

Towards a rational dynamics of plasmas

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**TOWARDS
A RATIONAL DYNAMICS
OF PLASMAS**

ROBERT BENACH

TOWARDS A RATIONAL DYNAMICS OF PLASMAS

Robert Benach

TOWARDS A RATIONAL DYNAMICS OF PLASMAS

PROEFSCHRIFT

ter verkrijging van de graad van doctor in de
technische wetenschappen aan de Technische
Hogeschool Eindhoven, op gezag van de rector
magnificus, prof.dr.ir. G. Vossers, voor een
commissie aangewezen door het college van
dekanen in het operbaar te verdedigen op
vrijdag 24 mei 1974 te 16.00 uur

door

Robert Benach

geboren te Cleveland

Dit proefschrift is goedgekeurd door de promotoren

Prof.dr. I. Müller *

en

Prof.dr. L.H.Th. Rietjens

* Department of Mechanics and Materials Science
The Johns Hopkins University
Baltimore, Maryland (U.S.A.)

"

Thou wert my guide, philosopher, and friend?
That urg'd by thee, I turn'd the tuneful art
From sounds to things, from fancy to the heart;
For Wit's false mirror held up Nature's light;
Shew'd erring Pride, whatever is, is right;
That Reason, Passion, answer one great aim;
That true Self-love and Social are the same;
That Virtue only makes our Bliss below;
and all our Knowledge is, ourselves to know."

Alexander Pope

To the memory of my mother, Estelle G. Benach, and to
Josje and Dave, my dearest companions, I dedicate with
deep respect and love this contribution to the knowledge
and hopefully the betterment of mankind.

PREFACE

The two general problems which stand in the background of this study are of an interdependent nature. On the one hand, of the "states of matter" from which a macrocosm may be composed, there must be a specification of the particular physical system to be considered here. On the other, there is the task of securing a suitable means for treating the behavior of at least that system.

Clearly, the first point is concerned with the problem of what the system in question is. Here I direct attention towards the "vague" concept of plasma which I assume is not a state of matter, but rather, a condition that may on occasion pervade those states of matter that are commonly recognized: viz., gases, liquids and solids.

The second point in turn is related to the complementary problem of describing what a (class of) system(s) does. Here too, as in the case of the first point, there are numerous basic issues of concern that require further investigation. Indeed, not only the subject matter of plasma physics but also the present means for its treatment appear to be fluid-like.

A theoretical treatment of problems relating to the gross (i.e. bulk) behavior of dynamic systems requires a complete set of macroscopic equations. For gaseous plasmas, a special class of electromagnetic fluids, there appears at this time to be essentially two general approaches by means of which the desired relationships are secured. The one is of a particle nature and is based upon a (non-)equilibrium statistical continuum development; and, the other is a classical (i.e. linear) continuum mechanical approach.

It is a foremost purpose of this study to consider the possibility of deriving the basic governing system equations from a "new" (for plasma physics) viewpoint: that being rational mechanics and rational thermodynamics¹. In order to ascertain the possible potential of a modern continuum mechanical treatment, the methodology of rational mechanics will be used to develop a particular non-equilibrium mixture theory for the case of a one-temperature, chemically reactive, non-relativistically moving, magnetizable and dielectric fluid medium.

Herein consideration is given specifically to the matter of securing for a plasma bulk a set of general constituent, mixture and electromagnetic field balance and jump balance relationships. And this, in addition to the proposal of a set of non-linear mechanical, electromagnetic and energetic constitutive equations which have been taken as being appropriate to the model of the system employed. On the basis of said model a set of equations emerges from the considerations here which is suitable for further theoretical and empirical investigation of a broad class of natural and laboratory plasma systems.

This study, in reflecting the tradition of rational mechanics, attempts to set down a preliminary conceptual framework suitable for the systematic investigation of plasmas. Although the theory developed here is of modest mathematical sophistication and treats mixtures of fluids with relatively simple physical properties, it is nonetheless directed towards the gaining of an eventual "breakthrough in understanding" of plasma systems.

¹ It is henceforth to be understood that these two aspects of modern continuum theory are meant with the mention of only the former. Note further that footnotes are subsequently placed at the end of the respective chapters to which they relate.

Further, while recognizing the value of well-motivated and executed special case studies, the purpose of this endeavor is taken to be served with the presentation of such results as usually form the starting point for those considerations. A treatment of these matters in greater depth or an investigation of their applicability must await the results of future developments.

Acknowledgements

It is not possible for me to express completely and honestly with words alone my deep sense of indebtedness to those who have contributed most meaningfully to the realization of this dissertation.

How to say thank you to Professor Müller for his encouragement, his untiring guidance and his innumerable kindnesses; to Dr. Alts for valuable discussions; to Professor Ericksen for helpful conversations; to Professor Truesdell for his interest and the example of scholarship he sets; and to my wife, Jos, for her love, forbearance and optimism - how indeed do I thank you all? To each of you I say simply, yet sincerely, thank you.

It is also a pleasure to express my appreciation to the "Technische Hogeschool Eindhoven" for the facilities placed at my disposal during the period in which I prepared the greater part of this work. To Mrs. H.K. van der Putten-Bosscher, Miss Th.J.M. van den Hurk and Mrs. E.E.F.M. Baselmans-Weijers I wish to state my gratefulness for their painstaking efforts taken in the typing of the final manuscript; and to Mr. H.F. Koolmees for the preparation of the figures.

Robert Benach

Eindhoven, 20 March 1974

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NOMENCLATURE

The following list of symbols is meant to be complete with the exception of almost all of the general material response coefficients and the invariants in terms of which they are derived. These quantities appear for the most part in Chapter C and can easily be recognized there from the context of the discussion on constitutive equations.

Symbol	Description	Place where defined or introduced
<u>Latin alphabet</u>		
a	A constant with the dimensions of time	(C3) ₂
a_i	Arbitrary polar vector	Sect. C.1.a.
A	Arbitrary field	Sect. D.1.
b_j	Arbitrary polar vector	Sect. C.1.a.
b_i a	Specific total external body force of a constituent	(B82) ₁
b_i E	Specific body force of the electromagnetic field in a mixture	(B91)
b_i 0	Specific non-electromagnetic body force of a mixture	(B88)
b_i $0a$	Specific non-electromagnetic body force of a constituent	(B83)
\mathcal{B}	mean normal "magnetic stress"	(E27) ₁
B a	Constituent body	Sect. B.a.
B_i	Magnetic flux density	Sect. B.5.

\mathbb{B}_{ij}	Magnetic stress deviator	(E27) ₂
$c(t)$	Singular line	Sect. B.c.ii.
c_a	Mass fraction of a constituent	Sect. B.b.
c_i	Arbitrary axial vector	Sect. C.1.a.
\bar{c}_a	Production rate density of mass for a constituent	Sect. B.3.
\bar{c}_a^S	Mass density supply by $s(t)$ for a constituent	Sect. B.3.
\bar{c}_a^{SC}	Charge density supply by $s(t)$ for a constituent	(B58)
\mathcal{C}_a	Material curve for a constituent	Sect. B.b.
\mathcal{C}_{ij}		(B153)
$d_i(t)$	Time-dependent vector with the dimensions of time	(C3) ₁
$d_{a^{kj}}$	Deformation rate tensor of a constituent	(B34) ₃
dl_i	Differential element of $c(t)$	Sect. B.c.ii.
D_i	Charge potential	(B44)
\mathcal{D}_i	Partial charge potential	(B45)
e	Fundamental unit of electric charge; Equilibrium relationship of e^I	Sect. B.b. Sect. E.2.
\hat{e}	Basis vector	Sect. B.a.
e^I		(D18)
\bar{e}_a	Production rate of total energy density of a constituent	(B128)
\bar{e}_E	Power expended by an electromagnetic field	(B141) ₃
$(\bar{e}_a^{SM}) \bar{e}_a^{SE}$	(Non-)electromagnetic supply by $s(t)$ of the total energy of a constituent	(B130)

\bar{e}^{SE}	Supply of ϵ by $s(t)$ E	(B142) ₂
E, EM	Regions of "low"-density gas plasmas	Sect. A.1.
E_i	Electric field	Sect. B.4.a.
ξ_a	Electromotive intensity of a constituent	(B48) ₁
ξ_i	Mixture electromotive intensity	(B48) ₂
f	Arbitrary scalar field on $s(t)$	(B80) ₁
\mathcal{F}	tensor function(a)	Chpt. C.
F_a^{α}	Deformation gradient of a constituent	(B3)
g	Arbitrary scalar field on $s(t)$	(B72) ₁
G	Region of "very low"-density gases	Sect. A.1.
G_i	"Abraham vector"	(B92)
h_k	Arbitrary vector field on $s(t)$	(B72) ₂
H_k	Current potential	(B67) ₁
\mathcal{H}_k	Partial current potential	(B67) ₂
.. i, j, k ..	Spatial vector component indices	Sect. B.a.
I	Reduced set of fundamental invariants	(C27)
I^*, II^*	Fundamental invariants	(C16)
I_α		(C12)
$l(t)$	Phase interface	Sect. B.c.i.
J a	Jacobian of a constituent	(B4)
J_a^M	Magnetization current density of a constituent	(B53)
J_a^P	Polarization current density of a constituent	(B54)

J_{aj}^T	Total current density of a constituent	(B57)
$\tilde{\sigma}_j$	Non-convective current density of a constituent	(B55)
J_j^F	Conduction current density of a mixture	(B62) ₁
J_j^M	Magnetization current density of a mixture	(B60)
J_j^P	Polarization current density of a mixture	(B61)
J_j^T	Total current density of a mixture	(B62) _{3,4}
$\tilde{\sigma}_j$	Non-convective current density of a mixture	(B62) ₂
J_i^*	Generalized conduction current of a mixture	(B70)
k_i	Arbitrary vector field on $s(t)$	(B80) ₂
K_b	Chemical potential in non-equilibrium of a constituent	(E42)
l_{ak}	Specific body couple of a constituent	Sect. B.7.a.
l_k	Specific body couple of a mixture	(B114) ₃
L	Characteristic geometric length	Sect. A.1.
L_{ak}	Specific total angular momentum of a constituent	(B99) ₁
L_k	Specific total angular momentum of a mixture	(B114) ₁
L_1, L_2, L_3	Pederson, scalar, Hall electrical conductivities	Footnote F21
m	Unit atomic mass	Sect. B.3.a.
\hat{m}	Unit binormal vector to $c(t)$	Sect. B.c.ii.
m_a	Mass of a constituent	Sect. B.b.
\hat{m}_i	Interaction force density of a constituent	(B82) ₂

\bar{m}_a^S	Supply by $s(t)$ of linear momentum for a constituent	(B84)
\bar{m}_i^S	Supply by $s(t)$ of mixture linear momentum	(B89)
\bar{m}_i^{SE}	Supply of G_i by $s(t)$	Sect. B.6.c.
$M, M - M^e$	Regions of "medium"-density gas plasmas	Sect. A.1.
L_a	Magnetic susceptibility of a constituent	(C32)
M_a^k	Magnetization density of a constituent	Sect. B.4.b.i.
M_l	Magnetization density of a mixture	(B59)
$m_{a^k j}$	Couple stress tensor of a constituent	(B99) ₂
$m_{k j}$	Couple stress tensor of a mixture	(B114) ₂
n	Total number of independent chemical reactions	Sect. B.3.a.
n_a	Number density of a constituent	Sect. B.b.
n_e	Electron number density	Sect. A.1.
\hat{n}	Unit normal vector	Sect. B.c.
\hat{n}_{a^k}	Production rate density of L_{a^k}	(B100) ₂
$\bar{n}_{a^k}^S$	Supply of L_{a^k} by $s(t)$	(B101)
\bar{n}_k^S	Supply of L_k by $s(t)$	(B117)
N_D	Number density of charged particles in a Debye sphere	Sect. A.1.
0	Origin of a co-ordinate system	Sect. B.a.
$0_{ij}(t)$	Time-dependent orthogonal tensor	(C3) ₁
p	Gas kinetic pressure; Pressure of a mixture and an electromagnetic field	Sect. A.1. (E29)

p_a	Pressure of a constituent in the absence of an electromagnetic field	(E26) ₂
p	Pressure of a mixture in the absence of an electromagnetic field	(E26) ₁
P_i	Volume polarization density of a constituent	Sect. B.4.
P_i	Volume polarization density of a mixture	(B43)
q_a	(Mass) specific charge	Sect. B.b.
q_k	Internal energy flux of a constituent	(B126)
q_j	Internal energy flux of a mixture	(B136)
q_j^I	Inner part of q_j	(B136) ₂
Q_a	Charge density of a constituent	(B47) ₂
Q_a^F	Volume density of free charge of a constituent	(B8)
Q_a^{SF}	Free charge density on $s(t)$ of a constituent	Sect. B.4.
Q	Charge density of a mixture	(B47)
Q^F	Volume density of free charge of a mixture	(B42) ₁
Q^S	Surface density of charge of a mixture	(B52)
Q^{SF}	Free charge density on $s(t)$ of a mixture	(B42) ₂
r_a	Specific body heating of a constituent	(B127)
r	Specific body heating of a mixture	(B137)
r_I	Inner part of r	(B152) ₂
R_a	Net production rate density of n_a	(B35)
R_3	Three-dimensional Euclidean space	Sect. B.a.
R_4	Euclidean space-time	Sect. B.a.
s	Total number of constituents	Sect. B.a.

s^*	Total number of electrically charged constituents	Sect. B.b.
$s(t)$	Singular surface	Sect. B.c.i.
S	Region of "high"-density gas plasmas; A scalar field; An absolute (polynomial) scalar invariant	Sect. A.1. (C1) ₁ (C11)
$\frac{S_i}{E_i}$	Poynting vector	(B141) ₂
$S^{(m)}$	A generic scalar	(C9)
S^*	A scalar (field) under a Euclidean transformation	(C4)
\mathcal{S}	A surface point	Sect. B.c.ii.
S_α	Invariant (polynomial) functions	(C12)
S_a	Material surface for a constituent	Sect. B.b.
t	Time	Sect. B.a.
\hat{t}	Unit tangent vector to $c(t)$	Sect. B.c.ii.
t^*	t under a Euclidean transformation	(C3) ₂
t_{ij} a_{ij}	Stress tensor of a constituent	(B81) ₂
t_{ij} E_{ij}	Maxwell stress tensor	(B93)
t_{ij}	Stress tensor of a mixture	(B87)
T	(Absolute) temperature	Sect. B.9.a.
$T, T - M^e$	Regions of "high-to-medium" density gas plasmas	Sect. A.1.
T_{ij}	A tensor field	(C1) ₃
T_{ij}^*	T_{ij} under a Euclidean transformation	(C4) ₄
u_i a_i	Diffusion velocity of a constituent	(B12)
u_n	Speed of displacement of $s(t)$	(B18)

\vec{u}	Velocity of propagation of $s(t)$	Sect. B.c.i.
v_i	Velocity of a constituent	(B9)
\bar{v}_i	Mass average velocity of a mixture	(B11)
v_a	Material volume for a constituent	Sect. B.b.
V_i	Velocity difference of a constituent	(B10)
V_i	A vector field	(C1) ₂
$V_k^{(n)}$	A generic vector	(C9)
V_i^*	A vector (field) under a Euclidean transformation	(C4)
$V_i^{(p)}$		Sect. C.1.
w_{ij}	Spin tensor of a constituent	Sect. D.1.
W_a	Molecular weight of a constituent	Sect. B.3.a.
x_i	Spatial co-ordinates	Sect. B.a.
\vec{x}	Position vector of a point in R_3	Sect. B.a.
x_i^*	x_i under a Euclidean transformation	(C3) ₁
$X_{a\alpha}$	Material co-ordinates of a constituent	Sect. B.a.
Z_a	Charge multiplicity of a constituent	Sect. B.b.

Greek alphabet

α	An independent homogeneous chemical reaction	Sect. B.3.a.
$\alpha, \beta, \gamma, \dots$	Material vector component indices	Sect. B.a.

β	Plasma beta	Sect. A.1.
β, β_e	Electron, ion Hall parameters	Sect. A.1.
γ_a	Specific entropy "production" rate of a constituent	(B147) ₂
γ_a^α	Stoichiometric coefficient of a constituent	Sect. B.3.a.
γ_a^S	Entropy supply density by $s(t)$ for a constituent	Sect. B.9.a.
γ	Specific entropy production of a mixture	(B150)
γ^S	Entropy density supply by $s(t)$ for a mixture	(B155)
$\hat{\delta}_{a k}$	Production rate of $\Omega_{a k}$	(B109) ₂
$\hat{\delta}_{a k}^S$	Supply of $\Omega_{a k}$ by $s(t)$	(B112)
$\hat{\delta}_k^S$	Supply of Ω_k by $s(t)$	(B118)
δ_{kj}	Kronecker delta	(B34) ₃
ϵ_a	Specific total energy of a constituent	(B125)
ϵ	Internal energy density of a mixture	(B135)
ϵ^I	Inner part of ϵ	(B135) ₂
ϵ^*	Specific (equilibrium) internal energy of a mixture in the absence of an electromagnetic field	(E21)
ϵ, E	Energy density of an electromagnetic field	(B141) ₁
ϵ_0	Vacuum permittivity	Sect. B.4.a.
ϵ_{ijk}	Alternator	(B4)
$\vec{\zeta}_a(\vec{X})$	Configuration of a constituent	Sect. B.a.

η_a	Specific entropy on a constituent	(B146)
η	Specific entropy of a mixture	(B152) ₁
η^*	Specific (equilibrium) entropy of a mixture in the absence of an electromagnetic field	(E18)
λ_e, λ_i	Electron, ion mean free path	Sect. A.1.
$\lambda_{Le}, \lambda_{Li}$	Electron, ion Larmor radius	Sect. A.1.
λ_D	Debye length	Sect. A.1.
Λ^α	α^{th} reaction rate density	Sect. B.3.a.
$\Lambda_{S_k}^{\xi_k}, \Lambda_{S_4}^{\xi_4}$ $\Lambda_{B_k}^{B_k}, \Lambda_{B_4}^{B_4}$	Lagrange multipliers	(D17) ₁₋₄
μ_b	Chemical potential in equilibrium of a constituent	(E43)
μ_0	Vacuum permeability	Sect. B.4.e.
ν_{ec}, ν_{ic}	Electron, ion collision frequency	Sect. A.1.
$\vec{\xi}_a(\vec{S}, t)$	Surface density of Ξ_a	(B21)
Ξ_a	Arbitrary surface-related field property of a constituent	(B21)
π	Mean pressure of a mixture	(D45)
ρ_a	Mass density of a constituent	(B5)
ρ	Mass density of a mixture	(B6)
σ_a^Ψ	Volume supply density of Ψ_a	(B26)

$\partial_V \psi_a$	Volume production rate of ψ_a	(B26)
$d_i \xi_a$	Surface supply density of ξ_a	(B28)
$\hat{\partial}_a \xi_a$	Surface production density of ξ_a	(B28)
\sum	Summation	(B6)
τ_e, τ_i	Electron, ion collision time	Sect. A.1.
u_e^c, u_i^c	Electron, ion collision frequency	Sect. A.1.
T_a^ψ	Total surface supply density by $s(t)$ of ψ_a	(B26)
T_a^ξ	Total line supply density by $c(t)$ of ξ_a	(B28)
Φ_a^ψ	Flux of ψ through a	(B26)
Φ_a^ξ	Flux of ξ through a	(B28)
Φ_M	Magnetic flux	(B73)
$\vec{X}_a^+(x, t)$	Continuous motion of a constituent	Sect. B.a.
$\psi_a^V(x, t)$	Volume density of ψ_a	Sect. B.c.i.
Ψ_a	Arbitrary volume-related field property of a constituent	(B16)
E_Ψ	Specific equilibrium free energy of a mixture	(E8)
Ψ^*	Specific (equilibrium) free energy of a mixture in the absence of an electromagnetic field	(E15)

$\omega_{e,c}, \omega_{i,c}$	Electron, ion cyclotron frequency	Sect. A.1.
ω_p	Plasma frequency	Sect. A.1.
Ω_k^a	Specific internal angular momentum of a constituent	(B109) ₁
Ω_k	Specific internal angular momentum of a mixture	(B122)

Subscripts

a, b, c, ...	Arbitrary constituent	Sect. B.a.
e, i	Electron, ion constituent	Sect. A.1.
A, P	Axial, polar vector	(C4) ₂₋₃

Superscripts

\sim	Material derivative of a constituent	(B14)
\cdot	Material derivative of a mixture	(B15)
\sim	Volume production rate density of a given field quantity for a constituent	Sect. B.3.a.
$(\sim)^S$	Surface supply density of a particular constituent field quantity	Sect. B.3.a.
$\bar{\quad}$	Average value of a field quantity	(D12) ₁
*	Field quantity considered in the absence of an electromagnetic field	(E15)
E	Field quantity in equilibrium	Sect. E.2.
L	Linearized field quantity	Footnote C35

Miscellaneous

$ $	Determinant	(C4) ₃
$[]$	Jump brackets	Sect. B.c.
$\frac{\partial}{\partial t}$	Derivative with respect to time	(B9)
$<$	Less than	(C34)
\ll	Much less than	Sect. A.1.
$=$	Equals	Sect. A.1.
$>$	Greater than	Footnote B72
\gg	Much greater than	Sect. A.1.
\geq	Greater than or equal to	(B151)
\propto	Proportional to	Sect. A.1.
∞	Infinity	Sect. B.a.
n	Intersection	(B17) ₂
$() \cdot ()$	Scalar product	(C26)
$() \times ()$	Vector product	(B54)

A. INTRODUCTION

A plasma¹ may be understood² to be any collection whatsoever of charged particles (e.g. electrons and ions) for which there exists cooperative behavior. This coherent material response is due essentially to the collective nature of the mutual space-charge (i.e. Coulomb) interactions between said particles.

A.1. On representative plasma systems

Both natural as well as laboratory plasmas find, at least in a statistical continuum description, a partial characterization which is based upon the assumed existence of certain "microscopic"³ parameters. The quantities which are commonly taken in particle models of plasmas include, among others, the respective particle number densities, the system related Debye and mean free path lengths, the Larmor radii and the charged particle cyclotron frequencies, and so forth.

To the extent that the theoretical concepts from which these various quantities derive are physically meaningful, one can utilize these parameters in terms of their relative magnitudes and numerical values as criteria for identifying the different plasma regimes. Indeed, it is possible in a general sense to note some of the different types of plasma systems by setting down, as in Table 1⁴, some of their characteristic property values; and, pictorially as is done in Fig. 1⁵.

The general topography of gaseous plasma physics having been given, it is now worthwhile to note further the possibility of classifying the various plasmas shown; and this most significantly in terms of the

Table 1. Natural and laboratory plasmas: selected characteristic properties.

	DIMENSION	ELECTRON DENSITY	NEUTRAL DENSITY	TEMPERATURE	MAGNETIC FIELD	PLASMA FREQUENCY	COLLISION FREQUENCY	ELECTRON CYCLOTRON FREQUENCY	DEBYE LENGTH	CONDUCTIVITY
	L	n_e	n	T	B	ω_p	ν_c	ω_H	λ_D	σ
	m	m^{-3}	m^{-3}	$^{\circ}K$	T	sec^{-1}	sec^{-1}	sec^{-1}	m	$ohm^{-1} m^{-1}$
LABORATORY										
Gas discharge	10^{-1}	10^{20}	10^{22}	10^4	10^{-1}	10^{12}	10^8	10^{10}	10^{-6}	10^4
MFD direct energy conversion plasma	10^{-1}	10^{21}	10^{24}	10^3	1	10^{12}	10^{12}	10^{11}	10^{-7}	1-10
Thermonuclear plasma	10^{-1}	10^{22}	0	10^8	10	10^{13}	10^5	10^{12}	10^{-5}	10^7
Liquid metal	10^{-1}	10^{29}	0	10^2	1	10^{16}	10^{14}	10^{10}	10^{-12}	10^6
SPACE										
Interplanetary plasma	10^{11}	10^7	0	10^5	10^{-8}	10^5	10^{-5}	10^3	10	10^5
Sunspot	10^7	10^{17}	10^{20}	$\frac{1}{2} \times 10^4$	1	10^{10}	10^6	10^{11}	10^{-5}	10^4
Earth interior	10^8	10^{29}	0	10^4	10^{-3}	10^{16}	10^{16}	10^{10}	10^{-13}	10^6

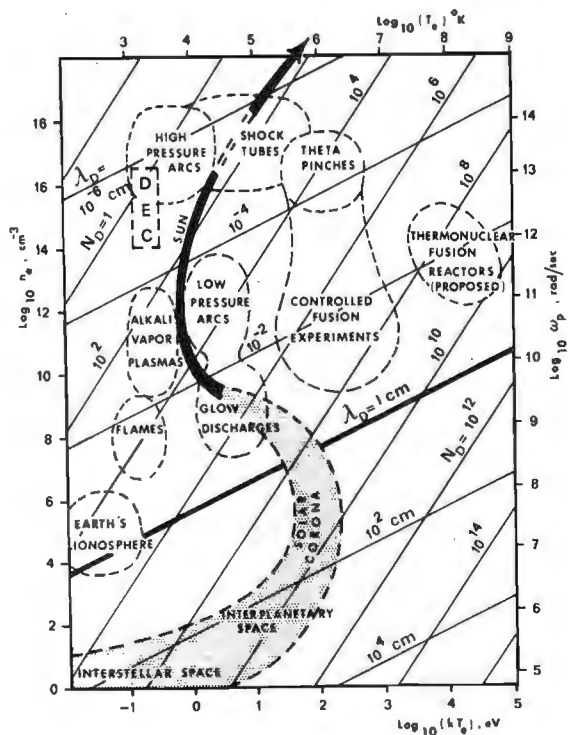


Fig. 1. Approximate regions of diverse important gaseous plasmas. (From [1966, 8, p. 8]. Used with permission of McGraw-Hill Book Company.)

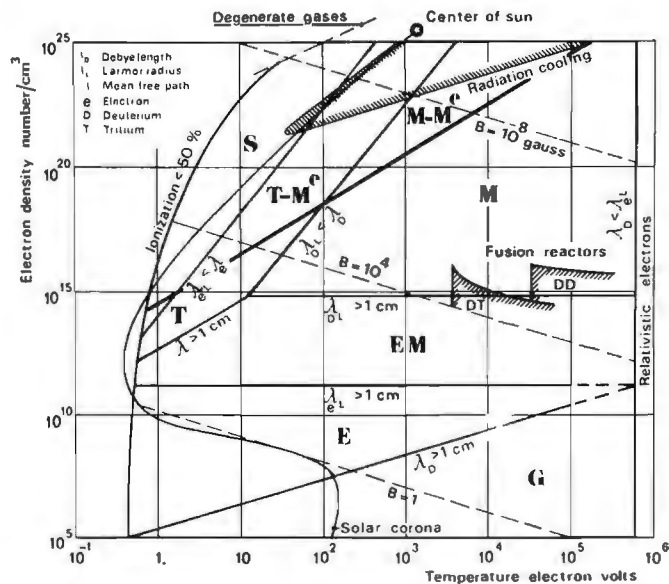


Fig. 2. A characteristic length classification of regions of deuterium gas magneto-fluid-dynamics. (By permission of A.R. Kantrowitz [1960, 4, p. 960].)

material response possibilities of said plasmas. Motivated by the needs of astrophysical and controlled thermonuclear fusion plasma research, Kantrowitz & Petschek⁶ prepared, on the basis of particle arguments, the particular "equilibrium" classification given in Fig. 2.

Because of their relative importance in influencing plasma behavior, the thermodynamic variables of electron number density (n) and temperature (T) were taken, together with the strength of the magnetic flux density (\vec{B}), as the parameters in terms of which this classification is expressed. The specific choice of deuterium here follows from practical advantages it offers: viz., it has only one valence electron and can thus form only one ion sort; and, it is suitable for investigation of high-temperature plasma behavior as occurs, e.g., in fusion research.

An additional important condition placed upon this classification is that of the ratio (β)⁷ of the ideal gas kinetic pressure ($p = nkT$) to that of the magnetic flux density field ($B^2/2\mu_0$) being equal to unity. Indeed, special emphasis has been placed upon this minimum limit case for plasmas of magneto-fluid-dynamics (MFD) wherein the coupling between the plasma and the electromagnetic field is strong and below which "useful" MFD is not to be expected⁸.

Let λ_D denote the Debye length for electrons⁹. This important plasma quantity defines for an equilibrium plasma a Debye sphere of radius λ_D about a given charged particle; which, given a "sufficient" number ($N_D = n\lambda_D^3$) of charged particles therein, shields that particle from the Coulomb force field due to other charged particles lying outside said sphere. The parameter λ_D is a measure for the relative importance of space-charge effects in a given system here and serves thus

as a criterion for identifying those systems which are (not) to be considered as plasmas. Conventionally, if L denotes a characteristic geometric length of a particular system (cf. Table 1), then only those systems for which $\lambda_D \ll L$ are called ¹⁰ gaseous plasmas. For the purpose of Fig. 2, L has been taken as being 1 cm

For a statistical continuum approach to the study of plasmas to be physically sensible, it is at least necessary that the averages of the various quantities involved (e.g. charged particle number densities) exist over a volume λ_D^3 . This requirement is customarily satisfied ¹¹ by the introduction into the considerations of such an approach, of the condition that $N_D \gg 1$. Further, using λ_D it is possible to introduce the (electron) plasma frequency ω_p ($\propto n_e^{1/2}$). This important parameter is characteristic of the rate of longitudinal oscillations of electrons in a plasma due to electrostatic fields; and, as such, is a measure of the number density of that particle.

In terms of models for the collisional interactions between particles it becomes possible to secure expressions for the approximate average distance traversed between particle encounters. These lengths are called mean free paths and they are important characteristic microscopic properties of a given system. Here, they are indicated respectively for the electrons and the one type of ion involved by λ_e and λ_i . The otherwise possible role played by neutral particles has, it may be observed, been taken in Fig. 2 as being negligible since the gas in question is "fully" ionized ¹². Related to these collisions of, but not necessarily between, electrons and ions are respectively the times τ_e ($= 1/\nu_e$) and τ_i ($= 1/\nu_i$); where ν_e and ν_i represent the indicated particle collision frequencies.

As is well-known, the trajectories of charged particles in a \vec{B} field are in general helices. As such, the motion of a charged particle can be decomposed into two parts. The one part is a motion parallel or anti-parallel to \vec{B} ; while, the other is a circular motion with a charge-dependent direction in a plane normal to \vec{B} . Characteristic of this latter motion for the case of electrons (ions) is the cyclotron frequency $\omega_{e(i)c}$ and its related Larmor radius $\lambda_{e(i)L}$. Lastly, I introduce here for the electrons and ions the not unimportant Hall parameters β ($= \omega_{e(i)c} \tau_e = \omega_{e(i)c} / \nu_{e(i)c} = \lambda / \lambda_{e(i)L}$)¹³ and β_i . These last quantities are (where τ_e defined) characteristic for a given plasma of, for example, the degree of anisotropy induced respectively in the electron and ion gases by the \vec{B} field related to that system.

Although Fig. 2 relates most directly to a deuterium plasma and not to any of the other gaseous plasmas found in the systems of Fig. 1, it is worthwhile nonetheless to consider both figures together. It is necessary in this regard to appreciate the fact that, at least in principle, it is possible to prepare for any of the other (e.g. laboratory) plasma regions of Fig. 1 a classification such as that of Fig. 2. At most it is to be expected that a certain shifting would occur of the lines of Fig. 2 which define the regions thereof to be discussed now. The qualitative features of that figure, which I wish to exploit for illustrative purposes here would, however, remain¹⁴.

"High"-density gas plasmas: region S

For such plasmas¹⁵, electron and ion collision processes dominate over those of a space-charge electric or magnetic field origin. The velocity distribution functions of these particles are maxwellian at the

"same" temperature; and, the isotropy of the system is evidenced in particular by the fact that with $\beta_e \ll 1$ and $\beta_i \ll 1$, its transport properties (e.g. electrical conductivity) are scalars.

"High-to-medium" density gas plasmas: regions T and T-M^e

With decreasing n and increasing T , the \vec{B} field assumes control over the electron motion while the ion behavior, as in region S, is influenced predominantly by collisions involving those particles. Here too, the respective particle types possess maxwellian velocity distribution functions, but not necessarily at the same temperature¹⁶. Noting in Fig. 2 the line dividing regions of unequal temperature, it is seen that T-M^e is the portion of region T where electrons and ions enjoy the same temperature.

In this case $\beta_e > 1$ while $\beta_i < 1$; the consequence of which is that the plasma has mixed transport properties. That is, the system now has anisotropic (i.e. tensor) material response with regard to those properties (e.g. electrical conductivity, diffusion and thermal conductivity) which are strongly dependent upon the electrons. On the other hand, however, the isotropy of the plasma is preserved here in terms of the still scalar ion-dependent properties (e.g. viscosity).

"Medium"-density gas plasmas: regions M and M-M^e

In the M region, for which n and T can respectively be yet lower and higher, the ions too succumb to the now irresistible influence of the \vec{B} field upon their motion. Thus, with $\beta_e \gg 1$ and $\beta_i \gg 1$, the anisotropy of the plasma induced by the \vec{B} field is total, for all of the transport properties of the system are now of a tensor character. Not unimportant

is the fact that the tendency which arose in region T for the electrons and ions to uncouple themselves energetically continues here. Aside from the region M-M^e where the rate of energy transfer between these gases is presumed sufficiently high to keep them at the same temperature, it is possible (at least in principle) that in region M each gas assumes a well-defined temperature of its own ¹⁷.

The extent to which this actually occurs, if indeed it does occur, is now dependent upon additional factors related to the plasma system. With the decline in importance of the collisional processes, there appears ¹⁸ the possibility of, e.g., non-linear wave-particle and wave-wave interactions which constitute a fundamentally different means of dissipating energy; and consequently of equilibrating the electron and ion temperatures.

"Low"-density gas plasmas: regions EM and E

For the EM region the \vec{B} field is still the major factor influencing the behavior of the electrons; and thus, with $\lambda_{Le} < L$, the condition $\beta \gg 1$ holds. Here, however, $\lambda_{Li} > L$ and the space-charge electric field has thus essentially taken over the role of controlling the ion motions. In this case the transport properties of a given system, if defined, remain anisotropic.

With a yet further diminishing of n_e the motion of the electrons, like that of the ions, becomes primarily dependent upon said electric field; it having replaced the hitherto important \vec{B} field. This is the region E for which $\lambda_{Le} > L$ and $\lambda_{Li} > L$.

"Very low"-density gases: region G

When n and T are such that $\lambda_D > L$, the system of interest has passed from a plasma condition to that of an ionized, but collisionless¹⁹, gas of very low density. Such systems lie in region G; but, due to the fact that a (statistical) continuum description of them is not possible, any consideration of these systems lies beyond the scope of this study.

Boundaries

The low-density (i.e. lower) boundary of this classification is, as discussed above, that line for which $\lambda_D > L$. Next, the low-temperature (i.e. left) boundary here occurs at the line denoting a 50 per cent degree of ionization²⁰. With increasing n the high-density (i.e. upper) boundary is defined essentially at all points where the plasma must be treated with quantum mechanical methods; this being due to the electron spin-spin interaction degeneracy which appears.

A further increase of n , although not shown here would²¹ indicate the passage of the system through the liquid and solid phases of matter, through a region of relativistic degeneracy, and eventually into the region of matter existing at nuclear densities; such as white giant and white dwarf stars, novae, etc. While it does not seem to be known if these latter circumstances of matter satisfy the definition of a plasma used here, plasma phenomena are known²² to exist under certain conditions in the former region of matter (i.e. liquids and solids) that is "somewhat" more condensed than the gaseous systems considered here. I am thus lead to remark that in general, plasma, rather than being a "fourth state" of matter²³, may perhaps be more validly thought of

only as a condition thereof. This contention is strengthened somewhat by the fact that even with the high-temperature (i.e. right) boundary of Fig. 2 given as that line for which the speed of electrons closely approximates that of light, it is possible ²⁴ to further treat the properties of that gas using the concepts of relativity theory. Although relativistic plasmas do exist ²⁵, they too, "like" the quantum plasmas mentioned above, are not considered here.

A.2. Position of the problem

A.2.a. Purpose

The formal description of the mechanics of a gaseous MFD plasma has developed essentially along lines analogous to those found in the classical mechanics of fluids. This more fully matured discipline ²⁶ offers two methods for attacking a particular problem: the particle approach and the classical continuum approach. An investigator of plasma behavior may find upon examining the physical conditions characteristic of a certain problem (e.g. the particle densities) that one or the other of these treatments is to be preferred.

Statistical continuum approach

A statistical continuum approach ²⁷ aims in part to determine, via a treatment of the presumed microscopic nature of a given system's material components, the quantities which enter into the macroscopic system equations (e.g. the so-called ensemble averaged parameters ²⁸). For the case of, say, gaseous plasmas, this approach concerns itself also with the physical composition of a given system, the "structure" of its ma-

terial components, and the microscopic nature of the interactions which involve these components.

When the plasma density, for example, is such that an assumption of "continuum" properties of the system is physically reasonable (such as in regions S, T, T-M^e, M and M-M^e here ²⁹) one can augment the individual particle approach (needed in regions E, EM and G ³⁰) by certain statistical considerations in order to secure the macroscopic equations of that system. Of importance in general, but in particular for this study, is the fact that in the area of gaseous plasma physics deemed suitable for laboratory MFD energy conversion purposes (cf. Fig. 1), research based upon such an approach is carried out ³¹ even though the condition $N_D \gg 1$ is clearly not satisfied.

With regard to plasma systems in general, partially ionized gases or otherwise, it is a no less significant point here that the present body of microphysical knowledge of, e.g. interparticle potential energy functions ³², non-adiabatic particle interactions ³³ and process cross-sections ³⁴, appears to be insufficient to permit the establishment of an adequate statistical continuum model for various plasmas of interest ³⁵. The consequences of such a situation are reflected, e.g., in the results of any classical mean free path theory ³⁶; which type of theory, it may be further noted, is itself open to doubt ³⁷. It seems, in addition, that the Chapman-Cowling formalism ³⁸ which is an important factor in contemporary plasma physics research, has also been questioned ³⁹ with regard to both its degree of relative generality as well as its applicability under particular circumstances.

Generally speaking, for this approach seen in regard to plasma physics, the concepts of (non-)equilibrium classical kinetic theory and statistical mechanics are thus seen to serve as one of the bases from which the conventional MFD system equations have been obtained.

Classical continuum mechanical approach

The statistical continuum approach was, most characteristically, predicated upon a presumed knowledge of the particulate nature of a material system. This is not, however, the only manner in which the macroscopic behavior of plasma systems may be studied.

The plasma investigator also has the option, under appropriate conditions, of attempting to describe a given system by means of a classical continuum mechanical approach⁴⁰. This alternative technique, which for MFD is neither more nor less general than the statistical continuum method⁴¹, provides for the examination of a given system by assuming from the beginning a macroscopic point of view. In "neglecting" the particle nature of matter, the so-called physical insight into the behavior of the system in question is taken to be relatable to a hopefully sufficient degree of carefully obtained macro-phenomenology.

Of special interest here is the fact that, regardless of the necessity of a multi-fluid description of MFD plasmas⁴², the classical continuum mechanical approach has established in this problem area essentially only a single-fluid treatment of plasmas⁴³. Further yet, when the development of the system mechanics is complemented by classical irreversible thermodynamics⁴⁴ considerations, serious objection has been raised⁴⁵ with regard to the derivation of the basic system equations.

For an MFD problem thus, a statistical continuum approach treats the plasma in terms of its various material components; attempting thereby to derive the macroscopic equations of a given system from considerations relating to the individual particle natures of said components. On the other hand, the method of classical continuum mechanics seeks the basic macroscopic relations by initially neglecting the discrete material nature of a system, then postulating the necessary kinematical and dynamical expressions, and finally supplementing the (semi-)empirical information relevant to a particular problem.

Both of these approaches, although differing in attitude, have been shown for MFD circumstances⁴⁶, to arrive at essentially the same macroscopic equations. Unfortunately, both approaches also appear⁴⁷ to be unable to provide a (adequate) general macroscopic description of the (non-)linear dissipative, i.e. irreversible thermodynamic, processes which are related to non-equilibrium plasma systems.

On the basis of these considerations, together with the earlier mentioned shortcomings of these two approaches to the study of dynamic plasma behavior, I conclude the following: viz., that it is desirable to examine the feasibility of securing a formalism that could offer for the same problem area a description with fewer limitations than that of the statistical continuum method, and a more detailed insight into the macro-physical character of a given plasma system than that given by the classical continuum mechanical method. This constitutes essentially the purpose of this study.

A.2.b. Problem

The general problem which this study attempts to provide a contribution towards solving is that of the establishment of a unified, self-consistent and "exact" theory of the non-linear macroscopic mechanics, electrodynamics and energetics of finite and bounded real multi-continua plasma dynamic systems. In terms of the actual systems concerned (cf. Fig. 1), this problem poses in general the necessity of describing the material behavior of real, chemically reactive, radiative, anisotropic, compressible, inhomogeneous, magnetizable, polarizable, non-equilibrium and non-linear electro-magneto-mechano-thermo-dynamic mixture systems.

I hasten to point out that this is not just an arbitrary hypothetical problem. It is concerned with real material systems and its solution in some physically more adequate sense than has been presented to date is not only highly desirable, but perhaps even urgent⁴⁸.

This problem unquestionably reflects a framework within which a generalization of the contemporary approaches to laboratory plasma physics may be sought. However, and not surprisingly, for numerous physical and mathematical reasons it does not appear to be possible to bring the problem as stated above to a "satisfactory" closure at this time. Hence, I have not attempted to consider said problem in the degree of generality expressed above; but, I propose instead to examine it here in the more restricted sense which I shall now discuss.

Rational mechanical approach

The discipline of rational mechanics, or modern natural philosophy as it is less commonly called, is in its modern form relatively young

(dating from the mid-1950's⁴⁹); but, its tradition lies essentially in the eighteenth, and to a lesser degree the nineteenth and early twentieth, century efforts of a handful of men⁵⁰. It is now, as an extension of geometry⁵¹, fundamentally a mathematical science with the purpose of establishing a rigorous theoretical foundation for the study of the physical behavior of material continua.

The basis here for a non-linear field theory treatment of gross irreversible material behavior rests essentially upon three general sets of assumptions. The first concerns itself with the topological character of the space involved; while the second and third relate, respectively, to the kinematics and dynamics of the class of systems whose behavior is to be described.

Based upon this foundation rational mechanics in particular strives, via use of an axiomatic approach to its subject matter, to secure physically well-motivated theoretical models of material systems which are explicit, self-consistent⁵², and complete⁵³ in their construction. This approach, as explained by Truesdell⁵⁴, consists of four elements: viz., primitive quantities, definitions, general axioms, and proved theorems. In terms of the physical primitive quantities which are defined only to the extent that the mathematical properties are given, the definitions of additional system-related quantities can be made. Relationships satisfied by these two elements are physical axioms ("laws") and here they are of two types: kinematical and dynamical.

Axioms of kinematics are represented by the set of balance equations proposed for the type of system under consideration; and, they are valid for the class of all such systems. On the other hand, the axioms of

dynamics are represented by the set of "appropriately invariant" and thermodynamically restricted constitutive equations postulated; which expressions make the theory system-specific in terms of its material response possibilities.

Experience and not, of course ⁵⁵, experiment is the guide which motivates those assumptions as are made. Rational mechanics draws its physical motivation from the body of common experience (e.g. "phenomenology") relevant to the specific problem area(s) under study; while, its theoretical motivation derives in part and to varying degrees from the conceptual backgrounds of those other disciplines which concern themselves with the same subject matter. Here (cf. Fig. 3) these are, e.g., statistical and classical continuum mechanics. With the establishment of the proved theorems and their related corollaries ⁵⁶ for a given theory, the formal structure of that particular rational mechanical description is taken to be complete.

In the relatively young discipline of rational mechanics there are a number of non-trivial open questions which serve to make this field a controversial one at the present time. For example, the number of basic principles involved and their precise formulation ⁵⁷, together with the (degree of) applicability of an axiomatization procedure to physical theories ⁵⁸ remain debatable subjects. Regarding both of these points, I shall endeavour to follow an enlightened course in the sequel ⁵⁹.

It is on the basis of two points that I propose here a rational mechanical approach to the problem of describing the general material behavior of plasma systems as a, if not the most, worthwhile of the presently available alternatives. The first point to be noted is the

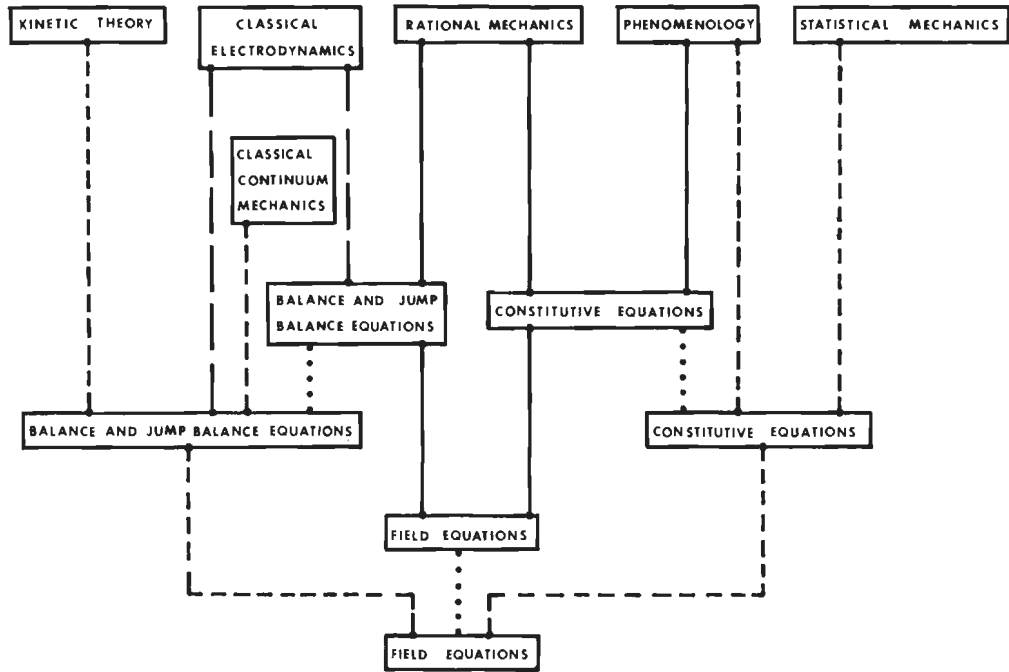


Fig. 3. Rational mechanics: its results and some of its relationships.

fact that a theory formulated thusly possesses in contrast to numerous, if not the majority, of plasma theories secured otherwise, not only a conceptual simplicity, clarity, rigor and a not undesirable aesthetic quality; but also, with regard to the eventual establishment of results of permanence⁶⁰, the capacity to aid in making itself as self-correcting as might possibly be expected⁶¹. The second, independent and here also important point is the recognition of the striking theoretical and experimental successes of this discipline in other areas of experience⁶²; and these with regard to problems the solution of which lies beyond the present capacity of the traditional approaches to provide⁶³.

Rational mechanics strives towards a unification via mathematics of the diverse physical sciences; and hence, it does not restrict its interests and efforts to any one of them in particular. It is in the above indicated sense that the word "rational" is employed here. I mean hereby no offence to those persons who by inclination or circumstance select to treat physical problems in terms of ad hoc models, approximative methods, "physical intuition", and soforth.

As observed earlier, the establishment of a theoretically "adequate" description of the gross non-linear irreversible thermodynamic behavior of fluid mixtures is a problem meriting study for its own sake; but also one deserving attention on the basis of practical considerations. With the applicability of rational mechanics to the investigation of gaseous plasma systems constituting a main purpose of this study, I now proceed to further delimit the scope of same. The restrictions placed upon this study concern, naturally enough, both its content and its generality.

Regarding content, as seen in relation to the aforementioned general description, I wish to present in greater or lesser degree the salient points pertinent to a particular rational mechanical theory: viz., that for a one-temperature, chemically reactive (but radiationless), non-relativistically moving, magnetizable dielectric fluid mixture. The point of view adopted here is that the basis for said treatment is given in the form of a complete set of system balance and jump balance equations together with the "appropriate" constitutive equations indicated therein. The solution of these relationships for the as yet undetermined field quantities, under the boundary and initial conditions that delimit the problem of interest, is taken to fulfil the main objective of a thermodynamic constitutive theory for the problem posed⁶⁴.

The content of the treatment of the constitutive equations will be based upon certain selected rational thermodynamic and modern constitutive theoretical principles. Due, however, to the as yet unresolved nature of various fundamental problems hereto related, the content of the rational mechanical treatment of constitutive equations will not have the "absolute" character of generality as found in the set of balance equations secured by employing the same formalism.

Among these problems are, e.g., the mixed invariance properties (Euclidean-Galilean/Lorentz invariance of the mechanical/electromagnetic field equations) of the system, and the problem of "correctly" identifying and using entropy concepts in a modern continuum theory⁶⁵. Regardless of these difficulties, the results that can be derived are, with all due respect for the various points of interpretation, of equal or greater macroscopic generality than those found from conventional approaches.

Liu & Müller⁶⁶ have recently examined the problem of describing the material behavior of single simple⁶⁷ heat conducting continua in an electromagnetic field. In this study I undertake the specific task of extending, in some respects, their study to a particular case of non-simple heat conducting Eulerian fluid mixtures which are also in an electromagnetic field.

