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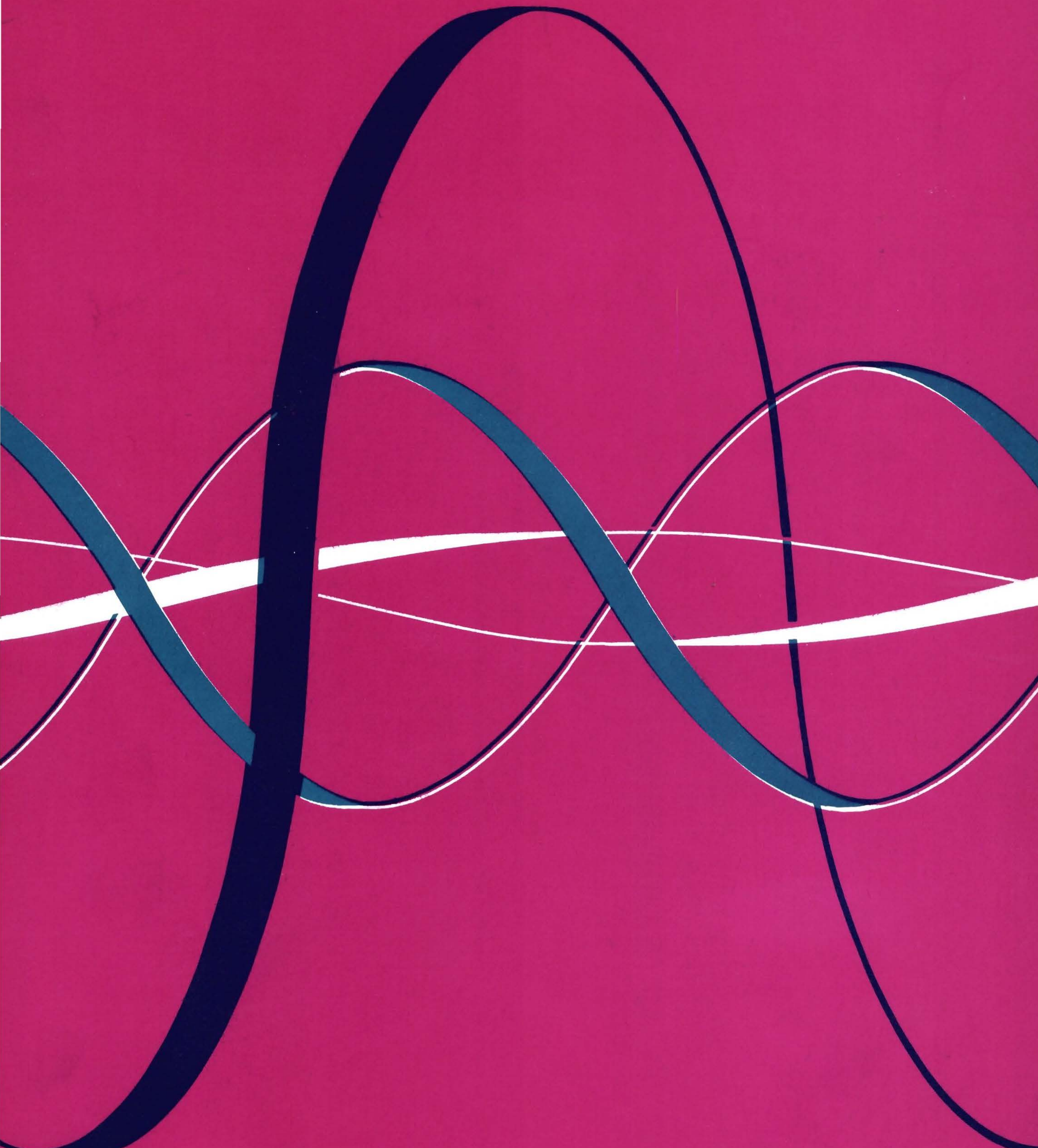

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A marginal-stability analysis

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PROEFSCHRIFT

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DRUKKERIJ BRONDER-OFFSET N.V.

ROTTERDAM

DIT PROEFSCHRIFT IS GOEDGEKEURD DOOR DE PROMOTOR

PROF.DR. H. BREMMER

"....peacefully solving differential equations"

Solzhenitsyn

Aan mijn ouders

Aan Antonia

CONTENTS

S U M M A R Y

Ideal magnetohydrodynamic instabilities, as occurring in simple plasma-vacuum systems, can be suppressed by replacing the vacuum by a force-free magnetic field, that is, a field satisfying the relation $\nabla\times\underset{\sim}{B} = \alpha\underset{\sim}{B}$. Force-free fields of constant α are investigated in particular. By proper choice of α the gravitational instabilities, of non-local and of surface-layer type, are absent in a plane plasma layer supported from below by a horizontal force-free magnetic field. For a sharp-pinch model of a dense plasma surrounded by a force-free field the same conclusion is reached with respect to kinks, surface-layer modes, and modes of the force-free region.

Complete stability criteria are derived from the marginal equation of motion and it is shown that this method is equivalent to the application of the energy principle. From the marginal-stability analysis the principle of exchange of stabilities is derived. This theory is applied to constant-pitch magnetic fields, which are shown to be necessarily unstable. Growth rates of the instabilities of these fields are calculated, correcting earlier results of Ware. Simple modifications of some constant-pitch models, namely Van der Laan's model of a constant-pitch force-free field and Alfvén's model of a constant-pitch field with parabolic pressure profile, prove to yield completely stable pinch configurations of a sharp or diffuse kind.

CHAPTER 1

INTRODUCTION

In magnetohydrodynamics a configuration consisting of a plasma, a current, and a magnetic field is called force-free if the Lorentz force vanishes, so that $\underset{\sim}{j}\times\underset{\sim}{B} = 0$ or $\nabla\times\underset{\sim}{B} = \alpha\underset{\sim}{B}$, where α is a scalar function of the position[1-10]). Although the magnetohydrodynamic stability of force-free magnetic fields has been extensively studied[11-19]), the problem of the influence of such fields on the stability of a neighbouring plasma seems to have escaped attention.

This problem is suggested by the long known instabilities that arise in simple plasma-vacuum systems[20-21]). A system of a homogeneous plasma separated from the outside world by a vacuum with the associated magnetic fields is about the most simple model one can set up for a hot plasma confined for thermonuclear purposes. It is attractive to conserve such a simple model as long as no discrepancies turn up with experimental data. For example, the growth rates of kink instabilities in pinches can be simply calculated from this model.

However, in experimental studies of the screw pinch the thus calculated growth rates and the measured containment times did not agree[22]), so that a modification of the model became necessary. This modification was provided by Van der Laan[23]), who pointed out the importance of the presence of a low-density plasma enclosing the central plasma core. During the period of formation of the pinch strong currents are induced in this tenuous plasma, resulting in a field which strongly deviates from a vacuum magnetic field. Because the density and the pressure of this tenuous plasma are very small, one can neglect the pressure gradient and the produced field must be approximately force-free. Next, the stabilizing influence of a constant-pitch force-free field on the kink instability of the dense plasma was demonstrated by Schuurman, Bobeldijk, and De Vries[24]). A force-free field with constant pitch in space will be formed in the outer region of the pinch if a pitch, constant in time, is applied at the wall by means of primary currents, assuming that the created field conserves its pitch during the inward motion of the field[23,25]). On the other hand, a force-free field with a varying pitch in space will be formed by the application of a field with a time-dependent pitch at the wall, when the primary currents are properly programmed. Therefore, the question arises as to which force-free field in the outer region of the pinch will be optimal for stability. This problem is treated in Chapter 6.

The purpose of this paper is twofold:

- To work out in detail the stabilizing influence of force-free fields on magnetohydrodynamic instabilities.
- To develop the marginal-stability analysis for simple cases to a level of rigour equivalent to the application of the energy principle.

The starting point will be the equations of ideal magnetohydrodynamics, as given by Bernstein et alii[26]) and Kruskal and Schwarzschild[20]) (Chapter 2). It will be shown, starting from the solution of the marginal equation of motion ($\omega^2 = 0$), how necessary and sufficient stability conditions can be derived for a plane plasma layer in the presence of gravity (Chapter 3), and in absence of the latter for a diffuse pinch (Chapter 5). For these cases the well-known stability criteria of Cowley[27]) and Newcomb[28]) are recovered, showing the equivalence of the application of the energy principle and that of the marginal-stability analysis. The demonstration of this equivalence was considered necessary because the marginal-stability analysis sometimes has been applied inaccurately and, therefore, has been placed in an unfavourable light[29].

These criteria are applied to the gravitational instability of a plane plasma layer, supported from below by a force-free magnetic field (Chapter 4), and to the instabilities of a homogeneous plasma cylinder, surrounded by a force-free magnetic field (Chapter 6). Force-free fields of constant α (as defined above) are investigated in particular. In some sense this represents an extension of the work of Kruskal and Schwarzschild[20] and of Kruskal and Tuck[21], both of which fit in our treatment by the substitution of $\alpha = 0$ (vacuum magnetic field). For an important range of values of α it is found that the instabilities described by these authors completely disappear.

In Chapter 7 some peculiarities with respect to stability are discussed for shearless magnetic fields and it is shown that these fields (force-free or not) are necessarily unstable. Next, the analysis given in Ref. 24 of a pinch surrounded by a constant-pitch force-free field (Van der Laan's model) is corrected using the derived formulation of the marginal-stability analysis. Because of the intrinsic instability of the constant-pitch force-free field the consideration in Chapter 6 of the larger class of force-free fields of constant α but varying pitch becomes important, theoretically as well as experimentally. Finally, a critical discussion is devoted to Alfvén's model of a plasma cylinder with a constant-pitch field in the presence of a pressure gradient.

Three appendices are included. The first contains a rigorous treatment of the singularities of the marginal equation of motion for the gravitational instability. The second one is devoted to the influence of resistivity on the stability of force-free magnetic fields, whereas in the last appendix toroidal effects on the stability of the pinch are considered.

Throughout this paper the MKSA system of units is used.

CHAPTER 2

BASIC EQUATIONS

Our treatise will be based on ideal magnetohydrodynamics, i.e. resistivity will be neglected (apart from a discussion in Appendix II). The equations of ideal magnetohydrodynamics are given in Refs. 20 and 26, where also the other assumptions and approximations underlying these equations may be found. For a static equilibrium these equations reduce to

$$\underset{\sim}{j} \times \underset{\sim}{B} - \nabla \underset{\sim}{p} + \rho \underset{\sim}{g} = 0 \,, \tag{1}$$

$$\nabla \times \underset{\sim}{B} = \mu_o \underset{\sim}{j} \,, \tag{2}$$

$$\nabla . \underset{\sim}{B} = 0 \,, \tag{3}$$

where $\underset{\sim}{j}$ is the current density, $\underset{\sim}{B}$ the magnetic field, p the pressure, ρ the density, and $\underset{\sim}{g}$ the acceleration due to gravity. In the following, $\underset{\sim}{g}$ will be taken constant. A gravitational field is included, since in many cases such a field accounts approximately for effects due to the curvature of the magnetic-field lines. The required boundary conditions for a plasma-pressureless plasma interface are

$$\langle p + B^2/2\mu_o \rangle = 0 \ , \tag{4}$$

$$\underset{\sim}{n}.\langle \underset{\sim}{B} \rangle = 0 \ , \tag{5}$$

the angular brackets denoting the jump of the quantity inside the brackets when crossing the surface and $\underset{\sim}{n}$ being the unit vector normal to the surface. The condition (5) also holds at the enclosing conducting wall.

In this paper a current-carrying pressureless plasma and a space with a force-free magnetic field will in general be treated as equivalent. Of course such a magnetic field can also exist in the presence of a finite pressure provided that the pressure gradient is negligible. However, it will turn out that the magnitude of the pressure has little influence on the stability criteria in some important cases (Chapters 4 and 6); the pressure will then be neglected altogether.

The stability of an equilibrium satisfying Eqs. (1) to (5) will be investigated by means of the equation of motion in Lagrangian coordinates as given by Bernstein et alii[26]):

$$\underset{\sim}{F}\{\underset{\sim}{\xi}\} = \rho \frac{\partial^2 \underset{\sim}{\xi}}{\partial t^2} \ , \tag{6}$$

in which the displacement vector $\underset{\sim}{\xi}$ of plasma elements is considered as a function of the initial position $\underset{\sim}{r}$ of such an element and of the time t and is supposed to be small. The force operator $\underset{\sim}{F}$ is a functional of $\underset{\sim}{\xi}$:

$$\underset{\sim}{F}\{\underset{\sim}{\xi}\} = \nabla(\gamma p \nabla.\underset{\sim}{\xi} + \underset{\sim}{\xi}.\nabla p) - \frac{1}{\mu_o} \underset{\sim}{Q}\times(\nabla\times\underset{\sim}{B}) - \frac{1}{\mu_o} \underset{\sim}{B}\times(\nabla\times\underset{\sim}{Q}) - \underset{\sim}{g}\nabla.(\rho\underset{\sim}{\xi}), \tag{6a}$$

where $\underset{\sim}{Q}$ represents the perturbation of the magnetic field at the unperturbed position of a plasma element (strictly speaking it is an Eulerian variable); $\underset{\sim}{Q}$ is given by

$$\underset{\sim}{Q} = \nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B}) \ . \tag{6b}$$

The boundary conditions at a plasma-pressureless plasma interface express the linearized pressure balance

$$-\gamma p \nabla \cdot \underset{\sim}{\xi}^{p} + \frac{1}{\mu_o} \underset{\sim}{B}^{p} \cdot \left[\underset{\sim}{Q}^{p} + \underset{\sim}{\xi}^{p} \cdot \nabla \underset{\sim}{B}^{p} \right] = \frac{1}{\mu_o} \underset{\sim}{B}^{t} \cdot \left[\underset{\sim}{Q}^{t} + \underset{\sim}{\xi}^{t} \cdot \nabla \underset{\sim}{B}^{t} \right] , \quad (7)$$

and the continuity of the displacement

$$\underset{\sim}{\xi}^{p} \cdot \underset{\sim}{n} = \underset{\sim}{\xi}^{t} \cdot \underset{\sim}{n} , \quad (8)$$

p and t referring to the central plasma and the tenuous pressureless plasma, respectively.
The boundary condition at the rigid wall reads:

$$\underset{\sim}{\xi} \cdot \underset{\sim}{n} = 0 . \quad (9)$$

In cylinder geometry one additional condition is needed:

$$\xi_r^p \text{ finite at } r = 0 . \quad (10)$$

In the cases to be dealt with inhomogeneity will be restricted to one dimension. Therefore, it will be sufficient to consider elementary solutions proportional to $\exp(i\underset{\sim}{k} \cdot \underset{\sim}{r} - i\omega t)$, where $\underset{\sim}{k}$ is the wavenumber perpendicular to the direction of inhomogeneity. For these elementary solutions Eq. (6) reduces to a single ordinary second-order differential equation for the component of $\underset{\sim}{\xi}$ in the direction of inhomogeneity. Equations (7) to (10) then fix three of the four integration constants of the components of $\underset{\sim}{\xi}^{p}$ and $\underset{\sim}{\xi}^{t}$ in this direction (one integration constant being fixed by normalization), and provide the characteristic equation.

Formally, the equations for an incompressible plasma can be obtained from the equations above by replacing $-\gamma p \nabla \cdot \underset{\sim}{\xi}$ by p_{1L}, where p_{1L} is the perturbation of the Lagrangian pressure. In the first-order approximation with respect to $\underset{\sim}{\xi}$ this latter quantity is connected with the Eulerian pressure $p = p_o + p_1$

according to $p_{1L} = p_1 + \underset{\sim}{\xi}.\nabla p_o$. In the case of an incompressible plasma the perturbation of the pressure cannot be found by merely integrating a conservation law like the adiabatic law and, therefore, the Lagrangian description does not provide us with an equation of motion in terms of the displacement vector $\underset{\sim}{\xi}$ alone. In order to be able to remove p_{1L} from the equations we then need the additional equation $\nabla.\underset{\sim}{\xi} = 0$.

CHAPTER 3

MARGINAL-STABILITY ANALYSIS OF A PLANE PLASMA LAYER UNDER THE INFLUENCE OF GRAVITY

In this chapter the marginal-stability analysis is developed for a plane plasma layer with a magnetic field $\tilde{B} = (B_x(y), 0, B_z(y))$, situated between two perfectly conducting walls at $y = y_1$ and $y = y_2$, under the influence of gravity: $\tilde{g} = (0, -g, 0)$. The coordinate system is the same as that represented in Fig. 1 of Sec. 4.1, following Kruskal and Schwarzschild[20]). In this example the inhomogeneity, at equilibrium, of the pressure, the density, and the magnetic field, is restricted to the y-direction. Differentiation of these quantities with respect to this direction is indicated by accents. The equilibrium equations (1) to (3) yield

$$(p + B^2/2\mu_o)' + \rho g = 0 \ . \qquad (11)$$

There exists an essential difference in the stability analysis of an incompressible and a compressible plasma. Therefore, both cases will be treated separately.

3.1. Incompressible plasma

Starting from Eq. (6), performing the substitutions for incompressible plasmas mentioned in Chapter 2, and assuming elementary perturbations for the magnetic field, pressure, and density of the same form as that for the plasma displacement, viz.

$$\underset{\sim}{\xi} = \hat{\underset{\sim}{\xi}}_{\omega,k_x,k_z}(y)\, e^{i(k_x x + k_z z - \omega t)} ,$$

four coupled ordinary differential equations are obtained. We henceforth drop the circumflex and indices ω, k_x, and k_z in order to avoid a too complicated notation so that, in what follows, we understand by $\underset{\sim}{\xi}$ expressions of the above form. Equation (6b) then gives the following relations between the components $\underset{\sim}{Q}$ and $\underset{\sim}{\xi}$ for these elementary solutions:

$$\begin{aligned} Q_x &= -(B_x\xi_y)' - ik_z(B_x\xi_z - B_z\xi_x) , \\ Q_y &= i(k_xB_x + k_zB_z)\xi_y , \\ Q_z &= -(B_z\xi_y)' + ik_x(B_x\xi_z - B_z\xi_x) . \end{aligned} \tag{12}$$

After some algebraic manipulations, also taking into account the equations (11) and (12) as well as the condition of incompressibility

$$i(k_x\xi_x + k_z\xi_z) = -\xi_y' , \tag{13}$$

the x- and z-component of the differential equation (6) finally yield the following relations:

$$i(B_x\xi_z - B_z\xi_x) = \frac{1}{k^2}(k_xB_z - k_zB_x)\xi_y' , \tag{14}$$

$$p_{1L} + \rho g\xi_y = \frac{\omega^2\rho}{k^2}\xi_y' , \tag{15}$$

where $k^2 = k_x^2 + k_z^2$. These relations are basic for our subsequent analysis. Two further redundant relations, linearly dependent on Eqs. (13), (14), and (15), read

$$i\omega^2\rho(B_x\xi_x + B_z\xi_z) = -(k_xB_x + k_zB_z)(p_{1L} + \rho g\xi_y) , \qquad (16a)$$

$$i(k_x\xi_z - k_z\xi_x) = 0 . \qquad (16b)$$

In the present incompressible case the direct connection between $i\xi_x$, $i\xi_z$, and ξ_y becomes very simple, viz.

$$i\xi_x = -\frac{k_x}{k^2}\xi_y' , \quad \text{and } i\xi_z = -\frac{k_z}{k^2}\xi_y' .$$

Substitution of Eqs. (14) and (15) in the remaining equation of motion in the y-direction yields

$$\left[\frac{\omega^2\rho-(k_xB_x+k_zB_z)^2/\mu_o}{k^2}\xi_y'\right]' - \left[\omega^2\rho-(k_xB_x+k_zB_z)^2/\mu_o+\rho'g\right]\xi_y = 0 . \qquad (17)$$

It should be remarked that, although the plasma is incompressible, we do not exclude the presence of pressure gradients. Thus, a consistent treatment of the thin surface layer for the sharp-boundary case with an incompressible plasma (Chapter 4) is made possible.

Equation (17) together with the boundary conditions of Eq. (9), $\xi_y(y_1) = \xi_y(y_2) = 0$, constitutes an eigenvalue problem for ω^2, the sign of ω^2 determining the stability of the concerned k_x,k_z mode(s). Now, we shall show that necessary and sufficient stability criteria can be obtained from the marginal equation of motion

$$\left[\frac{(k_xB_x+k_zB_z)^2/\mu_o}{k^2}\xi_{yo}'\right]' - \left[(k_xB_x+k_zB_z)^2/\mu_o - \rho'g\right]\xi_{yo} = 0 , \qquad (18)$$

which follows from Eq. (17) taking $\omega^2 = 0$.†)

Equation (17) can be written in the form $(f\xi_y')' - h\xi_y = 0$, where

$$f = \frac{-\omega^2\rho + (k_xB_x + k_zB_z)^2/\mu_o}{k^2} ,$$

$$h = -\omega^2\rho + (k_xB_x + k_zB_z)^2/\mu_o - \rho' g . \tag{19}$$

The functions f and h are to be real for the following reasons. First of all k_x and k_z should be real since we consider a plasma which is infinite in the x- and z-directions. Furthermore, it has been proved[26,31]) that ω^2 is real, as a consequence of the self-adjointness of Eq. (6) with the associated boundary conditions. For $\omega^2 < 0$ the functions f and h are monotonically increasing functions of $-\omega^2$. Then it follows from Sturm's fundamental theorem with the modification of Picone (see Ref.30) that ξ_y oscillates more slowly (if it oscillates at all) when $-\omega^2$ increases, i.e. the mutual distance between zero points of ξ_y increases when $-\omega^2$ increases. If, for specified values of k_x and k_z, ξ_{yo} represents the solution of Eq. (18) satisfying $\xi_{yo}(y_1) = 0$, then in general $\xi_{yo}(y_2) \neq 0$, and one has to choose $\omega^2 \neq 0$ in order to find a solution ξ_y of Eq. (17) satisfying both boundary conditions. For the time being, we exclude from the discussion singularities of Eq. (18) occurring when $k_xB_x + k_zB_z = 0$. In that case, if the function ξ_{yo} satisfying $\xi_{yo}(y_1) = 0$ has a zero on the open interval (y_1,y_2), then, according to Sturm's fundamental theorem, a solution to the full equation of motion (17) for $\omega^2 < 0$ exists satisfying $\xi_y(y_1) = \xi_y(y_2) = 0$. Thus, at least one unstable mode with wavenumbers k_x and k_z occurs. On the other hand, if ξ_{yo} has no zero points

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†) The index o marks the solution of the marginal equation of motion. In order to avoid a too complex notation this index will be dropped generally.

on (y_1,y_2), no solution of Eq. (17) will exist for $\omega^2 < 0$ that satisfies the boundary conditions, and the plasma is stable for these perturbations. As a consequence, the stability can be investigated indeed by only considering the marginal equation (18).

Next, the influence of the singularities in the marginal equation of motion (18), present where $k_xB_x + k_zB_z = 0$, will be discussed†). In the neighbourhood of these singular points, y_s say, the solution ξ_{yo} can become infinite; it is then more convenient to transform to the variable Q_y, which here becomes smaller than ξ_y by an order of magnitude (see Eq. (12)). In terms of Q_y Eq. (17) becomes

$$Q_y'' + \left[\frac{(\omega^2\rho - F^2/\mu_o)'}{\omega^2\rho - F^2/\mu_o} - 2\,\frac{F'}{F} \right] Q_y'$$

$$- \left[\frac{(\omega^2\rho - F^2/\mu_o)'}{\omega^2\rho - F^2/\mu_o}\,\frac{F'}{F} + \frac{F''}{F} - 2\left(\frac{F'}{F}\right)^2 + k^2 + \frac{\rho' g k^2}{\omega^2\rho - F^2/\mu_o} \right] Q_y = 0 \;, \tag{20}$$

where $F = k_xB_x + k_zB_z$. This equation will be studied for small negative ω^2, after which the limit $\omega^2 \to 0$ will be taken, following a procedure analogous to that of Kadomtsev[32] and Greene and Johnson[33] in their discussion of the diffuse pinch. The corresponding simplified treatment is represented here in view of its short derivations; a mathematical more rigorous analysis, to be given in Appendix I, proves to lead to the same results.

In the neighbourhood of a singular point $y = y_s$ in general $F \approx \lambda s$, where $s = y - y_s$, while $\lambda = F'(y_s)$ can be reduced (with the aid of the relation $F(y_s) = 0$) to the expression $\lambda = \left[(k/B)(B_xB_z' - B_zB_x') \right]_{y_s}$. We shall not consider cases in

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†) A study of the singularities of Eq. (17), where $\omega^2\rho = (k_xB_x + k_zB_z)^2/\mu_o$ can be avoided because we only consider negative values of ω^2, defining stability as the absence of solutions with $\omega^2 < 0$.

which F is not linearly dependent on s near s = 0, the generalization being straightforward. In a neighbourhood of y_s, which is large enough to contain a region where $-\omega^2\rho \ll \lambda^2 s^2/\mu_o$ and small enough for F to satisfy $F \approx \lambda s$, three regions can be distinguished:

I. $-\omega^2\rho \gg \lambda^2 s^2/\mu_o$. Equation (20) here becomes approximately

$$Q_y'' - \frac{2}{s} Q_y' + \frac{2}{s^2} Q_y = 0 ,$$

having the solutions $Q_{y1} \sim s^2$ and $Q_{y2} \sim s$.

II. $-\omega^2\rho \sim \lambda^2 s^2/\mu_o$. Neglecting the variation of ρ in Eq. (20), which is not quite correct but does not influence the argument, one roughly obtains

$$Q_y'' - \frac{1}{s} Q_y' + \frac{1}{s^2} Q_y = 0 ,$$

having the solutions $Q_{y3} \sim s$ and $Q_{y4} \sim s \ln s$.

III. $-\omega^2\rho \ll \lambda^2 s^2/\mu_o$. The approximation in this region reduces to the marginal equation of motion, which transforms for small s into

$$Q_y'' + \frac{\mu_o \rho' g k^2}{\lambda^2 s^2} Q_y = 0 ;$$

it has the solutions $Q_{y5} \sim s^{n_1}$ and $Q_{y6} \sim s^{n_2}$,

where $n_{1,2} = 1/2 \pm 1/2 \sqrt{1 - \frac{4\mu_o \rho' g k^2}{\lambda^2}}$ (plus sign for n_1).

The solutions to the marginal equation of motion behave differently according to whether the next inequality is satisfied or not:

$$1 - \frac{4\mu_o \rho' g k^2}{\lambda^2} > 0 \text{ or, equivalently, } \rho' g < \frac{(B_x B_z' - B_z B_x')^2}{4\mu_o B^2} . \qquad (21)$$

This is the well-known "Suydam" criterion for stability against the gravitational instability, in which the competition between the driving force of the instability (a density gradient in the presence of gravity) and the stabilizing influence of the shear of the magnetic-field lines is clearly seen. (See: Cowley[27]); the real Suydam criterion[34]) applies to pinch instabilities in cylinder geometry). If this criterion is not satisfied n_1 and n_2 become complex, so that the corresponding real solutions $Q_{yo}(s)$ and $\xi_{yo}(s)$ oscillate, and the more rapidly so if $s \to 0$. It then follows from Sturm's fundamental theorem that the plasma is unstable. The instabilities which arise here are localized interchange instabilities. If Suydam's criterion is satisfied, the solutions of the marginal equation of motion (18) in the neighbourhood of y_s can be written as $\xi_s \sim s^{n_1-1}$ and $\xi_\ell \sim s^{n_2-1}$, where the indices s (small) and ℓ (large) are introduced, following Newcomb's notation for the diffuse pinch[28]). In general both solutions tend to infinity as $s \to 0$, only the ratio ξ_s/ξ_ℓ is always small. Accordingly, Q_{y5} is always finite, whereas Q_{y6} can become infinite.

Let us assume that the inequality (21) is satisfied. If then the solutions of the regions I, II, and III are joined together by equating the functions and their first derivatives at the boundaries of the regions, taking the limit $\omega^2 \to 0$ thereafter, it turns out that both $Q_{y1} \sim s^2$ and $Q_{y2} \sim s$ in I change in II into $Q_{y3} \sim s$ and this solution in turn smoothly changes into Q_{y5} in III. Therefore, in the limit $\omega^2 \to 0$ both solutions of region I pass in region III into the "small" solution $Q_{y5} \sim s^{n_1}$ of the marginal equation of motion. Next, we take as new solutions half of the sum and half of the difference of these two solutions[32]). Because of the fact that the original solutions in region I were an even and an odd function, we now obtain one solution which is identically equal to zero to the left of the singular point (with the exception of a small region of a size tending to zero if $\omega^2 \to 0$) and behaving like ξ_s to the right, and a solution which is identically equal to zero to the right of the singular point and behaving like ξ_s to the left.

Thus, the effects of the singular points (if any between y_1 and y_2) on the marginal-stability analysis are twofold:

- The interval (y_1,y_2) is split into independent subintervals (i.e. intervals bounded by y_1,y_2, and the singular points), which are to be studied separately as far as stability is concerned. This results from the fact that the singularities are such that the solutions of the marginal equation of motion to the left and to the right of the singularity are uncoupled. Notice, however, that this does not imply that the solutions of the equation of motion for $\omega^2 \neq 0$ are localized in one of the independent subintervals. If $\omega^2 \neq 0$ region I is much larger than in limiting situations referring to $\omega^2 \to 0$, and this region is situated on either side of the singularity. However, it is true that for small ω^2 the perturbation can have a large amplitude in one independent subinterval and a small amplitude in the other intervals.

- The proper boundary condition to be posed at a singular point is that the solution ξ_{yo} of the marginal equation should be "small" there so that $\xi_{yo} \sim (y-y_s)^{n_1-1}$. According to Sturm's separation theorem[30]) the zero points of the different solutions of Eq. (18) alternate. Newcomb[28]) proved that the presence of a solution which is "small" at a singular point is equivalent to the existence of a zero of this solution at an ordinary point. In other words, Sturm's separation theorem also applies to intervals bounded by singular points, provided that a smallness of a solution at a singular point is counted as a zero point. In view of the separation property, a solution that is "large" at a singular point then will have a zero somewhere between this singularity and the next zero point of the "small" solution.

On the basis of the preceding analysis the stability criteria can be formulated, starting from the marginal equation of motion. It turns out that, quite generally, the marginal equation of motion is identical to the Euler-Lagrange equation following from a minimization of the energy, if no normalization is applied†). Therefore, it is obvious that we shall obtain the stability criteria for the gravitational instability, here under discussion for an incompressible plasma, in a form analogous to Newcomb's criteria for the diffuse linear pinch[28]), which were obtained by means of the energy principle. This will not be worked out here, since we only aimed at demonstrating that the application of the marginal-stability analysis can lead to rigorous results, just as well as the application of the energy principle. For that reason we shall not try to find an original formulation of the stability criteria and just adopt Newcomb's theorem 9 with the necessary modifications:

Theorem 1. For specified values of k_x and k_z a plane incompressible plasma layer under the influence of gravity in the y-direction is stable if and only if there exist no solutions to the marginal equation of motion (18) having more than one zero in the interval (y_1,y_2), or in the independent subintervals of it if singular points are present.

Theorem 1 is the general criterion. For applications, for example to the sharp boundary case of Chapter 4, it is more

†) This can easily be seen for the case of an infinite plasma, where δW is defined by the following integral over all space:

$$\delta W = -\frac{1}{2}\int \underset{\sim}{F}\{\underset{\sim}{\xi}\}.\underset{\sim}{\xi}\,d\tau ,$$

where $\underset{\sim}{F}$ is the force operator given by Eq. (6a). The variation of δW is written as

$$\delta(\delta W) = -\frac{1}{2}\int \left[\underset{\sim}{F}\{\delta\underset{\sim}{\xi}\}.\underset{\sim}{\xi}+\underset{\sim}{F}\{\underset{\sim}{\xi}\}.\delta\underset{\sim}{\xi}\right]d\tau = -\int \underset{\sim}{F}\{\underset{\sim}{\xi}\}.\delta\underset{\sim}{\xi}\,d\tau ,$$

where we have first used the linearity of the operator $\underset{\sim}{F}$ and next its self-adjointness. Then it is evident that, for arbitrary $\delta\underset{\sim}{\xi}$, δW reaches a minimum if $\underset{\sim}{F}\{\underset{\sim}{\xi}\} = 0$, which is identical to the marginal equation of motion following from the full equation of motion $\underset{\sim}{F}\{\underset{\sim}{\xi}\} = -\rho\omega^2\underset{\sim}{\xi}$ by putting $\omega^2 = 0$.

convenient to bring the criterion in the same form as Newcomb's theorem 12, which can here be formulated as the following statement:

Theorem 2. For specified values of k_x and k_z a plane incompressible plasma layer under the influence of gravity in the y-direction is stable in an independent subinterval (y_{s1},y_{s2}) if and only if, simultaneously:

1) "Suydam's" criterion (21) is fulfilled at the endpoints y_{s1} and y_{s2} if they are singular.

2) If ξ_{y1} and ξ_{y2} are the solutions to the marginal equation of motion (18) satisfying ξ_{y1} "small" at y_{s1}, ξ_{y2} "small" at y_{s2}, while $\xi_{y1} = \xi_{y2}$ at some interior point y_o of the interval, then ξ_{y1} should not vanish in (y_{s1},y_o) and ξ_{y2} should not vanish in (y_o,y_{s2}).

3) $\xi'_{y1}/\xi_{y1} > \xi'_{y2}/\xi_{y2}$ at $y = y_o$.

The second theorem is equivalent to the first one, but in addition it provides a clear-cut prescription for the investigation of the stability. In order of succession the three requirements of this theorem involve, when violated, a decreasing number of zero points of the solutions of the marginal equation of motion. Violation of "Suydam's" criterion even amounts to an infinity of zero points. The second item of theorem 3 is clear from the preceding analysis. The third item is necessary in order to prevent that the solution ξ_{y1} should have a zero point upon continuation in the interval (y_o,y_{s2}), and similarly for ξ_{y2}. The position of the interior point y_o is arbitrary, but the solutions ξ_{y1} and ξ_{y2} can be normalized such as to have equal amplitudes there.

3.2. Compressible plasma

The derivation of the required equations is fully analogous to that for the incompressible case. Starting from the equation of motion (6) and taking elementary solutions, again of the type considered in Sec. 3.1, three coupled ordinary differential equations are obtained. Two of the three equations, the x- and z-component of Eq. (6), yield expressions corresponding to Eqs. (13) and (14), viz.

$$i(B_x\xi_z - B_z\xi_x) = -(k_xB_z - k_zB_x)\,\frac{[\omega^2\rho(\gamma p + B^2/\mu_o) - \gamma pF^2/\mu_o]\,\xi_y' - \omega^2\rho^2 g\xi_y}{N}\,, \tag{22}$$

$$\nabla\cdot\underset{\sim}{\xi} = \frac{(\omega^2\rho - F^2/\mu_o)(\omega^2\rho\xi_y' - \rho g k^2\xi_y)}{N}\,; \tag{23}$$

here we have introduced the abbreviations

$N = \omega^4\rho^2 - \omega^2\rho k^2(\gamma p + B^2/\mu_o) + \gamma p k^2F^2/\mu_o$ and $F = k_xB_x + k_zB_z$.

Two further redundant relations, linearly depending on the preceding ones and corresponding to Eqs. (16), read:

$$i\omega^2\rho(B_x\xi_x + B_z\xi_z) = F(\gamma p\nabla\cdot\underset{\sim}{\xi} - \rho g\xi_y)\,, \tag{24a}$$

$$i(k_x\xi_z - k_z\xi_x) = -(k_xB_z - k_zB_x)F\,\frac{\omega^2\rho\xi_y' - \rho g k^2\xi_y}{N}\,, \tag{24b}$$

the first of which will be used in Sec. 7.1. The explicit expressions for $i\xi_x$ and $i\xi_z$ in terms of ξ_y are much more complicated than in the incompressible case and are omitted here since they are of no use for the further analysis.

