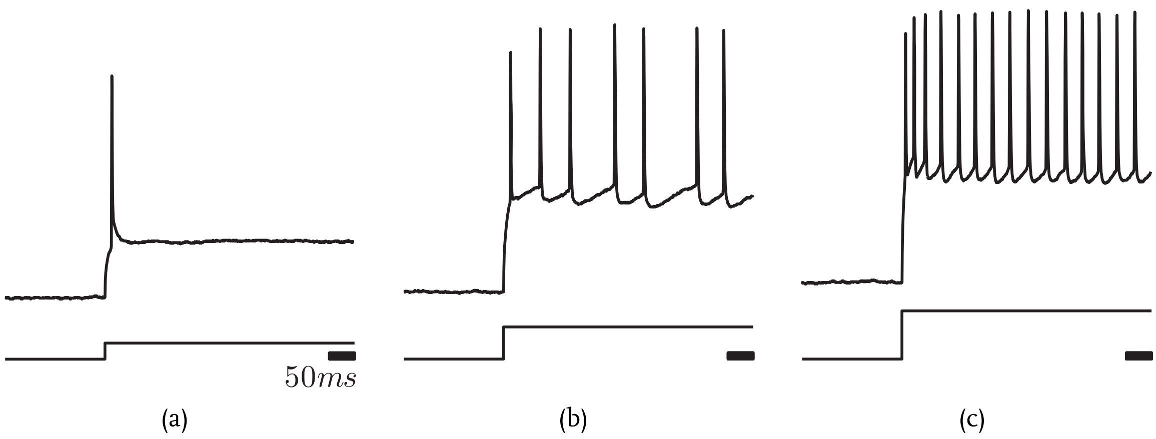


Figure 6.1. Current clamped measurements of the membrane potential of a neuron from the hippocampal area of a mouse for various input currents with a Heaviside function profile. For an initial input current of $0\ pA$, the cell is silent, i.e. the membrane potential is at a constant level. (a) For a final input current of $50\ pA$, the membrane potential shows a single action potential and remains constant afterwards. (b) A bursting solution with two action potentials per bursts for a final input current of $100\ pA$. (c) Continuous spiking for a final input current of $150\ pA$. The experimental data is provided by Dr. Alexey Semyanov and Dr. Inseon Song of the Semyanov Research Unit, Riken BSI.

output of a neuron and the input current be the only variable that influences the output, a model of a single neuron is a single-input-single-output system. Since, from a biophysical point of view, the amount of electrical energy that a neuron can produce itself is finite, every proper model of a single neuron has to be semipassive[1]. In this chapter it will be proved that the most important models of neural activity do have the semipassivity property. Four well-known models are selected from the many models of neuronal activity. The models that will be considered are the conductance based, biophysically meaningful, models of Hodgkin-Huxley [63] and Morris-Lecar [97], and the more abstract, mathematical, models derived by FitzHugh and Nagumo [48, 98] and Hindmarsh and Rose [62]. Despite the difference in the range of behavior that these models are capable to produce, these models have an important collective property; each model is semipassive.

It is well known that individual neurons in parts of the brain discharge their action potentials in synchrony. In fact, synchronous oscillations of neurons have been reported in the olfactory bulb, the visual cortex, the hippocampus and in the motor cortex [53, 138]. Presence or absence of synchrony in the brain is often linked to specific brain function or critical physiological state (e.g. epilepsy). Hence, understanding conditions that will lead to such behavior, exploring the possibilities to manipulate these conditions, and describe them rigorously is vital for further progress in neuroscience and related branches

[1] It is here assumed that every single neuron can be modeled by a set of ODEs.

of physics.

The exchange of information between neurons takes place at the so-called synapses. There are two types of synapses; *chemical synapses* and *electrical synapses* [73]. At chemical synapses, the pre-synaptic neuron releases neurotransmitters (as function of its membrane potential), and these neurotransmitters induce a synaptic current at the post-synaptic neuron. At the electrical synapses, which are also called *gap junctions*, there is a direct high conductance pathway between the pre-synaptic neuron and the post-synaptic neuron. It follows from Ohm's laws that the synaptic currents in case of electrical synapses are of the form $g \cdot (V_1(t) - V_2(t))$, where the constant g represents the synaptic conductance and $V_1(t) - V_2(t)$ denotes the difference in membrane potential of the neurons at the pre-synaptic side and the post-synaptic side at time t, respectively. Note that the electrical synapses are exactly the diffusive coupling functions that are introduced in chapter 3.

Recently it has been pointed out that electrical synapses play an important role in synchronization of individual neurons [21]. Several attempts have been made to understand when synchronization of neurons coupled via electrical synapses occurs. In [46, 80, 89, 105, 162] phase equations and phase response curves are used to analyze the dynamics of coupled neurons. The overall conclusion is that the neurons synchronize for sufficiently strong coupling. However, the use of phase equations is only justified when the coupling between the cells is relatively weak. In general, the results for strong coupling are rare [38]. In [38] Coombes uses a piecewise linear model of spiking neurons which allows to extend the results for weak coupling (using phase equations) to strong coupling. Chow and Kopell [34] used Integrate-and-Fire kind of models to investigate synchronization via electrical synapses. Using spike response functions (for the Integrate-and-Fire models an analytic expression for this function exists), it is shown that the oscillators synchronize for large coupling strength. Simulations indicate that the results also hold for more realistic models, however no rigorous mathematical proof is presented. In [77] conditions for synchrony in two coupled Hodgkin-Huxley neurons [63] are presented. It turns out that if the coupling between the neurons is strong enough, then the neurons will synchronize. In [109] synchronization for multiple interconnected chaotic Hindmarsh-Rose neurons [62] is discussed. Synchronization is witnessed for large coupling strength. Such results are not so surprising. Indeed, given that each model of neural activity does have the semipassivity property, it follows from the theory presented in chapter 3, in particular Corollary 3.11, that neurons coupled via electrical synapses synchronize for sufficiently strong coupling.

This chapter is organized as follows. In section 6.2 it will be proven that the four models mentioned above are all strictly semipassive. Next, in section 6.3 synchronization in networks of neurons is discussed. In particular, it will be demonstrated that ensembles of Hindmarsh-Rose and Morris-Lecar oscillators will end up in synchrony whenever the coupling between the neurons is large enough. In section 6.4 is will be shown that it is not obvious that systems being interconnected via diffusive coupling will have bounded solutions and eventually end up in synchrony. In particular, it is shown that two "dead" cells can become "alive" when being interconnected via diffusive coupling, i.e. the cells start to oscillate due to the interaction. Finally, section 6.5 concludes this chapter.

6.2 Semipassive neurons

In this section it will be proven that the neuronal models of Hodgkin-Huxley, Morris-Lecar, FitzHugh-Nagumo and Hindmarsh-Rose all have the semipassive property. Hence, by Theorem 3.3 in chapter 3, one can conclude that the solutions of these coupled oscillators exist and are bounded.

Hodgkin-Huxley model

The most important model in (computational) neuroscience is probably the Hodgkin-Huxley model [63]. In 1952, Hodgkin and Huxley proposed a model to describe the generation of action potentials in the giant axon of squid. (Hodgkin and Huxley received the Nobel Prize in Physiology or Medicine in 1963 for this work.) Their model consists of an equation for the membrane potential and equations for three ionic currents, viz. a sodium current, a potassium current and a leak current. The Hodgkin-Huxley model is considered as the first biophysically plausible model of a neuron. Many models of neural activity are closely related to the Hodgkin-Huxley model.

The Hodgkin-Huxley model is given by the following equations:

$$\begin{aligned} C\dot{y}(t) &= g_{Na} z_1^3(t) z_2(t) \left(E_{Na} - y(t)\right) + g_K z_3^4(t) \left(E_K - y(t)\right) \\ &\quad + g_L \left(E_L - y(t)\right) + E_m + u(t), \qquad (6.1a) \\ \dot{z}_i(t) &= \alpha_i(y(t)) \left(1 - z_i(t)\right) - \beta_i(y(t)) z_i(t), \quad i = 1, 2, 3, \qquad (6.1b) \end{aligned}$$

where $y \in \mathbb{R}$ is the membrane potential, $z_i \in \mathbb{R}$ are so-called *gating variables*, external input $u \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R})$, *positive* constants $g_{Na}, g_K, g_L, C \in \mathbb{R}_{>0}$ and constants $E_{Na}, E_K, E_L, E_m \in \mathbb{R}$. The terms $g_{Na} z_1^3(t) z_2(t), g_K z_3^4(t)$ and g_L denote the sodium conductance, the potassium conductance and the leak conductance, respectively, and E_{Na}, E_K and E_L are the corresponding reversal potentials. The constant C is the membrane capacity and E_m is a constant current. The functions $\alpha_j(\cdot)$ and $\beta_j(\cdot)$ are empirically de-

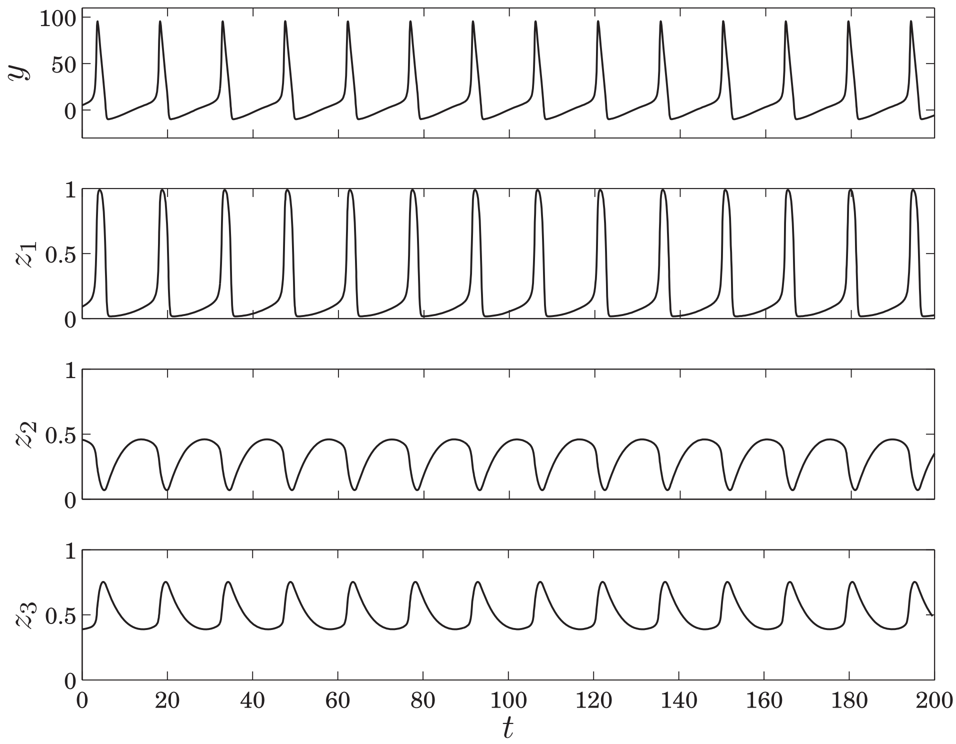


Figure 6.2. A solution of the Hodgkin-Huxley model with parameters values presented in the text.

termined to be

$$\alpha_1(s) = \frac{25 - s}{10\left(e^{(2.5 - s/10)} - 1\right)}, \qquad \beta_1(s) = 4e^{-s/18},$$

$$\alpha_2(s) = 0.07e^{-s/20}, \qquad \beta_2(s) = \frac{1}{e^{(3-s/10)} + 1},$$

$$\alpha_3(s) = \frac{10 - s}{100\left(e^{(1 - s/10)} - 1\right)}, \qquad \beta_3(s) = 0.125e^{-s/80}.$$

Figure 6.2 shows the responses of the Hodgkin-Huxley model for parameters

$$E_{Na} = 115, \quad E_K = -12, \quad E_L = 10.6, \quad g_{Na} = 120,$$
$$g_K = 36, \quad g_L = 0.3, \quad C = 1, \quad E_m = 10,$$

and $u(t) \equiv 0$.

Proposition 6.1. *The Hodgkin-Huxley model is strictly semipassive in $\mathcal{D}$, with*

$$\mathcal{D} = \{\mathrm{col}\,(y, z_1, z_2, z_3) \in \mathbb{R}^4 | 0 < z_i < 1,\ i = 1, 2, 3\}.$$

Proof of Proposition 6.1. First, it will be proved that for all $t_0 \leq t_1$, $t_0, t_1 \in \mathbb{R}$, (C1) $y(t)$ exists on the interval $t \in [t_0,\ t_1]$ and remains bounded if the input $u(t)$ is bounded;

(C2) for each $i = 1, 2, 3$, $z_i(t) \in (0, 1)$ for all $t \in [t_0,\ t_1]$ if $z_i(t_0) \in (0,\ 1)$.

Suppose that (C1) does not hold. Denote

$$u^* = \sup_{t \in [t_0, t_1]} |u(t)| . \tag{6.2}$$

According to assumptions of the proposition such u^* must exist. The right-hand side of (6.1) is locally Lipschitz, hence its solutions are at least defined over a finite time interval. Let $[t_0, T]$ be the maximal interval of their existence. Let $M \in \mathbb{R}_{>0}$ be an arbitrarily large constant. Then there should exist a time instant t_1' such that

$$|\xi(t)| \geq M, \quad \forall\, t \geq t_1', \tag{6.3}$$

with $\xi(t) := \mathrm{col}\,(y(t, z_1(t), z_2(t), z_3(t))$. Consider the dynamics

$$\dot{z}_i(t) = \alpha_i(y(t))\,(1 - z_i(t)) - \beta_i(y(t))z_i(t), \qquad i = 1, 2, 3. \tag{6.4}$$

One can easily verify that $\alpha_i(y(t)) > 0$ and $\beta_i(y(t)) > 0$ for any (bounded) $y(t)$. Hence on the boundary $z_i = 0$, $\dot{z}_i(t) > 0$, and at the boundary $z_i = 1$, $\dot{z}_i(t) < 0$. Thus $z_i(t)$ can not cross the boundaries $z_i = 0$ and $z_i = 1$. Hence the set $(0, 1)$ is positively invariant under the z_i-dynamics, i.e. for all $z_i(t_0) \in (0,\ 1)$,

$$0 < z_i(t) < 1, \quad \forall\, t \in [t_0, T]. \tag{6.5}$$

Then, according to (6.5), (6.1) the following holds

$$|\xi(t)| \leq e^{-\lambda(t-t_0)}|y(t_0)| + \rho + \frac{1}{\lambda}u^*, \ \ \forall\, t \in [t_0, T] \tag{6.6}$$

where ρ, λ are positive constants of which the value do not depend on M. Combining (6.3) and (6.6) gives

$$M \leq |\xi(t)| \leq e^{-\lambda(t-t_0)}|y(t_0)| + \rho + \frac{1}{\lambda}u^*, \ \ \forall\, t \in [t_1', T] \tag{6.7}$$

where M is arbitrarily large and ρ, $y(t_0)$, and $1/\lambda u^*$ are fixed and bounded. This is a contradiction, hence (C1) hold. This automatically implies that (C2) holds too.

To finalize the proof of strict semipassivity of (6.1) in $\mathcal{D}$, consider the positive definite storage function $V : \mathcal{D} \to \mathbb{R}_{\geq 0}$,

$$V = \tfrac{1}{2C}y(t)^2 + \tfrac{1}{2}\sum_{i=1}^{3} z_i^2(t). \tag{6.8}$$

A straightforward computation shows that

$$\begin{aligned}\dot{V} &= y(t)u(t) - \left(g_{Na}z_1^3(t)z_2(t) + g_K z_3^4(t) + g_L\right) y^2(t) \\ &\quad + \left(g_{Na}z_1^3(t)z_2(t)E_{Na} + g_K z_3^4(t)E_K + g_L E_L + E_m\right) y(t) \\ &\quad - \sum_{i=1}^{3} \left(\alpha_i(y(t)) \left(\left(z_i(t) - \tfrac{1}{2}\right)^2 - \tfrac{1}{4}\right) + \beta_i(y(t)){z_i}^2(t)\right). \end{aligned} \tag{6.9}$$

Because (C2) holds it follows that

$$\begin{aligned}\dot{V} \leq &y(t)u(t) - g_L y^2(t) + c_1 y(t) \\ &- \sum_{i=1}^{3} \left(\alpha_i(y(t)) \left(\left(z_i(t) - \tfrac{1}{2}\right)^2 - \tfrac{1}{4}\right) + \beta_i(y(t)){z_i}^2(t)\right), \end{aligned} \tag{6.10}$$

with constant

$$c_1 = \max_{d_1,d_2\in[0,1]} |d_1 g_{Na}E_{Na} + d_2 g_K E_K + g_L E_L + E_m| . \tag{6.11}$$

Given that (6.10) holds for all t, it follows that the Hodgkin-Huxley model is strictly semipassive in $\mathcal{D}$. □

Remark 6.1. The Hodgkin-Huxley model is strictly semipassive in $\mathcal{D}$. Although the results presented in chapters 3, 4 and 5 assume that the systems are strictly semipassive in $\mathbb{R}^n$, it is not difficult to verify that these results remain true when the systems are strictly semipassive in $\mathcal{D}$ with $\mathcal{D} = \mathbb{R} \times (0,1) \times \ldots \times (0,1)$. This is because the Hodgkin-Huxley model has the normal form

$$\dot{y}(t) = a(y(t), z(t)) + u(t), \tag{6.12a}$$
$$\dot{z}(t) = qz(t), y(t)), \tag{6.12b}$$

with $(y, z) \in \mathcal{D}$. Then, for any input signal $u(t)$ that depends only on the (possibly time-delayed) output signals $y(t)$ and satisfies $y(t)u(t) \leq 0$, the semipassivity property implies that the solutions of the closed-loop system are ultimately bounded. △

Morris-Lecar model

The Morris-Lecar model [97] describes the voltage oscillations in the barnacle giant muscle fiber. It is also used to describe the membrane potential in a variety of neural cells [73]. The model consists of an equation for the membrane potential which, like in the Hodgkin-Huxley model, depends on some ionic currents, namely a potassium current, a calcium current and a leak current. However, contrary to the Hodgkin-Huxley model, the calcium current is in the Morris-Lecar model a static function of the membrane potential. (In the Hodgkin-Huxley model the calcium current depends on two activation variables which are given by two differential equations that depend on the membrane potential). The Morris-Lecar model is given by the following equations:

$$\begin{aligned} C\dot{y}(t) &= g_L \left(E_L - y(t)\right) + g_{Ca}\alpha_\infty \left(y(t)\right) \left(E_{Ca} - y(t)\right) \\ &\quad + g_K z(t) \left(E_K - y(t)\right) + E_m + u(t), && (6.13a) \\ \dot{z}(t) &= \eta\left(y(t)\right)\left(\beta_\infty(y(t)) - z(t)\right), && (6.13b) \end{aligned}$$

with $y \in \mathbb{R}$ being the membrane potential, gating variable $z \in \mathbb{R}$, external input $u \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R})$, constant parameters $E_L,\ E_{Ca},\ E_K, E_m \in \mathbb{R}$ and positive constants $C, g_L,\ g_{Ca},\ g_K \in \mathbb{R}$. The terms $g_{Ca}\alpha_\infty\left(y(t)\right), g_K z(t)$ and g_L represent the sodium conductance, the potassium conductance and the leak conductance, respectively, E_{Na}, E_K and E_L are the corresponding reversal potentials, C is the membrane capacity and E_m is a constant current. The functions $\alpha_\infty(\cdot)$, $\beta_\infty(\cdot)$ and $\eta(\cdot)$ are defined as

$$\begin{aligned} \alpha_\infty(s) &= \frac{1}{2}\left(1 + \tanh\left(\frac{s - E_1}{E_2}\right)\right), \\ \beta_\infty(s) &= \frac{1}{2}\left(1 + \tanh\left(\frac{s - E_3}{E_4}\right)\right), \\ \eta(s) &= \bar{\eta}\cosh\left(\frac{s - E_3}{2E_4}\right), \end{aligned}$$

with constants $\bar{\eta} \in \mathbb{R}_{>0}$ and $E_1, E_2, E_3, E_4 \in \mathbb{R}$.

Figure 6.3 shows the responses of the Morris-Lecar model for parameters

$$\begin{aligned} &E_{Ca} = 100, && E_K = -70, && E_L = -50, && g_{Ca} = 4, \\ &g_K = 8, && g_L = 2, && C = 1, && E_m = 40, \\ &E_1 = -1, && E_2 = 15, && E_3 = 10, && E_4 = 14.5, \end{aligned}$$

and $u(t) \equiv 0$.

Proposition 6.2. *The Morris-Lecar model is strictly semipassive in $\mathcal{D}$, with*

$$\mathcal{D} = \{\mathrm{col}\left(y, z\right) \in \mathbb{R}^2 | 0 < z < 1\}.$$

Proof of Proposition 6.2. Notice that the set $(0,\ 1)$ is positively invariant under the z-dynamics. The proof can easily be deducted from the proof of strict semipassivity of the Hodgkin-Huxley model. □

Remark 6.2. Many biophysically meaningful neuronal models, i.e. conductance based models like the Hodgkin-Huxley and Morris-Lecar models, share the same structure, see for instance [63, 97, 154]. In particular, the evolution of the membrane potential is given by an equation of the form

$$C\dot{y}(t) = u(t) + \sum_{j=1}^{n} s_j(t) \tag{6.14}$$

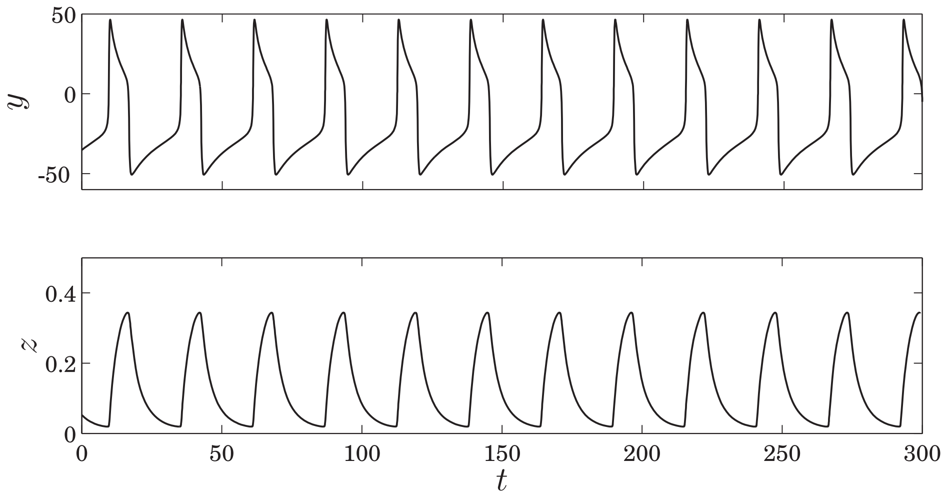


Figure 6.3. A solution of the Morris-Lecar model with parameters values presented in the text.

where $y \in \mathbb{R}$ denotes the membrane potential, $C \in \mathbb{R}_{>0}$ is the membrane capacity, $u \in \mathbb{R}$ is the input and $s_j(t)$ represent the ionic currents. Often $s_j(t) = g_j(t)(E_j - y(t))$ where $E_j \in \mathbb{R}$ is a constant reversal potential and $g_j(t) > 0$ are conductances which are such that $g_j(t) > 0$ for all t. In particular, the conductances are typically given as

$$g_j = \bar{g}_j \prod_{i=1}^{m} z_i^{p_{ij}} \tag{6.15}$$

with maximal conductance $\bar{g}_j \in \mathbb{R}_{>0}$, non-negative integers p_{ij} and voltage dependent gating variables $z_i(y(t))$. These gating variables satisfy $z_i(t) \in (0,\ 1)$ if $z_i(t_0) \in (0,\ 1)$ for all $t \geq t_0$.

All models of neuronal oscillators of this form are strictly semipassive (in $\mathbb{R} \times (0,\ 1) \times \ldots \times (0,\ 1)$), and the proof for strict semipassivity is similar to the proof presented for the Hodgkin-Huxley model. △

FitzHugh-Nagumo model

The FitzHugh-Nagumo model [48, 98] is one of the simplest models of the spiking dynamics of a neuron. The model is suggested by FitzHugh in 1961 [48] and, independently, by Nagumo, Arimoto and Yoshizawa in 1962 [98]. It gives a quantitative description of the excitable properties of the membrane potential of a neuron. The model is given by the following set of differential equations

$$\dot{y}(t) = y(t) - \frac{y^3(t)}{3} - z(t) + E_m + u(t), \tag{6.16a}$$
$$\dot{z}(t) = \phi\left(y(t) + a - bz(t)\right), \tag{6.16b}$$

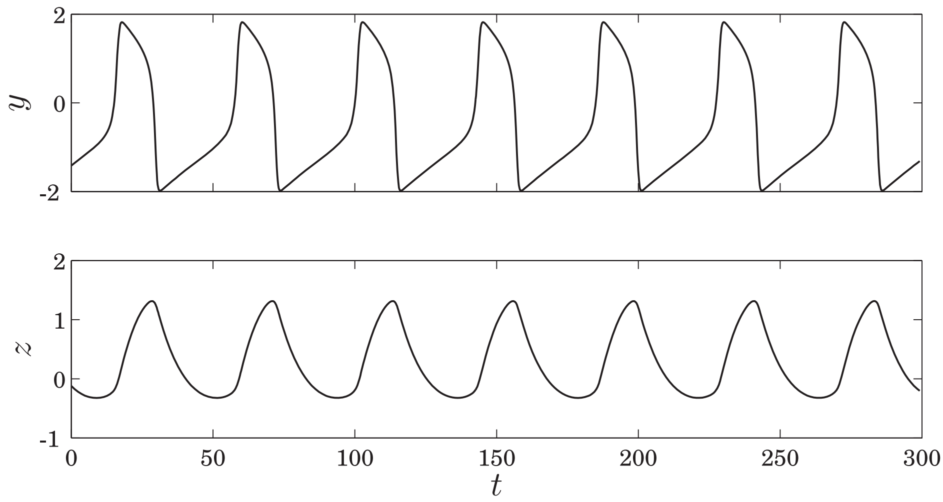


Figure 6.4. A solution of the FitzHugh-Nagumo model with parameters values presented in the text.

where $y \in \mathbb{R}$ represents the membrane potential, $z \in \mathbb{R}$ is an internal variable with no physical meaning, input $u \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R})$ and constants $a, b, \phi \in \mathbb{R}_{>0}$ and $E_m \in \mathbb{R}$.

Figure 6.4 shows the responses of the FitzHugh-Nagumo model for parameters

$$a = 0.7, \qquad b = 0.8, \qquad \phi = 0.08, \qquad E_m = 0.4,$$

and $u(t) \equiv 0$.

Proposition 6.3. *The FitzHugh-Nagumo model is strictly semipassivity.*

Proof of Proposition 6.3. The proof is easy and straightforward. Consider the positive definite storage function $V : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$

$$V = \frac{1}{2}\left(y^2(t) + \frac{1}{\phi}z^2(t)\right). \tag{6.17}$$

Then

$$\dot{V} = y(t)u(t) - \frac{y^4(t)}{3} + y^2(t) + E_m y(t) - bz^2(t) + az(t). \tag{6.18}$$

It is easy to see that $H(y(t), z(t)) := \frac{y^4(t)}{3} - y^2(t) - E_m y(t) + bz^2(t) - az(t)$ will be positive for large $|y(t)|$ and $|z(t)|$. Clearly the FitzHugh-Nagumo neuron is strictly semipassive. □

Hindmarsh-Rose model

In 1979 Hindmarsh and Rose participated in a project to model synchronization of two neurons of pond snail [60]. The biophysical models like the Hodgkin-Huxley model were

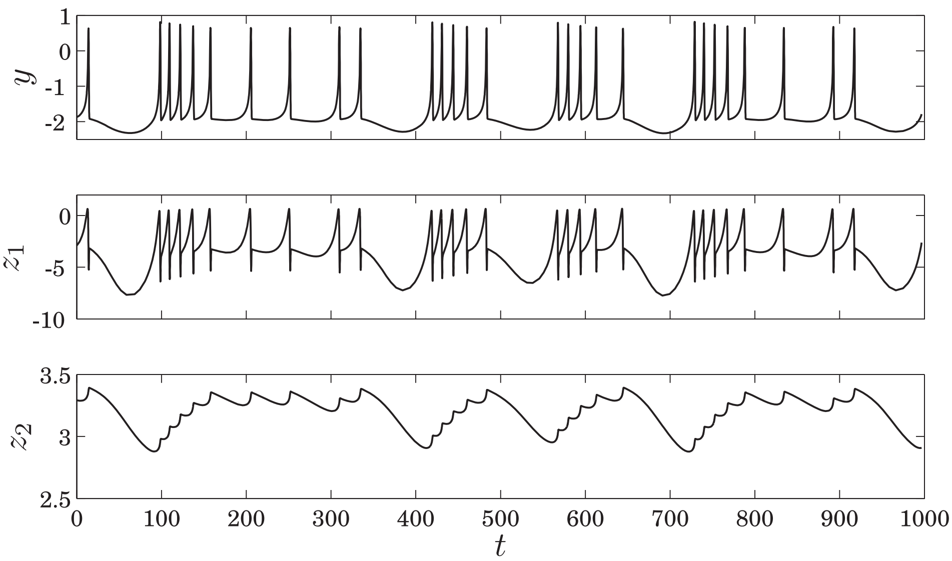


Figure 6.5. A solution of the Hindmarsh-Rose model with parameters values presented in the text.

too computationally expensive at that time, and the inexpensive FitzHugh-Nagumo model was not accurate enough for this project. Therefore Hindmarsh and Rose started to developed their own model. In 1982 Hindmarsh and Rose proposed a two-dimensional model [61] and, in 1984, they published the paper [62] with their famous three-dimensional model:

$$\dot{y}(t) = -ay^3(t) + by^2(t) + z_1 - z_2 + E_m + u(t), \tag{6.19a}$$

$$\dot{z}_1(t) = c - dy^2(t) - z_1(t), \tag{6.19b}$$

$$\dot{z}_2(t) = r\left(s\left(y(t) + y_0\right) - z_2(t)\right), \tag{6.19c}$$

where $y \in \mathbb{R}$ represents the membrane potential, internal nonphysical variables $z_1, z_2 \in \mathbb{R}$, input $u \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R})$ and constant parameters $a, b, c, d, r, s \in \mathbb{R}_{>0}$, $r \ll 1$, and $E_m, y_0 \in \mathbb{R}$. Depending on the choice of parameters, the model can produce persistent spike trains or bursting oscillations. Moreover, for some values of parameters it can even produce chaotic bursts. In Figure 6.5 the chaotic bursting is shown. The parameters that are used are

$$a = 1, \qquad b = 3, \qquad c = 1, \qquad d = 5,$$

$$r = 0.005, \qquad s = 4, \qquad y_0 = 1.618, \qquad E_m = 3.25,$$

and $u(t) \equiv 0$.

Proposition 6.4. *The Hindmarsh-Rose model is strictly semipassive.*

Proof of Proposition 6.4. The proof is adopted from [109]. Consider the positive definite storage function $V : \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$,

$$V = \tfrac{1}{2}\left(y^2(t) + \mu z_1^2(t) + \tfrac{1}{rs} z_2^2(t)\right), \tag{6.20}$$

with constant $\mu > 0$ that will be determined later. Hence

$$\begin{aligned}\dot{V} &= y(t)u(t) - ay^4(t) + by^3(t) + y(t)z_1(t) + E_m y(t)\\ &\quad + \mu c z_1(t) - \mu d y^2(t) z_1(t) - \mu z_1^2(t) + y_0 z_2(t) - \tfrac{1}{s} z_2^2(t).\end{aligned} \tag{6.21}$$

Let $\lambda_i \in (0,1)$, $i = 1, 2$, to obtain

$$\begin{aligned}&- ay^4(t) - \mu d y^2(t) z_1(t) =\\ &- a\lambda_1 y^4(t) - a(1-\lambda_1)\left(y^2(t) + \tfrac{\mu d}{2a(1-\lambda_1)} z_1(t)\right)^2 + \tfrac{\mu^2 d^2}{4a(1-\lambda_1)} z_1^2(t),\end{aligned} \tag{6.22}$$

and

$$\begin{aligned}&- \mu z_1^2(t) + y(t) z_1(t) =\\ &- \mu\lambda_2 z_1^2(t) - \mu(1-\lambda_2)\left(z_1(t) - \tfrac{1}{2\mu(1-\lambda_2)} y(t)\right)^2 + \tfrac{1}{4\mu(1-\lambda_2)} y^2(t).\end{aligned} \tag{6.23}$$

Combining (6.21), (6.22) and (6.23) yields

$$\begin{aligned}\dot{V} &= y(t)u(t) - a\lambda_1 y^4(t) + by^3(t) + \tfrac{1}{4\mu(1-\lambda_2)} y^2(t) + E_m y(t)\\ &\quad - \left(\mu\lambda_2 - \tfrac{\mu^2 d^2}{4a(1-\lambda_1)}\right) z_1^2(t) + \mu c z_1(t) - \tfrac{1}{s} z_2^2(t) + y_0 z_2(t)\\ &\quad - \mu(1-\lambda_2)\left(z_1(t) - \tfrac{1}{2\mu(1-\lambda_2)} y(t)\right)^2 - a(1-\lambda_1)\left(y^2(t) + \tfrac{\mu d}{2a(1-\lambda_1)} z_1(t)\right)^2.\end{aligned} \tag{6.24}$$

Take $\mu < \frac{4a\lambda_2(1-\lambda_1)}{d^2}$. Clearly, for sufficiently large $|y(t)|$, $|z_1(t)|$ and $|z_2(t)|$, the terms $a\lambda_1 y^4(t) - by^3(t) - \frac{1}{4\mu(1-\lambda_2)} y^2(t) - E_m y(t)$, $\left(\mu\lambda_2 - \frac{\mu^2 d^2}{4a(1-\lambda_1)}\right) z_1^2(t) - \mu c z_1(t)$ and $\frac{1}{s} z_2^2(t) - y_0 z_2(t)$ are positive. Moreover, since $\mu(1-\lambda_2)\left(z_1(t) - \frac{1}{2\mu(1-\lambda_2)} y(t)\right)^2$ and $a(1-\lambda_1)\left(y^2(t) + \frac{\mu d}{2a(1-\lambda_1)} z_1(t)\right)^2$ are non-negative, it follows that the Hindmarsh-Rose model is strictly semipassive. $\square$

6.3 Synchronization in networks of neurons

In the previous section it is shown that the models of Hodgkin-Huxley, the Morris-Lecar, the FitzHugh-Nagumo and the Hindmarsh-Rose are all strictly semipassive. Using the machinery presented in chapter 3, one can conclude that the solutions of the coupled systems are ultimately bounded. Moreover, if the models are minimum-phase, then a network of neurons will synchronize if the coupling is sufficiently strong and, if time-delays are present, the product of the coupling strength and time-delay is sufficiently small. Fortunately, this is not hard to prove that the models are minimum-phase. Indeed, the internal dynamics of all these models satisfy the Demidovich condition, i.e. Lemma 2.2, with $P = I$. Hence, for suitable values of the coupling strength and time-delay, the neurons will synchronize.

In this section it will be demonstrated, using computer simulations, that networks with these neurons indeed synchronize. In particular, simulation results will be presented for a network of Morris-Lecar neurons and a network of Hindmarsh-Rose neurons. The coupling will not contain time-delays and is assumed to be symmetric, i.e.

$$u_i(t) = \sigma \sum_{j \in \mathcal{E}_i} a_{ij}(y_j(t) - y_i(t)), \tag{6.25}$$

with $a_{ij} = a_{ji}$. Hence Theorem 3.1 can be used. The goal is of the simulations is not to determine the exact threshold values for which the network starts to synchronize. Such threshold values can be expressed in terms of the system parameters (see, for instance [109] or [16] for Hindmarsh-Rose neurons), or they can be determined by computing, for instance, the transversal Lyapunov exponents of the coupled systems [117]. Here, the goal is only to show that for large enough coupling the neurons will synchronize.

