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Citation for published version (APA):

Belitser, E., Andrade Serra, De, P. J., & Zanten, van, J. H. (2013). *Rate-optimal Bayesian intensity smoothing for inhomogeneous Poisson processes*. (arXiv.org; Vol. 1304.6017 [math.ST]). s.n.

Document status and date:

Published: 01/01/2013

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

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Rate-optimal Bayesian intensity smoothing for inhomogeneous Poisson processes

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April 23, 2013

Abstract

We apply nonparametric Bayesian methods to study the problem of estimating the intensity function of an inhomogeneous Poisson process. We exhibit a prior on intensities which both leads to a computationally feasible method and enjoys desirable theoretical optimality properties. The prior we use is based on B-spline expansions with free knots, adapted from well-established methods used in regression, for instance. We illustrate its practical use in the Poisson process setting by analysing count data coming from a call centre. Theoretically we derive a new general theorem on contraction rates for posteriors in the setting of intensity function estimation. Practical choices that have to be made in the construction of our concrete prior, such as choosing the priors on the number and the locations of the spline knots, are based on these theoretical findings. The results assert that when properly constructed, our approach yields a rate-optimal procedure that automatically adapts to the regularity of the unknown intensity function.

Keywords: Adaptive estimation; Bayesian nonparametric estimation; Contraction rate; Markov chain Monte Carlo; Poisson process; Splines.

1 Introduction

Poisson processes have a long-standing history and are some of the most widely used processes in statistics to study temporal and spacial count data, in diverse fields such as communication, meteorology, seismology, hydrology, astronomy, biology, medicine, actuary sciences and queueing, among others. In this paper we focus on inhomogeneous Poisson processes on the line with periodic intensity functions, which are models for count data in settings with a natural periodicity.

Nonparametric Bayesian methods, which are used more and more in many different statistical settings, have so far only been used on a limited scale to analyze such models. From the applied perspective they can be attractive for making inference about intensity

functions, for the same reasons they are appealing in other situations. Estimating the intensity essentially requires some sort of smoothing of the count data, and a nonparametric Bayesian approach can provide a natural way for achieving this. Using hierarchical priors we can automatically achieve a data-driven selection of the degree of smoothing. Moreover, Bayesian methods provide a way to quantify the uncertainty about the intensity using the spread of the posterior distribution. A typical implementation provides a computational algorithm that can generate a large number of (approximate) draws from the posterior. From this it is usually straightforward to construct numerical credible bands or sets.

The relatively small number of papers using nonparametric Bayesian methodology for intensity function smoothing have explored various possible prior distributions on intensities. An early reference is Møller et al. (1998), who consider log-Gaussian priors. Other papers employing Gaussian process priors, combined with suitable link functions, include Adams et al. (2009) and Palacios & Minin (2013). Kottas & Sansó (2007) consider kernel mixtures priors. See also the related paper DiMatteo et al. (2001), in which count data is analysed using spline-based priors.

The cited papers show that nonparametric Bayesian inference for inhomogeneous Poisson processes can give satisfactory results in various applications. On the theoretical side however the existing literature provides no performance guarantees in the form of consistency theorems or related results. It is by now well known that nonparametric Bayes methods may suffer from inconsistency, even when seemingly reasonable priors are used (e.g. Diaconis & Freedman (1986)). The purpose of this paper is therefore to exhibit a Bayesian approach to nonparametric intensity smoothing that is both computationally feasible and at the same time theoretically underpinned by results on consistency and related issues like convergence rates and adaptation to smoothness. Such theoretical results have in the last decade been obtained for various statistical settings, including density estimation, regression, classification, drift estimation for diffusions, etcetera (see e.g. Ghosal (2010) for an overview of some of these results). Until now, intensity estimation for inhomogeneous Poisson processes has remained largely unexplored.

As motivation and starting point for the paper we consider the problem of analysing count data from a call center. The same type of data were analyzed by frequentist methods in the paper Brown et al. (2005). We revisit the problem using a nonparametric Bayesian method employing a spline-based prior on the unknown intensity function. In addition to a single estimator of the intensity, this method provides credible bounds indicating the degree of uncertainty. In Section 3 we study theoretical properties of our procedure, namely consistency, posterior contraction rates and adaptation to smoothness. The results show that we have set up our procedure in such a way that we obtain consistent, rate-optimal estimation of the intensity and that the method adapts automatically to the unknown smoothness of the intensity curve, up to the level of the order of the splines that are used. Section 4 concludes with some remarks and directions for further research.

2 Analysis of call center data

2.1 Data and statistical model

The approach we propose and study is motivated by the wish to analyse a dataset consisting of counts of telephone calls arriving at a certain call center. The dataset was obtained from the website of the S.E.E. Center (<http://ie.technion.ac.il/Labs/Serveng/>) of the Faculty of Industrial Engineering and Management, Technion in Haifa, Israel. It consists of counts for calls arriving at a bank's 24 hour a day call center in the United States of America. We considered the records for the period from November 1, 2001 until December 31, 2001, covering a total of about 2.8 million incoming phone calls. These events are recorded in 30 second intervals with an average of about 32 calls per minute. The raw data are plotted in Figure 1.

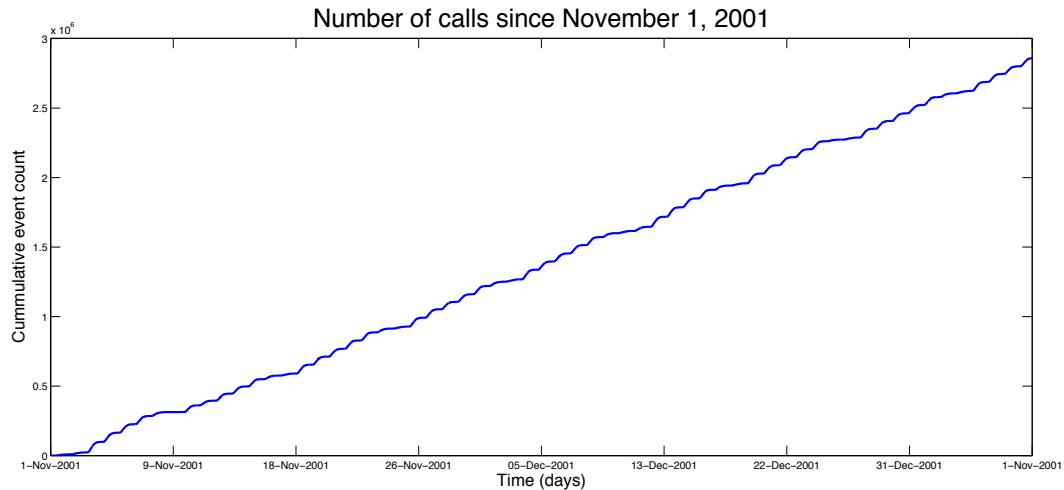


Figure 1: Number of incoming phone calls between November 1, 2001 and December 31, 2001.

We model the full count data as the realization of an inhomogenous Poisson process N with an intensity function λ that is periodic, the period being 24 hours (Daley & Vere-Jones (1988)). This Poisson assumption is natural and is investigated in some detail in Brown et al. (2005), who could not find significant evidence to the contrary in a similar dataset (same kind of data, but over a different time period). See also Belitser et al. (2012), who study the periodicity in the data.