Substitution of Eqs. (22) and (23) into the equation of motion for the y-direction yields

$$\left[\frac{(\omega^2\rho - F^2/\mu_o)\{\omega^2\rho(\gamma p + B^2/\mu_o) - \gamma pF^2/\mu_o\}}{N}\,\xi_y'\right]'$$

$$+\left[\omega^2\rho - F^2/\mu_o + \rho' g - \frac{k^2\rho^2 g^2(\omega^2\rho - F^2/\mu_o)}{N} - \left\{\frac{\omega^2\rho^2 g(\omega^2\rho - F^2/\mu_o)}{N}\right\}'\right]\xi_y = 0. \tag{25}$$

The marginal equation of motion here becomes

$$\left[\frac{(k_xB_x+k_zB_z)^2/\mu_o}{k^2}\,\xi'_{yo}\right]' - \left[(k_xB_x+k_zB_z)^2/\mu_o-\rho' g - \frac{\rho^2 g^2}{\gamma p}\right]\xi_{yo}=0. \tag{26}$$

In the marginal-stability analysis of the compressible case an essential difficulty arises. The equation of motion (25) is much more complicated than the corresponding equation of motion (17) for the incompressible case. In particular, the proof of the monotonic character in $-\omega^2$ of the functions f and h, corresponding to those of Eq. (19), is complicated because of the appearance of a derivative with respect to y in the last term of Eq. (25). Therefore, a straightforward application of Sturm's fundamental theorem, like in Sec. 3.1, seems rather intricate. For large values of $-\omega^2$ the term in question is negligible in comparison with the term $\omega^2\rho$; in that case it is easy to see that f and h are monotonically increasing functions of $-\omega^2$. Therefore, for large values of $-\omega^2$ the oscillating behaviour of ξ_y certainly will be monotonic in $-\omega^2$. In order to reach the same conclusion for smaller values of $-\omega^2$ it should be proved that $\partial\xi_y/\partial(\omega^2)$ cannot change its sign at a certain value of $-\omega^2$.

For that purpose we make use of the auxiliary theorem stating that the relation $k = k(\omega^2,\cdot\cdot)$, resulting from the solution of the eigenvalue problem of Eq. (25) with the boundary conditions (9), can have neither a minimum nor a maximum†); the dots in the brackets here mark the other parameters of the problem, like k_x/k_z, which should be kept constant. The proof of the auxiliary theorem is simple: Suppose that an extremum occurs for some values of the parameters k and ω^2, k_o and ω_o^2 say, so that $(\partial k/\partial(\omega^2))_o = 0$. Then, one obtains for small $\delta(\omega^2)$

$$k - k_o \approx \left(\frac{\partial^2 k}{\partial(\omega^2)^2}\right)_o \{\delta(\omega^2)\}^2 \;.$$

†) We owe this theorem and its proof to Dr. M.P.H. Weenink

Obviously, one can always choose $k - k_o$ such that $\{\delta(\omega^2)\}^2 < 0$ and then $\delta(\omega^2)$, and also $\omega^2 \approx \omega_o^2 + \delta(\omega^2)$ will be imaginary. However, this is in contradiction to the fact that ω^2 must be real, as has been proved by Bernstein et alii[26]) and Hain et alii[31]), making use of the self-adjointness of the operator $\underset{\sim}{F}\{\underset{\sim}{\xi}\}$.

From this theorem it follows that it is impossible to have two neighbouring values of $-\omega^2$ belonging to the same value of k. It then follows in turn that for fixed k and increasing $-\omega^2$ the mutual distances between the zero points of the solution ξ_y of Eq. (25) must increase monotonically, because otherwise (in view of the fixed size of the interval (y_1,y_2) and the boundary condition (9) applied at y_1 and y_2) neighbouring values of $-\omega^2$ should exist belonging to one value of k.

The proof of the splitting of the interval (y_1,y_2) in independent subintervals due to the possible presence of singular points (F = 0) here applies almost without change. The only difference with the incompressible case is the fact that $\nabla.\underset{\sim}{\xi} \neq 0$ if $\omega^2 = 0$ (as a result of the gravitation); the marginal equation of motion for the compressible case therefore differs from that for the incompressible case. As a result the stability criteria also will be different. This will turn out not to be the case for the pinch (Chapter 5). Moreover, the exponents characterizing the "small" and "large" solutions are different from those for the incompressible case, viz.

$$n_{1,2} = 1/2 \pm 1/2 \sqrt{1 - \frac{4\mu_o k^2}{\lambda^2}\left(\rho' g + \frac{\rho^2 g^2}{\gamma p}\right)} \;;$$

hence, "Suydam's" criterion here gets the well-known[27]) form

$$\rho' g + \frac{\rho^2 g^2}{\gamma p} < \frac{(B_x B_z' - B_z B_x')^2}{4\mu_o B^2} \,. \qquad (27)$$

Therefore, the theorems 1 and 2 of Sec. 3.1 also apply to the compressible case if the following substitutions are made: incompressible → compressible, marginal equation of motion (18) → (26), "Suydam's" criterion (21) → (27).

For a magnetic field having the same direction for all values of y the equation of motion (25), in contradistinction to Eq. (17), proves to develop a singularity for F = 0 over the whole interval (y_1, y_2) if k_x and k_z satisfy this condition. This degenerate case in the stability analysis was treated by Newcomb[35]) by means of the energy principle. The corresponding singularity for a constant-pitch field in cylinder geometry will be treated in detail in Sec. 7.1.

Now we have completed the marginal-stability analysis for the plane plasma layer. The formulation of the stability criteria could be provided without any reference to variational principles while the use of Hilbert's invariant integral, needed by Newcomb in his treatment of the diffuse pinch, could be avoided. The discussion of the diffuse pinch in Chapter 5 will not evoke essential new problems. Thus it is shown that the energy principle and the marginal-stability analysis are equivalent. There is one advantage associated with the application of the marginal-stability analysis, viz. the way in which the theorems 1 and 2 were obtained provides some insight in the form of the real solutions for $\omega^2 \neq 0$, if those for $\omega^2 = 0$ are known. It is clear, for example, that the most dangerous perturbation, i.e. the one having the largest value of $-\omega^2$, will have no zero points on the interval (y_1, y_2).

C H A P T E R 4

STABILIZATION OF THE GRAVITATIONAL INSTABILITY BY FORCE-FREE MAGNETIC FIELDS

The preceding analysis will now be applied to the gravitational instability of a plane incompressible plasma layer, supported from below by a current-carrying pressureless plasma associated with a force-free magnetic field. The plasma is taken incompressible because we are especially interested in the analogy with the stability of a compressible pinch. This analogy appears most clearly from the gravitational instability of an incompressible plasma.

4.1. Equilibrium

The configuration is shown in Fig. 1. Both a dense plasma and a tenuous plasma, separated at $y = 0$, are situated between two perfectly conducting walls at $y = a$ and $y = -b$. The gravitational force is directed along the negative y-axis. The dense upper plasma has a constant density ρ^p and the magnetic field $\underset{\sim}{B}^p$, parallel to the z-axis, has a constant magnitude, whereas in the lower tenuous plasma the magnetic field $\underset{\sim}{B}^t$ has a constant magnitude but changes its direction which, however, is always parallel to the xz-plane. Surface currents in the plane

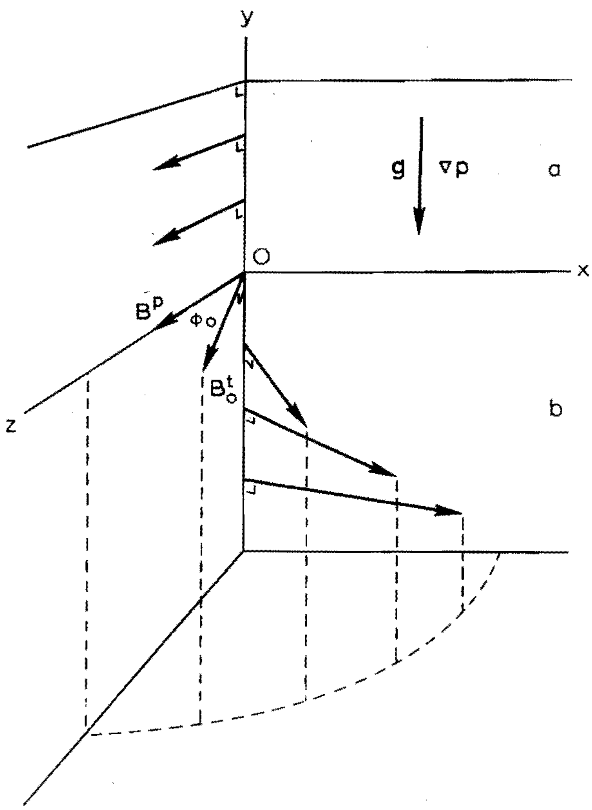

Fig. 1 Plane plasma layer supported by a force-free magnetic field.

y = 0 produce a jump in the magnitude and the direction of the magnetic field.

The equilibrium equation (11) provides the variation of the pressure in the dense plasma:

$$p = p_o - \rho^p g y \ . \qquad (28)$$

The pressure balance, Eq. (4), provides the magnitude of the jump of $\underset{\sim}{B}$ according to

$$p + (B^p)^2/2\mu_o = (B_o^t)^2/2\mu_o \ . \qquad (29)$$

The density of the tenuous pressureless plasma is supposed to be so small that the effect of the associated gravity force on the equilibrium can be neglected. It is true that the term

$\rho^t\omega^2$ should be taken into account in a normal-mode analysis, and then ρ^t should be finite. In the marginal-stability analysis, however, this term drops again ($\omega^2 = 0$) and, therefore, the neglect of ρ^t does not influence the form of the stability criteria (in some important cases, however, the neglect of ρ^t leads to unphysical results: see Sec. 6.4).

The condition for the magnetic field $\underset{\sim}{B}^t$ to be force-free reads $\nabla \times \underset{\sim}{B} = \alpha\underset{\sim}{B}$, or in components:

$$B_x' = -\alpha B_z \ , \quad B_z' = \alpha B_x \ . \tag{30}$$

Assuming α = constant the field components become

$$\begin{aligned} B_x^t &= B_o^t \sin(\phi_o - \alpha y) \ , \\ B_z^t &= B_o^t \cos(\phi_o - \alpha y) \ , \end{aligned} \tag{31}$$

where ϕ_o is the angle between $\underset{\sim}{B}_o^t$ and the z-axis. The parameter α simply represents the amount of rotation of $\underset{\sim}{B}^t$ with respect to the negative y-direction: $\phi' = -\alpha$. Although Eq. (30) can be solved easily for general α the assumption α = constant is of importance because it leads to a great simplification of the stability criteria.

4.2. Stability criteria

The general problem of the gravitational instability was studied by Kruskal and Schwarzschild[20]) for the special case of an infinitely extended compressible plasma, supported by a vacuum magnetic field. Meyer[36]) generalized this analysis to the case of crossed magnetic fields in the plasma and the vacuum and then found a pronounced stabilizing effect. However, long waves remained unstable. In the following analysis it will be shown, for the incompressible case, that a combination of conducting walls and crossed magnetic fields ($\phi_o \neq 0$) produces complete stability. Moreover, this stabilization is enhanced considerably if the vacuum magnetic field ($\alpha = 0$) is replaced by a force-free magnetic field with a properly chosen value of α.

For a stability analysis on the basis of theorem 2 the zeros of the solution ξ_{yo} (or Q_{yo}) of the marginal equation of motion, as well as the singular points ($F = k_x B_x + k_z B_z = 0$) should be determined on the following intervals: (y_i, a) for the dense plasma, (y_e, y_i) for the surface layer, and $(-b, y_e)$ for the tenuous plasma, $y = y_i$ and $y = y_e$ representing the upper and lower boundary of the transition layer. The surface between the dense and the tenuous plasma will be treated as a diffuse layer of thickness δ, taking the limit $\delta \to 0$, $y_e \to 0$, $y_i \to 0$ afterwards. For the sharp pinch surrounded by a vacuum magnetic field Rosenbluth[37]) showed that this procedure, owing to the possible existence of singular points in the surface layer, leads to more stringent conditions than a procedure based on the boundary conditions for a plasma-vacuum system as considered by Kruskal and Tuck[21]). For the plane plasma layer the same conclusion will be reached.

On the plasma interval (y_i, a) no singular points occur, with the exception of the case $k_z = 0$ which, however, follows trivially from the case $k_z \neq 0$ taking the limit $k_z \to 0$. The equation of motion (17) yields, for the situation assumed here, either the Alfvén wave: $\omega^2 \rho - k_z^2 (B^p)^2/\mu_o = 0$, which is of no importance for the stability analysis, or the equation

$$\xi_y^{p''} - k^2 \xi_y^p = 0 \ . \tag{32}$$

Hence, the equation of motion is the same as the marginal equation of motion. Its solution

$$\xi_y^p = C \sinh\left[(y-a)k \right] \tag{33}$$

has no zero points on the open interval (y_i, a).

For the pressureless plasma interval $(-b, y_e)$ it is more convenient to use Q_y. Substituting $\rho^t \omega^2 = 0$ and using Eq. (30)

the equation of motion (20) becomes

$$Q_y^{t''} + \left(\alpha^2 - k^2 + \frac{k_x B_z^t - k_z B_x^t}{k_x B_x^t + k_z B_z^t}\,\alpha'\right) Q_y^t = 0 \ . \tag{34}$$

The last term in this equation is added only in order to illustrate that the problem is much more complicated for a non-constant α. Dropping this term again we obtain the solution

$$Q_y^t = D_1 \sinh\left[\, y\sqrt{k^2-\alpha^2}\,\right] + D_2 \cosh\left[\, y\sqrt{k^2-\alpha^2}\,\right]. \tag{35}$$

For $k^2 < \alpha^2$ oscillating solutions are obtained from this expression by performing the substitutions: sinh $\rightarrow$ sin, cosh $\rightarrow$ cos, $\sqrt{k^2-\alpha^2} \rightarrow \sqrt{\alpha^2-k^2}$. The singular points of the marginal equation of motion in terms of ξ_y, Eq. (18), are determined by $F \equiv k_x B_x + k_z B_z = 0$ or, remembering (31), by

$$\cos\psi = \cos(\chi-\phi) = \cos(\chi-\phi_o+\alpha y) = 0 \ , \tag{36}$$

where χ is the angle between $\underset{\sim}{k}$ and the z-axis, ϕ the angle between $\underset{\sim}{B}^t$ and the z-axis, and ψ that between $\underset{\sim}{k}$ and $\underset{\sim}{B}^t$ (see Fig.2) It is obvious from Eqs. (35) and (36) that no unstable independent subintervals lying completely in the pressureless plasma region can exist, because the solution Q_y (and, consequently, also ξ_y) oscillates at most as rapid as $\sin \alpha y$ (namely, in case $k \rightarrow 0$), i.e. exactly in step with the singular points. Therefore, these subintervals are at most marginally unstable. At the same time it follows that a plane plasma layer with a force-free field (neglecting the gravity or the density) cannot be unstable. Therefore, only the solution on the interval $(-b^*, y_e)$ that is small at $y = -b^*$, viz.

$$Q_y^t = D \sinh\left[\, (y+b^*)\sqrt{k^2-\alpha^2}\,\right] \ , \tag{37}$$

is important for what follows; $-b^*$ here represents the position of the first singular point in the pressureless plasma or,

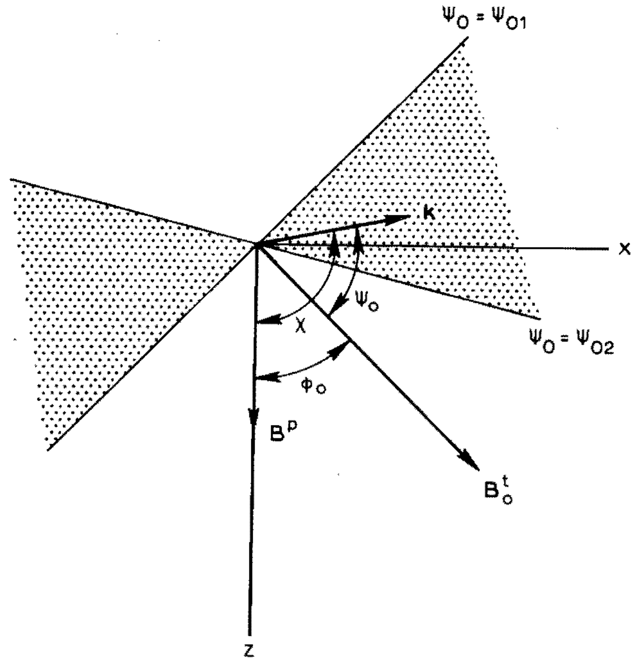

Fig. 2 Field components and direction of the perturbation (y=0). For $\underset{\sim}{k}$ lying in the shaded area a force-free magnetic field gives enhanced stability.

if there are no such points, the position of the wall (-b). From Eq. (36) it follows that $|\alpha| b^* \leq \pi$.

In the interval (y_e, y_i) of the surface layer the quantities ρ, p, and B^2 vary noticeably and, therefore, the logarithmic derivatives of these quantities are of the order $(\delta/a)^{-1}$. Thus the marginal equation of motion (18) becomes:

$$(f_o \xi_y^{1'})' - h_o \xi_y^1 = 0 \ , \tag{38}$$

where

$$f_o = (k_x B_x + k_z B_z)^2/(\mu_o k^2) \text{ and } h_o \approx -\rho' g \ . \tag{39}$$

(The upper index 1 refers to surface layer). Integrating Eq. (38) through the surface layer one obtains the result that ξ_y^1 is approximately constant in the layer, with the exception of a small ε-neighbourhood of a singular point where ξ_y^1 tends to infinity. The magnitude of ε is such that $(\delta/a)^2 \ll \varepsilon/a \ll \delta/a$. The proof of this property is completely analogous to that given by Newcomb[28] for the case of the surface layer of a sharp pinch and, therefore, will not be repeated here. In the ε-neighbourhood of the singular point the "small" solution is $\xi_y^1 \sim (y-y_s)^{n_1-1}$ where, according to Sec. 3.1,

$$n_1-1 = -1/2 + 1/2 \sqrt{1 - \frac{4\mu_o\rho' gk^2}{\lambda^2}} \approx - \frac{\mu_o\rho' gk^2}{\lambda^2} = -O(\delta/a) ,$$

which is a small, in general negative, power. The first part of the inequality above guarantees that ξ_y^1 is almost constant on $(y_e, y_s-\varepsilon)$ and $(y_s+\varepsilon, y_i)$, while the second part guarantees that the approximation of ξ by the "small" solution is valid on $(y_s-\varepsilon, y_s)$ and $(y_s, y_s+\varepsilon)$. We recall the important fact that the solutions on the two last mentioned intervals are independent of each other, so that the amplitude factors of the two "small" solutions are in general different.

Knowing the solutions of the marginal equation of motion everywhere, the application of theorem 2 is straightforward. "Suydam's" criterion (21) is trivially satisfied in both the plasma and in the pressureless plasma. This criterion also holds in the surface layer because the destabilizing term $\rho' g$ is of the order $(\delta/a)^{-1}$, and the stabilizing shear term of the order $(\delta/a)^{-2}$, which is one order of magnitude larger†). The solutions of the marginal equation of motion have no zeros on the intervals (y_i, a), (y_e, y_i), and $(-b^*, a)$; possible subintervals of $(-b, -b^*)$ are too small to contain more than one zero. Therefore, the only way in which instabilities can arise is by violation of the third item of theorem 2.

†) If the magnetic field changes its sense of rotation somewhere in the surface layer the shear term vanishes and consequently "Suydam's" criterion is not satisfied[28]). We exclude this case from the present analysis.

There are two possibilities:

(1) No singular points exist in the surface layer. The interval $(-b^*, a)$ then constitutes a single independent subinterval which we split at the point $y = y_i$, to be taken as the "point of comparison" y_o of item 3) of theorem 2. The surface-layer displacement ξ_y^1, the continuation ξ_y^t of which vanishes at $y = -b^*$, and the plasma displacement ξ_y^p vanishing at the other end $y = a$ of the single interval, should then be used as ξ_{y1} and ξ_{y2} when applying theorem 2. The value of $(\xi_y^{1'}/\xi_y^1)_{y_i}$ is found by integration of Eq. (38):

$$(f_o \xi_y^{1'})_{y_i} \approx \xi_y^1 \int_{y_e}^{y_i} h_o dy + (f_o \xi_y^{1'})_{y_e} .$$

Assuming that ξ_y and ξ_y' are continuous in y_e, the stability criterion becomes

$$\left(\frac{\xi_y^{1'}}{\xi_y^1}\right)_{y_i} \approx \frac{1}{f_o(y_i)} \left[-\rho g\right]_{y_e}^{y_i} + \frac{f_o(y_e)}{f_o(y_i)} \left(\frac{\xi_y^{t'}}{\xi_y^t}\right)_{y_e} > \left(\frac{\xi_y^{p'}}{\xi_y^p}\right)_{y_i} . \qquad (40)$$

Substituting the values of f_o from Eq. (39), transforming to Q_y by means of Eq. (12), making use of Eq. (30), and taking the limits $y_i \to 0$ and $y_e \to 0$, we obtain the stability criterion in terms of Q_y:

$$L \equiv -\frac{k_z^2 B^{p^2}}{\mu_o k^2} \frac{Q_y^{p'}}{Q_y^p} - \rho^p g >$$

$$-\frac{\alpha(k_x B_z^t - k_z B_x^t)(k_x B_x^t + k_z B_z^t)}{\mu_o k^2} - \frac{(k_x B_x^t + k_z B_z^t)^2}{\mu_o k^2} \frac{Q_y^{t'}}{Q_y^t} \equiv R . \qquad (41)$$

In this inequality the term $-\rho^t g$ has been neglected, in accordance with the equilibrium. Here and in what follows we understand by the symbols of the quantities in the stability criterion their values at the position of the boundary $(y = 0)$ without further additional indices. The expression for R also holds for not constant α, where the corresponding solution of Eq. (34) should be substituted for Q_y^t. Introducing the angles of Fig. 2 and the solutions (33) and (37), remembering that ξ_y^p and Q_y^p are identical apart from a constant factor (see Eq. (12)), the stability criterion becomes

$$L \equiv \frac{B^{p^2}}{\mu_o} \cos^2\chi . k \operatorname{cotgh}(ak) - \rho^p g >$$

$$- \frac{B^{t^2}}{\mu_o} \cos^2\psi_o \left[\sqrt{k^2-\alpha^2} \operatorname{cotgh} (b^* \sqrt{k^2-\alpha^2}) + \alpha \operatorname{tg}\psi_o \right] \equiv R. \qquad (42)$$

Here, of course, the expression for R is only valid if α = constant.

(2) One singular point y_s exists in the surface layer. (We exclude the situation in which more singular points exist, because this implies a magnetic field changing its sense of rotation). The interval $(-b^*,a)$ now consists of two independent subintervals $(-b^*,y_s)$ and (y_s,a).

(a) We split the interval $(-b^*,y_s)$ at $y = y_e$ (to be taken as y_o in item 3) of theorem 2)). Integration of Eq. (38) yields

$$(f_o \xi_y^{1'})_{y_e} \approx \xi_y^1 \int_{y_s-\varepsilon}^{y_e} h_o dy + (f_o \xi_y^{1'})_{y_s-\varepsilon} \approx \xi_y^1 \int_{y_s}^{y_e} h_o dy ,$$

where ξ_y^1 is the surface-layer solution that is "small" at y_s, to be compared with the tenuous-plasma solution ξ_y^t that vanishes at $y = -b^*$. The stability criterion becomes

$$\left(\frac{\xi_y^{t'}}{\xi_y^t}\right)_{y_e} > \left(\frac{\xi_y^{1'}}{\xi_y^1}\right)_{y_e} \approx \frac{1}{f_o(y_e)} \left[-\rho g\right]_{y_s}^{y_e} . \qquad (40a)$$

The expression analogous to Eq. (41) becomes

$$P \equiv -\rho^s g > - \frac{\alpha(k_x B_z^t - k_z B_x^t)(k_x B_x^t + k_z B_z^t)}{\mu_o k^2} - \frac{(k_x B_x^t + k_z B_z^t)^2}{\mu_o k^2} \frac{Q_y^{t'}}{Q_y^t} \equiv R ,$$
$$(41a)$$

where ρ^s is the density at the singular point.
The inequality corresponding to Eq. (42) here reads

$$P \equiv -\rho^s g > -\frac{B^{t^2}}{\mu_o}\cos^2\psi_o\left[\sqrt{k^2-\alpha^2}\ \mathrm{cotgh}(b^*\sqrt{k^2-\alpha^2})+\alpha\ \mathrm{tg}\psi_o\right] \equiv R. \tag{42a}$$

(b) We split the interval (y_s,a) at $y = y_i$. The expressions analogous to the Eqs. (40a), (41a), and (42a) then are, in succession:

$$\left(\frac{\xi_y^{1'}}{\xi_y^1}\right)_{y_i} \approx \frac{1}{f_o(y_i)}\left[-\rho g\right]_{y_s}^{y_i} > \left(\frac{\xi^{p'}}{\xi_y^p}\right)_{y_i}, \tag{40b}$$

$$L \equiv -\frac{k_z^2 B^{p^2}}{\mu_o k^2}\frac{Q_y^{p'}}{Q_y^p} - \rho^p g > -\rho^s g \equiv P, \tag{41b}$$

$$L \equiv \frac{B^{p^2}}{\mu_o}\cos^2\chi . k\ \mathrm{cotgh}(ak) - \rho^p g > -\rho^s g \equiv P. \tag{42b}$$

Notice again that the Eqs. (41a) and (41b) also hold for non-constant α.

The derived criteria (42) can further be simplified by noticing that the functions $k\ \mathrm{cotgh}(ak)$ and $\sqrt{k^2-\alpha^2}\ \mathrm{cotgh}(b^*\sqrt{k^2-\alpha^2})$ are monotonically increasing functions of k if $k^2 > \alpha^2$. We remind that $|\alpha| b^* \leq \pi$, so that for $k^2 < \alpha^2$ the latter function, transforming into $\sqrt{\alpha^2-k^2}\ \mathrm{cotg}(b^*\sqrt{\alpha^2-k^2})$, is also monotonically increasing with k^2. For a given direction of the perturbation (fixed by χ or ψ_o) only the wavenumber k appears in these functions, so that long-wavelength perturbations are the most dangerous ones. Therefore, a necessary and sufficient criterion for the stability of a plane incompressible plasma layer under the influence of gravity and supported from below by a force-free field, is obtained from the limit $k \to 0$ (indicated by the subscript o in L and R):

(1) In the absence of a singular point in the surface layer:

$$L_o \equiv \frac{B^{p^2}}{\mu_o} \frac{\cos^2 \chi}{a} - \rho^p g \quad > - \frac{B^{t^2}}{\mu_o} \cos^2 \psi_o . \alpha(\mathrm{cotg}\ \alpha b^* + \mathrm{tg}\ \psi_o) \equiv R_o . \tag{43}$$

(2) In the presence of a singular point in the surface layer:

$$L_o > P \equiv -\rho^s g \quad > \quad R_o . \tag{43a,b}$$

The latter, split, criterion is more stringent than the former.

4.3. Discussion

In a study of the sharp tubular pinch by means of the energy principle Newcomb and Kaufman[38]) distinguished between two types of instabilities:

- type-I instabilities: those instabilities for which the solution of the Euler-Lagrange equation, following from minimization of the energy, is continuous across the surface layer in the limit $\delta \to 0$,

- type-II instabilities: those instabilities for which this solution is not continuous in the limit $\delta \to 0$.

In our context this division corresponds to the violation of the criterion $L_o > R_o$, resulting in the appearance of type-I instabilities, and the violation of the criterion $L_o > P > R_o$, resulting in the appearance of type-II instabilities. Type-I instabilities can be treated without special knowledge of the structure of the surface layer. They also can be obtained from an application of the boundary conditions (7) and (8) (see Sec. 4.4). However, the splitting of the stability criterion cannot be obtained from these boundary conditions. The quantity P here depends on the density ρ^s at the singular point and, therefore, on the detailed structure of the surface layer. Hence, the presence of type-II instabilities which, for the pinch, were called surface-layer instabilities by Rosenbluth[37]) depends on this structure. It will be our task to demonstrate that realistic structures of the surface layer do exist that satisfy the split criterion (see below).

For a vacuum magnetic field ($\alpha = 0$) in the lower layer the criterion (43) for type-I instabilities becomes

$$\frac{B^{p^2}}{\mu_o}\frac{\cos^2\chi}{a} + \frac{B^{t^2}}{\mu_o}\frac{\cos^2\psi_o}{b} > \rho^p g . \qquad (44)$$

In this expression the stabilizing and the destabilizing terms are clearly distinguished. The right-hand side represents the driving force of the gravitational instability, the left-hand side consists of the first stabilizing term due to the internal field B^p and the conducting wall at $y = a$, and the second stabilizing term due to the external field B^t and the conducting wall at $y = -b$. Because $\chi = \psi_o + \phi_o$, it follows from (44) that for $\phi_o = 0$ (no field crossing) instability always occurs for perturbations with $\underset{\sim}{k} \perp \underset{\sim}{B}$, in agreement with the results of Kruskal and Schwarzschild[20]). Instabilities can neither be avoided for long-wavelength perturbations if a or b $\to \infty$, which is in agreement with Meyer's result[36]). Only the combined effects of field crossing ($\phi_o \neq 0$) and conducting walls can bring about complete stability.

From Eq. (44) it follows that the stabilization for a vacuum magnetic field is minimal if $\underset{\sim}{k}$ lies in the sector bounded by the directions $\chi = \pi/2$, for which the influence of the internal field vanishes, and $\psi_o = \pi/2$, for which the influence of the external field vanishes (see Fig. 2). However, this dangerous sector is covered by the sector for which the stabilization by the force-free field is more effective than that by the vacuum field. According to Eq. (43) a force-free field stabilizes more effectively than a vacuum field if $\underset{\sim}{k}$ lies in the sector bounded by $\psi_o = \psi_{o1} = \pi/2$ or $(3/2)\pi$ and by $\psi_o = \psi_{o2} = \mathrm{arctg}\{1/(\alpha b^*) - \mathrm{cotg}(\alpha b^*)\}$, that is, the shaded area in Fig. 2 for which $R_o(\alpha \neq 0) < R_o(\alpha = 0)$. Obviously, the reason for the enhanced stability is the fact that the perturbations with a wavenumber $\underset{\sim}{k}$ lying in this sector "see" a magnetic-field component parallel to $\underset{\sim}{k}$ which is larger in the lower layer, so that the perturbed magnetic energy is also larger there. Of course, we choose the sign of α such that the sense of rotation of $\underset{\sim}{B}$ in the force-free region agrees with that in the surface layer. Notice that the magnetic field in the force-free region has shear, but that its stabilization mechanism has nothing to do with the idea of shear (known from Suydam's criterion), in contrast with the stabilization mechanism for the surface layer.

We next investigate type-I as well as type-II instabilities in a special case. We assume a pressure exclusively due to gravity: $p = \rho^p g(a-y)$. Introducing $\beta = 2\mu_o p_o / B_o^{t^2}$, where $p_o = \rho^p ga$, and a structure factor $S = \rho^s/\rho^p$ for the surface layer, and applying Eq. (29), the quantities in Eq. (43) become:

$$L_o \sim (1-\beta)\cos^2\chi - \frac{1}{2}\beta ,$$

$$P \sim -\frac{1}{2}\beta S , \tag{45}$$

$$R_o \sim -\cos^2\psi_o . \alpha a\{\cot g(\alpha b^*) + tg\,\psi_o\} ,$$

where a common factor $B_o^{t^2}/\mu_o a$ has been dropped. The functions L_o, as well as L_o-P and P are monotonically decreasing functions of β, if the reasonable assumption $S \leq 1$ is made. Thus, with respect to type-I instabilities, as well as with respect to type-II instabilities (possibly equal) critical values of β exist above which instability sets in.

If $L_o > R_o$ type-I instabilities are absent. In Fig. 3 the quantities L_o and R_o from Eqs. (45) are plotted as a function of ψ_o, with the following choice of the parameters: $a = b$; $\phi_o = \pi/4$; $\alpha b = 0$, 0.5, and 1; $\beta = 0$ and 0.5. For $\beta = 0$ the curve L_o and the different curves R_o do not cross, while $L_o > R_o$ throughout, so that the configuration is stable against type-I instabilities. This is obvious, because the driving force of the instability is absent for $\beta = 0$. With

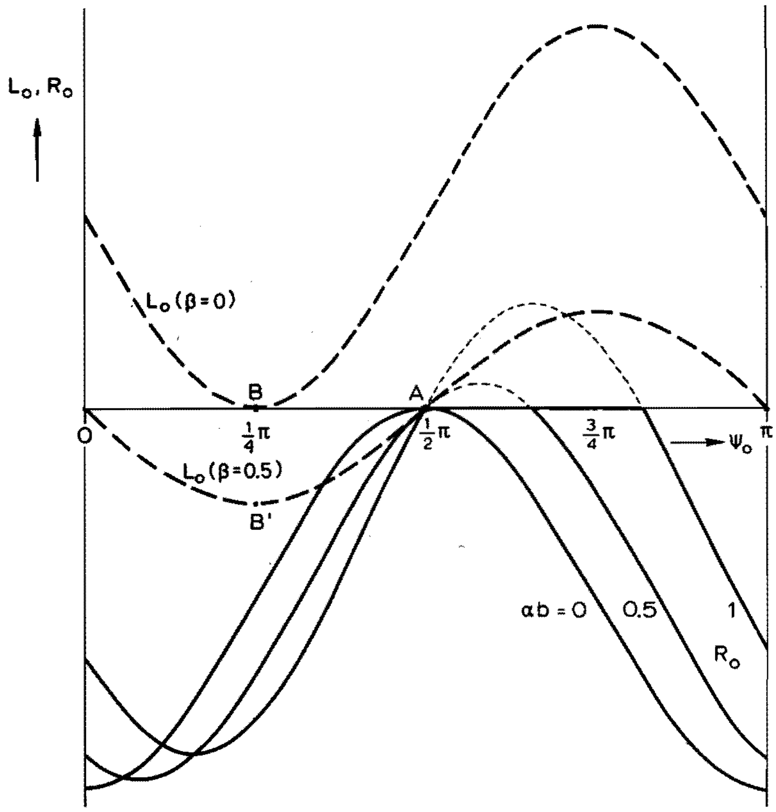

Fig. 3 Stability diagram of the plane plasma layer for $k \to 0$, $a = b$, $\phi_o = \pi/4$.

increasing values of β the curve L_o goes down and for $\beta = 0.5$ the curves L_o and R_o $(\alpha = 0)$ cross, the critical value of β for a vacuum magnetic field already being exceeded. If we consider non-vanishing values of α, it turns out that the curve R_o rotates around the fixed point A for which $R_o = 0$ and $\psi_o = \pi/2$, that is where $\underset{\sim}{k} \perp \underset{\sim}{B}{}_o^t$. For $\beta = 0.5$ the curve L_o just passes through this point and, therefore, in our numerical example this value of β is the maximal one admitting stability. Accordingly, one can choose α such that this value of β is marginally stable, i.e. the curves L_o and R_o touch in the point A. If the singular points of the pressureless plasma were not effective to the right of $\psi_o = \pi/2$ in Fig. 3, we would have $b^* = b$ and the dashed parts of the curves R_o would have physical significance. From the equation $\partial L_o/\partial\psi_o = \partial R_o/\partial\psi_o$ at $\psi_o = \pi/2$ it would follow that $\alpha b = 0.5$ is optimal for stability†). This method to determine the optimal α is not only incorrect in the present ideal theory (see, however, Appendix II), but also too pessimistic because the singular points in the pressureless plasma prove to play a stabilizing role, these points occurring for values of ψ_o between $\pi/2$ and $\pi/2 + \alpha b$. In fact, according to Eq. (36) the position $y = -b^*$ of a singular point is determined by $\psi = \psi_o - \alpha b^* = \pm\,\pi/2$. Therefore, if there are singular points present in the pressureless plasma we have $\mathrm{cotg}(\alpha b^*) = -\,\mathrm{tg}\,\psi_o$, so that $R_o \equiv 0$ and the dashed parts of R_o should be replaced by the solid horizontal parts. In Appendix II we will see that resistive effects destroy the stabilizing influence of the singular points; in that case the dashed parts of the curves R_o acquire physical significance. Within the frame of the present theory, however, they do not have this significance. If $\alpha b = \pi/2$ the horizontal part of R_o extends from $\psi_o = \pi/2$ to π, and as soon as $\alpha b > \pi/2$ instabilities appear for values of ψ_o immediately to the right of $\psi_o = 0$. These are not instabilities of the pressureless plasma (as we shall come across in Chapter 6 for cylindrical plasmas if α is large enough), but they are brought about by the fact that the above-mentioned stabilization mechanism becomes less effective if the layer in which the magnetic field is almost parallel to $\underset{\sim}{k}$ becomes too thin. Thus, the optimal values of αb lie somewhere in the region $0.5 < \alpha b < \pi/2$.