For systems of this type where the mixture may contain a ($a = 1, 2, \dots, s$) constituents, I take the foremost objective of a theory such as that contemplated above to be the following: viz., the calculation, as functions of position and time, of the field quantities

$$\rho_a(\vec{x}, t), \quad T(\vec{x}, t), \quad v_a(\vec{x}, t), \quad E_i(\vec{x}, t) \quad \text{and} \quad B_i(\vec{x}, t). \quad (1)$$

Here, the fields are respectively the mass density of constituent a , the absolute temperature of the constituents and the mixture, the velocity of constituent a , the total electric field taken with respect to a stationary observer, and the magnetic flux density.

In a field theory the macroscopic properties of a material system are considered to be field quantities. Generally speaking, all of the relationships which relate the different field quantities to one another are field equations. In this study (cf. Fig. 3) the term "field equations" shall be taken to mean the set of balance and jump balance equations for all the various macroscopic properties of a given system; together with a set of constitutive equations taken for a specific material system.

The possible meaningfulness of a transition from the more familiar approaches of plasma analysis to that of rational mechanics is not

necessarily made evident with the determination of a set of balance and jump balance equations (cf. Chpt B). Under less general circumstances than those considered here, said results when contrasted with those of more conventional approaches may give the false impression that rational mechanics has little new to offer.

All the (dis)similarities to other treatments notwithstanding, the statement above would constitute a prematurely formed conclusion by an individual who does not yet realize that any treatment whatsoever of only the kinematics of a system represents at best only a partial, albeit important, descriptive element of that system. Thus, Chpt. B shall present the essentials, for this study, of and related to the rational mechanical kinematics of a mixture of material and electromagnetic field continua. It becomes then in greater measure the burden most particularly of Chapters C and D treating, respectively, constitutive theory and the problem (introduced in Chpt. B) of entropy, together with the subsequent chapters, to make clear in some respects the possible favorability above the more traditional methods of a rational mechanical investigation of plasma continua behavior.

Footnotes to Chapter A

- ¹ The origin of this word seems to lie with the introduction of the word "protoplasm" in the nineteenth century by the biologist Purkinje. The meaning of "first plasma" was subsequently given to this initial term by von Mohl, a botanist. In the particular context of plasma physics here, the word "plasma" is commonly attributed to Tonks & Langmuir [1929, 1, p.196, footnote 5]; who, perhaps, in studying arc discharges observed the jelly-like behavior of the medium and were motivated to use this word.
- ² Kunkel [1966, 8, p.3]. Note, there exist other definitions of plasma in the literature (cf. for example Sutton & Sherman [1965, A, p.6]), but these are essentially special cases of that taken here (cf. Kunkel [1966, 8, p.5]).
- ³ Shkarofsky, Johnston & Bachynski [1966, 12, pp.2-3] make a finer distinction between "microscopic" and "macroscopic" than that which I employ here. On the basis of an argument related to the combination- and division-invariance of charges, the Debye length and the plasma frequency are considered to be macroscopic parameters there.
- ⁴ This table is adapted from that given by Bueren [1966, 2]. The values for magneto-fluid-dynamic plasmas added here were calculated from selected experimental results presented in [1966, 9].
- ⁵ Kunkel [1966, 8, p. 8]. Here too, the region denoted for magneto-fluid dynamics (MFD) direct energy conversion (DEC) systems has been added. It is perhaps worthwhile to observe that only gaseous plasmas are represented in this figure. Other known plasma systems which could supposedly be introduced into, and hence generalize, this overview include, among others, the following:

liquid metals (Kirko [1965, 10]), semi-conductors and semi-metals (Anker-Johnson [1966, 1]) and metals ([1965, 6]).

- ⁶ Kantrovitz & Petschek [1957, 3, p.5]. This classification, which is slightly adapted, is but one of a number of various types of classifications that exist in the literature of plasma physics.
- ⁷ The "plasma beta" is discussed, e.g., by Glasstone & Lovberg [1960, 5, pp.52-53] and Krall & Trivelpiece [1973, 5, p.7].
- ⁸ Sutton & Sherman [1965, 17, p.10]. I note that there exist at present several names for the subject area taken under consideration. Examples include plasma dynamics, magneto-gas-dynamics, magnetohydrodynamics, and so forth. I select the term "magneto-fluid-dynamics" for two reasons: first, it reflects all fluid and fluid-like media; and, it permits a distinction to be made with, e.g., magneto-solid-dynamics. This last mentioned subject which relates to, say, magneto-elastic media does not enter into the considerations of this study.
- ⁹ A Debye length is, of course, definable for ions. Further yet, with some quantum mechanical modifications, the concept of Debye length is adaptable to the case of solid-state plasmas.
- ¹⁰ Kunkel [1966, 8, p.4], Shkarofsky, Johnston & Bachynski [1966, 12, p.2] and Spitzer [1962, 8, p.22].
- ¹¹ Kunkel [1966, 8, p.6], Shkarofsky, Johnston & Bachynski [1966, 12, p.3] and Krall & Trivelpiece [1973, 5, p.4]. With regard to meaning of this "plasma approximation" condition, there appears to be some difference of opinion in the literature. The last two cited references seem to hold to the position that with diminishing N_D one can speak less and less of the systems as a plasma, and that a fluid description

of a medium with very low N_D , say less than or about equal to unity, is not valid. Gottlieb [1965, 8, p.46], on the other hand, expresses the view that the number of charged particles per cm^{-3} may be any number and the system in question is still a plasma as long as it satisfies the condition $\lambda_D \ll L$. I might further add that inasmuch as the degree of ionization does not enter into the definition of plasma employed here, it seems reasonable to state that if there are "enough" neutral particles in that cm^{-3} , a continuum description appears to be possible. Lastly, it may be noted that regardless of the relative importance of this point, not all authors (cf., e.g. Sutton & Sherman [1965, 17]) are explicit with regard to their introduction of it into their respective treatments.

- ¹² Alfven & Fälthammer [1963, 1, p.134 and p.180] state that a plasma is slightly (highly) ionized if the degree of ionization is less (greater) than 1 per cent.
- ¹³ Sutton & Sherman [1965, 17, p.156].
- ¹⁴ Petchek [1958, 4, p.967].
- ¹⁵ Cf., for example, Alfven & Fälthammer [1963, 1, pp.169-1970] with regard to the density-dependence of plasma properties.
- ¹⁶ Depending upon the relative (in)ability of the bulk electrons, ions and possibly neutral particles to exchange energy between themselves at a sufficiently high rate, the respective kinetic temperatures of these gases may be (un)equal; in which case the system in question is said to be in a condition of thermal (non-)equilibrium.

An awareness that multiple-temperature plasmas can exist is evidenced at least as early as 1929 (cf. Tonks & Langmuir [1929, 1, p. 201]). Kinetic theory considerations of this possibility were given

shortly thereafter (v. Engel & Steenbeck [1932, 1; 1934, 1]); and, numerous authors have since discussed (aspects) of this subject.

The first examination of this matter in rational mechanics seems to lie in the work of Truesdell [1957, 5, p.162]. Here also, this point has been taken under consideration by other researchers in this field (cf., e.g., Dunwoody & Müller [1968, 2]).

- 17 Clearly, the possibly important role played by, e.g., the radiation field with regard to the establishment and maintenance of individual component temperature fields should be given close attention in any particular problem. This study, while recognizing this fact, limits its scope to those systems for which the physical consequences of said radiation field on the material response of the plasmas involved are negligible.
- 18 This point is noted, e.g., by Fishman, Kantrowitz & Petschek [1960, 4, p.961ff], Sagdeev & Galeev [1969, 6] and Tsyтович [1970, 7].
- 19 Sutton & Sherman [1965, 17, p.12]. See also Kantrowitz & Petschek [1957, 3, p.12]; in addition to noting footnote 10 above.
- 20 As shown by Linhart [1969, 3, p.21], the difference between this curve and that one for, say, 10^{-2} per cent degree of ionization is that the latter lies slightly to the left of the former in such a representation.
- 21 Cf. Fermi [1966, 4, p.171].
- 22 Among others, Shkarofsky, Johnston & Bachynski [1966, 12, p.5] make mention of this fact.
- 23 These words which Crookes [1879, 1, p.164] used in drawing his conclusions of a particular investigation of glow discharges (region E)

probably constitute the first formal recognition of the unusual material properties (i.e. "plasma behavior") to be observed in specific systems under particular circumstances.

- ²⁴ Cf., for example, Hakim & Mangeny [1968, 3], Lichnerowicz [1967, 15], Bano & Balescu [1965, 3, pp.577-586] and Wasserman & Wells [1963, 11].
- ²⁵ Buneman & Pardo [1968, 1], Parker [1967, 16, pp.243-244] and Finkelstein [1961, 2].
- ²⁶ Grad [1962, 3, p.38] and Eringen [1962, 2, p.393].
- ²⁷ Examples of works treating plasma behavior from this point of view include those of Pai [1962, 4], Fried [1966, 5], Jancel & Kahan [1966, 6] and Klimontovich [1967, 13].
- ²⁸ See here, e.g., Jancel & Kahan [1966, 6] with regard to the thermostatic (i.e. equilibrium thermodynamics) approach where use is made of partition functions; and also, the kinetic theory approach which employs the concept of velocity distribution functions.
- ²⁹ Alfven & Fälthammer [1963, 1, p.171].
- ³⁰ Ibid.
- ³¹ For example, Sutton & Sherman [1965, 17]. Note again the remarks of footnote 11 above.
- ³² Chang [1967, 2], Sandri & Yates [1966, 11, p.49], Hirschfelder [1965, 9], Borkaw [1965, 5, p.265], Hirschfelder, Curtiss & Bird [1967, 9, p.18] and Clarke & McChesney [1964, 1, pp.386-390].
- ³³ Hirschfelder, Curtiss & Bird [1967, 9, pp.928-930], De Barbieri & Maroli [1967, 3, pp.331-332] and Clarke & McChesney [1964, 1, p.348].
- ³⁴ Verboom [1969, 8, p.907], Rudge [1968, 6, pp.588-589], Kieffer & Dunn [1966, 7, pp.32-33], Shkarofsky, Johnston & Bachynski [1966,

- 12, p.172] and Clarke & McChesney [1964, 1, p.164 and p.376].
- 35 Grad [1962, 3, p.37]. Examples include dense gaseous, liquid, and two-phase plasmas. For the issue of, in the first instance, dense gases see Hirschfelder, Curtiss & Bird [1967, 9, p.652]. Egelstaff [1967, 4, pp.149-150] discusses the situation regarding liquids; while a commentary upon some aspects of dense plasma behavior can be found in Robinson [1967, 18]. Lastly, the matter of two-phase liquid metal plasmas is taken into consideration by, e.g. Petrick [1966, 10].
- 36 Cf. Shkarofsky, Johnston & Bachynski [1966, 12, p.172] and Sutton & Sherman [1965, 17, p.84 and p.90].
- 37 Cohen [1962, 1, p.110] and Sandri & Yates [1966, 11].
- 38 Chapman & Cowling [1970, 2].
- 39 Alfvén & Fälthammer [1963, 1, p.135]. See also the more specific comments of Goldman & Sirovich [1967, 7], Sutton & Sherman [1965, 17, p.126], Cambel [1963, 2, pp.165-166] and Petschek [1958, 4, p.970].
- 40 As a means for examining plasma behavior, the continuum approach used thus far seems, according to Stix [1962, 9, p.1 and p.170], to have had remarkable success. An example of such a development is given in Shkarofsky, Johnston & Bachynski [1966, 12].
- 41 Sutton & Sherman [1965, 17, p.295]. Unfortunately, presentations such as that given there do not do justice to the possibilities of a classical continuum mechanical approach. Further, of course, such a statement is only true to the extent that a successful statistical continuum model utilizing essentially more realistic potential energy functions for the particles involved in any given, but otherwise arbitrary,

plasma problem does not exist. This latter approach is in principle the more general of the two.

⁴² Slawsky [1961, 4, p.115].

⁴³ Gross [1958, 2, p.47]. See also footnote 40 above.

⁴⁴ A description of this discipline is given by de Groot & Mazur [1963, 5]

⁴⁵ Cf. Aono [1972, 1]. Further discussion of essential shortcomings of the classical thermodynamics of irreversible processes is to be found in the references cited in footnote 5 of Chapter C.

⁴⁶ See, for example, Krall & Trivelpiece [1973, 5, Sects. 3.5.1. and 3.5.2] and Sutton & Sherman [1965, 17, Sect. 8.2].

⁴⁷ Among others, van Hove [1957, 2], Fishman, Kantrowitz & Petschek [1960, 4], Uhlenbeck [1962, 11], Frieman [1967, 6], Thom [1967, 19, pp.70-71], Neufeld [1969, 4] and Tsytovich [1970, 7]. This comment is, of course, not meant to deny the successes enjoyed by these methods; but rather, it is meant to serve as a recognition of the fact that as the limitations of these approaches become clearer, so does the need for other methods to accomplish what they are not able to do.

⁴⁸ Cf. Prigogine [1962, 7, p.2], Monroe [1973, 8, pp.564-565] and Robinson [1965, 13, p.2]. It is perhaps necessary to add here that in the spirit of rational mechanics (Truesdell & Toupin, henceforth to be denoted CFT, [1960, 9, Sect. 6]), the approach I take to this problem is that of attempting to establish a general framework for theories of plasmas. I do not consider this study to have the purpose of accounting for an ever increasing number of specific phenomena as such.

- ⁴⁹ Truesdell [1960, 10].
- ⁵⁰ CFT [1960, 9, Sect. 9].
- ⁵¹ Truesdell [1965, 20, p.27] and Truesdell [1966, 13, p.89].
- ⁵² Cf. Truesdell [1960, 10, p.10]; and, regarding the significance of self-consistency in the study of plasma behavior, see the discussion given in Grad [1962, 3, pp.60-61]. See also footnote 53 below.
- ⁵³ Aside from the fact (cf. Truesdell [1969, 7, p.6]) that there is no unique axiomatic basis, but an infinity of equivalent such bases, the impossibility of establishing a complete set of axioms (cf. Gödel's incompleteness theorem in Nagel & Newman [1956, 1] for any given base leads to the conclusion that such an approach is "incomplete". See, however, the remarks in Nagel & Newman [1956, 1, p.1695].
- ⁵⁴ Truesdell [1969, 7, pp.5-7]. I proceed here with the assumption that there is sufficient physical motivation to support the adoption of an axiomatic approach. See in this regard, e.g., Synge [1960, 8, p.5, footnote 1], Truesdell [1966, 13, p.94], Truesdell [1966, 14, p.3], Lindsay & Margenau [1957, 4] and Bunge [1967, 1].
- ⁵⁵ CFT [1960, 9, Sect. 3] and Slawsky [1961, 4, p.116].
- ⁵⁶ The literature of rational mechanics (cf. for example, Archive for Rational Mechanics and Analysis) contains numerous papers containing proved theorems and lemmas of a, say, mechanical and energetic character.
- ⁵⁷ Rivlin [1972, 7].
- ⁵⁸ Rivlin [1970, 6, p.134]. Note here further Truesdell [1966, 14, p.3].
- ⁵⁹ With regard to the principles of balance and constitutive nature I

note here in particular the erudite, and in many respects definitive treatises CFT [1960, 9] and Truesdell & Noll [1965, 19] which have become standard works in the field of rational mechanics. In addition to the literature cited, I mention also the work of Müller [1973, 9] upon the basis of which this effort also depends.

- ⁶⁰ Cf. Miles [1963, 7, p.317] with regard to CFT [1960, 9]; and note further Truesdell [1966, 14, p.3].
- ⁶¹ See Biot [1965, 4, pp.x-xi].
- ⁶² Note here, for example, Coleman, Markovitz & Noll [1966, 3] and Bell [1973, 1].
- ⁶³ This fact seems clearly to support the following point which is unfortunately so often forgotten in the professional sciences: viz. (Truesdell [1966, 14, p.4]), that "when you have the right basic concepts, the solving of problems becomes either easy or impossible. When you understand what the problem is, the solution is far easier to find, if it can be gotten at all".
- ⁶⁴ Clearly, a measure of the above mentioned adequacy is the extent of (dis)agreement between the predictions of the system model proposed and the available empirical knowledge relevant to that system. Thus, for the nature of a given system's physical response being found in the constitutive equations employed, the better the "physical insight" of the investigator with regard to the system under study, the more appropriate will be his constitutive equations and consequently the physically more adequate will be his theory. Note also the discussion of Sect. F.3.
- ⁶⁵ See Truesdell [1966, 13, pp.99-100], Grad [1961, 3, p.323] and footnote 57 above.

66 Liu & Müller [1972, 5].

67 The meaning in rational mechanics of the word "simple" is discussed in Truesdell & Noll [1965, 19, Sect. 28] and Müller [1973, 9, pp.43-44].

B. BALANCE EQUATIONS

Mechanics is the science which concerns itself with the motions of material systems and their causes. The two main complementary branches thereof which treat these aspects are called, respectively, kinematics¹ and dynamics. Kinematics examines the problem of describing motion per se. In doing so, it does not undertake to study the sources of motion nor the forces thereto related; the inquiry into these matters being the task of dynamics. Consequently the results of kinematics, known as balance equations, are applicable to any of a possibly large number of arbitrary physical systems whose respective motions can be shown to satisfy the principles upon which said equations rest. The dynamics of a continuous material system, on the other hand, is contained within the constitutive equations proposed for that particular system. Although the specific system under study may be thought of in general as belonging to a class of similar systems, the specific constitutive equations employed limit the diversity of material response possibilities to those believed most appropriate for the given system. Considerations pertinent to the establishing of rational mechanical constitutive equations will be given in the discussion of Chapter C.

It may be recognized on the basis of physical experience that, depending upon the physical character of the medium being considered, singularities in the various field quantities may occur. Consequently, a complete kinematical description of a mixture necessitates the establishing of two general types of integral balance relationships². The first is a balance equation valid for points of the medium and/or electromagnetic field in which the fields involved are regular; and second, is a jump balance equation for those points in which said fields suffer a finite discontinuity.

Since the problem at hand involves electromagnetic fields, one is obliged to consider volumes which may contain moving singular surfaces and surfaces which may incorporate singular lines. For regions of the system wherein the fields and field derivatives in question are suitably continuous³, the integral balance expression yields a general differential balance equation relationship that portrays the local kinematical description of that region⁴. Under additional assumptions regarding the possible boundedness of the mixture- and/or electromagnetic field-related fields, and the discontinuity surface(s) and line(s) themselves, a differential jump balance equation for both of the aforementioned cases can be gotten from their respective integral relations.

Once the general balance equations have been procured, the next step to be undertaken is the explicit identification of the particular meaning to be associated with the symbols therein representing a given field quantity, its flux, supply and production⁵. In doing so, the set of system balance equations to be used in the eventual calculation of the fields (A1) is generated. Here too one notes that it is again the physical nature of the system involved that motivates the particular selection of those parameters that is made.

Recognizing that the material fields thus found in principle by the procedure described above relate to any given yet arbitrary constituent, it is now desirable to remark upon the manner to be pursued here with regard to the determination of the mixture field quantities. For the sake of explicitness, I note the tacit employment to this point of the first part of Truesdell's⁶ second metaphysical principle. The set of three such principles read as follows:

1. All properties of the mixture must be mathematical consequences of properties of the constituents.
2. So as to describe the motion of a constituent, we may in imagination isolate it from the rest of the mixture, provided we allow properly for the actions of the other constituents upon it.
3. The motion of the mixture is governed by the same equations as is the single body."

The implementation of the third of these principles provides us with the desired mixture balance equations, together with the definitions of the mixture field quantities expressed in terms of the constituent properties: this being in accordance with the first principle.

That is what this chapter is about. I now proceed to secure those relationships which will be taken here as being both relevant and necessary to a complete kinematical description of the type of material system described above.

B.a. Co-ordinate systems

Since the theory given here falls within the realm of classical mechanics, I assume that the space manifold is R_3 ; a three-dimensional Euclidean vector point space with real co-ordinates ⁷.

Co-ordinate choice

A reference frame is a particular space R_3 . Upon selecting a basis \hat{e}_i ($i = 1, 2, 3$) and an origin, such a space may be represented by $(0, \hat{e})$. I further assume that a system of orthogonal Cartesian co-ordinates x_i suffices for the problem at hand; hence, $\vec{x} = x_i \hat{e}_i$ is the position vector of a point in the space.

Transformations in the common frame

For a theory to have a physical content independent of the choice of co-ordinates, it must to begin at least be invariant under general co-ordinate transformations. Such invariance is ensured by the use of tensor notation for the prescription of the physical quantities entering therein; said notation is thus forthwith employed here.

Now, there are two ways, both due to Euler ⁸, in which the continuous motion of material systems can be described. Motion is classically described in terms of point transformations. These are here viewed as taking place in the total manifold known as Euclidean space-time R_4 which consists of R_3 together with the one-dimensional manifold of the real (primitive) element time, denoted by t ($-\infty < t < \infty$), which is now introduced.

The term constituent means here any arbitrary yet uniquely identifiable atomic or molecular matter which admits a field representation. The particular body manifold composed of all the material points of a given constituent, say the a^{th} ($a = 1, 2, \dots, s$), and which satisfies the relations (1) below is defined to be a constituent body, denoted by B_a . For any constituent the center of its mass in a region of space is defined⁹ to be the material particle of that constituent.

Of the many possible co-ordinate systems one which is referred to as a common frame¹⁰ is singled out. It is convenient to select as the common frame one which may be taken as being fixed in space; e.g., the "laboratory frame"¹¹.

The first of two alternative choices of common frame co-ordinates involves the unique identification for all time t of each constituent material particle for any arbitrary, yet all, constituent bodies. This is accomplished by the introduction of the material co-ordinates X_{α}^a ($\alpha = 1, 2, 3$) which are assumed to be uniquely affixed to each material particle of constituent a . These co-ordinates may be thought of as always moving with their related material particles.

The second possibility concerns the employment of the spatial co-ordinates x_i ($i = 1, 2, 3$). These identify each point of the ambient space manifold and are referred to the places to, in and from which the particles \vec{X}_a may be motionally related.

A constituent configuration is represented by the specification of the positions of all the material particles of that constituent's body. It is defined by a continuous one-to-one mapping and, together with its inverse, is given by

$$\vec{x} = \underset{a}{\zeta}(\underset{a}{\vec{X}}) \quad \text{and} \quad \underset{a}{\vec{X}} = \underset{a}{\zeta}^{-1}(\vec{x}). \quad (1)$$

The point transformation which expresses a continuous motion (cf. Fig. 4.) is assumed to be a continuous sequence of configurations in time. It and its inverse are defined respectively by

$$\vec{x} = \underset{a}{\chi}(\underset{a}{\vec{X}}, t) \quad \text{and} \quad \underset{a}{\vec{X}} = \underset{a}{\chi}^{-1}(\vec{x}, t) \quad (2)$$

where \vec{x} is the place that the particle $\underset{a}{\vec{X}}$ occupies at time t .

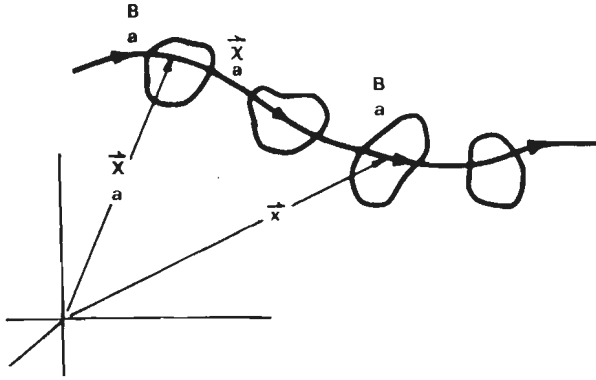


Fig. 4. A continuous motion for a single constituent.

The assumption made with regard to (1), the constituent deformation, and (2) have been set down by Truesdell¹² in a axiom of continuity: The relations (1) and (2) are assumed to be smooth single-valued functions. I now make the common ("Fick-Stefan"¹³) assumption that at each point of the system the mixture is considered to be the simultaneous superposition of the s constituent continua. Upon defining the constituent deformation gradient $F_{i\alpha}$ and the Jacobian J respectively by¹⁴

$$F_{i\alpha} = \frac{\partial x_i}{\partial X_\alpha} \quad (3)$$

and

$$J = \det \vec{F}_a = \frac{1}{6} \epsilon_{ijk} \epsilon_{\alpha\beta\gamma} F_{i\alpha} F_{j\beta} F_{k\gamma} \quad (4)$$

the assumption that J is non-singular is made to ensure that the inverse functions above exist.

By virtue of the axiom of continuity the development of a rational mechanical theory falls into two parts, each of which requires its own treatment. The one part is a theory for continuous fields and the other is a theory for the finite number of isolated point, line and surface singularities that may exist in a material system. The two corollaries which follow from this axiom are ¹⁶;

Corollary 1. Permanence of matter

A material constituent occupying a region of positive and finite volume cannot be deformed in such a manner that said region vanishes or becomes infinite.

It can be shown ¹⁷ that the condition on J which follows from this corollary is that $0 < J < \infty$.

Corollary 2. Principle of material impenetrability

Portions of matter never interpenetrate.

This is interpreted as meaning that for any time t there is at any given place in the system at most only one of each of the various particles \vec{x}_a . The fact that just one particle can occupy a given point at time t follows from the supposition that there is related to each and every constituent deformation one and only one particle \vec{x}_a possessing at that time t a given set of co-ordinates \vec{x} .

The selection of \vec{x}_a (for all a) and t as independent variables represents Euler's material description of motion; while, a choice of \vec{x} and t as variables is his spatial description¹⁸. This study utilizes the former.

B.b. Material systems

A material system is a manifold consisting of a set of material particles. It is thus a body in part or whole. In regarding such systems it is recognized that experience of the physical nature of a given system dictates which ones and how many of the known fundamental physical properties must be ascribed to the different mixture constituents. While recognizing further the existence of discrete material properties for matter considered on e.g. an atomic level, continuum mechanics in general restricts its attention essentially to those systems for which the physical characteristics may be assumed to possess continuous distributions.

Mass

The primitive quantities related to constituent particles are two in number: the mass and the (possible) charge of any given constituent. The molecular mass m_a of a constituent is associated with a density distribution of same (cf. e.g. (5)) and serves as a measure of that constituent's inertia as well as the amount of its matter in B_a .

With $e (= 1.602 \times 10^{-19}$ coulomb) the absolute value of electronic charge and Z the integer (negative, positive or zero) charge multiplicity, the constituent charge and (mass) specific charge are given for s^* ($0 \leq s^* \leq s$) constituents respectively by Z_e and $q = Z_e/m$.

Mass and charge density

Let n_a denote the constituent number density. In terms hereof I define the following initial set of material properties. The mass density of a constituent is given by

$$\rho_a = n_a m_a, \quad (5)$$

the total mass density by

$$\rho = \sum_{a=1}^s \rho_a, \quad (6)$$

the mass fraction by $c_a = \rho_a / \rho$ for which hold the condition that

$$\sum_{a=1}^s c_a = 1, \quad (7)$$

and the free charge density of a constituent defined by

$$Q_a^F = n_a Z_e = q_a n_a. \quad (8)$$

The constituent mass and charge densities are assumed to be continuous and additive quantities which, as absolute scalars, are invariant under the here non-relativistic motions of the system.

Material systems

Material systems¹⁹ come in three varieties: curves C_a , surfaces S_a and volumes \mathcal{V}_a . According to the meaning of a material system given above, each of these three cases is characterized by the fact that it always consists of the same set of material particles.

Linear velocities

It is on the basis of particle motion that velocity is defined. Thus, for a particle \vec{x}_a in a given place \vec{x} at the time t I define²⁰ the constituent velocity

$$v_a^i = \left. \frac{\partial x_i}{\partial t} \right|_{\vec{x}_a = \text{const.}} \quad (9)$$

Hereby follows that the velocity difference, mass average velocity and diffusion velocity are given respectively by

$$V_a^i = v_a^i - v_s^i \quad (a = 1, 2, \dots, s-1), \quad (10)$$

$$v_i = \frac{1}{\rho} \sum_{a=1}^s \rho_a v_a^i \quad (11)$$

and

$$u_a^i = v_a^i - v_i \quad (12)$$

which satisfies the condition that

$$\sum_{a=1}^s \rho_a u_a^i = 0. \quad (13)$$

Material derivatives

A description of material behavior often necessitates a knowledge of the time rates of change of the various fields involved in a given problem. It is worthwhile in this study to introduce the material derivatives following the constituent and mixture motions; these being ²¹, respectively,

$$(\dot{})^a = \frac{d^a}{dt} = \frac{\partial}{\partial t} + v_i()_{,i}$$

and

$$(\dot{}) = \frac{d}{dt} = \frac{\partial}{\partial t} + v_i()_{,i}$$

where the relation between $(\dot{})^a$ and $(\dot{})$ is clearly that

$$(\dot{})^a = (\dot{}) + u_i()_{,i} .$$

B.c. Material integral kinematics

The aim of this section is the presentation of expressions for the rate of change of general integrals for arbitrary volumes and surfaces containing respectively surfaces and lines of discontinuity.

B.c.i. Volume integrals

For a moving and continuous material volume $\mathcal{V}_a(t)$ bounded by a material surface $\mathcal{S}_a(t)$ as shown in Fig. 5, I now introduce the field Ψ represent-

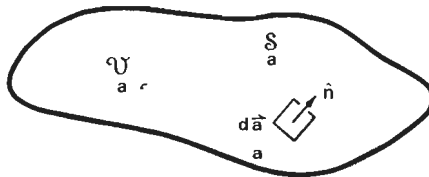


Fig. 5. Continuous material volume \mathcal{V}_a :

ing an arbitrary additive quantity of constituent a (i.e. a field property proportional to ρ). It is given by

$$\Psi_a = \int_{\mathcal{V}_a} \psi_a^v dv . \quad (16)$$

The parameter $\psi_a^v(\vec{x}, t)$ in the integrand of (16) represents here the volume density of a quantity Ψ of constituent a . This parameter is henceforth assumed to be a single-valued continuously differentiable tensor field defined in the volume over which the integral being considered is taken.

Phase interfaces

In general, the volumes with which one is concerned are not (in terms of the functions defined therein) continuous throughout the entire region they represent. It is thus desirable when studying the problem of describing fluid mixture behavior, to attempt an examination of the problems of and related to the (usually three-dimensional) interfacial regions which may exist in multicomponent and multiphase systems. Not only the material behavior of these regions themselves, but also the reciprocal influences upon material behavior between them and the bulk regions they separate may be of some interest.

A pictorial representation of a deformable and moving interfacial region, denoted by $I(t)$, is given in Fig. 6. The material volume \mathcal{V}_a associated

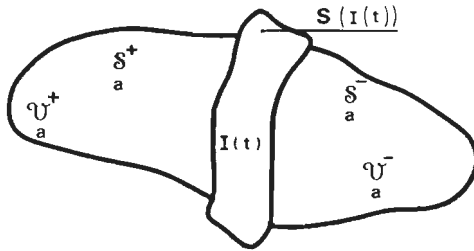


Fig. 6. Phase interface, $I(t)$.

with an arbitrary constituent is written as

$$\mathcal{V}_a = \mathcal{V}_a^+ + \mathcal{V}_a^- + \mathcal{V}(I(t)) ;$$

and, similarly, the material bounding surface \mathcal{S} of \mathcal{V} is expressed by

$$\mathcal{S}_a = \mathcal{S}_a^+ + \mathcal{S}_a^- + S(I(t)) .$$

The region $I(t)$ will in general require its own set of balance and constitutive equations; which fact in turn increases the difficulty of describing the behavior of the total system. This being the case, I make for the sake of expediency the not uncommon assumption that the fields of the material system for which the model developed here is to be applicable are such that the phase interfaces involved, if any, may be represented by surfaces of discontinuity $s(t)$. Under this assumption the following relations hold:

$$\mathcal{V}_a = \mathcal{V}_a^+ + \mathcal{V}_a^- + s(t) \quad \text{and} \quad \mathcal{S}_a = \mathcal{S}_a^+ + \mathcal{S}_a^- + (\mathcal{S}_a \cap s(t)). \quad (17)$$

Singular surfaces

As pointed out earlier, a consequence of the axiom of continuity is that a rational mechanical theory describes under general conditions the kinematics as well as the dynamics in both the sets of regular and singular points. Singular points of material media are those in which the function(s) representing one (or more) of the system related fields may not be taken as being continuous. The sets of such points may in general compose lines and surfaces of singularity. The problem of singular lines is of some importance in a study of material systems interacting with electromagnetic fields and shall be treated shortly. I begin, however, with a consideration of singular surfaces ²².

Whether a deformable and moving singular surface $s(t)$ exists in its own right or, as here, also as an idealization of a phase interface, is of no consequence to the following development. The intersection of a material volume \mathcal{V}_a by $s(t)$ is portrayed in Fig. 7.

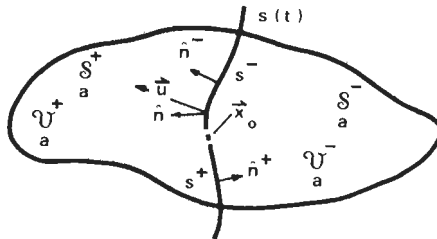


Fig. 7. Singular surface, $s(t)$.

There \hat{n} is a unit vector normal to $s(t)$ which is taken as being directed into \mathcal{V}_a^+ . \vec{x}_0 is a point upon $s(t)$ and \vec{u} is the velocity of propagation of that surface. Further, it is useful to introduce the speed of displacement u_n of $s(t)$ given by

$$u_n = u_i \hat{n}_i \tag{18}$$

and to denote respectively by ψ_a^+ and ψ_a^- the limiting values of the function $\psi_a(\vec{x}, t)$ as \vec{x} moves towards \vec{x}_0 from within \mathcal{V}_a^+ and \mathcal{V}_a^- . In terms of these parameters the material rate of change of (16) over a volume containing a singular surface is given by the well-known ²³ generalized Reynolds' transport theorem

$$\dot{\Psi}_a = \int_{\mathcal{V}_a^+ + \mathcal{V}_a^-} \frac{\partial \psi_a}{\partial t} dv + \int_{\mathcal{S}_a^+ + \mathcal{S}_a^-} \psi_a v_i da_i - \int_{s(t)} [[\psi_a]] u_n da \tag{19}$$

Here, the outward directed element of surface area is given by $da_i = \hat{n}_i da$ and the jump of ψ_a across $s(t)$ at a point \vec{x}_0 has been denoted by

$$[[\psi_a]] = \psi_a^+ - \psi_a^- \tag{20}$$

B.c.ii. Surface integrals

Consider for any given body B_a a material surface \mathcal{S}_a . For the case

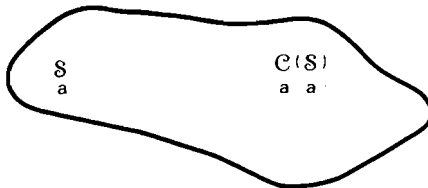


Fig. 8. Continuous material surface, \mathcal{S}_a .

depicted in Fig. 8 where that surface is continuous and bounded by the material curve $\mathcal{C}(\mathcal{S})_a$, I introduce the field parameter Ξ_a . It is given for all constituents by

$$\Xi_a = \int_{\mathcal{S}_a} \xi_a \, da_i . \quad (21)$$

At a surface point, denoted by the parameter $\vec{\xi}_a$, the integrand $\xi_a(\vec{\xi}_a, t)$ is termed the density of the quantity Ξ_a . It, like $\psi_a(\vec{x}, t)$ above, is assumed to satisfy the single-valuedness, continuity and multiple differentiability assumption; but here for fields defined in the surface for which the integral being considered is taken.

Singular lines

It is possible, as mentioned above in the discussion regarding singular surfaces, that the surface of a material system may evidence the presence of lines of discontinuity²⁴. For the case of a single line the geometry of such a situation is provided in Fig. 9. There \mathcal{S}_a is divided by the deformable

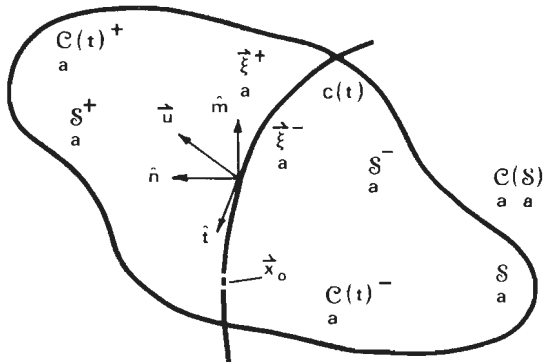


Fig. 9. Singular line, $c(t)$.

moving singular line $c(t)$ into two parts; it may be written as

$$\mathcal{S} = \mathcal{S}_a^+ + \mathcal{S}_a^- + c(t). \quad (22)$$

The line $c(t)$ moves with a velocity \vec{u} in the interior of $\mathcal{S}_a^+ + \mathcal{S}_a^-$, but also intersects $\mathcal{C}(\mathcal{S})$ which is given by

$$\mathcal{C}(\mathcal{S}) = \mathcal{C}(t)_a^+ + \mathcal{C}(t)_a^- + (\mathcal{C} \cap c(t)). \quad (23)$$

Lastly, \vec{x}_0 is a point on $c(t)$; while \hat{n} , $\hat{\xi}$ and $\hat{m} (= \hat{n} \times \hat{\xi})$ are respectively the unit principal normal, tangent and binormal vectors to $c(t)$.

The material derivative of (21) taken with respect to particles lying upon $c(t)$ and moving with velocity \vec{u} is, for $dl_i = \hat{\xi}_i dl$ and $\xi_i^{+(-)}$ the limiting value of the function ξ_i as here \vec{x} approaches \vec{x}_0 from within $\mathcal{S}_a^{+(-)}$, given by the also well-known²⁵ relation

$$\begin{aligned} \frac{d}{dt} \int_a \xi_i &= \int_{\mathcal{S}_a^+ + \mathcal{S}_a^-} \left[\frac{\partial \xi_i}{\partial t} + v_{aj} \xi_{i,j} \right] da_j + \int_{\mathcal{C}_a^+ + \mathcal{C}_a^-} (\hat{\xi}_a^{\pm} \times \vec{v}_a^{\pm})_i dl_j - \\ &- \int_{c(t)} \langle [\hat{\xi}_a^{\pm}] \times \vec{u} \rangle_i dl_j. \end{aligned} \quad (24)$$

Here, the jump of the field $\hat{\xi}_a^{\pm}$ across $c(t)$ at some point \vec{x}_0 is denoted as follows:

$$[\hat{\xi}_a^{\pm}] = \hat{\xi}_a^+ - \hat{\xi}_a^-. \quad (25)$$

B.1. Integral balances

The balance of an arbitrary constituent quantity with regard to some region of a system is here expressed initially by an integral relationship for that region between the material derivative of that quantity and its influx, supply and production rate ²⁶. Those quantities whose production terms vanish are said to be conserved and their balance equations become conservation relationships. These are at most special cases of the general equations which I now give.

B.1.a. Volume balances

For the first of the two regions of interest here, viz. a volume containing a surface of discontinuity (cf. Fig. 7), I assume that the balance equation involving (19) may be given by ²⁷

$$\dot{\Psi}_a = - \int_{S_a^+ + S_a^-} \phi_i^\psi da_i + \int_{V_a^+ + V_a^-} (\sigma_a^\psi + \partial_a^\psi) dv + \int_{s(t)} T_a^\Psi da \quad (26)$$

where ϕ_i^ψ is the flux through $S_a^+ + S_a^-$ of the constituent quantity, σ_a^ψ is that quantity's supply density (per unit volume) for said constituent being considered as a single continuum and ∂_a^ψ is the production rate density (per unit volume) of that quantity. The term T_a^Ψ is the total supply density (per unit area) by $s(t)$ of Ψ , and I would like now to comment upon its introduction into this study.

On singular surfaces and their balances

Dynamic transition regions are usually three-dimensional; and, depending upon the particular medium involved and its physical environment, also deformable and moving. As mentioned above, a description of the behavior of and within said regions, as well as the mutual influences between them and their adjacent bulk media, requires in general that in addition to the bulk relations an appropriate set of balance and constitutive equations be established for them.

For the sake of simplicity, however, the applicability of this study is limited to those systems for which phase interfaces, if any, may be described in terms of surfaces of discontinuity. For descriptive purposes I consider such a surface to be a smooth "thin" region moving in R_3 ; assuming further that the constitutive equations of the contiguous bulk media remain valid up to the surface.

Properties of connection are ascribed to $s(t)$ in the form of distributed surface source terms τ_a^ψ for each bulk quantity that is to be balanced. The discontinuous bulk field parameters are here then assumed, as functions of time and position upon $s(t)$, to change rapidly but in a suitably continuous manner across the surface.

An investigation of arbitrary physical singular surfaces and matters pertaining to the general kinematics and dynamics of them for systems of interest here lies beyond the scope of this initial study. Thus, the model of a plane and temporally persistent surface of discontinuity assumed here and employed later is taken primarily on the basis of its relative simplicity; it being presumed indicative of the function of such surfaces under more general circumstances.

I conclude the discussion on integral volume balances by limiting the considerations of this study to the particular case for which at most ψ and v_i are singular on $s(t)$. To this end I take ²⁸ the fields $\partial\psi/\partial t$, σ_a^ψ and $\bar{\sigma}_a^\psi$ to be bounded in the neighborhood of $s(t)$ and the limits of ϕ_i^ψ to be continuous functions of position. It follows then from (17) together with Green's transformation ²⁹ that the result of equating (19) and (26) is the volume integral balance equation; viz.,

$$\int_a \int_{V-s(t)} \left[\frac{\partial\psi}{\partial t} + (\psi_{v_i} v_i - \phi_i^\psi)_{,i} - \sigma_a^\psi - \bar{\sigma}_a^\psi \right] dv + \int_{s(t)} \left[\psi_{v_i} (v_i - u_i) \bar{n}_i + [\phi_i^\psi \bar{n}_i] - T_a^\psi \right] da = 0. \quad (27)$$

B.1.b. Surface balances

The development here closely parallels that above for the case of volume balances. For surfaces \mathcal{S} containing a moving singular line $c(t)$ (cf. Fig. 6) let ϕ_i^ξ , σ_a^ξ , $\bar{\sigma}_a^\xi$ and T_a^ξ denote respectively the influx through $\mathcal{C} = (\mathcal{C} \cap c(t))$, supply density (per unit area), its production rate density (per unit area) and its total supply density (per unit length) by $c(t)$. I now assume ³⁰ for such a surface that

$$\bar{\xi} = - \int_a \int_{\mathcal{C}-\mathcal{C} \cap c(t)} \phi_i^\xi d l_i + \int_{\mathcal{S}^+ + \mathcal{S}^-} (\sigma_a^\xi + \bar{\sigma}_a^\xi) da_i + \int_{c(t)} T_a^\xi d l_i \quad (28)$$

expresses the balance relationship which concerns (24). For the specific case where ξ_i and v_i can have a finite discontinuity upon $c(t)$, the parameters $\frac{\partial \xi_i}{\partial t}$, σ_i^ξ and δ_i^ξ are taken as bounded in the proximity of $c(t)$ while ϕ_i^ξ is assumed to be finite on the singular line. Equating (24) and (28) and then utilizing (22), (23) and Kelvin's transformation ³¹ leads to the result valid for any constituent that

$$\int_a \int_{S-c(t)} \left\{ \frac{\partial \xi_i}{\partial t} + v_i \xi_{1,l} + \epsilon_{ijk} [(\vec{\xi} \times \vec{v})_k + \phi_k^\xi]_{,j} - \sigma_i^\xi - \delta_i^\xi \right\} da +$$

$$+ \int_{c(t)} \{ [[\vec{\xi} \times (\vec{v} - \vec{u})]_i] + [\phi_i^\xi] + T_i^\Xi \} t_i dl = 0. \quad (29)$$

This expression shall be referred to as the surface integral balance equation.

B.2. Differential balances

Under assumptions concerning a region of interest and its related system fields, the integral balance equations (27) and (29) above will in this section each be reduced to two specific types of differential balance relationships. Since a consideration of a material system will thus no longer involve the region taken as a whole but rather with the points thereof which determine that region, the description of the system in question may be thought of as changing accordingly from a global to a local one.

B.2.a. Continuous system balances

Volume balances

From (27) it follows in the case of regular volumes for which may be assumed that $\dot{\psi}_a^v$, ψ_a^v , σ_a^ψ and $\bar{\sigma}_a^\psi$ are continuous and ϕ_a^ψ is continuously differentiable, that the volume differential balance equation is

$$\frac{\partial \psi_a^v}{\partial t} + (\psi_a^v v_i + \phi_a^\psi)_{,i} = \sigma_a^\psi + \bar{\sigma}_a^\psi. \quad (30)$$

Surface balances

Similarly, for circumstances involving regular surface regions and with the integrand of the residual equation (29) assumed continuous, the surface differential balance equation is

$$\frac{\partial \xi_i}{\partial t} + v_i \xi_{a,1} + \varepsilon_{ijk} [(\xi_a \times \vec{v})_k + \phi_{a,k}]_{,j} = \sigma_a^\xi + \bar{\sigma}_a^\xi. \quad (31)$$

B.2.b. Discontinuous system balances

Volume balances

The balance relationship for the surface of discontinuity region of a volume follows from the $s(t)$ -integral term of (27). For that term equal to zero to be valid in general, the assumed smooth integrand thereof must vanish; whence follows the volume jump balance equation

$$\llbracket \psi_a^v (v_i - u_i) \hat{n}_i \rrbracket + \llbracket \phi_a^\psi \hat{n}_i \rrbracket = \tau_a^\psi. \quad (32)$$

Surface balances

Finally, for the singular line portion of a surface region, the $c(t)$ -integral expression of (29) with its integrand taken to be smooth on $c(t)$ and vanishing yields the surface jump balance equation; viz.,

$$\llbracket \left[\frac{\vec{\xi}}{a} \times (\vec{v} - \vec{u}) \right]_i \mathbf{e}_i \rrbracket + \llbracket \Phi_i^\xi \mathbf{e}_i \rrbracket = T_i^\Xi \mathbf{e}_i . \quad (33)$$

With the securing of equations (30), (31), (32) and (33)³², it now becomes possible to proceed with the establishing of a more explicit model of the kinematical relationships needed for the problem at hand. Such expressions as will be given are based initially upon proposals (i.e. implicit and explicit assumptions) for the various field quantities, together with their respective fluxes, supplies and production rates. In this light these relations may be viewed as having been postulated.

B.3. Mass balance equations

This and the remaining sections of Chap. B have the purpose of proposing a set of differential balance and jump balance equations, for the material and electromagnetic fields, which I shall assume to govern the behavior of the class of fluid mixtures envisaged. These relationships are, beginning with those for mass, now given.

B.3.a. Constituents

Aside from diffusion the changes here of a constituent's mass density are a consequence only of chemical reactions which involve it. I assume³³ that $\psi_a = \rho$ while ϕ_a^ψ and σ_a^ψ (the mass flux and volume supply respectively)

both are zero. Further, $\hat{\sigma}_a^\psi = \hat{c}_a$, the production rate density for mass; while the supply density of mass by the surface $s(t)$ is denoted by $\hat{\tau}_a^\psi = \hat{c}_a^S$.

On the basis of these identifications (30) yields the following forms of the balance equation of mass density of a constituent:

$$\dot{\rho}_a + \rho_a v_{aa,j,j} = \hat{c}_a,$$

$$\dot{\rho}_a + \rho_a v_{aa,j,j} + u_{ja} \rho_{a,j} = \hat{c}_a \quad (34)$$

and

$$\dot{\rho}_a = \hat{c}_a - u_{ja} \rho_{a,j} - \rho_a \delta_{kj} d_{kj};$$

where (34)₂ follows from (34)₁ by means of (12) and $d_{kj} = v_{a(k,j)}$ is the deformation rate tensor of a constituent. Parentheses placed about indices denote in this study the symmetrization of the term concerned with respect to the enclosed indices.

On stoichiometry

Stoichiometry is taken here to be the phenomenological description of chemical processes in continuum mixtures. It does not concern itself with the causes of said processes and may thus ³⁴ be looked upon as the "kinematics" of chemical reactions. For any mixture constituent, of which s^* are charged, let there exist α ($\alpha = 1, 2, \dots, n$) possible independent chemical reactions. Further, denote by γ_a^α the stoichiometric coefficient of that constituent; and, by $\Lambda^\alpha(\vec{x}, t)$ its α^{th} reaction rate density. The first parameter indicates the number of constituent molecules participating in a given reaction α either as reactants or as products, while the latter represents the net rate of production per unit time and volume of the constituent number density for that reaction.

The relation for the net production rate density, R_a , for n may be given by

$$R_a = \sum_{\alpha=1}^n \gamma_a^\alpha \Lambda_a^\alpha. \quad (35)$$

In general it is necessary in the case of mixtures for which some or all of the constituents bear charge to account for the supply of both mass and charge through chemical reactions. Here I examine the question of mass supply, while in Sect. B.4.b. attention is directed towards the matter of chemical reactions and charge.