Let n be the number of days for which we have data ($n = 61$) and let T be the period (24 hours). Then the full call arrival counting process is given by $N = (N_t : t \in [0, nT])$, where N_t is the number of calls arriving in the time interval $[0, t]$. The Poisson assumption means that for every $0 \leq s \leq t$, the number of arrivals $N_t - N_s$ is independent of the history $(N_u : u \leq s)$ up till time s and that is has a Poisson distribution with mean

$$\int_s^t \lambda(u) du.$$

We will assume throughout that λ is at least a continuous function. The periodicity assumption then means that λ is a T -periodic function, i.e. $\lambda(t + T) = \lambda(t)$ for all $t \geq 0$. For $i = 1, \dots, n$ we define the counting process $N^{(i)} = (N_t^{(i)} : t \in [0, T])$ by

$$N_t^{(i)} = N_{(i-1)T+t} - N_{(i-1)T}, \quad t \in [0, T],$$

i.e. $N^{(i)}$ counts the number of arrivals during day i . Note that by the independence of the increments of the process N , the processes $N^{(i)}$ are independent inhomogenous Poisson processes which have the restriction of λ to $[0, T]$ as intensity function.

Our goal is to make inference about this function. Note that we do not observe the full process N . We only observe it at discrete times, namely every 30 seconds. On average about 16 calls arrive in a 30 second time interval, so we really only see aggregated counts. Let Δ be the time between observations (30 seconds) and let $m = T/\Delta$ be the number of counts per day that we have in our dataset ($m = 2880$ in our case). Then for every $i = 1, \dots, n$ and $j = 1, \dots, m$, the number of arrivals

$$C_{ij} = N_{j\Delta}^{(i)} - N_{(j-1)\Delta}^{(i)} \tag{1}$$

in the j th time interval on day i has a Poisson distribution with parameter

$$\lambda_j = \int_{(j-1)\Delta}^{j\Delta} \lambda(t) dt. \tag{2}$$

We denote the total available count data by $C^n = (C_{ij} : i = 1, \dots, n, j = 1, \dots, m)$. It follows that the likelihood is given by

$$p(C^n | \lambda) = \prod_{i=1}^n \prod_{j=1}^m \frac{\lambda_j^{C_{ij}} e^{-\lambda_j}}{C_{ij}!}. \tag{3}$$

In the following section we describe the prior we place on the intensity function λ .

2.2 Prior on the intensity function

There are different possible choices of priors for the function λ . A number of earlier considered options were already mentioned in the introduction (Gaussian processes, kernel mixtures, splines). The prior we investigate in this paper is motivated by the desire to have a computationally manageable procedure on the one hand and theoretical performance guarantees on the other. Still there will conceivably be more than one sensible choice meeting these requirements. Here we restrict ourselves to the investigation of a specific spline-based prior which is described in detail in this section.

More precisely we will employ a certain free-knot spline prior which is similar to priors considered earlier in different contexts (see for instance Smith & Kohn (1996), Denison et al. (1998), DiMatteo et al. (2001), or, more recently Sharef et al. (2010) and the references therein). Such priors have proven to be numerically attractive and capable of

capturing abrupt changes in functions of interest. This last point is relevant for our particular application, since we expect fluctuations during the day due to the varying activity of businesses over the day. Recently, several theoretical results were derived for spline-based priors in various setting as well (e.g. Ghosal et al. (2000), Ghosal et al. (2008), De Jonge & van Zanten (2012), Belitser & Serra (2013)). We will show in the next section that the procedure that we construct and implement has several desirable theoretical properties.

Background information on splines can be found, for example, in de Boor (2001) or Schumaker (2007). Let us fix some notations and terminology. A function is called a spline of order $q \in \mathbb{N}$, with respect to a certain partition of its support, if it is $q - 2$ times continuously differentiable and when restricted to each interval in the partition, it coincides with a polynomial of degree at most $q - 1$. Now consider $q \geq 2$. For any $j \in \mathbb{N}$, such that $j \geq q$ let $\mathcal{K}_j = \{(k_1, \dots, k_{j-q}) \in (0, T)^{j-q} : 0 < k_1 < \dots < k_{j-q} < 1\}$. We will refer to a vector $\mathbf{k} \in \mathcal{K}_j$ as a sequence of *inner knots*.

A vector $\mathbf{k} \in \mathcal{K}_j$ induces the partition $\{[k_0, k_1), [k_1, k_2), \dots, [k_{j-q}, k_{j-q+1}]\}$ of $[0, T]$, with $k_0 = 0$ and $k_{j-q+1} = T$. For $\mathbf{k} \in \mathcal{K}_j$, we denote by $\mathcal{S}^{\mathbf{k}} = \mathcal{S}_q^{\mathbf{k}}$ the linear space of splines of order q on $[0, T]$ with simple knots \mathbf{k} (see the definition of simple knots in, e.g., Schumaker (2007)). This space has dimension j and admits a basis of B-splines $\{B_1^{\mathbf{k}}, \dots, B_j^{\mathbf{k}}\}$. The construction of $\{B_1^{\mathbf{k}}, \dots, B_j^{\mathbf{k}}\}$ involves the knots $k_{-q+1}, \dots, k_{-1}, k_0, k_1, \dots, k_{j-q}, k_{j-q+1}, k_{j-q+2}, \dots, k_j$, with arbitrary extra knots $k_{-q+1} \leq \dots \leq k_{-1} \leq k_0 = 0$ and $T = k_{j-q+1} \leq k_{j-q+2} \leq \dots \leq k_j$. Usually one takes $k_{-q+1} = \dots = k_{-1} = k_0 = 0$ and $T = k_{j-q+1} = \dots = k_j$, and we adopt this choice as well. For $\mathbf{k} \in \mathcal{K}_j$ and $\boldsymbol{\theta} \in \mathbb{R}^j$ we denote by $s_{\boldsymbol{\theta}, \mathbf{k}}$ the spline in $\mathcal{S}^{\mathbf{k}}$ that has coefficient vector $\boldsymbol{\theta}$ relative to the basis $\{B_1^{\mathbf{k}}, \dots, B_j^{\mathbf{k}}\}$, i.e.

$$s_{\boldsymbol{\theta}, \mathbf{k}}(t) = \sum_{i=1}^j \theta_i B_i^{\mathbf{k}}(t), \quad t \in [0, T].$$

To define our prior Π on λ we first fix the order $q \geq 2$ of the splines that we use (cubic splines are popular, they correspond to the choice $q = 4$) and the minimum and maximum intensities $0 \leq M_1 < M_2$. Then a draw from the prior Π is constructed as follows:

1. (Number of B-splines): Draw $J \geq q$ according to a shifted Poisson distribution with mean μ .
2. (Location of the knots): Given $J = j$, construct a regular $1/j^2$ -spaced grid in $(0, T)$. Then uniformly at random, choose $j - q$ grid elements (without replacement) to form a sequence of inner knots \mathbf{k} .
3. (B-spline coefficients): Also given $J = j$, and independent of the previous step, draw a vector $\boldsymbol{\theta}$ of j independent, uniform $U[M_1, M_2]$ -distributed B-spline coefficients.
4. (Random spline): Finally, construct the random spline $s_{\boldsymbol{\theta}, \mathbf{k}}$ of order q corresponding to the inner knots \mathbf{k} and with B-spline coefficient vector $\boldsymbol{\theta}$.