Concerning type-I instabilities we recapitulate that those, occurring with a vacuum field if $\psi_o < \pi/2$, are suppressed in the case of a force-free field when α becomes large enough. The instabilities which threaten to appear for $\psi_o > \pi/2$

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†) Notice that this value of α is independent of the assumption of a constant α because in Eq. (41) $Q_y^{t\prime}/Q_y^t$ only appears in the term with the quadratic factor $(\underset{\sim}{k}.\underset{\sim}{B}^t)^2$ and in the vicinity of $\psi_o = \pi/2$ this term is negligible with respect to the linear term containing α explicitly.

if α becomes too large, are suppressed by the influence of the singular points in the pressureless plasma.

With respect to type-II instabilities we observe that the split criterion $L_o > P > R_o$ only makes sense for χ-values for which singular points do exist in the surface layer. Assuming that the direction of the magnetic field in the surface layer changes monotonically with y, this situation occurs for values of χ between $\pi/2$ and $\pi/2 + \phi_o$. Therefore, for $\phi_o = \pi/4$, the quantity P is only relevant in the interval $\pi/4 < \psi_o < \pi/2$. If $\psi_o = \pi/4$ the singular point lies at the upper boundary of the surface layer ($y_s=y_i$), so that S = 1 and P = L_o according to Eq. (45). If $\psi_o = \pi/2$ the singular point lies at the lower boundary of the surface layer ($y_s=y_e$), so that S = 0 and P = R_o according to Eq. (45). As a result the curve for P, if represented in Fig. 3, would connect the point B (if $\beta = 0$)or B' (if $\beta=0.5$) with the point A. A limiting condition restricting the variety of possible curves P is: $0 \leq S \leq 1$. If $\beta = 0$ the curve for P simply reduces to the horizontal line AB ($P \equiv 0$), so that $L_o > P > R_o$ for all α (this is trivial: there is no driving force for the instability). If $\beta = 0.5$ instabilities certainly arise for a vacuum field, because $L_o > R_o$ is already violated and no space is left between L_o and R_o for the curve P. This situation changes with a force-free field if $\alpha b > 0.5$. For instance, if the rate of change of the direction of $\underset{\sim}{B}$ is constant across the surface layer, the position of the singular point y_s is connected with ψ_o according to: $y_s = \frac{\pi/2-\psi_o}{\pi/2} 2\delta$. Then, it follows from Eq. (45) that a surface layer with $S=\rho^s/\rho^p>\sin\left\{\pi.\frac{y_s-y_e}{2\delta}\right\}$ satisfies $L_o > P$. If, at the same time, $\alpha b > 0.5$, enough space will be left for P to satisfy $P > R_o$. This very example suffices to show the existence of a very large class of realistic surface-layer structures which are stable against type-II instabilities in the presence of a force-free field.

Summarizing: force-free fields with properly chosen value of α can stabilize type-I as well as type-II instabilities up to a value of β which is higher than β_{crit} corresponding to a vacuum field. This result will turn out to be strongly analogous to the results to be obtained in Chapter 6 for the sharp pinch. In that case it will lead to more spectacular results, because a pinch surrounded by a vacuum field is already unstable for $\beta = 0$.

4.4. Growth rates

Finally, we shall show that the stability criteria and the growth rates of type-I instabilities can be obtained from the boundary conditions (7), (8), and (9), giving

$$p_{1L} + \frac{1}{\mu_o} \underset{\sim}{B}^p . [\underset{\sim}{Q}^p + \underset{\sim}{\xi}^p . \nabla \underset{\sim}{B}^p] = \frac{1}{\mu_o} \underset{\sim}{B}^t . [\underset{\sim}{Q}^t + \underset{\sim}{\xi}^t . \nabla \underset{\sim}{B}^t] \qquad (y=0), \quad (46)$$

$$\xi_y^p = \xi_y^t \qquad (y=0), \quad (47)$$

$$\left.\begin{aligned} \xi_y^p &= 0 \qquad (y=a), \\ \xi_y^t &= 0 \qquad (y=-b). \end{aligned}\right\} \quad (48)$$

By dividing Eq. (46) by Eq. (47), while using Eqs. (12) - (15), we obtain

$$-\rho^p g + \frac{\omega^2 \rho^p - k_z^2 B^{p^2}/\mu_o}{k^2} \frac{\xi_y^{p'}}{\xi_y^p} = - \frac{(k_x B_x^t + k_z B_z^t)^2}{\mu_o k^2} \frac{\xi_y^{t'}}{\xi_y^t} \qquad (y=0). \quad (49)$$

Here, ξ_y^p and ξ_y^t are the solutions of the complete equation of motion, which differ in general from the solutions ξ_{yo}^p and ξ_{yo}^t of the marginal equation of motion. It was shown for the plasma interval that the solutions coincide, $\xi_y^p = \xi_{yo}^p$ (Eq. (33)), because the magnetic field is homogeneous. In case the density ρ^t of the pressureless plasma is neglected the solutions ξ_y^t and ξ_{yo}^t also coincide, because ω^2 appears in the equation of motion only in the combination $\rho^t \omega^2$, so that it does not make any difference whether ρ^t or ω^2 is neglected. For the same reason the proof of the splitting of the pressureless plasma interval in independent subintervals is valid for $\rho^t \to 0$. Therefore, the solution ξ_y^t is obtained from Eq. (37), using Eq. (12). Substituting these solutions and the angles of Fig. 2 into Eq. (49) we obtain for the growth rates of type-I instabilities:

$$\omega^2 = - \frac{gk}{\mathrm{cotgh}(ak)} + \frac{k^2 B^{p^2}}{\mu_o \rho^p} \cos^2 \chi$$

$$+ \frac{k B_o^{t^2}}{\mu_o \rho^p} \cos^2 \psi_o \cdot \frac{\sqrt{k^2-\alpha^2}\ \mathrm{cotgh}(b^* \sqrt{k^2-\alpha^2}) + \alpha\ \mathrm{tg}\psi_o}{\mathrm{cotgh}(ak)}$$

$$= \frac{k}{\rho^p\ \mathrm{cotgh}(ak)} (L - R) \ . \qquad (50)$$

The corresponding approximation for small values of k reads

$$\omega^2 = \frac{ak^2}{\rho^p} (L_o - R_o) \ . \qquad (51)$$

The stability criterion $L_o > R_o$ for type-I instabilities follows immediately from this expression. At the same time it becomes clear how the conducting wall at y = a influences the stability. If $a \to \infty$ the growth rate of the gravitational instability is given by the well-known expression -gk. It is true that the growth rate is small for small k, but it dominates the magnetic terms which are proportional to k^2. Therefore, long-wavelength perturbations are always unstable if a tends to infinity. As a result of the introduction of a conducting wall the growth rate for small k becomes $-gak^2$, and now the magnetic terms can compete with the gravitational terms, thus making the stability of long waves possible.

We point out that the stability analysis by means of the boundary conditions is of limited applicability in comparison with the marginal-stability analysis. It is not possible to treat type-II instabilities in this way. Moreover, we were forced to neglect the density of the pressureless plasma in order to be able to solve the equation of motion. This approximation was also made in the preceding sections, but there it was only of importance for the equilibrium and did not play a role in the stability analysis. In the analogous treatment of the pinch (Chapter 6) the density of the pressureless plasma has no influence on the equilibrium and then it will not be necessary to neglect it.

CHAPTER 5

MARGINAL-STABILITY ANALYSIS OF A DIFFUSE LINEAR PINCH

The marginal-stability analysis will now be developed for a plasma with a distributed current in a cylinder with radius $\hat{R}$, surrounded by a perfectly conducting wall. The analysis is fully equivalent to that of Newcomb[28]), who started from the energy principle. Here, we can shorten the derivation of the stability criteria considerably , making use of the analogy with the plane case (Chapter 3).

The geometry of the configuration suggests to introduce cylinder coordinates r,θ, and z. Inhomogeneity of the pressure, the density, and the magnetic field will be restricted to the r-direction. Accents mark differentiation with respect to r. The equilibrium can be described in terms of $p = p(r)$ and $\underset{\sim}{B} = (0, B_\theta(r), B_z(r))$, which are connected with each other according to

$$\left(p + \frac{B^2}{2\mu_o}\right)' + \frac{B_\theta^2}{\mu_o r} = 0 \ . \tag{52}$$

In this case no essential difference exists between the stability analysis for a compressible and an incompressible plasma. Because the equations for the incompressible case can easily be derived from those for the compressible one, and not the other way round, we shall give the stability analysis for a compressible pinch.

Starting from Eq. (6) and taking elementary solutions of the form

$$\underset{\sim}{\xi} = (\hat{\xi}_{r,mk}(r),\ \hat{\xi}_{\theta,mk}(r),\ \hat{\xi}_{z,mk}(r))\, e^{i(m\theta+kz-\omega t)} ,$$

three coupled ordinary differential equations are obtained. Again we drop the circumflex as well as the lower indices m and k, and henceforth understand by $\underset{\sim}{\xi}$ expressions of the above form. The relation between $\underset{\sim}{Q}$ and $\underset{\sim}{\xi}$, as given by Eq. (6b), becomes

$$\begin{aligned} Q_r &= i(mB_\theta/r + kB_z)\xi_r , \\ Q_\theta &= -(B_\theta\xi_r)' - ik(B_\theta\xi_z - B_z\xi_\theta) , \\ Q_z &= -\frac{1}{r}(rB_z\xi_r)' + \frac{im}{r}(B_\theta\xi_z - B_z\xi_\theta) . \end{aligned} \tag{53}$$

With the aid of the Eqs. (52) and (53), the θ- and z-components of the equation of motion (6) can be reduced, after some tedious algebraic calculations, to

$$i(B_\theta\xi_z - B_z\xi_\theta) = -\frac{(mB_z/r-kB_\theta)\{\omega^2\rho(\gamma p+B^2/\mu_o)-\gamma pF^2/\mu_o\}(r\xi_r)'/r}{N} - \frac{2kB_\theta(\omega^2\rho B^2/\mu_o-\gamma pF^2/\mu_o)\xi_r/r}{N} , \tag{54}$$

$$\nabla\cdot\underset{\sim}{\xi} = \omega^2\rho\,\frac{(\omega^2\rho-F^2/\mu_o)(r\xi_r)'/r+2kB_\theta(mB_z/r-kB_\theta)\xi_r/(\mu_o r)}{N} ; \tag{55}$$

here, we introduced the abbreviations

$$F = mB_\theta/r + kB_z ,$$

and

$$N = \omega^4\rho^2 - \omega^2\rho(m^2/r^2+k^2)(\gamma p+B^2/\mu_o) + (m^2/r^2+k^2)\gamma pF^2/\mu_o \ .$$

Two further redundant relations, linearly depending on the preceding ones, read

$$i\omega^2\rho(B_\theta\xi_\theta + B_z\xi_z) = F\gamma p\nabla\cdot\underset{\sim}{\xi} \ , \qquad (56a)$$

$$i(m\xi_z/r - k\xi_\theta)$$

$$= \frac{F}{\mu_o}\,\frac{-\omega^2\rho(mB_z/r-kB_\theta)(r\xi_r)'/r-2kB_\theta\{\omega^2\rho-\gamma p(m^2/r^2+k^2)\}\xi_r/r}{N} \ , \qquad (56b)$$

the first of which will be used in Sec. 7.1.

Substitution of Eqs. (54) and (55) in the equation of motion in the r-direction yields

$$\left[\frac{(\omega^2\rho-F^2/\mu_o)\{\omega^2\rho(\gamma p+B^2/\mu_o)-\gamma pF^2/\mu_o\}}{N}\,\frac{1}{r}\,(r\xi_r)'\right]'$$

$$+\left[\omega^2\rho-F^2/\mu_o - \frac{2B_\theta}{\mu_o}\left(\frac{B_\theta}{r}\right)' - \frac{4k^2B_\theta^2}{\mu_o r^2}\,\frac{\omega^2\rho B^2/\mu_o-\gamma pF^2/\mu_o}{N}\right.$$

$$\left.+\ r\left\{\frac{2kB_\theta(mB_z/r-kB_\theta)}{\mu_o r^2}\,\frac{\omega^2\rho(\gamma p+B^2/\mu_o)-\gamma pF^2/\mu_o}{N}\right\}'\right]\xi_r = 0 \ . \qquad (57)$$

The boundary conditions associated with Eq. (57) are

$$r\xi_r = 0 \text{ at } r = 0 \text{ and at } r = \hat{R} \ . \qquad (58)$$

The equations (55) and (56) were previously derived by Ware[39]) and Eq. (57) by Hain and Lüst[40]). The equation of motion for the incompressible pinch, which was derived by Freidberg[41]), can be found heuristically from Eq. (57) by taking the limit $\gamma \rightarrow \infty$.

Again the equation of motion (57) can be represented by $(f\xi_r')' - h\xi_r = 0$. Just as for the compressible plasma layer (Sec. 3.2) the last term of Eq. (57) prevents a simple proof of the monotonic character in $-\omega^2$ of the functions f and h. The difficulty is removed in the same way as before by using the auxiliary theorem stating that the relation $k=k(\omega^2,..)$ or $m=m(\omega^2,..)$, following from the eigenvalue problem of the equations (57) and (58), can have neither a maximum nor a minimum. From this theorem it follows again that the zero points of ξ_r must move away from each other when $-\omega^2$ increases, at least if $\omega^2 < 0$. Also the splitting of the interval $(0,\hat{R})$ in independent subintervals can be proved along the lines of Sec. 3.1. Therefore, the applicability of the theorems 1 and 2 of the plane plasma layer can be extended to the diffuse pinch, if some necessary modifications are made (see the theorems 3 and 4 below).

The marginal equation of motion, following from Eq. (57), reads

$$(f_o\xi_{ro}')' - h_o\xi_{ro} = 0 \,, \tag{59}$$

where f_o and h_o can be written, with the aid of Eq. (52), in conformity with Newcomb's expressions[28]):

$$f_o = \frac{r^3F^2/\mu_o}{m^2+k^2r^2} \,, \tag{59a}$$

$$h_o = \frac{2k^2r^2}{m^2+k^2r^2}\,p' + rF^2/\mu_o - \frac{rF^2/\mu_o}{m^2+k^2r^2} - \frac{2k^2r^3(m^2B_\theta^2/r^2-k^2B_z^2)/\mu_o}{(m^2+k^2r^2)^2} \,. \tag{59b}$$

In the vicinity of a singular point r_s, $F \approx \lambda s$ in general, where $s = r - r_s$ and $\lambda = -\left[kB_z\mu'/\mu\right]_{r_s}$ with $\mu = B_\theta/(rB_z)$.

Just as in Sec. 3.1 the power of the "small" solution $\xi_s \sim s^{n_1-1}$ follows from the indicial equation belonging to Eq. (59), viz.

$$n_1 = 1/2 + 1/2\sqrt{1 + \frac{8\mu_o k^2 p'}{\lambda^2 r}} \ . \qquad (59c)$$

The well-known Suydam criterion[33]) requires that the expression underneath the square root sign should be positive:

$$p' + \frac{rB_z^2}{8\mu_o}\left(\frac{\mu'}{\mu}\right)^2 > 0 \ . \qquad (60)$$

The following theorems 3 and 4 are analogous to the preceding theorems 1 and 2 and almost identical to the theorems 9 and 12 of Newcomb.

Theorem 3. For specified values of m and k a diffuse linear pinch is stable if and only if there exist no solutions to the marginal equation of motion (59) having more than one zero in the interval $(0,\hat{R})$ or in the independent subintervals of it, if singular points occur.

Theorem 4. For specified values of m and k a diffuse linear pinch is stable in an independent subinterval (r_{s1},r_{s2}) if and only if:

1) Suydam's criterion (60) is fulfilled at the endpoints r_{s1} and r_{s2} if these are singular.

2) If ξ_{r1} and ξ_{r2} are the solutions of the marginal equation of motion (59) satisfying ξ_{r1} "small" at r_{s1}, ξ_{r2} "small" at r_{s2}, while $\xi_{r1} = \xi_{r2}$ at some interior point r_o of the interval, then ξ_{r1} should not vanish in the interval (r_{s1},r_o) and ξ_{r2} not in the interval (r_o,r_{s2}).

3) $\xi'_{r1}/\xi_{r1} > \xi'_{r2}/\xi_{r2}$ at $r = r_o$.

The marginal stability analysis, which was applied by Rosenbluth[37]) on the sharp pinch in a heuristic manner, has now acquired an exact basis. It is clear that it will lead to the same results as an application of the energy principle does.

Therefore, it cannot be subject to the criticism which was given some time ago by Tayler[29]) on a certain use of the principle of exchange of stabilities in the work of Dungey and Loughhead[42,43]). The latter principle (see Ref. 44 and Sec. 7.3) is a special method of applying the marginal-stability analysis, which leads to the same results. In Chapter 7, where constant-pitch magnetic fields are discussed, we shall have the opportunity to deal with Tayler's criticism. In the case of a constant-pitch magnetic field Eq. (57) develops a singularity when simultaneously $k = -\mu m$ (or $F \equiv 0$) and $\omega^2 = 0$, so that $N \equiv 0$ on the whole interval $(0,\hat{R})$. This is a degenerate case in the marginal-stability analysis, deserving special treatment.

It appears from Eq. (55) that $\nabla.\underset{\sim}{\xi} = 0$ if $\omega^2 = 0$. As a result the marginal equation of motion (59) is valid for compressible as well as for incompressible perturbations. Therefore, the stability criteria do not depend on the compressibility (for the present we make an exception for the degenerate case of a constant-pitch field, which will be treated in Chapter 7). For the compressible plane plasma layer under the influence of gravity, $\nabla.\underset{\sim}{\xi}$ is different from zero for the marginal modes (see Eq. (23)). If, in this case, we neglect gravity it follows that $\nabla.\underset{\sim}{\xi} = 0$ for $\omega^2 = 0$ and again the stability criteria are independent of the compressibility (Eq. (18) and (26) are identical if $g = 0$). In the above-mentioned criticism Tayler also objected to the fact that Loughhead[43]) claims to demonstrate that for a pinch with distributed current[42]) and for a plane plasma layer in the absence of gravity[43]) stability does not depend on the compressibility of the fluid. It is clear by now that this criticism is unjustified†). The stabil-

†) On the other hand, it is rather remarkable that Tayler did not notice the fact that the main conclusion of Loughhead's article[43]) is incorrect, viz. that a plane plasma layer in the absence of gravity can be unstable. This can be seen from the expression of the energy corresponding to Eq. (18) or (26) with $g = 0$:

ity criteria are influenced by compressibility only if gravity is introduced.

Some further remarks are to be made in connection with the compressibility of the diffuse pinch. The property that $\nabla.\underset{\sim}{\xi}$ vanishes for $\omega^2 = 0$ (marginal stability) corresponds to the fact that δW is minimized by $\nabla.\underset{\sim}{\xi} = 0$ in the application of the energy principle, without normalization of the functions. Owing to this, some authors are tempted to believe that incompressible modes are the most dangerous perturbations. However, from Eq. (55) it is evident that $\nabla.\underset{\sim}{\xi} \neq 0$ if $\omega^2 \neq 0$ (which is in general the case for the most dangerous modes, viz. those having the largest value of $-\omega^2$). It is true that minimization of the energy yields $\nabla.\underset{\sim}{\xi} = 0$, but in case the functions are not normalized one can only judge about the sign of δW and not about the "danger" of the perturbations. Another misconception is that an unstable incompressible pinch can be made stable by the introduction of compressibility, because $\nabla.\underset{\sim}{\xi}$ appears quadratically in δW. The transition from instability to stability takes place, however, via $\omega^2 = 0$ and then $\nabla.\underset{\sim}{\xi} = 0$. Therefore, compressibility can only alter the growth rates of the insta-

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$$\delta W \sim \int \left[\{(k_x B_x + k_z B_z)^2/k^2\}\xi_y'^2 + (k_x B_x + k_z B_z)^2 \xi_y^2\right] dy,$$

which can never be negative. Loughhead concludes to instability because he finds oscillating solutions to the marginal equation of motion. However, in Sec. 4.2 we mentioned that a plane plasma layer with a force-free field of constant α and in the absence of gravity cannot be unstable, because the singular points alternate more rapidly than the zero points of ξ_{yo} do. According to the above expression this conclusion turns out to be generally valid. Loughhead does not take into account the stabilizing influence of the singular points, however, and consequently his incorrect condition for instability (Eq. (54) of his paper) is just the condition which ensures the existence of singular points.

bilities, but not their unstable character. A more detailed and general discussion on compressibility, from the point of view of the energy principle, is given in Refs. 45 and 46.

C H A P T E R 6

STABILIZATION OF PINCH INSTABILITIES BY FORCE-FREE MAGNETIC FIELDS

The theory of Chapter 5 will be applied to a sharp pinch with a dense inner plasma, surrounded by a pressureless plasma associated with a force-free field. This problem is suggested by the experimental and theoretical work on the Jutphaas screw pinch[22-25]. Especially the results of this section represent an extension of Ref. 24. Some conclusions of this and the next section were published before[47,48].

6.1. Equilibrium

In Fig. 4 the configuration is shown. A dense plasma of radius r_o, embedded in a longitudinal magnetic field $\underset{\sim}{B}^p$, is surrounded by a tenuous plasma with a helical magnetic field $\underset{\sim}{B}^t$, occupying an annular region $r_o < r < r_1$. The wall at $r = r_1$ is perfectly conducting. The inner dense plasma has a constant density ρ^p, a constant pressure p, and the magnetic field $\underset{\sim}{B}^p$ has a constant magnitude and direction. In the pressureless plasma the magnitude as well as the direction of the magnetic field $\underset{\sim}{B}^t$ vary, but B_r vanishes throughout. As customary in the theory of the sharp pinch[21], infinitely large current densities are allowed at the position of the plasma surface $r = r_o$ (surface currents), which produce a jump in the magnitude and the direction of the magnetic field. The allowance of currents in the outer region of the pinch represents an extension of the theory of Kruskal and Tuck. This extension is justified and necessary on account of the experimental

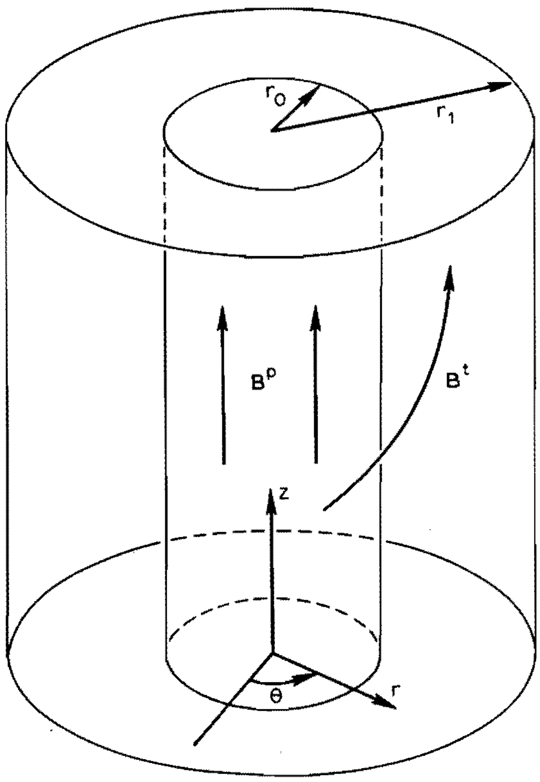

Fig. 4 Sharp linear pinch confined by a force-free magnetic field.

data[25]. The following quantities vanish at equilibrium: the electric field, the current density in the inner plasma, the charge density, and the velocity of the plasma.

The jump in the magnitude of $\underset{\sim}{B}$ is given by Eq. (4):

$$p + \frac{B^{p\,2}}{2\mu_o} = \frac{B_o^{t\,2}}{2\mu_o} , \tag{61}$$

where the index o in B_o^t marks the value at the place $r = r_o$. The quantity

$$\beta = \frac{2\mu_o p}{2\mu_o p + B^{p^2}} = \frac{2\mu_o p}{B_o^{t^2}} = 1 - \frac{B^{p^2}}{B_o^{t^2}} , \tag{62}$$

which characterizes the confining quality of the pinch, constitutes an important parameter.

The magnetic field in the outer region of the pinch has to be force-free because the density is so low that the pressure and, therefore, the pressure gradient can be neglected. For the following stability analysis, however, only the neglect of the pressure gradient is essential. In fact, in the marginal equation of motion (59) the pressure itself does not appear, so that the stability criterion does not depend on the magnitude of the pressure. The addition of a constant pressure in the inner and outer region of the pinch therefore does not change the following stability criteria, which will be derived while assuming $p^t = 0$. On the other hand, no force is associated with a finite density (like in the plane case with gravity) and, consequently, it will not be necessary to neglect the density in the pressureless plasma.

Force-free fields having translational and rotational symmetry ($\partial/\partial z = \partial/\partial\theta = 0$, $\partial/\partial r \neq 0$) can be fixed in three possible ways, viz. by prescribing the profile of one of the functions: 1) $\alpha = \alpha(r)$, 2) $\mu = \mu(r)$, or 3) $B^2 = q(r)$. Each of these three quantities has a physical significance: α is the ratio between the current and the magnetic field (apart from a factor μ_o), μ the reciprocal pitch of the field lines (apart from a factor 2π), and q the magnetic-energy density (apart from a factor $2\mu_o$). The field components B_θ and B_z can be derived from these quantities with the aid of the relations defining α, μ, and q, viz. $\nabla \times \underset{\sim}{B} = \alpha \underset{\sim}{B}$, $\mu = B_\theta/(rB_z)$, and $q = B_\theta^2 + B_z^2$. These relations involve the following ones, which are convenient for an explicit derivation of the magnetic field from one of the three profiles $\alpha(r)$, $\mu(r)$, and $q(r)$:

1) $B_z'' + \left(\frac{1}{r} - \frac{\alpha'}{\alpha}\right)B_z' + \alpha^2 B_z = 0$ and $B_\theta = -\frac{B_z'}{\alpha}$, (63)

2) $B_z' + \frac{\mu(\mu r^2)'}{1+\mu^2 r^2} B_z = 0$ and $B_\theta = \mu r B_z$, (64)

3) $B_z = \sqrt{q + \frac{1}{2}rq'}$ and $B_\theta = \sqrt{-\frac{1}{2}rq'}$. (65)

With the description in terms of q, which was introduced by Schlüter[3]), one must impose the conditions $q' \leq 0$ and $(r^2q)' \geq 0$. The following relations exist between α, μ, and q:

$$\alpha = \frac{2\mu+\mu' r}{1+\mu^2 r^2} , \quad \alpha = -\frac{3q'+rq''}{2\sqrt{-rq'(2q+rq')}} , \quad \mu = \frac{1}{r}\sqrt{-\frac{rq'}{2q+rq'}} . \qquad (66)$$

Here, the second equation is the correct relation between α and q, replacing the incorrect version reported in Ref. 3.

The three ways of description are equivalent. In addition to Eqs. (63) and (64), proper boundary conditions should be posed which, however, can be chosen as wanted. The description in terms of q seems to be the most attractive one because it does not require to solve a differential equation. The description in terms of μ was used in Ref. 24, in which the constant-pitch force-free field was treated. In that case, the choice of μ fixes the function $\alpha(r)$ as well as the value $\mu(r_o)$. In the present work we shall prescribe the magnitude and the direction of $\underset{\sim}{B}^t$ at $r = r_o$ (and therefore $\mu(r_o)$ and $q(r_o)$), just like in the plane case; we then still need a free parameter in order to enable an optimal choice for the force-free field. The simplest class of force-free magnetic fields fitting these requirements is that with α = constant. Given values of μ and q at $r = r_o$ then yield the necessary boundary conditions for Eq. (63), while α can be chosen freely. In this way, a class of fields is obtained having the same value of $\mu(r_o)$ and $q(r_o)$, but different values of α. Furthermore, the equations for the perturbed quantities prove to have analytical solutions if α = constant (Sec. 6.2).

If α = constant the solutions of Eq. (63) are

$$B_z = A_1 J_o(|\alpha| r) + A_2 N_o(|\alpha| r) \;,$$
$$B_\theta = (\alpha/|\alpha|)\left[A_1 J_1(|\alpha| r) + A_2 N_1(|\alpha| r)\right] \;, \qquad (67)$$

J and N being the Bessel and Neumann functions. For $A_2 = 0$ the well-known field of Lundquist[1,11]) is obtained. The stability of this field was investigated by Voslamber and Callebaut[17]). In our case the force-free field is situated in an annular region and, therefore, the Neumann functions need not be excluded, at the same time providing us with the required extra parameter. The pitch of these fields is given by

$$\mu = \frac{1}{r}\,\frac{\alpha}{|\alpha|}\,\frac{J_1(|\alpha| r)+(A_2/A_1)N_1(|\alpha| r)}{J_o(|\alpha| r)+(A_2/A_1)N_o(|\alpha| r)} \;. \qquad (68)$$

For given values of r_o, $\mu(r_o)$, and α the value A_2/A_1 follows from Eq. (68):

$$\frac{A_2}{A_1} = -\,\frac{\mu(r_o) r_o J_o(|\alpha| r_o)-(\alpha/|\alpha|)J_1(|\alpha| r_o)}{\mu(r_o) r_o N_o(|\alpha| r_o)-(\alpha/|\alpha|)N_1(|\alpha| r_o)} \;. \qquad (69)$$

The field and pitch distribution can now be calculated from Eqs. (67), (68), and (69). The magnitude of the magnetic field ($q(r_o)$) need not be given because the stability criteria only depend on $\mu(r_o)$, α, β, r_o, and r_1.

In Fig. 5 the pitch and field profiles are given for a certain choice of $\mu(r_o)$, r_o, and r_1, and for a number of values of the parameter α. The stability criteria, to be derived in Sec. 6.2, will be illustrated in Sec. 6.3 for this choice of the parameters. Here, $\alpha = 0$ represents the vacuum field (no currents in the outer region, but there may be a plasma present), $\alpha = -\,24.6\ m^{-1}$ gives a field which is close to a constant-pitch field (the dashed line in Fig. 5), $\alpha = -\,35\ m^{-1}$ represents the Lundquist field ($A_2/A_1 = 0$), while $\alpha = -\,50\ m^{-1}$ is chosen for reasons made clear later on.

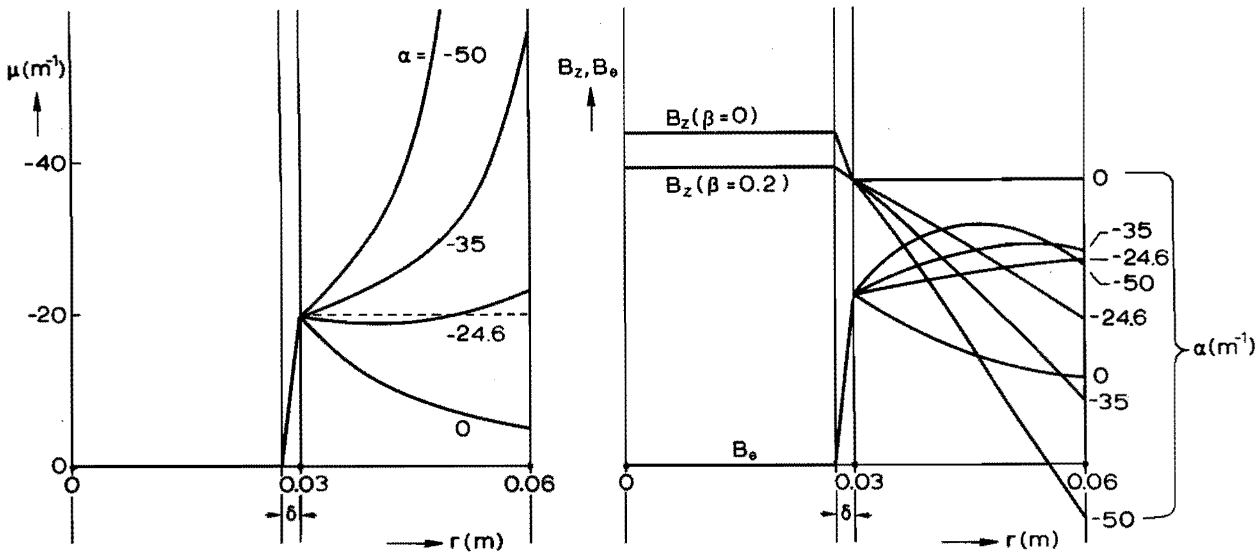

Fig. 5 Field configuration for α = constant, $r_o = 0.03$ m, $r_l = 0.06$ m, $\mu(r_o) = -20\ m^{-1}$.

Another reason exists why from all other possible force-free fields the choice of the fields with α = constant is interesting. Force-free fields of constant α have been studied extensively[1,4-7,13,14,17-19]), and in general authors attribute more physical significance to these fields than to those with non-constant α. Occasionally one gets the impression that the reason for this is the mathematical simplicity of these fields. However, in resistive magnetohydrodynamics, i.e. replacing Ohm's law $\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B} = 0$ by $\underset{\sim}{j} = \sigma(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B})$, a restriction for the possible choices of force-free fields is to be imposed. In the absence of plasma motion ($\underset{\sim}{v} = 0$) Ohm's law in the latter representation, with an isotropic and constant conductivity σ, leads for the resistive fields to an evolution equation for a force-free field $\underset{\sim}{B}(\underset{\sim}{r},\underset{\sim}{t})$:

$$\mu_o\sigma \frac{\partial \underset{\sim}{B}}{\partial t} = -\alpha^2 \underset{\sim}{B} - \nabla\alpha \times \underset{\sim}{B} , \tag{70}$$

and another such equation for $\alpha(\underset{\sim}{r},t)$:

$$\mu_o\sigma \frac{\partial\alpha}{\partial t} = \nabla^2\alpha + \frac{2\nabla\alpha.\nabla|B|}{|B|} . \tag{71}$$

From Eq. (70) it follows that only force-free fields with $\nabla\alpha = 0$ can decay exponentially without change of direction[1,9]. From Eq. (71) it is obvious that then also $\partial\alpha/\partial t = 0$. Conversely, one can prove that $\partial\alpha/\partial t = 0$ implies $\nabla\alpha = 0$. Therefore, in resistive magnetohydrodynamics only quasi-static force-free fields with either α = constant, or α a function of both $\underset{\sim}{r}$ and t remain possible[49,50].

Reference 50 brings a new element into the discussion of possible force-free fields. It is proved there that a field which is force-free at all times has to satisfy, in view of Eq. (70), the relation

$$\underset{\sim}{B} \times (\nabla\alpha.\nabla)\underset{\sim}{B} = 0 . \tag{72}$$

Together with the relations

$$\nabla \times \underset{\sim}{B} = \alpha\underset{\sim}{B} , \tag{73}$$

and

$$\nabla.\underset{\sim}{B} = 0 \quad \text{or} \quad \underset{\sim}{B}.\nabla\alpha = 0 , \tag{74}$$

Eq. (72) puts a strong restriction on the possible force-free fields with $\alpha \neq$ constant. For example, in cylindrical symmetry the Eqs. (72) and (73) lead to a field of the form

$$B_z = C\, r^{-A^2/(1+A^2)} , \quad B_\theta = A\, C r^{-A^2/(1+A^2)} , \tag{75}$$

where A and C may be functions of t. In Ref. 50 this field is rejected because it blows up at $r \to 0$. This argument does not hold for an annular region, but all the same this field can be excluded because the function α following from Eq. (75) does not satisfy Eq. (71) for any value of σ. Finally, the discussion about the possibility of force-free fields with α not constant is nicely brought to an end by Jette[51]), who quite generally proves that the Eqs. (72), (73), and (74) are incompatible in this case. Thus, "in resistive magnetohydrodynamics the only force-free magnetic fields $\underset{\sim}{B}$ which remain force-free in time are those for which α is constant in space and time".