Letting m be the unit atomic mass (e.g. 1/16 of the ^{16}O oxygen atom isotope mass) and $W_a = m/m_a$ be the constituent molecular weight, any arbitrary \bar{c}_a is expressible here in terms of its related mass by

$$\bar{c}_a = m R_a = \sum_{\alpha=1}^n \gamma_a^\alpha W_a m \Lambda_a^\alpha. \quad (36)$$

From (36) follows that the various physico-chemical processes available to the system are prescribed implicitly by the fields $\Lambda_a^\alpha(\vec{x}, t)$, and that a knowledge of all these parameters with regard to that system in turn determines \bar{c}_a . In a continuum theory the chemical kinetic considerations relevant to the specification of the $n \Lambda_a^\alpha$, or equivalently the $s \bar{c}_a$, are embodied in the formulation of constitutive equations for these fields. I shall return to this point in Chpt. C.

Equation (32) together with the assumption above for T_a^ψ provides the result

$$[[\rho_a(v_i - u_i) \bar{n}_i]] = \bar{c}_a^S \quad (37)$$

known as the jump balance equation for the mass density ρ_a .

B.3.b. Mixture

The conservation equation for the mass of the mixture derives from a summation over (34) for all constituents, use being made also of (11), and is given by

$$\dot{\rho} + \rho v_{j,j} = 0 . \quad (38)$$

Similarly it follows from (37) that

$$\llbracket \rho (v_i - u_i) \hat{n}_i \rrbracket = 0 \quad (39)$$

is the jump conservation equation for the mass of the mixture. The transition from the constituent to the mixture relationships has been accomplished by the introduction of the mixture conservation of mass assumptions (i.e. axioms) for \bar{c}_a and \bar{c}_a^S . These are, respectively,

$$\sum_{a=1}^s \bar{c}_a = 0 \quad \text{and} \quad \sum_{a=1}^s \bar{c}_a^S = 0 . \quad (40)$$

Further, I now introduce the first of two additional and important assumptions to be made here; it being ³⁵

$$\sum_{a=1}^s \gamma_a^\alpha w_a = 0 , \quad (41)$$

which expresses for the mixture the conservation of mass in each of the n homogeneous chemical reactions α . The restrictions on the stoichiometric coefficients following from (41) are complemented by those on the related to the coefficients of the charged constituents. These latter conditions will be given in Sect. B.4.b.

B.4. The charge-current equations

Here and in the next section I set down those electrodynamic relations which I assume to be basic to a continuum description of "slowly moving" (i.e. non-relativistic) magnetizable dielectric fluid mixtures. Since the development rests in part upon mixture theoretical considerations, the results secured evidence some limited generalization of the Maxwell electromagnetic theory of ponderable matter. Due to the important fact, among others, that emission, absorption and scattering of electromagnetic radiation cannot be treated at present by phenomenological electrodynamics, I consider in this study mixtures only under those conditions for which radiation is negligible.

On electrical conductors

Regardless of the region of interest, be it a bulk phase, surface of discontinuity or both, two types of charge densities may in general be attributed to it: viz. the so-called "free" and "bound" charges. In \mathcal{V} the free charge density of a constituent is given (cf. (8)) by Q_a^F , while on $s(t)$ the singular surface density of free charge is denoted by Q_a^{SF} . Free charges, it will be remembered, are the foremost source of the electromagnetic field. By virtue of their relative mobility they serve as the essential coupling factor between the mechanics of the mixture and the electromagnetic field which they generate and with which the mixture interacts.

The sum of each of the quantities Q_a^F and Q_a^{SF} for all constituents yield the definitions for the volume and singular surface densities Q^F and Q^{SF} of free charge of the mixture; these being respectively

$$Q^F = \prod_{a=1}^s Q_a^F \quad \text{and} \quad Q^{SF} = \prod_{a=1}^s Q_a^{SF} . \quad (42)$$

As electrical conductors I take here those material media for which the electrically charged constituents are relatively unconstrained in their displacements under the influence of the fields acting upon them. This point will be made more explicit in the discussion of the forces of constituent interaction below.

On dielectrics

Bound charges are characteristic of both electrically neutral constituents having structure (e.g. atoms and molecules) as well as those which may also possess a net charge (e.g. atomic and molecular ions). And further, for the microscopic structure of its material components being spherically symmetric or non-symmetric, a constituent may be referred to accordingly as being either non-polar or polar.

The polarization and magnetization of a constituent represent macroscopically the existence for that constituent of electric moments which reflect the distribution of its negative and positive bound charges with respect to each other, and magnetic moment which relate to said bound charge distributions as well as (possibly) to that of its intrinsic spins. I assume for the sake of simplicity that a dipole approximation is adequate in this study. The theory remains, of course, applicable to both non-polar and polar media.

Now let P_a be the polarization density in \mathcal{V}_a of a constituent, and in terms hereof let the related polarization density of the mixture P_i be defined by

$$P_i = \frac{s}{\sum_{a=1}^s} P_a \quad (43)$$

By the term dielectric I shall mean electrically polarizable media in general. This definition extends the one usually given which prescribes only the so-called perfectly electrically insulating media. For mixtures under the simultaneous influence of different fields it will be seen in Sect. E.6. that the generalization of definition to include real (i.e. imperfect) electrical insulators is worthwhile.

B.4.a. Coulomb's law: the charge flux equations

The material properties of a mixture system have been assumed to be determinable from those of the constituents. To prescribe the properties of the electromagnetic field I introduce a principle of superposition which states ³⁶ that:

The total electromagnetic field produced by a system of constituents is obtained by adding the fields of the individual constituents. Hereby is made clear that the origin of the total electromagnetic field lies in the simultaneous superposition of the internal and "external" contributions thereto of all the constituents of the system.

Since the "external" field, for example, may exist in regions of the system possibly unoccupied by any matter (i.e. vacuum), it seems physically unreasonable to attempt a formal apportioning of said total field to the constituents ³⁷ which may on occasion occupy that space. I assume instead that the total electromagnetic field and mixture fields (defined in terms of the constituents) are the natural complementary quantities in terms of which the full set of macroscopic system relationships may be secured.

For E_i denoting the total electric field taken with respect to a stationary observer and ϵ_0 ($= 8.854 \times 10^{-12}$ farad/m) the permittivity of vacuum, I follow Truesdell & Toupin³⁸ and introduce the first (cf. (67)) Maxwell-Lorentz aether relation: it being that for the charge potential D_i given by

$$D_i = \epsilon_0 E_i. \quad (44)$$

The partial charge potential of the mixture³⁹ is

$$\mathcal{D}_1 = D_1 + P_1 \quad (45)$$

and the well-known equation for electric flux density is written as

$$(D_1 + P_1)_{,1} = Q^F. \quad (46)$$

From this relation follows that the charge density Q of the mixture is given by

$$\begin{aligned} Q &= D_{1,1} \\ &= \sum_{a=1}^s Q_a \quad (Q_a = Q_a^F - P_{a,1} \text{ is the } \underline{\text{constituent charge density}}) \\ &= Q^F - P_{1,1} \end{aligned} \quad (47)$$

where the term $-P_{1,1}$ is called the polarization charge density of the mixture.

With a view towards the treatment of the constitutive equation later, another form of (46) will prove more useful. In terms of the magnetic flux density \vec{B} (cf. Sect. B.5.a.) let the electromotive intensity for a constituent and the mixture be defined respectively by

$$\xi_a^i = E_i + \epsilon_{ijk} v_j^B k^k \quad \text{and} \quad \xi_i = E_i + \epsilon_{ijk} v_j^B k^k . \quad (48)$$

Equation (46) may hereby be written in terms of an arbitrary reference constituent, say the s^{th} , in the form ⁴⁰

$$\xi_{s,j,j} + \epsilon_{ijk} v_s^i k^j k^k + \epsilon_{ijk} k^j v_s^i k^k - \frac{1}{\epsilon_0} (Q^F - P_{1,1}) = 0 . \quad (49)$$

For a region $s(t)$ across which \vec{D} has a finite discontinuity, the jump equation for electric flux that follows from (46) with Q^F bounded in the vicinity of $s(t)$ is

$$[[D_i + P_i] \hat{n}_i] = Q^{SF} ;$$

or equivalently ⁴¹ (50)

$$[[D_i + P_i] \hat{n}_i] = \epsilon_{ijk} h_j \hat{n}_k + Q^{SF} \hat{n}_i ,$$

where Q^{SF} is defined by (42)₂. Here (cf. Sect. B.4.c.) $h_j = \epsilon_{j p q} \hat{n}_p [[D_q]]$ is an arbitrary field defined on $s(t)$.

From (50)₂ it is possible to write that

$$[[D_i \hat{n}_i]] = Q^S \quad (51)$$

where the surface density of charge of the mixture is given by

$$Q^S = Q^{SF} - [[P_i \hat{n}_i]] \quad (52)$$

and $[[P_i \hat{n}_i]]$ is the mixture polarization density on $s(t)$. For developmental simplicity in the treatment given below, I make the (otherwise nonessential) assumption that Q^{SF} vanishes.

B.4.b. Charge balance equations

The balance of electric charge will, as in the case of mass, be given in two parts. The first deals with the constituents while the second concerns itself with the mixture as a whole.

B.4.b.i. Constituents

There exist ⁴² well-established models which describe to an extent deemed acceptable here the macroscopic electrodynamics of magnetizable, polarizable, and non-relativistically moving single material continua. Further, any mixture considered here is such that it reduces to a single continuum as the number of its constituents is reduced to unity. With the aforementioned metaphysical principles (particularly the third) in mind, I thus assume in analogy with the pure single continuum results alluded to above that the following definitions may be made.

The constituent magnetization density is represented by $M_{a,k}^M$; while, $J_{a,j}^M$ and $J_{a,j}^P$ are respectively the constituent magnetization and polarization current densities; these being here given, together with their sum, accordingly by

$$J_{a,j}^M = \epsilon_{jkl} M_{a,l,k}^M, \quad (53)$$

$$J_{a,j}^P = \frac{\partial P_{a,j}}{\partial t} + v_{a,l}^P P_{a,l} + \epsilon_{jkl} (\vec{P}_a \times \vec{v}_a)_{l,k} \quad (54)$$

and

$$\vec{J}_a = J_{a,j}^M + J_{a,j}^P. \quad (55)$$

With ${}^4_3 \psi_a = Q$, $\phi_a^\psi = \tilde{J}_a^T$, the charge density supply for constituents $\sigma_a^\psi = 0$ and the production rate density for the constituent charge density $\hat{\sigma}_a^\psi = q\hat{c}$, it follows from (30) that

$$\frac{\partial Q}{\partial t} + J_{a,j}^T = q\hat{c} \quad (56)$$

is the balance equation for the charge density of a constituent. $J_{a,j}^T$ is the constituent current density and is given by

$$J_{a,j}^T = Qv_{aa,j} + \tilde{J}_{a,j}^T \quad (57)$$

Similarly, with the charge density supply by the singular surface T_a^ψ , denoted by \hat{c}_a^{SC} , equation (32) yields the relationship

$$\llbracket Q(v_i - u_i)\hat{n}_i \rrbracket + \llbracket \tilde{J}_i^T \hat{n}_i \rrbracket = \hat{c}_a^{SC} \quad (58)$$

called the jump balance equation for constituent charge density.

B.4.b.ii. Mixture

From the definition of the mixture magnetization density M_1 given by

$$M_1 = \sum_{a=1}^s M_a \quad (59)$$

it can be observed that the magnetization current density J_j^M gotten from (53) is given by

$$J_j^M = \epsilon_{jkl} M_{l,k} \quad (60)$$

Likewise, from (54) and (12) comes the equation for the polarization current density

$$J_j^P = \frac{\partial P_j}{\partial t} + v_j^P P_{1,1} + \epsilon_{j,pq} \sum_{a=1}^s (\vec{p}_a \times \vec{v}_a)_{q,p} + \sum_{a=1}^s u_{aj}^P P_{a,1,1}. \quad (61)$$

The mixture conduction, non-convective and total current densities are here defined respectively by ⁴⁴

$$J_j^F = \sum_{a=1}^s Q_a^F u_{aj},$$

$$\delta_j = J_j^F + J_j^M + J_j^P - \sum_{a=1}^s u_{aj}^P P_{a,1,1} \quad (62)$$

$$J_j^T = \sum_{a=1}^s J_j^T$$

$$= Q v_j + \delta_j.$$

Charge balance expressions for the mixture follow from the summation over all constituents of the relations (56) and (58). The results of performing this operation on them are the conservation and jump conservation equations ⁴⁵ for the mixture charge density given respectively by

$$\frac{\partial Q}{\partial t} + J_{j,j}^T = 0 \quad \text{and} \quad [Q(v_i - u_i)\hat{n}_i] + [\delta_i \hat{n}_i] = 0. \quad (63)$$

Here use has been made of the definitions above; and, similar to (40), the mixture conservation of charge assumptions (i.e. axioms) made in passing from (56) and (58) to (63) are respectively that

$$\sum_{a=1}^s q_a \bar{c}_{aa} = 0 \quad \text{and} \quad \sum_{a=1}^s \bar{c}_a^{SC} = 0. \quad (64)$$

The condition of charge conservation in each of the homogeneous chemical reactions follows from the assumption, now introduced, that

$$\sum_{a=1}^s \gamma_a^\alpha q_a w_{aa} = 0. \quad (65)$$

Thus, as alluded to in Sect. B.3., the s^* electrically charged constituents represent important physico-chemical constraints on the chemical reaction processes of mixtures. It is clear, of course, that for those systems for which the additional condition of charge neutrality

$$\sum_{a=1}^s q_a \rho_{aa} = 0 \quad (66)$$

holds, that the specific charges q_a of (65) are themselves restricted.

B.4.c. Ampère-Maxwell's law: the current equations

The main aim of this section is the establishment of the macroscopic electrodynamic relations between the balances of electric charge and the electromagnetic field which derives from said charges. To this end, for μ_0 ($= 1.257 \times 10^{-7}$ henry/m) the vacuum permeability, I introduce the second Maxwell-Lorentz aether relation (cf. (44)) for the current potential H_k and also define the partial current potential \mathcal{H}_k respectively by the expressions ⁴⁶

$$H_k = \frac{1}{v_o} B_k \quad \text{and} \quad \mathcal{H}_k = H_k - M_k - \sum_{a=1}^s (\vec{P}_a \times \vec{v}_a)_k. \quad (67)$$

Consequent to these and earlier definitions, and the assumptions⁴⁷ that $\varepsilon_i = -D_i$, $\phi_k^\varepsilon = \mathcal{H}_k - (\vec{v} \times \vec{D})_k$, $\sigma_i^\varepsilon = J_i^F$ and $\bar{\sigma}_i^\varepsilon = 0$, it follows from (31) without subscripts that the current balance equation for the mixture may be written either as

$$\varepsilon_{ijk} \left[H_k - M_k - \sum_{a=1}^s (\vec{P}_a \times \vec{v}_a)_k \right]_{,j} = Q^F v_i + J_i^F + \frac{\partial}{\partial t} (D_i + P_i)$$

or, equivalently, as

(68)

$$\varepsilon_{ijk} H_{k,j} = Q v_i + \tilde{\delta}_i + \frac{\partial D_i}{\partial t}.$$

I note here another form of (68) secured with the help particularly of (12), (44), (48) and (69): it is⁴⁸

$$\begin{aligned} \dot{\delta}_i + \frac{1}{\varepsilon_o} \dot{P}_i - \varepsilon_{ijk} v_{sj} \frac{\partial B_k}{\partial t} - \varepsilon_{ijk} B_k \frac{\partial v_{sj}}{\partial t} - v_{js} \delta_{i,j} - \\ - \frac{1}{\varepsilon_o \mu_o} \varepsilon_{ijk} B_{k,j} + \frac{1}{\varepsilon_o} (Q^F - P_{1,1}) + \\ + \frac{1}{\varepsilon_o} (J_i^* + \varepsilon_{ijk} M_{k,j} - P_j v_{i,j} + P_i v_{j,j}) = 0 \end{aligned} \quad (69)$$

where

$$J_i^* = \sum_{a=1}^s [q_{pau_i} + \varepsilon_{ipq} (\vec{P}_a \times \vec{u})_{q,p}]. \quad (70)$$

These expressions will be utilized in the development of the constitutive equations to be proposed later.

Neglecting the possibility that a surface $s(t)$ may itself possess free, magnetization and polarization currents, I now set down the jump balance equation for the current density of the mixture across a surface $s(t)$ moving with the particular velocity $\vec{u} = u_n \hat{n}$. It follows from use of (33) without subscripts and the foregoing definitions for $\vec{\xi}$ and $\vec{\phi}^\xi$, and is

$$\epsilon_{ijk} \hat{n}_j \llbracket H_k - M_k - \sum_{a=1}^s (\vec{p}_a \times \vec{v}_a)_k \rrbracket + u_n \llbracket D_i + P_i \rrbracket = 0$$

or, equivalently ⁴⁹,

$$\llbracket H_k - M_k - \sum_{a=1}^s (\vec{p}_a \times \vec{v}_a)_k \rrbracket - g \hat{n}_k - u_n h_k = 0 .$$

The arbitrary fields of (71)₂, g and h_k , defined upon $s(t)$ are given by

$$g = \hat{n}_k \llbracket \mathcal{C}_k \rrbracket \quad \text{and} \quad h_k = \epsilon_{kpq} \hat{n}_p \llbracket \mathcal{D}_q \rrbracket ; \quad (72)$$

while, from (50)₁ under general circumstances and (72)₂ it may be seen that

$$\llbracket \mathcal{D}_q \rrbracket = \epsilon_{qrs} h_r \hat{n}_s + Q^{SF} \hat{n}_q . \quad (50)_2$$

B.5. Electromagnetic field equations

The equations introduced in this section are those which interrelate, in regions of volume as well as on surfaces of discontinuity, the non-relativistic dynamic electric and magnetic fields. With their presentation a indeterminate set of general electrodynamic relationships will have been secured. The matter of the under-determined character of these equations is, as it must be ⁵⁰, resolved with the specification of the electromagnetic field-material response properties of the system. I shall take this point up again in due course (cf. Chpt. C.).

B.5.a. Magnetic flux equations

For B_i the magnetic flux density, the magnetic flux ϕ_M is defined by

$$\phi_M = \int B_i da_i . \quad (73)$$

By virtue of the assumption (i.e. axiom)

$$\int_{S(t)} B_i da_i = 0 \quad (74)$$

introduced now, it may be concluded that the integrand must vanish; or, that in the volume of interest

$$B_{i,i} = 0 . \quad (75)$$

This well-known result is the conservation equation for \vec{B} .

When the region of volume contains a surface of discontinuity, the evaluation of (75) across that surface results in the jump conservation equation for this field ⁵¹

$$[[B_i \hat{n}_i]] = 0 \quad \text{or, equivalently,} \quad [[B_i]] = \epsilon_{ijk} k_j \hat{n}_k \quad (76)$$

where k_j , an arbitrary field upon $s(t)$, is defined by (80)₂.

B.5.b. Faraday's law of induction

There still remains something more to be said of a fundamental, as against a derived, nature concerning the dynamical macroscopic \vec{B} field and its relation to \vec{E} ; and this most particularly here for moving material systems interacting with an electromagnetic field in the case of a surface of discontinuity. To begin, however, I consider regular volumes.

For ⁵² $\xi_i = B_i$, $\phi_k^\xi = \frac{1}{k}$ and $\sigma_i^\xi = \delta_i^\xi = 0$, equation (31) without subscripts together with (75) yields the familiar result

$$\epsilon_{ijk} E_{k,j} = - \frac{\partial B_i}{\partial t} \quad (77)$$

known as Faraday's law of induction. It shall prove desirable to have a form of (77) which lends itself more easily to the development of the forthcoming constitutive equations. Use of (12) and (48)₁ herein leads to the desired alternative result; it being ⁵³

$$\dot{B}_k + u_s B_{k,j} - (B_j \delta_{jk} - B_k \delta_{1j}) v_{s1,j} + \epsilon_{kjl} \xi_{l,j} = 0. \quad (78)$$

It follows in a similar manner from a subscriptless equation (33) wherein the possibility of a singular line supply of \vec{B} , τ_k^B is neglected, that for a moving surface $s(t)$ the expressions ⁵⁴

$$\epsilon_{ijk} \hat{n}_j \llbracket E_k \rrbracket = u_n \llbracket B_i \rrbracket \quad \text{or, equivalently,} \quad \llbracket \epsilon_i \rrbracket = f \hat{n}_i - u_n k_i \quad (79)$$

represent the jump balance equation for \vec{E} . Here

$$f = \hat{n}_k \llbracket E_k \rrbracket \quad \text{and} \quad k_i = \epsilon_{ipq} \hat{n}_p \llbracket B_q \rrbracket \quad (80)$$

are arbitrary fields defined upon $s(t)$.

B.6. Linear momentum balance equations

The balance of total linear momentum for mixtures interacting with an electromagnetic field consists of two parts; one for the material portion of the system (i.e. the constituents and the mixture), and the other for the electromagnetic field. The presentation of the hereto related considerations of this study is the main purpose of this section.

B.6.a. Constituents

For any constituent let its density of linear momentum and the flux thereof, the constituent stress tensor, be given respectively by ⁵⁵

$$(\psi_v)_i = \rho v_i \quad \text{and} \quad \phi_{ij}^{\psi} = -t_{ij} . \quad (81)$$

The supply and production rate of the density of linear momentum for that constituent are called respectively the external (body) force density and the interaction force density and are given here by the relations

$$(\sigma_v)_i = \rho b_i \quad \text{and} \quad (\partial_v^{\psi})_i = \hat{m}_i . \quad (82)$$

In continuum mechanics the supply term is often presumed to be known ⁵⁶. This amounts, in fact, to the introduction here of an explicit relation for ρb_i . The selection of a "physically reasonable (electromagnetic) force expression" remains at present a non-trivial matter ⁵⁷. In this study I make the assumption that

$$\rho b_i = Q E_i + \epsilon_{ijk} J_j B_k + \rho b_i = Q \mathcal{E}_i + \epsilon_{ijk} \alpha_j B_k + \rho b_i \quad (83)$$

where ρb_i represents the non-electromagnetic forces that may in general be present and the remaining terms are those of the Lorentz force. The term \hat{m}_i relates, of course, to the momentum transfer resulting from constituent interactions. Further yet, let the supply from s(t) of linear momentum be denoted by

$$(T^{\psi})_i = \hat{m}_i^S . \quad (84)$$

In terms of these quantities it follows from (30) that the balance equation for the linear momentum of a constituent is

$$\rho_{aa} \dot{v}_i - t_{ij,j} - Q\delta_i - \epsilon_{ijk} \overset{\gamma}{\delta}_{aj}^B k - \rho_{a0a} b_i = \dot{m}_i - \overset{c}{v}_i \quad ,$$

while (32) yields the result

(85)

$$\llbracket \rho v_i (v_j - u_j) \hat{n}_j \rrbracket - \llbracket t_{ij} \hat{n}_j \rrbracket = \dot{m}_i^S .$$

This last relationship is the jump balance equation for its linear momentum.

8.6.b. Mixture

A summation over (85) for all constituents gives the related equations for the mixture. From (85)₁ comes Cauchy's first law of motion expressing the conservation of linear momentum of the mixture ⁵⁸

$$\rho \dot{v}_i - t_{ij,j} - Q\delta_i - \epsilon_{ijk} \overset{\gamma}{\delta}_{aj}^B k - \rho b_i = 0 ; \quad (86)_1$$

while (85)₂ gives the jump balance equation for that parameter, viz.

$$\llbracket \rho v_i (v_j - u_j) \hat{n}_j \rrbracket - \llbracket t_{ij} \hat{n}_j \rrbracket = \dot{m}_i^S . \quad (86)_2$$

Here, the stress tensor of the mixture is given by

$$t_{ij} = \sum_{a=1}^s (t_{ij}^a - \rho_{aa} u_i u_j) . \quad (87)$$

$$\rho b_i = \sum_{a=1}^s \rho_{a0a} b_i \quad (88)$$

is the total non-electromagnetic body force density. The supply by the singular surface for the linear momentum density of the mixture is

$$\hat{m}_i^S = \sum_{a=1}^S \hat{m}_a^S, \quad (89)$$

and the passage from $(85)_1$ to $(86)_1$ has taken place under the assumption (i.e. axiom) of linear momentum conservation of the mixture

$$\sum_{a=1}^S \hat{m}_a^S = 0. \quad (90)$$

B.6.c. Electromagnetic field

From $(86)_1$ it may be seen that the body force density of the electromagnetic field in the mixture is given by

$$\rho b_i^E = Q \mathcal{E}_i + \epsilon_{ijk} \mathcal{J}_j B_k. \quad (91)$$

It is possible, on the basis of this relation, to secure a linear momentum balance equation for the electromagnetic field. In doing so I introduce the following definitions:

first, the linear momentum density of the electromagnetic field is defined by the "Abraham vector" G_i where

$$G_i = \epsilon_{ijk} D_j B_k; \quad (92)$$

the Maxwell stress tensor for the electromagnetic field is

$$t_{ij}^E = D_i E_j + B_i H_j - \frac{1}{2} (D_k E_k + B_k H_k) \delta_{ij}; \quad (93)$$

and \hat{m}_i^{SE} denotes the supply by the singular surface of G_i .

From (91), (92) and (93) it thus follows that

$$\frac{\partial G_i}{\partial t} - t_{ij,j} = -\rho b_i \quad (94)$$

expresses the balance of the linear momentum density of the electromagnetic field. Further yet, the parameters above permit the use of (32) by means of which a jump balance equation for G_i can be gotten: it is given by

$$[[G_i]]u_n + [[t_{ij}\hat{n}_j]] = \hat{m}_i^{SE} \quad (95)$$

B.6.d. Mixture-electromagnetic field

From the foregoing considerations it appears clear that for the total system (matter plus electromagnetic field), that ⁵⁹

$$\frac{\partial}{\partial t} (\rho v_i + G_i) + (\rho v_i v_j - t_{ij} - t_{ij})_{,j} - \rho b_i = 0 \quad (96)$$

is its balance equation for linear momentum density. The coupled mixture and electromagnetic field density of linear momentum and the flux of same are given here respectively by $\rho v_i + G_i$ and $\rho v_i v_j - t_{ij} - t_{ij}$.

In addition, the jump balance equation is found to be of the form

$$[[\rho v_i (v_j - u_j)\hat{n}_j]] - [[G_i]]u_n - [(t_{ij} + t_{ij})\hat{n}_j] = \hat{m}_i^{SM}$$

or, upon using (39), equivalently as (97)

$$[[v_i]]\rho(v_j - u_j)\hat{n}_j - [[G_i]]u_n - [(t_{ij} + t_{ij})\hat{n}_j] = \hat{m}_i^{SM}$$

Here, the non-electromagnetic part of the interaction force density which is related to the surface $s(t)$ is written as

$$\hat{m}_i^{SM} = \hat{m}_i^S - \hat{m}_i^{SE} \quad (98)$$

B.7. Angular momentum balance equations

For fluid mixtures having a physical character thought to depend non-negligibly upon its polar constituents (whether atomic or molecular), it is necessary to establish relationships describing the behavior of the internal angular momenta of the system. And this, of course, even in the case when there is no electromagnetic field present. These are the matters of concern in this section, and they shall now be examined briefly.

B.7.a. Constituents

Let ⁶⁰ the density of total angular momentum together with its flux be given for any constituent by

$$(\psi_a^V)_k = \rho_{aa} L_k \quad \text{and} \quad (\phi_a^\psi)_{kj} = -\epsilon_{klij} x_l t_{aj} - m_{kj} \quad (99)$$

where m_{kj} is the couple stress tensor ⁶¹ of that particular constituent. Denoting by $\rho_{aa} l_k$ the assigned ⁶² body couple, the supply density of total angular momentum can be written. It and the production rate of $\rho_{aa} L_k$ are given here respectively by

$$(\sigma_a^\psi)_k = \epsilon_{klij} x_l \rho_{aa} b_i + \rho_{aa} l_k \quad \text{and} \quad (\bar{\sigma}_a^\psi)_k = \hat{n}_k. \quad (100)$$

It may be noted that the physical character of \hat{n}_k , like that of \hat{m}_i , is one of property transfer; this being due to the reactive and diffusive nature of the system. Lastly, the supply by a singular surface is denoted by

$$(\tau_a^\psi)_k = \hat{n}_k^S. \quad (101)$$

Use of the definitions above in (30) leads to the balance equation for the total angular momentum of a constituent; viz.

$$\begin{aligned} \frac{\partial \rho L_k}{\partial t} + \frac{(\rho L_k v_j)}{aa} ,j - m_{aj} ,j - \frac{\rho l_k}{aa} - \hat{n}_k &= \\ &= (\epsilon_{kli} x_l t_{ij}) ,j + \epsilon_{kli} x_l \rho b_i . \end{aligned} \quad (102)$$

The jump balance equation for this quantity follows from these parameters and (32). It is given here by the expression

$$\llbracket \frac{\rho L_k}{aa} (v_i - u_i) \hat{n}_i \rrbracket - \llbracket (\epsilon_{kli} x_l t_{ij} + m_{kj}) \hat{n}_j \rrbracket = \hat{n}_k^S . \quad (103)$$

Moment of linear momentum

Here ⁶³ the density of moment of linear momentum and its related flux are given respectively by

$$\left(\frac{\psi}{a} \right)_k = \epsilon_{kli} x_l \rho v_i \quad \text{and} \quad \left(\frac{\phi}{a} \right)_k = -\epsilon_{kli} x_l t_{ij} ; \quad (104)$$

while, the supply and production rate may be denoted as

$$\left(\frac{\sigma}{a} \right)_k = \epsilon_{kli} x_l \rho b_i \quad \text{and} \quad \left(\frac{\partial \psi}{a} \right)_k = \epsilon_{kij} t_{ij} + \epsilon_{kli} x_l \hat{m}_i . \quad (105)$$

The supply of that quantity by a surface s(t) is taken to be

$$\left(\frac{T}{a} \right)_k = \epsilon_{kli} x_l \hat{m}_i^S . \quad (106)$$

In terms of these parameters it follows from (30) that

$$\begin{aligned} \frac{\partial \rho(\epsilon_{kli} x_l v_i)}{\partial t} + [\rho(\epsilon_{kli} x_l v_i) v_j]_{,j} - \epsilon_{kij} t_{ij} - \epsilon_{kli} x_l \hat{m}_i &= \\ &= (\epsilon_{kli} t_{ij})_{,j} + \epsilon_{kli} x_l \text{ob}_i ; \end{aligned} \quad (107)$$

which expression is the equation for the balance of the moment of linear momentum of a constituent. Similarly, utilization of (32) yields the relationship

$$[(\epsilon_{kli} x_l v_i)_{aa} (v_j - u_j) \hat{n}_j] - [(\epsilon_{kli} x_l t_{ij} \hat{n}_j)] = \epsilon_{kli} x_l \hat{m}_i^S \quad (108)$$

known as the jump balance equation for said density of moment of linear momentum.

Internal angular momentum

The desired constituent expressions for internal angular momentum are secured by taking the difference between the relationships for the total angular momentum and those for the moment of linear momentum. For the specific internal angular momentum and the production rate of said quantity given for any constituent respectively by

$$\Omega_k = L_k - \epsilon_{kli} x_l v_i \quad \text{and} \quad \delta_k = \hat{n}_k - \epsilon_{kli} x_l \hat{m}_i, \quad (109)$$

the balance equation for the internal angular momentum of that constituent is found to be

$$\rho_{aa} \dot{\Omega}_k - m_{kj,j} - \rho_{aa} l_k + \epsilon_{kij} t_{ij} = \delta_k - \bar{\epsilon}_{aa} \Omega_k. \quad (110)$$

The jump balance equation of this quantity is seen from a similar treatment to be

$$[[\rho_{aa}^{\Omega} (v_j - u_j) \hat{n}_j]] - [[m_{kj} \hat{n}_j]] = \delta_a^S, \quad (111)$$

where the supply by s(t) for the internal angular momentum density of a constituent has been defined as

$$\delta_a^S = \hat{n}_a^S - \epsilon_{kli} x_l \hat{m}_i^S. \quad (112)$$

B.7.b. Mixture

Total angular momentum

The total angular momentum of the mixture has a balance equation gotten via a summation for all constituents over (102) and it is

$$\begin{aligned} \frac{\partial \rho L_k}{\partial t} + (\rho L_k v_j)_{,j} - m_{kj,j} - \rho l_k &= \\ &= (\epsilon_{kli} x_l t_{ij})_{,j} + \epsilon_{kli} x_l o_{ij}. \end{aligned} \quad (113)$$

Here, the new definitions involved are those for the total density of angular momentum, the couple stress tensor and the body couple of the mixture.

They are given respectively by

$$\rho L_k = \sum_{a=1}^s \rho_{aa}^L, \quad m_{kj} = \sum_{a=1}^s (m_{kj}^a - \rho_{aa}^L u_j^a) \text{ and } \rho l_k = \sum_{a=1}^s \rho l_k^a. \quad (114)$$

It may further be noticed that the transition from (102) to (113) has taken place under the assumption (i.e. axiom) that the total angular momentum in the bulk of the system is conserved; i.e., that

$$\sum_{a=1}^s \hat{n}_a^k = 0. \quad (115)$$

In a like manner a summation for all constituents over (103) yields the jump balance equation for the total angular momentum of the mixture. It reads as follows

$$\llbracket \rho L_k (v_j - u_j) \hat{n}_j \rrbracket - \llbracket (\epsilon_{klij} x_l t_{ij} + m_{kj}) \hat{n}_j \rrbracket = \hat{n}_k^S; \quad (116)$$

where

$$\hat{n}_k^S = \sum_{a=1}^s \hat{n}_k^S = \delta_k^S + \epsilon_{klij} x_l \hat{m}_i^S \quad (117)$$

is the singular surface supply of the total angular momentum of the mixture. The first term of (117)₂, the supply by s(t) of internal angular momentum of the mixture, is given by

$$\delta_k^S = \sum_{a=1}^s \delta_a^S. \quad (118)$$

Moment of linear momentum

From a summation over (107) for all constituents comes the relation

$$\begin{aligned} \frac{\partial}{\partial t} \epsilon_{klij} x_l \rho v_i + [\epsilon_{klij} x_l (\rho v_i v_j - t_{ij})]_{,j} - \\ - \epsilon_{kij} t_{ij} = \epsilon_{klij} x_l \rho b_i \end{aligned} \quad (119)$$

for the balance equation of the moment of linear momentum of the mixture.

Equation (118) in turn provides under these arguments the jump balance equation for this parameter; it being

$$\llbracket \epsilon_{klij} x_l \rho v_i (v_j - u_j) \hat{n}_j \rrbracket - \llbracket \epsilon_{klij} x_l t_{ij} \hat{n}_j \rrbracket = \epsilon_{klij} x_l \hat{m}_i^S. \quad (120)$$

Internal angular momentum

As was the case regarding the development of the constituent relations, here too a subtraction of the equations of (jump) balance for the moment of mixture linear momentum from those for the total angular momentum of the mixture gives the desired results. Hence, the conservation and jump balance equations for the internal angular momentum of the mixture are given respectively by ⁶⁴

$$\rho \dot{\Omega}_k - m_{kj,j} - \rho l_k + \epsilon_{kij} t_{ij} = 0$$

and

(121)

$$[[\rho \Omega_k (v_j - u_j) \hat{n}_j]] - [[m_{kj} \hat{n}_j]] = \delta_k^S.$$

Here, the density of internal angular momentum of the mixture has been defined as

$$\rho \Omega_k = \sum_{a=1}^s \rho_{aa} \Omega_k; \quad (122)$$

and by virtue of (90), (109)₂ and (115) it is seen that

$$\sum_{a=1}^s \delta_a^k = 0. \quad (123)$$

For the sake of simplicity in the description from this point on of the physical response of the system, I wish to make the following remarks. As can be easily seen from (111), the constituent stress tensors need not in general be symmetric. Such a statement is equally valid under general circumstances with regard to the mixture stress tensor: this on the basis of (121)₁. These anti-symmetric stress tensors make possible the treatment

of a broader class of fluid mixture systems than I wish to consider here. Hence I introduce at this time the assumption that (121)₁ be required to satisfy Cauchy's second law of motion⁶⁵. That is,

$$t_{ij} = t_{ji} ; \quad (124)$$

which relation expresses the fact that the stress tensor of the mixture is taken to be symmetric.

The general condition to be satisfied in order that said symmetry will exist follows from (121)₁ and can conceivably be met in a number of ways, each depending upon the physical nature of the particular system being investigated. I adopt here the point of view that under this assumption I am restricting the class of mixtures being treated to non-polar media or to polar mixtures of such a nature that the mixture stress tensor is symmetric. The prefix "non" as used with regard to the polarity of fluid mixtures should not be taken here in an absolute sense. But rather, those results established in this study are to be viewed as expressions assumed valid even for those fluid mixtures whose polarity, if any, may be taken as having a negligible role in the determination of the mechanical behavior of the medium. The manner in which this class of fluid mixtures complies with the symmetry condition (124) is, although unspecified, presumed to be such that all internal angular momentum related parameters and their governing equations ensure the validity of this assumption.

B.8. Energy balance equations

As was the case for linear momentum, the balance of energy for the total system reflects the contributions thereto of the constituents (and hence the mixture) and the electromagnetic field. To begin, I consider the first of these.

B.8.a. ConstituentsTotal energy

Equation (30) is again to be employed, and the definitions of the quantities therein are here taken to be the following⁶⁶. First,

$$\psi_a^v = \rho_a \left(\epsilon_a + \frac{1}{2} v_a^2 \right) \quad (125)$$

is the density of total energy of a constituent; where $\rho_a \epsilon_a$ and $\frac{1}{2} \rho_a v_a^2$ are respectively the densities for that constituent of internal and kinetic energy.

Next, the flux of the quantity (125) is given by

$$\phi_a^{\psi k} = -t_{ik} v_i^a + q_k^a \quad (126)$$

where q_k^a is the flux of internal energy of the constituent. Third,

$$\sigma_a^{\psi v} = \rho_a (b_i v_i^a + r_a) \quad (127)$$

represents the supply of total energy; with $\rho_a r_a$ being the density of body heating. And lastly,

$$\hat{\sigma}_a^{\psi v} = \hat{\epsilon}_a + J_i^T E_i \quad (128)$$

is the production rate for the constituent of its total energy density.

Here, $\hat{\epsilon}_a$ is the total energy production rate while the last term denotes the power expended by the electromagnetic field on that constituent.

From these definitions, together with (30), follows the relation

$$\begin{aligned} & \frac{\partial \rho (\epsilon + \frac{1}{2} v^2)}{\partial t} + [\rho (\epsilon + \frac{1}{2} v^2) v_j]_{,j} = \\ & = (t_{ij} v_i)_{,j} - q_{i,i} + \rho (b_i v_i + r) + \frac{\partial}{\partial t} + J_i^T E_i \end{aligned} \quad (129)$$

called the balance equation of total energy of a constituent. Upon introducing the supply by the surface s(t) of this quantity

$$\mathcal{T}_a^\Psi = \mathcal{e}_a^{SM} + \mathcal{e}_a^{SE}, \quad (130)$$

where the two terms denote respectively the non-electromagnetic and the electromagnetic parts of said supply, it follows from (32) that

$$[[\rho (\epsilon + \frac{1}{2} v^2) (v_i - u_i) \hat{n}_i]] + [[(q_i - t_{ji} v_j) \hat{n}_i]] = \mathcal{e}_a^{SM} + \mathcal{e}_a^{SE}. \quad (131)$$

This expression is termed the jump balance equation of the total energy.

Kinetic energy

Here the balance equation for the kinetic energy of a constituent derives from a multiplication of (85)₁ by v_i and it is given by

$$\begin{aligned} \rho \dot{v}_i v_i &= \rho (\frac{1}{2} v^2) \\ &= [t_{ij,j} + Q E_i + \epsilon_{ijk} J_j^T B_k + \rho b_i + (\hat{m}_i - \partial v_i)] v_i. \end{aligned} \quad (132)$$

Introduction of (132) into (129) now provides the desired result; viz.

$$\begin{aligned} \rho \dot{\epsilon}_{aa} - t_{ij} v_{i,j} + q_{j,j} - \rho r - \gamma_i \delta_i &= \\ &= \dot{\epsilon}_a - \dot{c} \left(\epsilon + \frac{1}{2} v^2 \right) - (\bar{m}_i - \bar{c} v_i) v_i \end{aligned}$$

or, equivalently,

(133)

$$\begin{aligned} \rho \dot{\epsilon}_{aa} + \rho \epsilon_{aa,j} u_j - t_{ij} v_{i,j} + q_{j,j} - \rho r - \gamma_i \delta_i &= \\ &= \dot{\epsilon}_a - \dot{c} \left(\epsilon + \frac{1}{2} v^2 \right) - (\bar{m}_i - \bar{c} v_i) v_i . \end{aligned}$$

These relations are termed the balance equations of the density of internal energy for a constituent.

B.8.b. Mixture

A summation over (133)₂ gives the equation of balance of the internal energy of the mixture

$$\rho \dot{\epsilon} - t_{ij} v_{i,j} + q_{j,j} - \rho r - \gamma_i \delta_i = 0 . \quad (134)$$

Hereto belong the following new definitions:

$$\rho \epsilon = \sum_{a=1}^s \rho_a \left(\epsilon_a + \frac{1}{2} u_a^2 \right) = \rho \epsilon^I + \sum_{a=1}^s \frac{1}{2} \rho_a u_a^2 \quad (135)$$

is the internal energy density of the mixture; with $\rho \epsilon^I$ its "inner part";

$$\begin{aligned} q_j &= \sum_{a=1}^s [q_{aj} - t_{ij} u_{ai} - \rho_a \left(\epsilon_a + \frac{1}{2} u_a^2 \right) u_{aj}] \\ &= q_j^I - \sum_{a=1}^s [t_{ij} u_{ai} + \rho_a \left(\epsilon_a + \frac{1}{2} u_a^2 \right) u_{aj}] \end{aligned} \quad (136)$$

represents the flux of internal energy of the mixture, its inner part being q_j^i ; and lastly, the density of body heating of the mixture is

$$\rho r = \sum_{a=1}^s \rho_a (r_a + b_{ij}^a u_j^i) . \quad (137)$$

It should be noted that the assumption (i.e. axiom) of conservation of energy of the mixture made in going from (133)₂ to (134) has been that

$$\sum_{a=1}^s \hat{e}_a = 0 . \quad (138)$$

Equation (131) when summed over for all constituents gives

$$\llbracket \rho(\epsilon + \frac{1}{2}v^2)(v_i - u_i)\hat{n}_i \rrbracket + \llbracket (q_i - t_{ji}v_j)\hat{n}_i \rrbracket = \hat{e}^{SM} + \hat{e}^{SE}$$

or, using (39), equivalently (139)

$$\llbracket (\epsilon + \frac{1}{2}v^2)\rho(v_i - u_i)\hat{n}_i + \llbracket (q_i - t_{ji}v_j)\hat{n}_i \rrbracket = \hat{e}^{SM} + \hat{e}^{SE} .$$

B.8.c. Electromagnetic field

The balance equation of energy for the electromagnetic field is here simply given by the well-known relationship

$$\frac{\partial \epsilon}{\partial t} + S_{Ei,i} = \frac{-\hat{e}}{E} ; \quad (140)$$

where, the energy density, flux of same (i.e. the Poynting vector) and the power expended on the sources (i.e. charges and currents) are given respectively by

$$\epsilon = \frac{1}{2}(D_k E_k + B_k H_k), \quad S_{Ei} = \epsilon_{ijk} E_j H_k \quad \text{and} \quad \hat{e} = J_i^T E_i . \quad (141)$$

Upon defining the quantity involved by $\psi_v = \epsilon$, and denoting formally its flux and its supply by a singular surface respectively as

$$\Phi_i^\psi = \frac{S_i}{E} - \epsilon v_i \quad \text{and} \quad T^\psi = \hat{e}^{SE}, \quad (142)$$

utilization of (32) provides the jump balance equation for the density of electromagnetic field energy: it is

$$\llbracket \frac{\epsilon}{E} u_n \rrbracket - \llbracket \frac{S_i \hat{n}_i}{E} \rrbracket = \hat{e}^{SE}. \quad (143)$$

B.8.d. Mixture-electromagnetic field

The developments of this section culminate with the merging of the results above in such a manner as to provide expressions for energy that are applicable to the total system. Here then, the conservation equation for the coupled densities of material and electromagnetic field internal energies is (for symmetric t_{ij})

$$\begin{aligned} \frac{\partial}{\partial t} (\rho\epsilon + \frac{\epsilon}{E}) + (\rho\epsilon v_i + \frac{S_i}{E})_{,i} - t_{ij} v_{i,j} + \\ + q_{j,j} - \rho r - \rho b_i v_i = 0. \end{aligned} \quad (144)$$

From (139) and (143) the jump balance equation here is seen to be either

$$\begin{aligned} \llbracket \rho(\epsilon + \frac{1}{2}v^2)(v_i - u_i)\hat{n}_i \rrbracket - \llbracket \frac{\epsilon}{E} u_n \rrbracket + \\ + \llbracket (q_i - t_{ij}v_j + \frac{S_i}{E})\hat{n}_i \rrbracket = \hat{e}^{SM}; \end{aligned}$$

or, again using (39), equivalently

$$(145)$$

$$\left[\epsilon + \frac{1}{2} v^2 \right] \rho (v_i - u_i) \bar{n}_i - \frac{[\epsilon]}{E} u_n + \left[(q_i - t_{ij} v_j - \frac{S_i}{E}) \bar{n}_i \right] = \dot{e}^{SM}.$$

B.9. Entropy balance equations

In a rational mechanical theory the selection and utilization of an entropy principle is of cardinal importance. This is so first with regard to the closure of the set of system balance equations, and then too in connection with the securing of the necessary constitutive equations.

Generally speaking, there does not seem to be complete unanimity among the practitioners of modern continuum mechanics regarding the formulation and interpretation of the various field parameters and/or their governing equations⁶⁷. An example of this situation may, in particular, be found with regard to the matter of the establishment of an entropy principle⁶⁸. For the problem at hand the considerations below and the entropy inequalities resulting therefrom have been introduced primarily on the basis of their compatibility with the study of Liu & Müller⁶⁹.

B.9.a. Constituents

Equation (30) serves once again as the departure point from which the development of the desired balance equations begins. Here⁷⁰, the quantity involved is the entropy density of a constituent; it and its flux being denoted respectively by

$$\psi_a^\psi = \rho \eta_{aa} \quad \text{and} \quad \phi_a^\psi = \frac{1}{T} (q_a + \epsilon_{ijk} \delta_{ja} M_k) . \quad (146)$$

The supply density and "production" rate ⁷¹ for that same constituent are in turn expressed as follows:

$$\sigma_{\mathbf{a}}^{\psi} = \frac{1}{T} \rho r_{\mathbf{a}\mathbf{a}} \quad \text{and} \quad \bar{\sigma}_{\mathbf{a}}^{\psi} = \rho \gamma_{\mathbf{a}\mathbf{a}} . \quad (147)$$

Here T is the (absolute) temperature of all constituents, and thus also of the mixture ⁷². The motivation for the selection of an entropy flux for constituents such as that introduced here is found in the more general result derived by Liu & Müller ⁷³ for the case of a single continuum. With a view towards looking eventually into the physical character of the mixture systems here when they are "close to equilibrium", I consider that the generalization introduced here of the linearized single-fluid result of Liu & Müller suggests itself as an acceptable possibility. Similarly, the constituent entropy supply relation proposed here follows from the single-fluid considerations of Liu ⁷⁴. On the basis of the discussion above, I deem the generalization (147)₁ of his derived non-relativistic fluid result to be a satisfactory choice here.

Given the identifications (146) and (147) above, it follows from (30) that the entropy "balance" equation for a constituent is

$$\rho \dot{\eta}_{\mathbf{a}\mathbf{a}} + \left[\frac{1}{T} (q_i + \epsilon_{ijk} \delta_j^i M_k) \right]_{,i} - \frac{1}{T} \rho r_{\mathbf{a}\mathbf{a}} + \bar{\sigma}_{\mathbf{a}\mathbf{a}} = \rho \gamma_{\mathbf{a}\mathbf{a}} . \quad (148)$$

For a supply by a surface $s(t)$ of the entropy density of a constituent denoted by $\tau_{\mathbf{a}}^{\psi} = \gamma_{\mathbf{a}}^{\mathbf{S}}$, equation (32) together with the definitions above yields the relation

$$\llbracket \rho n (v_i - u_i) \hat{n}_i \rrbracket + \llbracket \frac{1}{T} (q_i + \epsilon_{ijk} \delta_{j_a}^M \delta_{k_a}^M) \hat{n}_i \rrbracket = \gamma_a^S \quad (149)$$

termed here the jump "balance" equation of the constituent entropy density.

B.9.b. Mixture

The result of a summation for all constituents over (148), together with the assumption (i.e. axiom) that the entropy production density of the mixture

$$\rho \gamma = \sum_{a=1}^s \rho \gamma_{aa} \quad (150)$$

satisfies the condition that $\rho \gamma \geq 0$ ⁷⁵, is the relationship

$$\rho \dot{n}_i + \llbracket \frac{1}{T} (q_i + \sum_{a=1}^s \epsilon_{ijk} \delta_{j_a}^M \delta_{k_a}^M - \sum_{a=1}^s c_{aji} u_j) \rrbracket_{,i} - \frac{\rho r_i}{T} \geq 0. \quad (151)$$

This is the entropy balance equation for the mixture. The definitions which relate the constituent relations to this equation are as follows:

$$\rho n = \sum_{a=1}^s \rho n_{aa} \quad \text{and} \quad \rho r_i = \sum_{a=1}^s \rho r_{aa} \quad (152)$$

are respectively the entropy density and the inner part of the body heating of the mixture; and,

$$c_{aji} = \rho \left(\epsilon_{a a} + \frac{1}{2} u_a^2 - T n_a \right) \delta_{ji} - t_{aji}. \quad (153)$$

For this last relation it may be observed that

$$\sum_{a=1}^s C_{aji} = \rho(\epsilon - T\eta)\delta_{ji} - \sum_{a=1}^s t_{aji} \quad (154)$$

This means that for ϵ , η and t_{aji} given by constitutive equations (cf. Chpt. C), only $s-1$ of the C_{aji} are independent.