The specific choices made in the construction of the prior, like the Poisson distribution on J , choosing the knots uniformly at random from a grid, etcetera, are motivated by

the optimality theory that we derive in Section 3. The theory shows that there is some more flexibility, but for choices too far from the ones proposed above the performance guarantees brake down. Technically the prior on λ is the measure Π on the space $C[0, T]$ of continuous functions on $[0, T]$ given by the law, or distribution of the random spline $s_{\theta, \mathbf{k}}$ described above. The splines in $\mathcal{S}_q^{\mathbf{k}}$ are $q - 2$ times continuously differentiable, hence in this sense the choice of q determines the regularity of the prior. We will see in the next section that it also determines the maximal degree of smoothness of the true underlying intensity to which our procedure can adapt. In applications like the one we are interested in here, a sensible choice of the parameters M_1 and M_2 will typically be suggested by the average number of counts per time unit in the data. In Section 4 we comment on other possibilities, such as putting a prior on these parameters as well, in order to let their value be determined by the data automatically. This is possible, but will come at an additional computational cost. The construction of the grid in step 2. is non-standard compared to other spline-based priors proposed in the literature. It is motivated by recent work of Belitser & Serra (2013) and will allow us to derive desirable theoretical properties in the next section.

2.3 Posterior inference

For the data described in Section 2.1, with likelihood (3), and the spline prior Π described in Section 2.2, we implemented an MCMC procedure to sample from the corresponding posterior distribution of the intensity function λ of interest. The minimal and maximal intensity parameters M_1 and M_2 were set to 200 and 20000, respectively. These numbers were motivated by the range of the data (time is measured in hours). We took the order q of the splines equal to 4.

Since our prior is very similar to the ones used previously in for instance DiMatteo et al. (2001) or Sharef et al. (2010) in regression or hazard rate estimation settings, our computational algorithm is a rather straightforward adaptation of existing methods. A generic state of the chain is a $(2J - q + 1)$ -dimensional vector $(j, \mathbf{k}, \boldsymbol{\theta})$ where $j \in \mathbb{N}$, $j \geq q$ is the model index, $\mathbf{k} = \mathbf{k}_j \in (0, T)^{j-q}$ is a vector of inner knots and $\boldsymbol{\theta} = \boldsymbol{\theta}_j \in \mathbb{R}^j$ is a vector of B-spline coordinates. Together, these index a spline $s_{\theta, \mathbf{k}} = s_{\boldsymbol{\theta}_j, \mathbf{k}_j}^q \in \mathcal{S}_{\mathbf{k}_j}^q$. We will abbreviate the corresponding posterior distribution by $\pi(j, \mathbf{k}, \boldsymbol{\theta} | C^n)$. Since the splines involved are easy to evaluate and integrate we can compute the likelihood, and then the posterior, up to the normalization constant, without any approximations being needed.

We consider four different types of moves for the MCMC chain, namely: a) perturbing the coefficients, b) moving the location of one knot, c) birth of a new knot and d) death of an existing knot. Each of these moves is proposed, independently and respectively, with probabilities p_a , p_b , $p_c(j)$ and $p_d(j)$ where for each $j \geq q$, $p_a + p_b + p_c(j) + p_d(j) = 1$. In fact, we start by picking $0 < p_a + p_b < 1$ as parameters of the algorithm; if μ is the mean of the prior on J , then we take $p_c(q) = 1 - p_a - p_b$, $p_d(q) = 0$ and, for $j > q$, $p_c(j) = (1 - p_a - p_b)2^{-\frac{j-q}{\mu-q}}$ and $p_d(j) = (1 - p_a - p_b)(1 - 2^{-\frac{j-q}{\mu-q}})$. This choice results in $p_c(\mu) = p_d(\mu)$ if $j = \mu$, $p_c(\mu) > p_d(\mu)$ if $j < \mu$, $p_c(\mu) < p_d(\mu)$ if $j > \mu$.

When perturbing the coefficients we perform simple (Gaussian) random walk MCMC steps; the standard deviation of the random walk was chosen such that we obtained an

acceptance rate of roughly 23% for this type of move, as prescribed in Gelman et al. (1997). Let φ_j be the joint density of j i.i.d. standard normal random variables. Our proposals correspond to a move $(j, \mathbf{k}, \boldsymbol{\theta}) \rightarrow (j, \mathbf{k}, \boldsymbol{\theta} + \sigma \mathbf{u})$ which we accept with probability $\min(A(\mathbf{u}), 1)$, with

$$A(\mathbf{u}) = \frac{\pi(j, \mathbf{k}, \boldsymbol{\theta} + \sigma \mathbf{u} | C^n) p_a \varphi_j(-\sigma \mathbf{u})}{\pi(j, \mathbf{k}, \boldsymbol{\theta} | C^n) p_a \varphi_j(\sigma \mathbf{u})} = \frac{\pi(j, \mathbf{k}, \boldsymbol{\theta} + \sigma \mathbf{u} | C^n)}{\pi(j, \mathbf{k}, \boldsymbol{\theta} | C^n)}.$$

Moving a knot is also straightforward; one of the current $j - q$ knots, say k_i , is picked uniformly at random among those in \mathbf{k} , and we propose to change its location depending on how many of its neighboring position on the j^{-2} -spaced grid are free – we say that two knots k, k' are neighbors if $|k - k'| \leq j^{-2}$. This means that we propose a move $(j, \mathbf{k}, \boldsymbol{\theta}) \rightarrow (j, \mathbf{k}', \boldsymbol{\theta})$ where \mathbf{k} and \mathbf{k}' differ only at the i -th position: if k_i has two free neighboring positions, then it moves to either of them with equal probability $c_i = c_i(k_{i-1}, k_i, k_{i+1}) = 1/2$; if k_i only has one free neighboring position, then, with equal probability $c_i = 1/2$, it either moves to this free position or it does not move at all; if k_i has no free neighboring positions then it does not move, with probability $c_i = 1$. These particular choices assure the reversibility of the moves. We accept such a proposal with probability $\min(A(i), 1)$ where $A(i)$ is given by

$$A(i) = \frac{\pi(j, (k_1, \dots, k'_i, \dots, k_{j-q}), \boldsymbol{\theta} | C^n) p_b (j - q)^{-1} c_i}{\pi(j, (k_1, \dots, k_i, \dots, k_{j-q}), \boldsymbol{\theta} | C^n) p_b (j - q)^{-1} c_i} = \frac{\pi(j, \mathbf{k}', \boldsymbol{\theta} | C^n)}{\pi(j, \mathbf{k}, \boldsymbol{\theta} | C^n)}.$$