Jette's result implies that a force-free field with α varying in space and, therefore, also a constant-pitch force-free field, cannot exist. However, it is doubtful whether this result is relevant for the present discussion. The creation of a force-free field with $\alpha \neq$ constant in a pinch configuration cannot be excluded because Jette's theory only applies to plasmas at rest ($\underset{\sim}{v} = 0$). It is true that after the formation of the pinch only a force-free field with α = constant can exist in the presence of resistivity, but inevitably this field decays. As a result, the whole pinch configuration will desintegrate, so that the problem of the evolution of the dense inner region of the pinch is just as important as that of the evolution of the force-free outer region of the pinch. Therefore, as yet we do not think that more physical reality should be attached to force-free fields of constant α than to force-free fields with α varying in space (for example, the constant-pitch force-free field of Ref. 24), at least not for the case of the outer region of the pinch. However, we shall meet other reasons to prefer force-free fields of constant α rather than force-free fields of constant μ. In this paper we shall neglect resistivity altogether and our point of view will be that the class of force-free fields of constant α just provides a class of possible field distributions. The most stable of these distributions should be aimed at in order to obtain a pinch which is optimally stable during time intervals for which resistivity can be neglected.

6.2. Stability criteria

The kink instability of a sharp pinch was discussed first by Kruskal and Schwarzschild[20]. Later this analysis was extended to a pinch with an internal magnetic field and surrounded by conducting walls[21,52,53,54]. In these investigations a vacuum field was assumed between the plasma column and the wall. Schuurman, Bobeldijk, and De Vries[24] found a pronounced stabilizing effect on the kink instability by replacing this vacuum field by a constant-pitch force-free field. This effect will turn out to be conserved and even enhanced if force-free fields of constant α are considered.

For a stability analysis on the basis of theorem 4 the zeros of the solution ξ_{ro} (or Q_{ro}) of the marginal equation of motion, and the singular points ($F = mB_\theta/r + kB_z = 0$) should be determined for the following intervals: $(0,r_{oi})$ for the dense plasma, (r_{oi},r_{oe}) for the surface layer, and (r_{oe},r_1) for the tenuous plasma. Again, the boundary surface between the plasma and the pressureless plasma will be considered as a limiting case of a diffuse layer of thickness δ. For a sharp pinch Rosenbluth[37] showed that this procedure leads to stability criteria which cannot be satisfied for a pinch surrounded by a vacuum magnetic field. This result will be changed in a favourable sense if the vacuum field is replaced by a force-free field of constant α.

From the preceding remarks and from the experience with the plane case one could get the impression that force-free fields always stabilize. This is not the case, however. An important difference with the plane layer is that in a cylinder the pressureless plasma itself can be unstable. As an example we mention the instabilities of a Lundquist field if $\alpha\hat{R} > 3.176$, which were described by Voslamber and Callebaut[17]. Another example concerns the instabilities of a constant-pitch force-free field, which will be discussed in Sec. 7.4.

The solutions of the marginal equation of motion will be determined in succession for the three above-mentioned intervals.

No singular points (apart from k = 0) exist in the plasma interval $(0,r_{oi})$. Defining $v_s^2 = \gamma p/\rho^p$ and $v_A^2 = B^{p^2}/\mu_o\rho^p$, the equation of motion (57) can be solved even for the non-marginal case. It leads to the Alfvén wave: $\omega^2 = k^2v_A^2$, which is not important here, and to the differential equation

$$\left[\frac{r(r\xi_r^p)'}{m^2 + \dfrac{(k^2-\omega^2/v_A^2)(k^2-\omega^2/v_s^2)}{k^2-\omega^2/v_A^2-\omega^2/v_s^2}\,r^2}\right]' - \xi_r = 0 \; . \qquad (76)$$

The solution that satisfies the boundary condition at r = 0 was given by Kruskal and Tuck[21]):

$$\xi_r^p = C\, I_m' \left(\sqrt{\frac{(k^2-\omega^2/v_A^2)(k^2-\omega^2/v_s^2)}{k^2-\omega^2/v_A^2-\omega^2/v_s^2}}\; r\right) , \qquad (77)$$

where I_m is the modified Bessel function of the first kind†). This expression will be used in Sec. 6.4. For the stability analysis the solution of the marginal equation of motion is needed, viz.

$$\xi_r^p = C\, I_m'(kr) \; ; \qquad (78)$$

it has no zero points on the open interval $(0,r_{oi})$.

In the pressureless plasma interval (r_{oe},r_1) it proves to be more convenient to use $\underset{\sim}{Q}$. Moreover, we shall make the derivation somewhat more general than is needed for the present problem by starting from Eq. (6) with $\rho^t\omega^2 = 0$, $p = 0$, and $\underset{\sim}{g} = 0$. Using $\nabla \times \underset{\sim}{B} = \alpha\underset{\sim}{B}$ this equation becomes

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†) Notice that accents denote differentiation of the functions with respect to their complete arguments. Only for Bessel functions the argument, as indicated in the brackets, is different from r, so that the accent denotes something else than d/dr.

$$\nabla \times \underset{\sim}{Q} - \alpha \underset{\sim}{Q} = \lambda \underset{\sim}{B} , \tag{79}$$

where λ is a scalar function of $\underset{\sim}{r}$, which is of first order in $\underset{\sim}{\xi}$[18]). Taking the curl of Eq. (79), also using the equation itself, we obtain the wanted second-order differential equation for $\underset{\sim}{Q}$:

$$\nabla^2 \underset{\sim}{Q} + \alpha^2 \underset{\sim}{Q} + \nabla\alpha \times \underset{\sim}{Q} + 2\alpha\lambda \underset{\sim}{B} + \nabla\lambda \times \underset{\sim}{B} = 0 . \tag{80}$$

Taking the divergence of Eq. (79), using $\nabla . \underset{\sim}{B} = 0$ and $\nabla . \underset{\sim}{Q} = 0$, we get

$$- \nabla\alpha . \underset{\sim}{Q} = \nabla\lambda . \underset{\sim}{B} . \tag{81}$$

We return to cylinder symmetry, but still for a general r-dependence of α. Taking into account the θ- and z-dependence of λ, which is necessarily the same as for $\underset{\sim}{Q}$, Eq. (81) yields

$$\lambda = \frac{i\alpha' Q_r}{mB_\theta/r + kB_z} = i \frac{\alpha'}{F} Q_r . \tag{82}$$

The required marginal equation of motion is the r-component of Eq. (80), which must be written in terms of Q_r alone. The components Q_θ and Q_z can be expressed in terms of Q_r by using the r-component of Eq. (79) as well as $\nabla . \underset{\sim}{Q} = 0$:

$$Q_\theta = i \frac{k\alpha r^2 Q_r + m(rQ_r)'}{m^2+k^2r^2} ; \quad Q_z = i \frac{-m\alpha r Q_r + kr(rQ_r)'}{m^2+k^2r^2} . \tag{83}$$

The r-component of Eq. (80), with λ from Eq. (82) and Q_θ from Eq. (83), next yields

$$(rQ_r)'' + \frac{1}{r}\frac{m^2-k^2r^2}{m^2+k^2r^2}(rQ_r)' + \left(\alpha^2 - k^2 - \frac{m^2}{r^2} + \frac{2km\alpha}{m^2+k^2r^2} - \frac{mB_z - krB_\theta}{mB_\theta + krB_z}\alpha'\right) rQ_r = 0. \tag{84}$$

The stability of force-free fields with $\alpha \neq$ constant can be studied by means of this equation. For example, the basic equation (16) of Ref. 24 follows by introducing the value of α for μ = constant, that is, $\alpha = 2\mu/(1+\mu^2 r^2)$. The coefficients of Eq. (84) also can be written in terms of μ, μ', and μ'' instead of B_θ and B_z by using the Eqs. (64) and (66), or in terms of q, q', q'', and q''' by using the Eqs. (65) and (66). In the above form the components B_θ and B_z must be determined from Eq.(63). For non-constant α the equation (84) has a singularity when $mB_\theta + krB_z = 0$, which can be treated in the same way as done in Sec. 3.1. Only in the case of μ = constant this singularity gives rise to serious problems (see Sec. 7.4).

We now return to force-free fields with constant α. From Eq. (82) it then follows that $\lambda = 0$ while, according to Eq. (79), the perturbation of the magnetic field $\underset{\sim}{Q}$ of the marginal modes satisfies the equation for force-free fields. Of course this does not mean that the perturbed state is also force-free for the non-marginal modes, because this would exclude any perturbed motion. The marginal equation (84) is considerably simplified by the choice α = constant, because then the term with α' drops and the solution can be written in terms of Bessel functions. This solution can be found indirectly from the z-component of Eq. (80), viz.

$$Q_z'' + \frac{1}{r} Q_z' + \left(\alpha^2 - k^2 - \frac{m^2}{r^2}\right) Q_z = 0 \,. \tag{85}$$

This equation has the solution

$$Q_z = d_1 J_m(\sqrt{\alpha^2-k^2}\,r) + d_2 N_m(\sqrt{\alpha^2-k^2}\,r) \,, \tag{86}$$

from which the proper expressions for $\alpha^2 < k^2$ are found by the substitutions

$$\sqrt{\alpha^2-k^2} \rightarrow \sqrt{k^2-\alpha^2} \,, \quad J_m \rightarrow I_m, \quad N_m \rightarrow K_m \,, \tag{87}$$

where I_m and K_m are the modified Bessel functions of the first and second kind. The components Q_r and Q_θ follow from the r-and θ-components of Eq. (79), taking $\lambda = 0$:

$$Q_r = \frac{i}{\alpha^2-k^2}\left(\frac{m\alpha}{r} Q_z + kQ_z'\right), \quad Q_\theta = \frac{-1}{\alpha^2-k^2}\left(\frac{km}{r} Q_z + \alpha Q_z'\right). \qquad (88)$$

Therefore, the solution Q_r of the marginal equation for constant α reads

$$rQ_r^t = D_1\left[m\alpha J_m(\sqrt{\alpha^2-k^2}\,r) + kr\sqrt{\alpha^2-k^2}\, J_m'(\sqrt{\alpha^2-k^2}\,r)\right]$$

$$+ D_2\left[m\alpha N_m(\sqrt{\alpha^2-k^2}\,r) + kr\sqrt{\alpha^2-k^2}\, N_m'(\sqrt{\alpha^2-k^2}\,r)\right]. \qquad (89)$$

One of the constants D_1 and D_2 is fixed by the condition that the solution Q_r^t, needed for the application of theorem 4, should be "small" either at a singular point or at the wall. In a force-free region the "small" solution must have the form $Q_r^t \sim s^{n_1}$, where $n_1 = 1$, in view of $p' = 0$ in Eq. (59c), so that the condition "small" in this case really means small: $\xi_s \sim 1$ (ξ_s does not tend to infinity). The other constant is not important (it is fixed by normalization). The corresponding solution for ξ_r^t is found from Eq. (53):

$$\xi_r^t = Q_r^t / iF^t ,$$

where Q_r^t is given by Eq. (89), while Eqs. (67) and (69) yield

$$F^t = mB_\theta^t/r + kB_z^t$$

$$= A_1\left[\frac{\alpha}{|\alpha|}\frac{m}{r} J_1(|\alpha|r) + kJ_o(|\alpha|r)\right] + A_2\left[\frac{\alpha}{|\alpha|}\frac{m}{r} N_1(|\alpha|r) + kN_o(|\alpha|r)\right]$$

$$= A\left[\left\{\mu(r_o)r_oN_o(|\alpha|r_o) - \frac{\alpha}{|\alpha|}N_1(|\alpha|r_o)\right\}\left\{\frac{\alpha}{|\alpha|}\frac{m}{r}J_1(|\alpha|r) + kJ_o(|\alpha|r)\right\}\right.$$

$$\left. - \left\{\mu(r_o)r_oJ_o(|\alpha|r_o) - \frac{\alpha}{|\alpha|} J_1(|\alpha|r_o)\right\}\left\{\frac{\alpha}{|\alpha|}\frac{m}{r} N_1(|\alpha|r) + kN_o(|\alpha|r)\right\}\right].$$

(90)

The value of the constant A is irrelevant because the absolute magnitude of the magnetic field is not important for the stability criteria.

The solution ξ_r^1 in the surface layer (r_{oi},r_{oe}) is found in the same way as in Sec. 4.2. The logarithmic derivatives of ρ, p, and B^2 are all of the order $(\delta/r_o)^{-1}$. Therefore, the marginal equation of motion (59) becomes

$$(f_o\xi_r^{1'})' - h_o\xi_r^1 = 0 \; , \tag{91}$$

where

$$f_o = \frac{r^3(mB_\theta/r+kB_z)^2/\mu_o}{m^2+k^2r^2} \quad \text{and} \quad h_o \approx \frac{2k^2r^2}{m^2+k^2r^2}\,p' \; . \tag{91a}$$

Analogous to Newcomb's[28]) derivation one finds by integration of Eq. (91) through the surface layer that ξ_r^1 is approximately constant in the layer, with the exception of a small ε-neighbourhood of a possible singular point r_s, where the solution may be "small", that is, proportional to $(r-r_s)^{n_1-1}$. The magnitude of ε is such that $(\delta/r_o)^2 \ll \varepsilon/r_o \ll \delta/r_o$, and according to Eq. (59c) the exponent n_1-1 is given by

$$n_1-1 = -1/2 + 1/2\sqrt{1+\frac{8\mu_o\mu^2p'}{r_oB_z^2(\mu')^2}} \approx \frac{2\mu_o\mu^2p'}{r_oB_z^2(\mu')^2} = -0\left(\frac{\delta}{r_o}\right) \; ,$$

in view of the usually negative value of p' in the surface layer. The solution ξ_r^1 has no zero points in the surface layer if there is no singular point in the latter, because then ξ_r^1 is approximately constant. If, on the other hand, a singular point exists the corresponding "small" solutions neither have zero points in (r_{oi},r_s) nor in (r_s,r_{oe}), because these solutions consist of a constant part and a part which tends to infinity in the ε-neighbourhood of r_s.

The solutions of the marginal equation of motion now being known everywhere, the application of theorem 4 becomes straightforward. The first condition of theorem 4, Suydam's criterion, is satisfied trivially both in the plasma and in the pressureless plasma in view of the vanishing pressure gradient. Suydam's criterion is also satisfied in the surface layer because the destabilizing pressure gradient (order proportional to δ^{-1}) is dominated by the stabilizing shear term (order proportional to δ^{-2}) in the limit $\delta \to 0$.(We exclude cases in which the shear in the surface layer changes sign). The second condition of theorem 4 is satisfied for the plasma and for the surface layer. For the pressureless plasma it leads to conditions which refer to the stability of the force-free field itself. Here, an essential difference with the plane case turns up. In cylinder geometry, in contrast to the plane case, it is possible that for certain values of the parameters the zero points alternate more rapidly than the singular points. The third item of theorem 4, finally, leads to conditions analogous to (40), (41), and (42) of the plane case.

For the application of the items 2) and 3) of theorem 4 the independent subintervals and, therefore, the singular points must be determined first. In the pressureless plasma the singular points are determined from the equation $F^t(r) = 0$, where F^t is given by Eq. (90). The first singular point in the interval (r_{oe}, r_1) will be called r_1^*, while $r_1^* = r_1$ if no singular points exist. The solution of the marginal equation of motion on (r_{oe}, r_1^*) is given by Eq. (89), where rQ_r^t has to be "small" (in this case zero) at $r = r_1^*$ and therefore:

$$\frac{D_2}{D_1} = - \frac{m\alpha J_m(\sqrt{\alpha^2-k^2}\,r_1^*) + kr_1^*\sqrt{\alpha^2-k^2}\; J_m'(\sqrt{\alpha^2-k^2}\,r_1^*)}{m\alpha N_m(\sqrt{\alpha^2-k^2}\,r_1^*) + kr_1^*\sqrt{\alpha^2-k^2}\; N_m'(\sqrt{\alpha^2-k^2}\,r_1^*)} \; . \qquad (92)$$

The solution rQ_r^t can have zero points on the open interval (r_{oe}, r_1^*). Similarly, the solutions on the possibly existing interval (r_1^*, r_1), or on the independent subintervals of it, can show more than one zero point. According to item 2) of theorem 4 or according to theorem 3 this implies instability on account of the pressureless plasma itself. From Eq. (89) it is clear that these instabilities should be expected for $|k| < |\alpha|$ if αr

is large. This is in agreement with the results of Voslamber and Callebaut[17]) for the Lundquist field: the m = 1 mode is unstable for $|k/\alpha| \leq 0.272$ if $\alpha R > 3.176$. In our problem we have to do with an annular region and, consequently, an extra parameter in the problem. Therefore, we cannot give similar general results for the stability of the force-free field and a further discussion is postponed to Sec. 6.3. We shall find that instabilities of the force-free field may be absent for practical values of $\mu(r_o)$, α, r_o, and r_1.

The further application of theorem 4, viz. item 3) for the interval $(0,r_1^*)$, proceeds completely analogous to that for the plane case. The following possibilities arise:

(1) No singular point exists in the surface layer. The interval $(0,r_1^*)$ then constitutes an independent subinterval which we split at the point $r = r_{oi}$. The value of $(\xi_r^{1'}/\xi_r^1)_{r_{oi}}$ is found by integration of Eq. (91):

$$(f_o\xi_r^{1'})_{r_{oi}} \approx \xi_r^1 \int_{r_{oe}}^{r_{oi}} h_o dr + (f_o\xi_r^{1'})_{r_{oe}} \; .$$

Assuming that ξ_r and ξ_r' are continuous at r_{oe} for the relevant "small" solutions, substituting h_o from Eq. (91a), and taking r_o at r_{oi}, the third item of theorem 4 becomes

$$\left(\frac{\xi_r^{p'}}{\xi_r^p}\right)_{r_{oi}} > \left(\frac{\xi_r^{1'}}{\xi_r^1}\right)_{r_{oi}} \approx \frac{1}{f_o(r_{oi})} \left[\frac{2k^2r^2}{m^2+k^2r^2}\, p\right]_{r_{oe}}^{r_{oi}} + \frac{f_o(r_{oe})}{f_o(r_{oi})} \left(\frac{\xi_r^{t'}}{\xi_r^t}\right)_{r_{oe}} . \tag{93}$$

By transforming to rQ_r by means of Eqs. (53) and (73), substituting $f_o(r_{oe})$ and $f_o(r_{oi})$ from Eq. (91a), eliminating $p(r_{oi})-p(r_{oe})$ with the aid of Eq. (61), and the magnetic fields by applying Eqs. (62) and (64), and taking the limit $\delta \to 0$, we finally find

$$L \equiv (1-\beta)\,\frac{k^2r^2}{m^2+k^2r^2}\,\frac{(rQ_r^p)'}{rQ_r^p} >$$

$$> \frac{\mu^2 r}{1+\mu^2r^2} + \frac{\alpha r(k+\mu m)(k\mu r^2-m)}{(m^2+k^2r^2)(1+\mu^2r^2)} + \frac{r^2(k+\mu m)^2}{(m^2+k^2r^2)(1+\mu^2r^2)}\,\frac{(rQ_r^t)'}{rQ_r^t} \equiv R\ . \tag{94}$$

Of course the values of all quantities in this stability condition are to be taken at $r = r_o$. From Eq. (78) we obtain

$$\frac{k^2r^2}{m^2+k^2r^2}\,\frac{(rQ_r^p)'}{rQ_r^p} = k\,\frac{I_m(kr)}{I_m'(kr)} \tag{95}$$

for the expression entering in L.

Until now no use has been made of α = constant and, therefore, the condition (94) also holds for $\alpha \neq$ constant. In that case the solution rQ_r^t of Eq. (84) should be taken. For example, substitution of $\alpha = 2\mu/(1+\mu^2r^2)$ yields the stability criterion for the constant-pitch force-free field (Eq. (48) of Ref. 24).

For α = constant the Eqs. (89) and (92) give

$$\frac{(rQ_r^t)'}{rQ_r^t} = \Bigg[\left\{m\alpha N_m(\sqrt{\alpha^2-k^2}\,r_1^*)+kr_1^*\sqrt{\alpha^2-k^2}\;N_m'(\sqrt{\alpha^2-k^2}\,r_1^*)\right\}$$

$$\times\left\{-kr_o\left[\alpha^2-k^2-\frac{m^2}{r_o^2}\right]J_m(\sqrt{\alpha^2-k^2}\,r_o)+m\alpha\sqrt{\alpha^2-k^2}\;J_m'(\sqrt{\alpha^2-k^2}\,r_o)\right\}$$

$$-\left\{m\alpha J_m(\sqrt{\alpha^2-k^2}\,r_1^*)+kr_1^*\sqrt{\alpha^2-k^2}\;J_m'(\sqrt{\alpha^2-k^2}\,r_1^*)\right\}$$

$$\times\left\{-kr_o\left[\alpha^2-k^2-\frac{m^2}{r_o^2}\right]N_m(\sqrt{\alpha^2-k^2}\,r_o)+m\alpha\sqrt{\alpha^2-k^2}\;N_m'(\sqrt{\alpha^2-k^2}\,r_o)\right\}\Bigg]$$

$$\Big/\Bigg[\left\{m\alpha N_m(\sqrt{\alpha^2-k^2}\,r_1^*)+kr_1^*\sqrt{\alpha^2-k^2}\;N_m'(\sqrt{\alpha^2-k^2}\,r_1^*)\right\}\times$$

$$\times \left\{m\alpha J_m(\sqrt{\alpha^2-k^2}r_o)+kr_o\sqrt{\alpha^2-k^2}\ J_m'(\sqrt{\alpha^2-k^2}r_o)\right\}$$

$$-\left\{m\alpha J_m(\sqrt{\alpha^2-k^2}r_1^*)+kr_1^*\sqrt{\alpha^2-k^2}\ J_m'(\sqrt{\alpha^2-k^2}r_1^*)\right\}$$

$$\times \left\{m\alpha N_m(\sqrt{\alpha^2-k^2}r_o)+kr_o\sqrt{\alpha^2-k^2}\ N_m'(\sqrt{\alpha^2-k^2}r_o)\right\}\Bigg] , \qquad (96)$$

where r_1^* is either the first zero of $F^t(r)$, given by Eq. (90), or r_1. For $|k| > |\alpha|$ the substitutions (87) should be made.

(2) A single singular point r_s occurs in the surface layer. (We exclude structures of the surface layer having more than one singular point, because these structures violate Suydam's criterion). The interval $(0,r_1^*)$ then consists of two independent subintervals $(0,r_s)$ and (r_s,r_1^*).

(a) The interval $(0,r_s)$ can be split at r_{oi}. Integration of Eq. (91) yields

$$(f_o\xi_r^{1\prime})_{r_{oi}} \approx \xi_r^1 \int_{r_s-\varepsilon}^{r_{oi}} h_o\, dr + (f_o\xi_r^{1\prime})_{r_s-\varepsilon} \approx \xi_r^1 \int_{r_s}^{r_{oi}} h_o\, dr .$$

Substituting h_o from Eq. (91a) item 3 of the stability criterion of theorem 4 becomes

$$\left(\frac{\xi_r^{p\prime}}{\xi_r^p}\right)_{r_{oi}} > \left(\frac{\xi_r^{1\prime}}{\xi_r^1}\right)_{r_{oi}} \approx \frac{1}{f_o(r_{oi})}\left[\frac{2k^2r^2}{m^2+k^2r^2}\, p\right]_{r_s}^{r_{oi}} . \qquad (93a)$$

The expression analogous to Eq. (94) reads

$$L \equiv (1-\beta)\,\frac{k^2r^2}{m^2+k^2r^2}\,\frac{(rQ_r^p)'}{rQ_r^p} > \frac{k^2r}{m^2+k^2r^2}\,(1-\beta S) \equiv P , \qquad (94a)$$

where a structural factor S for the surface layer has been introduced, which is defined by

$$S = \frac{B_o^{t^2} - B^{s^2}}{B_o^{t^2} - B^{p^2}} ;$$

B^s represents the magnitude of the magnetic field at the singular point.

(b) The interval (r_s, r_1^*) will be split at r_{oe}. The expressions analogous to Eqs. (93a) and (94a) become

$$\left(\frac{\xi_r^{1'}}{\xi_r^1}\right)_{r_{oe}} \approx \frac{1}{f_o(r_{oe})} \left[\frac{2k^2r^2}{m^2+k^2r^2}\, P \right]_{r_s}^{r_{oe}} > \left(\frac{\xi_r^{t'}}{\xi_r^t}\right)_{r_{oe}} , \tag{93b}$$

$$P \equiv \frac{k^2 r}{m^2+k^2r^2}\,(1-\beta S) >$$

$$> \frac{\mu^2 r}{1+\mu^2 r^2} + \frac{\alpha r(k+\mu m)(k\mu r^2-m)}{(m^2+k^2r^2)(1+\mu^2r^2)} + \frac{r^2(k+\mu m)^2}{(m^2+k^2r^2)(1+\mu^2r^2)}\,\frac{(rQ_r^t)'}{rQ_r^t} \equiv R , \tag{94b}$$

where for constant α the last factor in R is given by Eq.(96).

Summarizing: the stability criteria for a sharp pinch, surrounded by a force-free field of constant α are:

- No solutions should exist for the marginal-mode equation for Q_r^t,

$$(rQ_r^t)'' + \frac{1}{r}\,\frac{m^2-k^2r^2}{m^2+k^2r^2}\,(rQ_r^t)' + \left(\alpha^2-k^2 - \frac{m^2}{r^2} + \frac{2km\alpha}{m^2+k^2r^2}\right) rQ_r^t = 0 , \tag{97}$$

having more than one zero in the outer region (r_o, r_1), or in the independent subintervals of it. Here,the solutions of Eq. (97) are given by Eq. (89), while the endpoints of the independent subintervals are the zeros of $F^t(r)$, F^t being given by Eq. (90), and r_1.

- The following inequalities should hold at the plasma boundary (r_o):

in the absence of a singular point in the thin surface layer: $L > R$, but if such a point does exist the criterion is split: $L > P > R$. Here the expressions L,P, and R are given by the Eqs. (94), (94a), (94b), (95), and (96).

For $\alpha \neq$ constant the following modifications should be made: Eq. (97) is to be replaced by Eq. (84), while the end-points of the independent subintervals are to be determined from a modified function F^t, which depends on the field distribution belonging to the chosen function $\alpha = \alpha(r)$. Further, in the expression R the solution Q_r^t of Eq. (97) is replaced by that of Eq. (84).

6.3. Discussion

The application of the above-mentioned stability criteria is considerably simplified by applying a theorem of Newcomb[28]) stating that a pinch is stable for all m and all k if and only if it is stable for $m = 0$, $k \to 0$ and for $m = 1$, all k. We noticed that Suydam's criterion is satisfied for the sharp pinch surrounded by a force-free field of constant α, so that the pinch is stable against interchanges. Thus, according to the mentioned theorem only the study of the sausage ($m = 0$) and of the kink ($m = 1$) instabilities remains necessary.

The study of the sausage instability can be restricted to $k \to 0$. Instabilities of the pressureless plasma itself do not arise because Q_r^t oscillates more slowly than F^t, which follows from an application of Sturm's comparison theorem (see: Ince[30])) to the equations for Q_r^t and F^t for the case $m = 0$, $k \to 0$. In view of Eq. (84) and an application of Eq. (63) to the definition of F these equations here read

$$Q_r^{t''} + \frac{1}{r} Q_r^{t'} + \left(\alpha^2 - \frac{1}{r^2}\right) Q_r^t = 0 \ ,$$

$$F^{t''} + \frac{1}{r} F^{t'} + \alpha^2 F^t = 0 \ .$$

For m = 0 no singular points occur in the surface layer, at least not if the magnetic field in the surface layer is assumed to have a uniform sense of rotation. Therefore, a consideration of the split criterion L > P > R can be omitted. Consequently, m = 0 instabilities can only arise in case the criterion

$$L = \frac{2(1-\beta)}{r_o} > \frac{\mu^2 r_o}{1+\mu^2 r_o^2} + \frac{\alpha\mu r_o}{1+\mu^2 r_o^2}$$

$$+ \frac{|\alpha|}{1+\mu^2 r_o^2} \frac{J_o(|\alpha| r_o) N_1(|\alpha| r_1^*) - N_o(|\alpha| r_o) J_1(|\alpha| r_1^*)}{J_1(|\alpha| r_o) N_1(|\alpha| r_1^*) - N_1(|\alpha| r_o) J_1(|\alpha| r_1^*)} = R \,, \quad (98)$$

is violated; here r_1^* equals r_1 or the first zero of B_z^t as given by the equations (67) and (69). In the special case of Fig. 5, with $\mu(r_o) = -20\ m^{-1}$, $r_o = 0.03$ m, and $r_1 = 0.06$ m the condition (98) provides the following critical values of β at which stability passes into instability for m = 0:

$\alpha(m^{-1})$	β_{crit}
0	100 %
-24.6	98.9%
-35	74.4%
-50	58.8%

The maximal allowable value of β thus decreases with increasing values of -α. The reason for this is clear from Fig. 5: at increasing values of -α the B_z-field in the outer region decreases and hence the stabilization of the sausage instability becomes less effective. It will turn out that β_{crit} for m = 1 is much smaller than β_{crit} for m = 0 (at least for the values of $\mu(r_o)$ which are considered here), so that the sausage instabilities do not constitute a serious threat to the stability of this configuration.

Just as for the plane case it is useful for the study of the kink instabilities to distinguish between type-I and type-II instabilities[38]), depending on whether L > R or L > P > R is violated. Moreover, a third type of instabilities can exist if the solutions of Eq. (97) have more than one zero in an independent subinterval. The latter type of instabilities will be called type-III instabilities. Usually the name kink is used for type-I instabilities. Type-II instabilities are also called surface-layer instabilities, whereas type-III instabilities are of the type which were studied by Voslamber and Callebaut[17]).

This classification of instabilities is based upon the behaviour of the marginal modes, which provides us roughly with the following picture. If $-\omega^2$ is small, i.e. in the vicinity of marginal stability, type-I instabilities cause a displacement of the plasma extending over the whole interval $(0,r_1)$ or $(0,r_1^*)$. In that case, type-II instabilities are localized in one of the intervals $(0,r_s)$ and (r_s,r_1^*) and, particularly, in the thin surface layer, whereas type-III instabilities are mainly localized in the interval (r_{oe},r_1). If $-\omega^2$ is not small there is no question of localization of the instabilities in one of the subintervals, and this classification of instabilities loses its sense. Especially in a diffuse pinch the difference between type-I and type-II instabilities disappears. At the same time this demonstrates that it is not justified to assume that type-II instabilities are unimportant in experimental circumstances.

Type-I instabilities are the most disastrous ones for plasma confinement in a pinch. For a vacuum magnetic field in the outer region (α = 0) the criterion (94), with (95) and (96), yields

$$L = (1-\beta)k\,\frac{I_m(kr_o)}{I_m'(kr_o)} > \frac{\mu^2 r_o}{1+\mu^2 r_o^2} + \frac{(k+\mu m)^2}{|k|(1+\mu^2 r_o^2)}\,\frac{I_m(|k|r_o)K_m'(|k|r_1) - K_m(|k|r_o)I_m'(|k|r_1)}{I_m'(|k|r_o)K_m'(|k|r_1) - K_m'(|k|r_o)I_m'(|k|r_1)} = R(\alpha=0). \qquad (99)$$

This is a well-known expression[53]) in which the stabilizing and destabilizing effects can clearly be distinguished. The left-hand side L is positive definite and shows the stabilizing influence of the plasma magnetic field, which vanishes if $\beta \to 1$ or $k \to 0$. The first term of the right-hand side R represents the driving force of the instability, viz. the curvature of the magnetic field lines at the plasma boundary. The second term of R is negative definite and represents the stabilizing influence of the vacuum magnetic field and the conducting wall; this contribution vanishes if $k \to -\mu m$. Just as for the plane plasma layer (Eq. (44)) there exists no direction of the perturbation for which both stabilizing terms vanish. An essential difference, however, is the fact that even for $\beta \to 0$ (vanishing plasma pressure) the driving force of the instability is present. For a vacuum magnetic field ($\alpha = 0$) this usually implies instability, even for vanishing small values of β. According to Eq. (99) the most dangerous perturbations have directions somewhere between those perpendicular to $\underset{\sim}{B}^p$ and to $\underset{\sim}{B}_o^t$ respectively. It will be shown that these instabilities can be stabilized by force-free fields with constant α.

In Fig. 6 the quantities $L(\beta = 0)$ and R according to Eq. (94) are plotted as functions of k for $m = 1$ and for the choice of the parameters of Fig. 5†). The curve L comes down with increasing β, as is evident from Eqs. (94) and (95). Hence, a critical value of β exists above which type-I instability shows up. For the vacuum field ($\alpha = 0$) the curves L and R cross each other, even for $\beta = 0$, so that a plasma surrounded by a vacuum field is unstable for all values of β. (The reason why part of the curve $R(\alpha = 0)$ has been dashed will become clear below). For increasing values of $-\alpha$ the curve R rotates counterclockwise around the fixed point A for which $k = -\mu$, where the value of R is independent of the choice of α. The best choice for the force-free field is therefore the one for which the mode $k=-\mu$ becomes the most dangerous one. Thus, substituting $k=-\mu$ and $m=1$

†) These numerical results were obtained by means of the computer program for constant μ (Ref. 24), which was modified for constant α.

in Eq. (94) the maximal allowable value of β for type-I instabilities turns out to be

$$\beta_{crit} = 1 - \frac{\mu r_o}{1+\mu^2 r_o^2} \frac{I_1'(\mu r_o)}{I_1(\mu r_o)} . \tag{100}$$

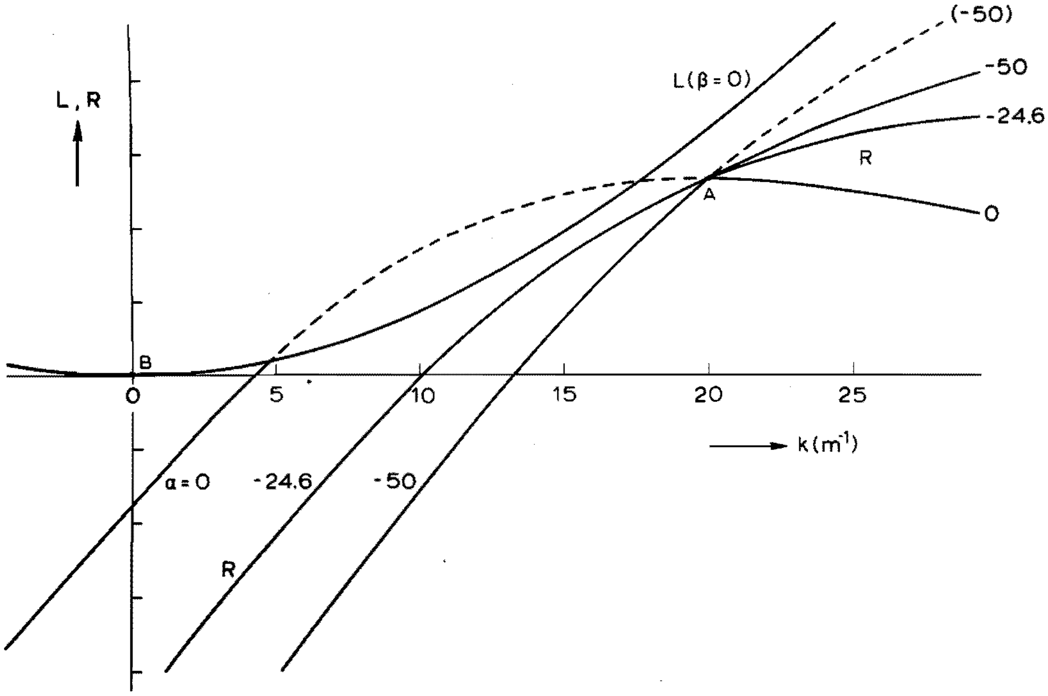

Fig. 6 Stability diagram for α = constant, m = 1, r_o = 0.03 m, r_1 = 0.06 m, $\mu(r_o)$ = -20 m^{-1}.

For $\mu(r_o)$ = - 20 m^{-1} and r_o = 0.03 m the value of β_{crit} is about 20%. For $\beta = \beta_{crit}$ the pinch is marginally stable with respect to type-I instabilities if α is properly chosen.