Upon summing (149) for all constituents and making the assumption (i.e. axiom) in doing so that the supply by a surface $s(t)$ of entropy density of the mixture

$$\gamma^S = \sum_{a=1}^s \gamma_a^S \quad (155)$$

fulfills the requirement that $\gamma^S \geq 0$, the following relation is found:

$$\llbracket \rho n(v_i - u_i)\hat{n}_i \rrbracket + \llbracket \frac{1}{T}(q_i + \sum_{a=1}^s \epsilon_{ijk} \delta_{ja}^k M_k - \sum_{a=1}^s C_{aji} u_j)\hat{n}_i \rrbracket \geq 0 \quad (156)$$

This is the jump balance equation for the mixture entropy density. Use once again of (39), together with the assumption that T is continuous across $s(t)$, allows (156) to be rewritten in the form

$$T \llbracket n \rho(v_i - u_i)\hat{n}_i + \llbracket q_i \hat{n}_i \rrbracket + \llbracket \sum_{a=1}^s \epsilon_{ijk} \delta_{ja}^k M_k \hat{n}_i \rrbracket - \llbracket \sum_{a=1}^s C_{aji} u_j \hat{n}_i \rrbracket \geq 0 \quad (157)$$

This relation shall be employed shortly in the subsequent development of the entropy principle here.

B.10. Selected equations: a recapitulation

For developmental purposes below it is worthwhile to collect here the set of equations which shall be referred to, directly or in terms of their equivalents, in the sequel to this chapter. These relationships are now given.

Balance and conservation equations

$$\dot{\rho}_a + \rho_{aa} v_{j,j} = \dot{c}_a \quad (158)$$

$$\rho_{aa} \dot{v}_i - t_{ij,j} - Q \delta_i - \epsilon_{ijk} \gamma_j^B B_k - \rho_{a0a} b_i = \hat{m}_i - \dot{c}_a v_i \quad (159)$$

$$t_{ij} = t_{ji} \quad (160)$$

$$\rho \dot{e} - t_{ij} v_{i,j} + q_{j,j} - \rho r - \gamma_i \delta_i = 0 \quad (161)$$

$$\rho \dot{h} + \left[\frac{1}{T} (q_i + \sum_{a=1}^s \epsilon_{ijk} \delta_j^M M_k - \sum_{a=1}^s C_{ajj} u_j^u) \right]_{,i} - \frac{1}{T} \rho r_i \geq 0 \quad (162)$$

$$D_{i,i} = Q^F \quad (163)$$

$$\epsilon_{ijk} \mathcal{C}_{k,j} = Q^F v_i + J_i^F + \frac{\partial D_i}{\partial t} \quad (164)$$

$$B_{i,i} = 0 \quad (165)$$

$$\epsilon_{ijk} E_{k,j} = - \frac{\partial B_i}{\partial t} \quad (166)$$

Jump balance equations

$$\llbracket \rho_a (v_i - u_i) \hat{n}_i \rrbracket = \hat{e}_a^S \quad (167)$$

$$\llbracket v_i \rrbracket \rho (v_j - u_j) \hat{n}_j - \llbracket G_i \rrbracket u_n - \llbracket (\tau_{ij} + t_{ij}) \hat{n}_j \rrbracket = \hat{m}_i^{SM} \quad (168)$$

$$\llbracket \epsilon + \frac{1}{2} v^2 \rrbracket \rho (v_i - u_i) \hat{n}_i - \llbracket \epsilon \rrbracket u_n + \llbracket (q_i - t_{ji} v_j - S_i) \hat{n}_i \rrbracket = \hat{e}^{SM} \quad (169)$$

$$\begin{aligned} T \llbracket \eta \rrbracket \rho (v_i - u_i) \hat{n}_i + \llbracket q_i \hat{n}_i \rrbracket + \llbracket \sum_{a=1}^S \epsilon_{ijk} \delta_{aj}^k M_a \hat{n}_i \rrbracket - \\ - \llbracket \sum_{a=1}^S c_{aj}^i u_j \hat{n}_i \rrbracket \geq 0 \end{aligned} \quad (170)$$

$$\llbracket \mathcal{D}_i \rrbracket = \epsilon_{ijk} h_j \hat{n}_k \quad (171)$$

$$\llbracket \mathcal{C}_k \rrbracket = g \hat{n}_k + u_n h_k \quad (172)$$

$$\llbracket B_i \rrbracket = \epsilon_{ijk} k_j \hat{n}_k \quad (173)$$

$$\llbracket E_i \rrbracket = f \hat{n}_i - u_n k_i \quad (174)$$

Footnotes to Chapter B

- ¹ Note CFT [1960, 9, Sect. 8].
- ² With regard to integral principles see, for example, CFT [1960, 9, Sect. 7].
- ³ The degree to which field quantities in a given region of a system are continuous is set down in an assumption of smoothness. In this study I assume whatever degree of smoothness is required to ensure the validity of the relationships concerned. This point is touched upon by Truesdell in CFT [1960, 9, Sect. 16].
- ⁴ Cf. Post [1962, 6, pp.25-26] and Bergmann [1949, 1, p.177].
- ⁵ Motivated by the work of Truesdell [1962, 10, p.2339] and Kelly [1964, 4, p.133], I too divide the total supply of a field quantity of a constituent into two parts. The one part relates to the constituents as though they were single continua; while the second part accounts for the density of the production rate (i.e. supply or transfer) of the given field quantity due to, e.g., the diffusive and reactive interaction of the constituents. This latter contribution to the total supply is generally not taken as being known a priori as is the former; but rather, it is most properly described in terms of appropriately formulated constitutive equations. The developments pertaining to balance equations in regions of volume and surface containing respectively singular surfaces and lines are given here in the sections beginning with Sect. B.1.
- ⁶ Truesdell [1969, 7, p.83].
- ⁷ CFT [1960, 9, Sect. 13].

- ⁸ I refer the reader to CFT [1960, 9, Sect. 66A] and the references given there for the historical aspects of this matter.
- ⁹ Cf. Alts [1970, 1, p.12]. The reader is cautioned not to confuse these "particles" of continuum mechanics with those of the constituents (e.g. electrons, atoms, etc.) in terms of which they are defined.
- ¹⁰ CFT [1960, 9, Sect. 15].
- ¹¹ Jaunzemis [1967, 12, p.67].
- ¹² CFT [1960, 9, Sects. 15 and 65].
- ¹³ CFT [1960, 9, Sect. 158].
- ¹⁴ Not altogether unusual, both direct and index notation shall be used in this study. For the latter Cartesian tensor notation is employed. As above, Greek letter indices relate to material co-ordinates while spatial co-ordinates are given by Latin letters. Both sets of indices range over 1,2,3 and, following Einstein, repeated indices indicate a summation.
- ¹⁵ The alternator (or permutation symbol) ϵ_{ijk} is a third-order tensor defined by $\epsilon_{ijk} = 1, -1$ or 0 if, respectively, the indices ijk are an even or odd permutation of 1,2,3 or if any two indices are equal.
- ¹⁶ CFT [1960, 9, Sects. 16 and 65].
- ¹⁷ Cf. for example CFT [1960, 9, (16.4)].
- ¹⁸ The use of the names Lagrangian and Eulerian for the material and spatial descriptions respectively has been shown by Truesdell (CFT [1960, 9, Sect. 66A and the references therein]) to be misnomers; both descriptions having resulted from work done by Euler.

¹⁹ CFT [1960, 9, Sect. 73].

²⁰ CFT [1960, Sect. 158].

²¹ Ibid.

²² CFT [1960, 9, Chpt. C].

²³ CFT [1960, 9, Sects. 81 and 192]. Cf. also Müller [1973, 9, p.24] and Kelly [1964, 4].

²⁴ I follow Kelly [1964, 4, pp.134-137] and Müller [1973, 9, pp.26-28] in this study in noting that the results of such considerations are significant with regard to a theory of material media interacting with an electromagnetic field. Truesdell & Toupin (CFT [1960, 9, Sect. 184]) note that "Singularities located upon lines seem not to have been studied from a general viewpoint."

²⁵ Cf. Kelly [1964, 4, p.136] and Müller [1973, 9, p.27].

²⁶ Note the comments of footnote 5 above.

²⁷ Cf. CFT [1960, 9, Sect. 193], Kelly [1964, 4] and Müller [1973, 9, p.24].

²⁸ CFT [1960, 9, Sect. 193] and Kelly [1964, 4, pp.133-134].

²⁹ Ericksen (CFT [1960, 9, Sect. App.26]) points out a historical motivation for using the terminology "Green's transformation" in place of, e.g., "Gauss' theorem", "divergence theorem" etc. Although writers in modern continuum mechanics seem divided in their choice of name for the mathematical results involved, I follow the CFT here. For a region of volume admitting singular surfaces the appropriate expression is given, e.g., by Jaunzemis [1967, 12, p.198].

³⁰ Cf. Kelly [1964, 4] and Müller [1973, 9, pp.26-27].

- ³¹ The so-called "Stokes' theorem" has been shown by Truesdell (cf. Ericksen, CFT [1960, 9, Sect. App. 28]) to have, in fact, derived from the studies of Kelvin. For the sake of proper attribution I here too follow the CFT and use the terminology "Kelvin's transformation". In the situation where the surface region of interest contains a singular line, the relevant expression here may be found in, e.g., Jaunzemis [1967, 12, p.199].
- ³² Cf., for example, Kelly [1964, 4].
- ³³ CFT [1960, 9, Sect. 159].
- ³⁴ Aris [1965, 2, p.81].
- ³⁵ Cf. Gyarmati [1970, 5, p.28] and Müller [1973, 9, p.192].
- ³⁶ The principle of superposition I introduce here is, with some slight modification, that given by Podolsky & Kunz [1969, 5, p.47]. It is to be noted that it too, like the mechanical principles of Truesdell (cf. footnote 6), is of a metaphysical nature. In this regard see p. 274 of the cited work by Podolsky & Kunz.
- ³⁷ This approach was taken by Kelly [1964, 4, pp.146-152].
- ³⁸ CFT [1960, 9, Sect. 279]. These relationships are postulated as holding in preferred Euclidean (or Galilean) frames of reference and they serve to couple the charge-current and the electromagnetic fields. Now, the Maxwell's equations for the electromagnetic field are known to be invariant under Lorentz transformations. I thus assume that since the mechanics of mixtures here is classical, and examined in terms of inertial Euclidean (or Galilean) frames, that there exists a frame of reference which is simultaneously Lorentzian and inertial for which the developments here are valid. See in this regard Müller

[1973, 9, p.130].

³⁹ CFT [1960, 9, (283.35)₂].

⁴⁰ Cf. Liu & Müller [1972, 5, (2.20)₂] and Müller [1973, 9, p.132].

⁴¹ Cf. CFT [1960, 9, p.689] and Liu & Müller [1972, 5, (4.4)₄].

⁴² See, for example, Jackson [1967, 11], Panofsky & Phillips [1962, 5], Penfield, Jr. & Haus [1967, 17], Stratton [1941, 1] and CFT [1960, 9, Chpt. F].

⁴³ Cf. Kelly [1964, 4, p.150].

⁴⁴ Cf. CFT [1960, 9, p.686].

⁴⁵ Cf. CFT [1960, 9, p.677].

⁴⁶ Note CFT [1960, 9, (279.1)₂ and (283.35)₁]; the latter equation cited being generalized here. This result has also been secured by Demiray & Eringen [1973, 3, (3.5)].

⁴⁷ Cf. Kelly [1964, 4, (4.110)].

⁴⁸ Cf. Liu & Müller [1972, 5, (2.20)₁] and Müller [1973, 9, p.132].

⁴⁹ This result generalizes, by virtue of its mixture character, that given by Truesdell & Toupin (CFT [1960, 9, p.639]), Liu & Müller [1972, 5, (4.4)₃] and Müller [1973, 9, p.126]. The formal inclusion of current densities upon $s(t)$ is a trivial matter. Such densities have been disregarded here for the sake of simplicity in the further development of this theory. Equation (71) including a surface current term has been given by Demiray & Eringen [1973, 3, (3.2)₂].

⁵⁰ See, for example, Sommerfeld [1963, 9, pp.19-20] and Kline & Kay [1965, 11, p.33]. Note here also footnote F7.

- 51 CFT [1960, 9, (278.9) and (278.12)], Liu & Müller [1972, 5, (4.4)₂] and Müller [1973, 9, p.120 and p.143].
- 52 Cf. Kelly [1964, 4, (4.85)] and Müller [1973, 9, p.120].
- 53 Cf. Liu & Müller [1972, 5, (2.19)₁] and Müller [1973, 9, p.131].
- 54 CFT [1960, 9, (278.8) and (278.11)], Liu & Müller [1972, 5, (4.4)₁] and Müller [1973, 9, p.121 and p.143].
- 55 With regard to (81) and (82) see, e.g., Kelly [1964, 4, p.139].
- 56 CFT [1960, 9, Sect. 157].
- 57 Cf. Penfield, Jr. & Haus [1967, 17] and CFT [1967, 9, Sects. 284 and 286].
- 58 CFT [1960, 9, Sect. 200]. The integral basis of this relationship derives here essentially from a constituent version of Euler's first law (ibid, Sect. 196) and the Euler-Cauchy stress principle (ibid, Sect. 200). In contrast to popular misconception (see, e.g., Sutton & Sherman [1965, 17, p.297] and Haase [1969, 2, p.228]) such a result is not the consequence of Newton's laws of motion which have been pointed out by Truesdell & Toupin (CFT [1960, 9, Sects. 2 and 196]) as being inadequate for this purpose; not to mention others.
- 59 Cf. Müller [1973, 9, p.130].
- 60 Cf. Kelly [1964, 4, p.141].
- 61 CFT [1960, 9, p.538] notes for single continua the introduction of the (anti-symmetric) couple stress tensor as a means of accounting for torques not arising from forces. Kelly [1964, 4], e.g., extended that development to the case of mixtures.
- 62 The body couples are, like the body forces, presumed known; and they

too must be specified. Due, however, to the assumption (124) they need not be made explicit in this study.

- ⁶³ These definitions are gotten from a multiplication of $(85)_1$ by $\epsilon_{k|i} x_i$ and are introduced solely for the formal convenience they offer.
- ⁶⁴ Cf. CFT [1960, 9, (205.10)] and Kelly [1964, 4, (4.37) and (4.39)]. Similar to the remarks made in footnote 58 for the case of linear momentum, $(121)_1$ is secured by means of Euler's second law (CFT [1960, 9, Sect. 196]) and the Euler-Cauchy (couple) stress principle (ibid, Sect. 200); both of which are expressed in a constituent form.
- ⁶⁵ CFT [1960, 9, p.546]. See here also Müller [1968, 9, (2.17)].
- ⁶⁶ Cf. CFT [1960, 9, Sect. 243], Kelly [1964, 4, pp.142-143], Truesdell [1969, 7, pp.84-85] and Müller [1973, 9, p.36].
- ⁶⁷ For example, see Truesdell [1969, 7, pp.93-94], Green & Naghdi [1969, 1] and Craine, Green & Naghdi [1970, 4].
- ⁶⁸ See, e.g., Green & Naghdi [1971, 4], Liu [1973, 6, p.27] and Demiray & Eringen [1973, 3, p.910].
- ⁶⁹ Liu & Müller [1972, 5]. Note also Müller [1973, 9, p.73].
- ⁷⁰ Cf. Truesdell [1969, 7, p.88].
- ⁷¹ For mixtures with constituents whose energetic coupling is non-negligible the term $\rho \gamma_{aa}$ may (cf. Truesdell, ibid) "be of any amount"; it being subject only to the restriction assumed for (150). The less general case wherein each constituent is accorded its own entropy inequality has been examined in various degrees by, e.g., Green & Naghdi [1971, 4] and Demiray & Eringen [1973, 3]. Such treatments

have, however, been pointed out by Dunwood & Müller [1968, 2, p.347] and Truesdell [ibid, p.93] as being physically deficient in that regard.

⁷² The temperature $T(\vec{x}, t)$ is a primitive (i.e. physically undefined) scalar field quantity in rational thermodynamics. (Cf. Truesdell [1968, 7] and Truesdell [1969, 7, Lecture 1].) Here $T(\vec{x}, t)$ is taken to be a function having positive values for all \vec{x} and t : i.e. $T > 0$. A many-temperature theory of mixtures would indeed prove both interesting and valuable. However, mixture continua behavior in general, and that of mixture continua interacting with an electromagnetic field in particular, represents a problem area barely investigated at present. I thus conclude (cf. Müller [1968, 4, p.6]) that it is worthwhile to limit the considerations here to single-temperature systems.

⁷³ Liu & Müller [1972, 5, p.169].

⁷⁴ Liu [1973, 7, pp.113-114]. Cf. also Müller [1968, 4, p.7].

⁷⁵ Cf. Truesdell [1969, 7, p.88].

C. CONSTITUTIVE EQUATIONS

The relationships secured thus far describe essentially the kinematics of a moving and chemically reactive mixture interacting with an electromagnetic field. Balance equations are, however, unable to provide any insight into the action of a system upon a portion of itself and the subsequent reaction (i.e. material response) of that part upon the system ¹. For example, the balance equations for physically similar material media (e.g. all noble gases) evidence no formal difference in their general structure. The distinction between physical systems comes with the characterization of their response properties.

A continuum description of material behavior is a phenomenological one. It is the knowledge and understanding of the phenomenology relevant to a given system that must lead to the constitutive equations which serve to give that system its own macro-identity. Constitutive equations are, as a result of analysis and abstraction of physical experience, a definition of ideal material systems: i.e., mathematical models of real media. The formulation of adequate constitutive equations is a most formidable task ²; shortcomings here, for example, in the case of matter in an electromagnetic field being generally considered as the origin of theoretical and experimental disagreement ³.

It is a twofold task that constitutive equations fulfil. They prescribe the material response of a system and in doing so serve to make the balance equations thereof determinate ⁴. The spirit of the approach taken here requires that these relationships be such that the second part of Truesdell's second metaphysical principle given above be satisfied.

As was the situation commented upon in connection with the entropy principle, the manner in which one "properly" allows for constituent interactions does not yet appear to be an entirely settled upon matter. The possible shortcomings and discrepancies notwithstanding, I assume that an adapted methodology of a rational mechanical constitutive theory such as that employed by Liu & Müller⁵ offers a reasonably secure way of deriving the constitutive equations for this problem.

The balance equations given earlier indicate that constitutive equations must be provided for the following "dependent"⁶ scalar (S), vector (V_i) and tensor (T_{ij}) fields:

$$S : \Lambda^{\alpha}, \epsilon^I, \eta \quad (a = 1, 2, \dots, s),$$

$$V_i : \hat{m}_a^I, q_a^I, P_{ai}, M_{ai} \quad (a = 1, 2, \dots, s)$$

and

$$T_{ij} : t_{aij}, C_{aij} \quad (a = 1, 2, \dots, s).$$

(1)

The "independent" scalar and vector field variables upon which these fields may depend have in this study been assumed to be the following⁷:

$$S : \rho_a, T \quad (a = 1, 2, \dots, s)$$

and

$$V_i : \rho_{a,k}, T_{,k}, v_{ak}, E_k, B_k \quad (a = 1, 2, \dots, s).$$

(2)

By assuming that the material response of any system is an intrinsic (i.e. observer independent) physical property of that system, it is desirable to establish constitutive equations that reflect this viewpoint. That is, for any given system the constitutive relations are

assumed to be invariant under co-ordinate transformations and form-invariant under observer transformations ⁸.

A change of reference frame (cf. Sect. B.a.), or observer transformation, is expressed here by a Euclidean transformation ⁹

$$x^*_i = 0_{ij}(t)x_j + d_i(t) \quad \text{and} \quad t^* = t - a \quad (3)$$

where the arbitrary quantities $0_{ij}(t)$, $d_i(t)$ and a are respectively a time-dependent orthogonal tensor and vector, and a constant. The scalar, vector and tensor constitutive field quantities (1) and the field variables upon which they are presumed to depend are called objective when they transform under (3) according to the following relations ¹⁰:

$$\begin{aligned} S^* &= S, \\ V_{P_i}^* &= 0_{ij}(t)V_{P_i} \quad (\text{for polar vectors}), \\ V_{A_i}^* &= |0_{pq}(t)|0_{ij}(t)V_{A_i} \quad (\text{for axial vectors}) \end{aligned} \quad (4)$$

and

$$T_{ij}^* = 0_{ik}(t)0_{jl}(t)T_{kl}.$$

Of the fields (1) and (2) above all are taken to be objective with the exception of \hat{m}_i , v_k and E_k . With a view towards the coming developments it is necessary to secure those combinations of variables involving these fields that are objective. With regard to \hat{m}_i I follow Müller ¹¹ and replace it by $\hat{m}_i - \hat{c}v_i$. Furthermore the objective vectors gotten here from v_k and E_k are, respectively, the velocity difference V_k and the electromotive intensity ξ_k ¹².

The set of objective and independent field variables is given by

$$\rho_a, \rho_{a,k}, T, T_{,k}, V_b, \delta_s \text{ and } B_k \quad (5)$$

and F is taken to denote any element of the set of constitutive fields

$$\Lambda_a^\alpha, \varepsilon^I, \eta, \hat{m}_i - \hat{c}v_{ia}, q_i, P_a, M_a, t_{ij} \text{ and } C_{ij}; \quad (6)$$

where, $a = 1, 2, \dots, s$ and $b = 1, 2, \dots, s-1$. The principle of equipresence¹³ as employed here leads to the result that the constitutive equation of any particular F is given implicitly by¹⁴

$$F = \mathcal{F}(\rho_a, \rho_{a,k}, T, T_{,k}, V_b, \delta_s, B_k); \quad (7)$$

where, in general, \mathcal{F} is appropriately a scalar, vector or (2^{nd} -order) tensor functional of its arguments. I assume¹⁵, however, that \mathcal{F} is representable by a function. On the basis of (7) this study is thus seen to concern itself with the bulk material behavior of mixtures of inviscid, chemically reactive, heat conducting, magnetizable and dielectric fluids.

Although the equations of motion are not invariant under changes of frame, the constitutive equations should be formulated in such a manner that the material response of a system is independent of any observer. The preservation under an observer transformation of the character of the functions \mathcal{F} is expressed in an axiom of form-invariance known as the principle of material frame-indifference¹⁶: it stating that $\mathcal{F}^* = \mathcal{F}$. In terms of the scalar, vector and tensor constitutive fields in (6) this means that the now isotropic functions \mathcal{F} are such that the following relations hold:

$$\begin{aligned}
 S(\rho, {}^0_{jk} \rho_{a,k}, T, {}^0_{jk} T_{,k}, {}^0_{jk} V_{b,k}, {}^0_{jk} \xi_k, |{}^0_{pq} | {}^0_{jk} B_k) &= \\
 &= S(\rho, \rho_{a,k}, T, T_{,k}, V_{b,k}, \xi_k, B_k), \quad (8)_1
 \end{aligned}$$

$$\begin{aligned}
 V_i(\rho, {}^0_{jk} \rho_{a,k}, T, {}^0_{jk} T_{,k}, {}^0_{jk} V_{b,k}, {}^0_{jk} \xi_k, |{}^0_{pq} | {}^0_{jk} B_k) &= \\
 &= {}^0_{ij} V_j(\rho, \rho_{a,k}, T, T_{,k}, V_{b,k}, \xi_k, B_k) \quad (8)_2
 \end{aligned}$$

and

$$\begin{aligned}
 T_{ij}(\rho, {}^0_{jk} \rho_{a,k}, T, {}^0_{jk} T_{,k}, {}^0_{jk} V_{b,k}, {}^0_{jk} \xi_k, |{}^0_{pq} | {}^0_{jk} B_k) &= \\
 &= {}^0_{ik} {}^0_{jl} T_{kl}(\rho, \rho_{a,k}, T, T_{,k}, V_{b,k}, \xi_k, B_k). \quad (8)_3
 \end{aligned}$$

The specification of constitutive equations in rational mechanics includes some mention of the symmetry restrictions to be imposed upon these relationships. Such conditions are inherent in the choice of the group of orthogonal transformations for which the constitutive equations are assumed valid. When it is the proper (full) orthogonal group which is involved, the system in question is said to possess hemihedral (holohedral) isotropy¹⁷; the former case is examined here.

I note in passing that the forms (7) of the various constitutive equations are used in obtaining the reduced entropy inequality which follows shortly. But before I take up that matter, it is necessary to secure the explicit relationships for the constitutive equations above.

C.1. Constitutive equation representations

Let $s^{(m)}$ ($m = 1, 2, \dots, s+1$) be a scalar and $v_k^{(n)}$ ($n = 1, 2, \dots, 2s+2$) be a generic vector in terms of which the argument fields of (7) may be expressed. In general, thus, (7) may now be written as

$$F_{i_1 i_2 \dots i_1} = \mathcal{F}_{i_1 i_2 \dots i_1}(s^{(m)}, v_k^{(n)}). \quad (9)$$

Under the principle of material frame-indifference it follows that

$$\begin{aligned} \mathcal{F}_{i_1 i_2 \dots i_1}(s^{(m)*}, v_k^{(n)*}) &= \\ &= 0_{i_1 j_1} 0_{i_2 j_2} \dots 0_{i_1 j_1} \mathcal{F}_{j_1 j_2 \dots j_1}(s^{(m)}, v_k^{(n)}); \end{aligned} \quad (10)$$

which relation is equivalent to (8). Now, the development of explicit form-invariant constitutive equations from (10) utilizes an invariant-theoretical methodology of rational mechanics. The particular approach I follow here is due to Pipkin & Rivlin¹⁸.

Let the $\mathcal{F}_{i_1 i_2 \dots i_1}$ be continuous single-valued functions which depend upon the now assumed uniformly bounded fields $s^{(m)}$ and $v_k^{(n)}$, and which possess whatever degree of smoothness may be required. Under these conditions it is possible¹⁹ without loss of generality to make the convenient, but otherwise non-essential²⁰, assumption that the $\mathcal{F}_{i_1 i_2 \dots i_1}$ are polynomial functions of the argument fields²¹.

For $\bar{V}_i^{(p)}$ ($p = 1, 2, \dots, l$) the components of an arbitrary generic vector which has the same transformation properties as does $v_k^{(n)}$ (cf. (4)), I form the absolute (polynomial) scalar invariant S under the group of proper orthogonal transformations: viz.

$$S = \bar{V}_i^{(1)} \bar{V}_i^{(2)} \dots \bar{V}_i^{(l)} \mathcal{F}_{i_1 i_2 \dots i_1}(s^{(m)}, v_k^{(n)}). \quad (11)$$

The polynomial character of the invariant S makes possible its expression as

$$S = \sum_{\alpha=1}^m S_{\alpha} I_{\alpha}. \quad (12)$$

Here S_{α} denotes the polynomial functions invariant under the group of proper orthogonal transformations which are elements of an integrity basis appropriate to the vectors $V_k^{(n)}$ in R_3 . The I_{α} are those invariants that are multilinear in $\bar{V}_i^{(p)}$ and which derive from an integrity basis for those vectors together with the $V_k^{(n)}$.

The integrity basis for the S_{α} is here (cf. Appendix C-1) that finite set of absolute scalar (polynomial) invariants formed²² from the absolute scalars ρ and T together with the inner products $(I_{\alpha}^{(1)}, I_{\beta}^{(2)})$, determinantal^a products (Δ_k^A) and products of these both (Δ_o^A) , of the vectors $V_k^{(n)}$. It depends upon the specific scalar, (polar or axial) vector and tensor nature of the $\mathcal{F}_{i_1 i_2 \dots i_l}$ what elements the integrity basis for the I_{α} will contain²³. The integrity bases related to the specific constitutive fields considered here are given in the appendices cited below.

An integrity basis should, in terms of the numbers of elements it contains, be minimal. It should further be such that no (polynomial) invariant member thereof is expressible in terms of any other(s); in which case it is said to be irreducible²⁴. Although of some importance, the non-trivial matters of proving minimality and demonstrating the irreducibility of the integrity bases given here lie beyond the scope of this study. I view this neglect of an otherwise essential point as being acceptable here on the grounds that rather than work, for example,

with the full non-linear constitutive equation representations, I shall employ in the sequel a reduced form of same for which the absence of such minimality and irreducibility considerations is of no consequence. It may be noted in passing that due regard has, of course, been given in the non-linear representations below to the well-known relation ²⁵

$$\epsilon_{ijk} \epsilon_{pqr} = \begin{vmatrix} \delta_{ip} & \delta_{jp} & \delta_{kp} \\ \delta_{iq} & \delta_{jq} & \delta_{kq} \\ \delta_{ir} & \delta_{jr} & \delta_{kr} \end{vmatrix}. \quad (13)$$

Thus, in closing this section, the general representation of any particular form-invariant (polynomial) constitutive equation here that follows from (12) is seen to have the so-called canonical form ²⁶

$$\mathcal{F}_{i_1 i_2 \dots i_l} (s^{(m)}, v_k^{(n)}) = \sum_{\alpha=1}^m S_{\alpha} \frac{\partial^l I_{\alpha}}{\partial \bar{v}_{i_1}^{(1)} \partial \bar{v}_{i_2}^{(2)} \dots \partial \bar{v}_{i_l}^{(l)}}. \quad (14)$$

The (polynomial) coefficients S_{α} are the material response functions which characterize, i.e. define, a given system. They relate to the various possible effects that may occur within that system under a particular set of circumstances. Rational mechanics can only establish, within a given context, the number of such coefficients; and, the thermodynamic restrictions upon them. Being of a phenomenological nature, it does not fall to rational mechanics to provide physical insight into the atomic or molecular background of these coefficients; this remaining the task of other disciplines.

C.1.a. Non-linear representations

Objective scalars

For scalars, i.e. 0th-order tensors, \mathcal{F} is seen from (10) to be an absolute (polynomial) scalar invariant of the fields $S^{(m)}$ and $V_k^{(n)}$.

The objective scalars, taken as functions of the variables denoted by these fields, can thus in general depend upon the sets of variables ²⁷ $I_\alpha^{(1)}$, $I_\beta^{(2)}$, Δ_k^A and Δ_o^A ; these being given in Appendix C-1.

Objective polar vectors

Let V_i denote the polar vector function the representation of which is desired. The introduction of an arbitrary polar vector, say a_i , makes possible the formation of an absolute scalar invariant $S = a_i V_i$ of the argument fields (2) together with a_i , which is linear in a_i . Here this means that

$$\begin{aligned}
 S_P &= \sum_{\gamma=1}^4 V_P^\gamma(I^*) I_\gamma^{(1)}(a_i) + \sum_{\delta=1}^4 V_P^\delta(I^*) I_\delta^{(2)}(a_i) + \\
 &+ \sum_{\lambda=1}^4 V_P^\lambda(I^*) E_\lambda^A(a_i); \quad (15)
 \end{aligned}$$

where, by virtue of (13), I have defined the two sets of invariants ²⁸

$$I^* = \{I_\alpha^{(1)}, I_\beta^{(2)}, \Delta_k^A, \Delta_o^A\} \quad \text{and} \quad II^* = \{I_\alpha^{(1)}, I_\beta^{(2)}\}. \quad (16)$$

From the expression (15) follows, upon differentiation with respect to the vector a_i , the implicit form of the vector V_i representation; it being

$$\begin{aligned}
 V_{P_i} = & \sum_{\gamma=1}^4 V_{P^\gamma}^{(1)}(I^*) \frac{\partial I_\gamma^{(1)}(a_i)}{\partial a_i} + \sum_{\delta=1}^4 V_{P^\delta}^{(2)}(I^*) \frac{\partial I_\delta^{(2)}(a_i)}{\partial a_i} + \\
 & + \sum_{\lambda=1}^4 V_{P^\lambda}^{(2)}(I^*) \frac{\partial E_\lambda^A(a_i)}{\partial a_i}. \quad (17)
 \end{aligned}$$

The terms $I_\gamma^{(1)}(a_i)$, $I_\delta^{(2)}(a_i)$ and $E_\lambda^A(a_i)$ are given respectively in Appendix C-2 (tables 1-3); and use thereof in (17) yields the result sought.

Thus, the general non-linear representation of an arbitrary polar vector which is taken to be a function of the variables (2) may be written as

$$\begin{aligned}
 V_{P_i} = & \sum_{a=1}^s \left[\frac{V_{Pa^1}^{(1)}(I^*)}{Pa^1} \delta_{ik} + \frac{V_{Pa^1}^{(3)}(I^*)}{Pa^1} \epsilon_{ijk} B_j + \frac{V_{Pa^1}^{(2)}(I^*)}{Pa^1} B_i B_k \right] \rho_{a,k} + \\
 & + \sum_{a=1}^{s-1} \left[\frac{V_{Pa^2}^{(1)}(I^*)}{Pa^2} \delta_{ik} + \frac{V_{Pa^2}^{(3)}(I^*)}{Pa^2} \epsilon_{ijk} B_j + \frac{V_{Pa^2}^{(2)}(I^*)}{Pa^2} B_i B_k \right] v_{a,k} + \\
 & + \left[\frac{V_{P^3}^{(1)}(I^*)}{P^3} \delta_{ik} + \frac{V_{P^3}^{(3)}(I^*)}{P^3} \epsilon_{ijk} B_j + \frac{V_{P^3}^{(2)}(I^*)}{P^3} B_i B_k \right] T_{,k} + \\
 & + \left[\frac{V_{P^4}^{(1)}(I^*)}{P^4} \delta_{ik} + \frac{V_{P^4}^{(3)}(I^*)}{P^4} \epsilon_{ijk} B_j + \frac{V_{P^4}^{(2)}(I^*)}{P^4} B_i B_k \right] \delta_{s,k}. \quad (18)
 \end{aligned}$$

From this expression it thus follows that the specific non-linear representations for the polar vectors $\bar{m}_i - \bar{c}v_i$, q_i^1 and P_i are given by the relationships below: viz.,

$$\begin{aligned}
\bar{m}_i - \bar{c}v_i &= \\
&= \sum_{b=1}^s \left[m_{ab}^{(1)}(I^*)\delta_{ik} + m_{ab}^{(3)}(II^*)\epsilon_{ijk}B_j + m_{ab}^{(2)}(I^*)B_iB_k \right] \rho_{b,k} + \\
&+ \sum_{b=1}^{s-1} \left[m_{ab}^{(1)}(I^*)\delta_{ik} + m_{ab}^{(3)}(II^*)\epsilon_{ijk}B_j + m_{ab}^{(2)}(I^*)B_iB_k \right] v_{b,k} + \\
&+ \left[m_a^{(1)}(I^*)\delta_{ik} + m_a^{(3)}(II^*)\epsilon_{ijk}B_j + m_a^{(2)}(I^*)B_iB_k \right] T_{,k} + \\
&+ \left[m_a^{(1)}(I^*)\delta_{ik} + m_a^{(3)}(II^*)\epsilon_{ijk}B_j + m_a^{(2)}(I^*)B_iB_k \right] \delta_{s,k} \quad (19)^{29}
\end{aligned}$$

for all a ;

$$\begin{aligned}
q_i^1 &= \sum_{a=1}^s \left[q_a^{(1)}(I^*)\delta_{ik} + q_a^{(3)}(II^*)\epsilon_{ijk}B_j + q_a^{(2)}(I^*)B_iB_k \right] \rho_{a,k} + \\
&+ \sum_{a=1}^{s-1} \left[q_a^{(1)}(I^*)\delta_{ik} + q_a^{(3)}(II^*)\epsilon_{ijk}B_j + q_a^{(2)}(I^*)B_iB_k \right] v_{a,k} + \\
&+ \left[q_3^{(1)}(I^*)\delta_{ik} + q_3^{(3)}(II^*)\epsilon_{ijk}B_j + q_3^{(2)}(I^*)B_iB_k \right] T_{,k} + \\
&+ \left[q_4^{(1)}(I^*)\delta_{ik} + q_4^{(3)}(II^*)\epsilon_{ijk}B_j + q_4^{(2)}(I^*)B_iB_k \right] \delta_{s,k}; \quad (20)^{30}
\end{aligned}$$

and,

$$p_a = \sum_{b=1}^s \left[p_{ab}^{(1)}(I^*)\delta_{ik} + p_{ab}^{(3)}(II^*)\epsilon_{ijk}B_j + p_{ab}^{(2)}(I^*)B_iB_k \right] \rho_{b,k} +$$

$$\begin{aligned}
& + \sum_{b=1}^{s-1} \left[P_{ab^2}^{(1)}(1^*) \delta_{ik} + P_{ab^2}^{(3)}(11^*) \epsilon_{ijk} B_j + P_{ab^2}^{(2)}(1^*) B_i B_k \right] V_b^k + \\
& + \left[P_a^{(1)}(1^*) \delta_{ik} + P_a^{(3)}(11^*) \epsilon_{ijk} B_j + P_a^{(2)}(1^*) B_i B_k \right] T_{,k} + \\
& + \left[P_a^{(1)}(1^*) \delta_{ik} + P_a^{(3)}(11^*) \epsilon_{ijk} B_j + P_a^{(2)}(1^*) B_i B_k \right] \delta_s^k \quad (21)^{31}
\end{aligned}$$

for all a .

Objective axial vectors

In a manner like that used to determine the representation of V_i , I now proceed to find for an axial vector function V_i its necessary representation. Here c_i serves as an arbitrary axial vector by means of which I set down the absolute scalar invariant $S_A = c_i V_i$ of the fields (2) that includes c_i , and is linear in c_i . It is here given by

$$\begin{aligned}
S_A & = \sum_{\zeta=1}^{17} V_{A^\zeta}^{(1)}(1^*) 11_\zeta^{(4)}(c_i) + \sum_{\nu=1}^8 V_{A^\nu}^{(2)}(11^*) E_\nu^A(c_i) + \\
& + \sum_{\pi=1}^4 \sum_{\psi=1}^4 V_{A^{\pi\psi}}^{(3)}(11^*) 11^{(9)}(c_i) + \sum_{\tau=1}^{12} \sum_{\sigma=1}^4 V_{A^{\tau/\sigma}}^{(4)}(11^*) 11_{\tau/\sigma}^{(5)}(c_i). \quad (22)
\end{aligned}$$

Differentiation of this relation with respect to c_i provides the following implicit representation here of the vector V_i :

$$\begin{aligned}
V_i & = \sum_{\zeta=1}^{17} V_{A^\zeta}^{(1)}(1^*) \frac{\partial 11_\zeta^{(4)}(c_i)}{\partial c_i} + \sum_{\nu=1}^8 V_{A^\nu}^{(2)}(11^*) \frac{\partial E_\nu^A(c_i)}{\partial c_i} + \\
& + \sum_{\pi=1}^4 \sum_{\psi=1}^4 V_{A^{\pi\psi}}^{(3)}(11^*) \frac{\partial 11^{(9)}(c_i)}{\partial c_i} + \sum_{\tau=1}^{12} \sum_{\sigma=1}^4 V_{A^{\tau/\sigma}}^{(4)}(11^*) \frac{\partial 11_{\tau/\sigma}^{(5)}(c_i)}{\partial c_i}. \quad (23)
\end{aligned}$$

Appendix C-3 (Tables 4-7) in turn provide the terms to be differentiated in (23) and the result of carrying out the indicated operation gives the vector V_i its explicit form here.

Consequent to (23) the non-linear representation for the axial objective vector M_i is thus of the form ³²

$$M_i = M_a^{(1)} (I^*) B_i + \sum_{b=1}^s M_{ab}^0 i_k \rho_{b,k} + \sum_{b=1}^{s-1} M_{ab}^V i_k V_{b,k} + M_a^T T_{,k} + M_{ik}^G G_{s,k} \quad (24)$$

where the last four coefficients in (24) are given, for all a , in Appendix C-4.

Objective tensors

The last general representation with which I am concerned here is that for a 2nd-order absolute tensor function T_{ij} . Following the same procedure ³³ employed above, I now take two arbitrary polar vectors a_i and b_j and construct an absolute scalar invariant $S_T = a_i T_{ij} b_j$ of the fields (2) together with a_i and b_i , which is bilinear in these two vectors. The scalar S_T may thus be written here in the form

$$S_T = t^{(1)} (I^*) \vec{a} \cdot \vec{b} + t^{(2)} (I^*) (\vec{a} \cdot \vec{B}) (\vec{b} \cdot \vec{B}) + t^{(3)} (II^*) \epsilon_{ijk} a_i b_j B_k + \sum_{\lambda=1}^8 t_{\lambda}^{(4)} (II^*) E_{\lambda}^P(a_i) (\vec{b} \cdot \vec{B}) + \sum_{\varphi=1}^8 t_{\varphi}^{(5)} (II^*) (\vec{a} \cdot \vec{B}) E_{\varphi}^P(b_i) +$$

$$\begin{aligned}
& + \sum_{\gamma=1}^4 \sum_{\epsilon=1}^4 t_{\gamma\epsilon}^{(6)} (I^*) ||_{\gamma\epsilon}^{(6)} (a_i, b_i) + \\
& + \sum_{\gamma=1}^4 \sum_{\mu=1}^4 t_{\gamma\mu}^{(7)} (II^*) ||_{\gamma\mu}^{(10)} (a_i, b_i) + \sum_{\lambda=1}^4 \sum_{\epsilon=1}^4 t_{\lambda\epsilon}^{(8)} (III^*) ||_{\lambda\epsilon}^{(11)} (a_i, b_i) + \\
& + \sum_{\delta=1}^4 \sum_{\epsilon=1}^4 t_{\delta\epsilon}^{(9)} (I^*) ||_{\delta\epsilon}^{(13)} (a_i, b_i) + \sum_{\gamma=1}^4 \sum_{\omega=1}^4 t_{\gamma\omega}^{(10)} (I^*) ||_{\gamma\omega}^{(14)} (a_i, b_i) + \\
& + \sum_{\delta=1}^4 \sum_{\mu=1}^4 t_{\delta\mu}^{(11)} (II^*) ||_{\delta\mu}^{(15)} (a_i, b_i) + \sum_{\omega=1}^4 \sum_{\lambda=1}^4 t_{\omega\lambda}^{(12)} (III^*) ||_{\omega\lambda}^{(16)} (a_i, b_i) .
\end{aligned}$$

A differentiation of this expression with respect to the vectors a_i and b_j brings forth the implicit representation of the required tensor; said relationship being here of the form

$$\begin{aligned}
T_{ij} = & \\
= & t^{(1)} (I^*) \delta_{ij} + t^{(2)} (I^*) B_i B_j + t^{(3)} (III^*) \epsilon_{ijkl} B_k B_l + \\
& + \sum_{\chi=1}^8 t_{\chi}^{(4)} (II^*) \frac{\partial E_{\chi}^P(a_i)}{\partial a_i} B_j + \sum_{\varphi=1}^8 t_{\varphi}^{(5)} (III^*) B_i \frac{\partial E_{\varphi}^P(b_j)}{\partial b_j} + \\
& + \sum_{\gamma=1}^4 \sum_{\epsilon=1}^4 t_{\gamma\epsilon}^{(6)} (I^*) \frac{\partial^2 ||_{\gamma\epsilon}^{(6)} (a_i, b_i)}{\partial a_i \partial b_j} + \\
& + \sum_{\gamma=1}^4 \sum_{\mu=1}^4 t_{\gamma\mu}^{(7)} (II^*) \frac{\partial^2 ||_{\gamma\mu}^{(10)} (a_i, b_i)}{\partial a_i \partial b_j} + \\
& + \sum_{\lambda=1}^4 \sum_{\epsilon=1}^4 t_{\lambda\epsilon}^{(8)} (III^*) \frac{\partial^2 ||_{\lambda\epsilon}^{(11)} (a_i, b_i)}{\partial a_i \partial b_j} +
\end{aligned}$$

$$\begin{aligned}
& + \sum_{\delta=1}^4 \sum_{\epsilon=1}^4 t_{\delta\epsilon}^{(9)}(I^*) \frac{\partial^2_{II}{}_{\delta\epsilon}{}^{(13)}(a_i, b_i)}{\partial a_i \partial b_j} + \\
& \qquad \qquad \qquad + \sum_{\gamma=1}^4 \sum_{\omega=1}^4 t_{\gamma\omega}^{(10)}(I^*) \frac{\partial^2_{II}{}_{\gamma\omega}{}^{(14)}(a_i, b_i)}{\partial a_i \partial b_j} + \\
& + \sum_{\delta=1}^4 \sum_{\mu=1}^4 t_{\delta\mu}^{(11)}(I^*) \frac{\partial^2_{II}{}_{\delta\mu}{}^{(15)}(a_i, b_i)}{\partial a_i \partial b_j} + \\
& \qquad \qquad \qquad + \sum_{\omega=1}^4 \sum_{\lambda=1}^4 t_{\omega\lambda}^{(12)}(I^*) \frac{\partial^2_{II}{}_{\omega\lambda}{}^{(16)}(a_i, b_i)}{\partial a_i \partial b_j}. \quad (25)
\end{aligned}$$

Upon introduction of the appropriately differentiated terms here, gotten with the help of Appendix C-5 (Tables 8-16), it is possible to rewrite (25) more explicitly.