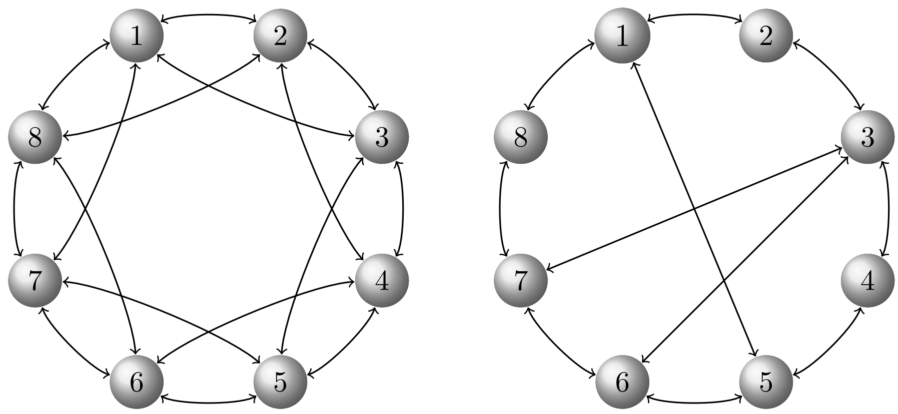
Example 6.1 (Synchronization of Hindmarsh-Rose oscillators). Consider the network depicted in 6.6(a) with eight coupled Hindmarsh-Rose neurons

$$\dot{y}_i(t) = -a y_i^3(t) + b y_i^2(t) + z_{i,1}(t) - z_{i,2}(t) + E_m + u_i(t) \tag{6.26a}$$
$$\dot{z}_{i,1}(t) = c - d y_i^2(t) - z_{i,1}(t) \tag{6.26b}$$
$$\dot{z}_{i,2}(t) = r\left(s\left(y_i(t) + y_0\right) - z_{i,2}(t)\right) \tag{6.26c}$$

where $i = 1, \ldots, 8$. The parameters that will be used are the same as presented before. The Laplacian matrix for the network is

$$L = \begin{pmatrix} 4 & -1 & -1 & 0 & 0 & 0 & -1 & -1 \\ -1 & 4 & -1 & -1 & 0 & 0 & 0 & -1 \\ -1 & -1 & 4 & -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 4 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 4 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 4 & -1 & -1 \\ -1 & 0 & 0 & 0 & -1 & -1 & 4 & -1 \\ -1 & -1 & 0 & 0 & 0 & -1 & -1 & 4 \end{pmatrix}, \tag{6.27}$$

and the smallest nonzero eigenvalue of L is $\lambda_2 \approx 2.58$. The simulations indicate that the neurons synchronize when the coupling strength $\sigma\lambda_2(L) \gtrapprox 0.387$, which corresponds to a synchronization threshold $\bar{\sigma} \approx 1.00$. (This agrees with the numerical results obtained in, for instance, [16], where it is stated that two diffusively coupled Hindmarsh-Rose neurons synchronize when the coupling strength $\sigma \geq 0.50$ which corresponds to a threshold



(a) Graph 1 (b) Graph 2

Figure 6.6. Eight diffusively coupled oscillators. Each interconnection has weight 1.

value $\bar{\sigma} = 1.00$.) Figure 6.7 shows the simulation results of the network of Hindmarsh-Rose oscillators. In the top panel, $\sigma = 0.2$, hence $\sigma\lambda_2(L) \approx 0.52 \leq \bar{\sigma}$. Indeed, the neurons do not synchronize. This becomes more clear in the graph in the bottom panel, where the error signals $\tilde{y}_j(t) := y_1(t) - y_j(t)$, $j = 2, \ldots, 8$, are plotted in gray. The middle panel shows a simulation with coupling $\sigma = 0.4$ such that $\sigma\lambda_2(L) \approx 1.03 > \bar{\sigma}$. The first 500 $[s]$ the systems are uncoupled and one observes the systems are not synchronized. The coupling becomes active when $t \geq 500$. It can be seen, especially in the bottom panel, that all systems rapidly synchronize. △

Example 6.2 (Synchronization of Morris-Lecar oscillators). This example discusses synchronization of eight Morris-Lecar oscillators on the graph depicted in Figure 6.6(b). The corresponding Laplacian matrix is given as

$$L = \begin{pmatrix} 3 & -1 & 0 & 0 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 3 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}. \tag{6.28}$$

The smallest nonzero eigenvalue of L is $\lambda_2 \approx 1.27$. Each Morris-Lecar oscillator is given by the following set of equations

$$C\dot{y}_i(t) = g_L\left(E_L - y_i(t)\right) + g_{Ca}\alpha_\infty\left(y_i(t)\right)\left(E_{Ca} - y_i(t)\right) + \tag{6.29a}$$

$$+ g_K z_i(t)\left(E_K - y_i(t)\right) + E_m + u_i(t), \tag{6.29b}$$

$$\dot{z}_i(t) = \eta\left(y_i(t)\right)\left(\beta_\infty(y_i(t)) - z_i(t)\right), \tag{6.29c}$$

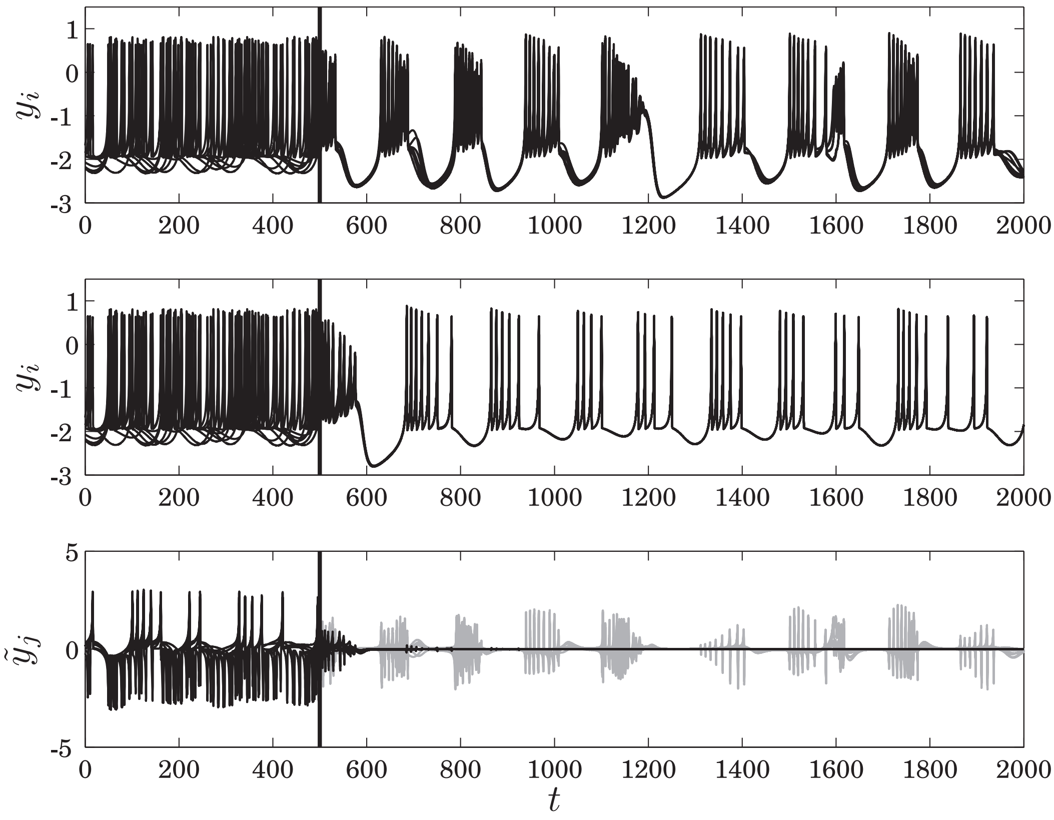


Figure 6.7. Synchronization of eight Hindmarsh-Rose neurons. The coupling becomes when $t \geq 500$. Top panel: no synchronization for $\sigma = 0.2$. Middle panel: synchronization for $\sigma = 0.4$. Bottom panel: error signals $\tilde{y}_j(t) := y_1(t) - y_j(t)$, $j = 2, \ldots, 8$, for $\sigma = 0.2$ (gray) and $\sigma = 0.4$ (black).

with $i = 1, \ldots, 8$. The parameters and functions $\alpha_\infty(\cdot)$, $\beta_\infty(\cdot)$. and $\eta(\cdot)$ are as presented in the previous section. The top panel of Figure 6.8 shows the simulation results for the eight diffusively coupled Morris-Lecar oscillators with $\sigma = 0.001$. The systems do not synchronize, which can be clearly seen in the bottom panel of Figure 6.8 in which the error signals $\tilde{y}_j(t) := y_1(t) - y_j(t)$, $j = 2, \ldots, 8$, are plotted in gray. The middle panel shows a simulation result with $\sigma = 0.05$. For $t < 250$ the oscillators are uncoupled and do not synchronize. Then, when $t \geq 250$ the coupling is turned on and all oscillators synchronize. The bottom panel shows the corresponding error signals in black. $\triangle$

6.4 Diffusion driven instabilities

Until now it is only shown that diffusive interaction can make systems synchronized. However, diffusive interaction of systems might also result in "instabilities". In this section the unstable behavior due to diffusive interaction will be discussed briefly. The main

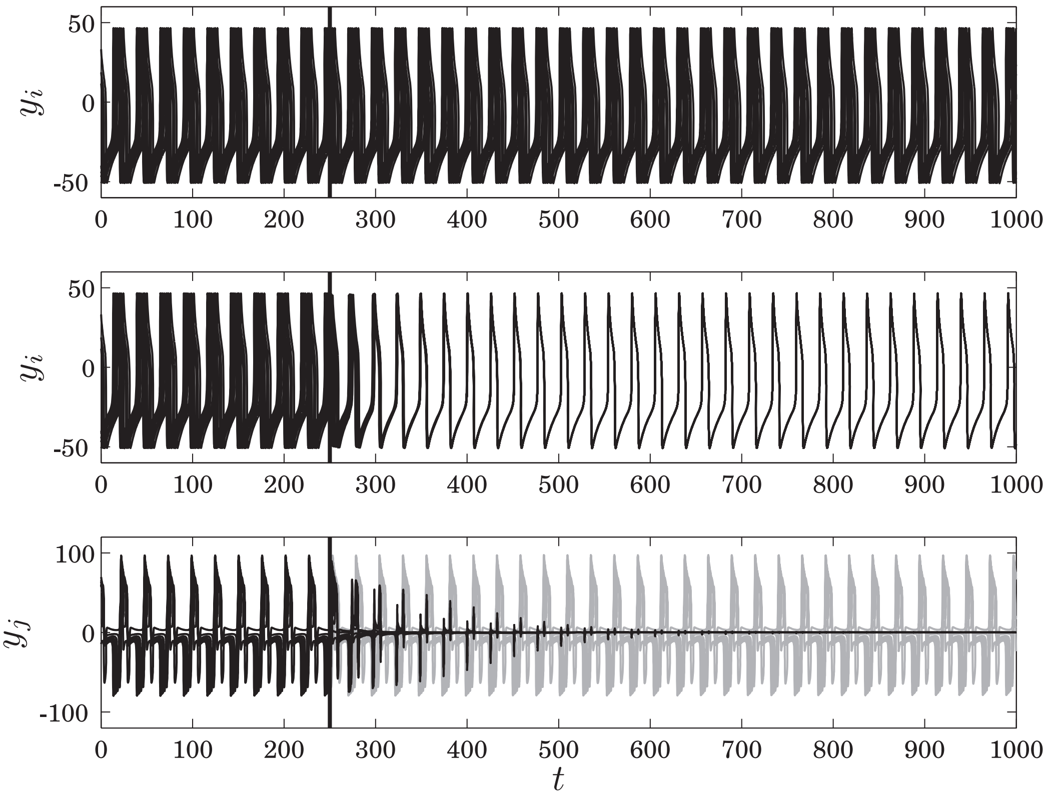


Figure 6.8. Synchronization of eight Morris-Lecar neurons. The coupling becomes when $t \geq 250$. Top panel: no synchronization for $\sigma = 0.001$. Middle panel: synchronization for $\sigma = 0.05$. Bottom panel: error signals $\tilde{y}_j(t) := y_1(t) - y_j(t)$, $j = 2, \ldots, 8$, for $\sigma = 0.001$ (gray) and $\sigma = 0.05$ (black).

reason to include this section in this chapter is that the diffusion driven instabilities have important applications in neuronal (and other biological) systems, see [84] and the references therein. In [84] it is shown that the diffusive interaction between initially silent cells is essential for generating stable oscillatory behavior. The main reason for the oscillations is that the internal variables, e.g. (in)activation particles, have the tendency to oscillate. If the cells are coupled with sufficiently strong interaction, the internal variables start to oscillate which makes the membrane potential to oscillate. The goal here is not to discuss the machinery for the generation of these oscillations in detail. These details can be found in [127, 140].

In this section it will be shown via two simple examples, inspired by [127], that it is not trivial that systems interacting via diffusive coupling have bounded solutions and possibly end up in synchrony for sufficiently strong coupling. In particular, it is demonstrated that diffusive coupling 1) can make the solutions of the interconnected systems to become unbounded, and 2) can make systems, which have an asymptotically stable equilibrium in isolation, to produce stable oscillations.

Example 6.3 (Unbounded solutions). Consider the linear non-minimum phase stable transfer function

$$H(s) = \frac{s^2 - s + 1}{s^3 + 2s^2 + 2s + 1}. \tag{6.30}$$

A possible state space realization for the system is

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \tag{6.31}$$

with

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ -4 & 2 & -3 \end{bmatrix}, \quad B = C^\top = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{6.32}$$

Consider now two diffusively coupled systems (6.31)

$$\begin{aligned} \dot{x}_1(t) &= Ax_1(t) + \sigma BC(x_2(t) - x_1(t)), \\ \dot{x}_2(t) &= Ax_2(t) + \sigma BC(x_1(t) - x_2(t)), \end{aligned} \tag{6.33}$$

Clearly the origin of each uncoupled system is globally asymptotically stable. However, the system is not semipassive. When $\sigma > 0.6512$ (for $\sigma = 0.6512$ the system undergoes a Poincaré-Andronov-Hopf bifurcation [127]) the solutions of the interconnected systems become unbounded. △

Example 6.3 shows how the diffusive coupling between two not semipassive systems might result in unbounded solutions. A similar phenomena is encountered in networks of diffusively coupled Chua circuits, cf. [161]. The piecewise linear model of the Chua circuit is not semipassive (the Chua attractor is not globally stable) and due to the interaction the trajectories of the systems can be driven outside the basin of attraction and the solutions grow unbounded.

The following example is taken from [127]. It shows how two systems, which both have an asymptotically stable equilibrium in absence of interaction, start to produce stable oscillations when the systems interact via diffusive coupling.

Example 6.4 (Diffusion driven oscillations). Consider two systems which interact via diffusive coupling:

$$\begin{aligned} \dot{x}_1(t) &= Ax_1(t)(1 + |x_1(t)|^2) + \sigma BC(x_2(t) - x_1(t)), \\ \dot{x}_2(t) &= Ax_2(t)(1 + |x_2(t)|^2) + \sigma BC(x_1(t) - x_2(t)), \end{aligned} \tag{6.34}$$

where matrices A, B and C are as presented above. Again the origin of an isolated system is asymptotically stable and when $\sigma = 0.6512$ the (linearized) system undergoes a Poincaré-Andronov-Hopf bifurcation. The coupled systems (6.34) start to produce stable oscillations whenever $\sigma > 0.6512$, see Figure 6.9 for simulation results. △

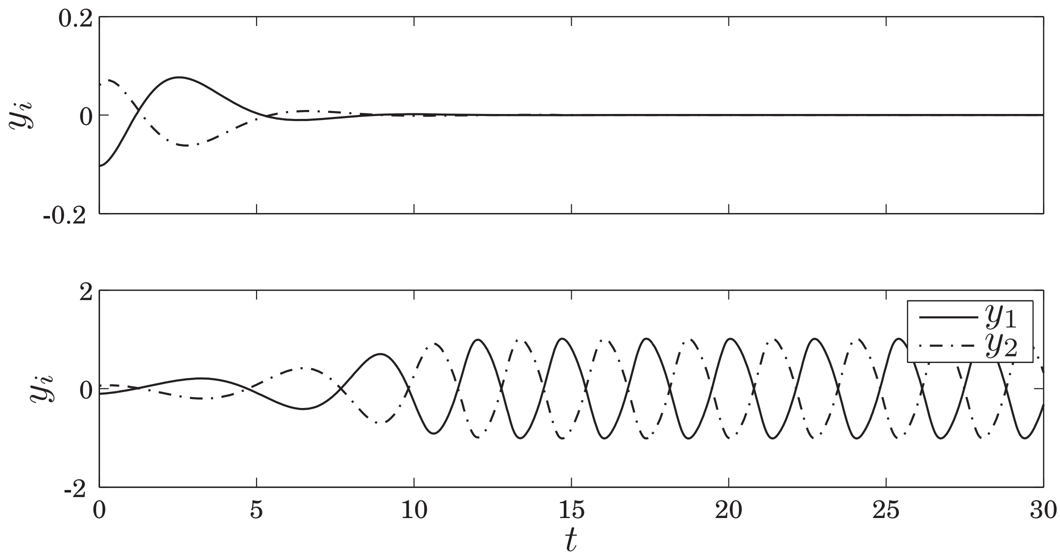


Figure 6.9. Diffusion driven oscillations. The top panel shows the outputs of the two uncoupled systems. In the bottom panel the outputs of two coupled systems with $\sigma = 2$ are shown.

The key mechanism for the oscillations is the Poincaré-Andronov-Hopf bifurcation and the non-minimum-phaseness of the systems. Note that the four models described above do have minimum-phase internal dynamics since the internal dynamics are convergent. Hence no "spontaneous" oscillations due to diffusive interaction will occur in networks of Hodgkin-Huxley, Morris-Lecar, FitzHugh-Nagumo and Hindmarsh-Rose neurons. Thus in a network consisting of silent "dead" neurons the diffusive interaction can not result in activity of the network.

6.5 Discussion

In this chapter it is proven that the Hodgkin-Huxley, Morris-Lecar, FitzHugh-Nagumo and Hindmarsh-Rose models for neural activity are strictly semipassive and have internal convergent dynamics. Then, using the theoretical framework presented in the first part of this thesis, it can be concluded that neurons that are coupled via electrical synapses always synchronize given that the coupling is sufficiently strong. (If time-delays on the interaction are taken into account, the product of the time-delay and the coupling strength should in addition be sufficiently small.) These results support the result of [77] and the simulation results of Chow and Kopell [34]. In [46] the authors discuss the emergence of clusters in networks of all-to-all coupled neurons as function of the coupling strength. Those clusters might emerge when the coupling is not strong enough to end up in synchrony. The emergence of clusters for diffusively coupled neurons can easily be explained by applying the theory presented in chapter 4 and this chapter.

In addition, it is shown in this chapter that diffusively coupled systems which are not semipassive might have unbounded solutions. A probably more interesting property, at least from the biological point of view, is that diffusively coupled non-minimum phase systems which are initially silent can start to produce stable oscillations.

It would be interesting to investigate the emergence of stable synchronization in networks of neurons that interact via chemical synapses. The approach to analyze synchronization of neurons in such networks is to use canonic models such as Integrate-and-Fire neurons and phase models. For instance, in [95] it is shown that certain Integrate-and-Fire neurons in a network with all-to-all connections synchronize for almost any initial condition. In [143, 144, 133] synchronization of more realistic neuronal oscillators in pulse-coupled networks is discussed. However, rigorous constructive results about what conditions the oscillators should satisfy and the effect of a particular network topology on the synchronization are not present nowadays. Even for systems that can be represented as the seemingly simple (pulse-)coupled Kuramoto oscillators, cf. [76], the problem of global synchronization is not tackled in full generality. Networks with strong interactions and/or chaotic regimes remain problematic.

Finally, it is important to realize that the semipassivity property of neurons can be very useful to analyze the dynamics in networks of chemically coupled neurons. Indeed, consider a collection of k neurons that interact via chemical synapses, where the chemical synapse is modeled as the Fast Threshold Modulation (FTM) coupling introduced in [143], i.e.

$$u_i(t) = \sum_{j=1}^{k} -g_{ij} H(y_j(t) - \theta)(E_{syn} - y_i(t)), \tag{6.35}$$

where $g_{ij} \in \mathbb{R}_{>0}$ denotes the synaptic conductance, $E_{syn} \in \mathbb{R}$ is the synaptic reversal potential which determines whether the synapse is inhibitory or excitatory, and the function $H(\cdot)$ is typically chosen as the Heaviside function such that neuron j will influence neuron i only if the membrane potential of neuron j exceeds the threshold $\theta \in \mathbb{R}$. It is not hard to verify that semipassive neuronal oscillators interconnected via chemical synapses have bounded solutions. (This follows from the fact that $\sum y_i(t)u_i(t) \leq 0$ outside some ball in $\mathbb{R}^k$, i.e. the "supplied energy" is bounded.)

Chapter seven

Synchronization in networks of diffusively coupled Hindmarsh-Rose neurons

Abstract. The goal of this chapter is to demonstrate and verify the theoretical results on full synchronization and partial synchronization that are presented in the first part of this thesis. In particular, the theoretical results are verified via computer simulations and real experiments in networks of diffusively coupled Hindmarsh-Rose neurons. Some of the experimental results presented in this chapter are published in [101]. Example 7.2 is taken from [146].

7.1 Introduction

As shown in the previous chapter, the Hindmarsh-Rose model of neural activity satisfies the semipassivity condition and has internal dynamics that are convergent. This makes the Hindmarsh-Rose neuron the perfect candidate to illustrate the theoretical results on synchronization that are presented in the first part of this thesis with. An advantage of the Hindmarsh-Rose model over the Hodgkin-Huxley and Morris-Lecar models, which are also strictly semipassive and minimum-phase, is that the Hindmarsh-Rose model can be relatively easy implemented on an electrical circuit board, see section 7.2. This makes it possible to do experiments on synchronization in networks of electronic Hindmarsh-Rose neurons. Of course, it is also relatively easy to produce a circuit realization of a FitzHugh-Nagumo neuron and do experiments with networks of FitzHugh-Nagumo neurons. However, the dynamical behavior that a Hindmarsh-Rose neuron is able to produce, e.g. bursting and even chaotic bursting, is much richer than that of a FitzHugh-Nagumo neuron, which makes it more interesting to consider networks of Hindmarsh-Rose neurons. Moreover, the Hindmarsh-Rose neurons is a more realistic model of a neuron than the FitzHugh-Nagumo model is. In appendix B it is shown that a Hindmarsh-Rose neuron is able to mimic the membrane potential of real neurons of a mouse.

Why is it interesting to do experiments on synchronization in networks of Hindmarsh-Rose neurons? The framework presented in chapters 3, 4 and 5 is derived for identical systems. However, in practice systems will never be completely identical, systems are at most almost identical. Hence, synchronization of the experimental systems implies some robustness against disturbances, e.g. measurement noise, and small deviations in the systems parameters. (There is no mathematical proof for this claim yet!) Besides, observations of interesting (synchronous) behavior of the systems in the experimental setup motivate the development of theory that can explain these observations. For instance, the theory about the scaling laws presented in chapter 5 is inspired by experimental observations.

In this chapter simulation results are presented for networks of chaotic Hindmarsh-Rose neurons,

$$\begin{aligned}
\dot{z}_{i,1}(t) &= 1 - 5y_i^2(t) - z_{i,1}(t), && (7.1a)\\
\dot{z}_{i,2}(t) &= 0.005\left(4\left(y_i(t) + 1.618\right) - z_{i,2}(t)\right), && (7.1b)\\
\dot{y}_i(t) &= -y_i^3(t) + 3y_i^2(t) + z_{i,1} - z_{i,2} + 3.25 + u_i(t), \quad i = \mathcal{I} := \{1, 2, \ldots, k\}, && (7.1c)
\end{aligned}$$

on a simple strongly connected graph that interact via non-delayed diffusive coupling

$$u_i(t) = \sigma \sum_{j \in \mathcal{E}_i} a_{ij}(y_j(t) - y_i(t)), \tag{7.2}$$

diffusive coupling type I

$$u_i(t) = \sigma \sum_{j \in \mathcal{E}_i} a_{ij}(y_j(t-\tau) - y_i(t)). \tag{7.3}$$

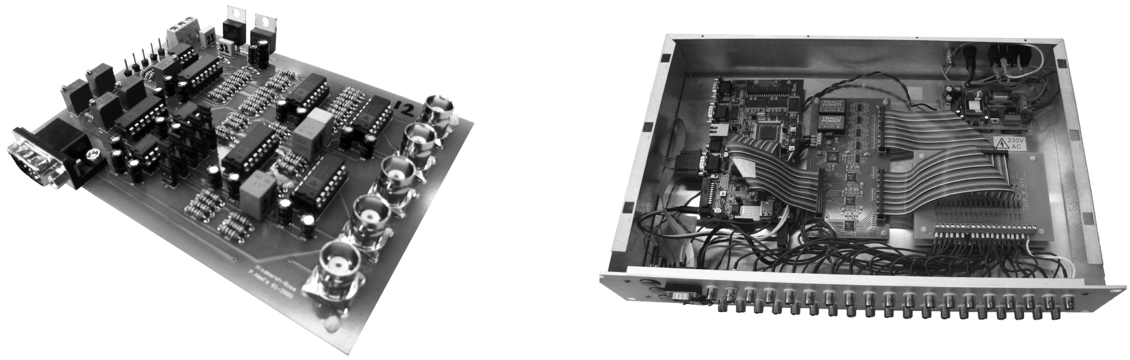
The results for the Hindmarsh-Rose neurons on a simple strongly connected graph that interact via diffusive coupling type II,

$$u_i(t) = \sigma \sum_{j \in \mathcal{E}_i} a_{ij}(y_j(t-\tau) - y_i(t-\tau)), \tag{7.4}$$

are supported by experiments. The reason why the experimental results are presented only for coupling type II becomes clear somewhat later.

The goal of this chapter is *not* to derive sharp bounds on the coupling strength and time-delays for which the neurons synchronize. The goal is to show that the framework presented in chapters 3, 4 and 5 does provide important (qualitative) insight in the collective dynamics of systems that interact via diffusive coupling. Note that, since the internal dynamics of the Hindmarsh-Rose neuron, i.e. the z_i-dynamics, are convergent[1], synchronization of the output signals $y_i(t)$ of the Hindmarsh-Rose neurons implies synchronization in the sense of Definition 3.1, i.e. the states of the Hindmarsh-Rose neurons will

[1] It is easy to see that the z_i-dynamics satisfy the Demidovich condition, see Lemma 2.2, with $P = I$.



(a) (b)

Figure 7.1. The experimental setup: (a) the electronic Hindmarsh-Rose neurons, and (b) the coupling interface.

asymptotically match. Hence, in this chapter, only the outputs signals $y_i(t)$ are plotted to show that the systems do (or do not) synchronize.

The remainder of this chapter is organized as follows. First, in section 7.2, the experimental setup with which the results for coupling type II will be demonstrated is briefly introduced. In section 7.3 results on full synchronization (chapter 3) are presented. Some remarkable examples on partial synchronization (chapter 4) are given in section 7.4. The results on the scaling laws in networks of systems that interact via coupling type II (chapter 5) are supported by experimental results that are presented in section 7.5. Section 7.6 concludes this chapter.

7.2 Experimental setup

In this section the experimental setup will be briefly introduced. For details about the setup the reader is referred to [100] and [101].

The experimental setup that is considered in this chapter consists of (up to) eighteen electronic equivalents of the Hindmarsh-Rose neuron and a coupling interface. A single electronic neuron is shown in Figure 7.1(a) and Figure 7.1(b) shows the coupling interface. The electronic Hindmarsh-Rose neurons are build using off-the-shelf components such as resistors, capacitors, analog voltage multipliers and operational amplifiers. Because of practical reasons, such as saturation of signals in the operational amplifiers, the Hindmarsh-Rose model (7.1) has to be modified slightly for circuit implementation.

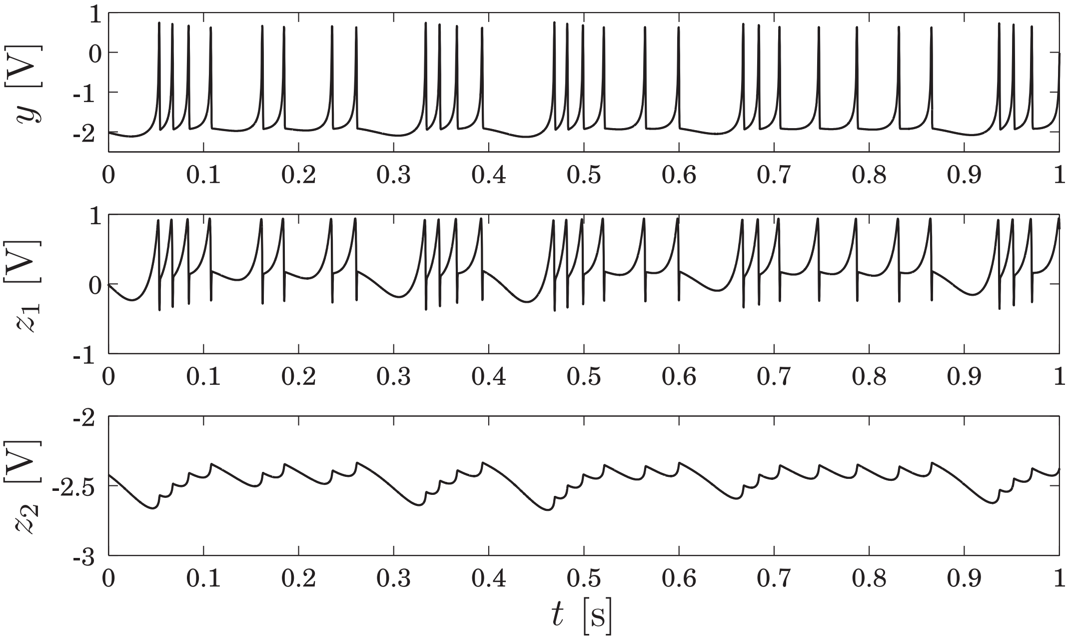


Figure 7.2. Solutions of the electronic Hindmarsh-Rose neuron for $u_i(t) \equiv 0$ V.

Consider the modified Hindmarsh-Rose model,

$$\tfrac{1}{1000}\dot{y}_i(t) = -y_i^3(t) + 3y_i(t) - 8 + 5z_{i,1} - z_{i,2} + 3.25 + u_i(t), \tag{7.5a}$$
$$\tfrac{1}{1000}\dot{z}_{i,1}(t) = -5y_i^2(t) - 2y_i(t) - z_{i,1}(t), \tag{7.5b}$$
$$\dot{z}_{i,2}(t) = 5\left(4\left(y_i(t) + 1.1180\right) - z_{i,2}(t)\right), \quad i = \mathcal{I}, \tag{7.5c}$$

where $y_i \in \mathbb{R}$ denotes still the neuron's membrane potential and $z_{1,i}, z_{2,i} \in \mathbb{R}$ are internal variables. This modified model (7.5) can be transformed into the original model (7.1) via the change of coordinates,

$$y \mapsto y + 1, \quad z_1 \mapsto 5z_1 - 4, \quad z_2 \mapsto z_2 + 6,$$

and a redefinition of time. (Note that the model (7.5) is a thousand times "faster" than the original model (7.1), that is, thousand time-units of the original model correspond to one time-unit for the modified model.) It follows that this modified model is strictly semipassive as well and has convergent internal dynamics. Each state of the electronic Hindmarsh-Rose model, i.e. the circuit realization of (7.5), can be measured as a voltage, and one time-unit of the model (7.5) corresponds to one second for the electric Hindmarsh-Rose neuron. The inputs $u_i(t)$ are also voltages. Figure 7.2 shows a measurement of the states of (7.5) for $u_i(t) \equiv 0$ V. The units V (for the states) and s (for time) are explicitly added in this figure since the measured states and time are real physical quantities.

Comparing Figure 7.2 and Figure 6.5, which shows a solution of the model (7.1), one concludes that the electronic Hindmarsh-Rose neuron mimics the behavior of the

Hindmarsh-Rose model pretty well. Indeed, both the model and the electronic neuron operate in the chaotic bursting regime, the shapes of the spikes are similar and the ranges of the signals match (taking the change of coordinates into account). The electronic neuron produces signals that are indeed thousand times "faster" than the signals of the original Hindmarsh-Rose model. Details about the electronic Hindmarsh-Rose neuron, including the electronic circuit layout, can be found in [100].

The topology of the network, the coupling functions and the time-delays are specified in the *coupling interface*. See Figure 7.1(b). With this interface it is possible to couple up to eighteen neurons. The most important component of the coupling interface is the Atmel ARM® Thumb®-based AT91SAM9260 microcontroller with an ARM926ej-s core that runs at 180 MHz. Custom made software allows to specify the coupling structure and control the time-delays[2] via simple c-code. The outputs $y_i(t)$ of the neurons are made available for the microcontroller via 16 bit analog-digital-converters with an input range of ± 10 V. The microcontroller computes the coupling functions $u_i(t)$ and these signals are made available to the electronic neurons via a 14 bit digital-analog-converter with an output limit of, again, ± 10 V. The analog-digital-converters sample simultaneously, which means that the outputs of all neurons that are coupled to the interface are acquired at the same time. The digital-analog-converter updates the signals $u_i(t)$ simultaneously. Clearly, the conversion of the signals and the computation of the coupling functions takes a small amount of time. This time-delay is estimated to be around 80 μs when all eighteen circuits are coupled. In addition, because of the simultaneous sampling and simultaneous updates of the interface, the time-delays will be approximately the same for every neuron. Thus coupling type II naturally emerges. This is the reason why experimental results are only shown for coupling type II and *not* for coupling type I. It has to be noted that the time-delay induced by the synchronization interface is so small (compared to the time-scale of the output signals of the electronic neurons) that it can almost be neglected. Hence it is also possible to do experiments that approximate the synchronous behavior in network of neurons that interact via non-delayed coupling or coupling type I. In fact, in [101] it is shown that the experimental results for networks of electronic Hindmarsh-Rose neurons that interact via diffusive coupling without additional delays, i.e. the best approximation of non-delayed coupling, are very close to the simulation results with such networks. Details about the synchronization interface can be found in [100].

7.2.1 Experimental synchronization of Hindmarsh-Rose neurons

Because of the tolerances on the electrical components and nonlinearities in the voltage multipliers and the operational amplifiers, every electronic Hindmarsh-Rose neuron will

[2] A time-delay is generated by buffering the measured output signals of the neurons. The size of the buffer corresponds to a certain amount of delay.

behave slightly different. This has important consequences for the synchronization of the electronic neurons. Indeed, a perfect asymptotic match of the outputs (and thus the states) of the neurons can not be guaranteed anymore. It can only be expected that the electronic neurons *practically* synchronize. Practical synchronization and practical partial synchronization are defined as follows.

Definition 7.1 (Practical synchronization of type II coupled Hindmarsh-Rose neurons). Consider k electronic HR neurons that interact via coupling type II. The interconnected systems are said to *practically synchronize* if, for every continuous initial history ϕ_i, ϕ_j, there is a sufficiently small positive number ϵ such that $\limsup_{t\to\infty} |y_i(t; t_0, \phi_i) - y_j(t; t_0, \phi_j)| < \epsilon$ for every $i, j \in \mathcal{I}$. △

Definition 7.2 (Practical partial synchronization of type II coupled Hindmarsh-Rose neurons). Consider k electronic HR neurons that interact via coupling type II. The interconnected systems are said to *practically partially synchronize* if, for every continuous initial history ϕ_i, ϕ_j, there is a sufficiently small positive number ϵ such that $\limsup_{t\to\infty} |y_i(t; t_0, \phi_i) - y_j(t; t_0, \phi_j)| < \epsilon$ for at least two but not all $i, j \in \mathcal{I}$. △

Remark 7.1. In Definitions 7.1 and 7.2 the neurons are said to practically synchronize when the outputs of the systems become sufficiently close and remain close as time increases. Note that the convergency property of the internal dynamics implies full-state practical synchronization. △

Remark 7.2. Definitions 7.1 and 7.2 are stated here only for Hindmarsh-Rose neurons that interact via coupling type II. A general definition for practical (partial) synchronization for any type of systems that interact via any type of coupling can easily be presented. It is important to note that the value of the number ϵ depends on the type of systems in the network and the type of coupling. △

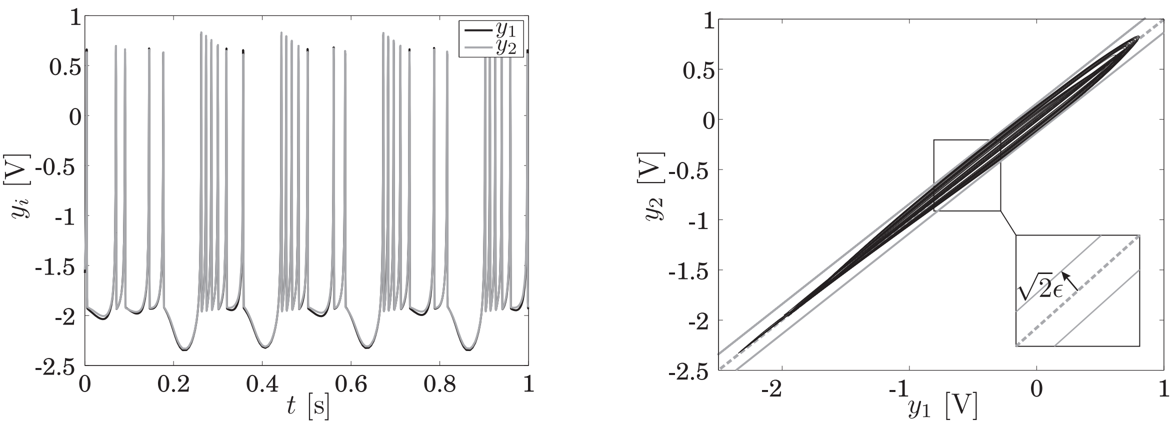
The following example shows practical synchronization of two type II coupled electronic Hindmarsh-Rose neurons.