Birth moves and death moves, where a new knot is respectively added and removed, are reverse moves of one another and so we will outline only how to perform the birth move. We propose a move $(j, \mathbf{k}, \boldsymbol{\theta}) \rightarrow (j + 1, \mathbf{k}', \boldsymbol{\theta}')$ where we add a new knot to the vector \mathbf{k} and a new coefficient to the vector $\boldsymbol{\theta}$. In doing so, a new B-spline is introduced to the B-spline basis and a new B-spline coefficient is generated. The new knot vector \mathbf{k}' contains all knots from \mathbf{k} rounded to the closest grid point on a $(j + 1)^{-2}$ spaced grid with the extra knot then picked uniformly at random among the remaining free positions; call it $k_{i-1} < k' < k_i$. Note that this construction does not prevent two knots in \mathbf{k}' from occupying the same position; such knot vectors have posterior probability 0, though, so that the probability of moving to such a state is zero. The coefficients on this basis are then picked as $\boldsymbol{\theta}' = f(\boldsymbol{\theta}, u) = (\theta_1, \dots, \theta_{m-1}, u, \theta_m, \dots, \theta_j)$ where f will be linear and invertible, and u is a random seed, a normally distributed random number with mean $\eta(\boldsymbol{\theta})$, to be picked later, and variance 1. The new knot will belong to the support of q B-splines, namely the i -th through $(i + q - 1)$ -th B-splines and we pick the index m in $\{i, \dots, i + q\}$ depending on the knot's position within the interval $[k_{i-1}, k_i]$; namely $m = i + \lfloor (q + 1)(k' - k_{i-1}) / (k_i - k_{i-1}) \rfloor$, where $\lfloor a \rfloor$ is the largest integer smaller or equal to a . The mean of the random seed u will be picked as a weighted mean of the coefficients $\boldsymbol{\theta}$, namely, $\eta(\boldsymbol{\theta}) = \sum_{i=1}^{m-1} w_i \theta_i + \sum_{i=m}^j w_{i-m} \theta_i$, where the weights w_i are normalized and

$$w_i \propto \int_0^T B_{\mathbf{k}'_j, m}^q(t) B_{\mathbf{k}'_j, i}^q(t) dt, \quad i = 1, \dots, j + 1.$$

With probability $\min(A(j, k', u), 1)$ we make the move $(j, \mathbf{k}, \boldsymbol{\theta}) \rightarrow (j+1, \mathbf{k}', \boldsymbol{\theta}')$, with $\boldsymbol{\theta}' = f(\boldsymbol{\theta}, u)$ and $\mathbf{k}' = (k_1, \dots, k_{i-1}, k', k_i, \dots, k_{j-q})$, where

$$A(j, k', u) = \frac{\pi(j+1, \mathbf{k}', f(\boldsymbol{\theta}, u) | C^n) (1 - p_a - p_b) p_d(j+1) (j-q+1)^{-1}}{\pi(j, \mathbf{k}, \boldsymbol{\theta} | C^n) (1 - p_a - p_b) p_c(j) (j^2 - j + q)^{-1} \varphi_1(u)} |J_f|$$

where $|J_f|$ is the Jacobian of the linear mapping described before.

Figure 2 summarizes the outcome of the analysis. In the top panel it shows the posterior and 95% point-wise credible intervals, based on 10,000 samples from posterior. The lower panel shows a histogram for the locations of the knots corresponding to the samples from the chain used to generate the top panel. Note that as expected, relatively many knots are placed in periods in which there are relatively many fluctuations in the intensity.

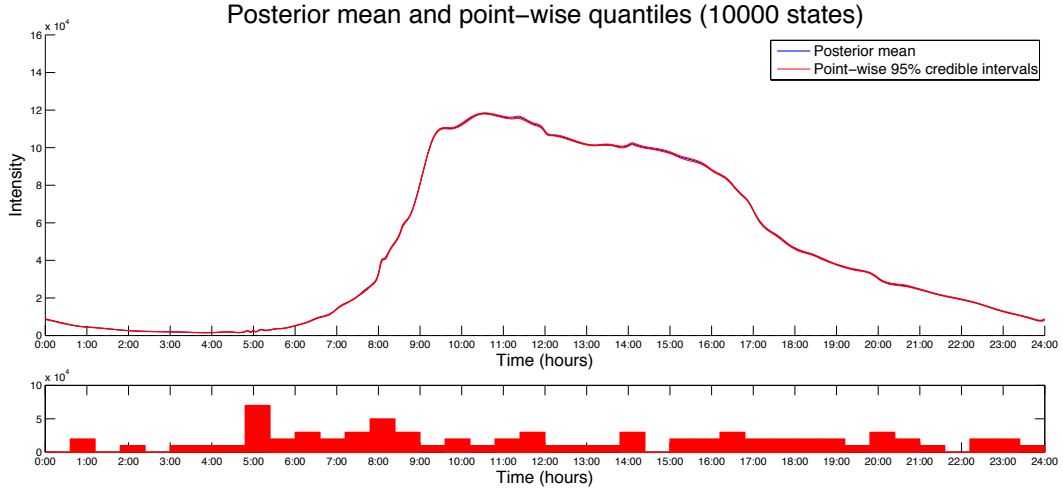


Figure 2: Top panel: posterior distribution of the intensity function λ based on the thinned data. (Blue: posterior mean, red: point-wise 95% credible intervals). Lower panel: posterior distribution of the knot locations (Histogram).

Due to the large amount of data (almost 3 million counts), the credible bands are very narrow. To illustrate the dependence on the amount of data we ran the analysis again with a thinned out dataset. We randomly removed counts, retaining about 1,000 counts. The same analysis then leads to the posterior plot given in Figure 3. In this case, the uncertainty in the posterior distribution becomes clearly visible.

We find that the prior that we defined in Section 2.2 is a computationally feasible choice for nonparametric Bayesian intensity smoothing in the context of this kind of periodic count data. In the next section we analyze its fundamental theoretical performance. See in particular Theorem 3 in Section 3.2.

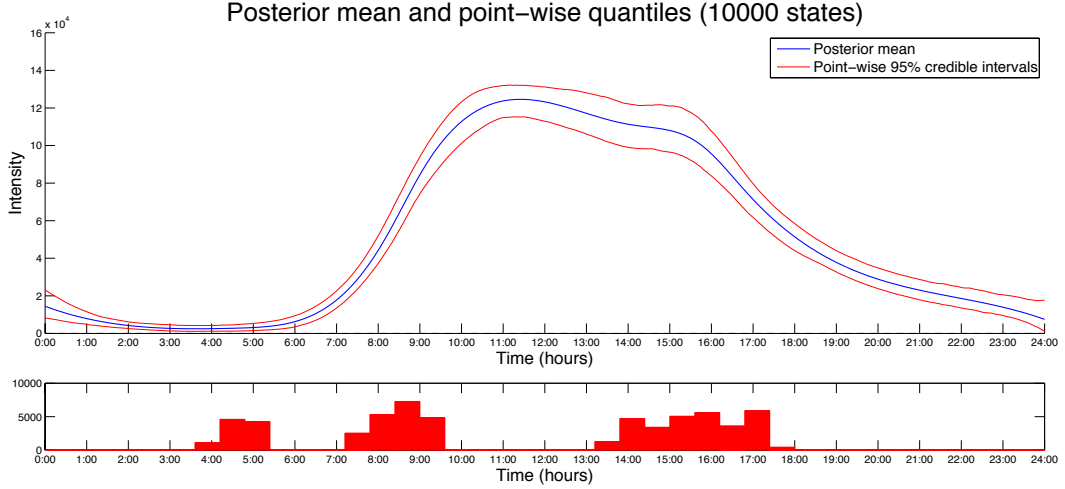


Figure 3: Top panel: posterior distribution of the intensity function λ based on the thinned data. (Blue: posterior mean, red: point-wise 95% credible intervals). Lower panel: posterior distribution of the knot locations (Histogram).