On the basis of the foregoing considerations, the general non-linear representations here for the objective (2^{nd} -order) tensors t_{ij} and C_{ij} can, for all a , now be given. Due to their complete similarity of form, I set down only the representation of t_{ij} ; it being ³⁴

$t_{ij} =$

$$\begin{aligned}
 & - t_a^{(1)} (1^*) \delta_{ij} + t_a^{(2)} (1^*) \delta_i \delta_j + t_a^{(3)} (1^*) \epsilon_{ijk} \delta_k + \\
 & + \sum_{b=1}^s \sum_{c=1}^{s-1} \tilde{t}_{abc}^{(4)} (1^*) \delta_{(i} (\vec{v}_\rho \times \vec{v})_j) + \sum_{b=1}^s \tilde{t}_{ab}^{(4)} (1^*) \delta_{(i} (\vec{v}_\rho \times \vec{T})_j) + \sum_{b=1}^s \tilde{t}_{ab}^{(4)} (1^*) \delta_{(i} (\vec{v}_\rho \times \vec{E})_j) + \\
 & + \sum_{b=1}^{s-1} \tilde{t}_{ab}^{(4)} (1^*) \delta_{(i} (\vec{V} \times \vec{T})_j) + \sum_{b=1}^{s-1} \tilde{t}_{ab}^{(4)} (1^*) \delta_{(i} (\vec{V} \times \vec{E})_j) + \tilde{t}_{ab}^{(4)} (1^*) \delta_{(i} (\vec{T} \times \vec{E})_j) + \\
 & + \sum_{b=1}^s \sum_{c=1}^s \tilde{t}_{abc}^{(4)} (1^*) \delta_{(i} (\vec{v}_\rho \times \vec{v}_\rho)_j) + \sum_{b=1}^{s-1} \sum_{c=1}^{s-1} \tilde{t}_{abc}^{(4)} (1^*) \delta_{(i} (\vec{V} \times \vec{V})_j) + \\
 & + \sum_{b=1}^s \sum_{c=1}^s \tilde{t}_{abc}^{(6)} (1^*) \rho_{b_i} \rho_{c_j} + \sum_{b=1}^{s-1} \sum_{c=1}^{s-1} \tilde{t}_{abc}^{(6)} (1^*) v_{b_i} v_{c_j} + t_{a33}^{(6)} (1^*) T_i T_j + t_{a44}^{(6)} (1^*) \epsilon_{ij} \epsilon_j + \\
 & + \sum_{b=1}^{s-1} \sum_{c=1}^s \tilde{t}_{abc}^{(6)} (1^*) v_{(i} \rho_{c_j)} + \sum_{b=1}^s \tilde{t}_{ab}^{(6)} (1^*) T_{(i} \rho_{b_j)} + \sum_{b=1}^s \tilde{t}_{ab}^{(6)} (1^*) \epsilon_{(i} \rho_{b_j)} + \\
 & + \sum_{b=1}^{s-1} \tilde{t}_{ab}^{(6)} (1^*) v_{(i} T_{j)} + \sum_{b=1}^{s-1} \tilde{t}_{ab}^{(6)} (1^*) v_{(i} \epsilon_{j)} + \tilde{t}_{a34}^{(6)} (1^*) T_{(i} \epsilon_{j)} + \\
 & + \sum_{b=1}^s \sum_{c=1}^s \tilde{t}_{abc}^{(8)} (1^*) \rho_{b_i} (\vec{v}_\rho \times \vec{E})_j) + \sum_{b=1}^{s-1} \sum_{c=1}^s \tilde{t}_{abc}^{(8)} (1^*) v_{b_i} (\vec{v}_\rho \times \vec{E})_j) + \sum_{b=1}^s \tilde{t}_{ab}^{(8)} (1^*) T_{(i} (\vec{v}_\rho \times \vec{E})_j) + \\
 & + \sum_{b=1}^s \tilde{t}_{ab}^{(8)} (1^*) \epsilon_{(i} (\vec{v}_\rho \times \vec{E})_j) + \sum_{b=1}^s \sum_{c=1}^{s-1} \tilde{t}_{abc}^{(8)} (1^*) \rho_{b_i} (\vec{V} \times \vec{E})_j) + \sum_{b=1}^{s-1} \sum_{c=1}^{s-1} \tilde{t}_{abc}^{(8)} (1^*) v_{(i} (\vec{V} \times \vec{E})_j) + \\
 & + \sum_{b=1}^{s-1} \tilde{t}_{ab}^{(8)} (1^*) T_{(i} (\vec{V} \times \vec{E})_j) + \sum_{b=1}^{s-1} \tilde{t}_{ab}^{(8)} (1^*) \epsilon_{(i} (\vec{V} \times \vec{E})_j) + \sum_{b=1}^s \tilde{t}_{ab}^{(8)} (1^*) \rho_{b_i} (\vec{T} \times \vec{E})_j) + \\
 & + \sum_{b=1}^{s-1} \tilde{t}_{ab}^{(8)} (1^*) v_{(i} (\vec{T} \times \vec{E})_j) + \tilde{t}_{a33}^{(8)} (1^*) T_{(i} (\vec{T} \times \vec{E})_j) + \tilde{t}_{a34}^{(8)} (1^*) \epsilon_{(i} (\vec{T} \times \vec{E})_j) + \\
 & + \sum_{b=1}^s \tilde{t}_{ab}^{(8)} (1^*) \rho_{b_i} (\vec{E} \times \vec{E})_j) + \sum_{b=1}^{s-1} \tilde{t}_{ab}^{(8)} (1^*) v_{(i} (\vec{E} \times \vec{E})_j) + \tilde{t}_{a43}^{(8)} (1^*) T_{(i} (\vec{E} \times \vec{E})_j) + \tilde{t}_{a44}^{(8)} (1^*) \epsilon_{(i} (\vec{E} \times \vec{E})_j) + \\
 & + \sum_{b=1}^s \left[\sum_{c=1}^s \tilde{t}_{abc}^{(10)} (1^*) (\vec{v}_\rho \cdot \vec{E}) + \sum_{c=1}^{s-1} \tilde{t}_{abc}^{(10)} (1^*) (\vec{V} \cdot \vec{E}) + \tilde{t}_{ab}^{(10)} (1^*) (\vec{T} \cdot \vec{E}) + \tilde{t}_{ab}^{(10)} (1^*) (\vec{E} \cdot \vec{E}) \right] \rho_{b_i} (i^B)_j) + \\
 & + \sum_{b=1}^{s-1} \left[\sum_{c=1}^s \tilde{t}_{abc}^{(10)} (1^*) (\vec{v}_\rho \cdot \vec{E}) + \sum_{c=1}^{s-1} \tilde{t}_{abc}^{(10)} (1^*) (\vec{V} \cdot \vec{E}) + \tilde{t}_{ab}^{(10)} (1^*) (\vec{T} \cdot \vec{E}) + \tilde{t}_{ab}^{(10)} (1^*) (\vec{E} \cdot \vec{E}) \right] v_{b_i} (i^B)_j) +
 \end{aligned}$$

$$\begin{aligned}
& + \left[\sum_{c=1}^s \tilde{\tau}_{ab\ 31}^{(10)}(1^*) (\vec{v}_\rho \cdot \vec{b}) + \sum_{b=1}^{s-1} \tilde{\tau}_{ab\ 32}^{(10)}(1^*) (\vec{v} \cdot \vec{b}) + \tilde{\tau}_{a\ 33}^{(10)}(1^*) (\vec{v} \cdot \vec{b}) + \tilde{\tau}_{a\ 34}^{(10)}(1^*) (\vec{e} \cdot \vec{b}) \right] T_{s,1}(\vec{b}_j) + \\
& + \left[\sum_{b=1}^s \tilde{\tau}_{ab\ 41}^{(10)}(1^*) (\vec{v}_\rho \cdot \vec{b}) + \sum_{b=1}^{s-1} \tilde{\tau}_{ab\ 42}^{(10)}(1^*) (\vec{v} \cdot \vec{b}) + \tilde{\tau}_{a\ 43}^{(10)}(1^*) (\vec{v} \cdot \vec{b}) + \tilde{\tau}_{a\ 44}^{(10)}(1^*) (\vec{e} \cdot \vec{b}) \right] \mathcal{E}_s(\vec{b}_j) + \\
& + \sum_{b=1}^s \left[\sum_{c=1}^s \tilde{\tau}_{abc\ 11}^{(12)}(11^*) (\vec{v}_\rho \cdot \vec{b}) + \sum_{c=1}^{s-1} \tilde{\tau}_{abc\ 21}^{(12)}(11^*) (\vec{v} \cdot \vec{b}) + \tilde{\tau}_{ab\ 31}^{(12)}(11^*) (\vec{v} \cdot \vec{b}) + \tilde{\tau}_{ab\ 41}^{(12)}(11^*) (\vec{e} \cdot \vec{b}) \right] B_{s,1}(\vec{v}_\rho \times \vec{b})_j + \\
& + \sum_{b=1}^{s-1} \left[\sum_{c=1}^s \tilde{\tau}_{abc\ 12}^{(12)}(11^*) (\vec{v}_\rho \cdot \vec{b}) + \sum_{c=1}^{s-1} \tilde{\tau}_{abc\ 22}^{(12)}(11^*) (\vec{v} \cdot \vec{b}) + \tilde{\tau}_{ab\ 32}^{(12)}(11^*) (\vec{v} \cdot \vec{b}) + \tilde{\tau}_{ab\ 42}^{(12)}(11^*) (\vec{e} \cdot \vec{b}) \right] B_{s,1}(\vec{v} \times \vec{b})_j + \\
& + \left[\sum_{b=1}^s \tilde{\tau}_{ab\ 13}^{(12)}(11^*) (\vec{v}_\rho \cdot \vec{b}) + \sum_{b=1}^{s-1} \tilde{\tau}_{ab\ 23}^{(12)}(11^*) (\vec{v} \cdot \vec{b}) + \tilde{\tau}_{a\ 33}^{(12)}(11^*) (\vec{v} \cdot \vec{b}) + \tilde{\tau}_{a\ 43}^{(12)}(11^*) (\vec{e} \cdot \vec{b}) \right] B_{s,1}(\vec{v} \cdot \vec{b})_j + \\
& + \left[\sum_{b=1}^s \tilde{\tau}_{ab\ 14}^{(12)}(11^*) (\vec{v}_\rho \cdot \vec{b}) + \sum_{b=1}^{s-1} \tilde{\tau}_{ab\ 24}^{(12)}(11^*) (\vec{v} \cdot \vec{b}) + \tilde{\tau}_{a\ 34}^{(12)}(11^*) (\vec{v} \cdot \vec{b}) + \tilde{\tau}_{a\ 44}^{(12)}(11^*) (\vec{e} \cdot \vec{b}) \right] B_{s,1}(\vec{e} \cdot \vec{b})_j + \\
& + \sum_{b=1}^s \sum_{c=1}^{s-1} \tilde{\tau}_{abc\ 1}^{(5)}(11^*) B_{[1, \vec{v}_\rho \times \vec{v}]_j} + \sum_{b=1}^s \tilde{\tau}_{ab\ 2}^{(5)}(11^*) B_{[1, \vec{v}_\rho \times \vec{v} \cdot \vec{v}]_j} + \sum_{b=1}^s \tilde{\tau}_{ab\ 3}^{(5)}(11^*) B_{[1, \vec{v}_\rho \times \vec{v} \cdot \vec{v}]_j} + \\
& + \sum_{b=1}^{s-1} \tilde{\tau}_{ab\ 4}^{(5)}(11^*) B_{[1, \vec{v} \times \vec{v} \cdot \vec{v}]_j} + \sum_{b=1}^{s-1} \tilde{\tau}_{ab\ 5}^{(5)}(11^*) B_{[1, \vec{v} \times \vec{v} \cdot \vec{v}]_j} + \tilde{\tau}_a^{(5)}(11^*) B_{[1, \vec{v} \cdot \vec{v} \cdot \vec{v}]_j} + \\
& + \sum_{b=1}^s \sum_{c=1}^s \tilde{\tau}_{abc\ 7}^{(5)}(11^*) B_{[1, \vec{v}_\rho \times \vec{v}_\rho]_j} + \sum_{b=1}^{s-1} \sum_{c=1}^{s-1} \tilde{\tau}_{abc\ 8}^{(5)}(11^*) B_{[1, \vec{v} \times \vec{v}]_j} + \\
& + \sum_{b=1}^{s-1} \sum_{c=1}^s \tilde{\tau}_{abc\ 12}^{(6)}(1^*) V_{[1, \rho, j]} + \sum_{b=1}^s \sum_{c=1}^s \tilde{\tau}_{abc\ 13}^{(6)}(1^*) T_{s,1}[\rho, j] + \sum_{b=1}^s \sum_{c=1}^s \tilde{\tau}_{abc\ 14}^{(6)}(1^*) \mathcal{E}_s[\rho, j] + \\
& + \sum_{b=1}^{s-1} \tilde{\tau}_{ab\ 23}^{(6)}(1^*) V_{[1, T, j]} + \sum_{b=1}^{s-1} \tilde{\tau}_{ab\ 24}^{(6)}(1^*) V_{[1, \mathcal{E}_s, j]} + \tilde{\tau}_{ab\ 34}^{(6)}(1^*) T_{s,1}[\mathcal{E}_s, j] + \\
& + \sum_{b=1}^s \sum_{c=1}^s \tilde{\tau}_{abc\ 11}^{(7)}(11^*) \rho_{b, [1, \vec{v}_\rho \times \vec{b}]_j} + \sum_{b=1}^{s-1} \sum_{c=1}^s \tilde{\tau}_{abc\ 12}^{(7)}(11^*) V_{[1, \rho, j]} + \sum_{b=1}^s \sum_{c=1}^s \tilde{\tau}_{abc\ 13}^{(7)}(11^*) T_{s,1}[\rho, j] + \\
& + \sum_{b=1}^s \tilde{\tau}_{ab\ 14}^{(7)}(11^*) \mathcal{E}_s[\rho, j] + \sum_{b=1}^s \sum_{c=1}^{s-1} \tilde{\tau}_{abc\ 21}^{(7)}(11^*) \rho_{b, [1, \vec{v} \times \vec{b}]_j} + \sum_{b=1}^{s-1} \sum_{c=1}^{s-1} \tilde{\tau}_{abc\ 22}^{(7)}(11^*) V_{[1, \rho, j]} + \\
& + \sum_{b=1}^{s-1} \tilde{\tau}_{ab\ 23}^{(7)}(11^*) T_{s,1}[\rho, j] + \sum_{b=1}^{s-1} \tilde{\tau}_{ab\ 24}^{(7)}(11^*) \mathcal{E}_s[\rho, j] + \sum_{b=1}^s \tilde{\tau}_{ab\ 31}^{(7)}(11^*) \rho_{b, [1, \vec{v} \cdot \vec{b}]_j} +
\end{aligned}$$

$$\begin{aligned}
& + \sum_{b=1}^{s-1} \left[\tilde{t}_{ab}^{(7)}(11^*) \tilde{v}_{[i}(\vec{v}_T \times \vec{B})_{j]} + \tilde{t}_{a33}^{(7)}(11^*) \tilde{T}_{[i}(\vec{v}_T \times \vec{B})_{j]} + \tilde{t}_{a34}^{(7)}(11^*) \tilde{E}_{s[i}(\vec{v}_T \times \vec{B})_{j]} + \right. \\
& + \sum_{b=1}^s \left[\tilde{t}_{ab}^{(7)}(11^*) \tilde{p}_{[i}(\vec{E} \times \vec{B})_{j]} + \sum_{b=1}^{s-1} \left[\tilde{t}_{ab}^{(7)}(11^*) \tilde{v}_{[i}(\vec{E} \times \vec{B})_{j]} + \tilde{t}_{ab}^{(7)}(11^*) \tilde{T}_{[i}(\vec{E} \times \vec{B})_{j]} + \tilde{t}_{ab}^{(7)}(11^*) \tilde{E}_{s[i}(\vec{E} \times \vec{B})_{j]} + \right. \\
& + \sum_{b=1}^s \left[\sum_{c=1}^s \left[\tilde{t}_{abc}^{(9)}(11^*) (\vec{v}_\rho \cdot \vec{B})_c + \sum_{c=1}^{s-1} \left[\tilde{t}_{abc}^{(9)}(11^*) (\vec{v} \cdot \vec{B})_c + \tilde{t}_{ab}^{(9)}(11^*) (\vec{v}_T \cdot \vec{B}) + \tilde{t}_{ab}^{(9)}(11^*) (\vec{E} \cdot \vec{B})_s \right] \tilde{E}_{[i} B_{j]} + \right. \\
& + \sum_{b=1}^{s-1} \left[\sum_{c=1}^s \left[\tilde{t}_{abc}^{(9)}(11^*) (\vec{v}_\rho \cdot \vec{B})_c + \sum_{c=1}^{s-1} \left[\tilde{t}_{abc}^{(9)}(11^*) (\vec{v} \cdot \vec{B})_c + \tilde{t}_{ab}^{(9)}(11^*) (\vec{v}_T \cdot \vec{B}) + \tilde{t}_{ab}^{(9)}(11^*) (\vec{E} \cdot \vec{B})_s \right] \tilde{v}_{[i} B_{j]} + \right. \\
& + \left[\sum_{b=1}^s \left[\tilde{t}_{ab}^{(9)}(11^*) (\vec{v}_\rho \cdot \vec{B})_b + \sum_{b=1}^{s-1} \left[\tilde{t}_{ab}^{(9)}(11^*) (\vec{v} \cdot \vec{B})_b + \tilde{t}_{a33}^{(9)}(11^*) (\vec{v}_T \cdot \vec{B}) + \tilde{t}_{a34}^{(9)}(11^*) (\vec{E} \cdot \vec{B})_s \right] \tilde{T}_{[i} B_{j]} + \right. \\
& + \left[\sum_{b=1}^s \left[\tilde{t}_{ab}^{(9)}(11^*) (\vec{v}_\rho \cdot \vec{B})_b + \sum_{b=1}^{s-1} \left[\tilde{t}_{ab}^{(9)}(11^*) (\vec{v} \cdot \vec{B})_b + \tilde{t}_{a33}^{(9)}(11^*) (\vec{v}_T \cdot \vec{B}) + \tilde{t}_{a34}^{(9)}(11^*) (\vec{E} \cdot \vec{B})_s \right] \tilde{E}_{s[i} B_{j]} + \right. \\
& + \sum_{b=1}^s \left[\sum_{c=1}^s \left[\tilde{t}_{abc}^{(11)}(11^*) (\vec{v}_\rho \cdot \vec{B})_c + \sum_{c=1}^{s-1} \left[\tilde{t}_{abc}^{(11)}(11^*) (\vec{v} \cdot \vec{B})_c + \tilde{t}_{ab}^{(11)}(11^*) (\vec{v}_T \cdot \vec{B}) + \tilde{t}_{ab}^{(11)}(11^*) (\vec{E} \cdot \vec{B})_s \right] \tilde{p}_{[i}(\vec{v}_\rho \times \vec{B})_{j]} + \right. \\
& + \sum_{b=1}^{s-1} \left[\sum_{c=1}^s \left[\tilde{t}_{abc}^{(11)}(11^*) (\vec{v}_\rho \cdot \vec{B})_c + \sum_{c=1}^{s-1} \left[\tilde{t}_{abc}^{(11)}(11^*) (\vec{v} \cdot \vec{B})_c + \tilde{t}_{ab}^{(11)}(11^*) (\vec{v}_T \cdot \vec{B}) + \tilde{t}_{ab}^{(11)}(11^*) (\vec{E} \cdot \vec{B})_s \right] \tilde{p}_{[i}(\vec{v} \times \vec{B})_{j]} + \right. \\
& + \left[\sum_{b=1}^s \left[\tilde{t}_{ab}^{(11)}(11^*) (\vec{v}_\rho \cdot \vec{B})_b + \sum_{b=1}^{s-1} \left[\tilde{t}_{ab}^{(12)}(11^*) (\vec{v} \cdot \vec{B})_b + \tilde{t}_{a33}^{(11)}(11^*) (\vec{v}_T \cdot \vec{B}) + \tilde{t}_{a33}^{(11)}(11^*) (\vec{E} \cdot \vec{B})_s \right] \tilde{B}_{[i}(\vec{v}_T \times \vec{B})_{j]} + \right. \\
& + \left[\sum_{b=1}^s \left[\tilde{t}_{ab}^{(11)}(11^*) (\vec{v}_\rho \cdot \vec{B})_b + \sum_{b=1}^{s-1} \left[\tilde{t}_{ab}^{(11)}(11^*) (\vec{v} \cdot \vec{B})_b + \tilde{t}_{a34}^{(11)}(11^*) (\vec{v}_T \cdot \vec{B}) + \tilde{t}_{a34}^{(11)}(11^*) (\vec{E} \cdot \vec{B})_s \right] \tilde{B}_{[i}(\vec{E} \times \vec{B})_{j]} \right. \quad (26)
\end{aligned}$$

The brackets (parentheses) placed about indices mean that it is the anti-symmetric (symmetric) component of that term, with regard to the enclosed indices, which is being considered.

C.1.b. Linearized representations

The results of the foregoing section, although interesting, do not lend themselves easily for purposes of a pragmatic nature. It is thus worthwhile to consider them initially in some restricted sense. Since a linearization process is a means by which the desired simplification can be effected, and since the results of such a process are assumed here to be meaningful for physical systems "close to equilibrium" (cf. Chpt. E), I now limit the applicability of the non-linear constitutive equations above to those (classes of) systems for which a linearization in the variables $\rho_{a,i}$, $V_{b,i}$, T_i and δ_s is sensible³⁵.

Objective scalars

The linearized objective scalars $\Lambda_\alpha^{L_1}$, ϵ^{L_1} and η^{L_1} are here (polynomial) functions of reduced set of variables I , where³⁶

$$I = \{\rho_a \ (a = 1, 2, \dots, s), T, B^2\}. \quad (27)$$

Objective vectors

The linearized representations for the objective polar vectors deriving from (19), (20) and (21) are, with all the material coefficients thereof now functions of I , given here respectively by the following relationships.

$$\frac{L}{\bar{m}_a} - \frac{L}{\bar{c}_{aa} v_a} = \sum_{b=1}^s (m_{ab}^{L_1} \delta_{ik} + m_{ab}^{L_2} \epsilon_{ijk} B_j + m_{ab}^{L_3} B_i B_k) \rho_{b,k} +$$

$$\begin{aligned}
& + \sum_{b=1}^{s-1} (m_{ab}^{L_4} \delta_{ik} + m_{ab}^{L_5} \epsilon_{ijk} B_j + m_{ab}^{L_6} B_i B_k) v_k + \\
& + (m_a^{L_7} \delta_{ik} + m_a^{L_8} \epsilon_{ijk} B_j + m_a^{L_9} B_i B_k) T_{,k} + \\
& + (m_a^{L_{10}} \delta_{ik} + m_a^{L_{11}} \epsilon_{ijk} B_j + m_a^{L_{12}} B_i B_k) \delta_k
\end{aligned} \tag{28}^{37}$$

for all a . Due to (B90) only $s-1$ of these terms are independent, which in turn means that the material coefficients in (28) are not all independent. The restrictions thereon are given by the following conditions:

$$\sum_{a=1}^s m_{ab}^{L_1} = 0 \quad \sum_{a=1}^s m_{ab}^{L_2} = 0 \quad \sum_{a=1}^s m_{ab}^{L_3} = 0 \tag{29}_1$$

with $b = 1, 2, \dots, s$;

$$\sum_{b=1}^s m_{ba}^{L_4} = -\frac{L}{a} \quad \sum_{b=1}^s m_{ba}^{L_5} = 0 \quad \sum_{b=1}^s m_{ba}^{L_6} = 0$$

with $a = 1, 2, \dots, s-1$;

$$\sum_{a=1}^s m_a^{L_7} = 0 \quad \sum_{a=1}^s m_a^{L_8} = 0 \quad \sum_{a=1}^s m_a^{L_9} = 0 \tag{29}_3$$

and

$$\sum_{a=1}^s m_a^{L_{10}} = 0 \quad \sum_{a=1}^s m_a^{L_{11}} = 0 \quad \sum_{a=1}^s m_a^{L_{12}} = 0. \tag{29}_4$$

$$\begin{aligned}
L_i q_i &= \sum_{a=1}^s (q_a^1 \delta_{ik} + q_a^2 \epsilon_{ijk} B_j + q_a^3 B_i B_k) \rho_{a,k} + \\
&+ \sum_{a=1}^{s-1} (q_a^4 \delta_{ik} + q_a^5 \epsilon_{ijk} B_j + q_a^6 B_i B_k) V_{a,k} + \\
&+ (q^7 \delta_{ik} + q^8 \epsilon_{ijk} B_j + q^9 B_i B_k) T_{,k} + \\
&+ (q^{10} \delta_{ik} + q^{11} \epsilon_{ijk} B_j + q^{12} B_i B_k) \delta_{s,k}. \tag{30}^{38}
\end{aligned}$$

It is worthwhile to pause here in order that the versatility of a rational mechanical constitutive theory may be illustrated. If (30), for example, is rewritten in the form

$$L_i q_i = \sum_{a=1}^s \kappa_{ik}^{\rho} \rho_{a,k} - \sum_{a=1}^{s-1} \kappa_{ik}^V V_{a,k} - \kappa_{ik}^T T_{,k} - \kappa_{ik}^{\delta} \delta_{s,k},$$

then the material coefficients κ_{ik}^V and κ_{ik}^T ($= q^7 \delta_{ik} + q^8 \epsilon_{ijk} B_j + q^9 B_i B_k$) are respectively the tensor Dufour³⁹ and "heat" conduction⁴⁰ coefficients for the mixture. The $q^7 \delta_{ik}$ term of κ_{ik}^T may be recognized as the thermal conductivity coefficient of the familiar (Biot-)Fourier law⁴¹; while the coefficient $q^8 \epsilon_{ijk} B_j$ relates to the Riggi-Leduc effect⁴². The $q^{11} \epsilon_{ijk} B_j$ term of κ_{ik}^{δ} relates in turn to the Ettinghausen effect⁴³.

The significant point here is that the presence of specific known effects, in addition to other perhaps unknown ones, in any given rational mechanical constitutive equation is a direct consequence of the selection of independent constitutive variables (cf. (2)). With increasingly

meaningful insight into the phenomenology of a given system, it becomes possible in rational mechanics to propose an also increasingly more satisfactory set of such variables ⁴⁴.

The methodology of this approach to the study of material behavior is thus seen to possess an inherent capacity for self-correction. It is at once both rigorous as well as flexible; providing (at least in principle) a means of successively approximating its theories to the physical experience which they attempt to describe. A continuously improving agreement between observation and explanation is, of course, the end result.

To continue, the linearized representation for the constituent polarization is given by ⁴⁵

$$\begin{aligned}
 \frac{L}{P}_i &= \sum_{b=1}^s (P_{ab}^1 \delta_{ik} + P_{ab}^2 \epsilon_{ijk} B_j + P_{ab}^3 B_i B_k)_{\rho, k} + \\
 &+ \sum_{b=1}^{s-1} (P_{ab}^4 \delta_{ik} + P_{ab}^5 \epsilon_{ijk} B_j + P_{ab}^6 B_i B_k)_{\nu, k} + \\
 &+ (P_{a ik}^7 \delta_{ik} - P_{a ik}^8 \epsilon_{ijk} B_j + P_{a ik}^9 B_i B_k)_{T, k} + \\
 &+ (P_{a ik}^{10} \delta_{ik} + P_{a ik}^{11} \epsilon_{ijk} B_j + P_{a ik}^{12} B_i B_k)_{\xi, k}; \quad (31)
 \end{aligned}$$

where, for example, the material coefficients of ξ_k are related to the electric susceptibility tensor of any given constituent. The appropriate such coefficient for the mixture follows from the use of the definition (B43) which involves this expression ⁴⁶.

The linearized representation which follows from (24) for the axial objective vector M_i is seen here to be of the form⁴⁷

$$\frac{L}{a} M_i = \frac{L}{a} M B_j \quad (32)$$

for all a . $\frac{L}{a} M$ may be thought of as the coefficient of magnetic susceptibility of a constituent; while, similar to the case of polarization, the use of (B59) provides the relation appropriate to the mixture⁴⁸.

Objective tensors

Lastly, the linearized representations for the 2nd-order tensors t_{ij} and c_{ij} derive from (26) and are, respectively,⁴⁹

$$\frac{L}{a} t_{ij} = \frac{L_1}{a} \delta_{ij} + \frac{L_2}{a} B_i B_j + \frac{L_3}{a} \varepsilon_{ijk} B_k$$

and

(33)

$$\frac{L}{a} c_{ij} = \frac{L_1}{a} \delta_{ij} + \frac{L_2}{a} B_i B_j + \frac{L_3}{a} \varepsilon_{ijk} B_k.$$

Once again, all of the material coefficients concerned are functions of the invariants I .

C.2. AppendicesAppendix C-1

$$\begin{aligned}
 I_{\alpha}^{(1)} \equiv & \{ \rho_a (a = 1, 2, \dots, s), T, \rho_{a,i} \rho_{b,j} (a < b), \rho_{a,i} V_{b,i} (b = 1, 2, \dots, s-1), \\
 & \rho_{a,i} T_{,i}, \rho_{a,i} \xi_{,i}, V_{a,i} V_{b,i} (a < b), V_{a,i} T_{,i}, V_{a,i} \xi_{,i}, (T_{,i})^2, \\
 & T_{,i} \xi_{,i}, (\xi_{,i})^2, (B_{,i})^2 \}; \tag{34}
 \end{aligned}$$

$$\begin{aligned}
 I_{\beta}^{(2)} \equiv & \{ (\rho_{a,k} B_{,k})^2 (a = 1, 2, \dots, s), (\rho_{a,k} B_{,k}) (V_{b,j} B_{,j}) (b = 1, 2, \dots, s-1), \\
 & (\rho_{a,k} B_{,k}) (T_{,j} B_{,j}), (\rho_{a,k} B_{,k}) (\xi_{,j} B_{,j}), (V_{a,k} B_{,k})^2, (V_{a,k} B_{,k}) (T_{,j} B_{,j}), \\
 & (V_{a,k} B_{,k}) (\xi_{,j} B_{,j}), (T_{,k} B_{,k})^2, (T_{,k} B_{,k}) (\xi_{,j} B_{,j}), (\xi_{,k} B_{,k})^2 \}; \tag{35}
 \end{aligned}$$

$$\begin{aligned}
 \Delta_{\kappa}^A \equiv & \{ \epsilon_{ijk} \rho_a^i V_{b,j} B_{,k} (a = 1, 2, \dots, s; b = 1, 2, \dots, s-1), \epsilon_{ijk} \rho_a^i T_{,j} B_{,k}, \\
 & \epsilon_{ijk} \rho_a^i \xi_{,j} B_{,k}, \epsilon_{ijk} V_a^i T_{,j} B_{,k}, \epsilon_{ijk} V_a^i \xi_{,j} B_{,k}, \epsilon_{ijk} T_{,i} \xi_{,j} B_{,k}, \\
 & \epsilon_{ijk} \rho_a^i \rho_{b,j} B_{,k} (a < b), \epsilon_{ijk} V_a^i V_{b,j} B_{,k} (a < b) \}; \tag{36}
 \end{aligned}$$

and,

Appendix C-2

Table 1. $II_Y^{(1)}(a_i)$: polar absolute scalars linear in a_i

- | | |
|--|----------------|
| 1. $\rho_{a,i} a_i$ ($a = 1, 2, \dots, s$) | 3. T_{i,a_i} |
| 2. $V_{i,a} a_i$ ($a = 1, 2, \dots, s-1$) | 4. $\xi_s a_i$ |

Table 2. $II_\delta^{(2)}(a_i)$: axial absolute scalars linear in a_i

- | | |
|---|----------------------------|
| 1. $(\rho_{a,k} B_k)(a_i B_i)$ ($a = 1, 2, \dots, s$) | 3. $(T_{,k} B_k)(a_i B_i)$ |
| 2. $(V_k B_k)(a_i B_i)$ ($a = 1, 2, \dots, s-1$) | 4. $(\xi_s B_k)(a_i B_i)$ |

Table 3. $E_\lambda^W(a_i)$: axial absolute scalar triple products linear in a_i

- | | |
|---|------------------------------------|
| 1. $\epsilon_{ijk} a_i \rho_{a,j} B_k$ ($a = 1, 2, \dots, s$) | 3. $\epsilon_{ijk} a_i T_{,j} B_k$ |
| 2. $\epsilon_{ijk} a_i V_a B_k$ ($a = 1, 2, \dots, s-1$) | 4. $\epsilon_{ijk} a_i \xi_s B_k$ |

Appendix C-3Table 4. $11_{\zeta}^{(4)}(c_i)$: axial absolute scalars linear in c_i .

- | | |
|---|---------------------------------------|
| 1. $B_i c_i$ | |
| 2. $(\rho_{a,k} B_k)(\rho_{b,i} c_i)$ ($a, b = 1, 2, \dots, s$) | 10. $(\rho_{a,k} B_k)(T_{i,c_i})$ |
| 3. $(V_{a,k} B_k)(\rho_{b,i} c_i)$ ($a = 1, 2, \dots, s-1$) | 11. $(V_{a,k} B_k)(T_{i,c_i})$ |
| 4. $(T_{i,k} B_k)(\rho_{b,i} c_i)$ | 12. $(T_{i,k} B_k)(T_{i,c_i})$ |
| 5. $(\xi_{s,k} B_k)(\rho_{b,i} c_i)$ | 13. $(\xi_{s,k} B_k)(T_{i,c_i})$ |
| 6. $(\rho_{a,k} B_k)(V_{b,i} c_i)$ | 14. $(\rho_{a,k} B_k)(\xi_{s,i} c_i)$ |
| 7. $(V_{a,k} B_k)(V_{b,i} c_i)$ | 15. $(V_{a,k} B_k)(\xi_{s,i} c_i)$ |
| 8. $(T_{i,k} B_k)(V_{b,i} c_i)$ | 16. $(T_{i,k} B_k)(\xi_{s,i} c_i)$ |
| 9. $(\xi_{s,k} B_k)(V_{b,i} c_i)$ | 17. $(\xi_{s,k} B_k)(\xi_{s,i} c_i)$ |

Table 5. $E_{\zeta}^A(c_i)$: axial absolute scalar triple products linear in c_i .

- | | |
|---|---|
| 1. $\epsilon_{ijk} c_i^{\rho_a} c_j^{\nu_b}$ ($a = 1, 2, \dots, s$;
$b = 1, 2, \dots, s-1$) | 5. $\epsilon_{ijk} c_i^{\nu_a} c_j^{\xi_k}$ |
| 2. $\epsilon_{ijk} c_i^{\rho_a} c_j^{\tau_{k,T}}$ | 6. $\epsilon_{ijk} c_i^{\tau_{j,T}} c_j^{\xi_k}$ |
| 3. $\epsilon_{ijk} c_i^{\rho_a} c_j^{\xi_k}$ | 7. $\epsilon_{ijk} c_i^{\rho_a} c_j^{\rho_b}$ ($a < b$) |
| 4. $\epsilon_{ijk} c_i^{\nu_a} c_j^{\tau_{k,T}}$ | 8. $\epsilon_{ijk} c_i^{\nu_a} c_j^{\nu_b}$ ($a < b$) |

Table 6. $11_{\pi\psi}^{(9)}(c_i)$: pseudoscalars independent of c_i -pseudoscalar triple products linear in c_i absolute scalars.

- | | |
|--|--|
| 1. $(\rho_{a,n} B_n)(\epsilon_{ijk} c_i B_j \rho_{d,k})$ ($a, d = 1, 2, \dots, s$) | 9. $(\rho_{a,n} B_n)(\epsilon_{ijk} c_i B_j T_k)$ |
| 2. $(V_{a,n} B_n)(\epsilon_{ijk} c_i B_j \rho_{d,k})$ ($a = 1, 2, \dots, s-1$) | 10. $(V_{a,n} B_n)(\epsilon_{ijk} c_i B_j T_k)$ |
| 3. $(T_{,n} B_n)(\epsilon_{ijk} c_i B_j \rho_{d,k})$ | 11. $(T_{,n} B_n)(\epsilon_{ijk} c_i B_j T_k)$ |
| 4. $(\xi_{s,n} B_n)(\epsilon_{ijk} c_i B_j \rho_{d,k})$ | 12. $(\xi_{s,n} B_n)(\epsilon_{ijk} c_i B_j T_k)$ |
| 5. $(\rho_{a,n} B_n)(\epsilon_{ijk} c_i B_j V_k)$ | 13. $(\rho_{a,n} B_n)(\epsilon_{ijk} c_i B_j \xi_k)$ |
| 6. $(V_{a,n} B_n)(\epsilon_{ijk} c_i B_j V_k)$ | 14. $(V_{a,n} B_n)(\epsilon_{ijk} c_i B_j \xi_k)$ |
| 7. $(T_{,n} B_n)(\epsilon_{ijk} c_i B_j V_k)$ | 15. $(T_{,n} B_n)(\epsilon_{ijk} c_i B_j \xi_k)$ |
| 8. $(\xi_{s,n} B_n)(\epsilon_{ijk} c_i B_j V_k)$ | 16. $(\xi_{s,n} B_n)(\epsilon_{ijk} c_i B_j \xi_k)$ |

Table 7. $11_{1/0}^{(5)}(c_i)$: triple products-pseudoscalar products absolute scalars linear in c_i .

1. $(\epsilon_{jkl}^{\rho} j_b^{\nu} k^T)_d (\rho, i, c_i)$	$(a = 1, 2, \dots, s; b = 1, 2, \dots, s-1)$	25. $(\epsilon_{jkl}^{\rho} j_b^{\nu} k^T)_d (T, i, c_i)$	
2. $(\epsilon_{jkl}^{\rho} j_b^{\nu} k_s^{\delta_1})_d (\rho, i, c_i)$		26. $(\epsilon_{jkl}^{\rho} j_b^{\nu} k_s^{\delta_1})_d (T, i, c_i)$	
3. $(\epsilon_{jkl}^{\rho} j_b^{\nu} k_s^{\delta_1})_d (\rho, i, c_i)$		27. $(\epsilon_{jkl}^{\rho} j_b^{\nu} k_s^{\delta_1})_d (T, i, c_i)$	
4. $(\epsilon_{jkl}^{\nu} j_b^{\rho} k_s^{\delta_1})_d (\rho, i, c_i)$		28. $(\epsilon_{jkl}^{\nu} j_b^{\rho} k_s^{\delta_1})_d (T, i, c_i)$	
5. $(\epsilon_{jkl}^{\rho} j_b^{\rho} k_c^{\rho})_d (\rho, i, c_i)$	$(a < b < c)$	29. $(\epsilon_{jkl}^{\rho} j_b^{\rho} k_c^{\rho})_d (T, i, c_i)$	$(a < b < c)$
6. $(\epsilon_{jkl}^{\rho} j_b^{\rho} k_c^{\nu})_d (\rho, i, c_i)$	$(a < b)$	30. $(\epsilon_{jkl}^{\rho} j_b^{\rho} k_c^{\nu})_d (T, i, c_i)$	$(a < b)$
7. $(\epsilon_{jkl}^{\rho} j_b^{\rho} k^T)_d (\rho, i, c_i)$	$(a < b)$	31. $(\epsilon_{jkl}^{\rho} j_b^{\rho} k^T)_d (T, i, c_i)$	$(a < b)$
8. $(\epsilon_{jkl}^{\rho} j_b^{\rho} k_s^{\delta_1})_d (\rho, i, c_i)$	$(a < b)$	32. $(\epsilon_{jkl}^{\rho} j_b^{\rho} k_s^{\delta_1})_d (T, i, c_i)$	$(a < b)$
9. $(\epsilon_{jkl}^{\nu} j_b^{\nu} k_c^{\nu})_d (\rho, i, c_i)$	$(a < b < c)$	33. $(\epsilon_{jkl}^{\nu} j_b^{\nu} k_c^{\nu})_d (T, i, c_i)$	$(a < b < c)$
10. $(\epsilon_{jkl}^{\nu} j_b^{\nu} k_c^{\rho})_d (\rho, i, c_i)$	$(a < b)$	34. $(\epsilon_{jkl}^{\nu} j_b^{\nu} k_c^{\rho})_d (T, i, c_i)$	$(a < b)$
11. $(\epsilon_{jkl}^{\nu} j_b^{\nu} k^T)_d (\rho, i, c_i)$	$(a < b)$	35. $(\epsilon_{jkl}^{\nu} j_b^{\nu} k^T)_d (T, i, c_i)$	$(a < b)$
12. $(\epsilon_{jkl}^{\nu} j_b^{\nu} k_s^{\delta_1})_d (\rho, i, c_i)$	$(a < b)$	36. $(\epsilon_{jkl}^{\nu} j_b^{\nu} k_s^{\delta_1})_d (T, i, c_i)$	$(a < b)$
13. $(\epsilon_{jkl}^{\rho} j_b^{\nu} k^T)_d (V_i, c_i)$		37. $(\epsilon_{jkl}^{\rho} j_b^{\nu} k^T)_d (\delta_s, c_i)$	
14. $(\epsilon_{jkl}^{\rho} j_b^{\nu} k_s^{\delta_1})_d (V_i, c_i)$		38. $(\epsilon_{jkl}^{\rho} j_b^{\nu} k_s^{\delta_1})_d (\delta_s, c_i)$	
15. $(\epsilon_{jkl}^{\rho} j_b^{\nu} k_s^{\delta_1})_d (V_i, c_i)$		39. $(\epsilon_{jkl}^{\rho} j_b^{\nu} k_s^{\delta_1})_d (\delta_s, c_i)$	
16. $(\epsilon_{jkl}^{\nu} j_b^{\rho} k_s^{\delta_1})_d (V_i, c_i)$		40. $(\epsilon_{jkl}^{\nu} j_b^{\rho} k_s^{\delta_1})_d (\delta_s, c_i)$	
17. $(\epsilon_{jkl}^{\rho} j_b^{\rho} k_c^{\rho})_d (V_i, c_i)$	$(a < b < c)$	41. $(\epsilon_{jkl}^{\rho} j_b^{\rho} k_c^{\rho})_d (\delta_s, c_i)$	$(a < b < c)$
18. $(\epsilon_{jkl}^{\rho} j_b^{\rho} k_c^{\nu})_d (V_i, c_i)$	$(a < b)$	42. $(\epsilon_{jkl}^{\rho} j_b^{\rho} k_c^{\nu})_d (\delta_s, c_i)$	$(a < b)$
19. $(\epsilon_{jkl}^{\rho} j_b^{\rho} k^T)_d (V_i, c_i)$	$(a < b)$	43. $(\epsilon_{jkl}^{\rho} j_b^{\rho} k^T)_d (\delta_s, c_i)$	$(a < b)$
20. $(\epsilon_{jkl}^{\rho} j_b^{\rho} k_s^{\delta_1})_d (V_i, c_i)$	$(a < b)$	44. $(\epsilon_{jkl}^{\rho} j_b^{\rho} k_s^{\delta_1})_d (\delta_s, c_i)$	$(a < b)$
21. $(\epsilon_{jkl}^{\nu} j_b^{\nu} k_c^{\nu})_d (V_i, c_i)$	$(a < b < c)$	45. $(\epsilon_{jkl}^{\nu} j_b^{\nu} k_c^{\nu})_d (\delta_s, c_i)$	$(a < b < c)$
22. $(\epsilon_{jkl}^{\nu} j_b^{\nu} k_c^{\rho})_d (V_i, c_i)$	$(a < b)$	46. $(\epsilon_{jkl}^{\nu} j_b^{\nu} k_c^{\rho})_d (\delta_s, c_i)$	$(a < b)$
23. $(\epsilon_{jkl}^{\nu} j_b^{\nu} k^T)_d (V_i, c_i)$	$(a < b)$	47. $(\epsilon_{jkl}^{\nu} j_b^{\nu} k^T)_d (\delta_s, c_i)$	$(a < b)$
24. $(\epsilon_{jkl}^{\nu} j_b^{\nu} k_s^{\delta_1})_d (V_i, c_i)$	$(a < b)$	48. $(\epsilon_{jkl}^{\nu} j_b^{\nu} k_s^{\delta_1})_d (\delta_s, c_i)$	$(a < b)$

Appendix C-4

$\frac{H^0}{ab} ik$

$$\begin{aligned}
 & \left(\sum_{c=1}^s H_{abc}^{(1)} (1^*) (\vec{v}_0 \cdot \vec{B}) + \sum_{c=1}^{s-1} H_{abc}^{(1)} (1^*) (\vec{V} \cdot \vec{B}) + H_{ab4}^{(1)} (1^*) (\vec{V} \cdot \vec{B}) + H_{ab5}^{(1)} (1^*) (\vec{E} \cdot \vec{B}) \right) \\
 & + \sum_{c=1}^s \sum_{d=1}^{s-1} H_{abcd}^{(4)} (11^*) [\vec{v}_0 \cdot (\vec{V} \times \vec{V} \cdot \vec{T})] + \sum_{c=1}^s \sum_{d=1}^{s-1} H_{abcd}^{(4)} (11^*) [\vec{v}_0 \cdot (\vec{V} \times \vec{E})] + \sum_{c=1}^s H_{ac}^{(4)} (11^*) [\vec{v}_0 \cdot (\vec{V} \times \vec{E})] + \\
 & + \sum_{c=1}^{s-1} H_{abc}^{(4)} (11^*) [\vec{V} \cdot (\vec{V} \times \vec{E})] + \sum_{c=1}^s \sum_{d=1}^s \sum_{f=1}^s H_{abcdf}^{(4)} (11^*) [\vec{v}_0 \cdot (\vec{v}_0 \times \vec{v}_0)] + \sum_{c=1}^s \sum_{d=1}^s \sum_{f=1}^{s-1} H_{cdf}^{(4)} (11^*) [\vec{v}_0 \cdot (\vec{v}_0 \times \vec{V})] \\
 & + \sum_{c=1}^s \sum_{d=1}^s H_{7/1}^{(4)} (11^*) [\vec{v}_0 \cdot (\vec{v}_0 \times \vec{V} \cdot \vec{T})] + \sum_{c=1}^s \sum_{d=1}^s H_{8/1}^{(4)} (11^*) [\vec{v}_0 \cdot (\vec{v}_0 \times \vec{E})] + \sum_{c=1}^{s-1} \sum_{d=1}^{s-1} \sum_{f=1}^{s-1} H_{9/1}^{(4)} (11^*) [\vec{V} \cdot (\vec{V} \times \vec{V})] + \\
 & + \left. \sum_{c=1}^{s-1} \sum_{d=1}^{s-1} \sum_{f=1}^s H_{10/1}^{(4)} (11^*) [\vec{V} \cdot (\vec{V} \times \vec{v}_0)] + \sum_{c=1}^{s-1} \sum_{d=1}^{s-1} H_{11/1}^{(4)} (11^*) [\vec{V} \cdot (\vec{V} \times \vec{V} \cdot \vec{T})] + \sum_{c=1}^{s-1} \sum_{d=1}^{s-1} H_{12/1}^{(4)} (11^*) [\vec{V} \cdot (\vec{V} \times \vec{E})] \right\} \delta_{ik} \\
 & + \sum_{c=1}^s H_{abc}^{(2)} (11^*) \epsilon_{ijk} \epsilon_{c'j} + \\
 & + \left[\sum_{c=1}^s H_{abc}^{(3)} (11^*) (\vec{v}_0 \cdot \vec{B}) + \sum_{c=1}^{s-1} H_{abc}^{(3)} (11^*) (\vec{V} \cdot \vec{B}) + H_{abc}^{(3)} (11^*) (\vec{V} \cdot \vec{B}) + H_{abc}^{(3)} (11^*) (\vec{E} \cdot \vec{B}) \right] \epsilon_{ijk} \beta_j \quad (38)
 \end{aligned}$$

$$\underline{M_{ab}^V}_{ijk} =$$

$$\begin{aligned}
& \left(\sum_{c=1}^s M_{abc}^{(1)}(1^*) (\vec{v}_c \cdot \vec{B}) + \sum_{c=1}^{s-1} M_{abc}^{(1)}(1^*) (\vec{V} \cdot \vec{B}) + M_{ab}^{(1)}(1^*) (\vec{v}_T \cdot \vec{B}) + M_{ab}^{(1)}(1^*) (\vec{B} \cdot \vec{B}) + \right. \\
& + \sum_{c=1}^s \sum_{d=1}^{s-1} M_{abcd}^{(4)} \frac{1}{2} (11^*) [\vec{v}_c \cdot (\vec{V} \times \vec{v}_T)] + \sum_{c=1}^s \sum_{d=1}^{s-1} M_{abcd}^{(4)} \frac{1}{2} (11^*) [\vec{v}_c \cdot (\vec{V} \times \vec{B})] + \sum_{c=1}^s M_{abc}^{(4)} \frac{1}{2} (11^*) [\vec{v}_c \cdot (\vec{v}_T \times \vec{B})] + \\
& + \sum_{c=1}^{s-1} M_{abc}^{(4)} \frac{1}{2} (11^*) [V \cdot (\vec{v}_T \times \vec{B})] + \sum_{c=1}^s \sum_{d=1}^s \sum_{f=1}^s M_{abcdf}^{(4)} \frac{1}{2} (11^*) [\vec{v}_c \cdot (\vec{v}_d \times \vec{v}_f)] + \sum_{c=1}^s \sum_{d=1}^s \sum_{f=1}^{s-1} M_{abcdf}^{(4)} \frac{1}{2} (11^*) [\vec{v}_c \cdot (\vec{v}_d \times \vec{V})] + \\
& + \sum_{c=1}^s \sum_{d=1}^s M_{abcd}^{(4)} \frac{1}{2} (11^*) [\vec{v}_c \cdot (\vec{v}_d \times \vec{v}_T)] + \sum_{c=1}^s \sum_{d=1}^s M_{abcd}^{(4)} \frac{1}{2} (11^*) [\vec{v}_c \cdot (\vec{v}_d \times \vec{B})] + \sum_{c=1}^{s-1} \sum_{d=1}^{s-1} \sum_{f=1}^{s-1} M_{abcdf}^{(4)} \frac{1}{2} (11^*) [\vec{V} \cdot (\vec{v}_c \times \vec{v}_d)] + \\
& + \left. \sum_{c=1}^s \sum_{d=1}^{s-1} \sum_{f=1}^s M_{abcdf}^{(4)} \frac{1}{2} (11^*) [\vec{V} \cdot (\vec{V} \times \vec{v}_d)] + \sum_{c=1}^{s-1} \sum_{d=1}^{s-1} M_{abcd}^{(4)} \frac{1}{2} (11^*) [\vec{V} \cdot (\vec{V} \times \vec{v}_T)] + \sum_{c=1}^{s-1} \sum_{d=1}^{s-1} M_{abcd}^{(4)} \frac{1}{2} (11^*) [\vec{V} \cdot (\vec{V} \times \vec{B})] \right\} \delta_{ik} + \\
& + \sum_{c=1}^s M_{abc}^{(2)}(11^*) \epsilon_{ijk} v_{c,j} + \sum_{c=1}^{s-1} M_{abc}^{(2)}(11^*) \epsilon_{ijk} v_{c,j} + \\
& + \left[\sum_{c=1}^s M_{abc}^{(3)}(11^*) (\vec{v}_c \cdot \vec{B}) + \sum_{c=1}^{s-1} M_{abc}^{(3)}(11^*) (\vec{V} \cdot \vec{B}) + M_{ab}^{(3)}(11^*) (\vec{v}_T \cdot \vec{B}) + M_{ab}^{(3)}(11^*) (\vec{B} \cdot \vec{B}) \right] \cdot \epsilon_{ijk} B_j, \quad (39)
\end{aligned}$$

$\underline{M_{ik}^T}$ =

$$\begin{aligned}
& \left(\sum_{b=1}^s M_{ab}^{(1)}(1^*) (\vec{v}_\rho \cdot \vec{\delta})_b + \sum_{b=1}^{s-1} M_{ab}^{(1)}(1^*) (\vec{v} \cdot \vec{\delta})_b + M_a^{(1)}(1^*) (\vec{v} \cdot \vec{\delta}) + M_a^{(1)}(1^*) (\vec{v} \cdot \vec{\delta})_s + \right. \\
& + \sum_{b=1}^s \sum_{c=1}^{s-1} M_{abc}^{(4)}(11^*) [\vec{v}_\rho \cdot (\vec{v} \times \vec{v} \cdot \vec{T})]_b c + \sum_{b=1}^s \sum_{c=1}^{s-1} M_{abc}^{(4)}(11^*) [\vec{v}_\rho \cdot (\vec{v} \times \vec{\delta})]_b c s + \sum_{b=1}^s M_{ab}^{(4)}(11^*) [\vec{v}_\rho \cdot (\vec{v} \cdot \vec{\delta})]_b s + \\
& + \sum_{b=1}^{s-1} M_{ab}^{(4)}(11^*) [\vec{v} \cdot (\vec{v} \cdot \vec{\delta})]_b s + \sum_{b=1}^s \sum_{c=1}^s \sum_{d=1}^s M_{abcd}^{(4)}(11^*) [\vec{v}_\rho \cdot (\vec{v}_\rho \times \vec{v}_\rho)]_b c d + \sum_{b=1}^s \sum_{c=1}^s \sum_{d=1}^{s-1} M_{abcd}^{(4)}(11^*) [\vec{v}_\rho \cdot (\vec{v}_\rho \times \vec{v})]_b c d + \\
& + \sum_{b=1}^s \sum_{c=1}^s M_{abc}^{(4)}(11^*) [\vec{v}_\rho \cdot (\vec{v}_\rho \times \vec{v} \cdot \vec{T})]_b c + \sum_{b=1}^s \sum_{c=1}^s M_{abc}^{(4)}(11^*) [\vec{v}_\rho \cdot (\vec{v}_\rho \times \vec{\delta})]_b c s + \sum_{b=1}^{s-1} \sum_{c=1}^{s-1} \sum_{d=1}^{s-1} M_{abcd}^{(4)}(11^*) [\vec{v} \cdot (\vec{v} \times \vec{v})]_b c d + \\
& + \left. \sum_{b=1}^{s-1} \sum_{c=1}^{s-1} \sum_{d=1}^s M_{abcd}^{(4)}(11^*) [\vec{v} \cdot (\vec{v} \times \vec{v}_\rho)]_b c d + \sum_{b=1}^{s-1} \sum_{c=1}^{s-1} M_{abc}^{(4)}(11^*) [\vec{v} \cdot (\vec{v} \times \vec{v} \cdot \vec{T})]_b c + \sum_{b=1}^{s-1} \sum_{c=1}^{s-1} M_{abc}^{(4)}(11^*) [\vec{v} \cdot (\vec{v} \times \vec{\delta})]_b c s \right\} \delta_{ik} + \\
& + \sum_{b=1}^s M_{ab}^{(2)}(11^*) c_{ijk}^p{}_{b,j} + \sum_{b=1}^{s-1} M_{ab}^{(2)}(11^*) c_{ijk}^v{}_{b,j} + \\
& + \left[\sum_{b=1}^s M_{ab}^{(3)}(11^*) (\vec{v}_\rho \cdot \vec{\delta})_b + \sum_{b=1}^{s-1} M_{ab}^{(3)}(11^*) (\vec{v} \cdot \vec{\delta})_b + M_a^{(3)}(11^*) (\vec{v} \cdot \vec{\delta}) + M_a^{(3)}(11^*) (\vec{\delta} \cdot \vec{\delta})_s \right] c_{ijk}^B{}_{j} , \quad (40)
\end{aligned}$$

and

$$\begin{aligned}
\frac{M_{ik}^G}{a} = & \left(\sum_{b=1}^s M_{ab}^{(1)}(11^*) \binom{1^*}{b} (\vec{\rho}_\rho \cdot \vec{\beta}) + \sum_{b=1}^{s-1} M_{ab}^{(1)}(11^*) \binom{1^*}{b} (\vec{V} \cdot \vec{\beta}) + M_a^{(1)}(11^*) (\vec{\sigma}_T \cdot \vec{\beta}) + M_a^{(1)}(11^*) \binom{1^*}{s} (\vec{\xi} \cdot \vec{\beta}) + \right. \\
& + \sum_{b=1}^s \sum_{c=1}^{s-1} M_{abc}^{(4)}(11^*) \binom{1^*}{b} \binom{1^*}{c} [\vec{\rho}_\rho \cdot (\vec{V} \times \vec{\sigma}_T)] + \sum_{b=1}^s \sum_{c=1}^{s-1} M_{abc}^{(4)}(11^*) \binom{1^*}{b} \binom{1^*}{c} [\vec{\rho}_\rho \cdot (\vec{V} \times \vec{\xi})] + \sum_{b=1}^s M_{ab}^{(4)}(11^*) \binom{1^*}{b} [\vec{\rho}_\rho \cdot (\vec{\sigma}_T \times \vec{\xi})] + \\
& + \sum_{b=1}^{s-1} M_{ab}^{(4)}(11^*) \binom{1^*}{b} [\vec{V} \cdot (\vec{\sigma}_T \times \vec{\xi})] + \sum_{b=1}^s \sum_{c=1}^s \sum_{d=1}^{s-1} M_{abcd}^{(4)}(11^*) \binom{1^*}{b} \binom{1^*}{c} \binom{1^*}{d} [\vec{\rho}_\rho \cdot (\vec{\sigma}_\rho \times \vec{\rho}_\rho)] + \sum_{b=1}^s \sum_{c=1}^s \sum_{d=1}^{s-1} M_{abcd}^{(4)}(11^*) \binom{1^*}{b} \binom{1^*}{c} \binom{1^*}{d} [\vec{\rho}_\rho \cdot (\vec{\sigma}_\rho \times \vec{V})] + \\
& + \sum_{b=1}^s \sum_{c=1}^s M_{abc}^{(4)}(11^*) \binom{1^*}{b} \binom{1^*}{c} [\vec{\rho}_\rho \cdot (\vec{\sigma}_\rho \times \vec{\sigma}_T)] + \sum_{b=1}^s \sum_{c=1}^s M_{abc}^{(4)}(11^*) \binom{1^*}{b} \binom{1^*}{c} [\vec{\rho}_\rho \cdot (\vec{\sigma}_\rho \times \vec{\xi})] + \sum_{b=1}^{s-1} \sum_{c=1}^{s-1} \sum_{d=1}^{s-1} M_{abcd}^{(4)}(11^*) \binom{1^*}{b} \binom{1^*}{c} \binom{1^*}{d} [\vec{V} \cdot (\vec{V} \times \vec{V})] + \\
& + \left. \sum_{b=1}^{s-1} \sum_{c=1}^{s-1} \sum_{d=1}^s M_{abcd}^{(4)}(11^*) \binom{1^*}{b} \binom{1^*}{c} \binom{1^*}{d} [\vec{V} \cdot (\vec{V} \times \vec{\rho}_\rho)] + \sum_{b=1}^{s-1} \sum_{c=1}^{s-1} M_{abc}^{(4)}(11^*) \binom{1^*}{b} \binom{1^*}{c} [\vec{V} \cdot (\vec{V} \times \vec{\sigma}_T)] + \sum_{b=1}^{s-1} \sum_{c=1}^{s-1} M_{abc}^{(4)}(11^*) \binom{1^*}{b} \binom{1^*}{c} [\vec{V} \cdot (\vec{V} \times \vec{\xi})] \right\} \delta_{ik} + \\
& + \sum_{b=1}^s M_{ab}^{(2)}(11^*) c_{ijk}^{\rho, j} + \sum_{b=1}^{s-1} M_{ab}^{(2)}(11^*) c_{ijk}^{\rho, j} + M_a^{(2)}(11^*) c_{ijk}^{\rho, j} + \\
& + \left. \left[\sum_{b=1}^s M_{ab}^{(3)}(11^*) \binom{1^*}{b} (\vec{\rho}_\rho \cdot \vec{\beta}) + \sum_{b=1}^{s-1} M_{ab}^{(3)}(11^*) \binom{1^*}{b} (\vec{V} \cdot \vec{\beta}) + M_{a34}^{(3)}(11^*) (\vec{\sigma}_T \cdot \vec{\beta}) + M_{a44}^{(3)}(11^*) \binom{1^*}{s} (\vec{\xi} \cdot \vec{\beta}) \right] c_{ijk}^{\rho, j} \right) \quad (41)
\end{aligned}$$

Appendix C-5Table 8. $E_X^P(a_i)$: pseudoscalar triple products linear in a_i .