Example 7.1 (Practical synchronization of two coupled neurons with minimal delay). Consider $k = 2$ electronic Hindmarsh-Rose systems which are coupled via

$$u_1(t) = \sigma(y_2(t-\tau) - y_1(t-\tau)), \tag{7.6a}$$

$$u_2(t) = \sigma(y_1(t-\tau) - y_2(t-\tau)), \tag{7.6b}$$

with minimal time-delay $\tau \approx 80$ μs. Experiments show that the two electronic Hindmarsh-Rose neurons practically synchronize with $\epsilon = 0.15$ V when $\sigma = 0.6$, see Figure 7.3. Figure 7.3(a) shows the outputs of two electronic Hindmarsh-Rose neurons as function of time t. The solutions are almost indistinguishable. In Figure (b) the outputs of the two neurons are plotted in the (y_1, y_2)-plane. It becomes now more clear that the systems do not perfectly synchronize since the signals are not confined to the diagonal $y_1 = y_2$. However, since the systems are practically synchronized the outputs remain close to the diagonal. △



(a) (b)

Figure 7.3. Practical synchronization of two electronic Hindmarsh-Rose neurons for $\sigma = 0.6$ and $\tau \approx 80$ μs: (a) practically synchronized outputs $y_1(t)$ and $y_2(t)$ as function of time t, and (b) practically synchronized outputs $y_1(t)$ and $y_2(t)$ in the (y_1, y_2)-plane.

Remark 7.3. In the example above the electronic Hindmarsh-Rose neurons are considered practically synchronized if the difference in outputs converges within a bound $\epsilon = 0.15$ V. This value of ϵ might look rather high at first sight. However, taking into account that the outputs signals of the Hindmarsh-Rose neuron are spiky, a small mismatch in the output signals easily results in relatively large differences between the output signals. As can be seen in Figure 7.3(a), with $\epsilon = 0.15$ V, the neurons produce their action potentials at the same time and the output signals are almost indistinguishable. It can be concluded from Figure 7.3(b) that the "synchronization error" is the largest when the action potentials are produced. Indeed, the deviation from the diagonal is the largest when the signals $y_1(t)$ and $y_2(t)$ are between -1 V and 0.5 V. △

7.3 Full synchronization

In this section simulations and experiments are presented that support the theoretical results of chapter 3. In Example 7.2 five neurons are considered that are coupled in a ring structure and interact via coupling type I. In Example 7.3 experimental results are shown for a network with four neurons that are all-to-all coupled via coupling type II.

Example 7.2 (Five type I coupled neurons in a directed ring. [146]). Consider $k = 5$ Hindmarsh-Rose systems that are uniformly coupled in a directed ring with coupling type I:

$$u_1(t) = \sigma(y_5(t-\tau) - y_1(t)), \tag{7.7a}$$
$$u_j(t) = \sigma(y_{j-1}(t-\tau) - y_j(t)), \quad j = 2, \dots, 5. \tag{7.7b}$$

See Figure 7.4(a). Clearly, the corresponding graph is simple and strongly connected. Since all time-delays are the same and the corresponding adjacency matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

it follows that assumption (H3.3) is satisfied[3]. Hence the synchronized state exists. Thus, by Theorem 3.10, the five coupled systems synchronize if σ is sufficiently large and $\sigma\tau$ is sufficiently small. The results (that are obtained via computer simulations) are shown in Figure 7.4. In Figure 7.4(b) the approximated values of σ and τ for which the systems synchronize are plotted. The systems synchronize if σ and τ belong to the dark shaded area. It can be concluded that the neurons indeed synchronize for sufficiently large sigma and sufficiently small $\sigma\tau$.

As mentioned before, it is important to note that coupling type I (with $\tau > 0$) generally does in not vanish when the systems are synchronized. In Figure 7.4(c) the synchronized solutions of the five systems are depicted for $\sigma = 2$ and $\tau = 0$. The synchronized solutions look like the solution of a free Hindmarsh-Rose neuron, i.e. a solution of a neuron with $u_i(t) \equiv 0$. This is because the coupling terms $u_i(t)$ vanish if $\tau = 0$ and the systems are synchronized. If $\tau > 0$, the synchronized solution is not a solution of a free Hindmarsh-Rose neuron anymore. See Figures 7.4(d) and 7.4(e) for simulation results with $\sigma = 3$, $\tau = 0.1$ and $\sigma = 4$, $\tau = 0.22$, respectively. △

Example 7.3 (A fully connected network with four type II coupled neurons). Consider $k = 4$ Hindmarsh-Rose neurons that interact via coupling type II. The four neurons are all-to-all connected and the coupling is uniform, i.e. all interconnection weights are the same and all time-delays are equal, and each neuron couples to every other neuron. See Figure 7.5(a). The coupling functions are

$$u_i(t) = \sigma \sum_{j=1, j\neq i}^{4} (y_j(t-\tau) - y_i(t-\tau)). \tag{7.8a}$$

Clearly all assumptions of Theorem 3.12 are satisfied, hence, by Theorem 3.12, (practical) synchronization is expected for sufficiently large σ and, at the same time, sufficiently small $\sigma\tau$. Figure 7.5(b) shows the values σ and τ for which the four electronic neurons are considered to be practically synchronized. The four neurons practically synchronize when σ and τ are in the dark shaded area of the (σ, τ)-plane. Again, it can be concluded that the neurons indeed synchronize for sufficiently large sigma and sufficiently small $\sigma\tau$. △

[3]assumption (H3.3) requires $\tau_{ij} = \tau$ and $\sum_{j\in\mathcal{E}_i} a_{ij} = 1$ for every $i, j \in \mathcal{I}$.

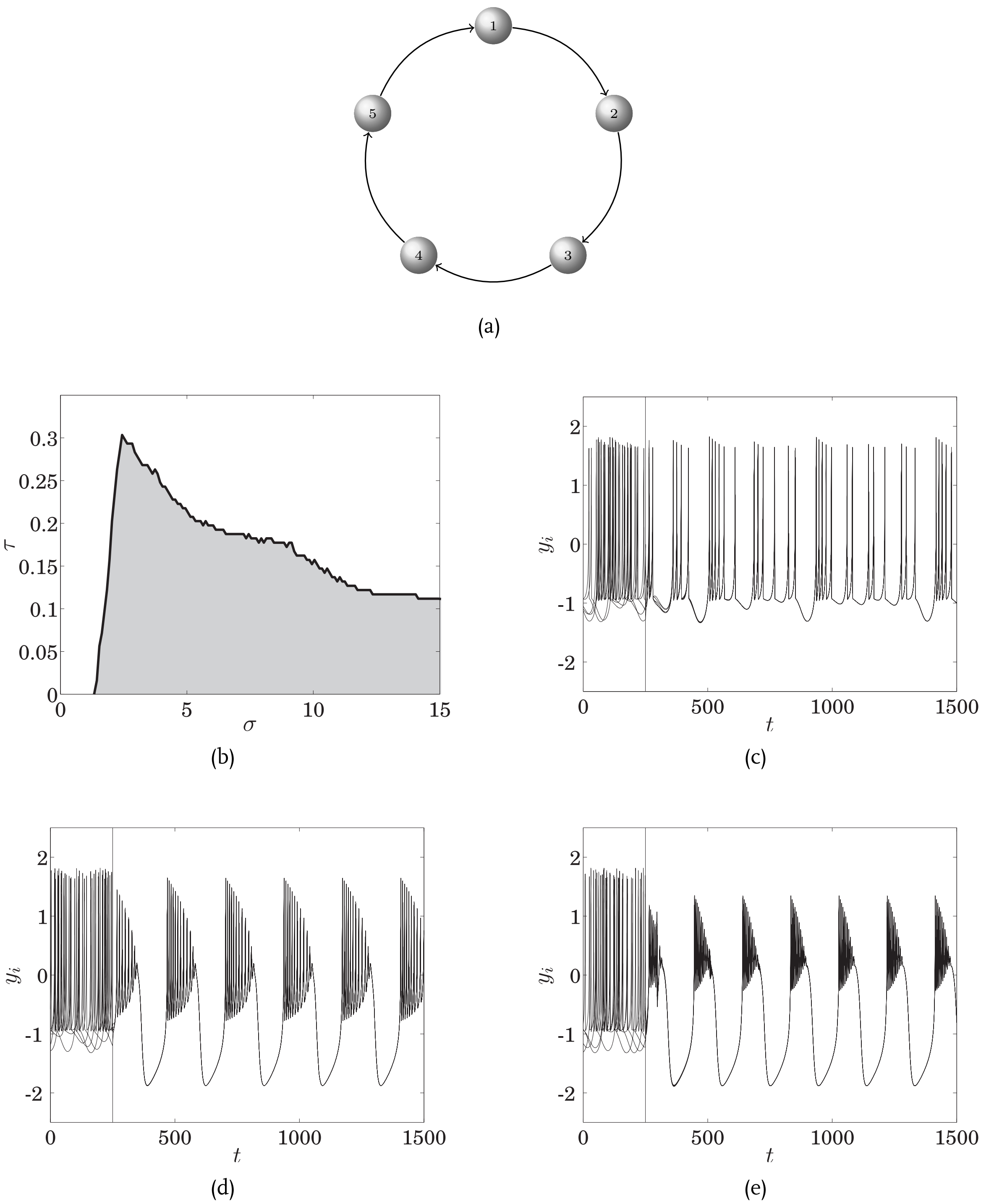


Figure 7.4. Synchronization of five type I coupled Hindmarsh-Rose neurons in a directed ring: (a) the network topology. (b) values of σ and τ for which the systems synchronize (shaded region). The time responses of the coupled neurons for (c) $\sigma = 2$, $\tau = 0$, (c) $\sigma = 3$, $\tau = 0.1$, (c) $\sigma = 4$, $\tau = 0.22$. For $t < 250$ the systems are uncoupled and do not synchronize, for $t \geq 250$ the coupling is active and the systems clearly synchronize.

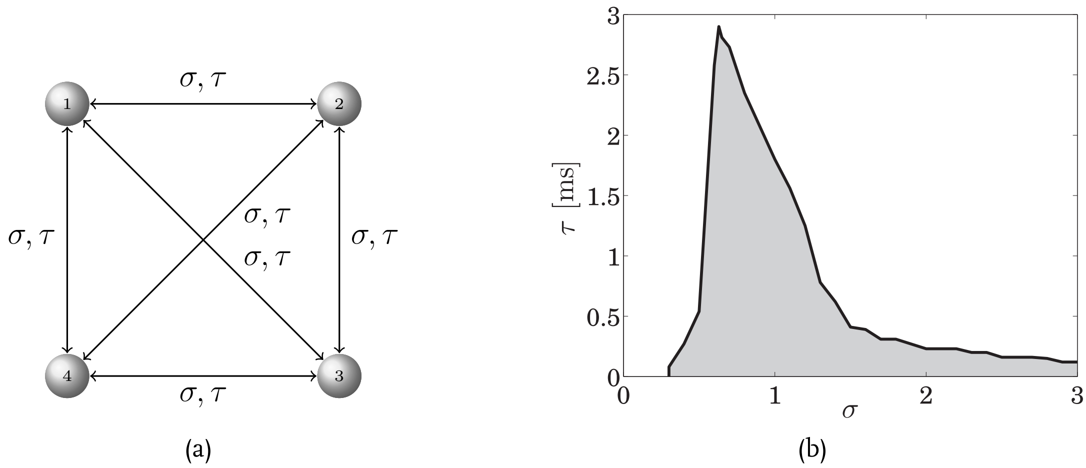


Figure 7.5. Practical synchronization of four all-to-all type II coupled neurons: (a) the network topology and (b) the values of σ and τ for which the four systems practically synchronize (shaded area).

7.4 Partial synchronization

In this section the theoretical results presented in chapter 4 are supported by simulations and experiments. First, in Example 7.4 the network of Example 4.1 is considered. Simulation results are presented for non-delayed interaction. Then, in Example 7.5 a network consisting of four neurons that interact via coupling type II is considered. The experimental results looks a bit surprising at first sight, but the result can be fully explained using the theory presented in chapter 4. Finally, Example 7.6 shows simulation results for four neurons that interact via coupling type I with multiple delays.

Example 7.4 (Partial synchronization in a directed network. See Example 4.1). Consider four Hindmarsh-Rose neurons that interact via non-delayed diffusive coupling. The network is as shown in Figure 7.6(a). As mentioned before, the Laplacian matrix for this network is given as

$$L = \begin{pmatrix} a_1 + 2a_3 & -a_1 & -a_3 & -a_3 \\ -a_1 & a_1 + 2a_3 & -a_3 & -a_3 \\ -a_2 & -a_2 & 2a_2 & 0 \\ -a_2 & -a_2 & 0 & 2a_2 \end{pmatrix}.$$

Assuming $a_1, a_2, a_3 > 0$ the graph is simple and strongly connected. Now Theorem 4.2 will be applied. Consider the permutation matrix

$$\Pi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

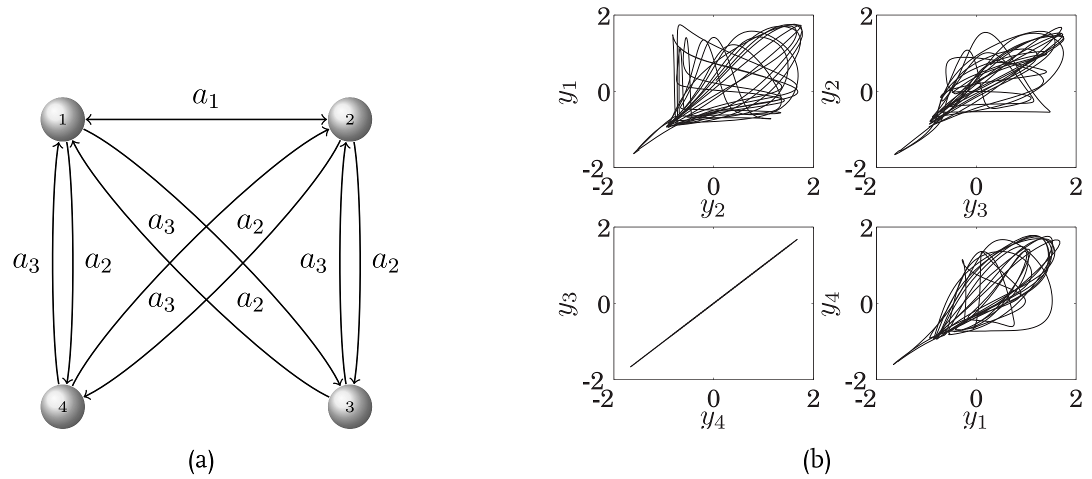


Figure 7.6. Four Hindmarsh-Rose neurons interacting via non-delayed diffusive coupling: (a) the network topology and (b) a simulation result with $\sigma = 1$, $a_1 = a_3 = 1$ and $a_2 = 5$.

which commutes with L, i.e. $\Pi L = L\Pi$. Hence $X = L$ is a solution of the matrix equation $(I - \Pi)L = X(I - \Pi)$ and a straightforward computation shows that $\lambda' = 2a_2$. This implies that the set $\ker(I_{kn} - \Pi)$ contains a globally attracting subset if σ is sufficiently large. If a_1 and a_3 are relatively small compared to a_2, then neurons 3 and 4 synchronize while there is no synchronization between the other neurons. This results is remarkable since neurons 3 and 4 do not interact directly with each other! Figure 7.6(b) shows a simulation result for $a_1 = a_3 = 1$, $a_2 = 5$ and $\sigma = 1$. As can de seen in the bottom left panel of this figure, outputs $y_3(t)$ and $y_4(t)$ synchronize (as they are on the diagonal in the (y_3, y_4)-plane). The other three figures show that there is no synchronization between the other neurons. △

Example 7.5 (Practical partial synchronization of four type II coupled neurons in a ring). Consider four Hindmarsh-Rose neurons that interact via coupling type II with a coupling configuration as shown in Figure 7.7(a). The surprising experimental results are shown in Figure 7.7(b). First it will be carefully explained what is going on. Then the results are explained using Corollary 4.8.

As expected, from Theorem 3.12, the four neurons fully practically synchronize when σ is sufficiently large and $\sigma\tau$ is sufficiently small. Those values of σ and τ for which the neurons fully practically synchronize are in the dark shaded area in Figure 7.7(b). Suppose that σ and τ are such that the neurons fully practically synchronize. If the value of σ decreases but the value of τ remains the same, full practical synchronization is lost. However, neurons 1 and 2 remain synchronous and neurons 3 and 4 are still practically synchronized. Of course, there is no synchronization when σ becomes very small. This result is probably expected. However, let σ and τ again be such that the neurons fully

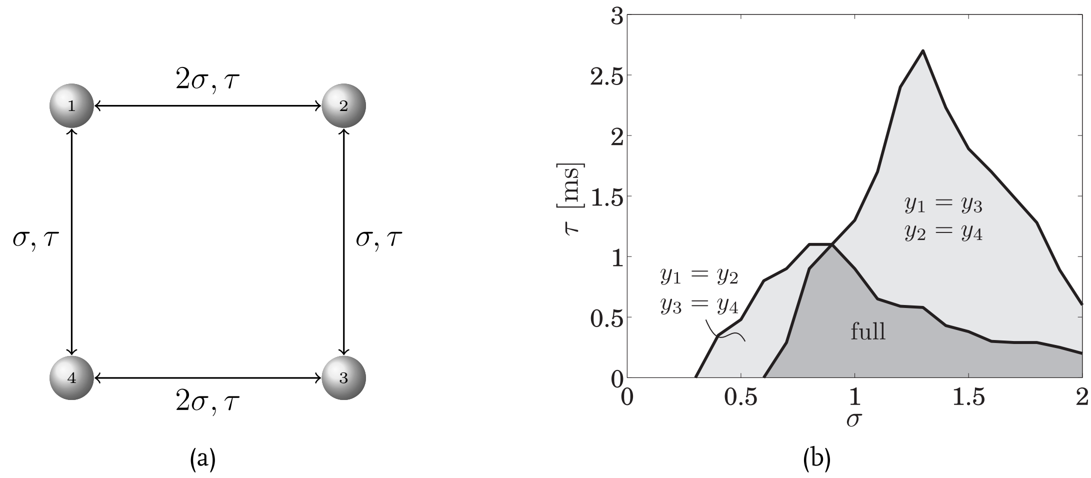


Figure 7.7. Four neurons that interact via coupling type II: (a) the network and (b) regions of practical partial synchronization as function of σ and τ.

practically synchronize, keep σ fixed but increase τ. What happens is that full practical synchronization is lost, but neurons 1 and 3, and neurons 2 and 4 remain practically synchronized. If τ becomes too large none of the neurons will synchronize anymore.

Although the latter result is unexpected, it can easily be explained using Corollary 4.8. The Laplacian matrix for this network is given as

$$L = \begin{pmatrix} 3 & -2 & 0 & -1 \\ -2 & 3 & -1 & 0 \\ 0 & -1 & 3 & -2 \\ -1 & 0 & -2 & 3 \end{pmatrix}, \tag{7.9}$$

Note that L is symmetric. Consider the permutation matrices

$$\Pi_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \Pi_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

It can easily be verified that both Π_1 and Π_2 commute with L, hence Corollary 4.8 can be used. The Laplacian matrix L has eigenvalues

$$\lambda_1 = 0, \quad \lambda_2 = 2, \quad \lambda_3 = 4, \quad \lambda_4 = 6,$$

and the corresponding eigenvectors are

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}.$$

Clearly, the eigenvectors of L that are in $\mathrm{range}\,(I-\Pi_1)$ are v_3 and v_4. Then Corollary 4.8 states that if $\sigma\lambda_3 = 4\sigma$ is sufficiently large and $\sigma\tau\lambda_4 = 6\sigma\tau$ is sufficiently small, the set $\ker(I-\Pi_1\otimes I)$ contains a globally attractive subset. In other words, neurons 1 and 2 synchronize and neurons 3 and 4 synchronize for sufficiently large 4σ and sufficiently small $6\sigma\tau$. On the other hand, the eigenvalues of L that are in $\mathrm{range}\,(I-\Pi_2)$ are v_2 and v_3 such that, again by Corollary 4.8, the set $\ker(I-\Pi_2\otimes I)$ contains a globally attractive subset if $\sigma\lambda_2 = 2\sigma$ is sufficiently large and $\sigma\tau\lambda_3 = 4\sigma\tau$ is sufficiently small. Thus neurons 1 and 3 synchronize and neurons 2 and 4 synchronize for sufficiently large 2σ and sufficiently small $4\sigma\tau$. It is not difficult to see that if $\sigma\lambda_2 = 2\sigma$ is sufficiently large and $\sigma\tau\lambda_4 = 6\sigma\tau$ is sufficiently small, then both sets $\ker(I-\Pi_1\otimes I)$ and $\ker(I-\Pi_2\otimes I)$ have an asymptotically stable subset, i.e. a subset of the set $\ker(I-\Pi_1\otimes I)\cap\ker(I-\Pi_1\otimes I)$ is asymptotically stable. But this implies full synchronization. To summarize, if σ is sufficiently large and $\sigma\tau$ is sufficiently small, then there is full synchronization. If σ is still sufficiently large but $\sigma\tau$ is not small enough, then a subset of the set $\ker(I-\Pi_2\otimes I)$ is asymptotically stable, i.e. neurons 1 and 3 synchronize and neurons 2 and 4 synchronize. If $\sigma\tau$ is sufficiently small but σ is not sufficiently large, then a subset of the set $\ker(I-\Pi_1\otimes I)$ is asymptotically stable, i.e. neurons 1 and 2 synchronize and neurons 3 and 4 synchronize. This is exactly what is shown in Figure 7.7(b). △

Example 7.6 (Partial synchronization in a ring of four type I coupled neurons with non-uniform time-delays). Consider the network of Example 4.3, i.e. consider four systems coupled in a ring as depicted in Figure 7.8(a). The coupling functions for this network are

$$u(t) = -\sigma y(t) + \tfrac{\sigma}{2}(\Pi_1\otimes I)y(t-\tau_1) + \tfrac{\sigma}{2}(\Pi_2\otimes I)y(t-\tau_2), \tag{7.10}$$

where $u(t) = \mathrm{col}\,(u_1(t), u_2(t), u_3(t), u_4(t))$, $y(t) = \mathrm{col}\,(y_1(t), y_2(t), y_3(t), y_4(t))$, $y(t-\tau_j) = \mathrm{col}\,(y_1(t-\tau_j), y_2(t-\tau_j), y_3(t-\tau_j), y_4(t-\tau_j))$, $j = 1, 2$, and

$$\Pi_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } \Pi_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

As already mention in Example 4.3, Theorem 4.10 implies that if σ is sufficiently large and $\sigma\tau_1$ is sufficiently small, then the set $\ker(I-\Pi_1\otimes I)$ has a globally asymptotically stable subset. If σ is sufficiently large and $\sigma\tau_2$ is sufficiently small, then the set $\ker(I-\Pi_2\otimes I)$ has a globally asymptotically stable subset. If σ is sufficiently large and both $\sigma\tau_1$ and $\sigma\tau_2$ are sufficiently small, then there is full synchronization. The simulation results confirm these statements. Figure 7.8(b) shows the values of σ and τ_1 for which neurons 1 and 2 synchronize, and neurons 3 and 4 synchronize. This results is independent of the value of τ_2. In Figure 7.8(c) the results of a simulation are shown where σ is sufficiently large and both $\sigma\tau_1$ and $\sigma\tau_2$ are sufficiently small. There is full synchronization as expected. If τ_2 becomes too large, full synchronization is lost but neurons 1 and 2 synchronize, and neurons 3 and 4 synchronize. See Figure 7.8(d). △

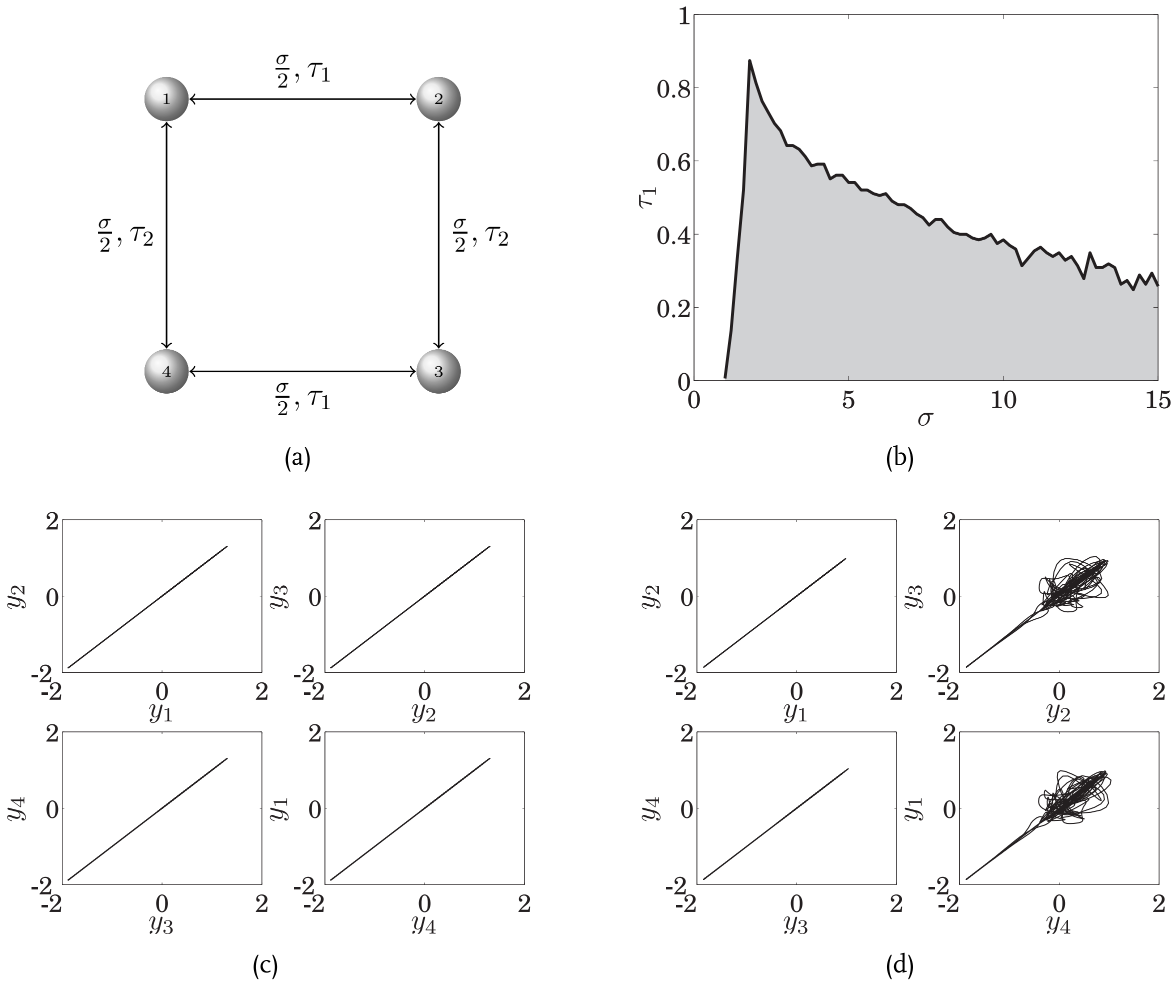


Figure 7.8. Four Hindmarsh-Rose neurons in a ring with non-uniform time-delays: (a) the network topology, (b) the values of σ and τ_1 for which neurons 1 and 2 and neurons 3 and 4 synchronize (independent of the value of τ_2!), (c) phase-space representation of full synchronization of the four neurons for $\sigma = 2$, $\tau_1 = 0.3$ and $\tau_2 = 0.6$, (d) phase-space representation of partial synchronization of neurons 1 and 2 and neurons 3 and 4 for $\sigma = 2, \tau_1 = 0.3$ and $\tau_2 = 5$.

7.5 Network topology

This section supports the results that are given in chapter 5, i.e. it will be shown how the values of the coupling strength and time-delay for which two neurons that interact via coupling type II synchronize can be used to determine the values of the coupling strength and time-delay for which any number of type II coupled neurons synchronizes. However, rather than the stability diagram of two coupled systems, the stability diagram of the four uniformly all-to-all coupled systems presented in Example 7.3 will be used to reconstruct the diagrams for other networks. The reason for this is that, because of saturation of the digital-analog-converters of the coupling interface, it is not possible to determine the stability diagram of two coupled systems for large values of σ. Since the coupling strength that is needed synchronize four all-to-all uniformly coupled neurons is only half the coupling strength required to synchronize two neurons[4], it is possible to measure the stability diagram for four systems for (relatively) large coupling strength. Note that the stability diagram is bounded by the line $\tau \equiv 0$ and a unimodal function. Hence, by Corollary 5.4, only the largest and the smallest nonzero eigenvalues of the Laplacian matrix have to be taken into account by applying the scaling. In Example 7.7 the stability diagram for three neurons coupled in a line will be reconstructed from the stability diagram of the four all-to-all uniformly coupled neurons, in Example 7.8 the same is done for four systems coupled in a ring.

Example 7.7 (Three neurons in a line). Consider three type II coupled neurons is a line. The Laplacian matrix of this network is

$$L = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}. \tag{7.11}$$

It is easy to verify that L has eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$ and $\lambda_3 = 3$. Recall the nonzero eigenvalues of the Laplacian matrix of the network of the four all-to-all coupled neurons are all equal to four. This implies that taking the intersection of the diagram depicted in Figure 7.5(b) scaled by a factor $4/\lambda_2 = 4$ and a factor $4/\lambda_3 = 4/3$ over the σ-axis gives the desired result. Indeed, as shown in Figure 7.9(a), the reconstructed stability diagram (dark shaded area) is pretty close to the measured stability diagram (bounded by the thick black line). △

Example 7.8 (Four neurons in a ring). Consider now four neurons in a network with Laplacian matrix

$$L = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}. \tag{7.12}$$

The Laplacian matrix has eigenvalues $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = 2$ and $\lambda_4 = 4$. Thus taking the intersection of the diagram depicted in Figure 7.5(b) scaled by a factor $4/\lambda_2 = 2$ and a factor $4/\lambda_4 = 1$ over the σ-axis gives the stability diagram for the four type II coupled neurons in a ring. The reconstructed stability diagram is shown in Figure 7.9(b) (dark shaded area), and, again, the reconstructed diagram and the measured stability diagram (bounded by the thick black line) show a pretty good resemblance. △

[4]This follows from Theorem 5.3 since, for four all-to-all uniformly coupled systems, $\lambda_2 = \lambda_3 = \lambda_4 = 4$.

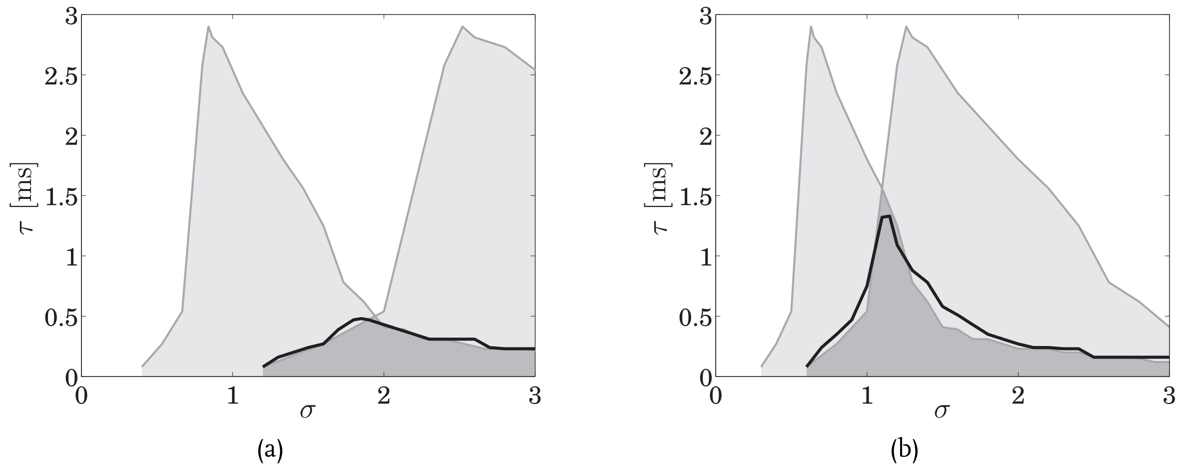


Figure 7.9. The stability diagrams for three neurons in a line (example 7.7) and four neurons in a ring (example 7.8). The diagrams are reconstructed from the diagram of four all-to-all uniformly type II coupled neurons. The scaled copies are given in the light shade, the intersection of the scaled copies is given in the darker shade. The thick black line indicates the boundary of the measured stability diagram. (a) three neurons in a line, and (b) four neurons in a ring.

7.6 Discussion

In this chapter results are presented of computer simulations and real experiments in networks of diffusively coupled Hindmarsh-Rose neurons. To quantify synchronization of the electronic Hindmarsh-Rose neurons in the experimental setup the notion of practical (partial) synchronization is introduced. Illustrative examples on full synchronization, partial synchronization and the role of the network topology are given for Hindmarsh-Rose neurons that interact via coupling type I, coupling type II or non-delayed diffusive coupling. In all examples the results support, or can be explained by, the theoretical framework given in chapters 3, 4 and 5.

The systems in the experimental setup are clearly not perfectly identical, but the results obtained with this experimental setup can still be explained by the theory that is developed. However, it is worth to investigate synchronization of non-identical systems in detail. In addition, there are some limitations in the experimental setup such as the saturation of the signals. Hardware improvements will make it possible to do more complicated experiments.

Chapter eight

Synchronization and activation in a model of a network of β-cells

Abstract. Islets of pancreatic β-cells are of utmost importance in the understanding of diabetes mellitus. In this chapter a model of a network of such pancreatic β-cells is considered. The cells are are globally coupled via gap junctions, i.e. all-to-all non-delayed diffusive coupling. Some of the cells in the islet are producing bursting oscillations while other cells are inactive. It is proven that the cells in the islet synchronize if the coupling is sufficiently large and all cells are active (or inactive). If the islet consists of both active and inactive cells and the coupling is sufficiently large, an active cluster and an inactive cluster emerge. It is shown that activity of the islet depends on the coupling strength and the number of active cells compared to the number of inactive cells. If too few cells are active the islet becomes inactive. The results presented in this chapter are published in [12].

8.1 Introduction

Diabetes mellitus is a problem of world wide concern [164, 175]. Dynamical analysis and control of pancreatic cells is one of its issues. The pancreas agglomerates cells in functional units called Langerhans islets. In particular, pancreatic β-cells play an important role in glucose homeostasis since they release insulin which is the hormone mainly responsible for the blood glucose regulation [68, 118]. Experimental studies show that the insulin secretion in β-cell is directly related to spiking/bursting electrical activity of the cell membrane. For instance, the absence of the spiking or bursting indicates that the insulin secretion is inhibited [119, 120, 137]. Synchronization of bursting activity in Langerhans islets is expected to play an important role in the insulin secretion [118, 137]. Moreover, there is experimental evidence that bursting electrical activity occurs when analyzing an islet as a whole, while when β-cells are analyzed in isolation, most of them are in an inhibited inactive state [118, 142]. On the other hand, if too many cells are inactive the islet might stop showing activity [120]. In this chapter a model of an islet with active and inactive β-cells is considered and it is studied how many cells need to be active the let the islet show activity such that the insulin secretion will not be inhibited. A single β-cell will be described by a model proposed by Pernarowski [120]. This model is capable of reproducing the inactive state, bursting oscillations and continuous spiking. The behavior of the model can be changed by varying a single parameter. Each cell will interact with all other cells via gap-junctions, i.e. a coupling given by the difference in membrane potential of the cells multiplied by the coupling strength. Using the machinery presented in chapter 3, see also [128, 124], it is proven that if the coupling is sufficiently strong an islet with all cells active (or inactive) synchronizes, and if the islet consists of both active and inactive cells it is proven that an active cluster and an inactive cluster emerge. It is shown that the network will be active as long as the islet contains a sufficient amount of active cells. It is well known that coupling between cells might influence the behavior of cells. In, for instance [127, 140], it is shown that certain systems that are inactive in isolation can produce stable oscillations when there are coupled. See also chapter 6, section 6.4. In [127, 140] the cells are assumed to be identical, whereas in this chapter non-identical cells are considered. The analysis shows that, depending on the coupling strength, the equilibrium of the islet changes from unstable to stable when not enough cells are active, which in turn implies that the electrical activity of the islet dies out and the insulin secretion is inhibited.