3 Theoretical results

3.1 Contraction rates for general priors

We derive our theoretical results for the particular prior we used in the Section 2 from general rate of contraction results that we present in this section. These are in the spirit of the general theorems about convergence rates of nonparametric Bayes procedures that were first developed for density estimation (Ghosal et al. (2000)) and later for various other statistical settings, see for instance van der Meulen et al. (2006), Ghosal & van der Vaart (2007), Panzar & van Zanten (2009). Here we complement this literature with general rate results regarding intensity estimation for inhomogenous Poisson processes. These results are not only applicable to the spline priors we consider in this paper, but may also be used to analyze contraction rates of other priors. Moreover, we formulate the theorems not just for the case that we have discrete observations of aggregated data, as in our data example, but also for the case that the full counting process is observed.

The setting is as in Section 2.1. We fix a period $T > 0$. In the full observations case we assume that for $n \in \mathbb{N}$, we observe an inhomogeneous Poisson process $N^n = (N_t^n : t \in [0, nT])$ up till time nT , with a T -periodic intensity function λ . Equivalently, we can say we observe n independent inhomogeneous Poisson processes $N^{(1)}, \dots, N^{(n)}$, indexed by $[0, T]$, and with a common intensity function λ , which is a positive, integrable function on $[0, T]$. It is well known that the law of N under the intensity function λ is equivalent to the law of a standard Poisson process and that the corresponding likelihood is given by

$$p(N^n | \lambda) = e^{-\int_0^{nT} (\lambda(t)-1) dt + \int_0^{nT} \log(\lambda(t)) dN_t^n}$$

(see for instance Jacod & Shiryaev (2003)). We consider prior distributions that charge strictly positive, continuous functions. Given such a prior Π_n on λ (which we allow to depend on n) we can then compute the corresponding posterior distribution $\Pi_n(\cdot \mid N^n)$ by Bayes' formula

$$\Pi_n(\lambda \in B \mid N^n) = \frac{\int_B p(N^n \mid \lambda) \Pi_n(d\lambda)}{\int p(N^n \mid \lambda) \Pi_n(d\lambda)}.$$

Formally we can view the prior Π_n as a measure on the space $\Lambda \subset C[0, T]$ of all continuous, strictly positive functions on $[0, T]$, endowed with its Borel σ -field. If we endow Λ with the uniform norm, the likelihood is a continuous function on Λ . Hence, the posterior is a well-defined measure on Λ .

The following theorem considers the frequentist setting in which the data are assumed to be generated by an unknown, “true” intensity function λ_0 . It gives conditions on the prior Π_n under which the posterior $\Pi_n(\cdot \mid N^n)$ contracts around the true λ_0 at a certain rate as the number of observed periods tends to infinity. The assumptions and conclusions of the theorem are formulated in terms of various distances on the intensity functions. For a continuous function f on $[0, T]$ we define the norms $\|f\|_2$ and $\|f\|_\infty$ as usual by

$$\|f\|_2^2 = \int_0^T f^2(t) dt, \quad \|f\|_\infty = \sup_{t \in [0, T]} |f(t)|.$$

For a set of positive continuous functions \mathcal{F} we write \mathcal{F}^c for its complement and $\sqrt{\mathcal{F}} = \{\sqrt{f} : f \in \mathcal{F}\}$. For $\varepsilon > 0$ and a norm $\|\cdot\|$ on \mathcal{F} , let $N(\varepsilon, \mathcal{F}, \|\cdot\|)$ be the minimal number of balls of $\|\cdot\|$ -radius ε needed to cover \mathcal{F} .

Theorem 1 (Contraction rate for full observations). *Assume that λ_0 is bounded away from 0. Suppose that for positive sequences $\tilde{\varepsilon}_n, \bar{\varepsilon}_n \rightarrow 0$ such that $n(\tilde{\varepsilon}_n \wedge \bar{\varepsilon}_n)^2 \rightarrow \infty$ as $n \rightarrow \infty$, and constants $c_1, c_2 > 0$ it holds that for all $c_3 > 1$, there exist subsets $\Lambda_n \subset \Lambda$ and a constant $c_4 > 0$ such that*

$$\Pi_n(\lambda : \|\lambda - \lambda_0\|_\infty \leq \tilde{\varepsilon}_n) \geq c_1 e^{-c_2 n \tilde{\varepsilon}_n^2}, \quad (4)$$

$$\Pi_n(\Lambda_n^c) \leq e^{-c_3 n \tilde{\varepsilon}_n^2}, \quad (5)$$

$$\log N(\bar{\varepsilon}_n, \sqrt{\Lambda_n}, \|\cdot\|_2) \leq c_4 n \bar{\varepsilon}_n^2. \quad (6)$$

Then for $\varepsilon_n = \tilde{\varepsilon}_n \vee \bar{\varepsilon}_n$ and all sufficiently large $M > 0$,

$$\mathbb{E}_{\lambda_0} \Pi_n(\lambda \in \Lambda : \|\sqrt{\lambda} - \sqrt{\lambda_0}\|_2 \geq M \varepsilon_n \mid N^n) \rightarrow 0 \quad (7)$$

as $n \rightarrow \infty$.

The proof of this theorem is given in Appendix A.1. The assumptions of the theorem parallel those of similar theorems obtained earlier for other settings including density estimation, regression, and classification. The first condition (4), the so-called *prior mass condition*, requires that the prior puts sufficient mass near the truth. Conditions (5)–(6) together require that most of the prior mass, quantified in the sense of the *remaining*

mass condition (5), is concentrated on *sieves* Λ_n which are “small” in the sense of metric entropy, quantified by the *entropy condition* (6).

The proof of the theorem shows that conditions (5)–(6) can in fact be slightly weakened, at the cost of using more complicated distance measures on the intensities. The conditions in the theorem are more easy to work with when studying concrete priors and are expected to give sharp results in many cases. We note that if under the prior all intensities are bounded away from 0, then the set $\sqrt{\Lambda_n}$ in (6) may be replaced by Λ_n . Moreover, if all intensities are uniformly bounded by a common constant under the prior, then the square-root norm $\|\sqrt{\cdot}\|_2$ in (7) may be replaced by the L^2 -norm $\|\cdot\|_2$ itself. In the next section we verify the conditions of the theorem for the spline priors used in Section 2.1.