- | | | |
|---|--|---|
| 1. $\epsilon_{ijk} a_i^\rho a_j^\rho V_b^k$ | ($a = 1, 2, \dots, s$;
$b = 1, 2, \dots, s-1$) | 5. $\epsilon_{ijk} a_i V_a^j \delta_k^s$ |
| 2. $\epsilon_{ijk} a_i^\rho a_j^T k$ | | 6. $\epsilon_{ijk} a_i^T a_j \delta_k^s$ |
| 3. $\epsilon_{ijk} a_i^\rho a_j \delta_k^s$ | | 7. $\epsilon_{ijk} a_i^\rho a_j^\rho k$ ($a < b$) |
| 4. $\epsilon_{ijk} a_i V_a^j T_k$ | | 8. $\epsilon_{ijk} a_i V_a^j V_b^k$ ($a < b$) |

Table 9. $E_\phi^P(b_i)$: pseudoscalar triple products linear in b_i .

- | | | |
|---|--|---|
| 1. $\epsilon_{ijk} b_i^\rho a_j^\rho V_b^k$ | ($a = 1, 2, \dots, s$;
$b = 1, 2, \dots, s-1$) | 5. $\epsilon_{ijk} b_i V_a^j \delta_k^s$ |
| 2. $\epsilon_{ijk} b_i^\rho a_j^T k$ | | 6. $\epsilon_{ijk} b_i^T a_j \delta_k^s$ |
| 3. $\epsilon_{ijk} b_i^\rho a_j \delta_k^s$ | | 7. $\epsilon_{ijk} b_i^\rho a_j^\rho k$ ($a < b$) |
| 4. $\epsilon_{ijk} b_i V_a^j T_k$ | | 8. $\epsilon_{ijk} b_i V_a^j V_b^k$ ($a < b$) |

Table 10. $\|_{\gamma \in}^{(6)}(a_i, b_i)$: polar absolute scalars bilinear in a_i and b_i .

- | | |
|---|---------------------------------------|
| 1. $(\rho_{a,i} a_i)(\rho_{b,j} b_j)$ $(a, b = 1, 2, \dots, s)$ | 9. $(\rho_{a,i} a_i)(T_{j,j} b_j)$ |
| 2. $(V_i a_i)(\rho_{b,j} b_j)$ $(a = 1, 2, \dots, s-1)$ | 10. $(V_i a_i)(T_{j,j} b_j)$ |
| 3. $(T_{i,i} a_i)(\rho_{b,j} b_j)$ | 11. $(T_{i,i} a_i)(T_{j,j} b_j)$ |
| 4. $(\xi_{i,i} a_i)(\rho_{b,j} b_j)$ | 12. $(\xi_{i,i} a_i)(T_{j,j} b_j)$ |
| 5. $(\rho_{a,i} a_i)(V_{b,j} b_j)$ | 13. $(\rho_{a,i} a_i)(\xi_{s,j} b_j)$ |
| 6. $(V_i a_i)(V_{b,j} b_j)$ | 14. $(V_i a_i)(\xi_{s,j} b_j)$ |
| 7. $(T_{i,i} a_i)(V_{b,j} b_j)$ | 15. $(T_{i,i} a_i)(\xi_{s,j} b_j)$ |
| 8. $(\xi_{i,i} a_i)(V_{b,j} b_j)$ | 16. $(\xi_{i,i} a_i)(\xi_{s,j} b_j)$ |

Table 11. $11_{\gamma\mu}^{(10)}(a_i, b_i)$: polar absolute scalars linear in a_i -axial absolute triple products linear in b_i absolute scalars.

- | | |
|--|---|
| 1. $(\rho_{a,i} a_i)(\epsilon_{jkl} b_{j_b}^{\rho} b_{k_l}^B)$ $(a, b = 1, 2, \dots, s)$ | 9. $(\rho_{a,i} a_i)(\epsilon_{jkl} b_{j_l}^T b_{k_l}^B)$ |
| 2. $(V_{a,i} a_i)(\epsilon_{jkl} b_{j_b}^{\rho} b_{k_l}^B)$ $(a = 1, 2, \dots, s-1)$ | 10. $(V_{a,i} a_i)(\epsilon_{jkl} b_{j_l}^T b_{k_l}^B)$ |
| 3. $(T_{a,i} a_i)(\epsilon_{jkl} b_{j_b}^{\rho} b_{k_l}^B)$ | 11. $(T_{a,i} a_i)(\epsilon_{jkl} b_{j_l}^T b_{k_l}^B)$ |
| 4. $(\delta_{a,i} a_i)(\epsilon_{jkl} b_{j_b}^{\rho} b_{k_l}^B)$ | 12. $(\delta_{a,i} a_i)(\epsilon_{jkl} b_{j_l}^T b_{k_l}^B)$ |
| 5. $(\rho_{a,i} a_i)(\epsilon_{jkl} b_{j_b}^V b_{k_l}^B)$ | 13. $(\rho_{a,i} a_i)(\epsilon_{jkl} b_{j_s} \delta_{k_l}^B)$ |
| 6. $(V_{a,i} a_i)(\epsilon_{jkl} b_{j_b}^V b_{k_l}^B)$ | 14. $(V_{a,i} a_i)(\epsilon_{jkl} b_{j_s} \delta_{k_l}^B)$ |
| 7. $(T_{a,i} a_i)(\epsilon_{jkl} b_{j_b}^V b_{k_l}^B)$ | 15. $(T_{a,i} a_i)(\epsilon_{jkl} b_{j_s} \delta_{k_l}^B)$ |
| 8. $(\delta_{a,i} a_i)(\epsilon_{jkl} b_{j_b}^V b_{k_l}^B)$ | 16. $(\delta_{a,i} a_i)(\epsilon_{jkl} b_{j_s} \delta_{k_l}^B)$ |

Table 12. $\Pi_{\lambda \epsilon}^{(11)}(a_i, b_i)$: axial absolute scalar triple products linear in a_i -polar absolute scalars linear in b_i absolute scalars.

- | | |
|--|---|
| 1. $(\epsilon_{ikl} a_i^\rho, k B_l)(\rho_a, j b_j)$ $(a, b = 1, 2, \dots, s)$ | 9. $(\epsilon_{ikl} a_i^\rho, k B_l)(T, j b_j)$ |
| 2. $(\epsilon_{ikl} a_i V_k B_l)(\rho_a, j b_j)$ $(a = 1, 2, \dots, s-1)$ | 10. $(\epsilon_{ikl} a_i V_k B_l)(T, j b_j)$ |
| 3. $(\epsilon_{ikl} a_i T, k B_l)(\rho_a, j b_j)$ | 11. $(\epsilon_{ikl} a_i T, k B_l)(T, j b_j)$ |
| 4. $(\epsilon_{ikl} a_i \xi_k B_l)(\rho_a, j b_j)$ | 12. $(\epsilon_{ikl} a_i \xi_k B_l)(T, j b_j)$ |
| 5. $(\epsilon_{ikl} a_i^\rho, k B_l)(V_j b_j)$ | 13. $(\epsilon_{ikl} a_i^\rho, k B_l)(\xi_j b_j)$ |
| 6. $(\epsilon_{ikl} a_i V_k B_l)(V_j b_j)$ | 14. $(\epsilon_{ikl} a_i V_k B_l)(\xi_j b_j)$ |
| 7. $(\epsilon_{ikl} a_i T, k B_l)(V_j b_j)$ | 15. $(\epsilon_{ikl} a_i T, k B_l)(\xi_j b_j)$ |
| 8. $(\epsilon_{ikl} a_i \xi_k B_l)(V_j b_j)$ | 16. $(\epsilon_{ikl} a_i \xi_k B_l)(\xi_j b_j)$ |

Table 13. $11_{\delta\epsilon}^{(13)}(a_i, b_j)$: axial absolute scalars linear in a_i -polar absolute scalars linear in b_j absolute scalars.

- | | | |
|--|---------------------------|--|
| 1. $(\vec{\nabla}_a \rho \cdot \vec{B})(a_i B_i) (\rho_{b,j} b_j)$ | $(a, b = 1, 2, \dots, s)$ | 9. $(\vec{\nabla}_a \rho \cdot \vec{B})(a_i B_i) (\tau_{j,b_j})$ |
| 2. $(\vec{V} \cdot \vec{B})(a_i B_i) (\rho_{b,j} b_j)$ | $(a = 1, 2, \dots, s-1)$ | 10. $(\vec{V} \cdot \vec{B})(a_i B_i) (\tau_{j,b_j})$ |
| 3. $(\vec{\nabla} \tau \cdot \vec{B})(a_i B_i) (\rho_{b,j} b_j)$ | | 11. $(\vec{\nabla} \tau \cdot \vec{B})(a_i B_i) (\tau_{j,b_j})$ |
| 4. $(\vec{\xi} \cdot \vec{B})(a_i B_i) (\rho_{b,j} b_j)$ | | 12. $(\vec{\xi} \cdot \vec{B})(a_i B_i) (\tau_{j,b_j})$ |
| 5. $(\vec{\nabla}_a \rho \cdot \vec{B})(a_i B_i) (v_j b_j)$ | | 13. $(\vec{\nabla}_a \rho \cdot \vec{B})(a_i B_i) (\xi_j b_j)$ |
| 6. $(\vec{V} \cdot \vec{B})(a_i B_i) (v_j b_j)$ | | 14. $(\vec{V} \cdot \vec{B})(a_i B_i) (\xi_j b_j)$ |
| 7. $(\vec{\nabla} \tau \cdot \vec{B})(a_i B_i) (v_j b_j)$ | | 15. $(\vec{\nabla} \tau \cdot \vec{B})(a_i B_i) (\xi_j b_j)$ |
| 8. $(\vec{\xi} \cdot \vec{B})(a_i B_i) (v_j b_j)$ | | 16. $(\vec{\xi} \cdot \vec{B})(a_i B_i) (\xi_j b_j)$ |

Table 14. $||_{\tau\omega}^{(14)}(a_i, b_i)$: polar absolute scalars linear in a_i -axial absolute scalars linear in b_i absolute scalars.

- | | |
|---|--|
| 1. $(\vec{v}_\rho \cdot \vec{B})_{\underset{a}{b}, i, a_i} b_j B_j$ $(a, b = 1, 2, \dots, s)$ | 9. $(\vec{v}T \cdot \vec{B})_{\underset{b}{b}, i, a_i} b_j B_j$ |
| 2. $(\vec{v}_\rho \cdot \vec{B})_{\underset{a}{b}, i, a_i} b_j B_j$ $(b = 1, 2, \dots, s-1)$ | 10. $(\vec{v}T \cdot \vec{B})_{\underset{b}{b}, i, a_i} b_j B_j$ |
| 3. $(\vec{v}_\rho \cdot \vec{B})_{\underset{a}{T}, i, a_i} b_j B_j$ | 11. $(\vec{v}T \cdot \vec{B})_{\underset{b}{T}, i, a_i} b_j B_j$ |
| 4. $(\vec{v}_\rho \cdot \vec{B})_{\underset{a}{s}, i, a_i} b_j B_j$ | 12. $(\vec{v}T \cdot \vec{B})_{\underset{s}{s}, i, a_i} b_j B_j$ |
| 5. $(\vec{V} \cdot \vec{B})_{\underset{a}{b}, i, a_i} b_j B_j$ | 13. $(\vec{E} \cdot \vec{B})_{\underset{s}{b}, i, a_i} b_j B_j$ |
| 6. $(\vec{V} \cdot \vec{B})_{\underset{a}{b}, i, a_i} b_j B_j$ | 14. $(\vec{E} \cdot \vec{B})_{\underset{s}{b}, i, a_i} b_j B_j$ |
| 7. $(\vec{V} \cdot \vec{B})_{\underset{a}{T}, i, a_i} b_j B_j$ | 15. $(\vec{E} \cdot \vec{B})_{\underset{s}{T}, i, a_i} b_j B_j$ |
| 8. $(\vec{V} \cdot \vec{B})_{\underset{a}{s}, i, a_i} b_j B_j$ | 16. $(\vec{E} \cdot \vec{B})_{\underset{s}{s}, i, a_i} b_j B_j$ |

Table 15. $\| \delta_{\mu}^{(15)}(a_i, b_i) \|$: axial absolute scalars linear in a_i -axial absolute scalar triple products linear in b_i absolute scalars.

- | | |
|---|--|
| 1. $(\vec{\nabla}_{\rho} \cdot \vec{B}) (a_i, B_i) \epsilon_{jkl} b_{j_c}^{\rho} b_{k_c}^{B_1}$ ($b, c = 1, 2, \dots, s$) | 9. $(\vec{\nabla}_{\rho} \cdot \vec{B}) (a_i, B_i) \epsilon_{jkl} b_{j^T, k}^{B_1}$ |
| 2. $(\vec{V} \cdot \vec{B}) (a_i, B_i) \epsilon_{jkl} b_{j_c}^{\rho} b_{k_c}^{B_1}$ ($b = 1, 2, \dots, s-1$) | 10. $(\vec{V} \cdot \vec{B}) (a_i, B_i) \epsilon_{jkl} b_{j^T, k}^{B_1}$ |
| 3. $(\vec{\nabla}_T \cdot \vec{B}) (a_i, B_i) \epsilon_{jkl} b_{j_c}^{\rho} b_{k_c}^{B_1}$ | 11. $(\vec{\nabla}_T \cdot \vec{B}) (a_i, B_i) \epsilon_{jkl} b_{j^T, k}^{B_1}$ |
| 4. $(\vec{\xi} \cdot \vec{B}) (a_i, B_i) \epsilon_{jkl} b_{j_c}^{\rho} b_{k_c}^{B_1}$ | 12. $(\vec{\xi} \cdot \vec{B}) (a_i, B_i) \epsilon_{jkl} b_{j^T, k}^{B_1}$ |
| 5. $(\vec{\nabla}_{\rho} \cdot \vec{B}) (a_i, B_i) \epsilon_{jkl} b_{j_c}^{\rho} b_{k_c}^{B_1}$ | 13. $(\vec{\nabla}_{\rho} \cdot \vec{B}) (a_i, B_i) \epsilon_{jkl} b_{j_s}^{\rho} b_{k_s}^{B_1}$ |
| 6. $(\vec{V} \cdot \vec{B}) (a_i, B_i) \epsilon_{jkl} b_{j_c}^{\rho} b_{k_c}^{B_1}$ | 14. $(\vec{V} \cdot \vec{B}) (a_i, B_i) \epsilon_{jkl} b_{j_s}^{\rho} b_{k_s}^{B_1}$ |
| 7. $(\vec{\nabla}_T \cdot \vec{B}) (a_i, B_i) \epsilon_{jkl} b_{j_c}^{\rho} b_{k_c}^{B_1}$ | 15. $(\vec{\nabla}_T \cdot \vec{B}) (a_i, B_i) \epsilon_{jkl} b_{j_s}^{\rho} b_{k_s}^{B_1}$ |
| 8. $(\vec{\xi} \cdot \vec{B}) (a_i, B_i) \epsilon_{jkl} b_{j_c}^{\rho} b_{k_c}^{B_1}$ | 16. $(\vec{\xi} \cdot \vec{B}) (a_i, B_i) \epsilon_{jkl} b_{j_s}^{\rho} b_{k_s}^{B_1}$ |

Table 16. $||_{\omega\lambda}^{(16)}(a_i, b_j)$: axial absolute scalars linear in b_j -axial absolute triple products linear in a_i ; absolute scalars.

- | | |
|---|---|
| 1. $(\vec{v}_\rho \cdot \vec{B}) (b_j B_j) \epsilon_{ikl} a_i^\rho a_c^{B_l}$ $(b, c = 1, 2, \dots, s)$ | 9. $(\vec{v}_\rho \cdot \vec{B}) (b_j B_j) \epsilon_{ikl} a_i^T a_c^{B_l}$ |
| 2. $(\vec{V} \cdot \vec{B}) (b_j B_j) \epsilon_{ikl} a_i^\rho a_c^{B_l}$ $(b = 1, 2, \dots, s-1)$ | 10. $(\vec{V} \cdot \vec{B}) (b_j B_j) \epsilon_{ikl} a_i^T a_c^{B_l}$ |
| 3. $(\vec{v}_T \cdot \vec{B}) (b_j B_j) \epsilon_{ikl} a_i^\rho a_c^{B_l}$ | 11. $(\vec{v}_T \cdot \vec{B}) (b_j B_j) \epsilon_{ikl} a_i^T a_c^{B_l}$ |
| 4. $(\vec{\xi} \cdot \vec{B}) (b_j B_j) \epsilon_{ikl} a_i^\rho a_c^{B_l}$ | 12. $(\vec{\xi} \cdot \vec{B}) (b_j B_j) \epsilon_{ikl} a_i^T a_c^{B_l}$ |
| 5. $(\vec{v}_\rho \cdot \vec{B}) (b_j B_j) \epsilon_{ikl} a_i^V a_c^{B_l}$ | 13. $(\vec{v}_\rho \cdot \vec{B}) (b_j B_j) \epsilon_{ikl} a_i \xi_k^{B_l}$ |
| 6. $(\vec{V} \cdot \vec{B}) (b_j B_j) \epsilon_{ikl} a_i^V a_c^{B_l}$ | 14. $(\vec{V} \cdot \vec{B}) (b_j B_j) \epsilon_{ikl} a_i \xi_k^{B_l}$ |
| 7. $(\vec{v}_T \cdot \vec{B}) (b_j B_j) \epsilon_{ikl} a_i^V a_c^{B_l}$ | 15. $(\vec{v}_T \cdot \vec{B}) (b_j B_j) \epsilon_{ikl} a_i \xi_k^{B_l}$ |
| 8. $(\vec{\xi} \cdot \vec{B}) (b_j B_j) \epsilon_{ikl} a_i^V a_c^{B_l}$ | 16. $(\vec{\xi} \cdot \vec{B}) (b_j B_j) \epsilon_{ikl} a_i \xi_k^{B_l}$ |

Footnotes to Chapter C

- ¹ With regard to material systems in motion and their interaction with an electromagnetic field, note the comments of Fano, Chu & Adler, [1960, 3, p.378].
- ² Note Truesdell & Noll [1965, 19, Sect. 20].
- ³ Post [1962, 6, p.2].
- ⁴ Cf. Jaunzemis [1967, 12, pp.281-282]. See here also footnote F7.
- ⁵ Liu & Müller [1972, 5]. There are, of course, other (e.g. linear) theoretical approaches to the problem of formulating constitutive equations. The motivation for the employment of the non-linear field theoretical approach used here lies in the particular significance of same as pointed out by Truesdell & Noll [1965, 19, pp.3-4]. The shortcomings of a classical thermodynamics of irreversible processes (TIP) approach have served to further strengthen the relative meaningfulness of the method utilized here. These matters are discussed, for example, in CFT [1960, 9, p.644, footnote 3], Truesdell [1966, 13, pp.49-52] and Truesdell [1969, 7, pp.111-149].
- ⁶ At present it appears that the process of denoting specific field variables as being either "dependent" or "independent" is not yet subject to a set of basic principles (cf. Eringen [1967, 5, pp.144-145]). Regardless of the questionable necessity for such principles, there seems to be some degree of common consent regarding "natural" choices of variables under certain conditions (cf. Müller [1968, 4, p.12]). In this study I adopt a preference for constitutive field identification compatible with that of Müller [1968, 4, p.9] and

Liu & Müller [1972, 5, pp.150-151]. See also Demiray & Eringen [1973, 2, (3.1) and (3.2)].

⁷ The importance of incorporating the fields $\rho_{a,k}$ in the set of independent constitutive variables was first recognized by Müller [1968, 4, pp.12-13]; and, Müller has termed "simple" those mixtures for which these variables are not taken into consideration. Analogous to the definition of non-simple single fluid continua given by Liu [1973, 6, p.27], I define non-simple fluid mixtures to be those materials having \bar{p}_a and $\rho_{a,k}$ as independent field variables. Exclusion of the former from (2) thus restricts this study to a special class of said mixtures. Further, I note that a desire to examine a model of reasonable simplicity has prompted the adoption of a less general velocity and temperature field dependence of the material response of the mixture (and hence the constituents) than was the case respectively in Müller [1968, 4] and Liu & Müller [1972, 5]. Specifically, the absence of the field variables $v_{a,k,j}$ will preclude dilatational and shear viscosity effects in this theory; and, exclusion of $\partial T/\partial t$ (or more essentially the objective variable \dot{T}) may be expected to have consequences in any consideration of, e.g., the speed of thermal waves in such a mixture as is treated here (cf. Liu & Müller [1972, 5, p.168]). It will become clear in the sequel that a generalization of this theory can be effected by, for example, including these neglected variables as well as possibly others in (2). I eschew from commenting upon the possible outcome of such a step.

⁸ Regarding these assumptions see, for example, CFT [1960, 9, p.700 and Sects. 2 and 196] and Truesdell [1966, 14, p.25].

- ⁹ CFT [1960, 9, p.453] and Truesdell [1966, 14, p.22]. It may be noted further that there would be no difference in the results here if the development involved a Galilean transformation (cf. CFT [1960, 9, p. 453]) instead of a Euclidean transformation. This point was made also by Liu & Müller [1972, 5, p.151, footnote]).
- ¹⁰ See, for example, Müller [1973, 9, p.17].
- ¹¹ Müller [1968, 4, p.12].
- ¹² For V_k the result is given by, e.g. Müller [1973, 9, p.165] while the result for the electromotive intensity follows an analogous development as given by Müller [ibid, pp. 118-119] for a single-continuum. The selection of a particular constituent as defining a reference fluid (in this case) is not uncommon; examples being found, among others, in Ingram & Eringen [1967, 10, p.293] and Müller [1968, 4, p.10].
- ¹³ Cf. Truesdell [1966, 13, pp.42-43 and p.45], CFT [1960, 9, pp.703-704] (which states said principle in an older form) and Jaunzemis [1967, 12, p.285]. The applicability of this principle to the study of mixtures was made clear by Müller [1968, 4, p.9]. I note that the substance, as well as the framing, of this (or a like) principle is still a matter of discussion at the present time. In his commentary hereon, Truesdell [1969, 7, p.84, footnote] has anticipated some of the criticism raised later by, e.g., Rivlin [1970, 6, pp.133-134] and Rivlin [1972, 7]. Although the principle of equipresence in its present form may suffer from ambiguity in one or more respects (Green & Laws [1967, 8, pp.274-275]), it has been shown by Fisher & Leitman [1970, 3, p.309] in a particular case that a violation of this

principle leads to an aphysical theory. The meaningfulness of the equipresence principle is, in addition, borne out by the stability considerations of Ziegler [1971, 6]. Bressan [1972, 2, p.6] has examined this point and, in strengthening this principle, has introduced the 'principle of strong equipresence'.

¹⁴ See, for example, Müller [1968, 4, p.9], Liu & Müller [1972, 5, p. 151] and Demiray & Eringen [1973, 2, p.893].

¹⁵ I follow Müller [1968, 4, p.9] in taking this step.

¹⁶ This principle is discussed by numerous writers on (aspects of) rational mechanics. References include, among other possibilities, the following: Noll [1958, 3], Noll [1959, 1, pp.280-281], CFT [1960, 9, pp.702-703], Truesdell [1966, 13, pp.5-6] and Müller [1973, 9, p.45, footnote]. The principle of material frame-indifference is, similar to that of equipresence, a matter still open to discussion.

Rivlin [1970, 6, p.121] has given an example of the difficulty that may arise through the use of this principle; while, on the basis of particular kinetic theory arguments first Müller [1972, 6] and then Edelen & McLennan [1973,4] have found grounds which may necessitate a reframing, or qualification, of the present principle of material frame-indifference.

It should be noted that in general one may treat not only absolute, but also relative, from-invariant function(al)s; this matter having been taken up by, e.g., Pipkin & Rivlin [1968, 5]. Here, I consider only the case of absolute invariants.

¹⁷ Pipkin & Rivlin [1960, 7, p.543]. Material systems are defined in rational mechanics with the postulation of specific constitutive

- equations. A subsequent classification of these systems can be made in terms of their respective symmetry properties. The investigation of these properties relates to the so-called isotropy group of a particular material system; a discussion of which may be found in CFT [1960, 9, p.701], Truesdell & Noll [1965, 19, Sect. 31], Truesdell [1966, 13, pp.9-14] and Truesdell [1966, 14, pp.56-64].
- ¹⁸ Pipkin & Rivlin [1959, 2]. This presentation parallels that of Spencer [1971, 5].
- ¹⁹ Green & Adkins [1960, 6, p.8, footnote 5].
- ²⁰ Cf. Pipkin & Winemann [1963, 8]. Note also the remarks of Truesdell & Noll [1965, 19, p.61, footnote 1] and Pipkin & Rivlin [1968, 5, p. 5, footnote (3)].
- ²¹ Cf. Pipkin & Rivlin [1960, 7, p.543, footnote 1].
- ²² Note in this regard Cauchy's theorem. An exposition hereof may be found in, e.g., Truesdell & Noll [1965, 19, Sect. 11].
- ²³ This may be seen, for example, in an often referred to paper by Smith [1965, 16].
- ²⁴ Pipkin & Rivlin [1968, 5, p.4, footnote (1)] point out the meaning of irreducibility as it is taken in the context of rational mechanics.
- ²⁵ For this relation as an expression in mathematics, as well as an element of a rational mechanical constitutive theory see, e.g., Pipkin & Rivlin [1959, 2, p.138].
- ²⁶ Wineman & Pipkin [1964, 7, p.186]. With the functions involved being viewed as polynomials, the operation of differentiation may, as noted by Spencer [1971, 5, p.259, footnote], be considered as an algebraic process.

- 27 Cf. Liu & Müller [1972, 5, p.162] and Müller [1973, 9, p.134].
- 28 In neither of the papers given as references in footnote 27 was this necessary (cf. Pipkin & Rivlin [1959, 2, p.138]) distinction made. The consequence of this oversight is that some of the material coefficients in the various constitutive equation representations given are looked upon as being somewhat more general than they are.
- 29 Such a term, although not given, is implicit in the paper of Müller [1968, 4]; but, due to the single-fluid nature of their study, Liu & Müller [1972, 5] cannot have such an expression.
- 30 This result "generalizes" that one which is implicit in the paper by Müller [1968, 4] and the single-fluid expression given by Liu & Müller [1972, 5, p.162, (5.2)₁].
- 31 Cf. Liu & Müller [1972, 5, p.162, (5.2)₂] and note that the definition (843) of polarization of the mixture relates this "more" general result to that of their single-fluid.
- 32 Cf. Liu & Müller [1972, 5, p.162, (5.3)] where, similar to the case of polarization, the definition (859) of the magnetization of the mixture yields a result directly comparable with that indicated above. Clearly, the absence of electromagnetic field considerations in the study by Müller precludes such a result as this, or that for the polarization.
- 33 It may be mentioned that an alternative procedure whereby the absolute scalar invariant is given by $S_T = A_{ij}T_{ij}$, with A_{ij} being an arbitrary tensor, can on occasion be used to advantage. See in this regard, e.g., Adkins [1960, 1]. The results of both approaches under general conditions are, however, according to Smith (private communi-

cation) equivalent.

- 34 This expression, while not stated in the study of Müller [1968, 4] is, however, given in a "less" general form by Liu & Müller [1972, 5, p.163, (5.4)]. It is, as a constituent relation, related to this latter single-fluid result through the definition (B87) of the stress tensor of the mixture.
- 35 In terms of the constitutive equations being considered as polynomials this means (cf. Pipkin & Rivlin [1960, 7, p.544]) that 2nd- and higher-order terms in the non-linear constitutive relations above which contain these field parameters are neglected. The order of approximation made, as well as the variables for which that operation is carried out, is a matter dictated by the nature of any given investigation. In the interest of securing tractable expressions in general, and examining the specific influence of a \vec{B} field upon the material response of fluid mixtures in particular, I have made the linearization and variable choices indicated. In a problem with an electrohydrodynamic (EHD) character the role of the electric field would be studied. For the more general case of magneto-electric (or bianisotropic) fluid mixtures, see the considerations of Benach & Müller [1974,1]. Lastly, I remark that in the development which follows, a superposed letter L denotes a linearized parameter.
- 36 Cf. Müller [1968, 4, pp.21-22] and Liu & Müller [1972, 5, p.165].
- 37 For the case of a binary mixture, note the result given by Müller [1968, 4, p.22]; and, for an s constituent mixture see Müller [1973, 9, p.172]. Observe also the relation of the conditions (29)₁₋₄ to those in the last cited reference.

³⁸ Cf. Pipkin & Rivlin [1960, 7, p.545], Müller [1968, 4, p.21] and Liu & Müller [1972, 5, p.164, (5.15)₂].

³⁹ The literature on the Dufour (or diffusion thermo) effect is somewhat unclear in expressing a unique meaning as to what in fact this effect is. Hirschfelder, Curtiss & Bird [1967, 9] state (p.8, footnote) that this effect constitutes the flux of energy resulting from concentration gradients; while later (pp.521-522) they mention this effect in terms of an expression wherein it relates to the constituent velocity differences. Although this is in agreement with the results of Chapman & Cowling [1970, 2, pp.142-143], it is not consistent with their first statement regarding this effect. Further yet, Slattery [1972, 8, p.472] in echoing Hirschfelder, Curtiss & Bird extends the meaning they gave to this effect so as to include its dependence also upon pressure gradients. Haase [1969, 2, p.359], on the other hand, subscribes to the opinion that this effect is defined in terms of mass fraction gradients; calling in essence the pressure gradient dependence of the internal energy flux the "pressure thermal" term.

Inasmuch as an expression for the difference of constituent velocities can depend on numerous terms, each of which may play a role in "driving" the relative diffusion(s) of concern, and of which the concentration gradients are only a single possible cause, I wish to suggest the following step towards clarification. It would be worthwhile to retain the definition of the Dufour effect as deriving from concentration gradients. The coefficient for this effect would then be a combination here, for example, of $\langle v_{ik}^p \rangle$ and that term, including $\langle v_{ik}^v \rangle$, expressing the dependence of v_{ik}^v on said gradients.

In this study, however, I follow common practice (cf. Müller [1973, 9, p.172] with regard to a rational mechanics development) and make the identification given.

⁴⁰ Cf., for example, Cambel [1963, 2, p.210] and Haase [1969, 2, p.464].

⁴¹ Haase [1969, 2, p.333, footnote].

⁴² Pipkin & Rivlin [1960, 7, p.545], Page & Adams [1965, 12, p.256] and Chapman & Cowling [1970, 2, p.369].

⁴³ Pipkin & Rivlin [1960, 7, p.545] and Page & Adams [1965, 12, p.256].

⁴⁴ Note here, in addition to the comments of Truesdell [1966, 14, pp. 3-4], the remarks of footnote 7 above.

⁴⁵ Cf. Liu & Müller [1972, 5, p.164, (5.15)₂].

⁴⁶ Benach & Müller [1974, 1, (4.9)₁].

⁴⁷ Cf. Liu & Müller [1972, 5, p.165, (5.17)] and Müller [1973, 9, p. 145, (104)₂].

⁴⁸ Note Liu & Müller [1972, 5, p.165] and Benach & Müller [1974, 1, (4.9)₂].

⁴⁹ Cf. Liu & Müller [1972, 5, p.165, (5.16)] with regard to the stress tensor $(33)_1$; and observe, that this relation like the other given above have material response coefficients which are less general (in terms of the temperature field) than those given in the reference cited.

D. A MATTER OF PRINCIPLE: ENTROPY

This chapter has the task of setting down a two-part principle of entropy for volumes comprising sets of regular and singular points ¹; and further, of establishing the direct results here (i.e. restrictions on the constitutive equations) of that principle.

D.1. Entropy principlesRegular volumes

To begin, (B152)₂ is eliminated between (B134) and (B151). Use now of (B15), and (B85)₁ to eliminate the body forces in this last result, gives the expression

$$\begin{aligned}
 & - \rho(\dot{\epsilon} + T\dot{\eta}) + \sum_{a=1}^s \rho_{aa} \dot{u}_i u_i - \left[\sum_{a=1}^s (t_{aij} + c_{aij}) u_i \right]_{,j} - \\
 & - \frac{1}{T} \left(q_i + \sum_{a=1}^s \epsilon_{ijk} \delta_{aj} M_k - \sum_{a=1}^s c_{aj} i u_j \right)_{T,i} - \sum_{a=1}^s (\hat{m}_i - \hat{c} v_i) u_i + \\
 & + \left(t_{ij} - \sum_{\gamma=1}^s t_{\gamma ij} \right) \frac{1}{\rho} \sum_{a=1}^s u_i \rho_{,j} + \\
 & + \sum_{a=1}^s \left[t_{aij} + \frac{\rho u_i u_j}{aa} + (t_{aij} - \sum_{\gamma=1}^s t_{\gamma ij}) \frac{\rho}{a} \right] d_{aij} + \\
 & + \sum_{a=1}^s t_{aij} w_{aj} + \left(\sum_{a=1}^s \epsilon_{ijk} \delta_{aj} M_k \right)_{,i} + \sum_{a=1}^s \gamma_i \delta_i \geq 0. \quad (1)
 \end{aligned}$$

Here, w_{aij} ($= v_{a[i,j]}$) is the spin tensor of constituent a . With ρb_i and (B152)₂ presumed known ², a thermodynamic process in V is defined to be a

solution for the fields (A1) of the equations (B158) to (B161) and (B163) to (B166) taken together with the constitutive equations (C7).

The entropy principle in V assumed here may be stated as follows:

For all thermodynamic processes in V the entropy inequality (D1), or equivalently (B151), must be valid.

Singular surfaces

A surface of discontinuity $s(t)$, here "plane", which permits the passage of a single mixture constituent is called a semipermeable wall. In terms of such walls the second part of the entropy principle, i.e. that on $s(t)$, is assumed:

For semipermeable walls in parallel motion, across which the temperature field is continuous, for which the densities of mixture entropy supply and production remain finite, and at which the tangential velocity components of all constituents vanish, the normal component of the entropy flux is assumed to be continuous; that is (cf. (B157)),

$$T[[\eta]]\rho(v_i - u_i)\hat{n}_i + [[q_i\hat{n}_i]] + \sum_{a=1}^s \epsilon_{ijk} \xi_{aj} M_{ak} \hat{n}_i - \sum_{a=1}^s C_{aji} u_j \hat{n}_i = 0. \quad (2)$$

It is desirable with a view towards the forthcoming developments to record an equivalent expression of (2) which will be more useful than that relation. The first step involved is the determination of $[[q_i\hat{n}_i]]$ from (B169) and (B170), with $\bar{e}^{SM} = \hat{m}_i^{SM} u_i$,⁴ followed by its substitution in (2). Performance of these operations leads to the expression

$$\begin{aligned}
& T[\rho](v_1 - u_1)\hat{n}_1 - [\epsilon + \frac{1}{2}v^2 - v_i u_i]\rho(v_1 - u_1)\hat{n}_1 + \\
& + [t_{ki}(v_k - u_k)\hat{n}_i] + \frac{[\epsilon]u_n}{E} - [G_i u_i]u_n - \frac{[S_i \hat{n}_i]}{E} - \\
& - \frac{[t_{ij} u_i \hat{n}_j]}{E} + \left[\sum_{a=1}^s \epsilon_{ijk} \delta_{a^j}^{M_a^k} \hat{n}_i \right] - \left[\sum_{a=1}^s c_{aki} u_a^k \hat{n}_i \right] = 0. \quad (3)
\end{aligned}$$

In order to further rewrite this relationship, I now make use of the following expressions:

$$\begin{aligned}
& - [\epsilon + \frac{1}{2}v^2 - v_i u_i]\rho(v_1 - u_1)\hat{n}_1 = \\
& = - [\epsilon + \frac{1}{2}(v_i - u_i)(v_i + u_i)]\rho(v_1 - u_1)\hat{n}_1, \quad (4)
\end{aligned}$$

$$[t_{ki}(v_k - u_k)\hat{n}_i] = \left[\frac{1}{\rho} t_{ki} \hat{n}_i \hat{n}_k \right] \rho(v_1 - u_1)\hat{n}_1, \quad (5)$$

$$\left[\sum_{a=1}^s \epsilon_{ijk} \delta_{a^j}^{M_a^k} \hat{n}_i \right] - \frac{[S_i \hat{n}_i]}{E} = \epsilon_{ijk} \left[\sum_{a=1}^s \delta_{a^j}^{M_a^k} - E_j H_k \right] \hat{n}_i, \quad (6)$$

$$- [G_i u_i]u_n = - [\epsilon_{ijk} D_j B_k] \hat{n}_i u_n^2, \quad (7)$$

and

$$\frac{[\epsilon]u_n}{E} - \frac{[t_{ij} u_i \hat{n}_j]}{E} = - [E_i D_j + B_i H_j] (\hat{n}_i \hat{n}_j - \delta_{ij}) u_n. \quad (8)$$

The last three results given above may, however, be yet further developed.

In doing so they become, respectively,

$$\begin{aligned}
& \epsilon_{ijk} \left[\sum_{a=1}^s \delta_{a^j}^{M_a^k} - E_j H_k \right] \hat{n}_i = \\
& = [(-\delta_k^p j + B_k^M j)(\hat{n}_j \hat{n}_k - \delta_{jk}) v_i \hat{n}_i] - [(\vec{v} \times \vec{B})_k^p v_i \hat{n}_i] -
\end{aligned}$$

$$\begin{aligned}
& - \epsilon_{ijk} \llbracket E_j \sum_{a=1}^s (\vec{p}_a \times \vec{u})_k \rrbracket \hat{n}_i - \epsilon_{ijk} \llbracket \sum_{a=1}^s (\vec{u} \times \vec{b})_j M_a^k \rrbracket \hat{n}_i - \\
& - \epsilon_{ijk} \bar{E}_j h_k \hat{n}_i u_n + \epsilon_{ijk} k_j \left[H_k - M_k - \sum_{a=1}^s (\vec{p}_a \times \vec{v})_a^k \right] \hat{n}_i u_n, \quad (9)
\end{aligned}$$

$$\begin{aligned}
& - \epsilon_{ijk} \llbracket D_j B_k \rrbracket \hat{n}_i u_n^2 = - \llbracket (\bar{D}_i + \bar{P}_i) k_j - \bar{B}_i h_j \rrbracket (\hat{n}_i \hat{n}_j - \delta_{ij}) u_n^2 + \\
& + \llbracket (\vec{p} \times \vec{b})_i \rrbracket \hat{n}_i u_n^2, \quad (10)
\end{aligned}$$

and

$$\begin{aligned}
& - \llbracket E_i D_j + B_i H_j \rrbracket (\hat{n}_i \hat{n}_j - \delta_{ij}) u_n = \\
& = - \llbracket (P_j + B_i M_j) \rrbracket (\hat{n}_i \hat{n}_j - \delta_{ij}) u_n - \\
& - \llbracket (\vec{v} \times \vec{b})_i P_j + B_i \sum_{a=1}^s (\vec{p}_a \times \vec{v})_j \rrbracket (\hat{n}_i \hat{n}_j - \delta_{ij}) u_n + \\
& + \llbracket (\bar{D}_i + \bar{P}_i) k_j - \bar{B}_i h_j \rrbracket (\hat{n}_i \hat{n}_j - \delta_{ij}) u_n^2 + \\
& + \epsilon_{klj} \bar{E}_k h_l \hat{n}_j u_n + \left[H_l - M_l - \sum_{a=1}^s (\vec{p}_a \times \vec{v})_a^l \right] \epsilon_{lkj} k_k \hat{n}_j u_n. \quad (11)
\end{aligned}$$

Here, use has been made of the relations for the average value of a field⁵ and of a product of fields which "suffer" a jump; these being given here, respectively, for arbitrary fields A and B by

$$\bar{A} = \frac{1}{2}(A^+ + A^-) \quad \text{and} \quad \llbracket AB \rrbracket = \bar{A} \llbracket B \rrbracket + \llbracket A \rrbracket \bar{B}. \quad (12)$$

Substitution of (4), (5), (9), (10) and (11) into (3) now yields a jump balance equation of mixture entropy of the form

$$\begin{aligned}
& \llbracket T\eta - \varepsilon - \frac{1}{2}(v_i - u_i)(v_i - u_i) + \frac{1}{\rho} t_{ki} \hat{n}_i \hat{n}_k + \\
& + (-\varepsilon_k \frac{P_j}{\rho} + B_k \frac{M_j}{\rho})(\hat{n}_j \hat{n}_k - \delta_{jk}) \rrbracket \rho (v_l - u_l) \hat{n}_l - \\
& - \llbracket \sum_{a=1}^s C_{aki} u_a \rrbracket \hat{n}_i - \varepsilon_{ijk} \llbracket E_j \sum_{a=1}^s (\vec{p}_a \times \vec{u}_a)_k \rrbracket \hat{n}_i + \\
& + \varepsilon_{ijk} \llbracket \sum_{a=1}^s (\vec{u}_a \times \vec{B})_j M_{ak} \rrbracket \hat{n}_i - \llbracket (\vec{v} \times \vec{B})_k P_k v_i \hat{n}_i \rrbracket + \\
& + \llbracket (\vec{p} \times \vec{B})_i \rrbracket \hat{n}_i u_n^2 - \\
& - \llbracket (\vec{v} \times \vec{B})_i P_j + B_i \sum_{a=1}^s (\vec{p}_a \times \vec{v})_j \rrbracket (\hat{n}_i \hat{n}_j - \delta_{ij}) u_n = 0. \quad (13)
\end{aligned}$$

This is the expression for the jump balance equation which derives from the second part of the entropy principle stated above. It is to this relationship that I shall return (cf. Chpt. E) after I have first examined more closely the entropy principle in V and its foremost consequences here; this I do now.