This chapter is organized as follows. In section 8.2 the model of a single cell is introduced and its dynamic behavior is briefly explained. Then in the next section the islet of globally coupled β-cells is introduced and some theoretical results concerning the synchronization of the activity in the islet are presented. Section 8.4 discuss when a islet stops showing activity and numerical simulations are presented that support the theoretical results. The results are further discussed in section 8.5.

8.2 A single β-cell

Consider a model of a β-cell [120]:

$$\dot{y}(t) = f(y(t)) - z_1(t) - z_2(t), \tag{8.1a}$$

$$\dot{z}_1(t) = w_\infty(y(t)) - z_1(t), \tag{8.1b}$$

$$\dot{z}_2(t) = \varepsilon\left(h(y(t)) - z_2(t)\right), \tag{8.1c}$$

where, $y \in \mathbb{R}$ denotes the membrane potential which is also the natural output of a cell, $z_1 \in \mathbb{R}$ a channel activation variable, $z_2 \in \mathbb{R}$ is related to the concentration of intracellular calcium and adenosine diphosphate (ADP), $\varepsilon \ll 1$ is a small positive parameter and the polynomials $f(y(t)) = -f_3 y^3(t) + f_2 y^2(t) + f_1 y(t)$, $w_\infty(y(t)) = w_3 y^3(t) + w_2 y^2(t) - w_1 y(t) - w_0$, $h(y(t)) = b\left(y(t) + y_0\right)$. In the sequel the following parameters will be used

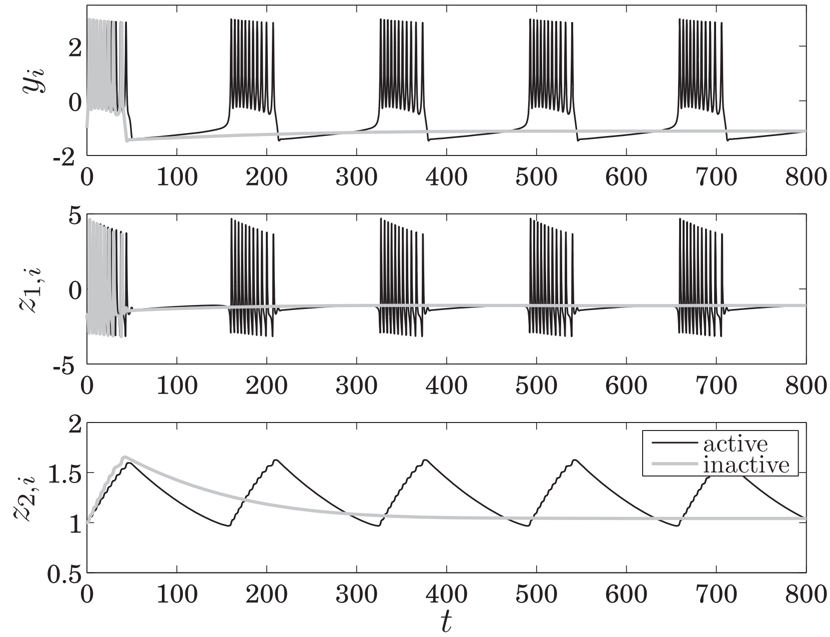


Figure 8.1. Numerical simulation of an isolated β-cell with the parameters presented in the text and initial conditions $(y_i(0), z_{1,i}(0), z_{2,i}(0)) = (-1, -2, 1)$. The black trajectories correspond to an active bursting cell ($y_0 = 0.954$), the gray trajectories represent an inactive cell ($y_0 = 1.375$). The single burst in the beginning is due to the initial conditions.

[120]: $f_3 = \frac{1}{12}$, $f_2 = \frac{3}{8}$, $f_1 = \frac{37}{64}$, $w_3 = \frac{11}{12}$, $w_2 = \frac{3}{8}$, $w_1 = 2\frac{27}{64}$, $w_0 = 3$, $\varepsilon = 0.0025$ and $b = 4$. If $y_0 = 0.954$ the cell bursts; the cell shows activity and the cell is called active. On the other hand, if $y_0 = 1.375$ the solutions of (8.1) converge to an equilibrium and the cell is called inactive. Figure 8.1 shows the state trajectories of an active cell and an inactive cell.

First, the fast-slow analysis of the system (8.1) presented in [120] will be summarized since it explains how the model generates the different behaviors depicted in Figure 8.1. See also, for instance, Chapter 6 of [71] and Section 11.4 of [72]. Consider an active cell, i.e. $y_0 = 0.954$. On the *fast t time scale*, the time scale that dominates during the bursts, the behavior of the cell is governed by letting $\varepsilon = 0$ such that

$$\dot{y}(t) = f(y(t)) - z_1(t) - z_2, \tag{8.2a}$$
$$\dot{z}_1(t) = w_\infty(y)(t) - z_1(t), \tag{8.2b}$$

where z_2 is now a constant parameter. The bifurcation diagram of this system is depicted in Figure 8.2. A family of stable limit cycles starts at a Hopf bifurcation indicated by point B in the diagram and terminates at the homoclinic bifurcation point C. The equilibria of the fast subsystem (8.2) lie on $z_2 = S(y) := f(y) - w_\infty(y)$. The equilibria located on the S-shaped curve $S(\cdot)$ between the left knee (point A) and the Hopf bifurcation point B are unstable, while all other equilibria are stable. On the *slow $t^* := \varepsilon t$ time scale*, the time in

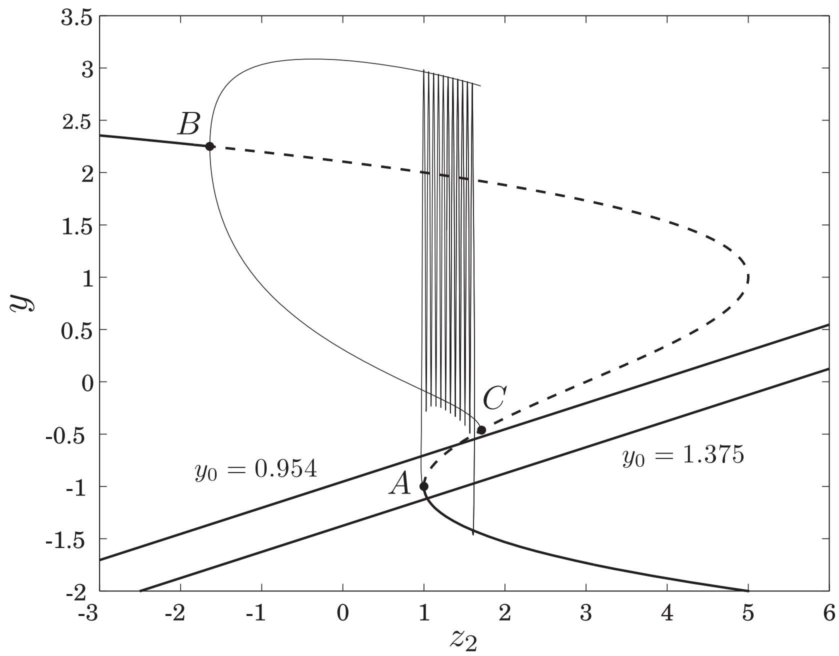


Figure 8.2. Fast-slow analysis and a projection of the trajectories of an active cell onto the (z_2, y)-plane.

between the bursts, the dynamics are given by the equations

$$z_2(t^*) = S(y(t^*)), \tag{8.3a}$$

$$\frac{\mathrm{d}z_2}{\mathrm{d}t^*}(t^*) = h(y(t^*)) - z_2(t^*). \tag{8.3b}$$

Between the bursts the solutions of (8.1a) follow the lower branch of the curve $z_2 = S(y)$ with z_2 slowly decreasing (since this branch lies below the nulcline $z_2 = h(y)$). The equilibrium of the system (8.1) is given by the intersection of the S-shaped curve with the nulcline of the slow system $h(y) = 0$. If $y_0 = 0.954$ the unique equilibrium is located at the unstable branch of $S(y)$. Suppose that the initial conditions of (8.1a) are chosen near the lower branch of $S(y)$. Then the solutions follow the lower branch with decreasing z_2 until the left knee (point A) is reached. At this point stability is lost and the fast subsystem starts to dominate. Hence the system starts to oscillate. During these oscillations z_2 will be slowly increasing such that at a certain moment a homoclinic bifurcation occurs (at point C), which forces the solutions back to near the lower branch of $S(y)$. This process repeats over and over resulting in the stable bursting behavior. On the other hand, if $y_0 = 1.375$ the intersection of $S(y)$ and $h(y)$ is on the lower branch of $S(y)$ which implies that the equilibrium of (8.1) is stable.

8.3 An islet of β-cells

Consider an islet consisting of k coupled β-cells

$$\dot{y}_i(t) = f(y_i(t)) - z_{1,i}(t) - z_{2,i}(t) + u_i(t), \tag{8.4a}$$

$$\dot{z}_{1,i}(t) = w_\infty(y_i(t)) - z_{1,i}(t), \tag{8.4b}$$

$$\dot{z}_{2,i}(t) = \varepsilon\left(b\left(y_i(t) + y_{0,i}\right) - z_{2,i}(t)\right), \tag{8.4c}$$

with $i = 1, \dots, k$ and $u_i \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R})$ is an input with which the cell is able to "communicate" with other cells. The islet consists of k_1 cells that are active while the remaining $k - k_1 =: k_2$ cells are inactive. Recall that the difference of a cell being active or inactive depends only on the value of $y_{0,i}$, i.e. $y_{0,i} = 0.954$ if a cell is active and $y_{0,i} = 1.375$ if the cell is inactive.

It is well known that β-cells couple via so-called gap junctions [137]. It will be assumed that the cells are globally (all-to-all) coupled with uniform coupling strength. Hence the coupling for the i^{th} cell is given by the equations

$$u_i(t) = \sigma \sum\nolimits_{j=1, j\neq i}^{k} (y_j(t) - y_i(t)), \tag{8.5}$$

with coupling strength $\sigma > 0$. The cells are called synchronized if, for every initial conditions, $\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0$ for all $i, j = 1, \dots, k$ with $x_i := \operatorname{col}(y_i, z_{1,i}, z_{2,i})$. The next two results follow from the machinery presented in chapter 3, see also [128, 147].

Lemma 8.1. *The solutions of the cells* (8.4) *coupled via* (8.5) *are ultimately bounded.*

Theorem 8.2. *Consider an islet with k cells* (8.4) *coupled via* (8.5). *There exists a constant $\bar{\sigma} > 0$ such that if $\sigma k > \bar{\sigma}$, then*

i. if all cells are active ($k_1 = k$), all cells show synchronized bursting oscillations;

ii. if all cells are inactive ($k_2 = k$), all cells are synchronized but there are no oscillations;

iii. if $k_1 < k$ cells are active and $k_2 < k$ cells are inactive, the active cells synchronize and the inactive cells synchronize, but the active cells do not synchronize with the inactive cells.

The proofs of Lemma 8.1 and Theorem 8.2 are provided in appendix A, section A.4. Lemma 8.1 states that all solutions of the interconnected cells enter some compact set in finite time and the solutions remain in that set thereafter. Note that this result is not trivial; it is well known that the solutions of interconnected systems might become unbounded even if the solutions of a system in isolation are bounded. This typically happens when the systems are non-minimum phase, see chapter 6, section 6.4. Theorem 8.2 states that if the coupling strength multiplied by the number of cells exceeds the threshold $\bar{\sigma}$, i.e. the coupling is sufficiently strong and/or the number of cells is sufficiently large, a cluster of synchronized active cells and a cluster of synchronized inactive cells emerge. If all cells are either active or inactive then all cells in the islet synchronize when the coupling is sufficiently strong.

Remark 8.1. Lemma 8.1 and Theorem 8.2 also hold for the biophysically plausible conductance based models of β-cells such as the models in [137, 71]. See chapter 6 for details. △

Remark 8.2. Lemma 8.1 is also true for a general network topology, Theorem 8.2 can be generalized for a general network topology in case that all cells are either active or inactive. See chapter 3 for details. △

8.4 An active or an inactive islet?

In this section an islet of coupled cells (8.4), (8.5) with k_1 active cells and k_2 inactive cells is considered. In the remainder it is assumed that $\sigma k \geq \bar{\sigma}$ such that (as follows from Theorem 8.2) a cluster of active cells and a cluster of inactive cells will emerge. Due to the interaction of the clusters two scenarios occur:

i. the active cluster "stimulates" the inactive cluster such that the cells in the inactive cluster start to produce oscillations;

ii. the inactive cluster suppresses the activity in the active cluster such that all activity in the islet dies out.

As one might imagine, there will be two parameters that determine whether the islet will be active or inactive, namely the coupling strength σk and the number of active cells relative to the number of inactive cells, i.e. the relative sizes of the clusters. Let η be the portion of active cells relative to the number of total cells, i.e. $\eta = \frac{k_1}{k}$. It follows that $1 - \eta$ represents the number of inactive cells relative to the number of total cells in the islet. In what follows estimates are presented of the critical portion $\eta^* = \eta^*(\sigma k)$ at which there is a change from activity to inactivity.

The dynamics of a cluster are given as

$$\dot{\zeta}_{1,m}(t) = f(\zeta_{1,m}(t)) - \zeta_{2,m}(t) - \zeta_{3,m}(t) + \nu_m(t), \tag{8.6a}$$
$$\dot{\zeta}_{2,m}(t) = w_\infty(\zeta_{1,m}(t)) - \zeta_{2,m}(t), \tag{8.6b}$$
$$\dot{\zeta}_{3,m}(t) = \varepsilon\left(b\left(\zeta_{1,m}(t) + y_{0,m}\right) - \zeta_{3,m}(t)\right), \tag{8.6c}$$

with $m = 1, 2$. Note that the equations describing the dynamics of the cluster are copies of the equations that describe the single cell. This is because the cells in a cluster are synchronized and share the same dynamics. Let $m = 1$ be the inactive cluster and $m = 2$ the active cluster, i.e. $y_{0,1} = 1.375$ and $y_{0,2} = 0.954$. It is not difficult to see that the coupling between the clusters is given by the equations $\nu_1(t) = \sigma k \eta(\zeta_{1,2}(t) - \zeta_{1,1}(t))$, $\nu_2(t) = \sigma k(1-\eta)(\zeta_{1,1}(t) - \zeta_{1,2}(t))$. The machinery presented in section 8.2 will now be used to determine the estimate of the critical portion η^*. Consider the change of coordinates: $\xi_{1,m}(t) = \zeta_{1,m}$, $\xi_{2,m}(t) = \zeta_{2,m}(t)$ and $\xi_{3,m}(t) = \zeta_{3,m}(t) + c_m$ with $c_m := b(1.250 - y_{0,m})$, i.e. $c_1 = 1.184$ and $c_2 = -0.5$. Hence

$$\dot{\xi}_{1,m}(t) = f(\xi_{1,m}(t)) - \xi_{2,m}(t) - \xi_{3,m}(t) + c_m + \tilde{\nu}_m(t), \tag{8.7a}$$

$$\dot{\xi}_{2,m}(t) = w_\infty(\xi_{1,m}(t)) - \xi_{2,m}(t), \tag{8.7b}$$

$$\dot{\xi}_{3,m}(t) = \varepsilon\left(b\left(\xi_{1,m}(t) + 1.250\right) - \xi_{3,m}(t)\right), \tag{8.7c}$$

and

$$\tilde{\nu}_1(t) = \sigma k \eta(\xi_{1,2}(t) - \xi_{1,1}(t)), \tag{8.8a}$$

$$\tilde{\nu}_2(t) = \sigma k(1-\eta)(\xi_{1,1}(t) - \xi_{1,2}(t)). \tag{8.8b}$$

Figure 8.3 depicts the S-shaped curves of the fast subsystem of the uncoupled clusters and the line $\xi_{3,m} = b(\xi_{1,m} + 1.250)$. Again, the equilibrium of the clusters are given by the intersection of the S-shaped curve and the line. Note that the location of the equilibrium of the original model (8.1) (and thus that of a uncoupled cluster) is changed by varying y_0. In Figure 8.2 this corresponds to shifting the line $z_2 = h(y) = b(y + y_0)$ up or down (as function of y_0) while keeping the S-shaped curve fixed. After the change of coordinates the location of the equilibrium still changes with y_0 since $c_m = c_m(y_0)$, but, as can be seen in Figure 8.3, this corresponds now to shifting the S-shaped curves to the left or right while keeping the line $\xi_{3,m} = b(\xi_{1,m} + 1.250)$ fixed. Consider two clusters (8.7) which are coupled via (8.8). Due to the interaction the location of the S-shaped curves of the active and the inactive clusters change, hence the locations (and thus stability) of the equilibria change. The estimate of the critical portion $\eta^*(\sigma k)$ will now be determined by estimating for which values of η, σk and the equilibrium of the inactive cluster the equilibrium of the active cluster is at the left knee of its S-shaped curve. In particular, consider the two following extreme cases:

Case 1. The location of the equilibrium of the inactive cluster $(\xi^o_{1,1}, \xi^o_{2,1}, \xi^o_{3,1})$ does not change due to the interaction with the active cluster. This is the case when the portion of active cells is small. Let $(\xi^o_{1,2}, \xi^o_{2,2}, \xi^o_{3,2})$ be the equilibrium of the active cluster. If the equilibrium is at the left knee, then

$$\begin{aligned} 0 = \tilde{S}(\xi^o_{1,2}) - b\left(\xi^o_{1,2} + c_2\right) \\ + \sigma k(1-\eta^*)(\xi^o_{1,1} - \xi^o_{1,2}), \end{aligned} \tag{8.9a}$$

$$0 = \tilde{S}'(\xi^o_{1,2}) - \sigma k(1-\eta^*),\ \tilde{S}''(\xi^o_{1,2}) > 0, \tag{8.9b}$$

where $\tilde{S}(\xi_{1,m}) := f(\xi_{1,m}) - w_\infty(\xi_{1,m}) + c_m$ and $'$ indicates the derivative with respect to $\xi_{1,m}$. Here (8.9a) is the equilibrium equation for the active cluster and (8.9b) is the

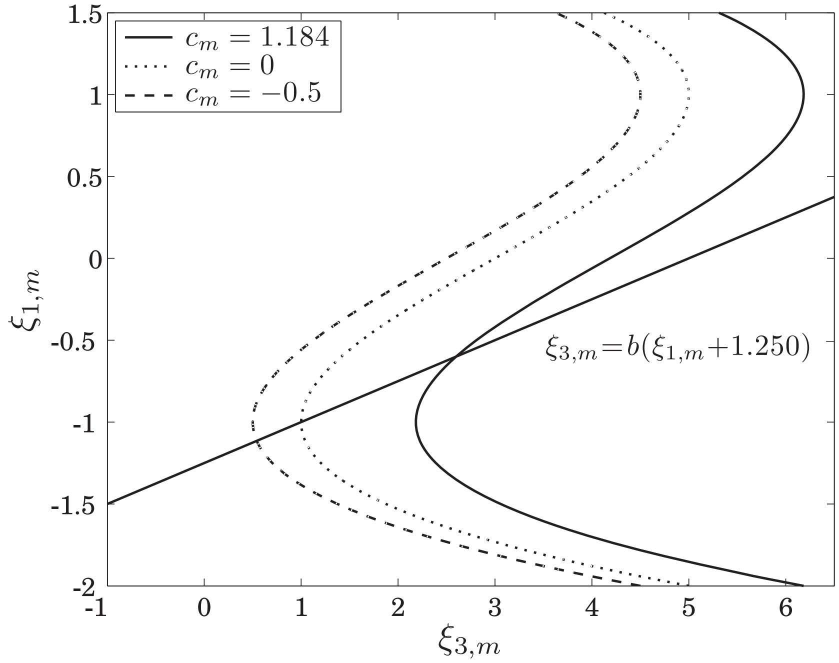


Figure 8.3. S-shaped curves of the uncoupled, i.e. $\tilde{\nu}_m(t) \equiv 0$, active cluster ($c_m = 1.184$) and inactive cluster ($c_m = -0.5$). Also presented are the S-shaped curve corresponding to $c_m = 0$ and the line $\xi_{3,m} = b(\xi_{1,m} + 1.250)$.

condition that guarantees the equilibrium to be at the left knee. Solving (8.9) for the given model parameters results in $\sigma k(1-\eta^*) = c$ with $c \approx 1.213$. Since $\eta^* \in [0,1]$ it follows that $\eta^* = \max(0, 1 - \frac{c}{\sigma k})$.

Case 2. The equilibria of both the active and inactive cluster are at the left knee of the S-shaped curve with $c_m = 0$. This happens if the coupling strength σk is large. In Figure 8.3 this corresponds to shifting the S-shaped curve of the active (inactive) cluster to the *left* (*right*) by an amount of c_1 (c_2) such that the S-shaped curves of the active and inactive cluster coincide with the S-shaped curve with $c_m = 0$. Thus

$$0 = \sigma k \eta^* (\xi^o_{1,2} - \xi^o_{1,1}) + c_1, \tag{8.10a}$$

$$0 = \sigma k (1-\eta^*)(\xi^o_{1,1} - \xi^o_{1,2}) + c_2, \tag{8.10b}$$

from which it follows that $\eta^* = \frac{c_1}{c_1 - c_2} \approx 0.297$.

Figure 8.4 summarizes the result. The estimated critical portion η^* is indicated by the thick gray line. The area in gray in the $(\sigma k, \eta)$ plane indicates the region where activity of the islet is guaranteed. For instance, for large σk at least 30% of the cells should be active to have any activity of the islet. The circles in Figure 8.4 indicate the critical portion obtained by numerical simulations of an islet with $k = 100$ cells. The analytical estimate approaches the numerical results well for small η and large σk.

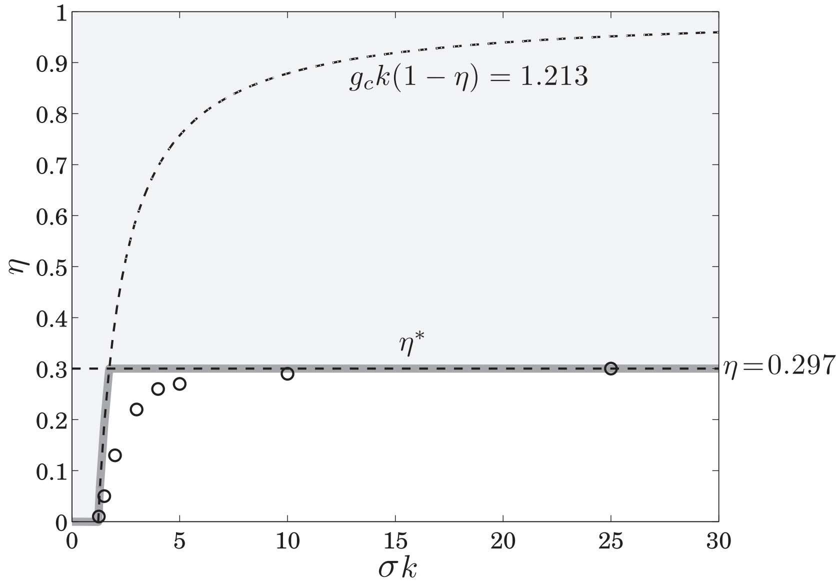


Figure 8.4. Analytical estimates of the critical portion (thick gray line) and the result of numerical simulations (circles).

Figure 8.5 shows the results of numerical simulations of a network consisting of $k = 7$ cells. The coupling strength $\sigma = 1$. In Figure 8.5(a), $k_1 = 3$ cells are active and $k_2 = 4$ cells are inactive. As expected two clusters emerge and the network shows activity. In Figure 8.5(b), $k_1 = 2$ cells are active and $k_2 = 5$ cells are inactive. Again, as expected, two clusters emerge but now all activity dies out.

8.5 Discussion

In this chapter a model of an islet of globally coupled β-cells is considered. Some cells are active and others cells are inactive. As stated in the introduction, the activity of an islet of β-cells is directly related to the blood glucose level, cf. [119, 120, 137]. It is investigated in this chapter to what extent it is possible that coupled β-cells ultimately may exhibit active or inactive behavior. First it is proven that the solutions of all cells in the islet are ultimately bounded. In addition, it is proven that if all cells are active or all cells are inactive, given that the coupling is sufficiently strong, all cells synchronize. If the islet consists of both active and inactive cells and the coupling is sufficiently strong, then an active cluster and an inactive clusters emerge. Using stability analysis of the equilibria of the clusters an estimate of the critical portion $\eta^*(\sigma k)$ is determined. If for some fixed coupling strength σk the portion of active cells $\eta > \eta^*$, the islet will still show some activity. Results of numerical simulations show that the estimates of $\eta^*(\sigma k)$ are accurate for small η and large σk.

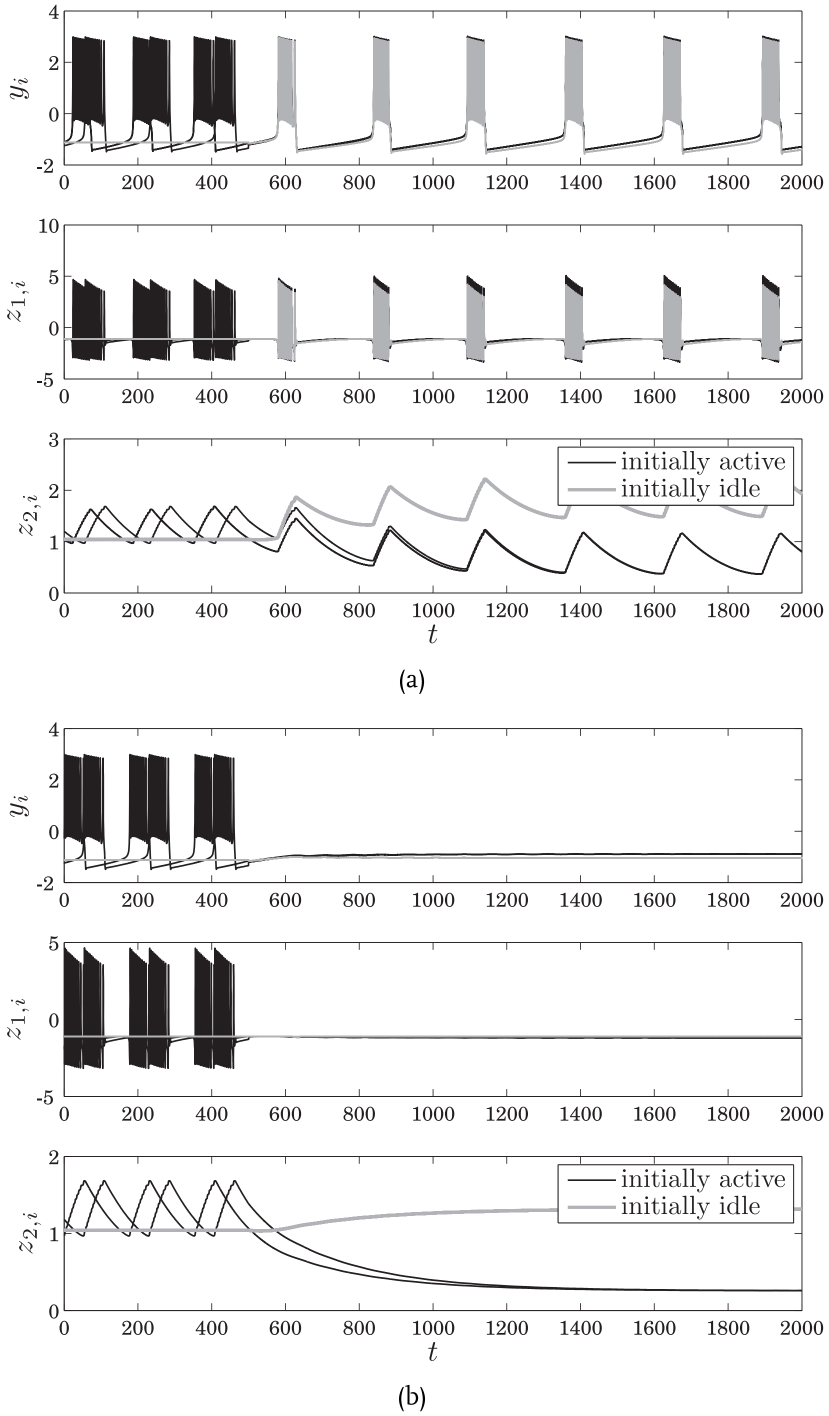


Figure 8.5. Numerical simulations of a network consisting of $k = 7$ cells coupled with strength $\sigma = 1$: (a) three active and four inactive cells, (b) two active and five inactive cells.

In [120], the critical portion for an islet consisting of a large number ($O(\frac{1}{\varepsilon})$) of cells that couple to their nearest neighbors is estimated to be 0.283. Although the analysis in [120] is different, the value of the estimated portion in the large islet with nearest neighbor coupling is close to the value that is estimated for an globally coupled islet consisting of an arbitrary number of cells. It would be interesting to study the influence of the topology of the network and the coupling strength in detail.

CHAPTER NINE

Controlled synchronization via nonlinear integral coupling

Abstract. The ideas presented in this chapter go beyond linear coupling. In particular, in this chapter the problems of controlled synchronization and regulation of oscillatory systems that interact via nonlinear coupling are considered. For a specific class of nonlinear systems, namely for minimum-phase systems with relative degree one, a systematic design procedure for finding nonlinear couplings between the systems is proposed. This nonlinear coupling guarantees asymptotic synchronization of the systems' states for arbitrary initial conditions. The corresponding coupling has the form of an integral and it can be considered as a generalized distance between the outputs of the coupled systems. It combines both the low-gain and the high-gain coupling design in one nonlinear function. The results are illustrated with simulations of coupled Hindmarsh-Rose neurons. The results presented in this chapter are published in [113].

9.1 Introduction

When considering synchronization of interconnected systems, one can distinguish two directions; synchronization *analysis* of interconnected systems with *given* couplings and interconnection structure, and *design* of interconnection couplings that guarantees synchronization of the systems. The first problem is discussed for diffusive coupling in the first part of this thesis. In this chapter the latter problem, also called *controlled synchronization*, is considered. It has to be noted that in this chapter no time-delays are taken into account.

The controlled synchronization problem is closely related to several control problems such as observer design and the output regulation problem, see, e.g. [114] for the connection between these problems. Browsing through the literature on synchronization of nonlinear systems, one encounters multiple results with linear, diffusive, couplings. While for analysis of interconnected systems with *given* linear couplings this is a normal approach, for the problem of couplings design, limiting oneself only to linear couplings may be too restrictive, especially for highly nonlinear systems. The main problem of linear couplings is that the coupling gains that guarantee synchronization are often large. This high-gain-feedback approach is needed to suppress the nonlinearities in the systems. At the same time, nonlinear couplings, which can be considered as couplings with varying gains, seem to be more natural for nonlinear systems; the gain should be high in the parts of the state space where the nonlinearities are essential and need to be suppressed, while it can be small in other parts of the state space. Synchronization in nonlinearly coupled systems has been considered mostly from the analysis point of view, see, e.g. [82, 57]. There are not so many results focusing on design aspects of synchronization through nonlinear coupling.

In this chapter a systematic approach is presented to design nonlinear coupling functions that guarantee synchronization of two unidirectionally or bidirectionally coupled systems and synchronization of multiple all-to-all coupled systems. In particular, in this chapter coupling functions are considered in the form of a definite integral of some non-negative weight function with the integral limits being the outputs of the systems. For two systems the magnitude of the coupling can be considered as a generalized distance between the systems' outputs. For the case of a constant weight function it leads to the conventional linear coupling. The introduction of a nonlinear weight function in such an integral coupling leads to greater flexibility, which may lead to reduced coupling gains as it will be demonstrated with an example. Lower coupling gains, in turn, lead to lower (measurement) noise sensitivity. Moreover, this form of coupling is very convenient for analysis and provides constructive design methods.

This approach has its roots in the semipassivity-based synchronization of systems with linear couplings on the one hand, and recent developments in the nonlinear output regulation problem [110, 114], on the other hand. In fact, as it has been pointed out in [114], the controlled synchronization problem, at least for the case of master-slave synchronization, can be considered as a particular case of the output regulation problem. Moreover, the methods that are used both in the controlled synchronization and the output regulation problems overlap in several aspects. Controllers developed within the output regulation problem often contain the so-called internal model [29, 90], that is, an auxiliary dynamical system which, being a part of the controller, guarantees the existence of a solution of the closed-loop system corresponding to zero regulation error. Whether or not this solution is stabilized depends on synchrony between the system and the internal model. This synchrony is achieved by another part of the controller, usually called a stabilizer. Apart from this conceptual similarity, there is also similarity in techniques and methods employed in these problems, see, for instance, [112, 124], and the works on the convergence property [111, 114], which is the key property in terms of stabilization in both the output regulation problem and the synchronization problem.

The remainder of this chapter is organized as follows. The controlled synchronization problem and the proposed design method are explained in section 9.2. In section 9.3 preliminary definitions and results on semipassivity and incremental passivity are presented. Section 9.4 contains the main theoretical results and section 9.5 illustrates the developed theory with an example of two nonlinearly coupled Hindmarsh-Rose oscillators. Section 9.6 concludes this chapter.

9.2 Controlled synchronization problem

Consider systems of the form

$$\dot{x}_i(t) = f(x_i(t), u_i(t)) \tag{9.1a}$$

$$y_i(t) = h(x_i(t)), \quad i = 1, \ldots, k, \tag{9.1b}$$

with state $x_i \in \mathbb{R}^n$, output $y_i \in \mathbb{R}$, input $u_i \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R})$ and sufficiently smooth functions $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$. The systems (9.1) are assumed to satisfy the standard assumptions on existence and uniqueness of solutions and exhibit, for $u_i(t) \equiv 0$, some bounded oscillatory dynamics. The problem of controlled synchronization of such systems that will be studied is to find coupling functions

$$u_i(t) = G_i(y_1(t), \ldots, y_k(t)), \quad i = 1, \ldots k, \tag{9.2}$$

that interconnect k identical systems of the form (9.1) such that for arbitrary initial conditions $x_1(0), \ldots x_k(0)$ all solutions of the closed-loop system (9.1), (9.2) are well-defined and synchronize in the sense of Definition 3.1, i.e. for all initial conditions,

$$|x_i(t) - x_j(t)| \to 0, \text{ as } t \to \infty, \quad \forall i, j. \tag{9.3}$$

Moreover, for identical outputs $y_1(t) = y_2(t) = \ldots = y_k(t) =: y_s(t)$ the coupling functions are supposed to vanish,

$$G_i(y_s(t), \ldots, y_s(t)) = 0, \quad i = 1, \ldots, k. \tag{9.4}$$

Of course, if the coupling functions all vanish when the systems are synchronized, then the k interconnected systems exhibit, in exact synchrony, the oscillatory dynamics of the original unforced system: (9.1) with $u_i(t) \equiv 0$.

First a particular case of this problem corresponding to synchronization of two interconnected systems (9.1) will be considered. In this case a coupling is proposed in the following integral form

$$u_1(t) = \int_{y_1(t)}^{y_2(t)} \lambda(s)\mathrm{d}s, \quad u_2(t) = \int_{y_2(t)}^{y_1(t)} \lambda(s)\mathrm{d}s, \tag{9.5}$$

with weight function $\lambda \in \mathcal{C}(\mathbb{R}, \mathbb{R}_{\geq 0})$. For a class of nonlinear systems, a design procedure will be proposed to find the weight function λ that will guarantee asymptotic synchronization of the coupled systems' states for arbitrary initial conditions. Note that for a constant function $\lambda(s) = \lambda$ for all $s \in \mathbb{R}$, the integral coupling (9.5) becomes the linear non-delayed diffusive coupling $u_1(t) = \lambda(y_2(t) - y_1(t))$, $u_2(t) = \lambda(y_1(t) - y_2(t))$. Linear coupling has the benefit that it is simple and uniform over various values of $y_1(t)$ and $y_2(t)$. But the systems' nonlinearities are not the same throughout the state space and, for some values of the outputs $y_1(t)$ and $y_2(t)$, one could use a lower gain λ than for the other values and still achieve asymptotic synchronization. The proposed integral coupling (9.5) overcomes the lack of versatility of the linear coupling. It allows one, through shaping the weight function $\lambda(s)$, to adjust the coupling gain depending on $y_1(t)$ and $y_2(t)$. (In this case the gain is understood as the ratio $\int_{y_1(t)}^{y_2(t)} \lambda(s)\mathrm{d}s/(y_2(t) - y_1(t))$.) A smart choice of λ may lead to gain reduction, at least in some average sense, which in turn may improve sensitivity of the closed-loop system to noise.