In the discrete observations case we only observe, for some $m \in \mathbb{N}$ and $\Delta = T/m$, aggregated counts C_{ij} for $i = 1, \dots, n$ and $j = 1, \dots, m$, given by (1). As before, we summarize these data using the notation $C^n = (C_{ij} : i = 1, \dots, n, j = 1, \dots, m)$. As explained in Section 2.1 the likelihood is in that case given by (3), where the λ_j ’s are defined as in (2). Consequently, the discrete-observations posterior is given by

$$\Pi_n(\lambda \in B \mid C^n) = \frac{\int_B p(C^n \mid \lambda) \Pi_n(d\lambda)}{\int p(C^n \mid \lambda) \Pi_n(d\lambda)}.$$

In this case it is clear that we can not consistently identify the whole intensity function λ from the data, but only the integrals $\lambda_1, \dots, \lambda_m$. In the following theorem, which deals with the convergence of the posterior distribution in the case of discrete observations, we therefore measure the convergence using a semi-metric that identifies intensity functions with the same integrals over time intervals in which we make observations. For $\lambda, \lambda' \in \Lambda$, we define the distance ρ by setting

$$\rho^2(\lambda, \lambda') = \sum_{j=1}^m \left(\sqrt{\lambda_j} - \sqrt{\lambda'_j} \right)^2 = \sum_{j=1}^m \left(\sqrt{\int_{(j-1)\Delta}^{j\Delta} \lambda(t) dt} - \sqrt{\int_{(j-1)\Delta}^{j\Delta} \lambda'(t) dt} \right)^2.$$

The theorem has exactly the same assumptions on the prior as Theorem 1 above, but gives a contraction rate relative to the distance ρ .

Theorem 2 (Contraction rate for discrete observations). *Assume that λ_0 is bounded away from 0. Suppose that for positive sequences $\tilde{\varepsilon}_n, \bar{\varepsilon}_n \rightarrow 0$ such that $n(\tilde{\varepsilon}_n \wedge \bar{\varepsilon}_n)^2 \rightarrow \infty$ as $n \rightarrow \infty$, and constants $c_1, c_2 > 0$ it holds that for all $c_3 > 1$, there exist subsets $\Lambda_n \subset \Lambda$ and a constant $c_4 > 0$ such that (4)–(6) hold. Then for $\varepsilon_n = \tilde{\varepsilon}_n \vee \bar{\varepsilon}_n$ and all sufficiently large $M > 0$,*

$$\mathbb{E}_{\lambda_0} \Pi_n(\lambda \in \Lambda : \rho(\lambda, \lambda_0) \geq M\varepsilon_n \mid C^n) \rightarrow 0$$

as $n \rightarrow \infty$.

The proof of the theorem is given in Appendix A.2

3.2 Contraction rates for our spline prior

Having the general rate of contraction results given by Theorems 1 and 2 at our disposal we can use them to study the performance of the spline-based prior defined in Section 2.2.

We fix the order $q \geq 2$ of the splines that are used. As before, let N^n be the a full path up till time nT of an inhomogenous Poisson process N with T -period intensity λ_0 and let C^n be the discrete-time counts $C^n = (C_{ij} : i = 1, \dots, n, j = 1, \dots, m)$, with C_{ij} as in (1).

The contraction rate of the posterior will depend on the regularity of the true intensity function, measured in Hölder sense. For $\alpha > 0$, let $C^\alpha[0, T]$ be the space of functions on $[0, T]$ with Hölder smoothness α . (For $\lfloor \alpha \rfloor$ the greatest integer strictly smaller than α , having $f \in C^\alpha[0, T]$ means that f has $\lfloor \alpha \rfloor$ derivatives and that the highest derivative $f^{(\lfloor \alpha \rfloor)}$ is Hölder-continuous of order $\alpha - \lfloor \alpha \rfloor$.)

Theorem 3 (Contraction rate for the spline prior). *Assume the true intensity function λ_0 belongs to $C^\alpha[0, T]$ for some $\alpha \in (0, q]$, and $M_1 \leq \lambda_0 \leq M_2$. Consider the prior Π constructed in Section 2.2. For all $p > 1$ and all sufficiently large $M > 0$ we have*

$$\mathbb{E}_{\lambda_0} \Pi_n(\lambda \in \Lambda : \|\lambda - \lambda_0\|_2 \geq M \left(\frac{n}{\log^p n} \right)^{-\frac{\alpha}{1+2\alpha}} \mid N^n) \rightarrow 0$$

and

$$\mathbb{E}_{\lambda_0} \Pi_n(\lambda \in \Lambda : \rho(\lambda, \lambda_0) \geq M \left(\frac{n}{\log^p n} \right)^{-\frac{\alpha}{1+2\alpha}} \mid C^n) \rightarrow 0$$

as $n \rightarrow \infty$.

Observe that up to a logarithmic factor, the rate of contraction in the theorem is the optimal rate $n^{-\alpha/(1+2\alpha)}$ for estimating an α -regular function. Moreover, the prior does not depend on α . Hence the procedure automatically adapts to the smoothness of the intensity function, up to the order of the splines that are used. The theorem deals with the case that we have known bounds M_1 and M_2 for the intensity. Generalisations to less restrictive settings are possible, see the remarks in Section 4.

4 Concluding remarks

In this paper we exhibit a specific spline-based prior for doing nonparametric Bayesian intensity smoothing for inhomogeneous Poisson processes. We show that the method is both practically feasible and is underpinned by theoretical performance guarantees in the form of adaptive rate-optimality results.

Extensions of our results in several directions are possible. In particular, with more work it is possible to drop the assumption that we know an a-priori bound on the unknown intensity, which may be undesirable or impossible in certain situations. An obvious extension is then to put a prior on the bound. Computationally this makes the procedure more demanding, but numerical investigations indicate it is still feasible. Theoretical results can be obtained for that more general setting as well. Among other things this involves an extension of Lemma 1. Having a prior on the upper bound for λ_0 may deteriorate the convergence rate however. The optimal rate will only be attained (up to lower order factors) if the prior on the bound has very thin tails.

Another desirable theoretical extension would be to obtain “local” rate of convergence results. Our present results deal with global norms on the intensity functions. It is

conceivable however that convergence is faster in regions where the intensity fluctuates relatively little, and faster in others. More work is necessary to derive theorems that describe this phenomenon.

A Proofs

A.1 Proof of Theorem 1

A useful observation is that we can view the statistical problem to which the theorem applies as a density estimation problem for functional data. Indeed, in the full observations case we observe a sample $N^{(1)}, \dots, N^{(n)}$, which are independent, identically distributed random elements in the Skorohod space $D[0, T]$ of càdlàg (right-continuous functions with left-hand limits) on $[0, T]$ (see Jacod & Shiryaev (2003), Chapter VI). Under the intensity function λ , the density p_λ of $N^{(1)}$ relative to the law of a standard Poisson process indexed by $[0, T]$ is given by

$$p_\lambda(N) = e^{-\int_0^T (\lambda(t)-1) dt + \int_0^T \log(\lambda(t)) dN_t}$$

(e.g. Jacod & Shiryaev (2003), Chapter III). Hence, the density estimation results of Ghosal et al. (2000), Ghosal & van der Vaart (2001) apply in our case.