The proper choice of α is complicated by the presence of singular points in the pressureless plasma. As a result the curves R are deflected (see for example the curve R(α = -50) to the right of k = 20 m^{-1}). In fact, the singular points cause a modification of the boundary condition for the solution rQ_r^t of Eq. (97), viz. $rQ_r^t = 0$ at r_1^*. If singular points occur, so that $r_1^* < r_1$, the wall is effectively displaced inward as far as stability is concerned. If we would neglect this effect by taking $r_1^* = r_1$ in the equations (94) and (96), the curves R would not be deflected and the optimal choice of $\alpha(\alpha_{opt})$ would be the one for which $L(\beta_{crit})$ would touch $R(\alpha_{opt})$ at $k = -\mu$. In that case α_{opt} would follow from

$$\left(\frac{\partial L(\beta_{crit})}{\partial k}\right)_{k=-\mu} = \left(\frac{\partial R(\alpha_{opt})}{\partial k}\right)_{k=-\mu} .$$

Using Eqs. (94) and (95) for m = 1 this equality would give

$$\alpha_{opt} = 2\mu+\mu^2 r_o \frac{I_1'(\mu r_o)}{I_1(\mu r_o)} - \frac{1+\mu^2 r_o^2}{r_o} \frac{I_1(\mu r_o)}{I_1'(\mu r_o)} . \tag{101}$$

For small μr_o Eq. (101) would imply that $\alpha_{opt} \approx 2\mu(r_o)$ and according to Eq. (66) a constant-pitch force-free field would be about the optimal choice for small μr_o. On the other hand, for the choice of the parameters of Fig. 5 ($\mu = -20\ m^{-1}$, $r_o = 0.03$ m) α_{opt} would become $-36.7\ m^{-1}$, i.e. close to the Lundquist field. Finally, the field distribution corresponding to $\alpha = \alpha_{opt}$ would be the only one allowing stability for values of β up to β_{crit}.

This picture is modified considerably in the correct treatment, in which the stabilizing effect of the singular points in the tenuous plasma (where $\mu = -k$ when m = 1) is taken into account. (See, however, Appendix II, where it is shown that this stabilizing effect is destroyed by resistivity). The position of the singular points for a certain value of k can be determined easily from Fig. 5 by drawing a horizontal line in the plot of $\mu = \mu(r)$ at the height of $\mu = -k$. The points of intersection of this line with the curve $\mu(r)$ give the positions of the singular points. For instance, for $\alpha = 0$ a singular point occurs for k-values from 5 to 20 m^{-1}. Consequently, for this region of k-values the curve R for a pressureless plasma without currents is considerably below the dashed curve of Fig.6, which gives the function $R(\alpha = 0)$, neglecting the singular points (that is, taking $r_1^* = r_1$). For $\alpha = 0$ the neglect of the singular points can have physical significance, because in this case one can consider the force-free region as a vacuum instead of a pressureless plasma without currents[26,28]. In the case of a vacuum $\underset{\sim}{\xi}$ has no physical significance (there is no plasma) and the singular points, entering in the problem only due to the differential equation for $\underset{\sim}{\xi}$, also can have no effect. The dashed part of the $\alpha = 0$ curve in Fig. 6, therefore, represents the function R for a vacuum. In the opposite case, if one considers the force-free region with $\alpha = 0$ as a pressureless plasma without currents, the dashed part of $R(\alpha = 0)$ should be replaced by a curve which is just above $L(\beta = 0)$ for k = 5 to 9.3 m^{-1} and below $L(\beta = 0)$ for k = 9.3 to 20 m^{-1}. The new curve $R(\alpha = 0)$ then has discontinuous tangents at k=5 and k=20 m^{-1} †). For k-values from 5 to 9.3 m^{-1} instability is not removed for all values of β, but the stabilizing influence of the singular points in the current-free pressureless plasma still provides a strong stabilizing effect. This effect might explain some (though not all) of the discrepancies found in pinch experiments with regard to the empirical containment times and the growth rates of kink instabilities, as calculated from the model of a sharp pinch surrounded by a vacuum.

For $\alpha = -24.6$ m^{-1} the curve R in Fig. 6 only displays effects due to singular points in a small neighbourhood of k = 20 m^{-1}, as can be understood as follows. According to Fig. 5 the pitch distribution for this value of α is close to that of a constant-pitch field, so that singular points only exist for k-values in the vicinity of k = 20 m^{-1}. As a result $R(\alpha=-24.6)$ has small dips on either side of k = 20 m^{-1}, which are not visible on the scale of Fig. 6. Notice that the singular points also involve that the last term of the expression R of Eq.(94) is no longer an order of magnitude smaller than the preceding term in the neighbourhood of $k + \mu m = 0$, $(rQ_r^t)'/rQ_r^t$ then becoming very large. Hence, the simple derivation of Eq. (101) is incorrect. From $\alpha = -36.7$ m^{-1} on the curve R is again smooth

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†) In order not to make the picture illegible this curve is not shown in Fig. 6. For the same reason the curve $R(\alpha = -35)$ is omitted.

to the left of $k = 20\ m^{-1}$, but the right part is deflected rather strongly. This effect is shown clearly for $R(\alpha=-50)$. The upper dashed part is the continuation of $R(\alpha=-50)$ to the right of $k = 20\ m^{-1}$, neglecting the singular points, whereas the solid line below represents the correct curve $R(\alpha=-50)$.

In Fig. 7 the curve $R(\alpha=-50)$ is shown once more, together with L for the critical value $\beta = 20\%$. It is clear that the same favourable situation arises as for the plane case: the type-I instabilities, which are present for a vacuum field for $k < 20\ m^{-1}$, are suppressed by increasing $-\alpha$ and the instabilities which then threaten to appear are suppressed by the influence of the singular points. Owing to this property, a pinch surrounded by a force-free field is stable against type-I instabilities for $\beta < \beta_{crit}$ not only when $\alpha = \alpha_{opt}$, as given by Eq. (101), but even for a range of values for which $-\alpha > -\alpha_{opt}$. For the parameter values of Fig. 6, $\alpha_{opt} = -36.7\ m^{-1}$. The value $\alpha = -50\ m^{-1}$ is chosen as representative for the range of α-values which are stable against type-I instabilities.

As to type-II instabilities, the split criterion $L>P>R$ (Eqs. (94a) and (94b)) only makes sense for values of k for which singular points exist in the surface layer. Assuming that the magnetic field has a uniform shear in the surface layer this is the case for k-values from 0 to $-\mu(r_{oe})$. For $k = 0$ the singular point lies at the inner side of the surface layer ($r_s = r_{oi}$), so that $S = 1$ and according to Eq. (94a): $P=0=L$. For $k = -\mu(r_{oe})$ the singular point lies at the outer side of the surface layer ($r_s = r_{oe}$), so that $S = 0$ and $P = \mu^2 r/(1 + \mu^2 r^2) = R$. As a result, in Fig. 6 the curve P connects the point A with the point B along a path which is determined by the special structure of the surface layer. The only limiting condition on the possible structures is: $S \geq 0$. If in addition we make the restriction that B^2 should be monotonous in the surface layer, then $S \leq 1$. In that case it is easily shown that the functions L-R, as well as P-R are monotonously decreasing functions of β, so that a critical value of β with respect to type-II instabilities exists. Notice that the definition of S has been chosen such that S is independent of β.

From Fig. 6 it is obvious that a pinch surrounded by a vacuum magnetic field, being unstable against type-I instabilities, has to be unstable against type-II instabilities as well. In fact, if L < R there simply is no space left for P to satisfy the split criterion L > P > R. This is the well-known result of Rosenbluth[37]). For increasing values of $-\alpha$, however, the curve R rotates counterclockwise around the fixed point A, thus giving, from a certain value of $-\alpha$ (viz. $-\alpha = 36.7\ m^{-1}$) on, passage to the curve P. Therefore, for values of $-\alpha > 36.7\ m^{-1}$, surface-layer structures exist which are stable against type-II instabilities. At the same time the stabilizing influence of the singular points in the tenuous plasma causes an absence of type-I instabilities for these higher values of $-\alpha$. An example of this situation is shown in Fig. 7. The depicted curve P for $\beta = 20\%$ is calculated while making the arbitrary assumption

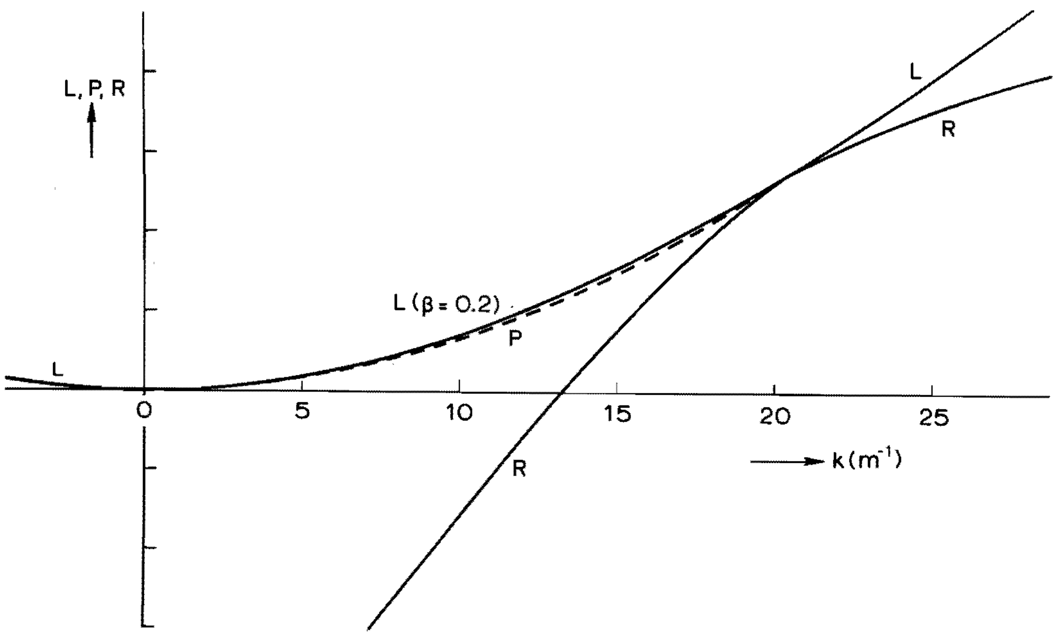

Fig. 7 Stability diagram for α = constant, $m = 1$, $r_o = 0.03$ m, $r_1 = 0.06$ m, $\mu(r_o) = -20\ m^{-1}$, $\alpha = -50\ m^{-1}$, $\beta = 20\%$.

that the fields vary linearly in the surface layer, as indicated in Fig. 5, and taking the limit $\delta \to 0$ afterwards. For this structure of the surface layer L > P > R holds for α=-50 m^{-1} and β = 20%. This one example is sufficient to show that in the presence of force-free fields with properly chosen α realistic surface-layer structures may exist for which type-II instabilities are absent. It is clear that the critical value of β for type-II instabilities is the same as β_{crit} for type-I instabilities (See Eq. (100)).

Finally, we are left with a discussion of possible type-III instabilities. To put it shortly: numerically it was checked that for the parameter values of Fig. 5, viz. $\mu(r_o)$=-20 m^{-1}, r_o = 0.03 m, r_1 = 0.06 m, α = 0, -24.6, -35, and -50 m^{-1}, the solution rQ_r^t of Eq. (97) has no zero points on the interval (r_o,r_1). This is more than sufficient for the stability of the pressureless plasma. Of course this method is too rough if such zero points occur on (r_o,r_1); the influence of the singular points must then be taken into account by investigating the behaviour of rQ_r^t on the independent subintervals of (r_o,r_1). However, if the values of αr are not too large it will often suffice to show the absence of type-III instabilities with the aid of the first mentioned rough method.

Summarizing, we have proved by numerical construction (the examples of Figs. 5,6, and 7) that a sharp pinch surrounded by a force-free field with a properly chosen constant α, is stable against sausage instabilities, kinks, surface-layer instabilities, and instabilities of the pressureless plasma[47]. An additional advantage of this configuration is the fact that the magnetic fields in the outer region of the pinch have shear. Therefore, Suydam's criterion will be satisfied for a refined model in which moderate pressure gradients are taken into account. This advantage is not shared by a constant-pitch force-free field (Sec. 7.4).

6.4. Growth rates

Just as for the plane case (Sec. 4.4), it is possible to derive the growth rates of type-I instabilities from the boundary conditions (7), (8), (9), and (10), viz.

$$-\gamma p \nabla . \underset{\sim}{\xi}^p + \frac{1}{\mu_o} \underset{\sim}{B}^p . \left[\underset{\sim}{Q}^p + \underset{\sim}{\xi}^p . \nabla \underset{\sim}{B}^p \right] = \frac{1}{\mu_o} \underset{\sim}{B}^t . \left[\underset{\sim}{Q}^t + \underset{\sim}{\xi}^t . \nabla \underset{\sim}{B}^t \right] \quad (r=r_o), \quad (102)$$

$$\xi_r^p = \xi_r^t \qquad (r=r_o), \quad (103)$$

$$r\xi_r^p = 0 \qquad (r=0) \; ,$$

$$r\xi_r^t = 0 \qquad (r=r_1) . \quad (104)$$

Dividing Eq. (102) by Eq. (103), while using the expressions (53), (54), and (55), the following characteristic equation is obtained:

$$\frac{(\omega^2 \rho^p - k^2 B^{p^2}/\mu_o)\{-\omega^2 \rho^p (\gamma p + B^{p^2}/\mu_o) + \gamma p k^2 B^{p^2}/\mu_o\}}{\omega^4 \rho^{p^2} - \omega^2 \rho^p (m^2/r^2 + k^2)(\gamma p + B^{p^2}/\mu_o) + \gamma p (m^2/r^2 + k^2) k^2 B^{p^2}/\mu_o} \frac{1}{r} \frac{(r\xi_r^p)'}{r\xi_r^p}$$

$$= \frac{B_\theta^{t^2}}{\mu_o r^2} - \frac{2kB_\theta^t}{\mu_o r^2} \frac{(mB_z^t/r - kB_\theta^t) B^{t^2}/\mu_o}{\omega^2 \rho^t - (m^2/r^2 + k^2) B^{t^2}/\mu_o}$$

$$- \frac{\{\omega^2 \rho^t - (mB_\theta^t/r + kB_z^t)^2/\mu_o\} B^{t^2}/\mu_o}{\omega^2 \rho^t - (m^2/r^2 + k^2) B^{t^2}/\mu_o} \frac{1}{r} \frac{(r\xi_r^t)'}{r\xi_r^t} \qquad (r=r_o). \quad (105)$$

Here ξ_r^p and ξ_r^t are the solutions to the complete equation of motion, subject to the boundary condition (104). The solution ξ_r^p is known: see Eq. (77). We do not know the analytical form of the solution ξ_r^t of Eq. (57) for a pressureless plasma with associated force-free field. Neglecting, however, the density ρ^t of the pressureless plasma ξ_r^t is again the known solution of the marginal equation of motion. The boundary condition (104) at $r = r_1$ must then be replaced by the condition that ξ_r^t is "small" at $r = r_1^*$, since for the proof of the splitting into independent subintervals it makes no difference whether $\omega^2=0$ or $\rho^t = 0$. Further, substituting $\omega^2\rho^t = 0$, ξ_r^p from Eq. (77), β from Eq. (62), μ from Eq. (64), v_A^2 and v_s^2 from Sec. 6.2 (just above Eq. (76)), and transforming to rQ_r^t by means of Eqs. (53) and (73), Eq. (105) becomes

$$(1-\beta)\sqrt{\frac{(k^2-\omega^2/v_A^2)(k^2-\omega^2/v_A^2-\omega^2/v_s^2)}{k^2-\omega^2/v_s^2}}\;\frac{I_m(\hat{k}r)}{I_m'(\hat{k}r)}$$

$$= \frac{\mu^2 r}{1+\mu^2 r^2} + \frac{\alpha r(k+\mu m)(k\mu r^2-m)}{(m^2+k^2r^2)(1+\mu^2r^2)} + \frac{(k+\mu m)^2 r}{(m^2+k^2r^2)(1+\mu^2r^2)}\;\frac{(rQ_r^t)'}{(rQ_r^t)}\,, \quad (r=r_o) \tag{106}$$

where $\hat{k} = \sqrt{\frac{(k^2-\omega^2/v_A^2)(k^2-\omega^2/v_s^2)}{k^2-\omega^2/v_A^2-\omega^2/v_s^2}}$, while $\frac{(rQ_r^t)'}{rQ_r^t}$ follows from Eq. (96) for profiles with constant α, and from the solution of Eq. (84) for $\alpha \neq$ constant.

The growth rates of type-I instabilities can be calculated from the implicit relation (106). For the incompressible pinch an explicit expression for ω^2 can be obtained from this relation by taking the limit $\gamma \to \infty$ (or $v_s^2 \to \infty$). The result reads

$$-\,\omega^2 = \frac{kv_A^2}{1-\beta}\,\frac{I_m'(kr_o)}{I_m(kr_o)}\,(R - L)\,, \tag{107}$$

where R and L are the known expressions of Eq. (94). The approximation of incompressibility is reliable if $-\omega^2 \ll k^2v_s^2$, so that the growth rate should not be too large. From Eq. (107) the well-known stability criterion $L > R$ is obtained again. In agreement with the statement in Chapter 5 the stability criterion is independent of the assumption of incompressibility, since the marginal equation corresponding to Eq. (106) is independent of v_s.

A stability analysis on the basis of Eq. (106) is necessarily restricted to type-I instabilities. Type-II instabilities cannot be obtained from an application of the boundary conditions (102) and (103), since these were derived from the assumption that ξ_r is continuous through the surface layer. A serious limitation also represents the fact that, in order to obtain a solution of the equation of motion, it was necessary to neglect the density of the pressureless plasma. This neglect is completely unjustified when type-III instabilities are present. In fact, a finite amount of perturbation energy is then available for the instability of the pressureless plasma, so that the neglect of ρ^t would imply an infinite value of $-\omega^2$ for a finite value of $-\omega^2\rho^t$. This is also brought about by Eq. (105), which contains ξ_r^t in a denominator of the right-hand side. If type-III instabilities show up ξ_r^t oscillates, so that the right-hand side blows up every time ξ_r^t has a zero at $r = r_o$. Therefore, when using Eq. (106) for the calculation of growth rates of type-I instabilities, one should make sure that the pressureless plasma itself is stable. We shall return to this point in Secs. 7.3 and 7.4.

CHAPTER 7

MARGINAL-STABILITY ANALYSIS OF SHEARLESS MAGNETIC FIELDS

The marginal-stability analysis, as given in Chapters 3 and 5, requires a further elaboration for magnetic fields of constant direction in the plane case and for magnetic fields of constant pitch in the cylindrical case. For these fields difficulties arise which are connected with the fact that the isolated singular points, which played an important role in the marginal-stability analysis, are absent here; instead, the whole interval becomes singular when $\underset{\sim}{k} \perp \underset{\sim}{B}$. The following discussion will concern the difference between kinks and interchanges for constant-pitch magnetic fields, and the associated significance of Suydam's criterion. Next, the growth rates of the instabilities of constant-pitch magnetic fields will be calculated in the local approximation (Sec. 7.2). The marginal-stability analysis will be reformulated in Sec. 7.3 in a discussion of the principle of exchange of stabilities, which principle can be applied in a simple way to constant-pitch fields. In Sec. 7.4 we come back to the sharp-pinch model of a dense plasma surrounded by a pressureless plasma associated with a force-free field, which then will have a constant pitch (Van der Laan's model). Finally, this model will be modified in Sec. 7.5 by providing the inner dense region of the pinch with a uniform current distribution associated with a parabolic pressure profile (Alfvén's model).

7.1. Discontinuities of the stability criterion

In the stability theory of the diffuse pinch usually localized interchanges are distinguished from non-localized kinks, where "localized" is meant in the sense of being effective over a limited domain in radial direction. This distinction can be based on theorem 4 for instance. The first item of theorem 4, Suydam's criterion, is obtained from the solution of the marginal equation of motion in the neighbourhood of a singular point. If Suydam's criterion is violated this solution has an infinite number of zeros in the vicinity of this singular point. Accordingly, the interchange modes are localized in the vicinity of such a singular point. Intuitively, this is clear from the fact that the wavevector $\underset{\sim}{k}$ of these modes is approximately perpendicular to the local direction of $\underset{\sim}{B}$, so that the field lines are little bent by the perturbation. The second and third item of theorem 4 concern the solutions of the marginal equation of motion on the whole independent subinterval. If any of these solutions has one or more zeros the pinch is unstable with respect to kink modes extending over the whole independent subinterval, i.e. they are much less localized than the interchange modes. Since the kinks are not localized in the neighbourhood of a singular point, the corresponding wavevector $\underset{\sim}{k}$ will in general have a direction differing from perpendicular to $\underset{\sim}{B}$, so that the field lines are bent considerably by the perturbation.

Similar considerations hold, in view of theorem 2, for a plane plasma layer under the influence of gravity. Therefore, gravitational instabilities can also be divided in localized gravitational interchanges if "Suydam's" criterion is violated, and non-localized gravitational instabilities, if the items 2) or 3) of theorem 2 are violated.

It will be clear that the distinction between kinks and interchanges on the basis of the localization of the modes is of limited applicability. Consider, in particular, a constant-pitch field; here Suydam's criterion is violated in the case of a negative pressure gradient (see Eq. (60)) and interchanges are possible which are not localized, because $\underset{\sim}{k}$ can be perpendicular to $\underset{\sim}{B}$ over the complete r-domain in which the pitch is constant. Apart from the difficulty that kinks and interchanges cannot be distinguished clearly in constant-pitch fields, another difficulty arises (associated with the first), viz. that the stability criteria for modes with $\underset{\sim}{k}.\underset{\sim}{B} = 0$ differ from those for which $\underset{\sim}{k}.\underset{\sim}{B} \to 0$. The reason for this discontinuity is that in the equation of motion for the diffuse pinch (Eq. (57)) the denominator N → 0 in the marginal-stability analysis ($\omega^2 \to 0$) if F → 0 (on the whole interval!). As a consequence, the stability criterion obtained depends on whether the limit $\omega^2 \to 0$ or F → 0 is taken first. The same exchange of limits arises in the plane compressible case for a field with constant direction, because the equation of motion (25) also contains a denominator N which tends to zero if both $\omega^2 \to 0$ and F → 0.

The above-mentioned difficulty does not arise in the plane incompressible case. The equation of motion (18) here leads to the following equations:

$$\text{in the limit } \omega^2 = 0,\ F \to 0: \quad \left(\frac{F^2/\mu_o}{k^2}\,\xi_y'\right)' + \rho' g \xi_y = 0\ , \tag{108}$$

$$\text{in the limit } F = 0,\ \omega^2 \to 0: \quad \left(\frac{\omega^2 \rho}{k^2}\,\xi_y'\right)' - \rho' g \xi_y = 0\ , \tag{109}$$

in which the indicated dominating contribution of the coefficients preserves the second-order character of the equation. In the first case the ordinary marginal equation of motion is obtained which shows, for small F, solutions with infinitely rapid oscillations if $\rho' g > 0$. Hence, the stability criterion here requires: $\rho' g < 0$. In the second case the equation of motion must be used in the limit of small ω^2, because the marginal equation of motion itself degenerates. For $\omega^2 < 0$ oscillating solutions then result if $\rho' g > 0$, so that the stability criterion again requires $\rho' g < 0$. In this case no question of discontinuity arises and the obtained stability criterion is simply "Suydam's" criterion (21) in the absence of shear.

Discontinuities of the stability criterion do show up when a finite compressibility is taken into account. This was shown by Newcomb[35]) for the plane compressible case with a magnetic field of constant direction. The equation of motion (25) then leads to the following equations:

in the limit $\omega^2 = 0$, $F \to 0$: $$\left(\frac{F^2/\mu_o}{k^2}\,\xi_y'\right)' + \left(\rho' g + \frac{\rho^2 g^2}{\gamma p}\right)\xi_y = 0\ , \tag{110}$$

in the limit $F = 0$, $\omega^2 \to 0$: $$\left(\frac{\omega^2 \rho}{k^2}\,\xi_y'\right)' - \left(\rho' g + \frac{\rho^2 g^2}{\gamma p + B^2/\mu_o}\right)\xi_y = 0\ . \tag{111}$$

In the first case the stability criterion amounts to "Suydam's" criterion (27) in the absence of shear, viz.

$$\rho' g + \frac{\rho^2 g^2}{\gamma p} < 0\ . \tag{112}$$

In the second case the well-known[35]) stability criterion for pure interchanges is found, viz.

$$\rho' g + \frac{\rho^2 g^2}{\gamma p + B^2/\mu_o} < 0\ . \tag{113}$$

Making use of an expansion with respect to a reciprocal scale length of the equilibrium Newcomb shows that the discontinuity of the stability criterion results from the existence of two types of modes for small $\underset{\sim}{k}.\underset{\sim}{B} \neq 0$. These modes are called type-1 and type-2 quasi-interchanges and correspond to the two possibilities admitted by Eq. (24a) if $F = \underset{\sim}{k}.\underset{\sim}{B} \to 0$:

1) $\omega^2 \neq 0$ and $\underset{\sim}{B}.\underset{\sim}{\xi} \to 0$, leading to type-1 modes which transform into pure interchanges if $F = 0$.

2) $\underset{\sim}{B}.\underset{\sim}{\xi} \neq 0$ and $\omega^2 \to 0$, leading to type-2 modes which transform into pure translations if F = 0.

It turns out (see Newcomb[35]) that for values of $\rho'g$ violating inequality (113) and, therefore, also inequality (112), type-1 modes (including the pure interchanges) are unstable and type-2 modes are stable. For values of $\rho'g$ satisfying Eq. (113) but violating Eq. (112), type-2 modes are unstable and type-1 modes are stable. The unstable type-2 quasi-interchanges have their largest growth rate for small but non-vanishing values of $\underset{\sim}{k}.\underset{\sim}{B}$, whereas they are marginally stable for $\underset{\sim}{k}.\underset{\sim}{B} = 0$ because then they are pure translations. Therefore, the discontinuities of the stability criteria for F = 0 are not of practical importance, since the growth rates do not exhibit a similar discontinuity.

The stability conditions for the compressible pinch with a constant-pitch magnetic field also show a discontinuity for F = 0, as has been observed by Tayler[55] and by Ware[56,57,58]. The equation of motion (57) of the diffuse pinch leads to the following equations:

in the limit $\omega^2 = 0$, $F \to 0$:

$$\left[\frac{rF^2/\mu_o}{m^2/r^2+k^2}\,\xi_r'\right]' - \frac{2k^2}{m^2/r^2+k^2}\,p'\xi_r = 0\ , \qquad (114)$$

in the limit $F = 0$, $\omega^2 \to 0$:

$$\left[\frac{\omega^2\rho}{m^2/r^2+k^2}\,\frac{1}{r}\,(r\xi_r)'\right]' + \frac{2B_\theta^2}{rB^2}\left[p' + \frac{2\gamma p B_\theta^2/\mu_o}{r(\gamma p+B^2/\mu_o)}\right]\xi_r = 0. \qquad (115)$$

In the first case Suydam's criterion (60) in the limit $\mu' \to 0$ is obtained:

$$p' > 0\ , \qquad (116)$$

in the second case the stability criterion is relaxed to

$$p' + \frac{2\gamma p B_\theta^2/\mu_o}{r(\gamma p+B^2/\mu_o)} > 0 \ . \qquad (117)$$

Notice that the latter criterion depends on the compressibility through the factor γp, whereas in all other cases the stability criteria of the diffuse pinch are independent of the compressibility. This is in agreement with Eq. (55), from which it follows that $\nabla.\underset{\sim}{\xi} \neq 0$ in the limit $F = 0$, $\omega^2 \to 0$. An analysis of the discontinuity of the stability criteria, similar to that given by Newcomb[35]) for the plane plasma layer, has been worked out by Ware[58]). Here, too, type-1 and type-2 modes appear in accordance with the two possible limiting cases of Eq. (56a) for $F \to 0$: either $\omega^2 \neq 0$ and $\underset{\sim}{B}.\underset{\sim}{\xi} \to 0$ or $\underset{\sim}{B}.\underset{\sim}{\xi} \neq 0$ and $\omega^2 \to 0$. If the inequality (116) is violated, whereas inequality (117) is satisfied, type-2 modes prove to be unstable and type-1 modes are stable. If Eq. (117) is violated, type-1 modes are unstable and type-2 modes are stable.

We now return to the starting point of the discussion, viz. the distinction between kinks and interchanges. For a constant-pitch field Ware[56,57]) introduced the following objective criterion as a definition for kinks and interchanges: $F = 0$ for "interchanges", and $F \neq 0$ for "kinks". On the basis of this definition inequality (116) is called the stability criterion for "kinks" and (117) that for "interchanges". Since criterion (116) is more stringent, one must expect that these "kinks" will turn up in pinch discharges with a constant-pitch magnetic field and a small negative pressure gradient. However, Ware arrives at a different conclusion from an analysis of the experimental results of Zeta and other slow pinch discharges. In these experiments the core of the discharge shows an approximately constant pitch. The outer region has a varying pitch, but this region is not taken into consideration.(As a matter of fact, this is not quite correct if one has to do with non-localized modes). Most remarkable about these experiments is their gross magnetohydrodynamic stability, i.e. violent kink instabilities associated with large field perturbations are absent. Moreover, according to a crude approximation the observed pressure gradients agree with negative values for which inequality (117) is marginally satisfied. Ware concludes from these data that "kinks" (i.e. type-2 modes) do not appear, whereas "interchanges" (i.e. type-1 modes) limit the pressure gradient to the negative value following from (117). An explanation for the absence of "kinks" would then be the small growth rate of these instabilities if F is small, so that finite Larmor-radius effects would enable a stabilization.

We raise some objections against Ware's nomenclature. First of all, type-1 modes also exist if $\underset{\sim}{k}.\underset{\sim}{B} \neq 0$; then they should be called "kinks", according to Ware's definition, although these modes still transform into the pure interchanges if $\underset{\sim}{k}.\underset{\sim}{B} \to 0$. Furthermore, the definition can be used only for a field with strictly constant pitch, since in a field having some shear the wavevector of the perturbation in general makes an angle with the field lines differing slightly from a right angle because of the finite localization width of the perturbation. Finally, it is not consistent to call interchanges the instabilities which are present in a shear field if Suydam's criterion is violated, and kinks those which are present in a shearless field under exactly the same conditions. For this reason we prefer Newcomb's designation of quasi-interchanges for type-1 as well as type-2 modes. Type-1 and type-2 instabilities only can be distinguished if $\underset{\sim}{k}.\underset{\sim}{B}$ is small and even then, strictly speaking, not very clearly because the unstable type-2 modes change continuously into the unstable type-1 modes if p' is lowered. Finite Larmor-radius stabilization then would apply for type-2 as well as for type-1 modes, or for neither of them. This argument will become more clear in Sec. 7.2, where we shall discuss in more detail the calculation of the growth rates of these modes.

Finally, the difficulties about the stability criterion completely disappear for fields having some shear. The old scheme then applies again and we talk about interchanges if

Suydam's criterion is violated. If we consider constant-pitch fields as limiting cases of fields having shear, Suydam's criterion applies in the limit $\mu' \to 0$ and the marginal-stability analysis of Chapter 5 holds without any change. The conclusion that the stability criteria of a diffuse pinch do not depend on compressibility then also remains valid. At the same time this point of view looks the most realistic one, because some shear always occurs in real experiments.

The mentioned point of view implies that Kadomtsev's treatment[32,59]) of the diffuse z-pinch should undergo a modification. In the z-pinch $B_z = 0$ and Kadomtsev obtains criterion (117) with $B = B_\theta$ for the $m = 0$ modes; this corresponds to the limit $F = 0$, $\omega^2 \to 0$. If $m \neq 0$ and $k \to \infty$ (so that $F \neq 0$) the following stability criterion is obtained from the marginal equation of motion:

$$p' + \frac{m^2 B_\theta^2}{2\mu_o r} > 0 \; . \tag{118}$$

It depends on the value of $2\mu_o p/B_\theta^2$ whether criterion (117) is more stringent than this one, $m = 0$ modes then being unstable, or whether criterion (118) is more stringent, involving unstable $m = 1$ modes (according to Kadomtsev). However, if we consider the z-pinch as a limiting case of a pinch with a small B_z-field and next take, for $m \neq 0$, the limit $B_z \to 0$ and $k \to \infty$ such that $F \to 0$, we again obtain the criterion (116), which is more stringent than both (117) and (118). Therefore, according to ideal magnetohydrodynamics a diffuse z-pinch with $p' < 0$ is always unstable with respect to quasi-interchanges with $m \neq 0$.

7.2. Growth rates of instabilities in a constant-pitch magnetic field

Our objections against Ware's treatment of constant-pitch magnetic fields mainly concerned the nomenclature, in which type-2 quasi-interchanges were called "kinks". We shall now show in addition that, if $-2\gamma p B_\theta^2/[\mu_o r(\gamma p+B^2/\mu_o)] < p' < 0$, quasi-interchanges arise which cannot be stabilized by finite Larmor-radius effects. As a matter of fact, the finite size of the ionic gyration radius in general favours stabilization when the growth rates of the instabilities are not too large. Ware's argument, that no instabilities arise in the mentioned p' interval, so that Suydam's criterion (116) should be replaced by the less stringent stability criterion (117), is based upon the fact that type-2 quasi-interchanges have small growth rates, enabling finite Larmor-radius stabilization. We shall show that the distinction between type-1 and type-2 modes makes no sense for the most important modes, viz. those having the largest growth rates; moreover, it will turn out that those modes have no small growth rate at all.

The growth rates of instabilities can be calculated from the complete equation of motion, that is, from Eq. (25) for the plane case and from Eq. (57) for the cylindrical case. In general, these equations can be solved only numerically. Newcomb[35]) calculated the growth rates of the quasi-interchanges for the plane case with a magnetic field of constant direction by introducing a local approximation, based on the scale length of the equilibrium. For the cylindrical case the growth rates of the quasi-interchanges for the constant-pitch field were calculated analogously by Ware[58]). Our treatment will be based on Eq. (57) and will differ from Ware's treatment in principle only by the fact that the analogy with Newcomb's paper is traced further. In doing so, however, we shall reach other conclusions than Ware.

The equilibrium equation (52) suggests to introduce a scale length L defined by

$$\frac{1}{L} = \frac{B_\theta^2/\mu_o}{r(\gamma p+B^2/\mu_o)} \ . \qquad (119)$$

From this equation it appears that r just as well as L can be used as a scale length if $B_\theta \gtrsim B_z$. Hence, the quantities

p, B_θ, and B_z vary appreciably over distances of the order $L \sim r$. Quasi-interchanges occur for small values of $F = k_{//}B$. Therefore, we shall assume that $k_{//} << k$. Furthermore, we restrict our attention to $kr >> 1$ and to local instabilities, i.e. solutions of Eq. (57) with $\omega^2 < 0$ which vary rapidly over distances much smaller than L, so that p, B_θ, and B_z may be considered as approximately constant. The equation (57) then has solutions of the form: $r\xi_r \sim \exp(iqr)$, with $qr >> 1$. The latter assumption will be justified on the basis of the final result. With these approximations ($k_{//} << k$, $kr >> 1$, $qr >> 1$) and assuming $|\omega^2\rho| << (m^2/r^2+k^2)B^2/\mu_o$ Eq. (57) gives

$$-q^2 - \frac{m^2/r^2+k^2}{\omega^2\rho - F^2/\mu_o}\left[\omega^2\rho - F^2/\mu_o - \frac{2B_\theta}{\mu_o}\left(\frac{B_\theta}{r}\right)' + \frac{4k^2(B_\theta^2/\mu_o)(\omega^2\rho B^2/\mu_o - F^2/\mu_o)}{(m^2+k^2r^2)\{\omega^2\rho(\gamma p+B^2/\mu_o)-\gamma pF^2/\mu_o\}} - r\left\{\frac{2kB_\theta(mB_z/r-kB_\theta)}{\mu_o(m^2+k^2r^2)}\right\}'\right] = 0. \tag{120}$$

From the equilibrium equation (52) and the condition of constant pitch it further follows that

$$B_z' = -\frac{B_z}{B^2/\mu_o}\left(p' + \frac{2B_\theta^2}{\mu_o r}\right),$$
$$B_\theta' = -\frac{B_\theta}{B^2/\mu_o}\left(p' + \frac{2B_\theta^2 - B^2}{\mu_o r}\right). \tag{121}$$

Using these relations, Eq. (120) yields, after some straightforward algebraic reductions, the following quadratic for $\rho\omega^2$:

$$\omega^4\rho^2 - \left[\frac{m^2+k^2r^2}{q^2r^2+m^2+k^2r^2}\frac{2B_\theta^2}{rB^2}p' + \frac{4k^2B_\theta^2/\mu_o}{q^2r^2+m^2+k^2r^2}\frac{\gamma p}{\gamma p+B^2/\mu_o}\right.$$
$$+ \frac{2\gamma p+B^2/\mu_o}{\gamma p+B^2/\mu_o}F^2/\mu_o - \frac{4mB_\theta p'F}{(q^2r^2+m^2+k^2r^2)B^2} - \frac{4k^2mrB_\theta F/\mu_o}{(q^2r^2+m^2+k^2r^2)(m^2+k^2r^2)} -$$

$$- \frac{4B_\theta^2 F^2/\mu_o}{(q^2r^2+m^2+k^2r^2)B^2}\Bigg]\omega^2\rho + \frac{\gamma p F^2/\mu_o}{\gamma p + B^2/\mu_o}\Bigg[\frac{m^2+k^2r^2}{q^2r^2+m^2+k^2r^2}\,\frac{2B_\theta^2}{rB^2}\,p'$$

$$+ F^2/\mu_o - \frac{4mB_\theta p' F}{(q^2r^2+m^2+k^2r^2)B^2} - \frac{4k^2 mrB_\theta F/\mu_o}{(q^2r^2+m^2+k^2r^2)(m^2+k^2r^2)}$$

$$- \frac{4B_\theta^2 F^2/\mu_o}{(q^2r^2+m^2+k^2r^2)B^2}\Bigg] = 0 \, . \qquad (122)$$

If $qr \sim kr$ the orders of magnitude of the various terms of this equation can be represented as follows:

$$\omega^4\rho^2 - B^2/(\mu_o r^2)\left[O(1)+O(1)+O(k_{//}^2 r^2)\right.$$

$$\left. +O(k_{//}/k)+O(k_{//}/k)+O(k_{//}^2/k^2)\right]\omega^2\rho$$

$$+ k_{//}^2 B^4/(\mu_o^2 r^2)\left[O(1)+O(k_{//}^2 r^2)\right.$$

$$\left. +O(k_{//}/k)+O(k_{//}/k)+O(k_{//}^2/k^2)\right] = 0; \qquad (123)$$

the derivation of the various terms assumes that the local values of p are of the order of B^2/μ_o, in accordance with situations realized in pinches. This equation shows which terms of Eq. (122) can be neglected for the different limiting cases.