9.3 Technical preliminaries

In this section some technical results are presented that will be useful for to derive the main result.

Lemma 9.1. *Consider two systems*

$$\dot{x}_1(t) = f_1(x_1(t), u_1(t)), \quad y_1(t) = h_1(x_1(t)), \tag{9.6}$$
$$\dot{x}_2(t) = f_2(x_2(t), u_2(t)), \quad y_2(t) = h_2(x_2(t)). \tag{9.7}$$

Suppose both systems are strictly C^1-semipassive with radially unbounded storage functions $V_1(x_1(t))$ and $V_2(x_2(t))$. Then all solutions of these systems interconnected with the integral coupling (9.5) *with a nonnegative continuous weight function $\lambda(s) \geq 0$, $s \in \mathbb{R}$, are defined and bounded on the infinite time interval $t \geq 0$. Moreover, there exists a radially unbounded nonnegative function $W(x_1(t), x_2(t))$ and a constant $c_* \geq 0$ such that for each $c \geq c_*$ the set $\{\mathrm{col}\,(x_1, x_2) \in \mathbb{R}^{2n} | W(x_1, x_2) \leq c\}$ is a compact positively invariant set with respect to* (9.6), (9.7), (9.5).

Proof of Lemma 9.1. Let $W(x_1(t), x_2(t)) := V_1(x_1(t)) + V_2(x_2(t))$, which is a radially unbounded function. Then the semipassivity assumption for the individual systems implies

$$\begin{aligned} \dot{W}(x_1(t), x_2(t)) &= \dot{V}_1(x_1(t)) + \dot{V}_2(x_2(t)) \\ &\leq y_1(t)u_1(t) - H_1(x_1(t)) + y_2(t)u_2(t) - H_2(x_2(t)) \\ &= (y_1(t) - y_2(t)) \int_{y_1(t)}^{y_2(t)} \lambda(s)\mathrm{d}s - H_1(x_1(t)) - H_2(x_2(t)). \end{aligned} \tag{9.8}$$

Notice that $\lambda(s) \geq 0$ for all $s \in \mathbb{R}$ implies $(y_1 - y_2)\int_{y_1}^{y_2} \lambda(s)\mathrm{d}s = -(y_2 - y_1)\int_{y_1}^{y_2} \lambda(s)\mathrm{d}s \leq 0$ for all $y_1, y_2 \in \mathbb{R}$. Hence,

$$\dot{W}(x_1(t), x_2(t)) \leq -H_1(x_1(t)) - H_2(x_2(t)) \leq 0, \forall\, |x_1(t)| \geq \rho_1,\ |x_2(t)| \geq \rho_2, \tag{9.9}$$

where $\rho_1 > 0$ and $\rho_2 > 0$ are the constants from the definition of the semipassivity property of systems (9.6) and (9.7), respectively. Hence, since the function W is radially unbounded, there exists $c_* > 0$ such that $\dot{W}(x(t)) \leq 0$ for all $x(t) = \mathrm{col}\,(x_1(t), x_2(t))$ satisfying $W(x(t)) \geq c \geq c_*$. Thus, the set $\{\mathrm{col}\,(x_1, x_2) \in \mathbb{R}^{2n} | W(x_1, x_2) \leq c\}$ is a compact positively invariant set. This implies, see e.g. [72], that all solutions $x(t)$ are defined for all $t \geq 0$ and bounded. □

Definition 9.1 (Incremental passivity [110]). A system (9.1) is called $\mathcal{C}^1$-incrementally-passive if there exists a function $\Delta V \in \mathcal{C}^1(\mathbb{R}^{2n}, \mathbb{R}_{\geq 0})$ such that

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\Delta V(x_1, x_2) &= \frac{\partial \Delta V}{\partial x_1} f(x_1, u_1) + \frac{\partial \Delta V}{\partial x_2} f(x_2, u_2) \\ &\leq (y_1 - y_2)^\top (u_1 - u_2) - \Delta W(x_1 - x_2), \end{aligned} \tag{9.10}$$

for all $x_1, x_2 \in \mathbb{R}^n$, $u_1, u_2, y_1, y_2 \in \mathbb{R}$ and some $\Delta W \in \mathcal{C}(\mathbb{R}^n, \mathbb{R}_{\geq 0})$. The system (9.1) is called strictly $\mathcal{C}^1$-incrementally-passive if $\Delta W(\cdot)$ is positive definite. △

The main property of incrementally-passive systems that will be used in this chapter is formulated in the next lemma.

Lemma 9.2. *Consider two identical systems*

$$\dot{x}_1(t) = \tilde{f}(x_1(t), v_1(t)), \quad y_1(t) = h(x_1(t)), \tag{9.11a}$$
$$\dot{x}_2(t) = \tilde{f}(x_2(t), v_2(t)), \quad y_2(t) = h(x_2(t)), \tag{9.11b}$$

that are interconnected through the integral coupling

$$v_1(t) = \int_0^{y_2(t)} \lambda(s)\mathrm{d}s, \quad v_2(t) = \int_0^{y_1(t)} \lambda(s)\mathrm{d}s, \tag{9.12}$$

with $\lambda(s) \geq 0$ for all $s \in \mathbb{R}$. Suppose that there is a constant $c_* > 0$ and a non-negative radially unbounded function $W(x_1(t), x_2(t)) \geq 0$ such that for each $c \geq c_*$ the set $W(x_1(t), x_2(t)) \leq c$ is compact and positively invariant with respect to the interconnected systems (9.11), (9.12)*. Then if each system* (9.11) *is strictly C^1-incrementally-passive with respect to input $v_i(t)$ and output $y_i(t)$, $i = 1, 2$, then each solution of the interconnected systems* (9.11), (9.12) *is defined and bounded for all $t \in [t_0, \infty)$ and satisfies*

$$x_1(t) - x_2(t) \to 0 \ \text{ as } \ t \to \infty. \tag{9.13}$$

Proof of Lemma 9.2. From the strict incremental-passivity of each system (9.11) it can be concluded that the time derivative of the function $\Delta V(x_1, x_2)$ along any solution (x_1, x_2) of the interconnected systems (9.11), (9.12) satisfies

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\Delta V \leq (y_1 - y_2)(v_1 - v_2) - \Delta W(x_1 - x_2) \tag{9.14}$$

for some positive definite function $\Delta W(\cdot)$. Note that since $\lambda(s) \geq 0$ in (9.12),

$$(y_1 - y_2)(v_1 - v_2) = -(y_1 - y_2)\int_{y_2}^{y_1} \lambda(s)\mathrm{d}s \leq 0 \tag{9.15}$$

for all y_1, y_2. Hence, since $\Delta W(x)$ is positive definite,

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\Delta V \leq -\Delta W(x_1 - x_2) < 0, \tag{9.16}$$

for all $x_1, x_2 \in \mathbb{R}^n$, $x_1 \neq x_2$. Thus, applying LaSalle's invariance principle [72] to the system (9.11), (9.12) in any compact positively invariant set $W(x_1, x_2) \leq c$ for arbitrary $c \geq c_*$, one concludes that any solution of the interconnected system (9.11), (9.12) is defined and bounded on the interval $t \geq t_0$. Moreover, it also implies that for any solution of the interconnected system (9.11), (9.12), $\Delta W(x_1(t) - x_2(t)) \to 0$ as $t \to +\infty$, which, in turn, implies (9.13). □

Remark 9.1. If the storage function $\Delta V(x_1, x_2)$ is taken in the form $\Delta V(x_1, x_2) = \tilde{V}(x_1 - x_2)$ for some positive definite function $\tilde{V} \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}_{\geq 0})$, one can easily show that for each $c \geq c_*$ there exists a class-$\mathcal{KL}$ function[1] $\beta_c(r, t)$ such that

$$|x_1(t) - x_2(t)| \leq \beta_c(|x_1(t_0) - x_2(t_0)|\,, t), \tag{9.17}$$

for every $x_1(t_0), x_2(t_0)$ satisfying $W(x_1(t_0), x_2(t_0)) \leq c$. In particular, if $\Delta V(x_1, x_2) = (x_1 - x_2)^\top P(x_1 - x_2)$ and $\Delta W(x_1 - x_2) = (x_1 - x_2)^\top R(x_1 - x_2)$ for some positive definite matrices $P = P^\top > 0$ and $R = R^\top > 0$ that do not depend on c, then there exist $\mu > 0$ and $\nu > 0$ such that any solution of (9.11), (9.12) is bounded and satisfies

$$|x_1(t) - x_2(t)| \leq \mu e^{-\nu t}\, |x_1(t_0) - x_2(t_0)|\,, \tag{9.18}$$

for all $t \geq t_0$. △

As a tool to determine incremental-passivity of a system, the following lemma, which is a minor modification of a result from [110], is provided.

[1] The function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{KL}$ function if, for each $t \geq t_0$, $\beta(\cdot, t)$ is nondecreasing and $\lim_{s\to 0} \beta(s, t) = 0$, and for each $s \geq 0$,$\beta(s, \cdot)$ is non-increasing and $\lim_{t\to\infty} \beta(s, t) = 0$. See, for instance, [72]

Lemma 9.3. *Consider the system*

$$\dot{x}(t) = \tilde{f}(x(t)) + Bv(t), \quad y(t) = Cx(t), \tag{9.19}$$

with state $x \in \mathbb{R}^n$, output $y \in \mathbb{R}$, input $u \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R})$ and function $\tilde{f} \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$. If there exist matrices $P = P^\top > 0$ and $R = R^\top > 0$ such that for every $x \in \mathbb{R}^n$,

$$P\frac{\partial \tilde{f}}{\partial x}(x) + \frac{\partial \tilde{f}^\top}{\partial x}(x)P \leq -R, \tag{9.20a}$$

$$PB = C^\top, \tag{9.20b}$$

then system (9.19) is incrementally-passive with $\Delta V(x_1, x_2) = (x_1 - x_2)^\top P(x_1 - x_2)$ and $\Delta W(x_1 - x_2) = (x_1 - x_2)^\top R(x_1 - x_2)$.

9.4 Controlled synchronization

This section considers the controlled synchronization problem for nonlinear systems of the form

$$\dot{x}_i(t) = f(x_i(t)) + Bu_i(t), \tag{9.21a}$$

$$y_i(t) = Cx_i(t), \tag{9.21b}$$

where $i = 1, 2, \dots, k$, state $x_i \in \mathbb{R}^n$, output $y_i \in \mathbb{R}$, input $u_i \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R})$, B and C are constant matrices of appropriate dimensions and function $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$.

9.4.1 Controlled synchronization of two systems

First, a network with only two coupled systems (9.21) is considered. The next theorem gives the main result.

Theorem 9.4. *Suppose that each system in (9.21) is strictly $\mathcal{C}^1$-semipassive and there exists a continuous function $\lambda(s) \geq 0$ for all $s \in \mathbb{R}$, such that*

$$\begin{aligned} P\frac{\partial f}{\partial x}(x) + \frac{\partial f^\top}{\partial x}(x)P - 2C^\top C\lambda(Cx) < -R, \\ PB = C^\top, \end{aligned} \tag{9.22}$$

for all $x \in \mathbb{R}^n$. Then all solutions of the systems (9.21) interconnected through integral coupling (9.5) with $\lambda(s)$ satisfying (9.22) are bounded and satisfy

$$|x_1(t) - x_2(t)| \leq \mu e^{-\nu t} |x_1(t_0) - x_2(t_0)|, \tag{9.23}$$

for all $t \geq t_0$ and some constants $\mu > 0$, $\nu > 0$.

Proof of Theorem 9.4. See appendix A.5.1. □

In the case when one coupling function is set to zero, the result of the previous theorem remains the same and synchronization is still achieved. It corresponds to the so-called controlled master-slave synchronization, which can be also considered as a variant of the output regulation problem, cf. [114].

Finding $\lambda(s)$ that satisfies condition (9.22) is, in general, not an easy task. In the next result a particular class of systems is considered for which one can find an explicit formula for $\lambda(s)$ satisfying the conditions of Theorem 9.4. Consider the following system in the normal form

$$\dot{z}(t) = q(z(t), y(t)), \tag{9.24a}$$

$$\dot{y}(t) = a(y(t), z(t)) + u(t), \tag{9.24b}$$

with output $y \in \mathbb{R}$, $z \in \mathbb{R}^{n-1}$, input $u \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R})$ and the functions $q(z(t), y(t))$ and $a(y(t), z(t))$ are continuously differentiable. Note that this system is of the form (9.21) with $x(t) = \operatorname{col}(z(t), y(t))$, $f(x(t)) = \operatorname{col}(q(z(t), y(t)), a(y(t), z(t)))$, $B = \begin{pmatrix} 0 & 1 \end{pmatrix}^\top$ and $C = \begin{pmatrix} 0 & 1 \end{pmatrix}$.

Theorem 9.5. *Consider the system* (9.24). *Suppose that there exist constant matrices* $Q = Q^\top > 0$ *and* $S = S^\top > 0$, $Q, S \in \mathbb{R}^{(n-1)\times(n-1)}$, *such that the inequality*

$$Q\frac{\partial q}{\partial z}(z, y) + \frac{\partial q}{\partial z}^\top(z, y)Q \leq -S \tag{9.25}$$

holds for all $z \in \mathbb{R}^{n-1}$ *and* $y \in \mathbb{R}$. *Suppose that there exists a continuous function* $\lambda(y) \geq 0$ *that satisfies*

$$\lambda(y) \geq \epsilon + \frac{\partial a}{\partial y}(y, z) + \frac{1}{2}\xi^\top(y, z)(S - \epsilon I_{n-1})^{-1}\xi(y, z) \tag{9.26}$$

$$\xi(y, z) := \left(Q\frac{\partial q}{\partial y}(z, y) + \frac{\partial a}{\partial z}^\top(y, z)\right) \tag{9.27}$$

for all $z \in \mathbb{R}^{n-1}$ *and* $y \in \mathbb{R}$ *and some constant* $\epsilon \in \mathbb{R}_{>0}$ *such that*

$$S - \epsilon I > 0. \tag{9.28}$$

Then $\lambda(y)$ *satisfies* (9.22).

Proof of Theorem 9.5. See appendix A.5.2. □

A function $\lambda(y)$ satisfying (9.26) can be found if the right-hand side of (9.26) is independent of z or can be bounded from above by a y-dependent function. Condition (9.25) guarantees that zero dynamics of system 9.24 (i.e. the z-dynamics) are convergent [111], which implies that for a given function y all solutions of the system $\dot{z}(t) = q(z(t), y(t))$ converge to a unique bounded globally asymptotically stable steady-state solution determined only by y. This property of the zero dynamics can be considered as a specific minimum-phase property of the overall system (9.24).

9.4.2 Controlled synchronization of multiple systems

This section considers k identical systems (9.21) which are interconnected by the all-to-all integral coupling of the form

$$u_i(t) = \sum_{j=i, j\neq i}^{k} \int_{y_i(t)}^{y_j(t)} \lambda(s)\mathrm{d}s, \quad i = 1, \dots, k. \tag{9.29}$$

The following results are counterparts of the corresponding results from the two-systems case.

Lemma 9.6. *Assume that each subsystem in (9.21) is strictly C^1-semipassive with a radially unbounded storage function V_i. Then all solutions of the network of k interconnected systems (9.21), (9.29) with $\lambda(s) \geq 0$ for all $s \in \mathbb{R}$ are defined and bounded over the infinite time interval $t \geq t_0$.*

The proof of this lemma follows the same line of reasoning as the proof of Lemma 9.1 with the overall storage function equal to $W := \sum_{i=1}^{k} V_i$.

Theorem 9.7. *Consider k identical strictly C^1-semipassive systems (9.21). Suppose there exist $P = P^\top > 0$ and $R = R^\top > 0$ such that for all $x \in \mathbb{R}^n$*

$$\begin{aligned} &P\frac{\partial f}{\partial x}(x) + \frac{\partial f}{\partial x}^\top(x)P - 2(k-1)C^\top C\lambda(Cx) \leq -R, \\ &PB = C^\top. \end{aligned} \tag{9.30}$$

Then all solutions of the k interconnected systems (9.21), 9.29 are bounded and satisfy

$$|x_i(t) - x_j(t)| \leq \mu e^{-\nu t}\, |x_i(t_0) - x_j(t_0)|\,, \quad \forall i, j \in \{1, 2, \dots, k\} \tag{9.31}$$

for all $t \geq t_0$ and for some $\nu > 0$, $\mu > 0$.

Proof of Theorem 9.7. The proof is presented in section A.5.3 of appendix A. □

Finally, to apply Theorem 9.7 to systems of the form (9.24), one can still use Theorem 9.5 with the only modification that in the case of k systems the right-hand side of (9.26) should be divided by $(k-1)$.

9.5 An example

As an example, consider again the Hindmarsh-Rose oscillator

$$\dot{z}_1(t) = c - dy^2(t) - z_1(t), \tag{9.32a}$$
$$\dot{z}_2(t) = r(s(y(t) + y_0) - z_2(t)), \tag{9.32b}$$
$$\dot{y}(t) = -ay^3(t) + by^2(t) + z_1(t) - z_2(t) + E_m + u(t). \tag{9.32c}$$

See chapter 6, section 6.2. Using the theory developed in the previous sections a *nonlinear* integral coupling (9.5) will be derived that guarantees global exponential synchronization of two identical oscillators (9.32). As proven in chapter 6, the Hindmarsh-Rose oscillator (9.32) is strictly $\mathcal{C}^1$-semipassive Moreover, this system is in the form (9.24). Hence Theorem 9.5 can be used to find the desired weight function $\lambda(y) \geq 0$.

First, note that condition (9.25) is satisfied with

$$Q = \frac{1}{2}\begin{pmatrix} \gamma & 0 \\ 0 & \eta \end{pmatrix}, \quad S = \begin{pmatrix} \gamma & 0 \\ 0 & \eta r \end{pmatrix}, \tag{9.33}$$

where γ and η are positive constants that will be determined somewhat later. The right-hand side of inequality (9.26) is independent of z. Hence one can find $\lambda(y) \geq 0$ from (9.26):

$$\lambda(y(t)) \geq \epsilon - 3ay^2(t) + 2by(t) + \frac{1}{2}\left(\frac{(1-\gamma dy(t))^2}{\gamma - \epsilon} + \frac{(\eta\epsilon s/2 - 1)^2}{2\eta r - \epsilon}\right), \tag{9.34}$$

where $\epsilon > 0$ is an arbitrary parameter satisfying $\epsilon < 2\eta r$ and $\epsilon < \gamma$ due to (9.28).

The gain function (9.34) will now be minimized. Minimization of $\lambda(y)$ is desirable to reduce noise sensitivity, for example. First, let $\eta = 2/(s\epsilon)$ such that (9.34) becomes

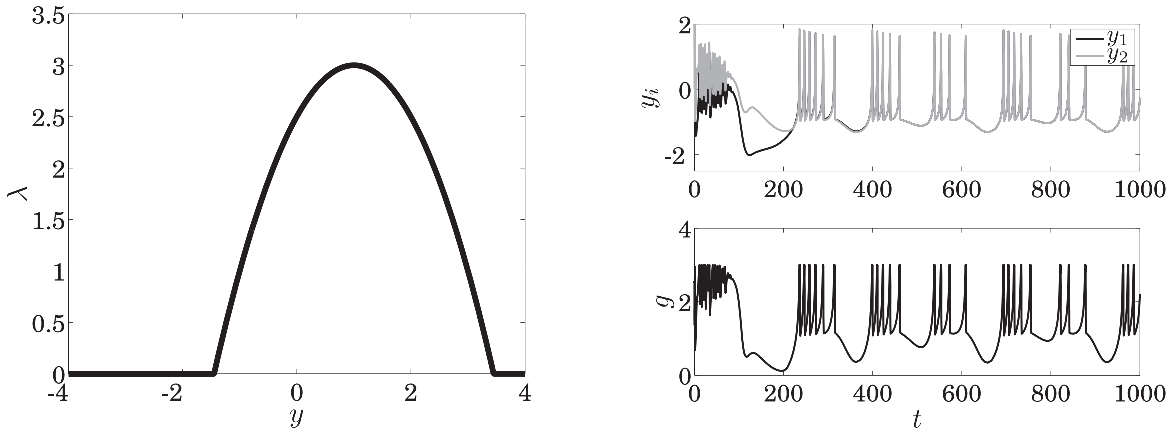
$$\lambda(y) \geq \epsilon - 3ay^2 + 2by + \frac{(1-\gamma dy)^2}{2(\gamma - \epsilon)}. \tag{9.35}$$

Note that the right-hand side of (9.35) is a quadratic function of y. Choose $0 < \gamma < 6a/d^2$, then for all sufficiently small $\epsilon > 0$ the right-hand side of (9.35) has a global maximum, which depends on γ. Further optimization of γ within the set $(0,\ 6a/d^2)$ allows minimization of this global maximum. This can be done analytically, but here the optimization is done using a simple MATLAB code. The parameter ϵ should be taken very small. After finding an optimal γ and choosing a small ϵ, $\lambda(y)$ is defined as

$$\lambda(y) = \max\left\{0, \epsilon - 3ay^2 + 2by + \frac{(1-\gamma dy)^2}{2(\gamma - \epsilon)}\right\}, \tag{9.36}$$

since $\lambda(y)$ has to be nonnegative.

For simulations of the two coupled Hindmarsh-Rose oscillators the parameters presented in chapter 6 will be used: $a = 1$, $b = 3$, $c = 1$, $d = 5$, $s = 4$, $E_m = 3.25$, $y_0 = 1.618$, $r = 0.005$. For these values of parameters the optimal value of γ (for $\epsilon = 0$) equals $\gamma = 0.2$. The corresponding $\lambda(y)$ is shown in Figure 9.1(a). In fact, as follows from (9.35), one can also choose $\lambda(y)$ to be constant, which corresponds to conventional linear diffusive coupling. See also chapter 6. As follows from Figure 9.1(a), the lower bound for constant λ that guarantees global exponential synchronization equals 3. This is already a significant improvement compared to the results from [109], where such a lower bound is computed to be equal to 10.75 (for the chosen values of parameters).



(a) (b)

Figure 9.1. Controlled synchronization of two Hindmarsh-Rose oscillators via nonlinear integral coupling: (a) nonlinear gain function $\lambda(y(t))$ and (b) synchronized outputs of the systems and the variable gain $g(t)$.

Still, the main benefit of the *nonlinear* integral coupling is the fact that it works like a coupling with a variable gain. It makes the gain high where necessary (e.g. where the nonlinearities are severe) and low where it is possible without compromising synchronization. This is clearly demonstrated in Figure 9.1(b), where simulation results for two coupled systems starting at the initial states $[3, 0, 2]^\top$ and $[10, 5, 0]^\top$ are presented. The upper plot depict the outputs of the systems, while the lower plot shows the variable gain $g(t)$ of the nonlinear integral coupling. (This gain is defined as $g(t) = \int_{y_1(t)}^{y_2(t)} \lambda(s)\mathrm{d}s/(y_2(t) - y_1(t))$.) The gain varies from 3 down to 0.055. The average value of the gain over the final 700 time units is 1.27. In fact, in a number of simulations performed for various initial conditions, the average gain computed over intervals longer than 3000 time units after synchrony had been achieved was never higher than 1.3. This value is even lower than an estimate of the lower bound for the constant coupling gain that guarantees *local* exponential synchronization found in [109], which equals 1.5. As follows from Figure 9.1(a), at some parts of the state space, e.g. when both y_1 and y_2 lie to the left or to the right of the parabola in Figure 9.1(a), no coupling is needed at all to maintain convergence of the system states to each other regardless of the distance between y_1 and y_2. This intriguing phenomenon has been observed in several simulations including one (with system parameters as in [109]) in which zero-coupling phenomenon occurred not only in transients, but even on the attractor. Lower gain means lower sensitivity to noise. This example demonstrates the advantages of the proposed approach to synchronization based on nonlinear integral coupling.

9.6 Discussion

In this chapter the controlled synchronization problem for a class of nonlinear systems is considered. It has been shown that the proposed nonlinear integral coupling guarantees global exponential synchronization of two systems (either unidirectionally or bidirectionally coupled) and of k systems with the all-to-all interconnection topology. A systematic procedure for finding such nonlinear couplings in the integral form is presented. The performance of the proposed method is successfully verified with simulations of two Hindmarsh-Rose oscillators. Through this case study it has been demonstrated that the nonlinear integral coupling may lead to lower (in average) coupling gains while preserving synchronization. This, in turn, may lead to improved noise sensitivity characteristics of the overall system. The results presented in this chapter hold when the systems are all-to-all connected. Of course, this assumption is rather restrictive and it would be interesting to extend the results to a (more) general network topology. In addition, it would be interesting to take possible time-delays into account.

Part III

Epilogue

Chapter ten

Conclusions and recommendations

Abstract. This final chapter summarizes the main results that are presented in this thesis. In addition, recommendations for future research are given.

10.1 Conclusions

In this thesis synchronization of coupled systems has been discussed. A theoretical explanation for synchronization in networks of diffusively time-delay coupled semipassive systems is presented. Diffusive time-delay interaction is an important type of coupling which is found in, for instance, networks of coupled neurons, networks of biological systems and coupled mechanical systems and electrical systems. Sufficient conditions for full synchronization and partial synchronization in such networks are derived, and the relation between the conditions for full synchronization and the topology of the network is studied. As a particular application of the theory, network of neurons which interact via so-called gap junctions have been considered. Synchronization in such networks is often related to specific brain functions or pathological tremors. The theory is also used, in combination with a fast-slow bifurcation analysis, to explain the stop of activity in networks of interacting “dead” and “alive” pancreatic β-cells. This problem is closely related to diabetes mellitus. In addition, using the same mathematical tools, a systematic design procedure is presented for finding nonlinear coupling functions that guarantee synchronization in networks of all-to-all coupled semipassive systems.

In the first part of this thesis, i.e. chapters 3, 4 and 5, the focus is on the theoretical framework that provides sufficient conditions for synchronization in networks of systems that interact via diffusive coupling. Two different types of interaction are considered: *coupling type I*, which is diffusive coupling in which time-delays appear only in the received output signals, and *coupling type II*, in which every signal in the coupling contains a time-delay. An important difference between the two types of coupling is that coupling type II vanishes if the systems are synchronized whereas coupling type I does not vanish in general. Sufficient conditions are given for both full synchronization and partial synchronization and a relation between the network topology and full synchronization is established for systems that interact via coupling type II.

In more detail, in chapter 3 sufficient conditions are given for full synchronization of systems that interact via coupling type I or coupling type II. It is proven that systems that are strictly semipassive and (nonlinear) minimum-phase always synchronize when the coupling is sufficiently strong and the product of the coupling strength and the (maximal) time-delay is sufficiently small. The result applies to both coupling type I and coupling type II.

In chapter 4 sufficient conditions for partial synchronization of strictly semipassive minimum-phase systems are presented. First, partial synchronization is discussed for coupling type I or coupling type II with uniform time-delays, i.e. all time-delays are the same. Exploiting the existence of symmetries in the network, it is shown that partial synchronization might happen when the coupling strength is smaller than the coupling strength that is required to have full synchronization. Partial synchronization might also be witnessed if the product of the time-delay and the coupling strength is larger than the value for which the systems fully synchronize. At the end of the chapter some results are presented for systems that interact via a particular type of coupling type I with non-uniform time-delays.

Chapter 5 discusses how the topology of the network influences the synchronization of systems that interact via symmetric coupling type II. A method is presented that uses the knowledge of the values of the coupling strength and the time-delay for which two systems *locally* synchronize to predict the values of the coupling strength and time-delay for which there is *local* synchronization in networks that consist of more than two systems. The results become global for strictly semipassive minimum-phase systems.

In the second part of this thesis the theoretical results that are derived in the first part are applied to network of neurons. First, in chapter 6, it is proven that four popular models of neuronal activity do have the strict semipassivity property. In addition, each model is minimum-phase. These results are important since they explain the (experimentally) observed synchronization of neurons that are coupled via electrical synapses (which can be modeled by diffusive coupling).

Chapter 7 presents examples in networks of synchronization of diffusively coupled Hindmarsh-Rose neurons. It is proven in chapter 6 that the Hindmarsh-Rose model is minimum-phase and strictly semipassive, hence the theoretical framework presented in the first part of this thesis can be applied to analyze synchronization in these networks. Simulations support the theory for neurons that interact via coupling type I. The results for coupling type II are verified using an experimental setup with electronic Hindmarsh-Rose neurons. Even though these electrical neurons are not completely identical, hence it can not be expected that they perfectly synchronize, the experimental results can be fully explained by the presented theory.

In chapter 8 a model of a network of pancreatic β-cells is considered. It is known that the regulation of the blood glucose level is related to the synchronized bursting activity of the diffusively coupled β-cells. The theory of the first part explains why the cells synchronize. In addition, results are presented for a network that consists of healthy cells, i.e. cells that show bursting activity, and dead cells, i.e. cells that show no activity. These results are closely related to what happens in the pancreas of people that suffer from diabetes mellitus. It is shown that all activity in the network dies out if the number of dead cells exceeds some threshold value (that depends on the coupling strength).

The results in chapter 9 go beyond linear diffusive coupling. In this chapter the *controlled synchronization* problem is discussed. A design procedure for the coupling functions is presented that guarantees synchronization in networks with all-to-all coupled strictly semipassive systems. The coupling functions that are derived have the form of a definite integral over a nonlinear weight function with the integration interval defined by the outputs of the coupled systems. These coupling functions have the property that they provide a high gain when necessary, e.g. to suppress nonlinearities, and low gain otherwise.

10.2 Recommendations

In this section some recommendations are given for future research. Specific recommendations that are related to the content of each chapter can already be found in the discussion section at the end of each chapter.

The main assumptions in this thesis are that the systems are identical, the coupling functions are static and linear (except for chapter 9), the topology of the network is fixed and the time-delays are constant. So the general recommendation would be to extend the results to time-varying network topologies, time-varying time-delays and general nonlinear dynamical coupling functions. However, it is unlikely that such a unifying general framework can be derived at once. In what follows, a couple of examples are presented that motivate future research in certain directions.

A platoon of vehicles. Consider as a first example a number of vehicles that have to move in a platoon. See, for instance, [99, 102]. As already mentioned in the introduction, letting vehicles riding in a platoon has the advantage that a significant reduction of aerodynamics drag is possible, resulting in lower fuel consumption, and, due to the smaller intervehicle distance, a larger number of vehicles can make use of the same piece of road. It is important to note that the vehicles are not identical. In [99] a platoon is considered in which each vehicle can communicate with (only) the preceding vehicle. Conditions for stable *platooning* are derived in the frequency domain and experimental results with two vehicles of the same type are presented. In [102] experimental results are presented for vehicles of different types, e.g. trucks and passenger cars. The structure of the network in these works is considered to be fixed. However, in practical situations, every now and then some of the vehicles have to take an exit while other vehicles are joining the platoon. This results in a network topology that is definitely not fixed. Thus this particular application demands conditions for synchronization of *non-identical* systems on a *time-varying* graph. In addition, the synchronization of these vehicles should be considered as a controlled synchronization problem since coupling functions that guarantee synchronization have to be designed. The question how the controllers should be synthesized such that synchronization of non-identical systems under switching topologies will be achieved is still not answered.

Networks of neurons that interact via chemical synapses. There are examples of coupled systems for which the interaction is certainly not linear. As already briefly mentioned in section 6.5 of chapter 6, an example worthwhile investigating is a network of chemically coupled neurons and/or a network of neurons that interact via both chemical coupling and electrical synapses. The chemical coupling functions are clearly nonlinear since a neuron can only influence the neurons to which it is connected if its membrane potential is sufficiently large, cf. [143]. Since neural synchrony is often linked to specific brain functioning or a critical physiological state such as epilepsy, it is very important to be able to explain why there is synchrony. In [74, 75] synchronization of such coupled neurons without delays is analyzed using singular perturbation techniques. However, much is still unclear. For instance, the question how the structure of the neuronal network influences synchronization is not completely understood. (Some results on the role of the network topology for chemically coupled Hindmarsh-Rose neurons (without time-delays) are presented in [15].) Hence for this application (with this particular type of coupling) it would certainly be relevant to have a framework which is similar to the framework as presented in chapters 3, 4 and 5. The neurons in such network are also not perfectly identical which motivates again to pay attention to synchronization of non-identical systems. In addition, it is questionable if the definition of synchronization that is used in this thesis is the right one when considering synchronization of neurons. Indeed, synchronization is in this thesis understood as asymptotic matching of the states of all systems. However, if neurons synchronize then the neurons show correlated behavior for a while but not forever. This motivates to investigate the synchronization problem with *finite time* (practical) convergence.

Synchronization under communication constraints In chapter 7, section 7.5, it is mentioned that saturations of the digital-analog-converters in the synchronization interface make it impossible to have synchronization of the electronic Hindmarsh-Rose neurons for large values of the coupling strength. This is a nice example of a communication constraint that limits the synchronization of coupled systems. Every application has its communication constraints, e.g. the limited bandwidth that is available for the communication between systems or saturation and/or quantization of the transmitted signals. Communication constraints have of course their influence on the synchronization of the systems (and this influence is most likely not positive). Hence, it is important to take these communication constraints into account when controllers are designed that are supposed to let the systems synchronize. In [49, 50] synchronization of systems is analyzed with limitations on the transmission of the coupling signals. Centralized controlled synchronization, that is, synchronization of systems via a centralized controller over some communication network, can be considered as a Networked Control System (NCS). Conditions for stability and stabilization of NCSs with all kinds of communication constraints can be found in [58] and the references therein. However, there are still many open problems about (controlled) synchronization under communication constraints. Note that one important communication constraint is already discussed in this thesis, that is, a finite transmission speed induces (a small amount of) time-delay. However, in this thesis, the time-delays are assumed not to vary in time, while in practical applications the time-delays will probably never be perfectly constant. A pretty straightforward extension of the results presented in this thesis is to allow time-varying time-delays in the proposed framework.

Of course, there are many more open problems and interesting applications related to synchronization than the ones mentioned in these three examples. It is for future research to exploit the potential of synchronization in real-life applications and unveil the many still existing mysteries surrounding this interesting phenomenon.

Appendix A

Proofs

A.1 Proofs chapter 3

A.1.1 Proof of Corollary 3.4

Theorem 3.3 implies that the solutions enter a compact set in finite time. By assumption, the systems are strictly semipassive with a radially unbounded storage function $V(x_i)$. Let $c^* = \sup_{H(\xi)=0} V(\xi)$ and let $\Omega_i := \{x_i \in \mathbb{R}^n | V(x_i) \le c^*\}$. Clearly the set $\cup_{i=1}^k \Omega_i$ is positively invariant under the given dynamics. Define

$$V^*(x_i) = \begin{cases} 0, & \text{for } x_i \in \Omega_i, \\ V(x_i) - c^*, & \text{otherwise,} \end{cases} \tag{A.1}$$

and let

$$W^*(x_t(\theta)) = \nu_1 V^*(x_1(t)) + \ldots + \nu_k V^*(x_k(t)) + \tfrac{\sigma}{2} \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{E}_i} \nu_i a_{ij} \int_{-\tau_{ij}}^{0} |y_j(t+s)|^2 \, \mathrm{d}s. \tag{A.2}$$

Note that W is a locally Lipschitz continuous functional and $x_{t,i}(\theta)$ are continuous and bounded. Hence

$$\dot{W}^*(\phi) = \limsup_{h\to 0^+} \tfrac{1}{h} \left(W^*(x_h(\phi)) - W^*(\phi) \right) \tag{A.3}$$

exists. From the arguments in the proof of Theorem 3.3 it follows that $\dot{W}^* \le 0$ on any compact set. Moreover, $\dot{W}^* = 0$ only on $\cup_{i=1}^k \Omega_i \cup \mathcal{D}$, with $\mathcal{D} = \{y_{t,i}(\theta), y_{t,j}(\theta) \in \mathbf{C} | y_i(t) = y_j(t - \tau_{ij}) \text{ for all } i, j \in \mathcal{I}\}$. Note that the dynamics on $\mathcal{D}$ are simply that of an uncoupled system $\dot{x}_i(t) = f(x_i(t))$. Then, using a LaSalle type of argument ([56] §5.3, Theorem 3.1 and Theorem 3.2), it can be concluded that all solutions converge to $\cup_{i=1}^k \Omega_i$.