D.2. Admissible thermodynamic processes in V

The entropy inequality (1) is now first rewritten by means of the introduction of the appropriate constitutive equations (C7), followed by the performing of the indicated differentiations and a regrouping of terms. The result of carrying out this operation is given by

$$\begin{aligned}
& - \rho \sum_{b=1}^s \left(\frac{\partial c}{\partial \rho} - T \frac{\partial n}{\partial \rho} - \frac{1}{\rho} \sum_{a=1}^s \xi_i \frac{\partial \rho_i}{\partial \rho} \right) \rho_b^* - \rho \sum_{b=1}^s \left(\frac{\partial c}{\partial \rho} - T \frac{\partial n}{\partial \rho} - \frac{1}{\rho} \sum_{a=1}^s \xi_i \frac{\partial \rho_i}{\partial \rho} \right) \rho_{b,k}^* - \\
& - \rho \sum_{a=1}^s \left(\frac{\partial c}{\partial v_a} - T \frac{\partial n}{\partial v_a} - \frac{\rho}{a} u_k - \frac{1}{\rho} \sum_{b=1}^s \xi_i \frac{\partial \rho_i}{\partial v_a} \right) v_{a,k}^* - \rho \left(\frac{\partial c}{\partial T} - T \frac{\partial n}{\partial T} - \frac{1}{\rho} \sum_{a=1}^s \xi_i \frac{\partial \rho_i}{\partial T} \right) T^* - \\
& - \rho \left(\frac{\partial c}{\partial T} - T \frac{\partial n}{\partial T} - \frac{1}{\rho} \sum_{a=1}^s \xi_i \frac{\partial \rho_i}{\partial T} \right) T_{,k}^* - \rho \left(\frac{\partial c}{\partial B_k} - T \frac{\partial n}{\partial B_k} - \frac{1}{\rho} \sum_{a=1}^s \xi_i \frac{\partial \rho_i}{\partial B_k} \right) B_{k,s}^* - \rho \left(\frac{\partial c}{\partial s_k} - T \frac{\partial n}{\partial s_k} - \frac{1}{\rho} \sum_{a=1}^s \xi_i \frac{\partial \rho_i}{\partial s_k} \right) s_{k,s}^* - \\
& - \frac{1}{T} \left[q_i - \sum_{a=1}^s C_{aj} u_{aj} + \sum_{a=1}^s \epsilon_{ijk} C_{aj} M_k + T \frac{\partial}{\partial T} \sum_{a=1}^s u_j (t_{aj} + C_{aj}) - T \sum_{a=1}^s u_i \xi_j \frac{\partial \rho_j}{\partial T} \right] T_{,i} - \\
& - \sum_{a=1}^s \left(\bar{m}_i - \bar{c}_{vi} \right) u_i - \sum_{b=1}^s \left[\frac{\partial}{\partial \rho} \sum_{a=1}^s u_i (t_{ij} + C_{ij}) - \frac{1}{\rho} (t_{ij} - \sum_{\gamma=1}^s t_{j\gamma}) u_{bi} - \sum_{a=1}^s u_j \xi_a \frac{\partial \rho_i}{\partial \rho} \right] \rho_{b,j} - \\
& - \left[\frac{\partial}{\partial T} \sum_{a=1}^s u_i (t_{ij} + C_{ij}) - \sum_{a=1}^s u_j \xi_i \frac{\partial \rho_i}{\partial T} \right] T_{,kj} - \sum_{b=1}^s \left[\frac{\partial}{\partial \rho} \sum_{a=1}^s u_i (t_{ij} + C_{ij}) - \sum_{a=1}^s u_j \xi_a \frac{\partial \rho_i}{\partial \rho} \right] \rho_{b,kj} + \\
& + \sum_{a=1}^{s-1} \left[t_{ij} + \rho u_i u_j + \frac{\rho}{a} (t_{ij} - \sum_{\gamma=1}^s t_{j\gamma}) - \frac{\partial}{\partial v_j} \sum_{b=1}^s u_k (t_{ki} + C_{ki}) + \right. \\
& \quad \left. + \sum_{b=1}^s u_i b_r \frac{\partial \rho_r}{\partial v_j} - (C_{iaj} p_j - \delta_{ij} \vec{c} \cdot \vec{p}) + (B_{iaj} M_j - \delta_{ij} \vec{B} \cdot \vec{M}) \right] d_{aj} + \\
& + \left[t_{ij} + \rho u_i u_j + \frac{\rho}{s} (t_{ij} - \sum_{\gamma=1}^s t_{j\gamma}) + \sum_{a=1}^{s-1} \frac{\partial}{\partial v_j} \sum_{b=1}^s u_k (t_{ki} + C_{ki}) - \right. \\
& \quad \left. - \sum_{b=1}^{s-1} \sum_{a=1}^s u_i a_r \frac{\partial \rho_r}{\partial v_j} - (C_{isj} p_j - \delta_{ij} \vec{c} \cdot \vec{p}) - \sum_{a=1}^{s-1} (B_{iaj} M_j - \delta_{ij} \vec{B} \cdot \vec{M}) \right] d_{sj} + \\
& + \sum_{a=1}^{s-1} \left[t_{ij} + \frac{\partial}{\partial v_j} \sum_{b=1}^s u_k (t_{ki} + C_{ki}) + \sum_{b=1}^s u_i \xi_b \frac{\partial \rho_r}{\partial v_j} - B_{iaj} M_j \right] (w_{ij} - w_{sj}) - \\
& - \sum_{a=1}^s \left[\frac{\partial u_i (t_{ij} + C_{ij})}{\partial B_k} - u_j \xi_i \frac{\partial \rho_i}{\partial B_k} + M_{aj} v_{aj} \right] B_{k,j} - \\
& - \sum_{a=1}^s \left[\frac{\partial u_i (t_{ij} + C_{ij})}{\partial s_k} - u_j \xi_i \frac{\partial \rho_i}{\partial s_k} - \epsilon_{ijk} M_j \right] s_{k,j} - \\
& - \sum_{a=1}^s C_{iaj} p_{aj} w_{ij} \geq 0.
\end{aligned}$$

The establishment in any given physical context of "appropriately" invariant and restricted constitutive equations is presumed here to be a main purpose of a constitutive theory. Regarding the first point, the principle of material objectivity has been employed (cf. Chpt. C) in the treatment of the invariance considerations here. For the second point it may be noted that the class of all possible thermodynamic processes of this study has been restricted by the entropy principle in V assumed above. It thus remains to examine its restrictive nature with regard to the constitutive equations (C7); which step I now take.

Elimination of restricted fields

To this end I now introduce a principle of consistency⁶ which states:

Constitutive equations proposed for a given system must be consistent with the balance equations of mechanics, electrodynamics and energetics taken for that system.

Here this means that only the equations (B158) and (B163) to (B166), or their equivalents, serve to constrain the fields (A1). This so because⁷ regardless of the prevailing conditions of the fields (A1), the supply terms $(B82)_1$ and $(B152)_2$ are considered to be determined by them and the balance equations in which they appear (i.e. $(B85)_1$ and (B134)) are satisfied.

I begin first by taking account, in a manner similar to that of Liu and Müller⁸, of the constraints on the fields (A1) due to the relations (B163) to (B166); that being the employment of the Lagrange multiplier method of Liu⁹ in the further evaluation of the entropy principle in V.

Using (B162) and the equivalent electrodynamic field equations (B78), (B75), (B69) and B(49), the condition to be satisfied may be written as

$$\begin{aligned}
 & \rho \dot{n} + \left[\frac{1}{T} (q_i + \sum_{a=1}^s \epsilon_{ijk} \xi_a^j M_a^k - \sum_{a=1}^s c_{aj} i_a^u) \right]_{,i} - \frac{1}{T} \rho r - \\
 & - \Lambda^{B_k} \{ \dot{B}_k + u_j B_{k,j} - (B_j \delta_{lk} - B_k \delta_{lj}) v_{l,j} + \epsilon_{kjl} \xi_{l,j} \} - \\
 & - \Lambda^{B_4} \{ B_{k,k} \} - \\
 & - \Lambda^{\xi_k} \left\{ \dot{\xi}_k + \frac{1}{\epsilon_0} \dot{p}_k - \epsilon_{kji} v_j B_{i,t} - \epsilon_{kji} B_i v_{j,t} - \right. \\
 & \quad \left. - v_j \xi_{k,j} - \frac{1}{\epsilon_0 \mu_0} \epsilon_{kji} B_{i,j} + \frac{1}{\epsilon_0} \left(\sum_{a=1}^s q_a - P_{j,j} \right) v_k + \right. \\
 & \quad \left. + \frac{1}{\epsilon_0} \sum_{a=1}^s q_a u_a^k + \epsilon_{kpq} \left(\frac{\vec{p}}{a} \times \frac{\vec{u}}{a} \right)_{q,p} + \right. \\
 & \quad \left. + \frac{1}{\epsilon_0} (\epsilon_{kjn} M_{n,j} - P_j v_{k,j} + P_k v_{j,j}) \right\} - \\
 & - \Lambda^{\xi_4} \left\{ \xi_{j,j} + \epsilon_{ijk} v_a^i B_{k,j} + \epsilon_{ijk} B_k v_{i,j} - \frac{1}{\epsilon_0} \left(\sum_{a=1}^s q_a - P_{j,j} \right) \right\} \\
 & \geq 0 ; \tag{15}
 \end{aligned}$$

where the Lagrange multipliers of Liu Λ^{B_k} , Λ^{B_4} , Λ^{ξ_k} and Λ^{ξ_4} may be functions of the variables (C2). Upon rewriting (15) in terms of (14) followed by a distribution of the Λ -terms throughout, the result that emerges is given by

$$\begin{aligned}
& - \sum_{b=1}^s \rho \left[\frac{\partial \epsilon}{\partial p} - T \frac{\partial \eta}{\partial p} - \frac{1}{\rho} \sum_{a=1}^s (\xi_a^i \frac{\partial p_i}{\partial p} - \Lambda^{sq} \frac{1}{\epsilon_0} \frac{\partial p}{\partial p}) \right]_{,b} - \\
& - \sum_{b=1}^s \rho \left[\frac{\partial \epsilon}{\partial p_{,k}} - T \frac{\partial \eta}{\partial p_{,k}} - \frac{1}{\rho} \sum_{a=1}^s (\xi_a^i \frac{\partial p_i}{\partial p_{,k}} - \Lambda^{sq} \frac{1}{\epsilon_0} \frac{\partial p}{\partial p_{,k}}) \right]_{,b,k} - \\
& - \sum_{a=1}^{s-1} \rho \left[\frac{\partial \epsilon}{\partial v_k} - T \frac{\partial \eta}{\partial v_k} - \frac{\rho}{\rho} u_k - \frac{1}{\rho} \sum_{b=1}^s (\xi_b^i \frac{\partial p_i}{\partial v_k} - \Lambda^{sq} \frac{1}{\epsilon_0} \frac{\partial p}{\partial v_k}) \right]_{,a,k} - \\
& - \rho \left[\frac{\partial \epsilon}{\partial T} - T \frac{\partial \eta}{\partial T} - \frac{1}{\rho} \sum_{a=1}^s (\xi_a^i \frac{\partial p_i}{\partial T} - \Lambda^{sq} \frac{1}{\epsilon_0} \frac{\partial p}{\partial T}) \right]_{,i} - \\
& - \rho \left[\frac{\partial \epsilon}{\partial T_{,k}} - T \frac{\partial \eta}{\partial T_{,k}} - \frac{1}{\rho} \sum_{a=1}^s (\xi_a^i \frac{\partial p_i}{\partial T_{,k}} - \Lambda^{sq} \frac{1}{\epsilon_0} \frac{\partial p}{\partial T_{,k}}) \right]_{,i,k} - \\
& - \rho \left\{ \frac{\partial \epsilon}{\partial B_k} - T \frac{\partial \eta}{\partial B_k} - \frac{1}{\rho} \left[\sum_{a=1}^s (\xi_a^i \frac{\partial p_i}{\partial B_k} + \Lambda^{sq} \frac{1}{\epsilon_0} \frac{\partial p}{\partial B_k}) - \Lambda^{sq} \epsilon_{kqi} v_i - \Lambda^{Bk} \right] \right\}_{,B_k} - \\
& - \rho \left\{ \frac{\partial \epsilon}{\partial \xi_k} - T \frac{\partial \eta}{\partial \xi_k} - \frac{1}{\rho} \left[\sum_{a=1}^s \xi_a^i \frac{\partial p_i}{\partial \xi_k} - \Lambda^{sq} (\delta_{qk} + \sum_{a=1}^s \frac{1}{\epsilon_0} \frac{\partial p}{\partial \xi_k}) \right] \right\}_{,s_k} - \\
& - \frac{1}{T} \left[q_i - \sum_{a=1}^s C_{aj} u_{,j} + \sum_{a=1}^s \epsilon_{ijk} \xi_{,j} u_{,k} + T \frac{\partial}{\partial T} \sum_{a=1}^s u_j (t_{,ji} + C_{,ji}) - \right. \\
& \quad \left. - T \sum_{a=1}^s \left\{ u_i \xi_{,j} \frac{\partial p_j}{\partial T} - \frac{1}{\epsilon_0} \left[\Lambda^{sq} \epsilon_{kin} \frac{\partial \eta}{\partial T} - v_k \frac{\partial p_i}{\partial T} + (u_i \delta_{ik} - u_k \delta_{ij}) \frac{\partial p_l}{\partial T} - \Lambda^{sq} \frac{\partial p}{\partial T} \right] \right\} \right]_{,i} - \\
& - \sum_{a=1}^s (\hat{m}_i - \bar{\epsilon} v_i) u_{,i} - \\
& - \sum_{b=1}^s \left[\frac{\partial}{\partial p} u_i (t_{,ij} + C_{,ij}) - \frac{1}{\rho} (t_{,ij} - \sum_{\gamma=1}^s t_{,ij}) u_{,i} - \right. \\
& \quad \left. - \sum_{a=1}^s \left(u_i \xi_{,j} \frac{\partial p_j}{\partial p} - \frac{1}{\epsilon_0} \left\{ \Lambda^{sq} \epsilon_{kjn} \frac{\partial \eta}{\partial p} - v_k \frac{\partial p_j}{\partial p} + \frac{1}{\rho} (p_j \delta_{ik} - p_k \delta_{ij}) u_i + (u_j \delta_{ik} - u_k \delta_{ij}) \frac{\partial p_l}{\partial p} + \Lambda^{sq} \frac{\partial p}{\partial p} \right\} \right) \right]_{,b,j} - \\
& - \left(\frac{\partial}{\partial T_{,k}} \sum_{a=1}^s u_i (t_{,ij} + C_{,ij}) - \sum_{a=1}^s u_j \xi_{,i} \frac{\partial p_j}{\partial T_{,k}} + \sum_{a=1}^s \frac{1}{\epsilon_0} \left\{ \Lambda^{sq} \epsilon_{qjn} \frac{\partial \eta}{\partial T_{,k}} - v_q \frac{\partial p_j}{\partial T_{,k}} + \right. \right. \\
& \quad \left. \left. + (u_j \delta_{rq} - u_q \delta_{rj}) \frac{\partial p}{\partial T_{,k}} + \Lambda^{sq} \frac{\partial p}{\partial T_{,k}} \right\} \right)_{,k,j} -
\end{aligned}$$

$$\begin{aligned}
& - \sum_{b=1}^s \left[\frac{\partial}{\partial p_{b,k}} \sum_{a=1}^s u_i(t_{ij} + c_{ij}) - \sum_{a=1}^s \left(u_j \xi_i \frac{\partial p_i}{\partial p_{b,k}} - \frac{1}{\epsilon_0} \left\{ \Lambda^s c_{qj} n \frac{\partial M_n}{\partial p_{b,k}} - v_q \frac{\partial p_j}{\partial p_{b,k}} + (u_j \delta_{rq} - u_q \delta_{rj}) \frac{\partial p_r}{\partial p_{b,k}} \right. \right. \right. \\
& \quad \left. \left. \left. - \Lambda^s \frac{\partial p_j}{\partial p_{b,k}} \right\} \right]_{p_{b,k,j}} + \\
& + \sum_{a=1}^{s-1} \left[t_{ij} + \rho u_i u_j + \frac{\rho}{\rho} (t_{ij} - \sum_{\gamma=1}^s t_{i\gamma}) - \frac{\partial}{\partial v_j} \sum_{b=1}^s u_k (t_{bki} + c_{bki}) - (\xi_i p_j - \delta_{ij} \vec{\xi} \cdot \vec{p}) + (B_i M_j - \delta_{ij} \vec{B} \cdot \vec{M}) + \right. \\
& \quad \left. + \sum_{b=1}^s \left(u_i \xi_r \frac{\partial p_r}{\partial v_j} - \frac{1}{\epsilon_0} \left\{ \Lambda^s c_{kj} n \frac{\partial M_n}{\partial v_j} - v_k \frac{\partial p_i}{\partial v_j} - \frac{\rho}{\rho} (p_j \delta_{ik} - p_i \delta_{jk}) + (u_j \delta_{vk} - u_k \delta_{vj}) \frac{\partial p_r}{\partial v_j} + \Lambda^s \frac{\partial p_b}{\partial v_j} \right\} \right) \right]_{d_{ij}} + \\
& + \left[t_{ij} + \rho u_i u_j + \frac{\rho}{\rho} (t_{ij} - \sum_{\gamma=1}^s t_{i\gamma}) + \sum_{a=1}^{s-1} \frac{\partial}{\partial v_j} \sum_{b=1}^s u_k (t_{bki} + c_{bki}) - (\xi_i p_j - \delta_{ij} \vec{\xi} \cdot \vec{p}) - \sum_{a=1}^{s-1} (B_i M_j - \delta_{ij} \vec{B} \cdot \vec{M}) - \right. \\
& \quad \left. - \sum_{a=1}^{s-1} \sum_{b=1}^s \left(u_i \xi_r \frac{\partial p_r}{\partial v_j} - \frac{1}{\epsilon_0} \left\{ \Lambda^s c_{kj} n \frac{\partial M_n}{\partial v_j} - v_k \frac{\partial p_i}{\partial v_j} - \frac{\rho}{\rho} (p_j \delta_{ik} - p_i \delta_{jk}) - (u_j \delta_{rk} - u_k \delta_{rj}) \frac{\partial p_r}{\partial v_j} - \Lambda^s \frac{\partial p_b}{\partial v_j} \right\} \right) \right. \\
& \quad \left. - \Lambda^s c_{ij} n B_k + \Lambda^s (B_i \delta_{jk} - B_k \delta_{ji}) \right]_{s_{ij}} + \\
& + \sum_{a=1}^{s-1} \left[t_{ij} + \frac{\partial}{\partial v_j} \sum_{b=1}^s u_k (t_{bki} + c_{bki}) - B_i M_j + \sum_{b=1}^s \left(u_i \xi_r \frac{\partial p_r}{\partial v_j} + \frac{1}{\epsilon_0} \left\{ \Lambda^s c_{kj} n \frac{\partial M_n}{\partial v_j} - v_k \frac{\partial p_i}{\partial v_j} + (u_j \delta_{rk} - u_k \delta_{rj}) \frac{\partial p_r}{\partial v_j} \right. \right. \right. \\
& \quad \left. \left. \left. + \Lambda^s \frac{\partial p_b}{\partial v_j} \right\} \right) \right] (w_{ij} - w_{ij}) - \\
& - \left[\sum_{a=1}^s \left[\frac{\partial u_i (t_{ij} + c_{ij})}{\partial B_k} - u_i \xi_i \frac{\partial p_i}{\partial B_k} + M_k v_j \right] + \Lambda^s \left[\sum_{a=1}^s \frac{1}{\epsilon_0} c_{qj} n \frac{\partial M_n}{\partial B_k} - v_q \frac{\partial p_j}{\partial B_k} + (u_j \delta_{rq} - u_q \delta_{rj}) \frac{\partial p_r}{\partial B_k} + \right. \right. \\
& \quad \left. \left. + c_{qjk} v_j - \frac{1}{\epsilon_0} c_{qjk} \right\} + \Lambda^s (c_{ijk} v_i + \sum_{a=1}^s \frac{1}{\epsilon_0} \frac{\partial p_j}{\partial B_k}) + \Lambda^s u_j + \Lambda^s \delta_{kj} \right] B_{k,j} - \\
& - \left[\sum_{a=1}^s \left[\frac{\partial u_i (t_{ij} + c_{ij})}{\partial s_k} - u_i \xi_i \frac{\partial p_i}{\partial s_k} - c_{ijk} M_i + \Lambda^s \frac{1}{\epsilon_0} c_{qj} n \frac{\partial M_n}{\partial s_k} - v_q \frac{\partial p_j}{\partial s_k} - (u_j \delta_{rq} - u_q \delta_{rj}) \frac{\partial p_r}{\partial s_k} \right] + \right. \\
& \quad \left. + \Lambda^s c_{qjk} + \Lambda^s v_j + \Lambda^s (\delta_{jk} + \sum_{a=1}^s \frac{1}{\epsilon_0} \frac{\partial p_j}{\partial s_k}) \right] \xi_{k,j} - \\
& - \sum_{a=1}^s (\xi_i p_j - \delta_{ij} \vec{\xi} \cdot \vec{p}) + \sum_{b=1}^s \Lambda^s \frac{1}{\epsilon_0} \frac{\rho}{\rho} p_j \delta_{ik} w_{ij} + (-\Lambda^s B_i \delta_{jk} - \Lambda^s c_{ijk} B_k) w_{ij} + \Lambda^s \frac{1}{\epsilon_0} Q^F - \\
& - \Lambda^s \left\{ \frac{1}{\epsilon_0} Q^F v_k + \sum_{a=1}^s \frac{1}{\epsilon_0} [\rho u_k + (p_a \delta_{ij} - p_j \delta_{ik}) u_{a,j}] - c_{kji} B_i v_{s,j}, t \right\} \geq 0.
\end{aligned}$$

Anticipating the reasoning noted below with regard to (20), the expressions for the Lagrange multipliers are (cf. Appendix D-1)

$$\Lambda^s_k = 0, \quad (17)_1$$

$$\Lambda^s_4 = 0, \quad (17)_2$$

$$\Lambda^{B_k} = M_k - \frac{1}{2} \epsilon_{kr1} \sum_{a=1}^s \left[\frac{\partial u_i (t_i[r] + C_i[r])}{\partial \delta_{s1}} - u_{[r]a} \delta_i \frac{\partial P_i}{\partial \delta_{s1}} \right] \quad (17)_3$$

and

$$\begin{aligned} \Lambda^{B_4} = & -\frac{1}{3} \sum_{a=1}^s \left[\frac{\partial u_i (t_{ij} + C_{ij})}{\partial B_j} - u_{aj} \delta_i \frac{\partial P_i}{\partial B_j} + M_j V_j \right] - \frac{1}{3} M_j u_j + \\ & + \frac{1}{6} \epsilon_{jr1} u_j \sum_{a=1}^s \left[\frac{\partial u_i (t_i[r] + C_i[r])}{\partial \delta_{s1}} - u_{[r]a} \delta_i \frac{\partial P_i}{\partial \delta_{s1}} \right]. \quad (17)_4 \end{aligned}$$

For notational simplicity I now let

$$e^l = \epsilon^l - \frac{1}{\rho} \sum_{a=1}^s \vec{\delta}_a \cdot \vec{P}_a. \quad (18)$$

Use of this relation and introduction of the results above, together with (B34)₃ which serves to take care of the constraint condition (B158), into (16) brings forth the following result; viz.,

$$\begin{aligned}
& - \sum_{b=1}^s \rho \left(\frac{\partial e^1}{\partial p_{b,k}} - T \frac{\partial \eta}{\partial p_{b,k}} \right) \dot{p}_{b,k} - \sum_{a=1}^{s-1} \rho \left[\frac{\partial (e^1 - T\eta)}{\partial v_{a,k}} - \frac{\rho}{a} u_{a,k} - \frac{1}{\rho} c_{kim} p_{ij} \dot{m}_{a,k} \right] \dot{v}_{a,k} - \rho \left(\frac{\partial e^1}{\partial T} - T \frac{\partial \eta}{\partial T} \right) \dot{T} - \\
& - \rho \left(\frac{\partial e^1}{\partial T} - T \frac{\partial \eta}{\partial T} \right) \dot{T}_{,k} - \rho \left(\frac{\partial e^1}{\partial B_k} - T \frac{\partial \eta}{\partial B_k} + \frac{1}{2\rho} c_{kr1} \sum_{a=1}^s \left[\frac{\partial u_i (t_{ij} [r_r + C_{ij} [r_r])}{\partial \xi_{ij}} - u_{a[r} \xi_{ij} \frac{\partial p_i}{\partial \xi_{ij}} \right] \right) + \frac{1}{\rho} \sum_{a=1}^s (M_{a,k} - c_{k1i} v_{ij} p_i) \dot{B}_{a,k} - \\
& - \rho \left(\frac{\partial e^1}{\partial s_k} - T \frac{\partial \eta}{\partial s_k} + \frac{1}{\rho} p_k \right) \dot{s}_k - \frac{1}{T} [q_i + c_{ij,k} \sum_{a=1}^s \xi_{ij} M_{a,k} - \sum_{a=1}^s C_{aj} u_{aj} + T \frac{\partial}{\partial T} \sum_{a=1}^s u_{aj} (t_{ij} + C_{aj})] - T \sum_{a=1}^s u_{aj} \xi_{ij} \frac{\partial p_j}{\partial T} T_{,i} - \\
& - \sum_{a=1}^s \left[(\bar{m}_i - c_{vi}) u_{ai} + \rho \left(\frac{\partial e^1}{\partial p} - T \frac{\partial \eta}{\partial p} - \frac{1}{\rho} \sum_{\gamma=1}^s \xi_{ij} p_{\gamma} \right) \dot{e} \right] - \\
& - \sum_{b=1}^s \left[\frac{\partial}{\partial p} \sum_b^s u_{ai} (t_{ij} + C_{ij}) - \frac{1}{\rho} (t_{ij} - \sum_{\gamma=1}^s t_{ij}) u_{bi} - \rho \left(\frac{\partial e}{\partial p} - T \frac{\partial \eta}{\partial p} \right) u_{bj} \right] \dot{p}_{b,j} - \\
& - \left(\frac{\partial}{\partial T} \sum_{a=1}^s [u_{ai} (t_{ij} + C_{ij}) - u_{aj} \xi_{ij} p_{ij}] \right) T_{,kj} - \left(\sum_{b=1}^s \frac{\partial}{\partial p} \sum_{a=1}^s [u_{ai} (t_{ij} + C_{ij}) - u_{aj} \xi_{ij} p_{ij}] \right) \dot{p}_{b,kj} + \\
& + \sum_{a=1}^{s-1} \left[t_{ij} + \rho u_{ij} u_{aj} + \frac{\rho}{a} (t_{ij} - \sum_{\gamma=1}^s t_{ij}) - \frac{\partial}{\partial v_{aj}} \sum_{b=1}^s u_{bk} (t_{ki} + C_{ki}) - (\xi_{ij} p_{ij} - \delta_{ij} \vec{c} \cdot \vec{p}) + \right. \\
& \quad \left. + (B_{ij} M_{aj} - \delta_{ij} \vec{B} \cdot \vec{M}) + \sum_{b=1}^s u_{bj} \xi_{ij} \frac{\partial p_r}{\partial v_{aj}} + \rho \left(\frac{\partial e}{\partial p} - T \frac{\partial \eta}{\partial p} - \frac{1}{\rho} \sum_{b=1}^s \xi_{ij} p_{b,r} \right) \delta_{ij} \right] \dot{v}_{aj} + \\
& + \left\{ t_{ij} + \rho u_{ij} u_{aj} + \frac{\rho}{s} (t_{ij} - \sum_{\gamma=1}^s t_{ij}) + \sum_{a=1}^{s-1} \frac{\partial}{\partial v_{aj}} \sum_{b=1}^s u_{bk} (t_{ki} + C_{ki}) - (\xi_{ij} p_{ij} - \delta_{ij} \vec{c} \cdot \vec{p}) + (B_{ij} M_{aj} - \delta_{ij} \vec{B} \cdot \vec{M}) - \right. \\
& \quad \left. - \sum_{a=1}^{s-1} \sum_{b=1}^s u_{bj} \xi_{ij} \frac{\partial p_r}{\partial v_{aj}} + \rho \left(\frac{\partial e}{\partial p} - T \frac{\partial \eta}{\partial p} - \frac{1}{\rho} \sum_{b=1}^s \xi_{ij} p_{b,r} \right) \delta_{ij} - \frac{1}{2} c_{kr1} (B_{ij} \delta_{jk} - B_{kj} \delta_{ji}) \sum_{a=1}^s \left[\frac{\partial u_a (t_{ij} [r_r + C_{ij} [r_r])}{\partial \xi_{ij}} - u_{a[r} \xi_{ij} \frac{\partial p_i}{\partial \xi_{ij}} \right] \right\} \dot{v}_{sj} + \\
& + \sum_{a=1}^{s-1} \left[t_{ij} + \frac{\partial}{\partial v_{aj}} \sum_{b=1}^s u_{bk} (t_{ki} + C_{ki}) - B_{ij} M_{aj} + \sum_{b=1}^s u_{bj} \xi_{ij} \frac{\partial p_r}{\partial v_{aj}} - \xi_{ij} p_{ij} \right] (w_{ij} - w_{sij}) - \\
& - \left[\sum_{a=1}^s \left[\frac{\partial u_i (t_{ij} + C_{ij})}{\partial B_k} - u_{aj} \xi_{ij} \frac{\partial p_i}{\partial B_k} + M_{aj} v_{ij} \right] + M_{aj} u_{ij} - \frac{1}{2} c_{kr1} u_{sij} \sum_{a=1}^s \left[\frac{\partial u_i (t_{ij} [r_r + C_{ij} [r_r])}{\partial \xi_{ij}} - u_{a[r} \xi_{ij} \frac{\partial p_i}{\partial \xi_{ij}} \right] - \right. \\
& \quad \left. - \left\{ \frac{1}{3} \sum_{a=1}^s \left[\frac{\partial u_i (t_{ij} + C_{ij})}{\partial B_1} - u_{aj} \xi_{ij} \frac{\partial p_i}{\partial B_1} + M_{aj} v_{ij} \right] + \frac{1}{3} M_{aj} u_{ij} - \frac{1}{6} c_{qr1} u_{sij} \sum_{a=1}^s \left[\frac{\partial u_i (t_{ij} [r_r + C_{ij} [r_r])}{\partial \xi_{ij}} - u_{a[r} \xi_{ij} \frac{\partial p_i}{\partial \xi_{ij}} \right] \right\} \right] \dot{B}_{k,j} - \\
& - \sum_{a=1}^s \left\{ \frac{\partial u_i (t_{ij} + C_{ij})}{\partial s_k} - u_{aj} \xi_{ij} \frac{\partial p_i}{\partial s_k} - \frac{1}{2} (\delta_{jr} \delta_{ki} - \delta_{ji} \delta_{kr}) \left[\frac{\partial u_i (t_{ij} [r_r + C_{ij} [r_r])}{\partial \xi_{ij}} - u_{a[r} \xi_{ij} \frac{\partial p_i}{\partial \xi_{ij}} \right] \right\} \dot{s}_{k,j} - \\
& - \sum_{a=1}^s \left(B_{ij} M_{aj} + \xi_{ij} p_{aj} + \frac{1}{2} c_{jk1} B_{ij} \left[\frac{\partial u_r (t_{ij} [r_r + C_{ij} [r_r])}{\partial \xi_{kj}} - u_{a[r} \xi_{ij} \frac{\partial p_i}{\partial \xi_{kj}} \right] \right) w_{sij} \geq 0. \tag{19}
\end{aligned}$$

Preliminary conditions from the entropy principle in V

Examination of (19) reveals an essential fact here; namely the linearity of that relationship in the particular variables

$$\begin{aligned} \dot{\rho}_{b,k}, \dot{V}_a, \dot{T}, \dot{T}_{,k}, \dot{B}_k, \dot{\xi}_{s,k}, T_{,kj}, \rho_{b,kj}, d_{aj}, \\ d_{sj}, w_{aj} - w_{sj}, B_{k,j}, \xi_{s,k,j} \text{ and } w_{ij}. \end{aligned} \quad (20)$$

Thus, the coefficients of these parameters are by virtue of the constitutive assumptions (C7) seen to be independent of them. Now, the entropy principle in V requires (1) (or equivalently (19)) to hold for all thermodynamic processes in V; and, to satisfy the inequality (19) under this condition it must be concluded here that the coefficients of the derivatives (20) vanish identically.

That being the case, the conditions on the non-linear constitutive equations which follow directly from the entropy principle in V are

$$\frac{\partial(e^I - T\eta)}{\partial \rho_{b,k}} = 0 \quad (b = 1, 2, \dots, s), \quad (21)$$

$$\frac{\partial(e^I - T\eta)}{\partial T} = -\eta, \quad (22)$$

$$\frac{\partial(e^I - T\eta)}{\partial T_{,k}} = 0, \quad (23)$$

$$\frac{\partial(e^I - T\eta)}{\partial V_a} = \frac{1}{\rho} \epsilon_{kim} P_a^i B_m^k \quad (a = 1, 2, \dots, s-1), \quad (24)$$

$$\frac{\partial(e^I - T_n)}{\partial \xi_k} = -\frac{1}{\rho} P_k, \quad (25)$$

$$\begin{aligned} \frac{\partial(e^I - T_n)}{\partial B_k} = & -\frac{1}{\rho} \sum_{a=1}^s (M_{a k} - \epsilon_{k l i} V_a P_i) - \\ & -\frac{1}{2\rho} \epsilon_{k r l} \sum_{a=1}^s \left[\frac{\partial u_a(t_i[r + C_i[r]])}{\partial \xi_l} - u_a[r] \xi_i \frac{\partial P_i}{\partial \xi_l} \right], \end{aligned} \quad (26)$$

$$\frac{\partial}{\partial T_{,k}} \sum_{a=1}^s [u_a(t_i(j + C_i(j)) - u_a(j) \xi_i P_i] = 0, \quad (27)$$

$$\frac{\partial}{\partial \rho_{b,k}} \sum_{a=1}^s [u_a(t_i(j + C_i(j)) - u_a(j) \xi_i P_i] = 0 \quad (b = 1, 2, \dots, s), \quad (28)$$

$$\begin{aligned} t_a(ij) = & -\rho \rho_a \frac{\partial(e - T_n)}{\partial \rho} \delta_{ij} + \frac{\partial}{\partial V_j} \sum_{b=1}^s u_b(t_k(i + C_{b k}(i)) + \\ & + \frac{\rho}{\rho} \sum_{b=1}^s \rho_{bb} u_{b j} - \rho_{aa} u_{a j} + (\xi_{(i} P_{a j)} - \vec{\xi} \cdot \vec{P}_a \delta_{ij}) - \\ & - (B_{(i} M_{a j)} - \vec{B} \cdot \vec{M}_a \delta_{ij}) - \sum_{b=1}^s u_b(i) \xi_{b r} \frac{\partial P_r}{\partial V_j} + \\ & + \frac{\rho}{\rho} \sum_{b=1}^s \xi_{b r} P_{b r} \delta_{ij} \quad (a = 1, 2, \dots, s-1), \end{aligned} \quad (29)$$

$$t_s(ij) = -\rho \rho_s \frac{\partial(e - T_n)}{\partial \rho} \delta_{ij} - \sum_{a=1}^{s-1} \frac{\partial}{\partial V_j} \sum_{b=1}^s u_b(t_k(i + C_{b k}(i)) +$$

$$\begin{aligned}
& + \frac{\rho}{\rho} \sum_{b=1}^s \frac{s}{\rho} \frac{\rho u_b u_b}{\rho} - \frac{\rho u_s u_s}{\rho} + (\xi_s^P(j) - \vec{\xi}_s \cdot \vec{P} \delta_{ij}) - \\
& - (B_{(iM_s j)} - \vec{B}_s \cdot \vec{M}_s \delta_{ij}) + \sum_{a=1}^{s-1} \sum_{b=1}^s u_b(i) \xi_b^r \frac{\partial P}{\partial V_a(j)} + \\
& + \frac{\rho}{\rho} \sum_{b=1}^s \xi_b^r \frac{\partial P}{\partial V_b} \delta_{ij} + \\
& + \frac{1}{2} \epsilon_{kr1} (B_i \delta_{jk} - B_k \delta_{ji}) \sum_{a=1}^s \left[\frac{\partial u_a^q (t_{a^q}(r) + C_{a^q}(r))}{\partial \xi_{1s}} - u_a^q(r) \xi_a^q \frac{\partial P}{\partial \xi_{1s}} \right], \quad (30)
\end{aligned}$$

$$\begin{aligned}
t_a^{[ij]} &= - \frac{\partial}{\partial V_a(j)} \sum_{b=1}^s u_b^k (t_{bk}^i + C_{bk}^i) + \xi_a^P(j) + B_{[iM_a j]} - \\
& - \sum_{b=1}^s u_b^i \xi_b^r \frac{\partial P}{\partial V_a(j)} \quad (a = 1, 2, \dots, s-1), \quad (31)
\end{aligned}$$

$$\begin{aligned}
& \sum_{a=1}^s \left[\frac{\partial u_a^i (t_{ij} + C_{ij})}{\partial B_k} - u_a^i \xi_a^i \frac{\partial P}{\partial B_k} + M_a^i (k_{a^i j}) \right] + M_a^i (k_{s^i j}) - \\
& - \frac{1}{2} \epsilon_{r1} (k_{s^i j}) \sum_{a=1}^s \left[\frac{\partial u_a^i (t_{i[r]r} + C_{i[r]r})}{\partial \xi_{1s}} - u_a^i \xi_a^i \frac{\partial P}{\partial \xi_{1s}} \right] - \\
& - \frac{1}{3} \left\{ \sum_{a=1}^s \left[\frac{\partial u_a^i (t_{i11} + C_{i11})}{\partial B_1} - u_a^i \xi_a^i \frac{\partial P}{\partial B_1} - M_a^i V_1 \right] + \frac{1}{3} M_1 u_1 \right\} - \\
& - \frac{1}{6} \epsilon_{qr1} u_a^q \sum_{a=1}^s \left[\frac{\partial u_a^i (t_{i[r]r} + C_{i[r]r})}{\partial \xi_{1s}} - u_a^i \xi_a^i \frac{\partial P}{\partial \xi_{1s}} \right] \delta_{ij} = 0, \quad (32)
\end{aligned}$$

$$\sum_{a=1}^s \left[\frac{\partial u_i(t_i[j] + C_{i[j]})}{\partial B_k} - u_{[j]a} \delta_i \frac{\partial P_i}{\partial B_k} - M_{[k]a}^V \right] + M_{[k]s}^U -$$

$$- \frac{1}{2} \epsilon_{r1} [k_s^U] \sum_{a=1}^s \left[\frac{\partial u_i(t_i[r] + C_{i[r]})}{\partial \delta_{1s}} - u_{[r]a} \delta_i \frac{\partial P_i}{\partial \delta_{1s}} \right] = 0, \quad (33)$$

$$\sum_{a=1}^s \left\{ \frac{\partial u_i(t_i(j) + C_{i(j)})}{\partial \delta_k} - u_{[j]a} \delta_i \frac{\partial P_i}{\partial \delta_k} - \right.$$

$$\left. - \frac{1}{2} (\delta_{jr} \delta_{kl} - \delta_{jl} \delta_{kr}) \left[\frac{\partial u_i(t_i[r] + C_{i[r]})}{\partial \delta_{1s}} - u_{[r]a} \delta_i \frac{\partial P_i}{\partial \delta_{1s}} \right] \right\} = 0, \quad (34)$$

$$\sum_{a=1}^s \left\{ \frac{\partial u_i(t_i[j] + C_{i[j]})}{\partial \delta_k} - u_{[j]a} \delta_i \frac{\partial P_i}{\partial \delta_k} - \right.$$

$$\left. - \frac{1}{2} (\delta_{r[j] \delta_{k]l}} - \delta_{l[j] \delta_{k]r}) \left[\frac{\partial u_i(t_i[r] + C_{i[r]})}{\partial \delta_{1s}} - u_{[r]a} \delta_i \frac{\partial P_i}{\partial \delta_{1s}} \right] \right\} = 0, \quad (35)$$

and

$$\sum_{a=1}^s \left\{ B_{[i]a}^M + \delta_{[i]a}^P + \right.$$

$$\left. + \frac{1}{2} \epsilon_{1k} [j] B_{i]a} \left[\frac{\partial u_q(t_{q[1]} + C_{q[1]})}{\partial \delta_k} - u_{[1]a} \delta_q \frac{\partial P_q}{\partial \delta_k} \right] \right\} = 0. \quad (36)$$

Preliminary results from the entropy principle in V

I now take note of the foremost direct consequences of the conditions above; which results represent the initial derived restrictions on the constitutive fields involved. Equations (21) and (22) indicate that e^I and η are independent of $\rho_{b,k}$ ($b = 1, 2, \dots, s$); while (22) and (23) mean that these fields are also independent of $T_{,k}$. Use of the identity

$$u_{a,k} = \sum_{d=1}^{s-1} \left(\delta_{ad} - \frac{\rho}{\rho} \frac{d}{d} \right) V_d^k. \quad (37)$$

introduced now, together with (22) and (24) leads to the conclusion that

$$\frac{\partial \eta}{\partial V_{a,k}} = - \frac{1}{\rho} \epsilon_{kim} \frac{\partial P_i}{\partial T} B_m \quad (a = 1, 2, \dots, s-1). \quad (38)$$

From (22) and (25) it follows that

$$\frac{\partial \eta}{\partial \xi_{s,k}} = \frac{1}{\rho} \frac{\partial P_k}{\partial T}; \quad (39)$$

and (22) together with (26) yields the result that

$$\begin{aligned} \frac{\partial \eta}{\partial B_k} &= \frac{1}{\rho} \sum_{a=1}^s \left(\frac{\partial M_a^k}{\partial T} - \epsilon_{k1i} V_a^i \frac{\partial P_i}{\partial T} \right) + \\ &+ \frac{1}{2\rho} \epsilon_{kr1} \sum_{a=1}^s \frac{\partial}{\partial T} \left[\frac{\partial u_i (t_{i[r} + C_{i[r}])}{\partial \xi_{s1}} - u_{a[r} \delta_{i[s]} \frac{\partial P_i}{\partial \xi_{s1}} \right]. \quad (40) \end{aligned}$$

It may be seen from (21) and (25) that the P_k are independent of $\rho_{b,k}$ ($b = 1, 2, \dots, s$); and further, from (23) and (25) that it is also independent of $T_{,k}$. Now, from (21) and (24) it follows that

$$\epsilon_{kim} \frac{\partial P_i}{\partial \rho_{b,l}} B_m = 0 \quad \begin{matrix} (a = 1, 2, \dots, s-1; \\ b = 1, 2, \dots, s) \end{matrix} \quad (41)$$

The expressions (23) and (24) lead to the result

$$\epsilon_{kim} \frac{\partial P_i}{\partial T_{,l}} B_m = 0 \quad (a = 1, 2, \dots, s) \quad (42)$$

Thus, on the basis of (41) and (42), it cannot be concluded in general that P_i , like P_i , is independent of the gradients of the constituent densities and the temperature gradient. A further result concerning the polarization is seen to derive from (24) and (25); it being that

$$\frac{\partial P_l}{\partial V_k} = - \epsilon_{kim} \frac{\partial P_i}{\partial s_l} B_m \quad (a = 1, 2, \dots, s-1) \quad (43)$$

Adding (29) and (30) gives, with regard to (B87) and (B124), the result

$$\begin{aligned} t_{ij} = & - \sum_{a=1}^s \rho_a \frac{\partial(e - T\eta)}{\partial \rho_a} \delta_{ij} - \sum_{a=1}^s \rho_{aa} u_a u_j + \sum_{a=1}^s \xi_a (i P_j) - \\ & - \sum_{a=1}^s (B_{(i} M_{j)}) - \vec{B} \cdot \vec{M}_a \delta_{ij} + \\ & + \frac{1}{2} \epsilon_{krl} (B_i \delta_{jk} - B_k \delta_{ji}) \sum_{a=1}^s \left[\frac{\partial u_a (t_{aq}(r + c_{aq}(r))}{\partial \xi_{l1}} - u_a (r) \xi_a \frac{\partial P_i}{\partial \xi_{l1}} \right] \end{aligned} \quad (44)$$

From this relationship a mean pressure of the mixture, $\pi = -1/3 t_{ii}$, may be introduced. It is here given by the expression

$$\pi = \sum_{a=1}^s \rho_a \frac{\partial(e - T\eta)}{\partial \rho_a} + \frac{1}{3} \sum_{a=1}^s \rho_{aa}^2 + \frac{1}{3} \sum_{a=1}^s \vec{\xi}_a \cdot \vec{p}_a + \frac{2}{3} \sum_{a=1}^s \vec{B} \cdot \vec{M}_a +$$

$$+ \frac{1}{3} \epsilon_{kr1} B_k \sum_{a=1}^s \left[\frac{\partial u_a^q (t_{aq}(r) + C_{aq}(r))}{\partial \xi_{1s}} - u_a(r) \delta_{ia} \frac{\partial p_i}{\partial \xi_{1s}} \right]. \quad (45)$$

Expressions of conditions on the fields C_{ij} as secured from (29), (30) and (31) are respectively found ¹⁰ to be

$$\begin{aligned} C_a(ij) = & \left[\rho \frac{\partial \rho (e^l - Tn)}{\partial \rho} + \frac{1}{2} \rho u^2 \right] \delta_{ij} - \sum_{b=1}^s u_b^k \frac{\partial (t_{bk}(i) + C_{bk}(i))}{\partial V_j} - \\ & - \frac{\rho}{a} \sum_{b=1}^s \rho u_b u_j + \rho u_a u_j - (\delta_a(i_a^P j) - \vec{\xi}_a \cdot \vec{P}_a \delta_{ij}) + \\ & + (B_a(i_a^M j) - \vec{B}_a \cdot \vec{M}_a \delta_{ij}) + \sum_{b=1}^s u_b(i_b^{\xi} \frac{\partial P_{bq}}{\partial V_j}) \quad (a = 1, 2, \dots, s-1), \quad (46) \end{aligned}$$

$$\begin{aligned} C_s(ij) = & \left[\rho \frac{\partial \rho (e^l - Tn)}{\partial \rho} + \frac{1}{2} \rho u^2 \right] \delta_{ij} + \sum_{a=1}^{s-1} \sum_{b=1}^s u_b^k \frac{\partial (t_{bk}(i) + C_{bk}(i))}{\partial V_j} - \\ & - \frac{\rho}{s} \sum_{b=1}^s \rho u_b u_j + \rho u_s u_j - (\delta_s(i_s^P j) - \vec{\xi}_s \cdot \vec{P}_s \delta_{ij}) + \\ & + (B_s(i_s^M j) - \vec{B}_s \cdot \vec{M}_s \delta_{ij}) - \sum_{a=1}^{s-1} \sum_{b=1}^s u_b(i_b^{\xi} \frac{\partial P_{bq}}{\partial V_j}) - \\ & - \frac{1}{3} \epsilon_{kr1} (B_i \delta_{jk} - B_k \delta_{ji}) \sum_{a=1}^s \left[\frac{\partial u_a^q (t_{aq}(r) + C_{aq}(r))}{\partial \xi_{1s}} - u_a(r) \delta_{aq} \frac{\partial p_a}{\partial \xi_{1s}} \right], \quad (47) \end{aligned}$$

and

$$\begin{aligned} C_a[ij] = & \sum_{b=1}^s u_b^k \frac{\partial (t_{bk}(i) + C_{bk}(i))}{\partial V_j} + \sum_{b=1}^s u_b(i_b^{\xi} \frac{\partial P_{bq}}{\partial V_j}) - \\ & - \delta_a[i_a^P j] - B_a[i_a^M j] \quad (a = 1, 2, \dots, s-1). \quad (48) \end{aligned}$$

I wish to note here that, due to the mixture nature of the problem at hand, it is not possible with the present degree of descriptive generality to give a lucid physical interpretation for each and every term given in the initial results above. These general results are nonetheless relevant in greater or lesser degree to the procuring of that eventual physical insight towards which attention is directed in this study. It is possible, however, to make yet further progress here without compromising said generality; this being accomplished below essentially by the exploitation of only some of the knowledge gained to this point.

Reduced entropy inequality

With the elimination of the restricted fields as necessitated by the entropy principle in V , a considerable simplification of (19) is achieved. The reduced entropy inequality in this study is thus found to be of the form

$$\begin{aligned}
 & \sum_{b=1}^s \left\{ \left[\rho \frac{\partial(\epsilon - T\eta)}{\partial \rho} \delta_{ij} + \frac{1}{\rho} (t_{ij} - \sum_{\gamma=1}^s t_{ij}^{\gamma}) \right] u_i - \right. \\
 & \quad \left. - \frac{\partial}{\partial \rho} \sum_{a=1}^s u_a (t_{aj} + C_{aj}) \right\} \rho_{,j} - \\
 & - \sum_{a=1}^s \left\{ \hat{m}_a u_i + \rho \left[\frac{\partial(e^I - T\eta)}{\partial a} + \frac{1}{\rho} \sum_{\gamma=1}^s \xi_{\gamma} p_{\gamma} - \frac{1}{\rho} v_a u_i \right] \hat{c}_a \right\} - \\
 & - \frac{1}{T} \left[q_i + \epsilon_{ijk} \sum_{a=1}^s \xi_{aj} M_k - \sum_{a=1}^s C_{aj} u_j - \right. \\
 & \quad \left. - T \frac{\partial}{\partial T} \sum_{a=1}^s u_a (t_{aj} + C_{aj}) - T \sum_{a=1}^s u_a \xi_{aj} \frac{\partial p_j}{\partial T} \right] T_{,i} \geq 0. \quad (49)
 \end{aligned}$$

Admissible thermodynamic processes in V

I note here that with ρ_{a0a} and $(B152)_2$ known, those solutions for the fields (A1) which follow from the equations (B158), (B159), (B161), (B163) to (B166), together with the constitutive equations (C7) which satisfy the reduced entropy inequality (49), are called ¹¹ admissible thermodynamic processes in V . Without making additional assumptions regarding the physical nature of the (class of) systems involved, it is not possible to go beyond (49). Such assumptions as may be made relate to specific physical systems the description of which is more limited than was the case to this point.