In the limit $F = k_{//}B \rightarrow 0$ the terms $O(k_{//}^2 r^2)$, $O(k_{//}/k)$, and $O(k_{//}^2/k^2)$ are negligible in comparison with the terms $O(1)$. Neglecting these terms and using the relation

$$k^2r^2/(m^2+k^2r^2) \approx B_\theta^2/B^2 \, , \qquad (124)$$

which holds if $k_{//} << k$, Eq. (122) provides the growth rates of type-1 and type-2 quasi-interchanges, viz.

$$\rho\omega_1^2 = \frac{m^2+k^2r^2}{q^2r^2+m^2+k^2r^2}\,\frac{2B_\theta^2}{rB^2}\left(p' + \frac{2\gamma p B_\theta^2/\mu_o}{r(\gamma p + B^2/\mu_o)}\right) \, , \qquad (125)$$

$$\rho\omega_2^2 = \frac{\gamma p}{\gamma p + B^2/\mu_o} \frac{p'}{p' + \dfrac{2\gamma p B_\theta^2/\mu_o}{r(\gamma p + B^2/\mu_o)}} F^2/\mu_o \ . \tag{126}$$

The latter expression is only valid for values of p' for which the denominator is not small. In the transition region where

$$p' + 2\gamma p B_\theta^2/\left[\mu_o r(\gamma p + B^2/\mu_o)\right] \approx 0 \tag{127a}$$

the two types of modes pass into each other, while the two values of $\rho\omega^2$ can be approximated there by:

$$\rho\omega^2 = \pm \sqrt{\frac{m^2 + k^2 r^2}{q^2 r^2 + m^2 + k^2 r^2}} \frac{2\gamma p B_\theta^2/\mu_o}{r(\gamma p + B^2/\mu_o)B} F \ . \tag{127b}$$

Equations (125) and (126) are the expressions derived by Ware for the growth rates of the quasi-interchanges, from which the stability criteria (116) and (117) follow immediately. If F is small $\rho\omega_2^2$ is also small, so that type-2 modes with small $k_{//}$ will in general be stabilized by finite Larmor-radius effects. However, Ware's argument that, as a consequence, the stability criterion (116) should be replaced by (117) (so that diffuse pinches with a constant-pitch magnetic field could be stable in the presence of a small negative pressure gradient) is incorrect. Equation (126) shows that the largest growth rates of type-2 quasi-interchanges are reached for large values of F, so that the approximations made above are not allowed for the modes with the largest growth rates.

We shall continue the analogy with Newcomb's paper at the point where Ware did break it off. Starting from Eq. (122) we ask for the value of $k_{//}$ for which the maximum growth rate is obtained. This will turn out to be the case for $k_{//}r = O(1)$, so that the terms $O(k_{//}/k)$ and $O(k_{//}^2/k^2)$ can be neglected in Eq. (122), but not the terms $O(k_{//}^2 r^2)$. The value of $k_{//}$ belonging to the maximum growth rate is found by differentiation with respect to F of the solutions of Eq. (122), with the mentioned approximations. This gives

$$F = 0 \ , \tag{128}$$

$$F^2 = -\frac{m^2+k^2r^2}{q^2r^2+m^2+k^2r^2}\,\frac{2B_\theta^2}{rB^2}\left[\frac{\gamma p+B^2/\mu_o}{B^2/\mu_o}\left\{p' + \frac{2\gamma p B_\theta^2/\mu_o}{r(\gamma p+B^2/\mu_o)}\,\frac{2\gamma p+B^2/\mu_o}{B^2/\mu_o}\right\} - \frac{2\gamma p+B^2/\mu_o}{B^2/\mu_o}\sqrt{\frac{2\gamma p B_\theta^2}{rB^2}\left(p' + \frac{2\gamma p B_\theta^2}{rB^2}\right)}\,\right] . \tag{129}$$

It appears from Eq. (129) that $k_{//}r = O(1)$ for the modes with the largest growth rate, while F^2 is positive and non-vanishing if

$$-\frac{2\gamma p B_\theta^2/\mu_o}{r(\gamma p+B^2/\mu_o)}\,\frac{2\gamma p+B^2/\mu_o}{\gamma p+B^2/\mu_o} < p' < 0 \ . \tag{130}$$

For this p' range the most dangerous modes are quasi-interchanges with a value of $k_{//} \neq 0$, following from Eq. (129). Substitution of F^2 from Eq. (129) in Eq. (122) yields the growth rates for these modes:

$$\rho\omega^2 = -\frac{m^2+k^2r^2}{q^2r^2+m^2+k^2r^2}\,\frac{4\mu_o\gamma^2p^2B_\theta^4}{r^2B^6}\left[1-\sqrt{\frac{rB^2}{2\gamma p B_\theta^2}\left(p'+\frac{2\gamma p B_\theta^2}{rB^2}\right)}\,\right]^2 . \tag{131}$$

This expression becomes extreme, just as the expression for the pure interchange modes (Eq. (125)), for $q^2r^2 \ll k^2r^2$.

Summarizing, we have derived:

if

$$-2\gamma p(2\gamma p + B^2/\mu_o)B_\theta^2/\left[\mu_o r(\gamma p+B^2/\mu_o)^2\right] < p' < 0 \ ,$$

quasi-interchanges, with $k_{//}$ given by Eq. (129), are the most dangerous modes. Their growth rate is given by

$$\rho\omega^2 = -\mu_o\left(\frac{2\gamma p B_\theta^2}{rB^3}\right)^2\left[1 - \sqrt{\frac{rB^2}{2\gamma p B_\theta^2}\left(p' + \frac{2\gamma p B_\theta^2}{rB^2}\right)}\right]^2 , \tag{132}$$

which reduces, for small negative p', or for incompressible ($\gamma \to \infty$)modes, to

$$\rho\omega^2 \approx -\mu_o(p'/2B)^2 . \tag{133}$$

If

$$p' < -2\gamma p(2\gamma p + B^2/\mu_o)B_\theta^2/\left[\mu_o r(\gamma p + B^2/\mu_o)^2\right] ,$$

pure interchanges ($k_{//} = 0$) are the most dangerous modes. Their growth rate is given by

$$\rho\omega^2 = \frac{2B_\theta^2}{rB^2}\left(p' + \frac{2\gamma p B_\theta^2/\mu_o}{r(\gamma p + B^2/\mu_o)}\right) . \tag{134}$$

We notice that the distinction between type-1 and type-2 modes can no longer be made for the modes with the largest growth rate. Furthermore, quasi-interchanges with $F \neq 0$ are the most dangerous modes for $p' \approx -2\gamma p B_\theta^2/\left[\mu_o r(\gamma p + B^2/\mu_o)\right]$ and this is still the case for more negative values of p', so that the inequality (117) loses its sense in this respect. Finally, it turns out that, even for small negative values of p' and practical values of ρ and B, the expression (132) in general yields such large growth rates that finite Larmor-radius effects can provide no effective stabilization mechanism. Thus, we are led to the same conclusion as that of the preceding section, viz. that the correct stability criterion for a diffuse pinch with a constant-pitch magnetic field is Suydam's criterion: $p' > 0$. Violation of this criterion leads in first instance to unstable quasi-interchange modes; pure interchange modes arise only if the criterion in question is strongly violated.

The assumption $qL \gg 1$ has been made in the preceding discussion, as well as in that of Newcomb[35]) and that of

Ware[58]). This is a fundamental and serious limitation of the treatment, considerably decreasing its reliability. The assumption was needed in order to be able to solve the equation of motion in a simple way. In this way growth rates of local instabilities are obtained. These are instabilities which appear also in a very thin plasma layer, since they correspond to solutions of the equation of motion which oscillate so rapidly that the boundary conditions can be satisfied, even if the walls are very close together. Accordingly, the precise position of the walls is not very important for these modes. In reality, however, we have to do with an interval which is so large that the solutions ξ_r that correspond to local instabilities have more than one zero in the interval. According to Sturm's fundamental theorem (see Chapters 3 and 5), other instabilities then exist for the same values of m and k which have a ξ_r without zero points in the open interval $(0,\hat{R})$ and, therefore, a larger growth rate than the local instabilities. However, exactly the instabilities having the largest growth rate and, therefore, not the local instabilities but those for which the precise position of the wall is important, are of interest to us. Using Newcomb's method the only information we get about these instabilities is that the growth rates are larger than the values following from the equations (132) and (134). For special cases of constant-pitch magnetic fields the stability of non-local modes will be discussed in Secs. 7.4 and 7.5.

7.3. The principle of exchange of stabilities

Before we return to the force-free magnetic fields, especially constant-pitch force-free fields, we shall discuss the principle of exchange of stabilities in connection with the marginal-stability analysis. This is of interest in view of the criticism which was given by Tayler[29]) on an improper use of this principle. Tayler's criticism is relevant to the modification of the stability analysis of Van der Laan's model[24]), which will be given in Sec. 7.4.

The principle of exchange of stabilities will be shortly described below (we base our discussion on Chandrasekhar's treatment[44])). An equilibrium or stationary state of a system is described by a number of time-independent parameters $X_1, X_2, \ldots X_j$. These parameters contain the equilibrium quantities like the pressure distribution (one of the parameters will be a functional of $p(\underset{\sim}{r})$), the geometry of the problem, the field components, etc. Next, the first-order quantities representing the perturbation are expanded in a set of normal modes, which are labelled with a vector $\underset{\sim}{k}$. The time dependence of these modes is assumed to be of the form: $\exp(-i\omega_{\underset{\sim}{k}} t)$. States which are marginal with respect to a mode $\underset{\sim}{k}$ are then to be determined from the condition:

$$\mathrm{Im}\ \omega_{\underset{\sim}{k}}(X_1, \ldots X_j) = 0 \,. \tag{135}$$

This condition fixes a locus in parameter space:

$$E_{\underset{\sim}{k}}(X_1, \ldots X_j) = 0 \,, \tag{136}$$

separating the states which are stable with respect to a special mode $\underset{\sim}{k}$ from the unstable ones.

Unstable dissipative systems may be divided into those for which, at onset of instability, Re $\omega_{\underset{\sim}{k}} \neq 0$ and those for which Re $\omega_{\underset{\sim}{k}} = 0$. In the former case the marginal states exhibit oscillatory motion and the system is said to be overstable. In the latter the marginal modes (Im $\omega_{\underset{\sim}{k}} = 0$) are non-oscillatory (Re $\omega_{\underset{\sim}{k}} = 0$), and the principle of exchange of stabilities holds; that is, the locus (136) can be found, instead of (135), from the condition

$$\omega_{\underset{\sim}{k}}(X_1, \ldots X_j) = 0 \,. \tag{137}$$

This implies a significant simplification of the stability analysis, because now one can find the locus (136) just by substituting $\omega = 0$ into the equations of motion, solving the boundary-value problem only for this specific value of ω.

For conservative (self-adjoint) systems ω^2 is always real[26,31]) and, consequently, ω_k is either purely imaginary or purely real. In this case the marginal modes are also non-oscillatory, so that the principle of exchange of stabilities, as formulated above, is always applicable. Marginal states with respect to a mode $\underset{\sim}{k}$ can then also be determined from a condition like (137), which in the same way fixes a locus in parameter space of the form (136). Finally, the locus separating the stable states from the unstable ones with respect to all modes is the envelope of the loci $E_{\underset{\sim}{k}}$:

$$E(X_1,..X_j) = 0 \ . \tag{138}$$

For continuous $\underset{\sim}{k}$ this locus consists of a single smooth curve, whereas for discrete $\underset{\sim}{k}$ it is composed of adjacent parts of smooth curves. Quite generally, it is the purpose of a stability analysis to determine the locus (138).

The determination of the locus $E_{k,m}(X) = 0$ (from now on X represents the whole set $X_1,..X_j$) for a diffuse pinch is not completely straightforward, as will become clear presently. Let us consider solutions ξ_r of the equation of motion (57) of the diffuse pinch on the interval $(0,r_1)$. The equations (57) and (58) constitute a boundary-value problem, fixing the eigenvalues of ω^2. We split the interval $(0,r_1)$ at an arbitrary point $r = r_o$; let $\xi_1(r)$ be a solution of Eq. (57) on $(0,r_o)$ satisfying the boundary condition (58) at $r = 0$, and $\xi_2(r)$ the solution of Eq. (57) on (r_o,r_1) satisfying (58) at $r = r_1$; the amplitudes of ξ_1 and ξ_2 may be chosen such that the two solutions join at $r = r_o$, so that $\xi_1(r_o) = \xi_2(r_o)$. For an arbitrary value of ω^2 the functions ξ_1 and ξ_2 in general cannot be joined together with a continuous tangent. If for a certain value of ω^2, $\xi_1'(r_o) = \xi_2'(r_o)$ holds in addition to $\xi_1(r_o) = \xi_2(r_o)$, then the composite function ξ_r is smooth on $(0,r_1)$, and ω^2 constitutes an eigenvalue. The conditions for continuity of ξ_r and ξ'_r can be represented by

$$D(\omega^2,k,m,X) = \frac{\xi'_1(r_o)}{\xi_1(r_o)} - \frac{\xi'_2(r_o)}{\xi_2(r_o)} = 0 \ . \qquad (139)$$

This characteristic equation contains solutions of the complete equation of motion and, therefore, it is difficult to solve in general. Sometimes, however, it can be solved for $\omega^2 = 0$; in any case the equation is simplified considerably by this substitution. For certain configurations (for example a plasma-vacuum system[60])) it can be proved that D is a monotonically decreasing function of ω^2. In those cases one arrives at the necessary stability criterion

$$D(\omega^2 = 0,k,m,X) > 0 \ , \qquad (140)$$

because then there exists no solution of Eq. (139) for $\omega^2 < 0$.

Comparing this criterion with theorem 4 of Chapter 5, it is clear that it cannot be complete for general cases. First of all, the singular points (F = 0) complicate the picture, but this effect can be taken into account in the same way as done in Chapters 3 and 5. The singular points are not essential for the present discussion and they will not be considered here. For example, one can think of a constant-pitch field where the singular points are absent and where the case $F \equiv 0$ is ignored in the same way as in Sec. 7.1 (one considers the limit $F \to 0$ only). A more serious defect of criterion (140) is the fact that it only yields item 3) of theorem 4. Thus, the items 1) and 2) of theorem 4 are overlooked. The reason for this is that $D(\omega^2,k,m,X)$, considered as a function of ω^2, tends to infinity every time ξ_1 or ξ_2 has a zero in $r = r_o$. Therefore, in general the function D has branches and it may be that D is monotonous in ω^2 for every individual branch, but the inequality (140) then no longer guarantees that Eq. (139) has no solution for $\omega^2 < 0$. The different branches of the function $D(\omega^2,k,m,X)$ can be labelled with a parameter n, which equals the total number of zero points of ξ_1 and ξ_2 on the open intervals $(0,r_o)$ and (r_o,r_1), respectively. The values of ω^2, k,m, and X being chosen, n is also fixed by means of the equation of motion (57) and the boundary conditions. From theorem 4 it follows that the criterion (140) is sufficient for stability (at least in the absence of singular points) if and only if the given values of k,m,X, while $\omega^2=0$, involve n = 0. For values of $n \geq 1$ the criterion (140) is not relevant and the pinch is unstable.

The connection of the preceding discussion with the principle of exchange of stabilities and with Tayler's criticism is the fact that the condition

$$D(\omega^2 = 0,k,m,X) = 0 \tag{141}$$

not simply provides the locus $E_{k,m}(X) = 0$ that separates the stable states (with respect to the modes k,m) from the unstable ones. Instead, a number of loci $E_{k,m,n}(X) = 0$ is obtained, of which only the locus $E_{k,m,n=0}(X) = 0$ separates the stable from the unstable states. Therefore, if we know a solution of Eq. (141) for certain values of k,m, and X, it is not certain whether there do not exist modes besides this one, belonging to the same values of k,m, and X (but with a lower value of n), which are solutions of Eq. (139) for $\omega^2<0$ and which, therefore, are unstable. This is essentially the warning of Tayler against a careless application of "the principle of exchange of stabilities" expressed by Eq. (141). The reason of this difficulty (Tayler fails to observe this and Chandrasekhar is not very explicit about this point) is the fact that in Eq. (141) a parameter is lacking, corresponding to a kind of wavenumber k_r in radial direction. This wavenumber cannot be defined for an inhomogeneous problem, but the parameter n plays a similar role for these problems. Thus, the locus $E_{k,m}(X) = 0$ is found from the condition

$$D(\omega^2 = 0,k,m,n = 0,X) = 0 \ , \tag{142}$$

which represents the true principle of exchange of stabilities. This formulation is in agreement with Chapters 3 and 5: the most unstable solution $\xi(\omega^2)$ of the boundary-value problem is that one for which $n = 0$ (n decreases with increasing $-\omega^2$).

Therefore, Tayler's criticism is fully taken into account by the introduction of the parameter n into the problem. If, in addition, the influence of the singular point is also properly taken into account, there can be no objection to the application of the principle of exchange of stabilities. The application of this principle thus proves to be identical to that of theorem 4.

7.4. Constant-pitch force-free magnetic fields (Van der Laan's model)

We return to the force-free magnetic fields and compare the stabilization of magnetohydrodynamic instabilities of a sharp screw pinch by force-free fields of constant α (Chapter 6) with the stabilization by force-free fields of constant μ, which was considered earlier in Ref. 24. In both cases the outer region of the pinch consists of a perfectly conducting plasma with a low density, in which currents are present. During the period of formation of the pinch these currents cause inward moving field lines, the pitch of which is constant in the moving coordinate system[23]. Thus, a force-free field of constant α can be produced by applying a field at the wall which varies in time in a properly prescribed way. In Van der Laan's model a constant-pitch force-free field is generated because the applied field has a pitch which is constant in time. In general, the latter field configuration can be realized easier than the former.

The field components of a constant-pitch force-free field are given by[24]

$$B_z^t = \frac{C}{1+\mu^2 r^2} , \quad B_\theta^t = \frac{C\mu r}{1+\mu^2 r^2} , \tag{143}$$

where the constant C is fixed by the required magnitude of the magnetic field at the plasma boundary. Chosing the same values for the parameters r_o, r_1, and $\mu(r_o)$ as in Fig. 5 for the fields of constant α, the equations (143) provide a single field for which μ = constant = $-20\ m^{-1}$ (the dashed line for μ in the left part of Fig. 5; the field components are the same as in the outer region of Fig. 9 of Sec. 7.5). On the average, this field deviates little from that with α = constant = $-24.6\ m^{-1}$. This will be reflected by the corresponding stability criteria.

Sections 7.1 and 7.2 led to the conclusion that, in practice ($p' < 0$), a constant-pitch field is always unstable with respect to quasi-interchange instabilities because Suydam's criterion is violated. However, in a constant-pitch force-free field $\mu' = 0$ as well as $p' = 0$, so that Suydam's criterion degenerates. In this case the question about the stability is still open.

Schuurman, Bobeldijk, and De Vries[24]) base their stability analysis of Van der Laan's model on the boundary condition for a plasma-pressureless plasma interface. This boundary condition (Eq. (35) of Ref. 24) may be obtained from our Eq.(106) by the substitution of the α-value for a constant-pitch force-free field, viz. $\alpha = 2\mu/(1+\mu^2 r^2)$. Equation (106) is of the form $D(\omega^2,k,m,X) = 0$ and, therefore, suitable for the application of the principle of exchange of stabilities. This equation has been derived from Eq. (105) and the latter can in turn be derived from Eq. (139) by starting from a diffuse pinch consisting of an inner region with a longitudinal magnetic field, a surface layer of thickness δ, and an outer region with a force-free field, and taking the limit $\delta \to 0$ afterwards (in the same way as in Sec. 6.2, but now using solutions of the complete equation of motion). Bobeldijk[60]) has shown that the left-hand side of Eq. (106) is a monotonically decreasing function of ω^2. Next, it is stated in Ref. 24 that the right-hand side of Eq. (106) is independent of ω^2, so that the stability criterion is represented by $D(\omega^2 = 0,k,m,X) > 0$, where $D \equiv L-R$, L and R being the expressions of Eq. (94).

On the other hand, in Sec. 7.3 we arrived at the conclusion that the criterion $D(\omega^2 = 0,k,m,X) > 0$ is incomplete in general, because it does not take into account the possibility that $D = D(\omega^2,k,m,X)$ as a function of ω^2 has branches if the functions ξ_1 and ξ_2 have zero points ($n \geq 1$). A comparison with the stability conditions of the sharp pinch surrounded by a force-free field with non-constant α, given at the end of Sec. 6.2, shows that this implies that the stability analysis of Ref. 24 overlooks instabilities of the pressureless plasma itself (type-III instabilities)†). The reason for this is that the starting point of Ref. 24, Eq. (106), only follows from Eq. (105) when neglecting the density ρ^t of the pressureless plasma. In Sec. 6.4 we pointed out, however, that this neglect is not allowed if type-III instabilities are present. In that case the starting point of the analysis must be the more correct Eq. (105), the right-hand side of which is a function of ω^2 and, therefore, leads to branches.

Obviously, the preceding discussion only makes sense if these type-III instabilities actually show up in Van der Laan's model. It had already been remarked in Ref. 24 that the quantity R tends to infinity for certain values of the parameters. From Eq. (94) it is obvious that this happens if rQ_r^t (and, therefore, ξ_r^t) has a zero at $r = r_o$, which implies that for those values of the parameters type-III instabilities really arise. We shall investigate these instabilities in more detail with the aid of the marginal equation of motion for a constant-pitch force-free field in terms of rQ_r^t, following from Eq.(84):

$$(rQ_r^t)'' + \frac{1}{r}\,\frac{m^2-k^2r^2}{m^2+k^2r^2}(rQ_r^t)' + \left(\alpha^2-k^2-\frac{m^2}{r^2}+\frac{2km\alpha}{m^2+k^2r^2}-\frac{1}{r}\quad\frac{m-k\mu r^2}{k+\mu m}\alpha'\right)rQ_r^t=0, \tag{144}$$

where μ = constant and $\alpha = 2\mu/(1+\mu^2r^2)$. This equation agrees with Eq. (16) of Ref. 24 but, since ρ^t was neglected, it had

†) Type-II instabilities also are not discussed in Ref. 24, but this is inherent to a stability analysis on the basis of the boundary conditions of a sharp pinch.

been used there as the complete equation of motion. It follows from Eq. (144) that an oscillating behaviour of the solutions of the marginal equation of motion and, therefore, the existence of type-III instabilities cannot be avoided in this model; this has been noticed already by Freidberg, Weitzner, and Weldon[61]). In fact, the last term of Eq. (144) contains the constant factor $k+\mu m$ in the denominator, so that this term tends to $+\infty$ or $-\infty$ depending on whether k tends to $-\mu m-0$ or to $-\mu m+0$ (we assume that $-\mu m > 0$). Thus, at one of the two sides of the point $k = -\mu m$ an interval of k-values always exists for which the solutions of Eq. (144) oscillate, and the more rapidly so according as k approaches $-\mu m$. The pressureless plasma is unstable for k-values in that interval.

We observe that the oscillating behaviour of the solutions of Eq. (144) is connected with the appearance of the constant factor $k+\mu m$ in the denominator of Eq. (144) for a constant-pitch field. This behaviour is not inherent to the fact that α is not constant. It is true that the factor in question also appears in the denominator for every other field for which $\alpha' \neq 0$ (and $\mu' \neq 0$), but in that case it is not constant and if $k+\mu m = 0$ we again have a singular point of the well-known type. The presence of shear then ensures that Suydam's criterion is satisfied in the neighbourhood of the singular point, so that for force-free fields with $\mu \neq$ constant the mentioned instabilities do not arise.

Notice that the type-III instabilities which arise in force-free fields with μ = constant can be regarded as a limiting case of Suydam-type instabilities (interchanges). The type-III instabilities which arise in force-free fields with constant α if the value of $|\alpha|$ is too large (described by Voslamber and Callebaut[17])), are of another kind, for which Suydam's criterion is not relevant and for which the name kinks is appropriate.

Although the stability criterion $D(\omega^2 = 0,k,m,X) > 0$ or $L > R$ is not correct with respect to instabilities of the pressureless plasma, a picture of L and R as a function of k can yet reveal these instabilities. This is because R blows up every time rQ_r^t has a zero point (see Eq. (94)) and, as a consequence, the plot of R as a function of k also exhibits branches. These branches of the function $R = R(k)$ can be labelled with the same parameter n as used for those of the function $D = D(\omega^2)$.

In Fig. 8 the corrected stability diagram of Van der Laan's model is given, showing $L(\beta = 0)$ and R as functions of

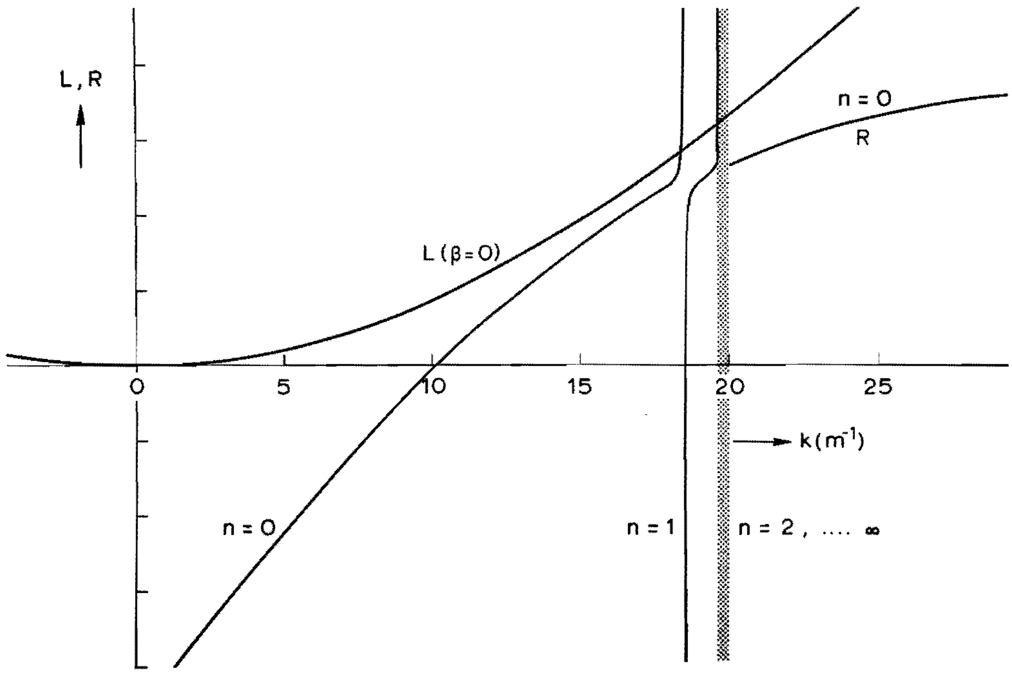

Fig. 8 Stability diagram for μ = constant, m = 1, r_o = 0.03 m, r_1 = 0.06 m, $\mu = -20\ m^{-1}$.

k for m = 1, $\mu = -20\ m^{-1}$, and the other parameters as in Fig.5. Due to the fact that the field with μ = constant = $-20\ m^{-1}$ deviates little from that with α = constant = $-24.6\ m^{-1}$, the stability diagram of Fig. 8 is almost identical to the diagram of Fig. 6 for $\alpha = -24.6\ m^{-1}$, at least in the region of k-values where type-III instabilities are absent. The function R exhibits branches for k-values from 18.4 to 20 m^{-1}, of which only the branches n = 0 and n = 1 are shown. An infinite number of branches accumulate immediately to the left of $k = 20\ m^{-1}$. The presence of these branches shows that the pressureless plasma is unstable, but the value of the quantity R - L by itself has no physical significance in this region†). So, we do not have transitions from infinite instability to infinite stability for the values of k where R blows up, but the whole region from $k = 18.4\ m^{-1}$ (the first point of intersection of $L(\beta = 0)$ with R) to $20\ m^{-1}$ is unstable for $\beta = 0$. For higher values of β this region is somewhat larger and, because of the presence of the branches, the critical value of β with respect to type-I instabilities is difficult to calculate. A comparison with $R(\alpha = -24.6)$ of Fig. 6 shows, however, that the critical value of β is certainly not 20%, which according to Eq. (100) could be reached by an optimal choice of the force-free field (for α = constant = $-24.6\ m^{-1}$: $\beta_{crit} = 16\%$). The assumption of Ref. 24 that "a constant-pitch region outside the plasma column is almost optimal for stability" turns out to be valid only if $\mu r_o \ll 1$ and if the instabilities of the pressureless plasma are neglected. Therefore, Fig. 3 of Ref. 24, where β_{crit} is plotted as a function of μr_o according to the relation (100), needs a correction for higher values of μr_o and is only applicable with respect to type-I instabilities. On the other hand, the picture provides the correct relation for the maximum allowable value of β for force-free fields in general, for example force-free fields with α = constant with properly chosen value of α.

The instabilities of the constant-pitch force-free field show up in the situation for which Suydam's criterion degenerates ($p' = 0$, $\mu' = 0$) and so one expects the growth rates to be

†) In the region of k-values where branches are absent the value of R - L is proportional to the growth rate of incompressible type-I instabilities according to Eq. (107).

small. Using again the local approximation we can calculate the growth rates in order to see whether this is justified. We start from Eq. (122) and drop terms with p'. Just as in Sec. 7.2 one can look either to the growth rates of the quasi-interchanges if $F = k_{//}B \rightarrow 0$, or to the growth rates of the most dangerous quasi-interchanges. In the first case, if $k_{//}r \rightarrow 0$, two modes are again obtained from Eq. (122): a stable type-1 mode with $\rho\omega_1^2$ given by Eq. (125) with p' = 0, and an unstable type-2 mode. The growth rate of the latter is given by a higher order term which is neglected in Eq. (126):

$$\rho\omega_2^2 = -\frac{mrF^3/\mu_o}{(m^2+k^2r^2)B_\theta} \approx \frac{B_z}{B}\left(\frac{k_{//}}{k}\right)^3 k^2B^2/\mu_o \,, \tag{145}$$

where Eq. (124) has been used for the approximation on the right-hand side. This growth rate is by an order $k_{//}/k$ smaller than the growth rate of the type-2 quasi-interchanges in the presence of a pressure gradient (Eq. (126)). In accordance with the stability diagram of Fig. 8 instability turns up only if $k_{//} < 0$, in contrast to the situation in the presence of a pressure gradient. We notice that both the property of the solutions of the marginal equation of motion to oscillate with an infinite rapidity if $k_{//} \rightarrow 0$ and the resulting accumulation of an infinite set of branches in the stability diagram of Fig. 8 to the left of $k = 20\ m^{-1}$, have no direct consequence for the growth rates. On the contrary, if $k_{//} \rightarrow 0$ also the growth rates tend to zero according to Eq. (145)! This one example illustrates the weakness of the marginal-stability analysis and, therefore, of the energy principle: one obtains stability criteria, but no information about the danger of the instabilities. The stability criteria can even be rather misleading, as is evident from Fig. 8.

The problem dealing with the $k_{//}$ value associated with the maximum growth rate is more important. This maximum turns out to occur for $k_{//}r = O(1/(kr))$, so that the terms $O(k_{//}^2r^2)$ and $O(k_{//}/k)$ in Eq. (122) become of the same order. Only the terms $O(k_{//}^2/k^2)$ and those containing p' then can be dropped in Eq. (122). By a differentiation of the solutions of Eq. (122) with respect to the variable F we then obtain the following values of $F = k_{//} B$ for which $\rho\omega^2$ has an extremum:

$$F = \frac{\gamma p + B^2/\mu_o}{2\gamma p + B^2/\mu_o} \frac{2k^2 m r B_\theta}{(q^2r^2+m^2+k^2r^2)(m^2+k^2r^2)} , \tag{146}$$

$$F^2 = 0, \tag{147}$$

$$F = \frac{3k^2 m r B_\theta}{(q^2r^2+m^2+k^2r^2)(m^2+k^2r^2)} . \tag{148}$$

The first value of F corresponds to the minimum of the curve of $\rho\omega^2$ as a function of $k_{//}$ for the stable modes. The second value of F corresponds to the inflection point in the curve of $\rho\omega^2$ as a function of $k_{//}$ for the unstable modes. The third value of F is the required one for which the growth rate of the unstable modes has a maximum. From Eq. (148) it follows that, for these modes, $k_{//} r \sim O(1/(kr))$. Substitution of F from Eq. (148) in the solution $\rho\omega^2$ of Eq. (122) gives the growth rate of the most dangerous quasi-interchanges:

$$\rho\omega^2 = \frac{-mr}{4\mu_o B_\theta (m^2+k^2r^2)} \left[\frac{3k^2 m r B_\theta}{(q^2r^2+m^2+k^2r^2)(m^2+k^2r^2)} \right]^3$$

$$\approx - \left(\frac{m^2+k^2r^2}{q^2r^2+m^2+k^2r^2} \right)^3 \frac{27 B_z^4 B_\theta^{12}}{4\mu_o k^4 r^6 B^{14}} , \tag{149}$$

where Eq. (124) has been used to obtain the approximation on the right-hand side. The maximum of this growth rate, obtained when $q^2r^2 << k^2r^2$, follows from

$$\rho\omega^2 = - \frac{27 B_z^4 B_\theta^{12}}{4\mu_o k^4 r^6 B^{14}} = - \frac{27(\mu r)^{12} B^2}{4\mu_o k^4 r^6 (1+\mu^2 r^2)^{10}} . \tag{150}$$

This expression for the growth rates of quasi-interchanges in a constant-pitch magnetic field has some unpleasant features which makes it of little practical use. First of all, the expression is strongly dependent on the local value of μr. If, for example, $\mu r \sim 0.1$ the growth rate is negligible for practical values of ρ and B, whereas for $\mu r \sim 1$ the growth rate can be rather large. Secondly, it follows from the presence of k^4 in the denominator of Eq. (150) that the growth rates are largest for small k, so that the local approximation here breaks down completely (see also the end of Sec. 7.3). Thirdly, the derivation of Eq. (150) strongly depends on the assumptions $p' = 0$ and $\mu' = 0$.