A.1.2 Proof of Theorem 3.6

The proof follows the lines of the proof of Theorem 3.3. Let

$$W(x(t)) = \sum_{i=1}^{k} \nu_i V_i(x_i(t)), \tag{A.4}$$

where ν_i are the entries of the vector ν which satisfies $\nu^\top L = 0$, and V_i are the storage functions of the systems. Then, by assumption,

$$\dot{W}(x(t)) \leq \sum_{i=1}^{k} \nu_i y_i^\top(t) u_i(t) - \nu_i H_i(x_i(t)), \tag{A.5}$$

with

$$\begin{aligned} \sum_{i=1}^{k} \nu_i y_i^\top(t) u_i(t) &= \sigma \sum_{i=1}^{k} \sum_{j \in \mathcal{E}_i} \nu_i a_{ij} y_i^\top(t) \left(y_j(t-\tau_{ij}) - y_i(t-\tau_{ji})\right) \\ &\leq \sigma \sum_{i=1}^{k} \sum_{j \in \mathcal{E}_i} \nu_i a_{ij} \left(|y_i(t)|^2 + \tfrac{1}{2} |y_j(t-\tau_{ij})|^2 + \tfrac{1}{2} |y_i(t-\tau_{ji})|^2\right). \end{aligned} \tag{A.6}$$

Define the functional

$$W_1(x_t(\theta)) = \tfrac{1}{2}\sigma \sum_{i=1}^{k} \sum_{j \in \mathcal{E}_i} \nu_i a_{ij} \left(\int_{-\tau_{ji}}^{0} |y_i(t+s)|^2 \, \mathrm{d}s + \int_{-\tau_{ij}}^{0} |y_j(t+s)|^2 \, \mathrm{d}s \right), \tag{A.7}$$

$\theta \in [-\tau^*, 0]$ with $\tau^* = \max_{i,j \in \mathcal{I}}(\tau_{ij})$, which is positive definite and

$$\dot{W}_1(x_t(\theta)) = \sigma \sum_{i=1}^{k} \sum_{j \in \mathcal{E}_i} \nu_i a_{ij} \left(|y_i(t)|^2 - |y_i(t-\tau_{ji})|^2 + |y_j(t)|^2 - |y_j(t-\tau_{ij})|^2\right). \tag{A.8}$$

Define the positive definite functional

$$W_2(x_t(\theta)) = W(x(t)) + \tfrac{1}{2} W_1(x_t(\theta)), \tag{A.9}$$

then it is easy to see that

$$\dot{W}_2(x_t(\theta)) \leq \sum_{i=1}^{k} \nu_i \left(-H_i(x_i(t)) + \sigma \sum_{j \in \mathcal{E}_i} a_{ij} \left(\tfrac{3}{2} |y_i(t)|^2 + \tfrac{1}{2} |y_j(t)|^2\right) \right). \tag{A.10}$$

Since $\nu^\top L = \nu^\top (D - A) = 0$ implies that $\nu^\top D \xi = \nu^\top A \xi$ for any vector $\xi \in \mathbb{R}^k$ it follows that

$$\dot{W}_2(x_t(\theta)) \leq \sum_{i=1}^{k} \nu_i \left(2\sigma d_i |y_i(t)|^2 - H_i(x_i(t))\right). \tag{A.11}$$

The result follows again from the arguments presented in [127].

A.1.3 Proof of Theorem 3.10

Assumption (H3.3) implies that the diagonal set $\mathcal{M}$ is invariant under the closed-loop dynamics. See Proposition 3.9. In what follows a Lyapunov-Razumikhin function will be constructed to prove that (a subset of) $\mathcal{M}$ is globally attracting.

First, assumption (H3.1) implies, by Theorem 3.3, that the closed-loop system is $\mathcal{L}$-dissipative. Let $M \in \mathbb{R}^{k\times k}$,

$$M = \begin{pmatrix} 1 & 0 \\ \mathbf{1} & -I \end{pmatrix}, \tag{A.12}$$

with $\mathbf{1} := \operatorname{col}(1,\dots,1)$, and define new coordinates $\hat{z}(t) = (M \otimes I)z(t)$, $\hat{y}(t) = (M \otimes I)y(t)$, with $z(t) = \operatorname{col}(z_1(t),\dots,z_k(t))$ and $y(t) = \operatorname{col}(y_1(t),\dots,y_k(t))$. Note that $\hat{z}_1(t) = z_1(t), \hat{z}_2(t) = (z_1 - z_2)(t),\dots,\hat{z}_k(t) = (z_1 - z_k)(t)$ and $\hat{y}_1(t) = y_1(t), \hat{y}_2(t) = (y_1 - y_2)(t),\dots,\hat{y}_k(t) = (y_1 - y_k)(t)$. Define $\tilde{z}(t) = \operatorname{col}(\hat{z}_2(t),\dots,\hat{z}_k(t))$ and $\tilde{y}(t) = \operatorname{col}(\hat{y}_2(t),\dots,\hat{y}_k(t))$ and note that $\tilde{z}(t) \equiv 0$ and $\tilde{y}(t) \equiv 0$ implies, together with the ultimate boundedness of solutions, that the systems synchronize.

Assumption (H3.2) implies that there exists a positive definite function $V_1 \in \mathcal{C}^2(\mathbb{R}^{(k-1)p}, \mathbb{R}_{\geq 0})$ such that

$$\dot{V}_1(\tilde{z}(t),\tilde{y}(t))\Big|_{\tilde{y}(t)\equiv 0} \leq -\alpha\,|\tilde{z}(t)|^2, \tag{A.13}$$

for some positive constant α. Then smoothness of the functions $q(\cdot,\cdot)$ and boundedness of the solutions of the closed-loop system implies

$$\dot{V}_1(\tilde{z}(t),\tilde{y}(t)) - \dot{V}_1(\tilde{z}(t),0) \leq c_1\,|\tilde{z}(t)| \cdot |\tilde{y}(t)|, \tag{A.14}$$

for some positive constant c_1.

Denote $u(t) = \operatorname{col}(u_1(t),\dots,u_k(t))$ such that

$$u(t) = -\sigma(I \otimes I)y(t) + \sigma(A \otimes I)y(t-\tau). \tag{A.15}$$

Using the continuity properties of the solutions, cf. [56], and Leibniz' rule, $y(t-\tau)$ can be written as

$$y(t-\tau) = y(t) - \int_{-\tau}^{0} \dot{y}(t+s)\mathrm{d}s. \tag{A.16}$$

Substitution of (A.16) into (A.15) yields

$$u(t) = -\sigma(L \otimes I)y(t) - \sigma(A \otimes I)\int_{-\tau}^{0} \dot{y}(t+s)\mathrm{d}s, \tag{A.17}$$

with Laplacian matrix $L = I - A$. Denote $\tilde{u}(t) = \operatorname{col}((u_1-u_2)(t),\dots,(u_1-u_k)(t))$, then

$$\tilde{u}(t) = -\sigma(\tilde{L} \otimes I)\tilde{y}(t) - \sigma(\tilde{A} \otimes I)\int_{-\tau}^{0} \dot{\tilde{y}}(t+s)\mathrm{d}s, \tag{A.18}$$

with

$$MLM^{-1} = \begin{pmatrix} 0 & * \\ \mathbf{0} & \tilde{L} \end{pmatrix}, \tag{A.19}$$

and $\tilde{A} = I - \tilde{L}$.Using (A.18) the closed-loop "error" system can be written as

$$\begin{aligned} \dot{\tilde{y}}(t) &= \tilde{a}(\tilde{y}(t), \tilde{z}(t), y_1(t), z_1(t)) - \sigma\tilde{y}(t) + \sigma(\tilde{A} \otimes I)\tilde{y}(t-\tau) && \text{(A.20)} \\ &= \tilde{a}(\tilde{y}(t), \tilde{z}(t), y_1(t), z_1(t)) - \sigma(L \otimes I)\tilde{y}(t) - \sigma(\tilde{A} \otimes I)\int_{-\tau}^{0} \dot{\tilde{y}}(t+s)\mathrm{d}s, && \text{(A.21)} \end{aligned}$$

where

$$\tilde{a}(\tilde{y}(t), \tilde{z}(t), y_1(t), z_1(t)) = \begin{pmatrix} a(y_1(t), z_1(t)) - a((y_1 - \tilde{y}_1)(t), (z_1 - \tilde{z}_1)(t)) \\ \vdots \\ a(y_1(t), z_1(t)) - a((y_1 - \tilde{y}_{k-1})(t), (z_1 - \tilde{z}_{k-1})(t)) \end{pmatrix}. \tag{A.22}$$

Substitution of (A.20) into (A.21) yields

$$\begin{aligned} \dot{\tilde{y}}(t) = {} & \tilde{a}(\tilde{y}(t), \tilde{z}(t), y_1(t), z_1(t)) - \sigma(L \otimes I)\tilde{y}(t) \\ & - \sigma(\tilde{A} \otimes I)\int_{-\tau}^{0} \tilde{a}(\tilde{y}(t+s), \tilde{z}(t+s), y_1(t+s), z_1(t+s))\mathrm{d}s \\ & + \sigma^2(\tilde{A} \otimes I)\int_{-\tau}^{0} \tilde{y}(t+s) - (\tilde{A} \otimes I)\tilde{y}(t+s-\tau)\mathrm{d}s. \end{aligned} \tag{A.23}$$

(This is the equivalent of (A.20) on the interval $t+\theta$, $\theta \in [-2\tau, 0]$.)

It follows directly from (A.19) that $\mathrm{spec}\left(\tilde{L}\right) = \mathrm{spec}\left(L\right) \setminus \{0\}$, hence $-\tilde{L}$ is a stable matrix, hence there exists a unique solution of the Lyapunov equation $(-\tilde{L})^\top P + P(-\tilde{L}) + Q = 0$ with $P = P^\top > 0$ and $Q = Q^\top > 0$. Let P be such that $\|P\| = 1$ and $(-\tilde{L})^\top P + P(-\tilde{L}) = -2\mu I$ for some constant $\mu > 0$, and consider the positive definite function

$$V_2(\tilde{y}(t)) = \tfrac{1}{2}\tilde{y}^\top(t)(P \otimes I)\tilde{y}(t). \tag{A.24}$$

Then

$$\dot{V}_2(\tilde{y}(t)) \leq \tfrac{1}{2}\tilde{y}^\top(t)(P \otimes I)\left(\tilde{a}(\tilde{y}(t), \tilde{z}(t), y_1(t), z_1(t)) + \tilde{u}(t)\right), \tag{A.25}$$

Ultimate boundedness of solutions and smoothness of the functions $a(\cdot,\cdot)$ implies that

$$\tfrac{1}{2}\tilde{y}^\top(t)(P \otimes I)\left(\tilde{a}(\tilde{y}(t), \tilde{z}(t), y_1(t), z_1(t))\right) \leq c_2\left|\tilde{z}(t)\right| \cdot \left|\tilde{y}(t)\right| + c_3\left|\tilde{y}(t)\right|^2, \tag{A.26}$$

for some positive constants c_2, c_3. Since $\left\|\tilde{A}\right\| \leq 1$ by construction and $\|P\| = 1$ by definition, it follows from (A.23) that if $\left|\tilde{y}_t(\theta)\right| < \kappa\left|\tilde{y}(t)\right|$ for $\theta \in [-2\tau, 0]$ with some constant $\kappa > 1$,

$$\dot{V}_2(\tilde{y}(t)) \leq c_2(1 + \kappa\sigma\tau)\left|\tilde{z}(t)\right| \cdot \left|\tilde{y}(t)\right| + \left(c_3(1 + \kappa\sigma\tau) + 2\kappa\sigma^2\tau - \sigma\mu\right)\left|\tilde{y}(t)\right|^2. \tag{A.27}$$

(Note that for some vector ξ of appropriate dimension $|A\xi| \leq \|A\| \cdot |\xi|$ since $\|\cdot\|$ is an induced norm.) Let $V_3(\tilde{z}(t), \tilde{y}(t)) = V_1(\tilde{z}(t)) + V_2(\tilde{y}(t))$ be a Lyapunov-Razumikhin function such that if $V_1(\tilde{z}(t)) + \kappa^2 V_2(\tilde{y}(t)) > V_1(\tilde{z}_t(\theta)) + \kappa^2 V_2(\tilde{y}_t(\theta))$, $\theta \in [-2\tau, 0]$

$$\begin{aligned} \dot{V}_3(\tilde{z}(t), \tilde{y}(t)) \leq & -\alpha \left|\tilde{z}(t)\right|^2 + (c_1 + c_2(1 + \kappa\sigma\tau)) \left|\tilde{z}(t)\right| \cdot \left|\tilde{y}(t)\right| \\ & + \left(c_3(1 + \kappa\sigma\tau) + 2\kappa\sigma^2\tau - \sigma\mu\right) \left|\tilde{y}(t)\right|^2 . \end{aligned} \tag{A.28}$$

Denote $\gamma = \sigma\tau$, then some basic algebra shows that (A.28) is negative definite if

$$\mu > \frac{1}{\sigma} \left(\frac{(c_1 + c_2(1 + \kappa\gamma))^2}{4\alpha} + c_3(1 + \kappa\gamma \right) + 2\kappa\gamma, \tag{A.29}$$

hence it can be concluded that (A.28) is negative definite if σ is sufficiently large and γ is sufficiently small. Thus there exist constants $\bar{\sigma}$ and $\bar{\gamma}$ such that (A.28) is negative definite if $\sigma \geq \bar{\sigma}$ and $\sigma\tau \leq \bar{\gamma}$. Then ultimate boundedness of solutions and the Lyapunov-Razumikhin theorem (Theorem 2.4) implies that the set $\{\mathrm{col}\,(\tilde{y}, \tilde{z}) = 0\}$ is a global attractor if $\sigma \geq \bar{\sigma}$ and $\sigma\tau \leq \bar{\gamma}$.

A.1.4 Proof of Corollary 3.11

It will be proven that the solutions of the systems that interact via non-symmetric diffusive coupling are ultimately bounded. Note that

$$u(t) = -\sigma(L \otimes I)y(t). \tag{A.30}$$

By Lemma 3.2 there exists a vector ν with all entries positive. Each system is strictly semipassive with a radially unbounded storage function V. Define, as in the proofs of Theorems 3.3 and 3.6,

$$W(z(t), y(t)) = \nu_1 V(z_1(t), y_1(t)) + \ldots + \nu_k V(z_k(t), y_k(t)), \tag{A.31}$$

such that, by assumption,

$$\dot{W}(z(t), y(t)) \leq -\nu_1 H(z_1(t), y_1(t)) - \ldots - \nu_k H(z_k(t), y_k(t)) + \sum_{i=1}^{k} \nu_i y_i^\top(t) u_i(t). \tag{A.32}$$

It now suffices to prove that $\sum_{i=1}^{k} \nu_i y_i^\top(t) u_i(t) \leq 0$. Using (3.14) it follows that

$$\sum_{i=1}^{k} \nu_i y_i^\top(t) u_i(t) = \sum_{i=1}^{k} \nu_i y_i^\top(t) \sum_{j \in \mathcal{E}_i} a_{ij}(y_j(t) - y_i(t)) \tag{A.33}$$

$$= -\sum_{i=1}^{k} \nu_i d_i \left|y_i(t)\right|^2 + \sum_{i=1}^{k} y_i^\top(t) \sum_{j \in \mathcal{E}_i} \nu_i a_{ij} y_j(t). \tag{A.34}$$

Since $\nu^\top L = \nu^\top(D - A) = 0$ implies $\nu^\top D\xi = \nu^\top A\xi$ for any vector $\xi \in \mathbb{R}^k$, it follows that $\sum_{i=1}^{k} \nu_i y_i^\top(t) u_i(t) \leq 0$, hence the closed-loop system is $\mathcal{L}$-dissipative.

The proof of convergence to $\mathcal{M}$ follows now with minor modifications from the proof of Theorem 3.10 using $\tau \equiv 0$.

A.1.5 Proof of Theorem 3.12

Assumption (H3.4) implies that the diagonal set $\mathcal{M}$ is invariant under the closed-loop dynamics, and assumption (H3.5) gives a sufficient condition for the closed-loop system being $\mathcal{L}$-dissipative, see Theorem 3.6.

This first part of the proof follows the line of the proof of Theorem 3.10. Let, again, $M \in \mathbb{R}^{k\times k}$,

$$M = \begin{pmatrix} 1 & 0 \\ \mathbf{1} & -I \end{pmatrix}, \tag{A.35}$$

and define new coordinates $\hat{z}(t) = (M \otimes I)z(t)$, $\hat{y}(t) = (M \otimes I)y(t)$, with $z(t) = \operatorname{col}(z_1(t), \dots, z_k(t))$ and $y(t) = \operatorname{col}(y_1(t), \dots, y_k(t))$. Note that $\hat{z}_1(t) = z_1(t), \hat{z}_2(t) = (z_1 - z_2)(t), \dots, \hat{z}_k(t) = (z_1 - z_k)(t)$ and $\hat{y}_1(t) = y_1(t), \hat{y}_2(t) = (y_1 - y_2)(t), \dots, \hat{y}_k(t) = (y_1 - y_k)(t)$. Define $\tilde{z}(t) = \operatorname{col}(\hat{z}_2(t), \dots, \hat{z}_k(t))$ and $\tilde{y}(t) = \operatorname{col}(\hat{y}_2(t), \dots, \hat{y}_k(t))$. Like before, Assumption (H3.2) implies that there exists a positive definite function $V_1 \in \mathcal{C}^2(\mathbb{R}^{(k-1)p}, \mathbb{R}_{\geq 0})$ such that

$$\dot{V}_1(\tilde{z}(t), \tilde{y}(t)) \leq c_1 |\tilde{z}(t)| \cdot |\tilde{y}(t)| - \alpha |\tilde{z}(t)|^2, \tag{A.36}$$

for some positive constants α, c_1. See the proof of Theorem 3.10 for details.

Like in the proof of Theorem 3.10 the Lyapunov-Razumikhin Theorem will be used to prove the claim. Since for the Lyapunov-Razumikhin Theorem the maximum time-delay interval $[-\tau^*, 0]$ has to be evaluated to determines stability, it will be assumed that all time-delays $\tau_{ij} \equiv \tau^*$. (This is a worst case scenario.) Denote $u(t) = \operatorname{col}(u_1(t), \dots, u_k(t))$, then

$$u(t) = -\sigma(L \otimes I)y(t - \tau^*), \tag{A.37}$$

which, using (A.16), can be written as

$$u(t) = -\sigma(L \otimes I)y(t) + \sigma(L \otimes I)\int_{-\tau^*}^{0} \dot{y}(t+s)\mathrm{d}s. \tag{A.38}$$

Denote $\tilde{u}(t) = \operatorname{col}((u_1 - u_2)(t), \dots, (u_1 - u_k)(t))$, then

$$\tilde{u}(t) = -\sigma(\tilde{L} \otimes I)\tilde{y}(t) + \sigma(\tilde{L} \otimes I)\int_{-\tau^*}^{0} \dot{\tilde{y}}(t+s)\mathrm{d}s, \tag{A.39}$$

with $\tilde{L}$ as in (A.19).

Using (A.39) the closed-loop "error" system

$$\dot{\tilde{y}}(t) = \tilde{a}(\tilde{y}(t), \tilde{z}(t), y_1(t), z_1(t)) - \sigma(\tilde{L} \otimes I)\tilde{y}(t) + \sigma(\tilde{L} \otimes I)\tilde{y}(t - \tau^*) \tag{A.40}$$

can be written as

$$\begin{aligned}\dot{\tilde{y}}(t) &= \tilde{a}(\tilde{y}(t), \tilde{z}(t), y_1(t), z_1(t)) - \sigma(L \otimes I)\tilde{y}(t) \\ &+ \sigma(\tilde{L} \otimes I)\int_{-\tau^*}^{0} \tilde{a}(\tilde{y}(t+s), \tilde{z}(t+s), y_1(t+s), z_1(t+s))\mathrm{d}s \\ &+ \sigma^2(\tilde{L}^2 \otimes I)\int_{-\tau^*}^{0} \tilde{y}(t+s)\mathrm{d}s. \end{aligned} \tag{A.41}$$

with $\tilde{a}(\tilde{y}(t), \tilde{z}(t), y_1(t), z_1(t))$ as in (A.22). Let $P = P^\top > 0$ be such that $\|P\| = 1$ and $(-\tilde{L})^\top P + P(-\tilde{L}) = -2\mu I$ for some constant $\mu > 0$. Such P exists since $-\tilde{L}$ is a stable matrix. Consider the positive definite function

$$V_2(\tilde{y}(t)) = \tfrac{1}{2}\tilde{y}^\top(t)(P \otimes I)\tilde{y}(t), \tag{A.42}$$

and its derivative along the solutions of the closed-loop “error” system

$$\dot{V}_2(\tilde{y}(t)) \leq \tfrac{1}{2}\tilde{y}^\top(t)(P \otimes I)\left(\tilde{a}(\tilde{y}(t), \tilde{z}(t), y_1(t), z_1(t)) + \tilde{u}(t)\right), \tag{A.43}$$

Ultimate boundedness of solutions and smoothness of the functions $a(\cdot,\cdot)$ implies that

$$\tfrac{1}{2}\tilde{y}^\top(t)(P \otimes I)\left(\tilde{a}(\tilde{y}(t), \tilde{z}(t), y_1(t), z_1(t))\right) \leq c_2 |\tilde{z}(t)| \cdot |\tilde{y}(t)| + c_3 |\tilde{y}(t)|^2, \tag{A.44}$$

for some positive constants c_2, c_3. Since $\left\|\tilde{L}\right\| \leq c_4$, with c_4 being the square root of the maximal eigenvalue of $\tilde{L}^\top\tilde{L}$, and $\|P\| = 1$ by definition, it follows from (A.23) that if $|\tilde{y}_t(\theta)| < \kappa |\tilde{y}(t)|$ for $\theta \in [-2\tau^*, 0]$ with constant $\kappa > 1$,

$$\dot{V}_2(\tilde{y}(t)) \leq c_2(1+\kappa c_4\sigma\tau^*) |\tilde{z}(t)| \cdot |\tilde{y}(t)| + \left(c_3(1+\kappa c_4\sigma\tau^*) + \kappa c_4^2\sigma^2\tau^* - \sigma\mu\right) |\tilde{y}(t)|^2. \tag{A.45}$$

Let $V_3(\tilde{z}(t), \tilde{y}(t)) = V_1(\tilde{z}(t)) + V_2(\tilde{y}(t))$ be a Lyapunov-Razumikhin function such that if $V_1(\tilde{z}(t)) + \kappa^2 V_2(\tilde{y}(t)) > V_1(\tilde{z}_t(\theta)) + V_2(\tilde{y}_t(\theta))$, $\theta \in [-2\tau^*, 0]$

$$\begin{aligned}\dot{V}_3(\tilde{z}(t), \tilde{y}(t)) \leq{}& -\alpha |\tilde{z}(t)|^2 + (c_1 + c_2(1+\kappa c_4\gamma)) |\tilde{z}(t)| \cdot |\tilde{y}(t)| \\ &+ \left(c_3(1+\kappa c_4\gamma) + \kappa\sigma c_4^2\gamma - \sigma\mu\right) |\tilde{y}(t)|^2. \end{aligned} \tag{A.46}$$

with $\gamma = \sigma\tau^*$. After some simple algebra it can be concluded that (A.46) is negative definite if σ is sufficiently large and γ is sufficiently small, i.e. there exist constants $\bar{\sigma}$ and $\bar{\gamma}$ such that (A.46) is negative definite if $\sigma \geq \bar{\sigma}$ and $\sigma\tau^* \leq \bar{\gamma}$.

A.2 Proofs chapter 4

A.2.1 Proof of Theorem 4.2

It follows from (the proof of) Corollary 3.11 that assumption (H4.1) implies that the solutions of the closed-loop system are ultimately bounded, and the bounds are independent of the network topology. The proof follows now almost immediately from the proof of the main result in [125].

If Π is a permutation matrix and there is a solution X of the matrix equation $(I-\Pi)L = X(I_k - \Pi)$, then the set $\ker(I_{kn} - \Pi \otimes I_n)$ defines a linear invariant manifold for the closed-loop system, see Lemma 4.1. Note that $x \in \ker(I_{kn} - \Pi \otimes I_n)$ defines equations of the form

$$x_i - x_j = 0, \tag{A.47}$$

for some $i, j \in \mathcal{I}$. Let $\mathcal{I}_\Pi$ be set of pairs (i, j) for which (A.47) holds. Assumption (H4.2) implies that there exists a function $V_1(z)$ such that

$$\begin{aligned}
\dot{V}_1(z(t)) &= \sum_{(i,j)\in\mathcal{I}_\Pi} \frac{\partial V_1(z_i - z_j)(t)}{\partial z}[q(z_i(t), y_i(t) - q(z_j(t), y_j(t))] \\
&\leq -\alpha \sum_{(i,j)\in\mathcal{I}_\Pi} |(z_i - z_j)(t)|^2 \\
&+ \sum_{(i,j)\in\mathcal{I}_\Pi} \frac{\partial V_1(z_i - z_j)(t)}{\partial z}[q(z_j(t), y_i(t) - q(z_j(t), y_j(t))] \\
&\leq -\alpha \sum_{(i,j)\in\mathcal{I}_\Pi} |(z_i - z_j)(t)|^2 + c_0 |(z_i - z_j)(t)| \cdot |(y_i - y_j)(t)|,
\end{aligned} \tag{A.48}$$

for some positive numbers α, c_0. Let

$$V_2(y(t)) = \tfrac{1}{2} |(I_{km} - \Pi \otimes I_m)y(t)|^2 = \tfrac{1}{2} \sum_{(i,j)\in\mathcal{I}_\Pi} |(y_i - y_j)(t)|^2 . \tag{A.49}$$

Then the time derivative of V_2 along the trajectories of the closed-loop system are given as

$$\dot{V}_2(y(t)) = -U(y(t)) + \sum_{(i,j)\in\mathcal{I}_\Pi} (y_i - y_j)^\top(t)[a(y_i(t), z_i(t)) + a(y_j(t), z_j(t))], \tag{A.50}$$

with

$$U(y(t)) = \sigma \tfrac{1}{2} y^\top(t)(I_k - \Pi)^\top (X + X^\top)(I_k - \Pi)y(t). \tag{A.51}$$

Let $V = V_1 + V_2$. Note that $V = 0$ on $\ker(I_{kn} - \Pi \otimes I_n)$ and positive everywhere else. Then using the ultimate boundedness of solutions, smoothness of the functions $q(\cdot, \cdot)$ and $a(\cdot, \cdot)$, and assumption (H4.3), it follows that

$$\begin{aligned}
\dot{V}(z(t), y(t)) &\leq - \sum_{(i,j)\in\mathcal{I}_\Pi} \alpha |(z_i - z_j)(t)|^2 + c_1 \sum_{(i,j)\in\mathcal{I}_\Pi} |(z_i - z_j)(t)| \cdot |(y_i - y_j)(t)| \\
&+ (c_2 - \sigma\lambda') \sum_{(i,j)\in\mathcal{I}_\Pi} |(y_i - y_j)(t)|^2 ,
\end{aligned} \tag{A.52}$$

for some constants c_1, c_2 and λ' is the largest number for which the inequality in (H4.3) holds. (See the proof of Theorem 3.10 for details.) After some simple linear algebra it follows that $\dot{V} < 0$ if $\sigma \geq \sigma^*$, with σ^* the smallest number such that

$$\sigma^* > \frac{c_1^2}{4\alpha\lambda'} + c_2. \tag{A.53}$$

Hence the set $\ker(I_{kn} - \Pi \otimes I_n)$ contains a globally asymptotically stable subset if $\sigma \geq \sigma^*$.

A.2.2 Proof of Theorem 4.4

Denote, for notational convenience, $\Xi_n = I_{kn} - \Pi \otimes I_n$. Assumption (H4.1) implies, by Theorem 3.3 and Corollary 3.4, that the solutions of the closed-loop system are ultimately bounded, and the bounds are independent of the network topology.

Lemma 4.3 implies that the set $\ker(\Xi_n)$ is invariant under the closed-loop dynamics. Like in the proof of Theorem 4.2, assumption (H4.2) implies that there exists a positive definite function $V_1(\Xi_p z(t))$ such that

$$\dot{V}_1(\Xi_p z(t), \Xi_m y(t)) \leq -c_0 \left|\Xi_p z(t)\right|^2 + c_1 \left|\Xi_p z(t)\right| \cdot \left|\Xi_m y(t)\right|, \tag{A.54}$$

for some positive constants c_0, c_1. Let

$$V_2(\Xi_m y(t)) = \tfrac{1}{2} \left|\Xi_m y(t)\right|^2, \tag{A.55}$$

such that

$$\dot{V}_2(\Xi_m y(t)) = y^\top(t)\Xi_m^\top \Xi_m \dot{y}(t). \tag{A.56}$$

Note that

$$\Xi_m \dot{y}(t) = \Xi_m a(y(t), z(t)) - \sigma \Xi_m y(t) + \sigma(X \otimes I)\Xi_m y(t-\tau), \tag{A.57}$$

where, with some abuse of notation,

$$a(y(t), z(t)) = \begin{pmatrix} a(y_1(t), z_1(t)) \\ \vdots \\ a(y_k(t), z_k(t)) \end{pmatrix}. \tag{A.58}$$

Using the continuity properties of $y(t)$ and Leibniz' rule,

$$\begin{aligned} \Xi_m \dot{y}(t) &= \Xi_m a(y(t), z(t)) \\ &\quad - \sigma((I - X) \otimes I)\Xi_m y(t) - \sigma(X \otimes I) \int_{-\tau}^{0} \Xi_m \dot{y}(t+s) \mathrm{d}s, \end{aligned} \tag{A.59}$$

or, explicitly,

$$\begin{aligned} \Xi_m \dot{y}(t) &= \Xi_m a(y(t), z(t)) - \sigma((I - X) \otimes I)\Xi_m y(t) \\ &- \sigma(X \otimes I) \int_{t-\tau}^{t} \Xi_m a(y(s), z(s)) - \sigma \Xi_m y(s) + \sigma(X \otimes I)\Xi_m y(s-\tau) \mathrm{d}s. \end{aligned} \tag{A.60}$$

It follows that for some $\kappa > 1$, if $\kappa^2 V_2(\Xi_m y(t)) > V_2(\Xi_m y_t(\theta))$ for $\theta \in [-2\tau, 0]$,

$$\begin{aligned}\dot{V}_2(\Xi_p z(t), \Xi_m y(t)) \leq{}& c_2(1+\kappa\sigma\tau c_4)\left|\Xi_p z(t)\right| \cdot \left|\Xi_m y(t)\right| \\ &+ \left(c_3(1+\kappa\sigma\tau c_4) + \kappa\sigma^2\tau c_4(1+c_4) - \sigma\lambda'\right)\left|\Xi_m y(t)\right|^2, \end{aligned} \tag{A.61}$$

with positive constants c_2, c_3, constant $c_4 = \|X\|$ and λ' is the largest number such that the inequality in Assumption (H4.3) holds. Let $V_1(\Xi_p z(t)) + V_2(\Xi_m y(t))$ be a Lyapunov-Razumikhin candidate. It follows from the arguments presented above that, if $\kappa^2 V_2(\Xi_m y(t)) > V_2(\Xi_m y_t(\theta))$ for $\theta \in [-2\tau, 0]$,

$$\begin{aligned}\dot{V}_1(\Xi_p z(t),&\Xi_m y(t)) + \dot{V}_2(\Xi_p z(t), \Xi_m y(t)) \\ &\leq -c_0\left|\Xi_p z(t)\right|^2 + (c_1 + c_2(1+\kappa\sigma\tau c_4))\left|\Xi_p z(t)\right| \cdot \left|\Xi_m y(t)\right| \\ &+ \left(c_3(1+\kappa\sigma\tau c_4) + \kappa\sigma^2\tau c_4(1+c_4) - \sigma\lambda'\right)\left|\Xi_m y(t)\right|^2. \end{aligned} \tag{A.62}$$

Hence, there exist constants σ^* and γ^* such that (A.62) is negative definite if $\sigma \geq \sigma^*$ and $\sigma\tau \leq \gamma^*$. Thus it follows from the Lyapunov-Razumikhin Theorem that the set $\ker(\Xi_n)$ is a global attractor given that $\sigma \geq \sigma^*$ and $\sigma\tau \leq \gamma^*$. This implies, together with the ultimately bounded solutions of the closed-loop system, that if $\sigma \geq \sigma^*$ and $\sigma\tau \leq \gamma^*$ there exists a globally asymptotically stable subset of the set $\ker(\Xi_n)$.

A.2.3 Proof of Theorem 4.10

Without loss of generality, it will be assumed that $\eta_1 > 0$ and $\Pi_1 \in \mathcal{P}_{\Pi(k)}$. Lemma 4.9 implies that the set $\ker(I_{kn} - \Pi_1 \otimes I_n)$ is invariant under the closed-loop dynamics. Assumption (H4.1) implies, by Theorem 3.3 and Corollary 3.4, that the solutions of the closed-loop system are ultimately bounded and, moreover, the bounds do not depend on the network topology.

It will now be proven that the set $\ker(I_{kn} - \Pi_1 \otimes I_n)$ is globally attracting for appropriate values of σ and τ_1. Rewrite (4.18) as

$$u(t) = \sigma\eta_1\left((\Pi_1 \otimes I)y(t-\tau_1) - Iy(t)\right) + \sigma\sum_{\ell\setminus\{1\}}\eta_\ell\left((\Pi_\ell \otimes I)y(t-\tau_\ell) - Iy(t)\right). \tag{A.63}$$

Denote, for notational convenience, $\Xi_n = I_{kn} - \Pi_1 \otimes I_n$. Like the proofs of Theorems 4.2, 4.4 and 4.7, a Lyapunov-Razumikhin function of the form

$$V(\Xi_p z(t), \Xi_m y(t)) = V_1(\Xi_p z(t)) + V_2(\Xi_m y(t)) \tag{A.64}$$

will be constructed. Assumption (H4.2) implies that the function V_1 can be chosen such that

$$\dot{V}_1(\Xi_p z(t)) \leq -c_0\left|\Xi_p z(t)\right|^2 + c_1\left|\Xi_p z(t)\right| \cdot \left|\Xi_m y(t)\right|, \tag{A.65}$$

for some positive constants c_0, c_1. Let $V_2(\Xi_m y(t)) = \frac{1}{2} y^\top(t)\Xi_m^\top \Xi_m y(t)$ such that, using argument like in the proof of Theorem 4.2,

$$\dot{V}_2(\Xi_M y(t)) \leq c_2 |\Xi_p z(t)| \cdot |\Xi_m y(t)| + c_3 |\Xi_m y(t)|^2 + y^\top(t)\Xi_m^\top \Xi_m u(t), \tag{A.66}$$

for some positive constants c_2, c_3. Note that

$$\begin{aligned} y^\top(t)\Xi_m^\top \Xi_m u(t) &= \sigma\eta_1 y^\top(t)\Xi_m^\top (\Xi_m(\Pi_1 \otimes I)y(t-\tau_1) - \Xi_m y(t)) \\ &\quad + \sigma y^\top(t)\Xi_m^\top \sum_{\ell\backslash\{1\}} \eta_\ell (\Xi_m(\Pi_\ell \otimes I)y(t-\tau_\ell) - \Xi_m y(t)). \end{aligned} \tag{A.67}$$

Write, using Leibniz' rule,

$$y(t-\tau_1) = y(t) - \int_{-\tau_1}^{0} \dot{y}(t+s)\mathrm{d}s. \tag{A.68}$$

Then

$$\begin{aligned} y^\top(t)\Xi_m^\top \Xi_m u(t) &= -\sigma\eta_1 y^\top(t)\Xi_m^\top \Xi_m y(t) - \sigma\eta_1 y^\top(t)\Xi_m^\top(\Pi_1 \otimes I)\Xi_m \int_{\tau_1}^{0} \dot{y}(t+s)\mathrm{d}s \\ &\quad + \sigma y^\top(t)\Xi_p^\top \sum_{\ell\backslash\{1\}} \eta_\ell (\Xi_m(\Pi_\ell \otimes I)y(t-\tau_\ell) - \Xi_m y(t)). \end{aligned} \tag{A.69}$$

Suppose that $\kappa |y(t)| > |y_t(\theta)|$ with $\theta \in [-2\tau^*, 0]$ and $\kappa > 1$, then

$$\begin{aligned} y^\top(t)\Xi_m^\top \Xi_m u(t) &\leq -2\sigma\eta_1 |\Xi_m y(t)|^2 + \sigma\kappa^* \cdot |\Xi_m y(t)|^2 \\ &\quad + \kappa\sigma\tau_1\eta_1 \left(c_2 |\Xi_p z(t)| \cdot |\Xi_m y(t)| + (c_3 + 2\sigma) |\Xi_m y(t)|^2\right), \end{aligned} \tag{A.70}$$

where $\kappa^* = -1 + \sum_{\ell\backslash\{1\}} \eta_\ell \kappa$. Note that $\kappa > 1$ implies $\kappa^* > 0$. It follows that if $\kappa^2 V(\Xi_p z(t), \Xi_m y(t)) > V(\Xi_p z_t(\theta), \Xi_m y_t(\theta))$, $\theta \in [-2\tau^*, 0]$,

$$\begin{aligned} \dot{V}(\Xi_p z(t), \Xi_m y(t)) &\leq \left(c_3(1+\kappa\sigma\tau_1\eta_1) + 2\kappa^2\sigma\tau_1\eta_1 - \sigma(2\eta_1 - \kappa^*)\right) |\Xi_m y(t)|^2 \\ &\quad (c_1 + c_2(1+\kappa\sigma\tau_1\eta_1)) |\Xi_p z(t)| \cdot |\Xi_m y(t)| - c_0 |\Xi_p z(t)|^2. \end{aligned} \tag{A.71}$$

Again, for sufficiently large σ and sufficiently small $\sigma\tau$ there is $\kappa^* > 0$ such that (A.71) is negative definite. Hence there exist positive constants σ' and γ' such that if $\sigma \geq \sigma'$ and $\sigma\tau_1 \leq \gamma'$ there exists a globally attractive subset of the set $\ker(I_{kn} - \Pi_1 \otimes I_n)$.