We want to apply Theorem 2.1 of Ghosal & van der Vaart (2001). This gives conditions for posterior contraction rates in terms of the Hellinger distance on densities and other, related distance measures. The Hellinger distance $h(p_\lambda, p_{\lambda'})$ is in our case given by $h^2(p_\lambda, p_{\lambda'}) = 2(1 - \mathbb{E}_{\lambda'} \sqrt{p_\lambda(N)/p_{\lambda'}(N)})$, where \mathbb{E}_λ is the expectation corresponding to the probability measure \mathbb{P}_λ under which the process N is a Poisson process with intensity function λ . The other relevant distance measures are the Kullback-Leibler divergence $K(p_\lambda, p_{\lambda'}) = -\mathbb{E}_{\lambda'} \log(p_\lambda(N)/p_{\lambda'}(N))$ between p_λ and $p_{\lambda'}$ and the related variance measure $V(p_\lambda, p_{\lambda'}) = \mathbb{V}_{\lambda'} \log(p_\lambda(N)/p_{\lambda'}(N))$. For a Poisson process N with intensity λ and a bounded, measurable function f , we have

$$\begin{aligned} \mathbb{E} \int_0^T f(t) dN_t &= \int_0^T f(t) \lambda(t) dt, \\ \mathbb{V} \int_0^T f(t) dN_t &= \int_0^T f^2(t) \lambda(t) dt, \\ \mathbb{E} e^{\int_0^T f(t) dN_t} &= e^{-\int_0^T (1 - \exp(f(t))) \lambda(t) dt}. \end{aligned}$$

Using these relations it is straightforward to verify that we have

$$\begin{aligned} h^2(p_\lambda, p_{\lambda'}) &= 2(1 - e^{-\frac{1}{2} \int_0^T (\sqrt{\lambda(t)} - \sqrt{\lambda'(t)})^2 dt}), \\ K(p_\lambda, p_{\lambda'}) &= \int_0^T (\lambda(t) - \lambda'(t)) dt + \int_0^T \lambda'(t) \log \frac{\lambda'(t)}{\lambda(t)} dt, \\ V(p_\lambda, p_{\lambda'}) &= \int_0^T \lambda'(t) \log^2 \frac{\lambda'(t)}{\lambda(t)} dt, \end{aligned}$$

respectively.

The following lemma relates these statistical distances between densities to certain distances between intensity functions. We denote the minimum and maximum of two numbers a and b by $a \wedge b$ and $a \vee b$, respectively.

Lemma 1. *We have the inequalities*

$$\begin{aligned} \frac{1}{\sqrt{2}} \left(\|\sqrt{\lambda} - \sqrt{\lambda'}\|_2 \wedge 1 \right) &\leq h(p_\lambda, p_{\lambda'}) \leq \sqrt{2} \left(\|\sqrt{\lambda} - \sqrt{\lambda'}\|_2 \wedge 1 \right), \\ K(p_\lambda, p_{\lambda'}) &\leq 3 \|\sqrt{\lambda} - \sqrt{\lambda'}\|_2^2 + V(p_\lambda, p_{\lambda'}), \\ \|\sqrt{\lambda} - \sqrt{\lambda'}\|_2^2 &\leq \frac{1}{4} \int_0^T (\lambda(t) \vee \lambda'(t)) \log^2 \frac{\lambda(t)}{\lambda'(t)} dt. \end{aligned}$$

Proof. The inequalities for h follow from the fact that $(1/4)(x \wedge 1) \leq 1 - \exp(-x/2) \leq x \wedge 1$ for $x \geq 0$.

For the Kullback-Leibler divergence we have

$$K(p_\lambda, p_{\lambda'}) = \int_0^T \lambda'(t) f(\lambda(t)/\lambda'(t)) dt,$$

for $f(x) = x - 1 - \log x$. By Taylor's formula, $|f(x)|$ is bounded by a constant times $(\sqrt{x} - 1)^2$ in a neighborhood of 1. Since $|f(x)|$ is bounded by $|x|$ for $x \geq 1$ and $|x|/(\sqrt{x} - 1)^2 \rightarrow 1$ as $x \rightarrow \infty$, we have in fact $|f(x)| \leq 3(\sqrt{x} - 1)^2$ for all $x \in (1/e, \infty)$, say. For $(0, 1/e)$ we have $|f(x)| \leq |\log x|$. It follows that

$$K(p_\lambda, p_{\lambda'}) \leq 3 \int_{\lambda/\lambda' \geq 1/e} (\sqrt{\lambda(t)} - \sqrt{\lambda'(t)})^2 dt + \int_{\lambda/\lambda' \leq 1/e} \lambda'(t) \left| \log \frac{\lambda(t)}{\lambda'(t)} \right| dt.$$

The first term on the right is bounded by $3\|\sqrt{\lambda} - \sqrt{\lambda'}\|_2^2$. For the second term we note that for $\lambda/\lambda' \leq 1/e$, we have $\log |\lambda/\lambda'| \geq 1$ and hence $\log |\lambda/\lambda'| \leq \log^2 |\lambda/\lambda'|$. The statement of the lemma follows.

To prove the last inequality, write $\|\sqrt{\lambda} - \sqrt{\lambda'}\|_2^2$ as the sum of an integral over the set $\{\lambda' \leq \lambda\}$ and an integral over the set $\{\lambda' > \lambda\}$ and use the fact that $1 - x \leq |\log x|$ for $x \in (0, 1)$. \square

To connect assumptions (4)–(6) to the corresponding assumptions of Theorem 2.1 of Ghosal & van der Vaart (2001) we first note that since λ_0 is bounded away from 0 and infinity by assumption, the same holds for any $\lambda \in \Lambda$ that is uniformly close enough to λ_0 . The lemma and the definition of V therefore imply that for λ uniformly close enough to λ_0 , both $K(p_\lambda, p_{\lambda_0})$ and $V(p_\lambda, p_{\lambda_0})$ are bounded by a constant times the uniform norm $\|\lambda - \lambda_0\|_\infty$. It follows that for n large enough, the Kullback-Leibler-type ball

$$B(\varepsilon_n) = \{\lambda \in \Lambda : K(p_\lambda, p_{\lambda_0}) \leq \tilde{\varepsilon}_n^2, V(p_\lambda, p_{\lambda_0}) \leq \tilde{\varepsilon}_n^2\}$$

is larger than a multiple of the uniform ball $\{\lambda \in \Lambda : \|\lambda - \lambda_0\|_\infty \leq \tilde{\varepsilon}_n\}$. The lemma also implies that the covering number $N(\tilde{\varepsilon}_n, \{p_\lambda : \lambda \in \Lambda_n\}, h)$ is bounded by $N(\tilde{\varepsilon}_n/\sqrt{2}, \sqrt{\Lambda_n}, \|\cdot\|)$.

$\|_2$). Hence, assumptions (4)–(6) imply that the conditions of Theorem 2.1 of Ghosal & van der Vaart (2001) are fulfilled. This theorem states that for M large enough, $\mathbb{E}_{\lambda_0} \Pi_n(\lambda : h(p_\lambda, p_{\lambda_0}) \geq M\varepsilon_n) \rightarrow 0$. To complete the proof, note that by the fact that $M\varepsilon_n \leq 1$ for n large enough and the first inequality of the lemma, it holds, for n large enough, that $\|\sqrt{\lambda} - \sqrt{\lambda_0}\|_2 \geq \sqrt{2}M\varepsilon_n$ implies that $h(p_\lambda, p_{\lambda_0}) \geq M\varepsilon_n$.