The latter point concerns the properties of the model itself and, therefore, it is not very useful to try to remove the first two points of criticism by a more refined calculation of the growth rates (for example numerically). Van der Laan's model is very successful in explaining the gross stability of the screw pinch, i.e. the absence of kinks (type-I instabilities), but the situation with respect to the instabilities of the force-free outer region compells us to make a minor modification of the model. Two modifications are possible, depending on whether one finds experimentally instabilities for which one wishes to calculate the growth rates, or whether one wishes to design a theoretically stable model which one tries to realize experimentally:

1) One drops the restriction $p' = 0$ and considers an outer region of constant pitch, where a small negative pressure gradient gives rise to instabilities for which the growth rates are given by Eq. (133) in local approximation. The small pressure gradient will not result in a strong deviation of the field from a force-free field, so that the conclusions with respect to type-I instabilities are conserved.

2) One drops the restriction $\mu' = 0$ and considers other force-free fields, e.g. with α = constant. A comparison of Fig. 6 with Fig. 8 shows that the small shear of the field with $\alpha = \text{constant} = -24.6\ m^{-1}$ is sufficient to remove the in-

stabilities of the pressureless plasma. Here one should remember that the force-free fields with α = constant also yield results which are strongly dependent on the assumption $p' = 0$ if μ' vanishes anywhere in the interval. This happens, for example, in the case $\alpha = -24.6\ m^{-1}$. Because this field has a mimimum in the function $\mu = \mu(r)$ (see Fig. 5) Suydam-type instabilities will develop at the position of the minimum, even in the presence of a very small pressure gradient. For these instabilities Eq. (133) is again a good approximation for the growth rates. On the other hand, force-free fields of constant α having a larger value of $-\alpha$ are more favourable with respect to Suydam's modes, because these fields satisfy Suydam's criterion more than marginally. Moreover, in Sec. 6.3 we came to the conclusion that these field configurations (e.g. $\alpha = -50\ m^{-1}$) are also the most favourable ones with respect to type-I and type-II instabilities.

Summarizing, we consider the instabilities of a constant-pitch force-free magnetic field as a limiting case of Suydam-type instabilities, which can easily be removed by replacing the constant-pitch force-free field by a force-free field of constant α.

7.5. Constant-pitch magnetic fields in the presence of a pressure gradient (Alfvén's model)

One special model with a constant-pitch magnetic field and a negative pressure gradient (therefore, not force-free!) is known which can be solved analytically without making use of the local approximation. This is Alfvén's model, for which the stability was analyzed by Dungey and Loughhead[42]). The paper of Dungey and Loughhead was one of the first stability calculations in magnetohydrodynamics by means of the normal-mode analysis. Unfortunately, this paper was criticized by Tayler[29]) because of an incorrect use of the principle of exchange of stabilities. We shall see, however, that this criticism was unjustified. Since Alfvén's model provides some additional insight in the stability of constant-pitch fields, we shall consider it briefly. Moreover, we shall use this model to replace the rather unrealistic rectangular pressure profile of the sharp-pinch model by a more diffuse profile.

The model of Alfvén was introduced as a possible mechanism for the generation of cosmic magnetic fields[9,62,63]. The idea is that in a plasma tube with an initial longitudinal magnetic field (B_z), an azimuthal field (B_θ) will develop as a result of motion of the plasma. This motion may be, e.g., a differential rotation due to convection in a rotating plasma. The resulting motion of the magnetic-field lines is analogous to the twisting of an elastic string. This analogy suggests that the magnetic field as a whole will form a loop if the twist of the field lines exceeds a certain limit. Alfvén showed that such a loop may cause an amplification of the initial B_z-field. If this process is repeated several times a longitudinal field B_{zf} is finally created having an order of magnitude given by

$$\frac{B_{zf}^2}{2\mu_o} \sim \frac{\frac{1}{2}\rho v^2}{(\mu_c \hat{R})^2} , \tag{151}$$

where v is the differential velocity of the plasma, $\hat{R}$ the radius of the plasma tube, and μ_c the critical value of μ (which is a measure of the twist of the field lines) above which loop formation sets in.

Alfvén's model will be considered as static as far as stability is concerned. In particular, the tube with the magnetic-field lines is given a uniform twist in this model. Furthermore, the longitudinal current producing the B_θ-field is also assumed uniform, so that B_θ varies linearly with r. Consequently, the various equilibrium quantities are given by

$$\rho = \text{constant}, \quad j_z = \text{constant}, \quad j_\theta = 0 ,$$

$$B_\theta = Ar, \quad B_z = \text{constant} , \tag{152}$$

$$p(r) = p(0) - A^2 r^2/\mu_o .$$

The pressure decreases parabolically to zero at $r = \hat{R}$, where the tube of field lines is assumed to be bounded. Some material pressure must balance the magnetic pressure there (the origin of this material pressure, however, is not discussed in Alfvén's papers).

Dungey and Loughhead investigated the stability of this model describing loop formation as the simultaneous presence of an $m = 1$ and an $m = -1$ instability. The criterion for loop formation according to this definition is: $\mu\hat{R} > 2$. This result was found earlier by Lundquist[11]) for the instability of a twisted magnetic field with respect to a specific type of displacement. Substitution of the critical value of $\mu\hat{R}$ from this criterion in Eq. (151) yields a reasonable value of B_{zf}. Alfvén used the result of Dungey and Loughhead in his later papers[9,63]) without noticing, however, that the criterion for loop formation gives a finite value of $\mu\hat{R}$ indeed, but that Dungey and Loughhead also found that other instabilities arise in Alfvén's model for every value of μ and, therefore, already for infinitesimal small μ (see also Ref. 64). By now these instabilities are very well known to us: they are, in the local approximation, the quasi-interchange instabilities which are present in a constant-pitch field in the presence of a negative pressure gradient for k-values in the neighbourhood of $k = -\mu m$. Therefore, the mechanism proposed by Alfvén for the generation of cosmic magnetic fields cannot work in the described way, if the tube of field lines is given a uniform twist. If one wishes to conserve the essential properties of Alfvén's model for the generation of cosmic magnetic fields it will be necessary either to take into account the development of the quasi-interchange instabilities during their non-linear phase or to drop the restriction of uniform twist and to consider fields which are twisted such that Suydam's criterion is satisfied.

The analysis of Dungey and Loughhead is restricted to incompressible perturbations of the equilibrium fixed by the equations (152). The required equation of motion for incompressible perturbations can be derived heuristically from

Eq. (57) by taking the limit $\gamma \to \infty$. In this way one finds the equation derived by Freidberg[41]). Substituting the quantities of Eq. (152) this equation proves to read

$$\left[\frac{\omega^2\rho-F^2/\mu_o}{m^2+k^2r^2}\, r(r\xi_r)'\right]' - \left[\omega^2\rho-F^2/\mu_o + \frac{4k^2mr^2AF/\mu_o}{(m^2+k^2r^2)^2} - \frac{4k^2r^2A^2F^2/\mu_o^2}{(m^2+k^2r^2)(\omega^2\rho-F^2/\mu_o)}\right]\xi_r=0 \ , \qquad (153)$$

where $F = mA + kB_z = \text{constant}$. (154)

Using the expressions of Dungey and Loughhead the solution of this equation that satisfies the boundary condition at r = 0 can be written:

$$r\xi_r = C\left[k^*r\ I_m'(k^*r) - \frac{2AmF/\mu_o}{\omega^2\rho-F^2/\mu_o}\ I_m(k^*r)\right] \ , \qquad (155)$$

where

$$k^* = k\sqrt{1 - \frac{4A^2F^2/\mu_o^2}{(\rho\omega^2-F^2/\mu_o)^2}} \ . \qquad (156)$$

The solution of the marginal equation of motion contains Bessel functions of the form $J_m(kr\sqrt{4A^2/F^2-1})$ and it is obvious that for $F \to 0$ this solution oscillates with an infinite rapidity, thus excluding stability of this model (as known from Secs. 7.1 and 7.2). Here, as opposed to the constant-pitch force-free field, instabilities arise for k-values on either side of $k = -\mu m$. If the plasma is bounded by a metallic wall at $r = r_1$ one obtains from Eq. (155) the growth rates of the instabilities from

$$r_1\xi_r(r_1) = 0 \ . \qquad (157)$$

Of course, for astrophysical plasmas the assumption of a metallic wall is undesirable. In those cases, instead of Eq. (157), one should impose the condition that $\xi_r(r)$ vanishes at infinity.

Then, the following discussion applies without change. Equation (157) can only be satisfied if $r\xi_r$ contains oscillating Bessel functions, so that the next inequality must hold:

$$4A^2F^2/\mu_o^2 \;>\; (\rho\omega^2 - F^2/\mu_o)^2 \;.$$

For $F \to 0$ this implies $-\rho\omega^2 < 2|A||F|/\mu_o$ and it follows that the growth rate tends to zero if $F \to 0$ (just as in the constant-pitch force-free field). This is in agreement with the expression (127b) for the local instabilities. In fact, equation (127b) applies to Alfvén's model because $p' + 2B_\theta^2/(\mu_o r) = 0$, so that p' exactly has the value for which incompressible type-1 and type-2 modes transform into each other according to the limit $\gamma \to 0$ for the expression (127a). For $qr \ll kr$ and $\gamma \to \infty$ Eq. (127) provides the growth rates of the local quasi-interchanges in the limit $F \to 0$:

$$\rho\omega^2 = \pm \frac{2B_\theta^2/\mu_o}{rB} F \;. \qquad (158)$$

This expression shows that instabilities arise for k-values on either side of $k = -\mu m$. Owing to the property that the modes change their character when F passes through zero (the stable mode becomes unstable and vice versa) the curve of $\rho\omega^2$ as a function of k for the unstable modes exhibits a typical break (discontinuous tangent). This break was noticed previously by Tayler[53]), who remarked that this effect is the result of the assumption of incompressibility. This is only true in so far as the break vanishes for Alfvén's model if compressible perturbations are considered, since then the condition (127a) is no longer satisfied (so that Eq. (125) or (126) gives the growth rate in the limit $F \to 0$); on the other hand, the break remains present for the compressible perturbations of other equilibria satisfying this condition.

An important limitation for the determination of the growth rates from the equations (155) and (157) is the restriction to incompressible modes. As a matter of fact, it appears from the local approximation that the growth rates of the compressible modes for constant-pitch fields are larger than those of the incompressible ones. This can be verified from an expansion of Eq. (132) in orders of $1/\gamma$. The zero order expression (133) gives the growth rates of the incompressible modes and the next order term, accounting for the influence of the compressibility on the growth rate, is negative. Hence, in this case compressibility destabilizes.

Tayler[53]) used Alfvén's model and the solutions of Eq. (155) for a model of a diffuse pinch consisting of an inner region ($r < r_o$) with a uniform current distribution and a parabolic pressure profile and an outer region ($r > r_o$ and extending up to ∞) with a vacuum magnetic field. We shall modify this model in the same way as that of Kruskal and Tuck[21]) (Chapter 6), viz. by replacing the vacuum outer region by a force-free field bounded by a conducting wall at $r = r_1$. Just as in Sec. 6.4, the characteristic equation for this configuration can be derived by starting from the boundary conditions (7)-(10). Performing the substitution $-\gamma p \nabla . \underset{\sim}{\xi}^p \rightarrow p_{1L}$ for the incompressible inner region we obtain Eq. (105) with a modified left-hand side:

$$-A^2/\mu_o + \frac{2mAF/\mu_o}{m^2+k^2r^2} + \frac{r(\omega^2\rho^p - F^2/\mu_o)}{m^2+k^2r^2}\,\frac{(r\xi_r^p)'}{r\xi_r^p}$$

$$= \frac{B_\theta^{t\,2}}{\mu_o r^2} - \frac{2kB_\theta^t}{\mu_o r^2}\,\frac{(mB_z^t/r - kB_\theta^t)B^{t\,2}/\mu_o}{\omega^2\rho^t - (m^2/r^2+k^2)B^{t\,2}/\mu_o}$$

$$- \frac{\{\omega^2\rho^t - (mB_\theta^t/r + kB_z^t)^2/\mu_o\}B^{t\,2}/\mu_o}{\omega^2\rho^t - (m^2/r^2+k^2)B^{t\,2}/\mu_o}\,\frac{1}{r}\,\frac{(r\xi_r^t)'}{r\xi_r^t}\,,\ (r=r_o) \qquad (159)$$

where A is given by Eq. (152), F by Eq. (154), $r\xi_r^p$ by Eq. (155), and $r\xi_r^t$ by the solution corresponding to the solutions rQ_r^t of Eq. (84), satisfying the usual boundary condition $r\xi_r^t$ "small" at $r = r_1^*$ (or zero at $r = r_1$).

Neglecting the density of the outer region (as done in Sec. 6.4; however, observe the remark made there with respect to type-III instabilities) we obtain the following characteristic equation†) replacing Eq. (106):

$$\frac{(\omega^2\rho^p - F^2/\mu_o)(k^{*2}/k^2)I_m(k^*r)}{k^*rI_m'(k^*r) - \dfrac{2AmF/\mu_o}{\omega^2\rho^p - F^2/\mu_o} I_m(k^*r)} - \frac{A^2}{\mu_o}$$

$$= -\frac{B_o^{t^2}}{\mu_o r}\left[\frac{\mu^{t^2} r}{1+\mu^{t^2} r^2} + \frac{\alpha r(k+\mu^t m)(k\mu^t r^2 - m)}{(m^2+k^2r^2)(1+\mu^{t^2} r^2)} + \frac{(k+\mu^t m)^2 r}{(m^2+k^2r^2)(1+\mu^{t^2} r^2)}\,\frac{(rQ_r^t)'}{rQ_r^t}\right]$$

$$\equiv -\frac{B_o^{t^2}}{\mu_o r} R \,, \qquad (160)$$

where R is the known expression for force-free fields (Eq.(94)). The following stability criterion of the form L > R results from Eq. (160) after the substitution $\omega^2 = 0$ in the left-hand side:

$$L \equiv \frac{B_o^{p^2}}{rB_o^{t^2}}\left[\frac{r^2\{(k+\mu^p m)^2 - 4\mu^{p^2}\}}{1+\mu^{p^2} r^2}\;\frac{I_m(k_o^* r)}{k_o^* rI_m'(k_o^* r) + \dfrac{2\mu^p m}{k+\mu^p m} I_m(k_o^* r)} + \frac{\mu^{p^2} r^2}{1+\mu^{p^2} r^2}\right]$$

$$> \frac{\mu^{t^2} r}{1+\mu^{t^2} r^2} + \frac{\alpha r(k+\mu^t m)(k\mu^t r^2 - m)}{(m^2+k^2r^2)(1+\mu^{t^2} r^2)} + \frac{(k+\mu^t m)^2 r}{(m^2+k^2r^2)(1+\mu^{t^2} r^2)}\,\frac{(rQ_r^t)'}{rQ_r^t} \equiv R, \qquad (161)$$

where $\mu^p = A/B_z$ and $k_o^* = k\sqrt{1 - \dfrac{4\mu^{p^2}}{(k+\mu^p m)^2}}$.

Here too, like in Sec. 7.4, the criterion L > R is not sufficient for stability. In fact, zero points of $\xi_r^p(\omega^2=0)$ may occur in the open interval $(0, r_o)$ in addition to the zero points of ξ_r^t $(\omega^2=0)$ in the open interval (r_o, r_1), so that the function L = L(k) can have branches in the stability diagram

†) This equation was derived earlier by Dr. W. Schuurman (unpublished).

just as well as the function $R = R(k)$. Therefore, the stability criterion $L > R$ should be supplemented with the conditions that $\xi_r^p(\omega^2=0)$ and $\xi_r^t(\omega^2=0)$ should have no zero points in the open intervals $(0,r_o)$ and (r_o,r_1), respectively. Notice that, consequently, the division of the instabilities in type-I, type-II, and type-III instabilities, made in Sec. 6.3, must undergo a further splitting. Type-I instabilities can now be divided into two kinds, depending on whether they are connected with a violation of $L > R$ or with oscillations of the function $\xi_r^p(\omega^2=0)$.

Dungey and Loughhead[42] implicitly applied this enlarged, correct stability criterion in their treatment of the pure Alfvén model. In this model no magnetic field is assumed in the outer region ($B_o^t = 0$), so that the relevant criterion follows from Eq. (161) by putting $R = 0$. The corresponding equation $L = 0$ (the dispersion equation for this model in which $\omega^2 = 0$ is substituted) was used by Dungey and Loughhead as a starting point of their stability calculations. For given values of μ^p, k, and m this equation has a solution for a number of values of the radius $\hat{R}$ of the plasma tube, "the smallest of which will be called $\hat{R}_o$" (quoting the authors). Next, it is stated that "the condition for instability is expressed by the inequality $\hat{R} > \hat{R}_o$". However, the determination of the smallest value of $\hat{R}$ is fully equivalent to the determination of the solution $\xi_r^p(\omega^2=0)$ of the marginal equation of motion that has no zero points (n=0) on the open interval $(0,\hat{R})$. This shows that Dungey and Loughhead took the required care in the application of the principle of exchange of stabilities and that Tayler's criticism with respect to this paper is unjustified. Tayler illustrates his criticism with the special case $B_z = 0$, showing that the m = 1 modes are always unstable. Indeed, from Fig. 2 of Dungey and Loughhead's paper it appears that $\hat{R}_o = 0$ for $B_z = 0$, $m = 1$, and arbitrary k, so that the same conclusion is reached!

The inequality (161) contains, apart from the parameters k and m characterizing the perturbation, the following equilibrium quantities which can be chosen arbitrarily: r_o, r_1, μ^p, μ^t,

B_o^p/B_o^t, and possibly†) α. Thus, comparing with the model treated in Chapter 6, one additional parameter enters, viz. μ^p. The model of Chapter 6 fits in the present treatment by the choice $\mu^p = 0$, and that of Tayler[53]) by taking $\alpha = 0$. The parameter B_o^p/B_o^t fixes the jump of the magnetic field at the plasma boundary ($r = r_o$) and, therefore, also the jump of the pressure by means of the equilibrium condition (4). The functions p(r) and $\beta(r)$ are then fixed according to

$$p = \frac{B_o^{p^2}}{2\mu_o}\left[\frac{B_o^{t^2}}{B_o^{p^2}} - 1 + \frac{2\mu^{p^2} r_o^2(1-r^2/r_o^2)}{1+\mu^{p^2} r_o^2}\right], \tag{162}$$

$$\beta = \frac{p}{p+B^2/(2\mu_o)} = 1 - \frac{1+\mu^{p^2} r^2}{B_o^{t^2}/B_o^{p^2}(1+\mu^{p^2} r_o^2)+\mu^{p^2} r_o^2(1-r^2/r_o^2)}. \tag{163}$$

A completely diffuse model is obtained by taking $\mu^p = \mu^t$ and $B_o^p/B_o^t = 1$. In this case the pressure decreases to zero at $r = r_o$ and surface currents producing a jump in the magnetic-field components at $r = r_o$ are absent. This model seems to represent a good approximation to many diffuse profiles found experimentally. On the other hand, theoretically, it has the disadvantage to be unstable for a large region of k-values. Moreover, the singular behaviour due to the modes of the inner plasma cannot be distinguished in the stability diagram from the singular behaviour due to the modes of the force-free region.

In view of these remarks it looks most instructive to treat an example of a mixed sharp-diffuse pinch, displaying a jump of the pressure and of the field components at $r = r_o$. In order to make the model maximally pathological we shall also

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†) The parameter $\alpha(r_o)$ is fixed if μ^t = constant and it can be chosen arbitrarily if α = constant. In the other cases, where $\alpha \neq$ constant and $\mu^t \neq$ constant, the expression R depends, in addition to the parameters $\alpha(r_o)$ and $\mu^t(r_o)$, also on the complete profile $\alpha = \alpha(r)$ via Q_r^t.

assume a constant pitch for the field lines in the outer region. For this model p, B_θ, and B_z are shown as a function of r in Fig. 9 for the following choice of the parameters:

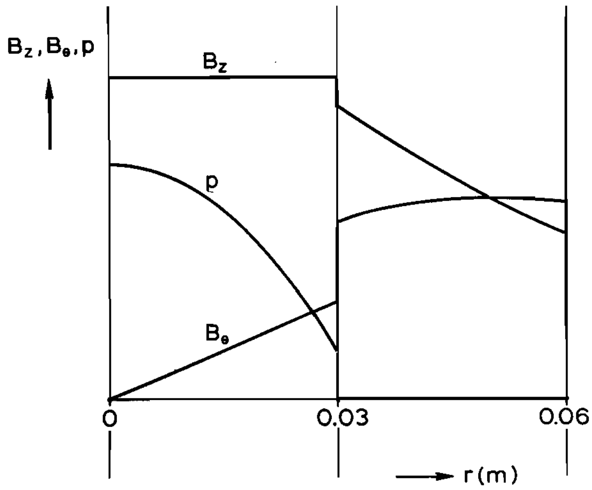

Fig. 9 Configuration for μ^p and μ^t = constant, r_o = 0.03 m, r_1 = 0.06 m, $\mu^p = -10\ m^{-1}$, $\mu^t = -20\ m^{-1}$, $B_o^p/B_o^t = 0.98$.

$\mu^p = -10\ m^{-1}$, $B_o^p/B_o^t = 0.98$, and the other parameters as in the model of Sec. 7.4. The stability diagram for m = 1 is given in Fig. 10 (L dashed, R drawn). The value of the parameter B_o^p/B_o^t has been chosen such that L = R for $k = 20\ m^{-1}$ (the interchange value of k for the outer region). According to Eq. (163) the above value of B_o^p/B_o^t gives a β decreasing from β(0) = 18.3% to $\beta(r_o)$ = 4%, i.e. on the average considerably below the value of 20% which was allowed for the stability of the sharp-pinch model of Chapter 6.

Figure 10 displays three unstable regions: 1) the region of k-values around $k = -\mu^p = 10\ m^{-1}$, 2) the region of k-values from $15\ m^{-1}$ to $18\ m^{-1}$, 3) the limited region of k-values to the left of $k = -\mu^t = 20\ m^{-1}$. The instability of the first and the third region is caused by the fact that the pitch of the magnetic-field lines is constant. A comparison with Fig. 6 shows that the instability of the second re-

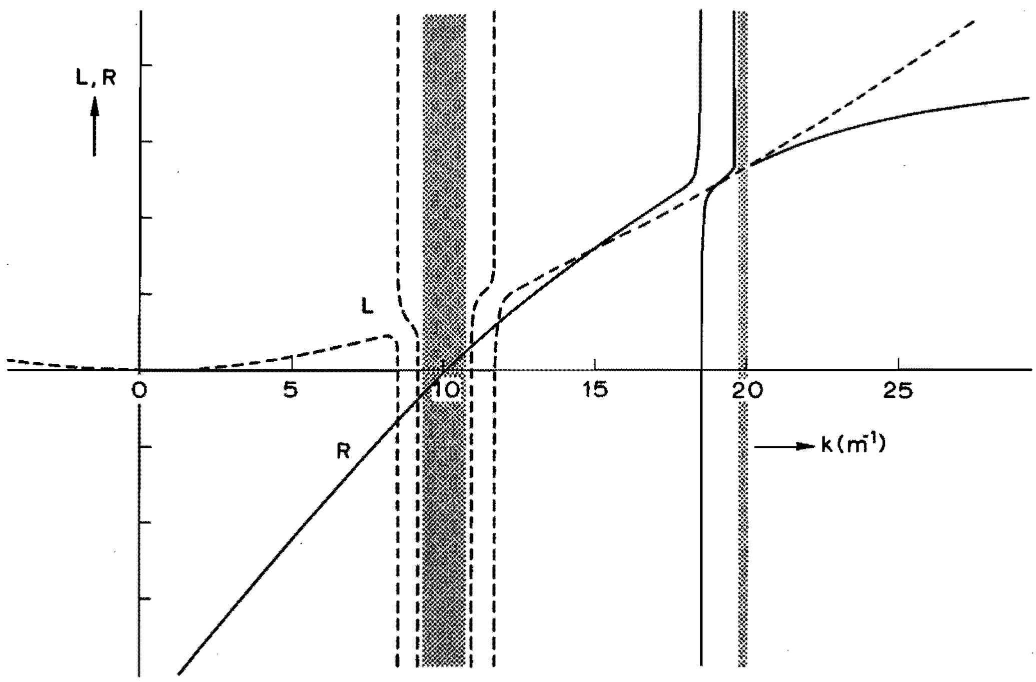

Fig. 10 Stability diagram for μ^p and μ^t = constant, m = 1, r_o = 0.03 m, r_1 = 0.06 m, μ^p = -10 m^{-1}, μ^t = -20 m^{-1}, B_o^p/B_o^t = 0.98.

gion is the result of a too small current density in the force-free region (for increasing values of $-\alpha$ the curve R rotates counterclockwise around the point with abscissa k = 20 m^{-1}) and also that the instability of the second and third region of k-values can be simply removed by replacing the constant-pitch force-free field by a force-free field with a properly chosen value of α = constant (e.g. α = - 50 m^{-1}). Thus, only the instability of the first unstable region remains and this suggests that this instability will also be removed if the field and pressure distribution of the inner region of Fig. 9 are modified in such a way as to satisfy Suydam's criterion. (It is not difficult to see how this must be done). This assumption is justified by the fact that the pressure and field profiles thus obtained look very similar to the diffuse profiles observed in Zeta during the period of improved stability[65]; it was proved numerically that these profiles are completely stable according to Newcomb's stability criteria. The construction of the high-beta toroidal-pinch experiment of Culham[66] is based on this fact.

Finally, we want to emphasize that the choice of the parameters of the examples of Chapters 6 and 7 was such ($\mu r \sim 1$) that the optimal choice of the force-free field with respect to type-I and type-II instabilities gave a field which strongly deviates from a constant-pitch force-free field. In the present screw-pinch experiments at Jutphaas the value of $\mu r \sim 0.1$, so that the optimal choice of the force-free field deviates little from a constant-pitch field. Then, one should expect quasi-interchange instabilities of the constant-pitch force-free magnetic field to play an unfavourable role. However, for these low values of μr it becomes important to remember that the actual experimental configuration is not an infinitely long cylinder but a torus (see Appendix III).

CHAPTER 8

CONCLUSIONS

In this paper some problems of the stability theory of plasmas were treated while starting from the equations of ideal magnetohydrodynamics, i.e. the equations of a fluid of infinite conductivity in the presence of a magnetic field. Two classical examples were considered, viz. a plane plasma layer in a gravitational field and a cylindrical plasma confined by means of the pinch effect. For these cases the linearized magnetohydrodynamic equations were reduced to one differential equation for the component of the displacement vector $\underset{\sim}{\xi}$ in the direction of inhomogeneity. This equation (called "the equation of motion") contains all information about waves and instabilities for the mentioned cases. The stability was investigated by means of the marginal equation of motion which follows from the general equation of motion by substituting $\omega^2 = 0$, where ω fixes the time-dependence of the normal modes. It was shown that this equation is equivalent to the Euler-Lagrange equation following from the minimization of the energy. As a consequence, the singular points of the marginal equation of motion have the same influence as they have in the energy principle and the stability criteria for the marginal-stability analysis can be represented analogous to Newcomb's stability criterion for the diffuse pinch.

Using the thus formulated stability criteria the stability was investigated for a plane plasma layer in a gravity field, supported from below by a force-free field of constant α. The gravitational instabilities which can arise are of two types: type-I instabilities involving a displacement of the plasma as a whole, and type-II instabilities which are more or less localized in the thin surface layer which separates the plasma from the force-free region. Both types of instabilities occur in a layer which is supported by a vacuum field ($\alpha = 0$) and they can be stabilized by a force-free field with a properly chosen value of α = constant. Here, an increase of $|\alpha|$ (higher current density in the force-free region) and the influence of the singular points cooperate favourably.

The same favourable situation arises in a similar model for the screw pinch, viz. a homogeneous plasma cylinder surrounded by a force-free field of constant α. Although the equations and stability criteria show a strong analogy with the plane case, one additional type of instabilities is possible in this case, namely kink instabilities of the force-free region itself. It was shown how these type-III instabilities can be stabilized by a proper choice of α, just as well as type-I and type-II instabilities. This model of a sharp-screw pinch thus proved to be completely magnetohydrodynamically stable up to a β of the order of 20%. This conclusion was reached while continuing a line of research started by Van der Laan[23]) and by Schuurman, Bobeldijk, and De Vries[24]).

It was shown that constant-pitch magnetic fields have some peculiarities which are partly of a physical and partly of a mathematical nature. The stability criteria for constant-pitch fields (not necessarily force-free) show a discontinuity for values of the wavenumber in the vicinity of the interchange value (wave vector perpendicular to the applied magnetic field). Contrary to Ware's results, we showed that from the two obtained stability criteria (viz. for $k_{//} \to 0$ and for $k_{//} = 0$, respectively) only one is physically significant, viz. Suydam's crterion in the limit $\mu' \to 0$ (being the criterion for $k_{//} \to 0$).

This latter criterion, viz. simply $p' > 0$, is nearly always violated for constant-pitch fields in experimental conditions. The growth rates of the instabilities then arising (quasi-interchange instabilities) were calculated in a local approximation and it was shown that they may be rather large. Even in the idealized case of a constant-pitch force-free field, where $p' = 0$, instabilities arise, which can be considered as limiting cases of Suydam-type instabilities. However, these remaining instabilities can be removed rather simply by replacing the constant-pitch force-free field by a force-free field of constant α. Finally, Alfvén's model was discussed. This model has a parabolic pressure profile and a constant-pitch magnetic field and, as a consequence, it involves quasi-interchange instabilities. Because of the presence of the latter the mechanism for the generation of cosmic magnetic fields, proposed by Alfvén on the basis of this model, cannot work. However, for laboratory plasmas a stable model can be obtained by modifying the pressure profile and the magnetic field such that Suydam's criterion is satisfied, and possibly by surrounding the whole configuration by a force-free field of constant α.†)

Summarizing: the sharp-pinch model of a dense plasma surrounded by a force-free magnetic field of constant α is com-

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†) Here, we want to emphasize that the basic difference between astrophysical and laboratory plasmas, as far as magnetohydrodynamic stability is concerned, results from the presence of a conducting wall for the latter situations. Throughout the present paper force-free fields only acted stabilizing when accompanied by such a wall. It was shown by Anzer[67,68]) that force-free fields which smoothly transform into vacuum fields extending up to infinity are necessarily unstable. Hence, the presently discussed stabilizing role of force-free fields is only relevant for laboratory plasmas. However, this does not exclude the fact that force-free fields, even if unstable, may be important in astrophysical situations.

pletely magnetohydrodynamically stable if α is properly chosen. The model contains enough free parameters to serve as a useful approximation for experimental situations, e.g. the high-beta toroidal-pinch experiment of Culham ($\mu r \sim 1$ and a force-free magnetic field with large shear) and the screw pinch of Jutphaas ($\mu r \sim 0.1$ and a force-free magnetic field with small shear).

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A P P E N D I C E S

Some mathematical and physical approximations, applied in the text, will be justified or replaced in the following appendices. First of all, the treatment of the marginal equation of motion for the plane incompressible layer (Sec. 3.1) will be given in a mathematically more rigorous form. Next, the main physical approximations, viz. the neglect of resistivity and toroidicity, will be dropped partly and it will be investigated to what extent the obtained solutions are relevant for a resistive plasma and for a toroidal configuration.

APPENDIX I

Exact treatment of the marginal equation of motion in the neighbourhood of a singular point

In this appendix the influence on the stability analysis of a singular point ($F = \underset{\sim}{k}.\underset{\sim}{B} = 0$) of the marginal equation of motion will be treated more rigorously than in Sec. 3.1. Here, we shall use the property that, in the neighbourhood of $F = 0$, the complete equation of motion for the plane incompressible plasma layer can be solved in terms of known special functions, so that the joining conditions for the solutions in the three regions I, II, and III can be obtained exactly (see also Ref. 33). In the neighbourhood of a singular point $y = y_s$ the solutions of the marginal equation of motion (18)

$$\{(F^2/\mu_o)\xi'_{yo}\}' - k^2(F^2/\mu_o - \rho' g)\xi_{yo} = 0 \qquad (I.1)$$

can be obtained by introducing the local approximations

$$\rho' g \approx c \quad \text{and} \quad F^2/\mu_o \approx bs^2 , \qquad (I.2)$$

with the constants c and $b = \lambda^2/\mu_o$, while $s = y - y_s$ (see Sec. 3.1). The solutions $s^{-\frac{1}{2}} I_{\nu+\frac{1}{2}}(ks)$ are proportional to s^ν near the singular point, ν being defined as a root of the index equation

$$\nu(\nu + 1) + \bar{c} = 0 ,$$

where

$$\bar{c} = \frac{k^2 c}{b} = \frac{\mu_o \rho' g B^2}{(B_x B'_z - B_z B'_x)^2} . \qquad (I.3)$$

In accordance with Eq. (12) the indices ν_1 and ν_2 of ξ_{yo} are smaller by 1 than the indices n_1 and n_2 of Q_{yo}, as introduced in Sec. 3.1:

$$\nu_{1,2} = -1/2 \pm 1/2\sqrt{1 - 4\bar{c}} = n_{1,2} - 1. \quad \text{(upper sign for } \nu_1) \qquad (I.4)$$

Here and in the following, we assume that "Suydam's" criterion (21) is satisfied, so that $1 - 4\bar{c} > 0$, involving real indices ν_1 and ν_2. Hence, the "small" and the "large" solution of the marginal equation of motion are

$$\xi_s \sim s^{\nu_1} \quad \text{and} \quad \xi_\ell \sim s^{\nu_2} . \qquad (I.5)$$

These solutions can be brought into a dimensionless form by the introduction of a small distance L_ε in the problem, which we define as the distance up to which the approximations (I.2) are justified. This distance is a small fraction ε of the scale length L of inhomogeneities of the magnetic field B and the density $\rho(L_\varepsilon = \varepsilon L)$. The general solution of Eq. (I.1) then becomes as follows in the neighbourhood of $y = y_s$:

$$\xi_{yo} = A\left|\frac{s}{L_\varepsilon}\right|^{\nu_1} + B\left|\frac{s}{L_\varepsilon}\right|^{\nu_2} . \tag{I.6}$$

This expression is reliable in the regions to the left and to the right of the singular point in so far as the approximations (I.2) hold, i.e. up to distances of the order L_ε. The basic problem then concerns the determination of the constants A^+ and B^+ for a solution to the right of the singular point, once A^- and B^- for the solution to the left have been given, and vice versa.

Rather than trying to find the joining conditions from analytic continuation of the solutions we take a physical point of view and use the additional information that Eq. (I.1) is obtained from the complete equation of motion (17), viz.

$$\left[(\omega^2\rho - F^2/\mu_o)\xi_y'\right]' - k^2\left[\omega^2\rho - F^2/\mu_o + \rho' g\right]\xi_y = 0 , \tag{I.7}$$

by taking $\omega^2 = 0$. However, it is known that the limit $\omega^2 \to 0$ gives results different from those obtained when taking $\omega^2 = 0$ right away, if singular points $F = 0$ occur. In a sufficiently small neighbourhood of such a point the term $\omega^2\rho$ will always dominate the term F^2/μ_o, even if $\omega^2\rho$ is very small. Therefore, in the presence of singularities of the marginal equation of motion it is more appropriate to study, instead of Eq. (I.1), Eq. (I.7) in the limit $\omega^2 \to 0$. In a vicinity $|s| \lesssim L_\varepsilon$ around $y = y_s$ the approximations (I.2) hold, while we may also write there

$$-\omega^2\rho \approx a , \tag{I.8}$$

in which a may be considered as a positive constant since we are looking for instabilities. Because a and bs^2 are small in the domain under discussion, we have $a + bs^2 \ll |c|$ and Eq. (I.7) becomes to a good approximation

$$\left[(a + bs^2)\xi_y'\right]' + k^2 c\, \xi_y = 0 \, . \tag{I.9}$$

In terms of the dimensionless quantities $t = \sqrt{b/a}\, s$ and $\bar{c} = k^2 c/b$ this equation can be represented by

$$(1+t^2)\, \frac{d^2\xi_y}{dt^2} + 2t\, \frac{d\xi_y}{dt} + \bar{c}\, \xi_y = 0 \, . \tag{I.10}$$

Here, it should be noticed that $\bar{c} = O(1)$, as follows from the equations (I.3) and (11), and that the regions I, II, and III, which were introduced in Sec. 3.1, correspond to $t \ll 1$, $t \sim 1$, and $t \gg 1$, respectively. Passing to the complex variable $z = it$ and substituting $\bar{c} = -\nu(\nu+1)$, we finally obtain Legendre's differential equation

$$(1-z^2)\, \frac{d^2\xi_y}{dz^2} - 2z\, \frac{d\xi_y}{dz} + \nu(\nu+1)\xi_y = 0 \, . \tag{I.11}$$

The solutions of this equation are the well-known[69]) Legendre functions of the first and second kind, which are exactly suitable for our purpose.