A.3 Proofs chapter 5

A.3.1 Proof of Theorem 5.5

Without loss of generality it will be assumed that $\max_i \sum_{j\in\mathcal{E}_i} a_{ij} = 1$. Gerschgorin's Theorem implies that each eigenvalue $\lambda_j \leq 2$. Let $(\sigma', \tau') \in \bar{\mathcal{S}}$, then $(\sigma, \tau) \in \bar{\mathcal{S}}_j$ implies

$2\sigma/\lambda_j = \sigma'$, from which it can be concluded that $\sigma \leq \sigma'$. Then assumption (H5.4) implies, by Corollary 3.7, that the solutions of the closed-loop system are ultimately bounded. Moreover, the bounds of the solutions of a network with $k > 2$ systems can be chosen identical as the bounds of the solutions for a network consisting of two systems.

The proof follows now almost immediately from the proof of Theorem 3.12. Obviously, as the delays in the coupling are all the same, the set $\mathcal{M} = \{\mathrm{col}\,(z_1, \ldots, z_k, y_1, \ldots, y_k) \in \mathbb{R}^{k(p+m}|z_i = z_j \text{ and } y_i = y_j \text{ for all } i, j \in \mathcal{I}\}$ is invariant under the closed loop dynamics (5.27), (5.8).

Let $M \in \mathbb{R}^{k\times k}$,

$$M = \begin{pmatrix} 1 & 0 \\ \mathbf{1} & -I \end{pmatrix}. \tag{A.72}$$

Define coordinates $\hat{z}(t) = (M \otimes I)z(t)$, $\hat{y}(t) = (M \otimes I)y(t)$, with $z(t) = \mathrm{col}\,(z_1(t), \ldots, z_k(t))$ and $y(t) = \mathrm{col}\,(y_1(t), \ldots, y_k(t))$. Note that $\hat{z}_1(t) = z_1(t), \hat{z}_2(t) = (z_1 - z_2)(t), \ldots, \hat{z}_k(t) = (z_1 - z_k)(t)$ and $\hat{y}_1(t) = y_1(t), \hat{y}_2(t) = (y_1 - y_2)(t), \ldots, \hat{y}_k(t) = (y_1 - y_k)(t)$. Define $\tilde{z}(t) = \mathrm{col}\,(\hat{z}_2(t), \ldots, \hat{z}_k(t))$ and $\tilde{y}(t) = \mathrm{col}\,(\hat{y}_2(t), \ldots, \hat{y}_k(t))$.

Observe that

$$MLM^{-1} = \begin{pmatrix} 0 & * \\ 0 & \tilde{L} \end{pmatrix}, \tag{A.73}$$

with $\tilde{L}$ having eigenvalues $\lambda_2, \ldots, \lambda_k$. In these new coordinates, the closed-loop system (5.27), (5.8) can be written as

$$\dot{\tilde{z}}(t) = \tilde{q}(z_1(t), y_1(t), \tilde{z}(t), \tilde{y}(t)), \tag{A.74a}$$

$$\dot{\tilde{y}}(t) = \tilde{a}(y_1(t), z_1(t), \tilde{y}(t), \tilde{z}(t)) - \sigma(\tilde{L} \otimes I)\tilde{y}(t-\tau), \tag{A.74b}$$

where

$$\tilde{q}(z_1(t), y_1(t), \tilde{z}(t), \tilde{y}(t)) := \begin{pmatrix} q(z_1(t), y_1(t)) - q(z_1(t) - \tilde{z}_2(t), y_1(t) - \tilde{y}_2(t)) \\ \vdots \\ q(z_1(t), y_1(t)) - q(z_1(t) - \tilde{z}_k(t), y_1(t) - \tilde{y}_k(t)) \end{pmatrix}, \tag{A.75}$$

and

$$\tilde{a}(y_1(t), z_1(t), \tilde{y}(t), \tilde{z}(t)) := \begin{pmatrix} a(y_1(t), z_1(t)) - q(y_1(t) - \tilde{y}_2(t), z_1(t) - \tilde{z}_2(t)) \\ \vdots \\ a(y_1(t), z_1(t)) - q(y_1(t) - \tilde{y}_k(t), z_1(t) - \tilde{z}_k(t)) \end{pmatrix}. \tag{A.76}$$

There exists a nonsingular matrix $U \in \mathbb{R}^{(k-1)\times(k-1)}$ such that $U\tilde{L}U^{-1} = \Lambda$ with Λ a diagonal matrix with the eigenvalues of $\tilde{L}$, hence the nonzero eigenvalues of L, as entries. It will be assumed that $\|U\| = 1$. (If $\|U\| \neq 1$, then there is always a positive number c such that $\|cU\| = 1$.) Introduce new coordinates $\bar{y}(t) = (U \otimes I)\tilde{y}(t)$ and, for consistency of notation, $\bar{z}(t) = \tilde{z}(t)$. Note that $\|U\| = 1$ implies that the bounds on $\bar{y}(t)$ are the same as the bounds on $\bar{y}(t)$. Note in addition that $\bar{y}(t) = 0$ and $\bar{z}(t) = 0$ as $t \to \infty$ implies synchronization. In new coordinates the closed-loop system (5.27), (5.8) becomes

$$\dot{\bar{z}}(t) = \bar{q}(z_1(t), y_1(t), \bar{z}(t), \bar{y}(t)), \tag{A.77a}$$
$$\dot{\bar{y}}(t) = \bar{a}(y_1(t), z_1(t), \bar{y}(t), \bar{z}(t)) - \sigma(\Lambda \otimes I)\bar{y}(t-\tau), \tag{A.77b}$$

where $\bar{a}(y_1, z_1, \bar{y}, \bar{z}) = (U \otimes I)\tilde{a}(y_1, z_1, (U^{-1} \otimes I)\bar{y}, \bar{z})$ and $\bar{q}(z_1, y_1, \bar{z}, \bar{y}) = \tilde{a}(z_1, y_1, \bar{z}, (U^{-1} \otimes I)\bar{y})$. It will now be proven, using the Lyapunov-Razumikhin theorem, that there are values for σ and τ for which the origin of (A.77) is asymptotically stable. First, using Leibniz' rule,

$$\bar{y}(t-\tau) = \bar{y}(t) - \int_{-\tau}^{0} \dot{\bar{y}}(t+s)\mathrm{d}s, \tag{A.78}$$

the equivalent system for (A.77) on the interval $[t_0 - 2\tau, \infty)$ can be derived

$$\dot{\bar{z}}(t) = \bar{q}(z_1(t), y_1(t), \bar{z}(t), \bar{y}(t)), \tag{A.79a}$$
$$\dot{\bar{y}}(t) = \bar{a}(y_1(t), z_1(t), \bar{y}(t), \bar{z}(t)) - \sigma(\Lambda \otimes I)\bar{y}(t) + \sigma(\Lambda \otimes I)\int_{-\tau}^{0} \dot{\bar{y}}(t+s)\mathrm{d}s. \tag{A.79b}$$

Consider the positive definite function

$$V_0(\bar{z}(t)) = \bar{z}^\top(t)(P \otimes I)\bar{z}(t), \tag{A.80}$$

with positive definite matrix P as in assumption (H5.5). Then ultimate boundedness of the solutions, smoothness of the function $q(\cdot,\cdot)$ and assumption (H5.5) implies that

$$\dot{V}_0(\bar{z}(t), \tilde{y}(t)) \le -c_0 \left|\bar{z}(t)\right|^2 + c_1 \left|\bar{z}(t)\right| \cdot \left|\bar{y}(t)\right|, \tag{A.81}$$

for some positive numbers c_0 and c_1 which do not depend the network topology. Let now the positive definite function

$$V(\bar{z}(t), \bar{y}(t)) = V_0(\bar{z}(t)) + \bar{y}^\top(t)\bar{y}(t), \tag{A.82}$$

be a Lyapunov-Razumikhin function candidate. Note that ultimate boundedness of the solutions and smoothness of the function $a(\cdot,\cdot)$ implies that

$$\bar{y}^\top(t)\bar{a}(y_1(t), z_1(t), \bar{y}(t), \bar{z}(t)) \le c_2 \left|\bar{z}(t)\right| \cdot \left|\bar{y}(t)\right| + c_3 \left|\bar{y}(t)\right|^2, \tag{A.83}$$

for some positive numbers c_2 and c_3 which do not depend the network topology. Then the Lyapunov-Razumikhin theorem implies that if $V(\bar{z}(t), \bar{y}(t)) \ge \kappa^2 V(\bar{z}_t(\theta), \bar{y}_t(\theta))$ on $\theta \in [-2\tau, 0]$ for some number $\kappa > 1$, and

$$\begin{aligned}\dot{V}(\bar{z}(t), \bar{y}(t)) \le &- c_0 \left|\bar{z}(t)\right|^2 + (c_1 + c_2(1 + \kappa\sigma\tau\lambda_k)) \left|\bar{z}(t)\right| \cdot \left|\bar{y}(t)\right| \\ &+ (c_3(1 + \kappa\sigma\tau\lambda_k) + \kappa\sigma^2\tau\lambda_k^2 - \sigma\lambda_2) \left|\bar{y}(t)\right|^2,\end{aligned} \tag{A.84}$$

λ_2 and λ_k are the smallest nonzero and largest eigenvalues of L, respectively, is negative definite, then the origin of (A.77) is asymptotically stable, which, in combination with boundedness of solutions implies synchronization.

Denote $\xi(\bar{z}(t), \bar{y}(t)) = \mathrm{col}\left(|\bar{z}(t)|, |\bar{y}(t)|\right)$ and $\gamma = \sigma\tau$, then (A.84) can be written as

$$-\xi^\top(\bar{z}(t), \bar{y}(t))Q\xi(\bar{z}(t), \bar{y}(t)), \tag{A.85}$$

with the symmetric matrix

$$Q = \begin{pmatrix} c_0 & -\frac{c_1 + c_2(1+\kappa\gamma\lambda_k)}{2} \\ -\frac{c_1 + c_2(1+\kappa\gamma\lambda_k)}{2} & \sigma\lambda_2 - (c_3(1+\kappa\gamma\lambda_k) + \kappa\sigma\gamma\lambda_k^2) \end{pmatrix}. \tag{A.86}$$

Observe that $Q > 0$ if σ is sufficiently large and γ is sufficiently small, i.e. there exist constants $\bar{\sigma}$ and $\bar{\gamma}$ such that if $\sigma \geq \bar{\sigma}$ and $\sigma\tau \leq \bar{\gamma}$, the matrix $Q > 0$. For two coupled systems with $a_{12} = a_{21} = 1$ it follows directly that $\lambda_2 = \lambda_k = 2$. Let $\bar{\sigma}$ and $\bar{\gamma}$ be such that the right hand side of (A.84) is negative definite if $\sigma \geq \bar{\sigma}$ and $\sigma\tau = \gamma \leq \bar{\gamma}$. Thus the Lyapunov-Razumikhin theorem and boundedness of solutions implies that the two systems synchronize when $\bar{\mathcal{S}} = \{(\sigma, \tau) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} | \sigma \geq \bar{\sigma} \text{ and } \sigma\tau \leq \bar{\gamma}\}$. It is easy to see that if the network consists of k systems, $(\sigma, \tau) \in \bar{\mathcal{S}}_2 \cap \bar{\mathcal{S}}_k$ with $\bar{\mathcal{S}}_j := \left\{(\sigma, \tau) | \, (2\sigma/\lambda_j, \tau) \in \bar{\mathcal{S}}\right\}$ implies that the right hand side of (A.84) is negative definite.

A.4 Proofs chapter 8

A.4.1 Proof of Lemma 8.1

The proof follows almost immediately from the proof of Theorem 3.6 and the proof of strict semipassivity of the Hindmarsh-Rose neuron. Let $x_i(t) := \mathrm{col}\left(y_i(t), z_{1,i}(t), z_{2,i}(t)\right)$ and consider the positive definite storage function $V_i(x_i(t)) = \frac{1}{4}y_i^4(t) + \frac{1}{2w_3}z_{1,i}^2(t) + \frac{\mu}{2\varepsilon}z_{2,i}^2(t)$ with a positive constant μ. Let the constants $\lambda_j \in (0,1)$, $j = 1, \ldots, 4$, constant $\mu = \frac{1}{4f_3\lambda_1\lambda_3(1-\lambda_4)}$, then $\dot{V}_i(x_i(t)) = y_i^3(t)u_i(t) - H_i(x_i(t))$ with the function

$$\begin{aligned} H_i(x_i(t)) &= \frac{\lambda_2}{w_3}\left(z_{1,i}(t) + \frac{w_1}{2\lambda_2}y_i(t) - \frac{w_2}{2\lambda_2}y_i^2(t)\right)^2 \\ &+ \lambda_1 f_3\left(y_i^3(t) + \frac{1}{2\lambda_1 f_3}z_{2,i}(t)\right)^2 + \mu\lambda_3\lambda_4\left(z_{2,i}(t) - \frac{b}{2\lambda_4}y_i(t)\right)^2 \\ &+ (1-\lambda_3)\mu z_{2,i}^2(t) - \mu b y_{0,i}z_{2,i}(t) + (1-\lambda_2)\frac{1}{w_3}z_{1,i}^2(t) + \frac{w_0}{w_3}z_{1,i}(t) \\ &+ (1-\lambda_1)f_3 y_i^6(t) - f_2 y_i^5(t) - f_1 y_i^4(t) - \frac{\mu\lambda_3 b^2}{4\lambda_4}y_i^2(t) \\ &- \frac{1}{4w_3\lambda_2}\left(w_1 y_i(t) - w_2 y_i^2(t)\right)^2. \end{aligned} \tag{A.87}$$

It is not difficult to see that $H_i(x_i(t))$ is positive for sufficiently large $|x_i(t)|$. Let $W(x(t)) := \sum_{i=1}^k V_i(x_i(t))$, $x(t) := \operatorname{col}(x_1(t), \dots, x_k(t))$, then $\dot{W}(x(t)) = \sum_{i=1}^k y_i(t)u_i(t) - H_i(t)$. Note that $\sum_{i=1}^k y_i^3(t)u_i(t) \le \frac{\sigma}{2}\sum_{i=1}^k \sum_{j=1,j\neq i}^k y_i^2(t)(y_j^2(t) - y_i^2(t))$. Since $\sum_{i=1}^k \sum_{j=1,j\neq i}^k (y_j^2(t) - y_i^2(t)) \le 0$ and thus $\sum_{i=1}^k y_i^3(t)u_i(t) \le 0$ it follows that $\dot{W}(x(t)) = -\sum_{i=1}^k H_i(x_i(t)) \le 0$ for sufficiently large $|x|$. The function W is radially unbounded since each function V_i is radially unbounded, hence there exists a constant c^* such that $\dot{W} < 0$ for each constant c and all x satisfying $W \ge c > c^*$. Thus the set $\{x \in \mathbb{R}^{3k} : W \ge c\}$ is a positively invariant compact set under the dynamics (8.4), (8.5) and all solutions exist and are bounded.

A.4.2 Proof of Theorem 8.2

Without loss of generality it is assumed that the cells $i \in \{1, \dots, k_1\}$ are active and the cells $i \in \{k_1+1, \dots, k\}$ are inactive. It will be proven that the active cells will synchronize with each other even in presence of coupling to the inactive cells. First, for notational convenience, define $z_i(t) := \operatorname{col}(z_{1,i}(t), z_{2,i}(t))$, $\dot{y}_i(t) = a(y_i(t), z_i(t)) + u_i(t)$, with $a(y_i(t), z_i(t)) = f(y_i(t)) - z_{i,i}(t) - z_{2,i}(t)$ and $\dot{z}_i(t) = q(y_i(t), z_i(t)) := \operatorname{col}(w_\infty(y_i(t)) - z_{1,i}(t), \varepsilon\,(b\,(y_i(t) + y_{0,i}) - z_{2,i}(t)))$. Define $\tilde{y}_1(t) = y_1(t)$, $\tilde{y}_j(t) := y_1(t) - y_{j+1}(t)$, $\tilde{z}_1(t) = z_1(t)$, $\tilde{z}_j(t) = z_1(t) - z_{j+1}(t)$, $j = 2, \dots, k_1$, then $\dot{\tilde{y}}_j(t) = a(y_1(t), z_1(t)) - a(y_1(t) - \tilde{y}_j(t), z_1(t) - \tilde{z}_j(t)) + u_1 - u_j$ and $\dot{\tilde{z}}_j(t) = q(z_1(t), y_1(t)) - q(z_1(t) - \tilde{z}_j(t), y_1(t) - \tilde{y}_j(t))$. Consider the Lyapunov function $V = \frac{1}{2}\tilde{y}^\top(t)\tilde{y}(t) + \frac{1}{2}\tilde{z}^\top(t)P\tilde{z}(t)$ with $\tilde{y}(t) = \operatorname{col}(\tilde{y}_2(t), \dots, \tilde{y}_{k_1}(t))$, $\tilde{z}(t) = \operatorname{col}(\tilde{z}_2(t), \dots, \tilde{z}_{k_1}(t))$ and

$$P = \tilde{P} \otimes I, \quad \tilde{P} = \frac{1}{\varepsilon}\begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}. \tag{A.88}$$

Note that $a(y_1(t), z_1(t)) - a(y_j(t), z_j(t)) = (a(y_1(t), z_1(t)) - a(y_j(t), z_1(t))) + (a(y_j(t), z_1(t)) - a(y_j(t), z_j(t)))$. Using the ultimate boundedness of the states of all systems (Lemma 8.1), the triangle inequality and Lipschitz continuity of $a(\cdot,\cdot)$, it follows that there exist constants $c_0, c_1 \in \mathbb{R}_{>0}$ such that $\tilde{y}_j(t)(a(y_1(t), z_1(t)) - a(y_j(t), z_j(t))) \le c_0\,|\tilde{y}_j(t)| \cdot |\tilde{z}_j(t)| + c_1\,|\tilde{y}_j(t)|^2$. It can easily be verified that the internal dynamics of the β-cell satisfy the Demidovich condition. Hence $\tilde{z}_j^\top(t)\tilde{P}(q(z_1(t), y_1(t)) - q(z_j(t), y_j(t))) \le -\,|\tilde{z}_j(t)|^2 + c_0\,|\tilde{y}_j(t)| \cdot |\tilde{z}_j(t)|$ for some constant $c_2 \in \mathbb{R}_{>0}$. Thus there exist positive constants C_0, C_1 such that $\dot{V} \le -\,|\tilde{z}(t)|^2 + C_0\,|\tilde{z}(t)| \cdot |\tilde{y}(t)| + C_1\,|\tilde{y}(t)|^2 + \tilde{y}^\top(t)\tilde{u}(t)$ with $\tilde{u} := \operatorname{col}(u_1 - u_2, \dots, u_1 - u_{k_1})$. Note that the constants C_0 and C_1 only depend on the bounds on the trajectories $y_i(t)$ and $z_i(t)$ and the functions $a(\cdot,\cdot)$, $q(\cdot,\cdot)$ and not on the number of cells. Since the coupling is global, it follows that

$$u_1(t) = \sigma(y_j(t) - y_1(t)) + \sigma \sum\nolimits_{\ell=2,\ell\neq j}^{k} (y_\ell(t) - y_1(t)), \tag{A.89a}$$

$$u_j(t) = \sigma(y_1(t) - y_j(t)) + \sigma \sum_{\ell=2,\ell\neq j}^{k} (y_\ell(t) - y_j(t)), \tag{A.89b}$$

such that $\tilde{u}_j(t) = -\sigma k \tilde{y}_j(t)$. Hence $\tilde{y}^\top(t)\tilde{u}(t) = -\sigma k\,|\tilde{y}(t)|^2$ such that if $\sigma k \geq \bar{\sigma} := \frac{C_0^2}{4} + C_1$ there exists a constant $\epsilon > 0$ such that $\dot{V} \leq -\epsilon V$. It follows that $\int_{t_0}^{t} -\dot{V}(\tau)\mathrm{d}\tau = V(t_0) - V(t) \leq V(t_0) < \infty$. Hence, using Barbalat's lemma (note that $\dot{V}$ is uniformly continuous), it can be concluded that the active cells synchronize. Using the same machinery one can easily prove that the inactive cells will also synchronize with each other. On the other hand, the active cells will not synchronize with the inactive cells since the linear manifold corresponding to synchronization $\mathcal{M} := \{\mathrm{col}\,(x_1, \ldots, x_k) \in \mathbb{R}^{3k} | x_1 = \ldots = x_{k_1} = x_{k_1+1} = \ldots = x_k\}$, $x_i := \mathrm{col}\,(y_i, z_{1,i}, z_{2,i})$, is not invariant under the closed loop dynamics (8.4), (8.5). It follows immediately that all cells in the islet synchronize whenever $\sigma k \geq \bar{\sigma}$ provided that all cells are active ($k_1 = k$) or all cells are inactive ($k_2 = k$) such that $\mathcal{M}$ is invariant under the given dynamics.

A.5 Proofs chapter 9

A.5.1 Proof of Theorem 9.4

Firstly, since $\lambda(s)$ is nonnegative and each system (9.21) is strictly $\mathcal{C}^1$-semipassive, Lemma 9.1 implies that there is a constant $c_* > 0$ and a radially unbounded nonnegative function $W(x_1(t), x_2(t))$ such that the set $\{\mathrm{col}\,(x_1, x_2) \in \mathbb{R}^{2n} | W(x_1, x_2) \leq c\}$ for any $c \geq c_*$ is compact and positively invariant with respect to (9.21), (9.5). Secondly, note that the coupling (9.5) can be decomposed as

$$\int_{y_1(t)}^{y_2(t)} \lambda(s)\mathrm{d}s = \int_{y_1(t)}^{0} \lambda(s)\mathrm{d}s + \int_{0}^{y_2(t)} \lambda(s)\mathrm{d}s. \tag{A.90}$$

Hence, the interconnected systems (9.21), (9.5) can be equivalently written as the systems

$$\dot{x}_i(t) = \tilde{f}(x_i(t)) + Bv_i(t), \quad y_i(t) = Cx_i(t), \tag{A.91}$$

with $i = 1, 2$, $\tilde{f}(x(t)) = f(x(t)) + B\int_{Cx(t)}^{0} \lambda(s)\mathrm{d}s$, and

$$v_1(t) = \int_{0}^{y_2(t)} \lambda(s)\mathrm{d}s, \quad v_2(t) = \int_{0}^{y_1(t)} \lambda(s)\mathrm{d}s. \tag{A.92}$$

Note that

$$\frac{\partial \tilde{f}}{\partial x}(x(t)) = \frac{\partial f}{\partial x}(x(t)) - BC\lambda(Cx(t)). \tag{A.93}$$

Since $PB = C^\top$, condition (9.22) implies that $\tilde{f}(x)$ satisfies condition (9.20) of Lemma 9.3. Hence the system

$$\dot{x}(t) = \tilde{f}(x(t)) + Bv(t), \quad y(t) = Cx(t) \tag{A.94}$$

is strictly $\mathcal{C}^1$-incrementally-passive with $\Delta V(x_1, x_2) = (x_1(t) - x_2(t))^\top P(x_1 - x_2)$ and $\Delta W(x_1 - x_2) = (x_1 - x_2)^\top R(x_1 - x_2)$. Applying Lemma 9.2 one concludes that any solution of systems (A.91) interconnected through (A.92) is defined and bounded for $t \geq t_0$ and, moreover, the states $x_1(t)$ and $x_2(t)$ asymptotically synchronize. In particular, due to Remark 9.1, the convergence is exponential, i.e. (9.23) holds.

A.5.2 Proof of Theorem 9.5

Choose the matrices P and R in (9.22) as

$$P := \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix}, \quad R := \begin{pmatrix} \epsilon I_{n-1} & 0 \\ 0 & 2\epsilon \end{pmatrix}. \tag{A.95}$$

Notice that this P satisfies the equality $PB = C^\top$. By combining all the terms in the first inequality in 9.22 in the right-hand side, one can see that for the chosen $P = P^\top > 0$ and $R = R^\top > 0$ this matrix inequality is equivalent to

$$J := \begin{pmatrix} A & M \\ M^\top & N \end{pmatrix} \geq 0, \tag{A.96}$$

where

$$A = -Q\frac{\partial q}{\partial z}(z, y) - \frac{\partial q^\top}{\partial z}(z, y)Q - \epsilon I_{n-1}, \tag{A.97}$$

$$M = -Q\frac{\partial q}{\partial y}(z, y) - \frac{\partial a^\top}{\partial z}(y, z), \tag{A.98}$$

$$N = -2\frac{\partial a}{\partial y}(y, z) + 2\lambda(y) - 2\epsilon. \tag{A.99}$$

Due to (9.25), inequality A.96 holds if

$$\tilde{J} := \begin{pmatrix} S - \epsilon I_{n-1} & M \\ M^\top & N \end{pmatrix} > 0. \tag{A.100}$$

Recall that $\tilde{J}(z, y)$ is positive definite if and only if $S - \epsilon I_{n-1} > 0$ and $N - M^\top(S - \epsilon I_{n-1})^{-1}M > 0$. The first inequality is guaranteed by (9.28), while the last one holds due to the choice of $\lambda(y)$ satisfying (9.26).

A.5.3 Proof of Theorem 9.7

Boundedness of solutions follows from Lemma 9.6. To prove synchronization of systems' states, rewrite the systems (9.21), (9.29) as follows:

$$\dot{x}_i(t) = \tilde{f}(x_i(t)) + Bv_i(t), \quad y_i = Cx_i(t), \tag{A.101}$$

where $\tilde{f}(x(t)) = f(x(t)) + (k-1)B\int_{Cx(t)}^{0}\lambda(s)\mathrm{d}s$ and

$$v_i(t) = \sum_{j=1, j\neq i}^{k} \int_0^{y_j(t)} \lambda(s)\mathrm{d}s. \tag{A.102}$$

Let $\zeta_1(t) := \operatorname{col}(x_1(t), x_1(t), \ldots, x_1(t))$ and $\zeta_2 := \operatorname{col}(x_2(t), x_3(t), \ldots, x_k(t))$. Consider the following incremental storage function

$$\Delta V = (\zeta_1 - \zeta_2)^\top (I_{k-1} \otimes P)(\zeta_1 - \zeta_2). \tag{A.103}$$

Then

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\Delta V &= 2(\zeta_1 - \zeta_2)^\top (I_{k-1} \otimes P)\left(F(\zeta_1) - F(\zeta_2)\right) && \text{(A.104)} \\ &\quad - \sum_{i=2}^{k} (y_1 - y_i)^\top \int_{y_i}^{y_1} \lambda(s)\mathrm{d}s, && \text{(A.105)} \end{aligned}$$

where

$$\begin{aligned} F(\zeta_1) &= \begin{pmatrix} \tilde{f}^\top(x_1) & \tilde{f}^\top(x_1) & \cdots & \tilde{f}^\top(x_1) \end{pmatrix}^\top, \\ F(\zeta_2) &= \begin{pmatrix} \tilde{f}^\top(x_2) & \tilde{f}^\top(x_3) & \cdots & \tilde{f}^\top(x_k) \end{pmatrix}^\top. \end{aligned}$$

Note that $-\sum_{i=2}^{k}(y_1 - y_i)^\top \int_{y_i}^{y_1} \lambda(s)\mathrm{d}s \le 0$ since $\lambda(s) \ge 0$ for all $s \in \mathbb{R}$. Then it follows that (9.30) implies (see, e.g. [111])

$$\begin{aligned} &2(\zeta_1 - \zeta_2)^\top (I_{k-1} \otimes P)\left(F(\zeta_1) - F(\zeta_2)\right) \\ &\le -(\zeta_1 - \zeta_2)^\top (I_{k-1} \otimes R)(\zeta_1 - \zeta_2) < 0. \end{aligned} \tag{A.106}$$

This, in turn, implies exponential convergence of $\zeta_2(t)$ to $\zeta_1(t)$ and concludes the proof of the theorem.

Appendix B

Reconstructing dynamics of spiking neurons from input-output measurements

Abstract. In this appendix a method is proposed to estimate the state and parameters of models of neural dynamics from current clamped measurements of the membrane potential. The content of this appendix has been published in [148] and [155]. Generalized results are presented in [156].

B.1 Introduction

Mathematical modeling of neural dynamics is essential for understanding the principles behind neural computation. Since the introduction of clamping techniques, which made it possible to measure the membrane potential and currents of single neurons [73], and inspired by the pioneering works of Hodgkin and Huxley [63], a large number of models describing action potential generation of neural cells have been developed (see [69] for a review). These models offer a *qualitative* description of the mechanisms of spike generation in neural cells. To study the specific behavior of neural cells, e.g. the dynamic fluctuations of the membrane potential, a rigid *quantitative* evaluation of these models against empirical data is needed. For the dynamical models this amounts to the identification of the model's states and parameter values from input-output measurements in the presence of noise.

Which of the many available models is the most suitable one for this goal? In general, models of neural dynamics can be classified as biophysically plausible or as purely mathematical. The biophysically realistic conductance based neuronal models describe the generation of the spikes as a function of the individual ionic currents flowing through the neuron's membrane. Although being time consuming, the parameters of these models can, in principle, be partially obtained through measurements. However, complete and accurate estimation of their parameters for a single *living* cell is hardly practicable.

Because of these complications, a number of mathematical models that mimic the spiking behavior of real neurons are introduced throughout the years, e.g. the Hindmarsh-Rose [62] and Fitzhugh-Nagumo [48, 98] neuronal models. These models are simpler in structure and in the number of parameters. Their parameters, however, have no immediate physical interpretation. Hence, they cannot be measured explicitly in experiments. It is showed by Izhikevich [70] that the mathematical models can, depending on their specific parameters, cover a wide range of the dynamics that have been observed in real neurons. Furthermore, they have the advantage of simplicity. This makes model identification an easier task.

Here, the aim is to provide a method that allows a successful mapping of mathematical neuronal models to the vast collection of available empirical data. However, fitting these models to given input-output data is a hard technical problem. This is because the any information of the internal, non-physical, states of the system is not available, and the input-output information that is available is often deficient. Yet, to successfully model the measured data one needs to reconstruct the unknown states and estimate the parameters of the system simultaneously.

The problem of estimating the state and parameter vectors for a given nonlinear system from input-output data is a well established field in system identification [83] and adaptive control [135]. It has a broad domain of relevant applications in physics and engineering, and efficient recipes for solving practical problems are available. In most cases, when state and parameter identification is required, these methods apply to a rich class of systems that can be transformed into the so-called *canonical adaptive observer form* [13]:

$$\dot{x}(t) = Ax(t) + \varphi(t, y(t))\theta + g(t), \tag{B.1a}$$
$$y(t) = C^\top x(t), \tag{B.1b}$$

with state $x(t) \in \mathbb{R}^n$, output $y(t) \in \mathcal{Y} \subset \mathbb{R}$, $\theta \in \mathbb{R}^d$ is a vector with *unknown* parameters, *known* functions $g : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $\varphi : \mathbb{R} \times \mathcal{Y} \to \mathbb{R}^n \times \mathbb{R}^d$, and

$$A = \begin{pmatrix} 0 & k^\top \\ 0 & F \end{pmatrix} \text{ and } C^\top = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}.$$

where $k = \operatorname{col}(k_1, \ldots, k_{n-1})$ is a vector of *known* constants and F is a *known* $(n-1) \times (n-1)$ matrix, usually diagonal, with eigenvalues in the open left half-plane of the complex domain. Algorithms for the asymptotic recovery of the state variables and the parameter vector θ can be found in, for instance, [13, 91, 92].

Models of neural dynamics, however, typically are not in the form (B.1), and, more important, they cannot be transformed into this specific form. Consider, for instance, the

Hindmarsh-Rose model [62]

$$\dot{x}_1(t) = \theta_{1,3}x_1^3(t) + \theta_{1,2}x_1^2(t) + \theta_{1,0} + x_2(t) - x_3(t) + g(t), \tag{B.2a}$$
$$\dot{x}_2(t) = -\lambda_2 x_2(t) + \theta_{2,2}x_1^2(t) + \theta_{2,0}, \tag{B.2b}$$
$$\dot{x}_3(t) = -\lambda_3 x_3(t) + \theta_{3,1}x_1(t) + \theta_{3,0}, \tag{B.2c}$$

and the FitzHugh-Nagumo model [48, 98]

$$\dot{x}_1(t) = \theta_{1,3}x_1^3(t) + \theta_{1,1}x_1(t) - x_2(t) + \theta_{1,0} + g(t), \tag{B.3a}$$
$$\dot{x}_2(t) = -\lambda_2 x_2(t) + \theta_{2,1}x_1(t) + \theta_{2,0}. \tag{B.3b}$$

The parameters $\theta_{i,j}$ and λ_i are in general unknown. This implies that the matrix A in (B.1), in particular the matrix F in A, is uncertain. So these models are not in the observer canonical form. Hence new methods for estimating the *unknown* $\theta_{i,j}$, λ_i for the relevant classes of systems (B.2), (B.3) are required.

In this appendix the focus is, in particular, on the estimation of the parameters of the Hindmarsh-Rose model. First a slight modification of the model (B.2) is presented and some basic properties of this model are summarized. Second, a procedure allowing successful fitting of the model to measured data will be developed for this modified model. Third, it is demonstrated how this approach can be used for the reconstruction of the spiking dynamics of single neurons in slices of hippocampal tissue in vitro. In section B.2 the modified Hindmarsh-Rose model is introduced and this section contains the formal statement of the identification problem. In section B.3 the parameter estimation procedure is described and sufficient conditions for convergence of the estimates are given. Section B.4 describes the details of the application of this procedure to the problem of reconstructing the spikes of hippocampal neurons from a mouse. Section B.5 concludes this appendix.

B.2 Preliminaries

Consider the following slight modification of the Hindmarsh-Rose equations (B.2):

$$\dot{x}_1(t) = \theta_{1,3}x_1^3(t) + \theta_{1,2}x_1^2(t) + \theta_{1,1}x_1(t) + \theta_{1,0} + x_2(t) - x_3(t) + g(t), \tag{B.4a}$$
$$\dot{x}_2(t) = -\lambda_2 x_2(t) + \theta_{2,2}x_1^2(t) + \theta_{2,1}x_1(t) + \theta_{2,0}, \tag{B.4b}$$
$$\dot{x}_3(t) = -\lambda_3 x_3(t) + \theta_{3,1}x_1(t) + \theta_{3,0}, \tag{B.4c}$$

where $\theta_{i,j}$ are unknown constant parameters and λ_2, λ_3 are the unknown time constants of the internal states. The state x_1 represents the membrane potential and is also the (natural) output of the neuron, x_2 is a fast internal variable, x_3 is a slow variable ($\lambda_3 \ll 1$), and $g(t)$ is an external clamping current. The system (B.4) has, compared to the original equations (B.2), a full third order polynomial in x_1 in the first equation and a full order second order polynomial in x_1 in the second equation. The modified model can adapt to arbitrary time-scales and has less restrictions on the shape of the spikes.

The specific behavior of the Hindmarsh-Rose model can be analyzed by decomposition into fast and slow subsystems (see for instance [18, 152]), where the fast subsystem is composed by the x_1 and x_2 dynamics, and the x_3 dynamics define the slow subsystem. The following properties hold for the Hindmarsh-Rose system:

i. the shape of the spikes is mainly determined by the fast subsystem,

ii. the firing frequency of the spikes in absence of the slow subsystem, i.e. $x_3 = 0$, is dictated by the amplitude of the external current $g(t)$,

iii. the third equation, i.e. the slow variable, perturbs the input $g(t)$ and modulates the firing frequency such that, depending on the parameters, the model can produce periodic bursts, aperiodic bursts or spiking behavior with adaptable firing frequency.