A.2 Proof of Theorem 2

The proof is similar as the proof of Theorem 1, but this time we start from the observation that in the discrete-observations case, the data constitute a sample of n independent, identically distributed random vectors $C^{(1)}, \dots, C^{(n)}$ in \mathbb{R}^m , where

$$C^{(i)} = (C_{i1}, \dots, C_{im})$$

and C_{ij} is given by (1). The coordinates C_{ij} of $C^{(i)}$ are independent Poisson variables with mean λ_j given by (2).

Again we apply Theorem 2.1 of Ghosal & van der Vaart (2001). In this case the Hellinger distance h_m , Kullback Leibler divergence K_m and variance measure V_m are easily seen to be given by

$$\begin{aligned} h_m^2(\lambda, \lambda') &= 2(1 - e^{-\frac{1}{2} \sum (\sqrt{\lambda_j} - \sqrt{\lambda'_j})^2}), \\ K_m(\lambda, \lambda') &= \sum (\lambda_j - \lambda'_j) + \sum \lambda'_j \log \frac{\lambda'_j}{\lambda_j}, \\ V_m(\lambda, \lambda') &= \sum \lambda'_j \log^2 \frac{\lambda'_j}{\lambda_j}, \end{aligned}$$

respectively. These quantities satisfy the same bounds as in Lemma 1, but with the integrals replaced by the corresponding sums. Moreover, by expanding the square and using Cauchy-Schwarz we see that

$$\sum (\sqrt{\lambda_j} - \sqrt{\lambda'_j})^2 \leq \|\sqrt{\lambda} - \sqrt{\lambda'}\|_2^2,$$

and hence also

$$V_m(\lambda, \lambda') \leq 4 \sum \frac{\lambda'_j}{\lambda_j \wedge \lambda'_j} (\sqrt{\lambda_j} - \sqrt{\lambda'_j})^2 \leq 4 \frac{\|\lambda'\|_\infty}{\inf_t |\lambda(t)| \wedge \inf_t |\lambda'(t)|} \|\sqrt{\lambda} - \sqrt{\lambda'}\|_2^2.$$

Using these relations the proof can be completed exactly as in Section A.1.

A.3 Proof of Theorem 3

Under the prior Π , the number of knots J has, by construction, a shifted Poisson distribution. By Stirling's approximation, this implies that for large j ,

$$\mathbb{P}(J > j) \lesssim e^{-c_1 j \log j}, \quad \mathbb{P}(J = j) \gtrsim e^{-c_2 j \log j}$$

for some $c_1, c_2 > 0$. For the sequence of inner knots \mathbf{k} constructed in the definition of the prior we have that the mesh width $M(\mathbf{k}) = \max\{|k_j - k_{j-1}|\}$ and the sparsity $m(\mathbf{k}) = \min\{|k_j - k_{j-1}|\}$ satisfy

$$\mathbb{P}(m(\mathbf{k}) < j^{-2} \mid J = j) = 0, \quad \mathbb{P}(M(\mathbf{k}) \leq 2/j \mid J = j) \gtrsim e^{-j \log j}.$$

The first of these facts follows trivially from the construction, the second one by bounding the probability of interest from below by the probability that every of the consecutive intervals of length T/j contains at least one knot. For the B-spline coefficients we have, by independence,

$$\mathbb{P}(\|\boldsymbol{\theta} - \boldsymbol{\theta}_0\|_\infty \leq \varepsilon \mid J = j) \gtrsim \varepsilon^j$$

for all $\boldsymbol{\theta}_0 \in [M_1, M_2]^j$. Theorem 1 of Belitser & Serra (2013) deals exactly with this situation. In the present setting the theorem asserts that if $\lambda_0 \in C^\alpha[0, T]$ and $M_1 \leq \lambda_0 \leq M_2$, then for $J_n, \bar{J}_n > q$ and positive $\varepsilon_n \geq \bar{\varepsilon}_n$ such that $\varepsilon_n \rightarrow 0$, $n\bar{\varepsilon}_n^2 \rightarrow \infty$ and

$$2\left(\frac{\bar{\varepsilon}_n}{\|\lambda_0\|_{C^\alpha}}\right)^{-1/\alpha} \leq \bar{J}_n, \quad \log \bar{J}_n \lesssim \log \frac{1}{\bar{\varepsilon}_n}, \quad J_n \log \frac{J_n^3}{\varepsilon_n} \lesssim n\varepsilon_n^2, \quad n\bar{\varepsilon}_n^2 \leq J_n \log J_n, \quad (8)$$

then there exist function spaces (of splines) Λ_n and a constant $c > 0$ such that

$$\Pi(\lambda : \|\lambda - \lambda_0\|_\infty \leq 2\bar{\varepsilon}_n) \gtrsim e^{-c\bar{J}_n \log \frac{1}{\bar{\varepsilon}_n}}, \quad (9)$$

$$\Pi(\lambda \notin \Lambda_n) \lesssim e^{-c_1 n\bar{\varepsilon}_n^2}, \quad (10)$$

$$\log N(\varepsilon_n, \Lambda_n, \|\cdot\|_2) \lesssim n\varepsilon_n^2. \quad (11)$$

Now observe that the first two inequalities in (8) hold for

$$\bar{\varepsilon}_n = n^{-\frac{\alpha}{1+2\alpha}} \log^p n, \quad \bar{J}_n = K n^{\frac{1}{1+2\alpha}} \log^q n,$$

provided K is large enough and $q \geq -p/\alpha$. The third and fourth inequalities then hold for

$$J_n = L n^{\frac{1}{1+2\alpha}} \log^r n, \quad \varepsilon_n = n^{-\frac{\alpha}{1+2\alpha}} \log^s n$$

if L is large enough and $2p \leq r+1 \leq 2s$. To complete the proof we have to link (9)–(11) to the conditions (4)–(6) of Theorems 1 and 2. Note that since (5) should hold for all $c_3 > 0$, we need to have

$$\bar{J}_n \log \frac{1}{\bar{\varepsilon}_n} \ll n\bar{\varepsilon}_n^2.$$

For our choices of \bar{J}_n and $\bar{\varepsilon}_n$ this holds if $2p > q+1$. This amounts to choosing $p > \alpha/(1+2\alpha)$. Then if we define

$$\tilde{\varepsilon}_n = \sqrt{\frac{\bar{J}_n}{n} \log \frac{1}{\bar{\varepsilon}_n}}$$

the right-hand side of (9) equals $\exp(-2n\tilde{\varepsilon}_n^2)$. Moreover, it holds that $\tilde{\varepsilon}_n \sim n^{-\alpha/(1+2\alpha)}(\log n)^{(q+1)/2}$, so if we make sure that $p > (q+1)/2$, the desired inequality (4) holds. The considerations above imply that (5) then holds as well, for any $c_3 \geq 1$. Recall that we found that the

entropy condition holds for $\varepsilon_n \sim n^{-\alpha/(1+2\alpha)}(\log n)^s$, provided $s > p$. This means that we should choose p, q, r and s above such that

$$p > \frac{\alpha}{1+2\alpha}, \quad r = 2p - 1, \quad s > p, \quad q = -\frac{1}{1+2\alpha}.$$

Since the intensities in Λ_n are uniformly bounded by a common constant (see the proof of Theorem 1 of Belitser & Serra (2013)), (11) implies that (6) is fulfilled.

Acknowledgement

Research supported by the Netherlands Organization for Scientific Research NWO.

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