Two independent solutions of Eq. (I.11) are given by the Legendre functions of the first kind:

$$P_\nu(z) = F(-\nu,\ \nu+1;\ 1;\ \tfrac{1}{2} - \tfrac{1}{2} z) \ ,$$

$$P_\nu(-z) = F(-\nu,\ \nu+1;\ 1;\ \tfrac{1}{2} + \tfrac{1}{2} z) \ ,$$

F being the hypergeometric series, which here converges for $|z-1| < 2$ and $|z+1| < 2$, respectively. Both functions thus are well defined along the sections between $z = i\sqrt{3}$ and $z = -i\sqrt{3}$ of the imaginary z-axis, that is, for $|t| < \sqrt{3}$.

Two other independent solutions are the following Legendre functions of the second kind, the power series for each of which converges when $|z| > 1$:

$$Q_\nu(z) = 2^{-\nu-1}\sqrt{\pi}\,\frac{\Gamma(\nu+1)}{\Gamma(\nu+\frac{3}{2})}\,z^{-\nu-1}\;F(\tfrac{1}{2}\nu+1,\ \tfrac{1}{2}\nu+\tfrac{1}{2};\ \nu+\tfrac{3}{2};\ z^{-2})\ ,$$

$$Q_{-\nu-1}(z) = 2^{\nu}\sqrt{\pi}\,\frac{\Gamma(-\nu)}{\Gamma(-\nu+\frac{1}{2})}\,z^{\nu}\quad F(-\tfrac{1}{2}\nu+\tfrac{1}{2},\ -\tfrac{1}{2}\nu;\ -\nu+\tfrac{1}{2};\ z^{-2}).$$

Since $\nu_1 = -\nu_2-1$ and $\nu_2 = -\nu_1-1$, the following relations exist between the Legendre functions: $P_{\nu_1} = P_{-\nu_1-1} = P_{\nu_2}$; $Q_{\nu_1} = Q_{-\nu_2-1}$; $Q_{\nu_2} = Q_{-\nu_1-1}$.

We require real solutions ξ_y which suggests to take the real parts of the preceding solutions. In our case, however, $z = it$, so that the real parts of the Legendre functions of the first kind are not independent: $\mathrm{Re}\{P_\nu(it)\} = \mathrm{Re}\{P_\nu(-it)\}$. Therefore, we take as the second independent function the imaginary part of $P_\nu(it)$, which turns out to be odd: $\mathrm{Im}\{P_\nu(it)\} = -\mathrm{Im}\{P_\nu(-it)\}$. We thus arrive at the following independent real solutions of Eq. (I.10) for $|t| < \sqrt{3}$:

$$\mathrm{Re}\{P_\nu(it)\} = \mathrm{Re}\{F(-\nu_1,\ -\nu_2;\ 1;\ \tfrac{1}{2} - \tfrac{1}{2}\,i|t|)\}\ ,$$

$$\mathrm{Im}\{P_\nu(it)\} = \frac{t}{|t|}\,\mathrm{Im}\{F(-\nu_1,\ -\nu_2;\ 1;\ \tfrac{1}{2} - \tfrac{1}{2}\,i|t|)\}\ , \qquad \text{(I.12)}$$

which can be supplemented, for $|t| > 1$, by the other solutions

$$\mathrm{Re}\{Q_{\nu_2}(it)\}$$

$$=2^{\nu_1}\sqrt{\pi}\,\frac{\Gamma(-\nu_1)}{\Gamma(-\nu_1+\frac{1}{2})}\cos(\tfrac{1}{2}\pi\nu_1)\,|t|^{\nu_1}\,F(-\tfrac{1}{2}\nu_1+\tfrac{1}{2},\ -\tfrac{1}{2}\nu_1;\ -\nu_1+\tfrac{1}{2};\ -|t|^{-2}),$$

$$\mathrm{Re}\{Q_{\nu_1}(it)\}$$

$$=2^{\nu_2}\sqrt{\pi}\,\frac{\Gamma(-\nu_2)}{\Gamma(-\nu_2+\frac{1}{2})}\cos(\tfrac{1}{2}\pi\nu_2)\,|t|^{\nu_2}\,F(-\tfrac{1}{2}\nu_2+\tfrac{1}{2},\ -\tfrac{1}{2}\nu_2;\ -\nu_2+\tfrac{1}{2};\ -|t|^{-2}).$$

(I.13)

Obviously, the latter functions behave, respectively, like $|t|^{\nu_1}$ and $|t|^{\nu_2}$ when $t >> 1$, i.e. like the "small" and the "large" solution of the marginal equation of motion. It should be noticed that the expressions (I.5) for the "small" and "large" solutions are only valid if $s \lesssim L_\varepsilon$, so that t is restricted to $t \lesssim \sqrt{b/a}\, L_\varepsilon$. This does not imply, however, that $t >> 1$ is impossible, because we are discussing small values of a and, finally, we shall even take the limit $a \to 0$, while keeping b and L_ε fixed.

In the present treatment the solutions in the regions I and II, introduced in Sec. 3.1, are given by Eq. (I.12), and those in region III by Eq. (I.13). In view of the existence of an overlapping of the domains of convergence of the corresponding series we can match the two sets of solutions (e.g. at $t = 1.5$) by applying the known[69]) connection formula

$$P_\nu(z) = \frac{tg(\pi\nu)}{\pi} \left[Q_\nu(z) - Q_{-\nu-1}(z) \right] \quad . \tag{I.14}$$

It has yet to be shown that in the limit $\omega^2 \to 0$ (or $a \to 0$) both solutions of the inner region ($|t| < \sqrt{3}$) transform into the "small" solution of the outer region ($|t| > 1$). The solutions (I.12) of the inner region are even and odd respectively; we shall label them ξ^e and ξ^o accordingly. It then follows from Eqs. (I.14) and (I.13) that ξ^e and ξ^o tend to the following asymptotic expressions when $t >> 1$:

$$\xi^e = \mathrm{Re}\{P_\nu(it)\} \to \frac{tg(\pi\nu_2)}{\sqrt{\pi}} \left[2^{\nu_1} \frac{\Gamma(-\nu_1)}{\Gamma(-\nu_1+\frac{1}{2})} \cos(\tfrac{1}{2}\pi\nu_1)|t|^{\nu_1} - 2^{\nu_2} \frac{\Gamma(-\nu_2)}{\Gamma(-\nu_2+\frac{1}{2})} \cos(\tfrac{1}{2}\pi\nu_2)|t|^{\nu_2} \right] , \tag{I.15}$$

$$\xi^o = \mathrm{Im}\{P_\nu(it)\} \to \frac{t}{|t|} \frac{tg(\pi\nu_2)}{\sqrt{\pi}} \left[2^{\nu_1} \frac{\Gamma(-\nu_1)}{\Gamma(-\nu_1+\frac{1}{2})} \sin(\tfrac{1}{2}\pi\nu_1)|t|^{\nu_1} - 2^{\nu_2} \frac{\Gamma(-\nu_2)}{\Gamma(-\nu_2+\frac{1}{2})} \sin(\tfrac{1}{2}\pi\nu_2)|t|^{\nu_2} \right] . \tag{I.16}$$

We next transform back to the variable s, which measures the geometrical distance from the singular point and, therefore, is not affected by the limiting process $a \to 0$. In terms of s the even solution ξ^e may be written in the domain $s > \sqrt{a/b}$ as

$$\xi^e \to A^e\left|\frac{s}{L_\varepsilon}\right|^{\nu_1} + B^e\left|\frac{s}{L_\varepsilon}\right|^{\nu_2} ,$$

where

$$\frac{A^e}{B^e} = 2^{\nu_1-\nu_2}\,\frac{\Gamma(-\nu_1)\Gamma(-\nu_2+\frac{1}{2})}{\Gamma(-\nu_2)\Gamma(-\nu_1+\frac{1}{2})}\,\frac{\cos(\frac{1}{2}\pi\nu_1)}{\cos(\frac{1}{2}\pi\nu_2)}\left(\sqrt{\frac{b}{a}}\,L_\varepsilon\right)^{\nu_1-\nu_2} . \quad \text{(I.17)}$$

Next, we take the limit $\omega^2 \to 0$ $(a \to 0)$; since $\nu_1-\nu_2=\sqrt{1-4\bar{c}} > 0$, we find $\lim\limits_{a\to 0}\frac{A^e}{B^e} = \infty$, so that $\xi^e \to A^e\left|\frac{s}{L_\varepsilon}\right|^{\nu_1}$. Likewise, the odd solution also proves to transform into the "small" solution if the limit $\omega^2 \to 0$ is taken: $\xi^o \to A^o\,\frac{s}{|s|}\left|\frac{s}{L_\varepsilon}\right|^{\nu_1}$.

Incidentally, we notice that the limit $\omega^2 \to 0$ of the expression A^e/B^e in Eq. (I.17) is mainly determined by $(\sqrt{b/a}\,L_\varepsilon)^{\nu_1-\nu_2}$, because the factor in front of it is in general of the order unity. However, if $\bar{c} = 0$ the indices become $\nu_2 = -1$ and $\nu_1 = 0$, so that $\Gamma(-\nu_1)$ and, consequently, A^e/B^e blows up already before the limit $a \to 0$ is taken. This difficulty is associated with the fact that for $\bar{c} = 0$ the approximation $a + bs^2 << |c|$ is incorrect, so that Eq. (I.9) no longer holds. One can get around this difficulty by considering small values of $\bar{c}$ and still smaller values of $a + bs^2$, subsequently taking the limit $a \to 0$ and next the limit $c \to 0$. Alternatively, one could try to solve the correct equation of motion for $\bar{c} = 0$ right away.

Finally, we choose as new independent solutions half the sum and half the difference of the, properly normalized, even and odd solution:

$$\xi^+ = \frac{1}{2}\left(\frac{\xi^e}{A^e} + \frac{\xi^o}{A^o}\right) \to \frac{1}{2}\left(1 + \frac{s}{|s|}\right)\left|\frac{s}{L_\varepsilon}\right|^{\nu_1} , \quad \text{(I.18)}$$

$$\xi^- = \frac{1}{2}\left(\frac{\xi^e}{A^e} - \frac{\xi^o}{A^o}\right) \to \frac{1}{2}\left(1 - \frac{s}{|s|}\right)\left|\frac{s}{L_\varepsilon}\right|^{\nu_1} . \qquad (I.19)$$

The right-hand sides of the equations (I.18) and (I.19) are the solutions of Eq. (I.7) in the limit $\omega^2 \to 0$, which thus represent independent solutions of the (nearly) marginal equation of motion. These solutions hold for $|t| \gg 1$, or $\sqrt{a/b} \ll |s| \lesssim L_\varepsilon$, where the left inequality no longer presents a restriction since $a \to 0$. Therefore, in this limit, the size of the inner region shrinks to zero and the following behaviour results for the solutions of the marginal equation of motion: the solution ξ^+ only differs from zero to the right of the singular point ($s > 0$), where it behaves like ξ_s, whereas ξ^- only differs from zero to the left of this singular point ($s < 0$), also behaving there like ξ_s. This can be interpreted by stating that a singular point of the marginal equation of motion splits the interval (y_1,y_2) in two independent subintervals (y_1,y_s) and (y_s,y_2), which are to be studied separately as far as stability is concerned. Stated differently: the two intervals (y_1,y_s) and (y_s,y_2) are separated by a virtual wall, where the boundary condition ξ_{y_o} is "small" should be posed.

APPENDIX II

Resistive effects

The addition of a small resistivity to the plasma model strongly modifies the stability results[70,71]. This situation is in many respects similar to that for the marginal equation of motion, where the addition of the small inertia term $\omega^2\rho$ led to the important effects described in Sec. 3.1 and Appendix I, viz. a splitting of the plasma interval and the presence of a kind of virtual wall at the singularity ($F = 0$). In the marginal equation of motion the displacement ξ_y (for the plane layer) or ξ_r (for the pinch) may tend to infinity in the vicinity of a singular point of this equation. Therefore, since all perturbations of the physical quantities involved can be expressed in terms of ξ, it is not surprising that the perturbation of the current also may blow up there. It is obvious that a resistivity η, no matter how small, will then play the important role of keeping the current finite. Thus, like in the ideal case for small ω^2, one can define, in the resistive case for small η, an inner region (now called the resistive layer) centered around the singular point ($F = 0$) where both inertia ($\omega^2\rho$) and resistivity (η) are important, and an outer region where these effects can be neglected. The problem of solving the resistive equations in the limit of small η now amounts to joining the solutions of the inner resistive layer with those of the outer ideal region.

The full resistive problem requires an analysis well beyond the present one, but one important result of the resistive theory can be applied straightforwardly to the configurations dealt with in Chapters 4 and 6. Here, we have in mind the theorem, proved in Refs. 72 and 73, that the stability criterion for a constant-pressure plasma with respect to resistive tearing modes (new non-localized modes appearing as a consequence of the finite resistivity) is the same as that which would be found from the energy principle of ideal MHD if the resistive layer were replaced by a vacuum. Intuitively, this is clear from the fact that resistivity allows the lines of force to break and join, which process naturally occurs in a vacuum region. Thus, starting from the mentioned theorem, a simple method may be obtained to determine the stability of force-free fields with respect to tearing modes[74,75]). Since these modes are closely related to kinks, their investigation appears to be particularly important for the type of pinch configuration considered in Chapter 6.

In order to illustrate what happens to the singular points of the marginal equation of motion when a finite resistivity is introduced, let us consider a plane plasma layer with a force-free field situated between two conducting walls at $y = y_1$ and

$y = y_2$ (the model of Chapter 4 without the upper layer; there is no gravity). Here, we follow the analysis given in Ref. 75, omitting details. The force-free field is assumed to have a constant value of α, so that its components are given by Eq. (31), in which ϕ_o may be chosen arbitrarily. Let us look at a mode, specified by k_x and k_z, and let us assume that the interval (y_1,y_2) contains only one singular point y_s. It can then be deduced[72,73]) from the above-mentioned theorem that the stability criterion for the (k_x,k_z) tearing mode is given, in the limit $\eta \to 0$, by

$$\left(\frac{Q'_{y1}}{Q_{y1}}\right)_{y_s-0} > \left(\frac{Q'_{y2}}{Q_{y2}}\right)_{y_s+0} , \qquad (II.1)$$

where Q_{y1} and Q_{y2} are solutions of the marginal equation of motion (34) with $\alpha' = 0$, that satisfy the boundary conditions $Q_{y1} = 0$ at $y = y_1$ and $Q_{y2} = 0$ at $y = y_2$.

Comparing the inequality (II.1) with item 3) of theorem 2 (Sec. 3.1) we notice that it has the same form, apart from the fact that ξ_y is replaced by Q_y and that the point of comparison y_o now is a singular point y_s. Because we have restricted the discussion to force-free fields of constant α, the solutions Q_y, as given by Eq. (35), are well-behaved in the whole interval (y_1,y_2). In fact, the force-free fields of constant α are the only ones for which the singularities at $F = k_xB_x+k_zB_z = 0$ disappear from the marginal equation of motion in terms of Q_y, as can be seen from Eq. (34). Therefore, there is no difficulty in continuously extending the solution Q_{y1} (being defined only in (y_1,y_s)) to the interval (y_s,y_2) and vice versa for Q_{y2} to (y_1,y_s). The condition (II.1) then implies that none of these continued solutions possesses a zero in the open interval (y_1,y_2). Provided that both Q_{y1} and Q_{y2}, including their mentioned continuations, do not have zeros in the open interval (y_1,y_2), it follows from Sturm's separation theorem[30]) that no solutions whatever will exist which have more than one zero in (y_1,y_2). We then can state the following theorem.

Theorem. For specified values of k_x and k_z a plane plasma layer with a force-free magnetic field of constant α is stable with respect to tearing modes, in the limit of low resistivity, if and only if there exist no solutions to the marginal equation of motion in terms of Q_y which have more than one zero in the interval (y_1, y_2).

In this way, operating in terms of the variable Q_y instead of ξ_y, the singular points lose their significance for the stability analysis and one can simply ignore them. We had to restrict the above theorem to force-free fields of constant α; it is not quite clear how it should be generalized for more general fields. However, for the present purpose the restriction is not too serious.

The solutions Q_y, given in Eq. (35), only oscillate if $k < |\alpha|$ and, consequently, only long-wavelength perturbations can be unstable. From the above theorem and Eq. (35) we now immediately obtain the criterion for stability with respect to tearing modes:

$$\sqrt{\alpha^2 - k^2}\, b < \pi \,, \qquad \text{(II.2)}$$

with $b = y_2 - y_1$. In view of the k-dependence this criterion simply reduces to

$$|\alpha| b < \pi \,. \qquad \text{(II.3)}$$

Remembering that α is the ratio between the current density and the magnetic field, we observe that this relation puts a limit to the total current flowing along the field lines, above which the layer is unstable due to long-wavelength tearing modes. We recall that there was no such a limitation in the ideal case and that the layer was perfectly stable in the absence of gravity (see footnote Chapter 5 and Ref. 75).

Having obtained a simple method to get rid of the singularities for plane force-free fields of constant α, we return to the problem of the stabilization of the gravitational instability by these fields (Chapter 4). If we admit tearing modes in the force-free field supporting the gravitating plasma layer (Fig. 1), the stability analysis of Sec. 4.2 is slightly simplified. In fact, ignoring the singular points in the lower interval (0,-b), the boundary condition $Q_y^t = 0$ for the solutions (35) should no longer be posed at $y = -b^*$, but at $y = -b$. As a result, b^* should be replaced by b in the expressions R entering the stability conditions (42) and (43). This amounts to a strongly destabilizing effect, as can be seen from Fig. 3 where the dashed parts of R_o were obtained by assuming this modified boundary condition (which was not justified in the context of Sec. 4.3).

In Sec. 4.3 a discussion was already devoted to the value of α for which stability of the configuration of Fig. 1 becomes optimal. It was stated there that this value can be found from the condition that the curves $L_o(\beta_{crit})$ and R_o touch in the point A of Fig. 3. Because the curves R_o were not smooth in the ideal theory this led to a range of values of α for which stability is obtained. In the present discussion the dashed parts of the curves R_o have got physical significance, so that the relevant curves R_o are smooth. As a result, a well-fixed value of α is obtained for which stability is optimal. For the numerical example of Fig. 3 this α is given by $\alpha b = 0.5$ and it is clear that stability is still possible, in spite of the destabilizing influence of the resistivity of the force-free region.

For the cylindrical force-free field of constant α a stability criterion similar to (II.1) can be obtained, replacing Q_y by Q_r and y_s by r_s. This leads to a stability theorem of the same form as that for the plane case. The only important difference is the fact that a cylindrical force-free field of constant α may already be unstable in the absence of resistivity. Therefore, one cannot decide from an oscillating behaviour of Q_r alone whether one has to do with an ordinary kink (of the type described by Voslamber and Callebaut[17]) or with a tearing mode. Here, the close relationship between kinks and tearing modes becomes particularly manifest. Mathematically, the relevant difference between the two types of modes concerns the property that tearing modes do not respect singular points and independent subintervals, whereas kinks do. Physically, both types of modes can be distinguished on the basis of their time-behaviour; tearing modes have a growth rate proportional to $\eta^{3/5}$, whereas this growth rate is of course independent of η for kinks. This property implies that the stability criteria obtained on the basis of ideal MHD do not become entirely irrelevant if resistivity is included, because the resistive tearing modes grow much slower than kinks if η is small. On the other hand, for the time being experimental values of η in pinches are not yet such as to justify the neglect of tearing modes.

Next, we reconsider the stability criteria obtained in Chapter 6 in order to include a finite resistivity of the outer force-free region of the pinch. Again, ignoring the singular points there, we find that the proper boundary condition to be posed for the solutions rQ_r^t, given by Eq. (89), reads $rQ_r^t = 0$ at $r = r_1$. This implies that r_1^* should be replaced by r_1 in the expression (96) for $(rQ_r^t)'/rQ_r^t$ entering in the stability condition (94). For the stability diagram of Fig. 6 this means that, e.g., the curve $R(\alpha = -50)$ is no longer deflected to the right of the point A, so that the dashed part of it (labelled with (-50)) acquires physical significance. As a consequence, the new (smooth) curve $R(\alpha = -50)$ will cross the curve $L(\beta = 20\%)$ and a pinch, consisting of a dense inner region with $\beta = 20\%$, surrounded by a force-free field of α = constant = $-50\ m^{-1}$, will be unstable with respect to tearing modes. We remark that these modes are made possible by the resistivity of the outer region, but that they are by no means entirely localized in that region. In fact, they involve a motion of the dense plasma column as a whole (in this respect they are just kinks), but in the outer region they induce strong currents in the resistive layer centered around the singular point, so that the growth rate is reduced to that given by the resistive theory and vanishes in the limit $\eta \to 0$. Nevertheless, it is clear that those modes should be taken serious and that it is not safe to rely upon the stabilizing influence of the singular points.

Taking the resistive effects into account in this way, we are led to a well-fixed value of α giving optimal stability of the pinch. This α is given by Eq. (101); for the numerical example of Fig. 5 this equation gives $\alpha_{opt} = -36.7\ m^{-1}$. For this value of α the pinch is stable, up to $\beta = 20\%$, with respect to ideal type-I modes and, in addition, to tearing modes. It is clear, however, that this choice for the optimal α spoils the advantageous situation with respect to type-II instabilities, since the curves L and R now touch each other and very little space is left between them. On the other hand, once having given a more refined picture of the plasma motion about the singular points in the outer region, one should also modify the treatment of the surface-layer instabilities, since these also involve singular points. For the surface layer, however, we do not have available a simple theorem like that stated in Refs. 70 and 71 and we shall leave this point to future calculations. Moreover, it is known[71]) that other types of resistive instabilities (viz. resistive interchanges) are present if $p' < 0$, because resistivity makes the shear term in Suydam's criterion ineffective. Consequently, the surface layer will always be unstable with respect to resistive interchanges. However, this layer being thin, single-particle effects like finite gyration radii should be taken into account also.

Another, completely different, resistive effect concerns the influence of the resistivity of the conducting wall surrounding the plasma[54]). In general, the resistivity of the wall is neglected in stability calculations, so that the perturbation of the normal component of the magnetic field should vanish ($\underset{\sim}{Q}.\underset{\sim}{n} = 0$). Taking into account the resistivity of the conducting wall this boundary condition no longer holds and the field lines can be pushed through the wall. This effect is a strongly destabilizing one. It will be treated elsewhere[76]).

APPENDIX III

Toroidal effects

In this appendix we shall discuss the relevance of the stability results obtained for a cylindrical pinch to the stability of a toroidal plasma. Although the neglect of toroidicity might introduce errors of the same order of magnitude as those made in neglecting resistivity, the former neglect is much more conspicuous since even a glance at an experimental device suffices to notice the difference with an infinite cylinder. In fact, it is even rather surprising that infinite-cylinder theory can be applied at all to tori.

The most simple and also most important toroidal correction to infinite-cylinder theory is the introduction of a periodicity condition[54]) for the longitudinal wavelength of the modes, so that one only considers cylindrical modes which would fit the circumference of the torus if it were straightened out. If R_t represents the major radius of the torus, this leads to the condition

$$k = \frac{n}{R_t} , \qquad \text{(III.1)}$$

where n is an integer (having nothing to do with the parameter n used in Sec. 7.4 to label the branches of the curve R!). This condition can be represented in the stability diagram of Fig. 6 by drawing vertical lines (labelled with n) with abscissae given by (III.1). For a torus the stability condition $L > R$ only needs to be satisfied for those discrete k-values, so that L should be compared with R at the vertical lines. It is clear that this represents a strongly stabilizing effect. This procedure was applied by De Vries at alii[77]) for a pinch consisting of a dense plasma surrounded by a constant-pitch force-free field. It was shown that sharp maxima occur in a plot of β_{crit}

against $|\mu|R_t$ for values of $|\mu|R_t$ such that k-values, admitted by Eq. (III.1), occur on either side of $k = -\mu$. For example, at $|\mu|R_t = 1.3$ a value of β_{crit} of some percents, allowed by infinite cylinder theory, was found to increase to more than 20%. Applying the periodicity condition (III.1) to the constant-α model of Chapter 6 approximately the same results as described in Ref. 77 will be found, since for low values of $|\mu|R_t$ (where toroidal corrections are important) the difference between a constant-pitch field and the optimal constant-α field is negligible (see Eq. (101)). On the other hand, for higher values of $|\mu|R_t$, better stability results can be obtained with the constant-α model, but in that case toroidal effects are less important.

The well-known Kruskal-Shafranov limit[54]) (the maximal toroidal current admitting stability in connection with the twisting of the magnetic-field lines around the magnetic axis) results from an application of the periodicity condition to a configuration consisting of a dense plasma surrounded by a vacuum field. Needless to say then that this limit has no sense for a screw pinch. On the other hand, the model is applicable to a Tokamak, because there the presence of limiters prevents force-free currents to flow in the outer region. The Kruskal-Shafranov limit can be understood on the basis of the stability diagram of Fig. 6. For a vacuum field ($\alpha = 0$) the unstable k-region is situated there to the left of $k = -\mu$. Therefore, in order to get rid of the dangerous $m = 1$ instabilities, one should choose the value of $|\mu|R_t$ such that the first k-value admitted by the periodicity condition (III.1) lies to the right of $k = -\mu$. This gives exactly the Kruskal-Shafranov limit:

$$|\mu|R_t < 1. \qquad \text{(III.2)}$$

Since μ is related to the surface currents this condition puts a limit to the total current I flowing in the z-direction, i.e. along the torus. If this current is assumed to be concentrated on a shell with radius r_o, it will be connected with μ according to the relation $I = 2\pi r_o B^t_{\theta o}/\mu_o \approx 2\pi r_o^2 \mu B^p_z/\mu_o$, where we have put $B^t_{zo} \approx B^p_z$ (as justified by Fig. 5). Thus, in terms of the current I, the condition (III.2) becomes

$$\frac{\mu_o I R_t}{2\pi r_o^2 B_z} < 1 . \qquad \text{(III.3)}$$

This restriction on the current flowing along the torus is important for Tokamaks, because it limits the temperature to be obtained by means of Ohmic heating. The value of μ to be substituted in Eq. (III.2) for a diffuse configuration is of course rather vaguely defined. Therefore, in general one takes care that all local values of μ across the tube satisfy Eq. (III.2).

Although posing the periodicity condition is the most straightforward and obvious way to take into account the toroidal geometry, it seems to be a rather crude method to neglect the more refined effects connected with toroidal curvature altogether. Yet there exists strong evidence that this procedure yields reliable results for the stability theory of a pinch, at least as far as non-localized kinks are concerned. To support this statement we refer to Refs. 78,79, and 80. Ware[78]) divided modes occurring in a torus into those driven by $j_\perp$ (or ∇p) and those driven by $j_{//}$. This classification corresponds more or less to the usual division into localized interchanges and non-localized kinks. Starting from the energy principle he found that toroidal effects on the stability of the $j_{//}$-driven modes disappear to first order in the small parameter r_1/R_t (the inverse aspect ratio of the torus). Next, it was shown by Shafranov[79]) on the basis of some rough calculations involving the forces driving the instabilities and those associated with toroidal curvature, that toroidal effects only act as a small correction on the stability of kinks. Finally, on the basis of a normal-mode analysis, it was shown by Schuurman and Bobeldijk[80]) that there is no change in the growth rate of a kink mode, as calculated from cylinder theory, up to first order in r_1/R_t. All these investigations suggest that it is a good approximation to study kink instabilities with the aid of solutions obtained from infinite-cylinder theory, taking into account the toroidicity by posing the periodicity condition.

On the other hand, no such conclusion holds for localized interchange modes. Here, toroidal curvature is extremely important. This can be seen from the generalized Suydam criterion for toroidal plasmas. This generalization was obtained first by Mercier[81]) and a lot of work has been done since then by many workers. For our purpose, however, the clear representation of the toroidal criterion for localized interchanges as given by Shafranov and Yurchenko[82]) is most suitable:

$$\frac{1}{4}\left(\frac{q'}{q}\right)^2 + \frac{2\mu_o p'}{rB_z^2}(1 - q^2) > 0 , \qquad \text{(III.4)}$$

where

$$q = \frac{1}{\mu R_t} = \frac{rB_z}{R_t B_\theta} . \qquad \text{(III.5)}$$

Here, the coordinates r,θ, and z represent, respectively, the distance to, the azimuthal angle around, and the distance along the magnetic axis. On purpose we used this specific notation, since it transforms into that of the ordinary cylinder coordinates in the limit $R_t \to \infty$. In this limit $q^2 \to 0$ and criterion (III.4) reduces to the cylindrical Suydam criterion (60). Because, in general, $p' < 0$ the following sufficient criterion for stability against interchanges is obtained:

$$q^2 > 1 . \qquad \text{(III.6)}$$

Close to the magnetic axis this criterion is necessary because there the first term of Eq. (III.4) is negligible compared with the second term (at least for reasonable current distributions). The remarkable feature of Eq. (III.6) is the stabilization of interchanges by toroidal curvature, independent of the magnitude of p'. Another remarkable property of this criterion is that it coincides with the Kruskal-Shafranov limit (III.2), although both conditions are related to completely distinct toroidal effects.

Our interest in the toroidal Suydam criterion here concerns the comparison with the cylindrical case for which we have found that the sharp pinch consisting of a dense plasma surrounded by a constant-pitch force-free field (Van der Laan's model, Sec. 7.4) has to be unstable with respect to quasi-interchanges. These quasi-interchanges occurred in the situation that Suydam's criterion was degenerate. The same degeneracy is seen to arise for toroidal constant-pitch force-free fields ($q' = 0$, $p' = 0$). The unfavourable property of the constant-pitch field to have no shear turns out to be less dangerous in the toroidal case, since criterion (III.4) suggests that this field can yet be stable with respect to interchanges if $q^2>1$, even in the presence of a large pressure gradient. Even above the Kruskal-Shafranov limit the term with the pressure gradient is less effective in the toroidal case than it would be in the cylindrical case. This might explain why the instabilities of the constant-pitch field, which are clearly predicted by infinite-cylinder theory (Sec. 7.4), are less unequivocally observed experimentally.

Finally, we remark that criterion (III.4) only holds for toroidal plasmas for which the magnetic surfaces have a circular cross section. This will be approximately true if the cross section of the conducting wall is circular and the aspect ratio of the torus is not too large. However, one can show that the local stability criterion is modified considerably if the magnetic surfaces are given a non-circular cross section[83,84]), e.g. by properly shaping the conducting wall enclosing the plasma. For example, by triangularly shaping the magnetic surfaces, the magnetic well can be deepened substantially[85]), so that the condition (III.4) is to be replaced by a much less stringent condition. This is important for Tokamaks, since it implies that the Kruskal-Shafranov limit may be surpassed without evoking localized interchange instabilities. The influence of shaping the surfaces on the non-localized kinks is not yet investigated. For the screw pinch, on the other hand, there is no need to introduce this complication, because the Kruskal-Shafranov limit is not relevant for this plasma-confinement scheme, at least not if the force-free field in the outer region of the pinch is taken into account.

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Curriculum vitae

Na het behalen van het H.B.S.-B diploma aan het Christelijk Lyceum voor Zeeland te Goes, liet ik mij in september 1957 inschrijven aan de Technische Hogeschool te Delft. De onjuiste keuze van de afdeling der elektrotechniek herstelde ik in januari 1960, toen ik mij liet overschrijven naar de afdeling der technische natuurkunde. Mijn afstudeerwerk verrichtte ik in de werkgroep voor lage temperaturen onder leiding van prof.dr. B.S. Blaisse. Het betrof het meten van zeer lage temperaturen met behulp van magnetische pulsresponsie. Het diploma van natuurkundig ingenieur werd mij in juni 1965 uitgereikt.

Van augustus 1965 tot juni 1967 vervulde ik de militaire dienstplicht, sedert januari 1966 gedetacheerd bij het Medisch Biologisch Laboratorium RVO-TNO te Rijswijk. In de werkgroep radio-biofysica onder leiding van prof.dr. Joh. Blok hield ik mij bezig met het onderzoek aan de fluorescentie van met γ-straling bestraalde DNA-oplossingen.

In augustus 1967 trad ik in dienst van de Stichting voor Fundamenteel Onderzoek der Materie bij de werkgroep TN 1 (theoretische plasmafysica), die is ondergebracht in het FOM-Instituut voor Plasmafysica te Jutphaas. Het werk aldaar betrof het in dit proefschrift beschrevene. Daarbij waren de belangstelling en de hulp van prof.dr. H. Bremmer en dr. M.P.H. Weenink, achtereenvolgens werkgroepleiders van TN 1, bijzonder stimulerend. Als zodanig dient ook het contact met de pinchgroep aldaar genoemd te worden.

S T E L L I N G E N

I

De bewering van Rosenbluth, dat oppervlakte-instabiliteiten noodzakelijkerwijs optreden in een scherpe pinch waarin het externe en interne longitudinale magneetveld dezelfde richting hebben, geldt voor het geval dat het buitengebied een vacuüm is. Indien het vacuüm wordt vervangen door een krachtvrij veld bestaan er realistische veldconfiguraties die stabiel zijn.

Dit proefschrift, Sec. 6.3.

II

De door Schmidt gegeven afleiding van het stabiliteitscriterium voor krachtvrije magneetvelden, alsmede het criterium zelf, zijn principieel onjuist.

G. Schmidt, Physics of High Temperature Plasmas, New York (1966) p. 141.

III

De stelling van Anzer, dat het Lundquist-veld niet bruikbaar is voor stationaire astrofysische problemen omdat het instabiel is voor kinks, is ongegrond zolang de groeisnelheden van deze instabiliteiten niet in de beschouwing worden betrokken.

U. Anzer, Solar Phys. 4 (1968) 101.

IV

Het model van Hasegawa voor de gedurende auroras waargenomen instabiliteiten is innerlijk tegenstrijdig.

A. Hasegawa, Phys.Rev.Letters 24 (1970) 1162.

J.P. Goedbloed and R.Y. Dagazian, ICTP, Trieste (1970), IC/70/119.

V

Oplossingen van DNA die bestraald zijn met Co-60 gammastraling vertonen een karakteristieke fluorescentie, die afkomstig is van twee componenten welke voor bestraling niet aanwezig zijn.

J.P. Goedbloed and J.J. van Hemmen, Int.J.Radiat.Biol. 14 (1968) 351.

VI

De uitspraak van Bell, "we shall consider Pascal primarily as a highly gifted mathematician who let his masochistic proclivities for self-torturing and profitless speculations on the sectarian controversies of his day degrade him to what would now be called a religious neurotic", bevat meer informatie over Bell dan over Pascal.

E.T. Bell, Men of Mathematics, New York (1937).

VII

Hoewel Guardini over het geheel genomen een evenwichtig beeld schetst van Pascal, zijn zijn speculaties over Pascal's levenseinde aan bedenkingen onderhevig.

R. Guardini, Christliches Bewusstsein, München (1962).

VIII

Het verdient aanbeveling in het onderwijs meer aandacht aan de expressievakken te schenken.

IX

Een merkwaardig soort filosofie die sommige natuurkundigen ertoe brengt het object van hun studie, nature, te schrijven als "Nature"!

P.A.M. Dirac, The Principles of Quantum Mechanics, Oxford (1958) p. 11.
L.D. Landau and E.M. Lifshitz, Fluid Mechanics, Oxford (1959) p. 102.

X

De afwezigheid van aansluiting van het hoger technisch onderwijs op het technisch hoger onderwijs en de moeilijkheden die een H.T.S.-er in de weg worden gelegd om later alsnog het ingenieursdiploma te verwerven, zijn onbillijk voor de betrokkenen en dienen in hun belang te verdwijnen.

XI

De militaire dienst is een noodzakelijk kwaad. Het is ongerijmd hiervan een positieve invloed te verwachten op de persoonlijkheidsvorming van dienstplichtigen.

XII

Het gebruik van academische titels in het maatschappelijk verkeer dient vermeden te worden.

Johan Peter Goedbloed 17 december 1970