B.2.1 Problem Formulation

Consider the following class of nonlinear neuronal models:

$$\dot{x}_1(t) = \theta_1^\top \phi_1(t, x_1(t)) + \sum_{i=2}^{n} x_i(t), \tag{B.5a}$$

$$\dot{x}_i(t) = -\lambda_i x_i(t) + \theta_i^\top \phi_i(t, x_1(t)), \tag{B.5b}$$

with continuous functions $\phi_j : \mathbb{R} \times \mathcal{Y} \to \mathbb{R}^{d_j}$, $d_j \in \mathbb{N}\backslash\{1\}$, $i = 1, 2, \ldots, n$. Variable $x_1(t)$ in system (B.5) represents the dynamics of the cell's membrane potential, and variables $x_i(t)$, $i \geq 2$, are internal states that can be associated with the ionic currents flowing in the cell. The parameters $\theta_i \in \mathbb{R}^{d_i}$, $\lambda_i \in \mathbb{R}_{>0}$ are constant. Clearly, the models (B.2)–(B.4) belong to the particular class of systems (B.5).

The values of the variable $x_1(t)$ are assumed to be available at any instance of time and the functions $\phi_i(t, x_1(t))$ are supposed to be known. The variables x_i, $i = 2, 3, \ldots, n$, however, are not assumed to be available. The actual values of the parameters $\theta_1, \ldots, \theta_n$, $\lambda_2, \ldots, \lambda_n$, are unknown *a-priori*. (Note that θ_i can be vectors with parameter values.) It will be assumed that the domains of admissible values of θ_i, λ_i are known or can, at least, be estimated. In particular, it will be assumed that $\theta_{i,j} \in [\theta_{i,j,\min}, \theta_{i,j,\max}]$, $\lambda_i \in [\lambda_{i,\min}, \lambda_{i,\max}]$, and the values of $\theta_{i,j,\min}$, $\theta_{i,j,\max}$, $\lambda_{i,\min}$, $\lambda_{i,\max}$ are available.

Let, for notational convenience, $\theta = \mathrm{col}\,(\theta_1, \ldots \theta_n)$ and $\lambda = \mathrm{col}\,(\lambda_2, \ldots, \lambda_n)$. The vectors $\hat{\theta}$ and $\hat{\lambda}$ denote the estimations of θ and λ. The domains of θ, λ are given by the symbols Ω_θ and Ω_λ, respectively.

The problem is how to derive an algorithm which is capable of reconstructing the states and estimate the unknown parameters of the system (B.5) solely depending on the signal $x_1(t)$. In the present work this problem is considered within the framework of designing an observer for the dynamics and parameters of (B.5) that is driven by the measured signal $x_1(t)$ and has dynamics of the form:

$$\dot{\hat{x}}(t) = f(t, \hat{x}(t), z(t), x_1(t)), \tag{B.6a}$$

$$z(t) = h(\hat{x}(t)), \tag{B.6b}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the approximation of states of the system (B.5) and $z(t) = \operatorname{col}\left(\hat{\theta}(t), \hat{\lambda}(t)\right)$ contains estimates of the parameters of the system. Hence, the goal is to find conditions such that for some given small numbers $\delta_x, \delta_z \in \mathbb{R}_{>0}$ and all $t_0 \in \mathbb{R}_{\geq 0}$ the following properties hold:

$$\exists\, t' \geq t_0 \quad \text{s.t.} \quad \forall\, t \geq t' \quad : \begin{cases} |\hat{x}(t) - x(t)| \leq \delta_x, \\ |z(t) - \vartheta| \leq \delta_z. \end{cases} \tag{B.7}$$

where $\vartheta = \operatorname{col}(\theta, \lambda)$ is the vector with the actual parameters of the system (B.5).

B.3 Main Result

This section presents the main results. A "classical" observer will be designed which estimates the state $x(t)$ and parameters θ. An auxiliary system is introduced to estimate the values of λ. These results are based on the concepts of weakly attracting sets [94, 52][1] and non-uniform small gain theorems [157].

First, for notational convenience, the following vector function is introduced

$$\phi(t, x_1(t), \lambda) = \begin{pmatrix} \phi_1(t, x_1(t)) \\ \int_{t_0}^t e^{-\lambda_2(t-\tau)} \phi_2(s, x_1(s), \tau) \mathrm{d}s \\ \vdots \\ \int_{t_0}^t e^{-\lambda_n(t-\tau)} \phi_n(s, x_1(s), \tau) \mathrm{d}s \end{pmatrix}, \tag{B.8}$$

which is a concatenation of $\phi_1(\cdot)$ and the integrals

$$\int_{t_0}^t e^{-\lambda_i(t-s)} \phi_i(s, x_1(s)) \mathrm{d}s, \quad i = 2, \ldots, n. \tag{B.9}$$

Then, using (B.8), the system (B.5) can be written in the more compact form:

$$\dot{x}_1(t) = \theta^\top \phi(t, x_1(t), \lambda). \tag{B.10}$$

[1]The closed-loop system will not be stable in Lyapunov sense. The closed-loop system will have an attracting set in the sense of Definition 2.3.

Given that functions $\phi_i(\cdot,\cdot)$ are known and that the values of $x_1(t^*)$, $t^* \in [t_0, t]$ are available, the integrals (B.9) can be calculated explicitly as functions of λ_i and t.

Taking into account that the time variable t can be arbitrarily large, explicit calculation of integrals (B.9) is expensive in the computational sense and, in principle, requires infinitely large memory. For this reason an approximation of the function $\phi(t, x_1(t), \lambda)$ will be used. For instance, in the case that the signal $x_1(t)$ is periodic, bounded, and the functions $\phi_i(t, x_1(t))$ are locally Lipschitz in x_1 and periodic in t with the same period as $x_1(t)$, the functions $\phi_i(t, x_1(t))$ can be expressed in a Fourier series expansion:

$$\phi_i(t, x_1(t)) = \frac{a_{i,0}}{2} + \sum_{j=1}^{\infty} \left(a_{i,j}\cos(\omega_j t) + b_{i,j}\sin(\omega_j t)\right). \tag{B.11}$$

Taking a finite number N of members from the series expansion (B.11) the following approximation of (B.9) holds:

$$\begin{aligned}\int_{t_0}^{t} e^{-\lambda_i(t-s)}\phi_i(s, x_1(s))\mathrm{d}\tau \approx & \frac{a_{0,i}}{2\lambda_i} + \sum_{j=1}^{N} \frac{a_{i,j}}{\lambda_i^2+\omega_j^2}\left(\sin(\omega_j t)\omega_j + \lambda_i\cos(\omega_j t)\right) \\ & + \sum_{j=1}^{N} \frac{b_{i,j}}{\lambda_i^2+\omega_j^2}\left(-\cos(\omega_j t)\omega_j + \lambda_i\sin(\omega_j t)\right) + \epsilon(t),\end{aligned} \tag{B.12}$$

where $\epsilon(t) : \mathbb{R} \to \mathbb{R}$ is an exponentially decaying term. In the case that the signal $x_1(t)$ is not periodic in t or the functions $\phi_i(t, x_1(t))$ are not periodic in t, the integrals (B.9) can be approximated as:

$$\int_{t_0}^{t} e^{-\lambda_i(t-s)}\phi_i(s, x_1(s))\mathrm{d}s \approx \int_{t-T}^{t} e^{-\lambda_i(t-s)}\phi_i(s, x_1(s))\mathrm{d}s + \epsilon(t), \tag{B.13}$$

where $T \in \mathbb{R} > 0$ is sufficiently large.

Let the function $\bar{\phi}(t, x_1(t), \lambda)$ be the computationally realizable approximation of (B.8) such that

$$\left|\bar{\phi}(t, x_1(t), \lambda) - \phi(t, x_1(t), \lambda)\right| \leq \Delta, \tag{B.14}$$

for all $t \in \mathbb{R}_{>0}$ and some small $\Delta \in \mathbb{R}_{>0}$.

Consider the following "classical" observer that estimates the states and the parameters θ of the systems (B.10):

$$\dot{\hat{x}}_1(t) = -\alpha \cdot (\hat{x}_1(t) - x_1(t)) + \hat{\theta}^\top(t)\bar{\phi}(t, x_1(t), \hat{\lambda}), \tag{B.15a}$$

$$\dot{\hat{\theta}}(t) = -\gamma_\theta \cdot (\hat{x}_1(t) - x_1(t)) \cdot \bar{\phi}(t, x_1(t), \hat{\lambda}(t)), \tag{B.15b}$$

with constants $\gamma_\theta, \alpha \in \mathbb{R}_{>0}$. In (B.15), $\hat{x}_1(t)$, $\hat{\theta}(t)$ and $\hat{\lambda}$ denote the estimates of $x_1(t)$, θ and λ, respectively. (A definition of $\hat{\lambda}$ follows somewhat later.) Define $q(t) = \mathrm{col}\left(\hat{x}_1(t) - x_1(t), \hat{\theta}(t) - \theta\right)$, then the closed-loop system (B.10), (B.15) can be written as

$$\dot{q}(t) = A(t, x_1(t), \hat{\lambda})q(t) + b\, u(t, x_1(t), \lambda, \hat{\lambda}), \tag{B.16}$$

where $b = \begin{pmatrix}1 & 0 & \dots & 0\end{pmatrix}^\top$,

$$A(t, x_1(t), \hat{\lambda}) = \begin{pmatrix} -\alpha & \bar{\phi}(t, x_1(t), \hat{\lambda})^\top \\ -\gamma_\theta \bar{\phi}(t, x_1(t), \hat{\lambda}) & 0 \end{pmatrix},$$

and

$$u(t, x_1(t), \hat{\lambda}, \lambda) = \theta^\top(\bar{\phi}(t, x_1(t), \hat{\lambda}) - \bar{\phi}(t, x_1(t), \lambda)) + \theta^\top(\bar{\phi}(t, x_1(t), \lambda) - \phi(t, x_1(t), \lambda)).$$

The closed loop system (B.16) consists of the time-varying linear system $\dot{q}(t) = A(t, x_1(t), \hat{\lambda})q(t)$ which is perturbed by the function $u(t, x_1(t), \hat{\lambda}, \lambda)$. Note, in addition, that

$$\limsup_{\hat{\lambda} \to \lambda} \left| u(t, x_1(t), \lambda, \hat{\lambda}) \right| \leq |\theta| \, \Delta. \tag{B.17}$$

The control problem is now, in terms of (B.7), to find values $\hat{\lambda}$ close to λ, and conditions such that $\lim_{t\to\infty} |q(t)| \leq \delta_q$, with small $\delta_q \in \mathbb{R}_{>0}$.

The value of λ will be estimated using an auxiliary system. Consider the system

$$\dot{\xi}_{1,i}(t) = \gamma_i \cdot \sigma(|x_1(t) - \hat{x}_1(t)|_\varepsilon) \cdot \left(\xi_{1,i}(t) - \xi_{2,i}(t) - \xi_{1,i}(t)\left(\xi_{1,i}^2(t) + \xi_{2,i}^2(t)\right)\right), \tag{B.18a}$$
$$\dot{\xi}_{2,i}(t) = \gamma_i \cdot \sigma(|x_1(t) - \hat{x}_1(t)|_\varepsilon) \cdot \left(\xi_{1,i}(t) + \xi_{2,i}(t) - \xi_{2,i}(t)\left(\xi_{1,i}^2(t) + \xi_{2,i}^2(t)\right)\right), \tag{B.18b}$$

with initial conditions $\xi_{1,i}^2(t_0) + \xi_{2,i}^2(t_0) = 1$ and $i = \{2, \dots, n\}$. The function $\sigma : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is supposed to satisfy $|\sigma(s)| \leq |s|$ for all $s \in \mathbb{R}$, and $\sigma(\cdot)$ is bounded. The constants $\gamma_i \in \mathbb{R}_{>0}$ are *rationally-independent*, i.e.:

$$\sum \gamma_i k_i \neq 0, \ \forall \, k_i \in \mathbb{Z}. \tag{B.19}$$

The systems (B.18) with initial conditions $\xi_{1,i}^2(t_0) + \xi_{2,i}^2(t_0) = 1$ are positively invariant on the manifold $\xi_{1,i}^2(t) + \xi_{2,i}^2(t) = 1$. Taking into account that the constants γ_i are rationally-independent, it can be concluded that trajectories $\xi_{1,i}(t)$ densely fill an invariant n-dimensional torus [11]. In other words, the system (B.18) with initial conditions $\xi_{1,i}^2(t_0) + \xi_{2,i}^2(t_0) = 1$ is Poisson-stable in $\Omega_x = \{\xi_{1,i}, \xi_{2,i} \in \mathbb{R}^{2n} | \xi_{1,i} \in [-1, 1]\}$. Furthermore, notice that trajectories $\xi_{1,i}(t)$, $\xi_{2,i}(t)$ are globally bounded and that the right-hand side of (B.18) is locally Lipschitz in $\xi_{1,i}$, $\xi_{2,i}$. The values of λ_i will be estimated as follows:

$$\hat{\lambda}_i(\xi_{1,i}(t)) = \lambda_{i,\min} + \frac{\lambda_{i,\max} - \lambda_{i,\min}}{2}(\xi_{1,i}(t) + 1). \tag{B.20}$$

Clearly, the following estimate holds:

$$\left| \dot{\hat{\lambda}}(t) \right| \leq \gamma^* M, \quad M \in \mathbb{R}_{>0}, \ \gamma^* = \max_i \{\gamma_i\}. \tag{B.21}$$

The systems (B.16) and (B.18) are considered as two interconnected systems, denoted by $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. The system $\mathcal{S}_2$ provides values $\hat{\lambda}(t)$ from the compact domain Ω_λ

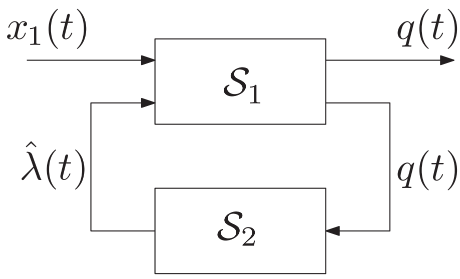


Figure B.1. The interconnected systems $\mathcal{S}_1$ and $\mathcal{S}_2$.

as function of the output of the system $\mathcal{S}_1$. These values $\hat{\lambda}(t)$ are, in turn, injected into the system $\mathcal{S}_1$. The system $\mathcal{S}_1$ is driven by the measured data and the estimates $\hat{\lambda}(t)$ and will provide estimates of the state $x_1(t)$ and the parameters θ. A schematic representation of the structure of these interconnected systems is provided in Figure B.1.

Sufficient conditions for convergence of the solutions of the system $\mathcal{S}_1$ to an invariant attracting set in the neighborhood of the origin will be given next. In particular, it will be showed that the systems (B.15), (B.18) serve as the desired observer (B.6) for the class of systems specified by equations (B.5), i.e. the properties of (B.7) will be satisfied. First the notion of λ-uniform persistency of excitation has to be introduced.

Definition B.1 (λ-uniform persistency of excitation [87]). Let function $\varphi : \mathbb{R} \times \mathcal{D}_0 \times \mathcal{D}_1 \to \mathbb{R}^{n\times m}$ be continuous and bounded. Then $\varphi(t, x(t), \lambda)$ is λ-uniformly persistently exciting (λ-uPE) if there exist $\mu, L \in \mathbb{R}_{>0}$ such that for each $x \in \mathcal{D}_0$, and each $\lambda \in \mathcal{D}_1$,

$$\int_t^{t+L} \varphi(s, x(s), \lambda)\varphi^\top(s, x(s), \lambda)\mathrm{d}s \geq \mu I, \quad \forall t \geq t_0.$$

△

The latter notion, in contrast to the conventional definitions of persistency of excitation, allows to deal with the parameterized regressors $\varphi(t, x(t), \lambda)$. This is essential for deriving the asymptotic properties of the interconnected $\mathcal{S}_1$, $\mathcal{S}_2$ systems. These properties are formulated in the theorem below:

Theorem B.1. *Let the systems* (B.10), (B.15), (B.18) *be given. Assume that function* $\bar{\phi}(t, x_1(t), \lambda)$ *is bounded, i.e.* $\left|\bar{\phi}(t, x_1(t), \lambda)\right| \leq B$, $B \in \mathbb{R}_{>0}$, *for all* $t \geq 0$ *and* $\lambda \in \Omega_\lambda$, *$\lambda$-uPE with constants μ and L as in Definition B.1, and Lipschitz in λ:*

$$\left|\bar{\phi}(t, x_1(t), \lambda) - \bar{\phi}(t, x_1(t), \lambda')\right| \leq D\left|\lambda - \lambda'\right|,$$

for some $D \in \mathbb{R}_{>0}$. Then there exist a number γ^* satisfying

$$\gamma^* = \frac{\mu}{4BDLM},$$

and a constant $\varepsilon > 0$ such that for all $\gamma_i \in (0, \gamma^*]$, *i. the trajectories of the closed loop system* (B.15), (B.18) *are bounded;*

ii. there exists a vector $\lambda^* \in \Omega_\lambda$*:* $\lim_{t\to\infty} \hat{\lambda}(t) = \lambda^*$*;*

iii. there exist positive constants $\kappa = \kappa(\alpha, \gamma_0)$ *and* δ *such that the following estimates hold:*

$$\limsup_{t\to\infty} \left|\hat{\theta}(t) - \theta\right| < \kappa(D\delta + 3\Delta),$$
$$\lim_{t\to\infty} \left|\hat{\lambda}_i(t) - \lambda_i\right| < \delta,$$
$$\lim_{t\to\infty} \left|\hat{x}_1(t) - x_1(t)\right|_\varepsilon = 0.$$

The proof of Theorem B.1 is based on Theorem 1 and Corollary 4 in [157]. See also [158]. A generalization of Theorem B.1 will be made available in [156].

Theorem B.1 assures that the estimates $\hat{\theta}(t)$, $\hat{\lambda}(t)$ converge to a neighborhood of the actual values θ, λ asymptotically. Given that $\left|\hat{x}_1(t) - x_1(t)\right|_\varepsilon \to 0$ as $t \to \infty$, the size of this neighborhood can be specified as a function of the parameter ε. The value of ε in turn depends on the amount of noise in the driving signal, and the values of Δ and γ_i (the smaller the Δ, γ_i the smaller the ε) such that the former, taking the presence of noise into account, can in principle be made sufficiently small.

B.4 Experimental validation

It this section it is demonstrated how the results can be applied to the problem of estimating the parameters of a neuronal model from *in vitro* measurements of single neurons. In particular, an algorithm is constructed that allows fitting the modified Hindmarsh-Rose model (B.4) to a spike train recorded from real neural cells in slices of hippocampal tissue of mouse[2]. Since the measured signal contains solely spiking dynamics the third equation of the Hindmarsh-Rose model, i.e. the slow variable, is neglected. Hence, the problem reduces to finding the parameters $\theta_{1,0}$, $\theta_{1,1}$, $\theta_{1,2}$, $\theta_{1,3}$, $\theta_{2,0}$, $\theta_{2,1}$, $\theta_{2,2}$, λ_2 of the reduced version of (B.4):

$$\dot{x}_1(t) = \theta_{1,3}x_1^3(t) + \theta_{1,2}x_1^2(t) + \theta_{1,1}x_1(t) + \theta_{1,0} + x_2(t) + g(t), \tag{B.22a}$$
$$\dot{x}_2(t) = -\lambda_2 x_2(t) + \theta_{2,2}x_1^2(t) + \theta_{2,1}x_1(t) + \theta_{2,0}. \tag{B.22b}$$

In the experimental data the input function $g(t)$ was a constant current such that $g(t)$, in this case, can be absorbed into the parameter $\theta_{1,0}$. Notice also that the value of $\theta_{2,0}$ can

[2] The data of the the single neuron recordings is provided by Dr. Alexey Semyanov and Dr. Inseon Song of the Semyanov Research Unit, Riken BSI.

be aggregated into the parameter $\theta_{1,0}$. Thus instead of (B.22) one obtains the following equations:

$$\dot{x}_1(t) = \theta_{1,3}x_1^3(t) + \theta_{1,2}x_1^2(t) + \theta_{1,1}x_1(t) + \theta_{1,0}^* + x_2(t), \tag{B.23a}$$

$$\dot{x}_2(t) = -\lambda_2 x_2(t) + \theta_{2,2}x_1^2(t) + \theta_{2,1}x_1(t). \tag{B.23b}$$

Using (B.15), (B.18) and Theorem B.1, the following system is capable of estimating the unknown parameters of (B.23):

$$\dot{\hat{x}}_1(t) = -\alpha(\hat{x}_1(t) - x_1(t)) + \hat{\theta}^\top(t)\bar{\phi}(t, x_1(t), \hat{\lambda}_2(t)), \tag{B.24a}$$

$$\dot{\hat{\theta}}(t) = -\gamma_\theta \cdot (\hat{x}_1(t) - x_1(t)) \cdot \bar{\phi}(t, x_1(t), \hat{\lambda}_2(t)), \tag{B.24b}$$

$$\dot{\xi}_{1,1}(t) = \gamma_1 \cdot \sigma(|\hat{x}_1(t) - x_2(t)|_\varepsilon) \cdot \left(\xi_{1,1}(t) - \xi_{2,1}(t) - \xi_{1,i}(t)\left(\xi_{1,1}^2(t) + \xi_{2,1}^2(t)\right)\right), \tag{B.24c}$$

$$\dot{\xi}_{2,1}(t) = \gamma_1 \cdot \sigma(|\hat{x}_1(t) - x_1(t)|_\varepsilon) \cdot \left(\xi_{1,1}(t) + \xi_{2,1}(t) - \xi_{2,1}(t)\left(\xi_{1,1}^2(t) + \xi_{2,1}^2(t)\right)\right), \tag{B.24d}$$

$$\hat{\lambda}_2(t) = \lambda_{1,\min} + \frac{\lambda_{1,\max} - \lambda_{2,\min}}{2}(\xi_{1,1}(t) + 1), \tag{B.24e}$$

with $\sigma(\cdot) = \arctan(\cdot)$. In (B.24) the vector $\hat{\theta}$ is the estimate of $\theta = (\theta_{0,0}^*, \theta_{0,1}, \theta_{0,2}, \theta_{0,3}, \theta_{1,1}, \theta_{1,2})^\top$, and $\hat{\lambda}_1$ is the estimate of λ_1. The function $\bar{\phi}_0(x_0(t), \hat{\lambda}_1, t)$ in (B.24) is the computationally realizable approximation of

$$\phi(t, x_1(t), \hat{\lambda}_2(t)) = \begin{pmatrix} 1 \\ x_1(t) \\ x_1^2(t) \\ x_1^3(t) \\ \int_{t_0}^t e^{-\hat{\lambda}_2(s)(t-s)} x_1(s)\mathrm{d}s \\ \int_{t_0}^t e^{-\hat{\lambda}_2(s)(t-s)} x_1^2(s)\mathrm{d}s \end{pmatrix} \tag{B.25}$$

The measured signal $x_1(t)$ is periodic, hence the Fourier-series expansion (B.12) is used to approximate (B.25). The domain Ω_λ is defined as $\Omega_\lambda = [0.5, 2.5]$ with $\lambda_{\min} = 0.5$ and $\lambda_{\max} = 2.5$, respectively. The Fourier-approximation (B.12) of (B.25) is persistently exciting for all $\hat{\lambda}_2 \in \Omega_\lambda$. In simulations the following set of parameters is used: $\gamma_\theta = 3$, $\gamma_1 = 0.02/\pi$, $\alpha = 20$, and $\varepsilon = 0.12$. The trajectories of the estimates $\hat{\lambda}_1(t)$ for various initial conditions are shown in the top panel of Figure B.2(a). The trajectories $\hat{\lambda}_2(t)$ converge to a bounded domain in the interval $[2, 2.4]$. For each value of $\hat{\lambda}_2$ the estimates $\hat{\theta}$ converge to a bounded domain as well. For example, for the trajectory starting at $\hat{\lambda}_2(0) = 0.5$ the following estimates are computed:

$$\hat{\theta}_{1,3} \in [-10.4, -10.25], \qquad \hat{\theta}_{1,2} \in [-4.45, -4.3], \qquad \hat{\theta}_{1,1} \in [6.6, 6.75],$$
$$\hat{\theta}_{1,0}^* \in [0.75, 0.95], \qquad \hat{\theta}_{2,2} \in [-32.5, -32.4], \qquad \hat{\theta}_{2,1} \in [-32.2, -32.1].$$

The range of these estimates correspond to the amount of uncertainty of in the system (B.23). After some manual tuning it is found that the following choice of parameters $\hat{\theta}$,

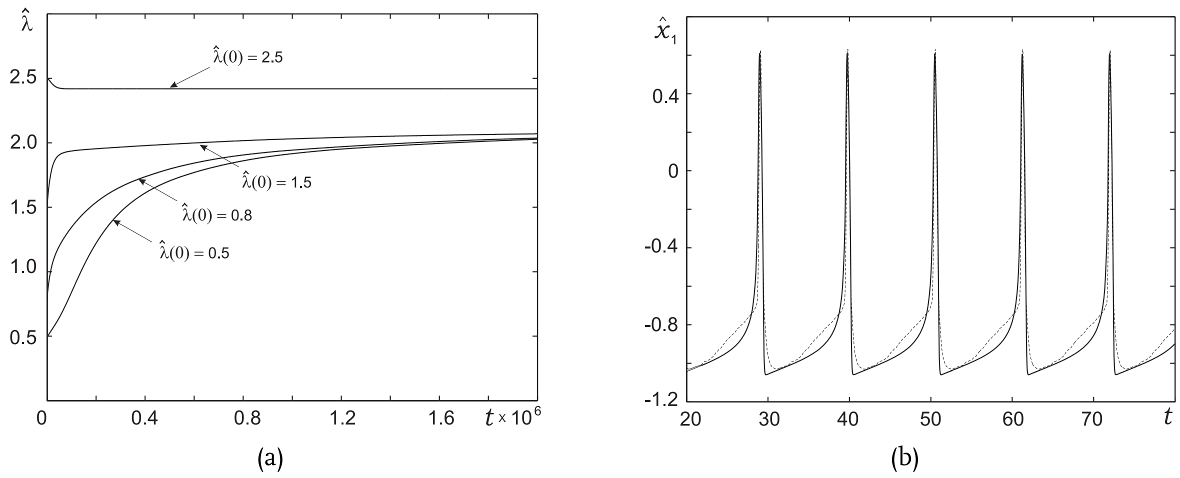


Figure B.2. (a) Trajectories $\hat{\lambda}(t)$ as functions of time for different values of initial conditions. (b) Trajectory $\hat{x}_1(t)$ of system (B.23) with parameters (B.26) (solid line) plotted against the actual data (dashed line).

$\hat{\lambda}_2$ results in rather accurate fitting:

$$\begin{aligned} &\hat{\theta}_{1,3} = -10.4, \quad &&\hat{\theta}_{1,2} = -4.35, \quad \hat{\theta}_{1,1} = 6.65, \quad &&\hat{\theta}^*_{1,0} = 0.9125, \\ &\hat{\theta}_{2,2} = -32.45, \quad &&\hat{\theta}_{2,1} = -32.15, \quad \hat{\lambda}_2 = 2.027. && \end{aligned} \tag{B.26}$$

The reconstructed trajectory $\hat{x}_1(t)$ with the parameters (B.26) is shown in the bottom panel of Figure B.2(b). Notice that despite the presence of small mismatches along the trajectories, the amplitude and the shape of the spikes do closely follow the measured response of the hippocampal neuron.

B.5 Discussion

In this appendix a method is presented with which the parameters of systems that can not be transformed into the observer canonical form can be estimated. The proposed method can be applied to systems that are of the class (B.5), such as (mathematical) models that mimic neuronal behavior. The method is demonstrated by a successful reconstruction of the states and estimations of the parameters of a modified Hindmarsh-Rose model driven by spikes recorded from a single neuron in vitro. In particular, it is shown that the spiking dynamics measured from a single neuron from the hippocampal area of mouse can be reasonably accurate reconstructed with the modified Hindmarsh-Rose model (B.4). Moreover, the estimated parameters of the model converge to small bounded domains. The size of these domains can, in principle, be decreased by assigning a smaller value to the parameter ε. However, it might be possible that the model is not accurate enough to describe the spikes with such precision. The fact that the modified model's parameters

$\theta_{1,1}, \theta_{2,1} \neq 0$ indicates that the equations of the original Hindmarsh-Rose model are too restricted for proper parameter fitting and our choice to use the modified model is justified. This appendix considered a simplified case where the clamping current applied to the neuron was constant and the neuron produced simple spiking behavior. In general, the output function of neurons is more complicated. Bursting sequences, for instance, are noticed in neurons of the pond snail *Lymnaea* [62] and firing frequency adaptation often occurs when the neuron is stimulated with block shaped currents. In order to mimic this more complicated behavior, the full set of equations of the modified Hindmarsh-Rose model should be taken into account.

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Samenvatting

Synchroon gedrag in netwerken van gekoppelde systemen

Synchronisatie in netwerken van gekoppelde dynamische systemen is een interessant fenomeen dat voorkomt in de natuur en verschillende takken van de wetenschap en techniek. Voorbeelden omvatten het gelijktijdig oplichten van duizenden vuurvliegjes, het synchrone vuren van zogenaamde “action potentials” door groepen van neuronen, coöperatief gedrag van robots en synchronisatie van chaotische systemen met toepassingen in de beveiligde communicatie. Hoe kan het dat systemen in een netwerk te synchroniseren? Om te kunnen synchroniseren is het noodzakelijk dat de systemen informatie over hun toestand communiceren met de systemen waaraan ze gekoppeld zijn. De vraag is dan hoe de communicatie structuur eruit moet zien en hoe de systemen in het netwerk moeten reageren op de ontvangen informatie om synchronisatie te bereiken. Met andere woorden, welke netwerk structuren en wat voor een koppelingsfuncties resulteren in synchronisatie van de systemen? En aangezien het uitwisseling van informatie tijd kost, kunnen systemen in netwerken synchroniseren in aanwezigheid van tijdsvertragingen in de koppelingen?

Het eerste deel van dit proefschrift richt zich op de synchronisatie van *identieke* systemen die interactie hebben via *diffusieve koppeling*. Dat is een koppeling bestaande uit het gewogen verschil van de uitgangssignalen van de systemen. Deze koppeling mogen tijdsvertragingen bevatten. Twee soorten diffusieve tijdsvertraagde koppelingen worden beschouwd: *koppeling type I* is een diffusieve koppeling waarin alleen de gezonden signalen onderhevig zijn aan tijdsvertragingen, en *koppeling type II* is een diffusieve koppeling waarin alle signalen zijn vertraagd. Het is bewezen dat netwerken bestaande uit zogenaamde *strikt semipassieve* systemen die interactie hebben via een diffusieve tijdsvertraagde koppeling begrensde oplossingen hebben. Daarnaast is het bewezen dat zogenaamde *minimum-fase* strikt semipassieve systemen die diffusief gekoppeld zijn altijd synchroniseren op voorwaarde dat de koppelingssterkte voldoende groot is. Als de koppelingen tijdsvertraagde signalen bevatten, dan dient daarnaast ook nog het product van de tijdsvertraging en de koppelingssterkte voldoende klein te zijn om te synchroniseren.

Vervolgens wordt de specifieke rol van de structuur van het netwerk in relatie tot synchronisatie van de diffusief gekoppelde systemen besproken. Ten eerste worden condities gegeven voor het bestaan van zogenaamde *lineaire invariante variëteiten* voor netwerken van diffusieve tijdsvertraagde gekoppelde systemen. Deze condities hangen af van het bestaan van symmetrieën in het netwerk. Het is bewezen dat de oplossingen van de diffusief gekoppelde systemen die strikt semipassief en minimum-fase zijn convergeren naar een dergelijke lineaire invariante variëteit op voorwaarde dat de koppelingssterkte voldoende groot is en het product van de tijdsvertraging en de koppelingssterkte voldoende klein is. Het netwerk toont *gedeeltelijke synchronisatie* indien niet alle, maar slechts enkele systemen in een netwerk synchroniseren voor deze waarden van de koppelingssterkte en tijdsvertraging. Ten tweede wordt voor systemen die interactie hebben via *symmetrische koppeling type II* aangetoond dat de waarden van de koppelingssterkte en tijdsvertraging waarvoor *elk* netwerk synchroniseert kunnen worden afgeleid uit de structuur van het netwerk en de waarden van de koppelingssterkte en de tijdsvertraging waarvoor twee gekoppelde systemen synchroniseren.

In het tweede deel van het proefschrift wordt de theorie uit het eerste deel gebruikt om synchronisatie in netwerken van gekoppelde zenuwcellen uit te verklaren. Allereerst is het bewezen dat vier belangrijke modellen die de dynamische activiteit van zenuwcellen bechrijven, namelijk het *Hodgkin-Huxley* model, het *Morris-Lecar* model, het *Hindmarsh-Rose* model en het *Fitzhugh-Nagumo* model, allen strikt semipassief zijn. Omdat al deze modellen ook nog eens de minimum-fase eigenschap bezitten, zullen netwerken van diffusief gekoppelde zenuwcellen synchroniseren indien de koppeling voldoende sterk is en het product van de tijdsvertraging en de koppelingssterkte voldoende klein is. Numerieke simulaties met verschillende netwerken van diffusief gekoppelde Hindmarsh-Rose zenuwcellen ondersteunen deze theoretische bevindingen. Daarnaast zijn de theoretische bevindingen gevalideerd met behulp van een experimentele opstelling bestaande uit type II gekoppelde elektronische Hindmarsh-Rose zenuwcellen.

Het proefschrift gaat verder met een studie van netwerken bestaande uit *pancreatische β-cellen*. Het is bekend dat deze β-cellen diffusief gekoppeld zijn en synchroniseren. De gesynchroniseerde activiteit van een netwerk van β-cellen is gerelateerd aan de afscheiding van insuline wat het hormoon is dat de suikerspiegel reguleert. Als het netwerk bestaat uit gezonde β-cellen en dode β-cellen kan het voorkomen dat de activiteit van het netwerk afneemt of zelfs stopt. Dit resulteert in een sterk verminderde afscheiding van insuline. (Patiënten die onvoldoende insuline aanmaken door het afsterven van β-cellen lijden aan *diabetes type 1*.) Of een netwerk activiteit toont of niet hangt af van het aantal gezonde cellen in verhouding tot het aantal dode cellen. Een bifurcatie analyse geeft een schatting van uit hoeveel gezonde cellen het netwerk dient te bestaan ten opzichte van het aantal dode cellen om het goed functioneren van het netwerk te kunnen garanderen.

Als laatste wordt het *geregelde synchronisatie probleem* voor gekoppelde strikt semipassieve systemen beschouwd. Een systematische procedure wordt gepresenteerd voor het ontwerpen van koppelingsfuncties die synchronisatie garanderen in een netwerk waarin alle systemen interactie met elkaar hebben. De koppelingsfuncties hebben de vorm van een bepaalde integraal over een scalaire niet-negatieve functie op een interval bepaald door de uitgangen van de systemen. Het voordeel van deze koppelingsfuncties is dat de koppelingssterke alleen hoog is als dat nodig is, bijvoorbeeld in de delen van de toestandsruimte van het netwerk waar niet-lineariteiten dienen te worden onderdrukt. Numerieke simulaties in netwerken van Hindmarsh-Rose zenuwcellen ondersteunen de theoretische bevindingen.

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Curriculum Vitae

Erik Steur was born on December 22, 1982 in Tiel, the Netherlands. He completed his secondary school at RSG Lingecollege, Tiel, in 2001, after which he started his study Mechanical Engineering at Eindhoven University of Technology. As part of his study he carried out an international internship at RIKEN Brain Science Institute, Wako-shi, Japan, on the topic “Parameter Estimation Techniques for Hindmarsh-Rose Neurons”. In september 2007 he received his Master’s degree (cum laude) in Mechanical Engineering from Eindhoven University of Technology on the thesis “On Synchronization of Electromechanical Hindmarsh-Rose Oscillators” which was awarded the KIVI NIRIA regeltechniek prijs 2008.

In oktober 2007 he started as a Ph.D. student in the Dynamics and Control group at the department of Mechanical Engineering of Eindhoven University of Technology. The main topic of his research was the development of theory for synchronization of systems which interact via time-delay coupling. In addition it was investigated how these results could be applied to networks of interacting neurons. In september 2010 he spent a month at University of Leicester, department of Mathematics. The main results of his Ph.D. research are presented in this thesis.