Under general circumstances, the relationships as given above lend themselves neither to simple interpretation nor to an easy derivation of more definite results. In the following chapters I shall consider this point more closely.

D.3. Appendix D-1. The Lagrange multipliers Λ^{δ_k} , Λ^{δ_4} , Λ^{β_k} and Λ^{β_4} .

Let

$$\Delta_{jk} = \delta_{jk} + \sum_{a=1}^s \frac{1}{\epsilon_0} \frac{\partial P_j}{\partial \delta_k^a}, \quad (50)$$

$$A_{qk} = \epsilon_{kqi} \frac{v_i}{s} - \frac{1}{\epsilon_0} \frac{\partial P_q}{\partial \beta_k}, \quad (51)$$

$$A_k = \rho \left(\frac{\partial \epsilon}{\partial \beta_k} - T \frac{\partial \eta}{\partial \beta_k} - \frac{1}{\rho} \sum_{a=1}^s \delta_a \frac{\partial P_i}{\partial \beta_k^a} \right), \quad (52)$$

$$\begin{aligned} C_{qjk} &= \sum_{a=1}^s \frac{1}{\epsilon_0} \left[\epsilon_{qjn} \frac{\partial M_n}{\partial \beta_k^a} - v_q \frac{\partial P_j}{\partial \beta_k^a} + (u_j \delta_{rq} - u_q \delta_{rj}) \frac{\partial P_r}{\partial \delta_k^a} \right] + \\ &+ \epsilon_{q1k} \frac{v_1 v_j}{s} - \frac{1}{\epsilon_0 \mu_0} \epsilon_{qjk}, \end{aligned} \quad (53)$$

$$D_{jk} = \epsilon_{ijk} \frac{v_i}{s} + \sum_{a=1}^s \frac{1}{\epsilon_0} \frac{\partial P_j}{\partial \beta_k^a}, \quad (54)$$

$$E_{jk} = \sum_{a=1}^s \left[\frac{\partial u_i (t_{ij} + C_{ij})}{\partial \beta_k^a} - u_a \delta_i \frac{\partial P_i}{\partial \delta_k^a} - M_k \frac{v_j}{a^j} \right], \quad (55)$$

$$F_k = \rho \left(\frac{\partial \epsilon}{\partial \delta_k^s} - T \frac{\partial \eta}{\partial \delta_k^s} - \frac{1}{\rho} \sum_{a=1}^s \delta_a \frac{\partial P_i}{\partial \delta_k^a} \right), \quad (56)$$

$$G_{qjk} = \sum_{a=1}^s \frac{1}{\epsilon_0} \left[\epsilon_{qjn} \frac{\partial M_n}{\partial \delta_k^a} - v_q \frac{\partial P_j}{\partial \delta_k^a} - (u_j \delta_{rq} - u_q \delta_{rj}) \frac{\partial P_r}{\partial \delta_k^a} \right] \quad (57)$$

and

$$H_{jk} = \sum_{a=1}^s \left[\frac{\partial u_i (t_{ij} + c_{ij})}{\partial \xi_{sk}} - u_j \delta_i \frac{\partial P_a}{\partial \xi_{sk}} - \epsilon_{jki} M_a \right] \quad (58)$$

In terms of these parameters I now rewrite the \dot{B}_k , $B_{k,j}$, $\dot{\xi}_{sk}$ and $\xi_{s,k,j}$ coefficients of (16), each of which must identically vanish, in the form

$$\Lambda^B_k + \Lambda^{\xi_{sq}} A_{qk} + A_k = 0$$

$$\Lambda^B_{k u_j} + \Lambda^{B_4} \delta_{jk} + \Lambda^{\xi_{sq}} C_{qjk} + \Lambda^{\xi_4} D_{jk} + E_{jk} = 0$$

$$- \Lambda^{\xi_{sq}} \Delta_{qk} + F_k = 0$$

$$\Lambda^B_q \epsilon_{qjk} + \Lambda^{\xi_{sq}} G_{qjk} + \Lambda^{\xi_k} v_j + \Lambda^{\xi_4} \Delta_{jk} + H_{jk} = 0 ;$$

which equations shall be referred to respectively as I, II, III and IV.

On Λ^{ξ_k} and Λ^{ξ_4}

From III, (50) and (56) it follows that $\Lambda^{\xi_{sq}}$ is independent of v_i .

Therefore, noting that

$$v_i = v_i - \frac{\rho}{s} \sum_{a=1}^{s-1} \frac{\rho}{\rho} v_a ,$$

a differentiation with respect to v_i of I, II and IV yields respectively the results

$$\frac{\partial \Lambda^B_k}{\partial v_i} + \Lambda^{\xi_{sq}} \epsilon_{kqi} = 0 , \quad (59)_1$$

$$\begin{aligned} \frac{\partial \Lambda^B_k}{\partial v_i} u_j + \frac{\partial \Lambda^{B_4}}{\partial v_i} \delta_{jk} + \Lambda^{\xi_{sq}} \left[- \frac{1}{\epsilon_0} \delta_{qi} \frac{\partial P_j}{\partial B_k} + \epsilon_{q1k} (\delta_{li} v_j + v_l \delta_{ji}) \right] + \\ + \frac{\partial \Lambda^{\xi_4}}{\partial v_i} D_{jk} + \Lambda^{\xi_4} \epsilon_{ijk} = 0 \end{aligned} \quad (59)_2$$

$$\frac{\partial \Lambda^{B_k}}{\partial v_i} \epsilon_{qjk} + \Lambda^S \left[-\frac{1}{\epsilon_0} \frac{\partial p_j}{\partial \xi_k} \right] + \Lambda^S \delta_{ji} + \frac{\partial \Lambda^S}{\partial v_i} \Delta_{jk} = 0. \quad (59)_3$$

Differentiation with respect to v_n of (59)₁, then of (59)₃ and lastly of (59)₂ leads respectively to $\partial^2 \Lambda^{B_k} / \partial v_n \partial v_i = 0$, $\partial^2 \Lambda^S / \partial v_n \partial v_i = 0$ and

$$\frac{\partial^2 \Lambda^{B_4}}{\partial v_n \partial v_i} \delta_{jk} + \Lambda^S \epsilon_{qik} (\delta_{li} \delta_{jn} + \delta_{ln} \delta_{ji}) + \frac{\partial \Lambda^S}{\partial v_i} \epsilon_{njk} + \frac{\partial \Lambda^S}{\partial v_n} \epsilon_{ijk} = 0. \quad (60)$$

Contraction on the jk -indices yields $\partial^2 \Lambda^{B_4} / \partial v_n \partial v_i = 0$, so (60) now reads

$$\Lambda^S \epsilon_{qik} (\delta_{ln} \delta_{ji} + \delta_{jn} \delta_{li}) + \frac{\partial \Lambda^S}{\partial v_i} \epsilon_{njk} + \frac{\partial \Lambda^S}{\partial v_n} \epsilon_{ijk} = 0. \quad (61)$$

For $i \neq j \neq n$ this equation becomes

$$\frac{\partial \Lambda^S}{\partial v_i} \epsilon_{njk} + \frac{\partial \Lambda^S}{\partial v_n} \epsilon_{ijk} = 0 \quad (62)$$

whence follows that $\partial \Lambda^S / \partial v_i = 0$ since $\partial \Lambda^S / \partial v_3 = 0$, $\partial \Lambda^S / \partial v_1$ and $\partial \Lambda^S / \partial v_2 = 0$ for, respectively, the combinations of indices ($i=1, j=2, n=3$ and $k=3$), ($i=1, j=2, n=3$ and $k=3$) and ($i=2, j=3, n=1$ and $k=1$). Thus (61) implies that

$$\Lambda^S \epsilon_{ik} = 0. \quad (63)$$

With this result (59)₁ shows that $\partial \Lambda^{B_k} / \partial v_i = 0$; and since (59)₂ becomes

$$\frac{\partial \Lambda^{B_4}}{\partial v_i} \delta_{jk} + \Lambda^S \epsilon_{ijk} = 0, \quad (64)$$

it may be concluded that

$$\Lambda^S \epsilon_{ik} = 0 \quad \text{and} \quad \frac{\partial \Lambda^{B_4}}{\partial v_i} = 0. \quad (65)$$

On Λ^{B_k}

Under (62)₁ and (65)₁, it follows from the condition that the coefficient of $\delta_{s,k,j}$ vanish, together with (B59), that

$$(\Lambda^{B_q} - M_q)\epsilon_{qjk} = - \sum_{a=1}^s \left[\frac{\partial u_i (t_{ij} + C_{ij})}{\partial \delta_{s,k}} + u_j \delta_i \frac{\partial P_i}{\partial \delta_{s,k}} \right].$$

Upon multiplying this expression by ϵ_{rjk} it follows directly that, with a change of indices, the desired relationship is given here by

$$\Lambda^{B_k} = M_k - \frac{1}{2} \epsilon_{kr1} \sum_{a=1}^s \left[\frac{\partial u_i (t_{i[r} + C_{i[r}])}{\partial \delta_{s,1}} - u_{[r} \delta_i \frac{\partial P_i}{\partial \delta_{s,1}} \right]. \quad (66)$$

On Λ^{B_4}

The last Lagrange multiplier relation sought is that for Λ^{B_4} . It follows from the coefficient of the $B_{k,j}$ term in (16). Using (62)₁, (65)₁ and (66) there in, the expression found is

$$\begin{aligned} \Lambda^{B_4} = & - \frac{1}{3} \sum_{a=1}^s \left[\frac{\partial u_i (t_{ij} + C_{ij})}{\partial B_j} - u_j \delta_i \frac{\partial P_i}{\partial B_j} + M_i V_{aj} \right] - \frac{1}{3} M_j u_j + \\ & + \frac{1}{6} \epsilon_{jr1} u_j \sum_{a=1}^s \left[\frac{\partial u_i (t_{i[r} + C_{i[r}])}{\partial \delta_{s,1}} - u_{[r} \delta_i \frac{\partial P_i}{\partial \delta_{s,1}} \right]. \quad (67) \end{aligned}$$

Footnotes to Chapter D

- ¹ Whereas Müller [1968, 4] developed the properties of a mixture of non-electromagnetic fluids in terms of an entropy principle set forth only for regular points, Liu & Müller [1972, 5, Sect. 4] and Müller [1973, 9, pp.142-143] utilize in their single-fluid treatments a particular material surface of discontinuity in order to exploit the character of singular points in their systems with regard to securing additional insight into the entropy flux properties of those systems. For the mixture problem at hand, I too adopt the singular surface approach mentioned; but, of necessity, I consider the more general case where such a surface is not material for all of the mixture constituents.
- ² Cf. Coleman & Noll [1963, 3, p.171, footnote 1].
- ³ Ibid [1963, 3, p.169]. It is important to note here that in contrast especially to classical thermodynamics (cf. Truesdell [1966, 14, pp. 234-235] and Truesdell [1969, 7, p.4]) where only homogeneous, i.e. position-independent, processes are treated, such a development as this has a foundation intimately connected with material motions and as such describes thermodynamic processes as functions of both position and time.
- ⁴ Müller [1973, 9, p.131].
- ⁵ $(12)_1$ may perhaps be called "Friedrichs' mean value" since he defined and subsequently used it, together with Kranzer [1958, 1], in the study of shock waves in MFD. See in this regard, e.g., Sutton & Sherman [1965, 17, pp.324-326].
- ⁶ CFT [1960, 9, p.700].

- ⁷ Coleman & Noll [1963, 3, p.171].
- ⁸ Liu & Müller [1972, 5, Sect. 3].
- ⁹ Liu [1972, 4].
- ¹⁰ Cf. Müller [1973, 9, pp.172-173] with regard to the presence of the term $\frac{1}{2}\rho u_{aa}^2$ in (46) and (47)
- ¹¹ Coleman & Noll [1963, 3, p.168].

E. ON EQUILIBRIUM PROPERTIES OF FLUID MIXTURES INTERACTING WITH AN ELECTRO-MAGNETIC FIELD

The considerations above have been seen to lead to rather general results which are applicable in principle to a broad class of mixture systems. Aside from the complexity of said results, additional motivation for further simplification of the model they represent is found here in the lack at present of a comparable theory from disciplines other than rational mechanics.

Results of an increasingly specific nature can be secured only upon the introduction of an also increasing number of limitations on the model being developed. Of particular significance are restrictions on admissible thermodynamic processes in V ; and it is to these that attention is now directed.

E.1. On equilibrium

A fluid mixture equilibrium in V is defined here to be an admissible thermodynamic process in V for which the following conditions hold¹: first, the constituent velocities are equal, uniform and time-independent; next, the temperature field is uniform and time-independent; and lastly, the reaction rate densities all vanish identically. The implications of the first condition are clearly the equality of all electromotive intensities and the vanishing of the velocities V_i and u_i .

With due regard to the degree of smoothness assumed, I note that constitutive functions and their representations can be expanded about an equilibrium at any point in V . Systems for which higher than second-order terms

of an expansion in an assumed set of variables are physically negligible are said here to be "close to equilibrium" ² in V . Under these conditions I assume that it is prudent to employ linearized constitutive equations. Matters pertaining to the stability of equilibria in V , although of some importance, fall beyond the scope of this study ³.

E.2. Equilibrium and the entropy principle in V

Consequent to the definition given above, an evaluation of the relations (D21)-(D36) and (D44)-(D47) reveals that in equilibrium (denoted by $|_E$), for which $e^I = e$, the new relations not satisfied identically are given by:

$$\frac{\partial(e - T\eta)|_E}{\partial V_k^a} = \frac{1}{\rho} \epsilon_{kim} P_a^i |_E B_m^B \quad (a = 1, 2, \dots, s-1), \quad (1)$$

$$\frac{\partial(e - T\eta)|_E}{\partial \xi_k^s} = -\frac{1}{\rho} P_k |_E, \quad (2)$$

$$\frac{\partial(e - T\eta)|_E}{\partial B_k} = -\frac{1}{\rho} M_k |_E, \quad (3)$$

$$B_{[i} M_{j]} |_E + \xi_{[i} P_{j]} |_E = 0, \quad (4)$$

$$t_{ij} |_E = - \sum_{a=1}^s \rho_a \frac{\partial(e - T\eta)|_E}{\partial \rho_a} \delta_{ij} + \xi_{(i} P_{j)} |_E - (B_{(i} M_{j)} |_E - \vec{B} \cdot \vec{M} |_E \delta_{ij}), \quad (5)$$

$$\pi \Big|_E = \sum_{a=1}^s \rho \frac{\partial (\epsilon - T\eta) \Big|_E}{\partial \rho} + \frac{1}{3} \vec{\epsilon} \cdot \vec{P} \Big|_E + \frac{2}{3} \vec{B} \cdot \vec{M} \Big|_E \quad (6)$$

and

$$\begin{aligned} C_{ij} \Big|_E &= \rho \frac{\partial (\epsilon - T\eta) \Big|_E}{\partial \rho} \delta_{ij} - (\delta_{ij} P_a \Big|_E - \vec{\epsilon} \cdot \vec{P} \Big|_E \delta_{ij}) + \\ &+ (B_{ja} M_i \Big|_E - \vec{B} \cdot \vec{M} \Big|_E \delta_{ij}) \quad (a = 1, 2, \dots, s) . \end{aligned} \quad (7)$$

On magnetizable fluid mixtures

The class of fluid mixtures described thus far may evidence material behavior dependent upon both the magnetization and polarization of the constituents involved. As such, the model here includes two sub-classes of media each of which may be considered in its own right; these being those mixtures for which either M_i or P_i may be disregarded.

For the sake of ease in the exploitation of the conditions above, I make the otherwise non-essential assumption that $P_i \Big|_E$ vanishes⁴. The evaluation in equilibrium of the constitutive fields shall now be denoted where useful by means of a superposed letter "E".

With the specific free energy of the mixture in equilibrium given by

$$\Psi = e - T\eta . \quad (8)$$

Equations (1) and (2) thus become respectively

$$\frac{\overset{E}{\partial \Psi}}{\partial V_k} = 0 \quad (a = 1, 2, \dots, s-1) \quad \text{and} \quad \frac{\overset{E}{\partial \Psi}}{\partial \xi_k} = 0 ; \quad (9)$$

which relations in turn indicate the fact that $\overset{E}{\Psi}$ is not a function of the velocity differences or the electromotive intensity. Thus, in agreement with (C27),

$$\overset{E}{\Psi} = \overset{E}{\Psi}(\rho, T, B^2) \quad (b = 1, 2, \dots, s) . \quad (10)$$

Examination of (3) under the same conditions gives

$$\frac{\overset{E}{\partial \Psi}}{\partial B_k} = - \frac{1}{\rho} \overset{E}{M}_k . \quad (11)$$

Using (10) and (C32) I note that (11) can be rewritten as follows

$$\frac{\overset{E}{\partial \Psi}}{\partial B_k} = 2 \frac{\overset{E}{\partial \Psi}}{\partial B^2} B_k = - \frac{1}{\rho} \overset{E}{M}_k = - \frac{1}{\rho} \overset{E}{M} B_k ; \quad (12)$$

where

$$\overset{E}{M} = \sum_{a=1}^s \overset{E}{M}_a . \quad (13)$$

From (12) it is thus found that

$$\frac{\overset{E}{\partial \Psi}}{\partial B^2} B_k = - \frac{1}{2\rho} \overset{E}{M} B_k \quad (14)$$

which when integrated yields the result that

$$\overset{E}{\Psi} \Big|_E = \overset{E}{\Psi}_b(\rho, T) - \frac{1}{2\rho} \int_0^{B^2} \overset{E}{M} \Big|_E dB^2 . \quad (15)$$

Here $\overset{*}{\Psi}_b(\rho, T)$ is the specific free energy of the mixture in the absence of an electromagnetic field ⁵.

The entropy inequality condition (D22) when evaluated in equilibrium becomes

$$\eta \Big|_E = - \frac{\partial \Psi}{\partial T} . \quad (16)$$

Use herein of (15) provides upon integration the result for the specific entropy of the mixture in equilibrium, viz.

$$\eta \Big|_E = \overset{*}{\eta}_b(\rho, T) + \frac{1}{2\rho} \int_0^{B^2} \frac{\partial M \Big|_E}{\partial T} dB^2 ; \quad (17)$$

where the specific entropy of the mixture in the absence of an electro-magnetic field is given by

$$\overset{*}{\eta}_b(\rho, T) = - \frac{\partial \overset{*}{\Psi}}{\partial T} . \quad (18)$$

Lastly, because

$$\epsilon \Big|_E = \frac{E}{\epsilon} = \Psi + T\eta , \quad (19)$$

it follows from the use of (15) and (17) in (19)₂ that

$$\epsilon \Big|_E = \overset{*}{\epsilon}_b(\rho, T) - \frac{1}{2\rho} \int_0^{B^2} (M \Big|_E - T \frac{\partial M \Big|_E}{\partial T}) dB^2 . \quad (20)$$

The term $\varepsilon_b^*(\rho, T)$ denotes the specific internal energy of the mixture for the case of no electromagnetic field; it being defined by

$$\varepsilon_b^*(\rho, T) = \bar{\psi} - T \frac{\partial \bar{\psi}}{\partial T}. \quad (21)$$

E.3. Stress tensor

From (C33)₁ it follows under equilibrium conditions that

$$t_{ij}|_E = \sum_{a=1}^s t_{a|ij}|_E = {}^L t_1 \delta_{ij} + {}^L t_2 B_i B_j \quad (22)$$

where, having taken account of (B124),

$${}^L t_1 = \sum_{a=1}^s {}^L t_a, \quad {}^L t_2 = \sum_{a=1}^s {}^L t_a^2 \quad \text{and} \quad {}^L t_3 = \sum_{a=1}^s {}^L t_a^3 = 0. \quad (23)$$

On magnetizable fluid mixtures

For the special case of mixtures for which P_a^E is zero it also follows from (5), with due regard to (7), (C32) and (19) that

$$t_{ij}|_E = - \sum_{a=1}^s \rho_p \frac{\partial \psi|_E}{\partial \rho_a} \delta_{ij} - (B_j M_i|_E - \vec{B} \cdot \vec{M}|_E \delta_{ij}); \quad (24)$$

or, with the use of (15),

$$t_{ij}|_E = \left[-\bar{p}_b^*(\rho, T) + \sum_{a=1}^s \frac{1}{2} \rho_p \int_0^{B^2} \frac{\partial}{\partial \rho} \left(\frac{M|_E}{\rho} dB^2 \right) \delta_{ij} - (B_j M_i|_E - \vec{B} \cdot \vec{M}|_E \delta_{ij}) \right]. \quad (25)$$

Here

$$\overset{*}{p}_b(\rho, T) = \sum_{a=1}^s \overset{*}{p}_{a b}(\rho, T) \quad \text{and} \quad \overset{*}{p}_{a b}(\rho, T) = \rho_a \frac{\partial \overset{*}{\psi}}{\partial \rho_a} \quad (26)$$

(with $a, b = 1, 2, \dots, s$) denote, respectively, the pressure of the mixture in the absence of an electromagnetic field and the pressure, suggested by that relationship, of a constituent under the same conditions.

I now wish to consider the deviatoric expansion of the last term of (25). For this purpose I denote the mean normal "magnetic stress" and the magnetic stress deviator respectively by

$$\mathfrak{B} = \frac{1}{3} B_k B_k \quad \text{and} \quad B_{ij} = B_i B_j - \mathfrak{B} \delta_{ij} . \quad (27)$$

Upon introduction into (25) of (27), (C32) and (13), together with the expansion of the integral term therein, it is seen that the stress of the mixture in equilibrium may be written as

$$t_{ij} \Big|_E = -p \delta_{ij} - M \Big|_E B_{ij} \quad (28)$$

where the pressure of the mixture and the electromagnetic field is

$$p = \overset{*}{p} - \frac{2}{3} M \Big|_E B^2 + \frac{1}{2} \int_0^{B^2} (M \Big|_E - \sum_{a=1}^s \rho_a \frac{\partial M \Big|_E}{\partial \rho_a}) dB^2 . \quad (29)$$

E.4. Chemical potential

In order to determine the material behavior of a fluid mixture of variable composition, it is in general necessary to examine the role played by both the homogeneous and the heterogeneous chemical reactions involving that mixture. Although appreciating the importance under certain conditions of the latter type of reactions, I further restrict this study to a more

explicit investigation of only the former. In this section I take up in particular the problem of determining the chemical potentials for a mixture of fluids interacting with an electromagnetic field.

The entropy principle on $s(t)$ revisited

Surfaces of two kinds are of particular interest here. The first to be mentioned prohibits the passage of matter and thus serves to enclose some region of volume. Such a surface defines a mechanical boundary of that region and is representable as a material surface. In contrast, the second type of surface is porous to some extent and hence permits the transfer of mass. A surface with this character has earlier been termed a semi-permeable membrane. It is in terms of such a membrane that I now consider further the entropy principle on $s(t)$; and, following an approach due to Müller⁶, thereby determine the chemical potential of a mixture constituent.

A plane membrane in parallel motion is a material surface for all constituents a except that arbitrary one, say b , for which it is taken to be permeable. Upon considering such a membrane as separating two mixtures of the type being considered here, the selective nature of its permeability is expressible in terms of the system velocities: viz.

$$v_i = u_i \quad \text{for} \quad a \neq b ; \quad (30)$$

and thus

$$\rho \frac{v_k - u_k}{b} = \rho (v_k - u_k) \quad \text{and} \quad u_k = \left(\frac{1}{\rho} \delta_{ab} - \frac{1}{\rho} \right) \rho (v_k - u_k) . \quad (31)$$

In addition to satisfying the conditions prescribed by the entropy principle on $s(t)$, this membrane is also assumed to have negligible (if any) surface tension, charges and currents. With (30) and (31) I first continue here the general development of the jump balance relation of entropy (D13); after which, the specific result found here for magnetizable mixtures will be noted. To begin, however, it may be noted that

$$\begin{aligned} -\llbracket \sum_{a=1}^s c_{aki} u_{ak} \rrbracket \hat{n}_i &= -\llbracket \sum_{a=1}^s \left(\frac{1}{\rho} \delta_{ab} - \frac{1}{\rho} c_{aki} \hat{n}_i \hat{n}_k \right) \rrbracket \rho (v_1 - u_1) \hat{n}_1 = \\ &= \llbracket \frac{1}{\rho} \sum_{a=1}^s c_{aki} \hat{n}_i \hat{n}_k - \frac{1}{\rho} c_{bki} \hat{n}_i \hat{n}_k \rrbracket \rho (v_1 - u_1) \hat{n}_1, \quad (32) \end{aligned}$$

$$\begin{aligned} -\epsilon_{ijk} \llbracket E_j \sum_{a=1}^s (\vec{p}_a \times \vec{u}_a)_k \rrbracket \hat{n}_i &= -\llbracket \epsilon_k \frac{P_i}{\rho} (\hat{n}_j \hat{n}_k - \delta_{jk}) \rrbracket \rho (v_1 - u_1) \hat{n}_1 - \\ &- \llbracket \epsilon_k \frac{P_j}{\rho} (\hat{n}_j \hat{n}_k - \delta_{jk}) \rrbracket \rho (v_1 - u_1) \hat{n}_1 + \\ &+ \llbracket (\vec{v} \times \vec{B})_k \left(\frac{P_k}{\rho} - \frac{P_k}{\rho} \right) \rrbracket \rho (v_1 - u_1) \hat{n}_1 \quad (33) \end{aligned}$$

and

$$\begin{aligned} \epsilon_{ijk} \llbracket \sum_{a=1}^s (\vec{u}_a \times \vec{B})_j M_{ak} \rrbracket \hat{n}_i &= -\llbracket B_k \frac{M_i}{\rho} (\hat{n}_j \hat{n}_k - \delta_{jk}) \rrbracket \rho (v_1 - u_1) \hat{n}_1 + \\ &+ \llbracket B_k \frac{M_k}{\rho} (\hat{n}_j \hat{n}_k - \delta_{jk}) \rrbracket \rho (v_1 - u_1) \hat{n}_1. \quad (34) \end{aligned}$$

Introduction of these relationships into (D13) yields

$$\begin{aligned}
 & \left[-\frac{1}{\rho} C_{ki} \hat{n}_i \hat{n}_k + \left(-\mathcal{E}_k \frac{P_j}{b} + B_k \frac{M_j}{b} \right) (\hat{n}_j \hat{n}_k - \delta_{jk}) + \right. \\
 & \quad + \frac{1}{2} (v_i - u_i)(v_i - u_i) - (\epsilon - Tn) + \\
 & \quad + \frac{1}{\rho} (t_{ki} + \sum_{a=1}^s C_{ki} \hat{n}_i \hat{n}_k) \rho (v_l - u_l) \hat{n}_l + \\
 & \quad + \left[(\vec{v} \times \vec{B})_k \left(\frac{P_k}{\rho} - \frac{b^k}{\rho} \right) \right] \rho (v_l - u_l) \hat{n}_l - \\
 & \quad - \left[(\vec{v} \times \vec{B})_k P_k v_i \hat{n}_i \right] + \left[(\vec{P} \times \vec{B})_i \right] \hat{n}_i u_n^2 - \\
 & \quad - \left[(\vec{v} \times \vec{B})_i P_j + B_i \sum_{a=1}^s \left(\vec{P} \times \vec{v} \right)_j \right] (\hat{n}_i \hat{n}_j - \delta_{ij}) u_n = 0.
 \end{aligned} \tag{35}$$

A further revision of (35) is possible and is secured from an evaluation of the last four terms therein.

To begin, use of (B12) leads to the following expression

$$B_i \sum_{a=1}^s \left(\vec{P} \times \vec{v} \right)_j = B_i (\vec{P} \times \vec{v})_j + B_i \sum_{a=1}^s \left(\vec{P} \times \vec{v} \right)_j \tag{36}$$

and the substitution of this equation into (35) implies that there are two sets of terms which may now be considered. The first is

$$\begin{aligned}
\llbracket \mathbf{B}_i \sum_{a=1}^s (\vec{p}_a \times \vec{u}_a)_i \rrbracket u_n &+ \llbracket (\vec{v} \times \vec{b})_k \left(\frac{p_k}{\rho} - \frac{b_k}{\rho} \right) \rrbracket \rho (v_1 - u_1) \hat{n}_1 = \\
&= \llbracket (\vec{b} \times \frac{\vec{p}}{\rho})_i (v_i - u_i) \rrbracket \rho (v_1 - u_1) \hat{n}_1 - \\
&\quad - \llbracket (\vec{b} \times \frac{b}{\rho})_i (v_i - u_i) \rrbracket \rho (v_1 - u_1) \hat{n}_1 ; \quad (37)
\end{aligned}$$

and second, the remaining terms may be given as

$$\begin{aligned}
-\llbracket (\vec{v} \times \vec{b})_k p_k v_i \hat{n}_i \rrbracket &+ \llbracket (\vec{p} \times \vec{b})_i \rrbracket \hat{n}_i u_n^2 - \\
&- \llbracket (\vec{v} \times \vec{b})_i p_j + B_i (\vec{p} \times \vec{v})_j \rrbracket (\hat{n}_i \hat{n}_j - \delta_{ij}) u_n = \\
&= -\llbracket (\vec{v} \times \vec{b})_k \frac{p_k}{\rho} \rrbracket \rho (v_1 - u_1) \hat{n}_1 + \\
&+ \llbracket (\vec{b} \times \frac{\vec{p}}{\rho})_i u_i \rrbracket \rho (v_1 - u_1) \hat{n}_1 . \quad (38)
\end{aligned}$$

Addition of (37) and (38) gives the result

$$-\llbracket (\vec{b} \times \frac{b}{\rho})_i (v_i - u_i) \rrbracket \rho (v_1 - u_1) \hat{n}_1 . \quad (39)$$

Now, upon recognizing the fact that by (B87) and (B154)

$$\frac{1}{\rho} (t_{ki} + \sum_{a=1}^s c_{ki}^a) \hat{n}_i \hat{n}_k = \epsilon - T\eta - \frac{1}{\rho} \sum_{a=1}^s \rho \frac{(\vec{u} \cdot \hat{n})^2}{a} , \quad (40)$$

a substitution of (39) and (40) into (35) brings the jump balance equation for entropy into the following form

$$\llbracket \frac{K}{b} \rrbracket = 0 . \quad (41)$$

Here, the quantity K which is seen to be continuous across $s(t)$ is called the chemical potential in non-equilibrium of constituent b ; it being given by

$$\begin{aligned}
 K_b &= \frac{1}{\rho_b} C_{bki} \hat{n}_i \hat{n}_k + \left(\mathcal{E}_k \frac{P_{bj}}{\rho_b} - B_k \frac{M_j}{\rho_b} \right) (\hat{n}_j \hat{n}_k - \delta_{jk}) + \\
 &+ \left(\vec{\mathcal{B}} \times \frac{\vec{b}}{\rho_b} \right)_i \hat{n}_i [(v_j - u_j) \hat{n}_j] - \frac{1}{2} (v_i - u_i) (v_i - u_i) + \\
 &+ \frac{1}{\rho} \sum_{a=1}^s \rho_a \frac{(\vec{u}_a \cdot \vec{n})^2}{a} \quad (b = 1, 2, \dots, s). \quad (42)
 \end{aligned}$$

The chemical potential of any constituent b in equilibrium, μ_b , follows from (42); it being given in general here by the expression

$$\begin{aligned}
 \mu_b &\equiv K_b \Big|_E = \\
 &= \frac{1}{\rho_b} C_{bki} \Big|_E \hat{n}_i \hat{n}_k + \left(\mathcal{E}_k \frac{P_{bj} \Big|_E}{\rho_b} - B_k \frac{M_j \Big|_E}{\rho_b} \right) (\hat{n}_j \hat{n}_k - \delta_{jk}). \quad (43)
 \end{aligned}$$

On magnetizable fluid mixtures

Under the additional assumption that $P_j \Big|_E$ vanishes (43) becomes

$$\mu_b = \frac{1}{\rho_b} C_{bki} \Big|_E \hat{n}_i \hat{n}_k - B_k \frac{M_j \Big|_E}{\rho_b} (\hat{n}_j \hat{n}_k - \delta_{jk}), \quad (44)$$

while from (7) and (8) it is found that

$$\frac{1}{\rho} C_{bki} \Big|_E \hat{n}_i \hat{n}_k = \frac{\partial \rho \Psi \Big|_E}{\partial \rho} + B_k \frac{M_j \Big|_E}{\rho} (\hat{n}_j \hat{n}_k - \delta_{jk}) . \quad (45)$$

Introduction of (45) into (44) thus yields the relationship

$$\mu_b = \frac{\partial \rho \Psi \Big|_E}{\partial \rho} ; \quad (46)$$

where, upon using (15) it follows that for any constituent b the (magneto-) chemical potential is here given by

$$\mu_b = \mu_{ba}^*(\rho, T) - \frac{1}{2} \int_0^{B^2} \frac{\partial M \Big|_E}{\partial \rho} dB^2 \quad (47)$$

for $a, b = 1, 2, \dots, s$. The chemical potential of constituent b in the absence of an electromagnetic field, μ_b^* , in (47) is written as

$$\mu_{ba}^*(\rho, T) = \frac{\partial \rho \Psi^*}{\partial \rho} . \quad (48)$$

From (7), (8) and (46) it is thus seen, e.g., that for any constituent b in a magnetizable mixture the following general relation holds: viz.,

$$C_{bij} \Big|_E = \rho \mu_{bb} \delta_{ij} + (B_j M_i \Big|_E - \vec{B} \cdot \vec{M} \Big|_E \delta_{ij}) . \quad (49)$$

It is of some consequence that μ_b is, at least in principle, a measurable quantity ⁷. Clearly, together with a knowledge of M , it permits the empirical determination of the constitutive quantity C_{bij} in equilibrium. Further yet is its role in e.g. the chemical reactions and diffusion processes of the mixture, to which points I shall return shortly. But first I wish to set down another expression wherein the chemical potentials of the mixture constituents in equilibrium are also important.

E.5. Gibbs-Duhem equation

In the thermodynamics of mixtures a Gibbs-Duhem equation is the means by which ⁸ the composition dependence of μ and the constituent pressures can be examined. From (47) it may be seen that

$$\sum_{b=1}^s \frac{\rho}{\rho} \frac{b}{b} \mu_b = \sum_{b=1}^s \frac{\rho}{\rho} \frac{b}{b} \mu_b^* - \frac{1}{2} \int_0^{B^2} \sum_{b=1}^s \frac{\rho}{\rho} \frac{b}{b} \frac{\partial M|_E}{\partial \rho} dB^2. \quad (50)$$

Using (26) and (48) herein gives in turn that

$$\sum_{b=1}^s \frac{\rho}{\rho} \frac{b}{b} \mu_b = \frac{p}{\rho} + \Psi - \frac{1}{2} \int_0^{B^2} \sum_{b=1}^s \frac{\rho}{\rho} \frac{b}{b} \frac{\partial M|_E}{\partial \rho} dB^2; \quad (51)$$

which, upon introduction of (15) and (29) becomes

$$\sum_{b=1}^s \frac{\rho}{\rho} \frac{b}{b} \mu_b = \frac{p}{\rho} + \Psi + \frac{2}{3} \frac{E}{\rho} B^2. \quad (52)$$

The relations above are seen to be empirically determinable from the properties of the individual ("pure") constituents ⁹. Thus, an additional significant point here is that with the realization of the Gibbs-Duhem equation (52), with M known, a means for establishing the function Ψ has been secured. It is by virtue of this result that a further development of the equilibrium conditions (1)-(6) becomes possible.

E.6. Chemically (non-)reacting fluid mixtures

For mixtures in which there are no chemical reactions the terms \hat{c}_a are zero; and, hence, the reduced entropy inequality (049) assumes the form

$$\begin{aligned}
& \sum_{a=1}^s \left\{ \left[\rho \frac{\partial(\epsilon - T\eta)}{\partial \rho} \right] \delta_{ij} + \frac{1}{\rho} (t_{ij} - \sum_{\gamma=1}^s t_{\gamma ij}) \right] u_i - \\
& - \frac{\partial}{\partial \rho} \sum_{a=1}^s u_a (t_{aj} + c_{aj}) \Big] \rho_{,j} - \sum_{a=1}^s \hat{m}_a u_i - \\
& - \frac{1}{T} [q_i + \epsilon_{ijk} \sum_{a=1}^s \delta_{aj} M_k - \sum_{a=1}^s c_{aj} u_j - \\
& - T \frac{\partial}{\partial T} \sum_{a=1}^s u_a (t_{aj} + c_{aj}) - \\
& - T \sum_{a=1}^s u_a \delta_j \left[\frac{\partial p}{\partial T} \right]_{T,i} \geq 0. \tag{53}
\end{aligned}$$

For single-phase (i.e. homogeneous) mixtures composed of s constituent fluids, a criterion of equilibrium is ¹⁰ that the entropy production γ (cf. (B150)) of those mixtures vanish. In general said production is here the left-hand-side of the reduced entropy inequality (D49), i.e.

$$\gamma \geq 0;$$

and, this in turn reduces for the case where chemical reactions are negligible to the expression (53).

Thus, upon denoting by X_A the variables T, k and V_a^k which vanish in equilibrium, it is seen that γ which is generally a function of the variables (C6) satisfies the relationship

$$\gamma \Big|_E = \gamma(\rho_a, \rho_a^k, T, 0, 0, \epsilon_k, B_k) = 0. \tag{54}$$

That is, γ equals zero (its minimum value) in equilibrium.

The well-known conditions ¹¹ which ensure (54) are that the first and second derivatives of γ taken with respect to X_A be respectively zero and positive; these requirements being expressed by

$$\left. \frac{\partial \gamma}{\partial X_A} \right|_E = 0 \quad \text{and} \quad \left\| \left. \frac{\partial^2 \gamma}{\partial X_A \partial X_B} \right|_E \right\| \text{ is positive semi-definite.} \quad (55)$$

Investigation of (53) under (55)₁ with $X_A = T_{,k}$ yields the result that the flux of internal energy in equilibrium, rather than being zero, is given by ¹²

$$q_i \Big|_E = -\epsilon_{ijk} \delta_j^M \Big|_E \quad \text{or by} \quad q_i \Big|_E = -M \epsilon_{ijk} \delta_j^B \Big|_E \quad (56)$$

where use has been made of (C32) and (13).

The consequence of applying (55)₁ to (53) with $X_A = V_{,a}$ is that the interaction force density in equilibrium may be written as

$$\hat{m}_i \Big|_E = \frac{\partial \rho (\epsilon - Tn) \Big|_E}{\partial \rho_a} \rho_{,i} - \sum_{b=1}^s \frac{\partial (t_{ij} + c_{ij}) \Big|_E}{\partial \rho_b} \rho_{,b,j} ; \quad (57)$$

On magnetizable fluid mixtures

With regard to plasmas the special case of magnetizable fluids is once again of particular interest. Hence, although (57) can also be examined for situations of non-vanishing $\rho_{,i}^L$ ¹³, I do not consider this point here.

From (8), (46) and (49) the relation (57) can be written in the form

$$\hat{m}_i \Big|_E = \sum_{b=1}^s \left[\left[-\rho \frac{\partial \mu}{\partial \rho_b} + \vec{B} \cdot \frac{\partial \vec{M}}{\partial \rho_b} \Big|_E \right] \delta_{ij} - \frac{\partial (t_{ij} \Big|_E + B_j^M \Big|_E)}{\partial \rho_b} \right] \rho_{,b,j} . \quad (58)$$

It may in addition be noticed from (57) together with (B154) that (cf. (B90))

$$\sum_{a=1}^s \hat{m}_i \Big|_E = 0. \quad (59)$$

A comparison of (C33)₁, (22)₁ and (25) suggests the decomposition

$$t_{ij} \Big|_E + B_j M_i \Big|_E = \left\{ -\frac{\star}{a} + \frac{\vec{B} \cdot \vec{M}}{a} + \rho \frac{\partial I^M}{\partial \rho} \right\} \delta_{ij} + t^3 \Big|_E \epsilon_{ijk} B_k$$

where

(60)

$$I^M = \frac{1}{2} \int_0^{B^2} M \Big|_E dB^2.$$

Differentiation of this expression with respect to ρ leads to

$$\begin{aligned} \frac{\partial (t_{ij} \Big|_E + B_j M_i \Big|_E)}{\partial \rho} \Big|_b &= \left\{ -\frac{\partial \star}{\partial \rho} \Big|_b + B \cdot \frac{\partial \vec{M}}{\partial \rho} \Big|_b + \frac{\partial}{\partial \rho} \left[\rho \frac{\partial}{\partial \rho} \left(\frac{I^M}{\rho} \right) \right] \right\} \delta_{ij} + \\ &+ \frac{\partial t^3}{\partial \rho} \Big|_b \epsilon_{ijk} B_k; \end{aligned} \quad (61)$$

and, using (47) and this last result in (58) gives

$$\hat{m}_i \Big|_E = \sum_{b=1}^s \left\{ \left[\frac{\partial \star}{\partial \rho} \Big|_b - \rho \frac{\partial}{\partial \rho} \Big|_b - \rho \left(\delta_{ab} - \frac{\rho}{b} \frac{\partial}{\partial \rho} \left(\frac{I^M}{\rho} \right) \right) \right] s_{ij} - \frac{\partial t^3}{\partial \rho} \Big|_b \epsilon_{ijk} B_k \right\} \rho_{,j}. \quad (62)$$

Linearized constitutive equations

Under equilibrium conditions evaluation of the representation (C30) of the internal energy flux gives that

$$\begin{aligned} L_i \Big|_E = & \sum_{a=1}^s (q_a^1 \Big|_E \delta_{ik} + q_a^2 \Big|_E \epsilon_{ijk} B_j + q_a^3 \Big|_E (B_i B_k)_\rho \Big|_E)_{a,k} + \\ & + (q^{10} \Big|_E \delta_{ik} + q^{11} \Big|_E \epsilon_{ijk} B_j + q^{12} \Big|_E (B_i B_k) \delta_k). \end{aligned} \quad (63)$$

Comparison of this result with (57) leads to the conclusion that

$$L_i \Big|_E = q_a^2 \Big|_E = q_a^3 \Big|_E = q^{10} \Big|_E = q^{12} \Big|_E = 0 \text{ while } q^{11} \Big|_E = M. \quad (64)$$

Hence, in general q_i^L is here such that ¹⁴

$$\begin{aligned} L_i \Big|_E = & \sum_{a=1}^{s-1} (q_a^4 \delta_{ik} + q_a^5 \epsilon_{ijk} B_j + q_a^6 (B_i B_k)_\rho \Big|_E)_{a,k} + \\ & + (q^7 \delta_{ik} + q^8 \epsilon_{ijk} B_j + q^9 (B_i B_k)_T)_{,k} + \\ & + M \epsilon_{ijk} B_j \delta_k. \end{aligned} \quad (65)$$

Remembering that \hat{c} has been taken to vanish, it is possible in a similar fashion to that above to consider the linearized representation (C28) for \hat{m}_i under equilibrium conditions. The resulting expression is for all constituents of the form

$$\begin{aligned} L_i \Big|_E = & \sum_{b=1}^s (m_{ab}^1 \Big|_E \delta_{ik} + m_{ab}^2 \Big|_E \epsilon_{ijk} B_j + m_{ab}^3 \Big|_E (B_i B_k)_\rho \Big|_E)_{b,k} + \\ & + (m_a^{10} \Big|_E \delta_{ik} + m_a^{11} \Big|_E \epsilon_{ijk} B_j + m_a^{12} \Big|_E (B_i B_k) \delta_k). \end{aligned} \quad (66)$$

From (62) the conclusion drawn regarding (66) is that

$$L_{m_{ab}} \Big|_E = \frac{\partial p^*}{\partial \rho} \Big|_b - \rho \frac{\partial \mu^*}{\partial \rho} \Big|_a - \rho \left(\delta_{ab} - \frac{\rho}{a} \frac{\partial}{\partial \rho} \left(\frac{1}{\rho} \right) \right) \quad (67)$$

$$L_{m_{ab}}^2 = - \frac{\partial t^3}{\partial \rho} \Big|_b, \quad L_{m_{ab}}^3 = 0$$

and that

$$\text{either } L_{m_a^{10}} \Big|_E = L_{m_a^{11}} \Big|_E = L_{m_a^{12}} \Big|_E = 0 \text{ or } \xi_k = 0. \quad (68)$$

In general, for an equilibrium in V the electromotive intensity ξ_k need not vanish; neither here by definition nor empirically. I thus assume henceforth that the restrictions (68)₁ on the interaction force density coefficients represent the physically more reasonable conditions to be satisfied.

Two meaningful implications of the assumption on non-zero ξ_k in equilibrium may now be mentioned. First, the definition given earlier for dielectrics (cf. Sect. B.4.) is seen to be more general than that usually given. And second, regardless of the capacity of the mixture to conduct electricity, the flux of internal energy is seen from (56) to also be non-zero.

Lastly here, it follows from (628), together with (67) and (68)₁, that here

$$L_{\hat{m}_i^a} = \frac{s}{\sum_{b=1}^s} \left\{ \left[\frac{\partial p^*}{\partial \rho} \Big|_b + \rho \frac{\partial \mu^*}{\partial \rho} \Big|_a - \rho \left(\delta_{ab} - \frac{\rho}{a} \frac{\partial}{\partial \rho} \left(\frac{1}{\rho} \right) \right) \right] \delta_{ij} - \right.$$

$$\left. - \frac{\partial t^3}{\partial \rho} \Big|_b \varepsilon_{ijk} B_k \Big|_{b,j} \right\} +$$

$$\begin{aligned}
& + \sum_{b=1}^{s-1} (m_{ab}^4 \delta_{ik} + m_{ab}^5 \epsilon_{ijk} B_j + m_{ab}^6 B_i B_k) V_k + \\
& + (m_a^7 \delta_{ik} + m_a^8 \epsilon_{ijk} B_j + m_a^9 B_i B_k) T_{,k} . \quad (69)
\end{aligned}$$

E.6.a. Law of mass action

For chemically reacting mixtures the reduced entropy inequality (D49) leads, under the definition of thermodynamic equilibrium in V , to additional conditions on the system. Using (B36) and (B40)₁, the expression in (D49) involving \hat{c} assumes after some manipulation the form

$$-\sum_{a=1}^s (\hat{m}_i^a - \hat{c} v_i^a) u_i^a - \sum_{a=1}^s \sum_{\alpha=1}^n \gamma_{aa}^{\alpha W} \left[\frac{\partial \rho}{\partial \rho} (e^I - T\eta) \right] \Lambda^\alpha . \quad (70)$$

An equilibrium evaluation of (D49) with (70) therein under the conditions

$$\Lambda^\alpha \Big|_E = 0 \quad \text{and} \quad \frac{\partial \gamma}{\partial \Lambda^\alpha} \Big|_E = 0 \quad (71)$$

gives the result that

$$\sum_{a=1}^s \gamma_{aa}^{\alpha W} \frac{\partial \rho (e^I - T\eta)}{\partial \rho} \Big|_E = 0 . \quad (72)$$

On magnetizable fluid mixtures

With the help of (19) and (46) this equilibrium condition becomes

$$\sum_{a=1}^s \gamma_{aa}^{\alpha W} \Big|_E = 0 ; \quad (73)$$

which result is an expression of the so-called law of mass action for a mixture interacting with an electromagnetic field. Introduction herein of the relation (47) for the (magneto-)chemical potential yields

$$\sum_{a=1}^s \gamma_a \omega_a^* - \frac{1}{2} \int_0^{B^2} \frac{\partial M}{\partial \rho} \Big|_E dB^2 = 0 . \quad (74)$$

E.7. Gibbs' equation

In classical non-equilibrium thermodynamics ¹⁵ the local balance equation for the entropy of a mixture follows from an assumption of macroscopic state variables in terms of which a differential of entropy is established. Said entropy is expressible as a function of the state parameters (e.g. internal energy, mass fractions, etc.) and their differentials in such a manner that substitution of the balance equations of these quantities for their counterparts in that functional relation yields the desired result.

It is not my purpose here to attempt an examination of the premises from which such results are derived in that discipline. Rather, I shall show in this section that the essential result (i.e. the differential of entropy equation) with which the classical approach begins follows here as a result of considerations of a very limited special case.

On magnetizable fluid mixtures

It follows from (D18) and (D22) that in an equilibrium for which $P_a \Big|_E$ vanishes

$$T \frac{\partial \eta}{\partial T} \Big|_E = \frac{\partial \epsilon}{\partial T} \Big|_E ; \quad (75)$$

where (19) gives that

$$T\eta|_E = \epsilon|_E - \Psi|_E. \quad (76)$$

Differentiation of (76) with respect to ρ and B^2 yields, respectively, the results

$$T \frac{\partial \eta|_E}{\partial \rho_b} = \frac{\partial \epsilon|_E}{\partial \rho_b} - \frac{\partial \Psi|_E}{\partial \rho_b} \quad \text{and} \quad T \frac{\partial \eta|_E}{\partial B^2} = \frac{\partial \epsilon|_E}{\partial B^2} - \frac{\partial \Psi|_E}{\partial B^2}. \quad (77)$$

Now, since $\eta|_E = \eta(\rho, T, B^2)|_E$,

$$T d\eta|_E = \sum_{b=1}^s T \frac{\partial \eta|_E}{\partial \rho_b} d\rho_b + T \frac{\partial \eta|_E}{\partial T} dT + T \frac{\partial \eta|_E}{\partial B^2} dB^2$$

or, using (77),

$$T d\eta|_E = d\epsilon|_E - \sum_{b=1}^s \frac{\partial \Psi|_E}{\partial \rho_b} d\rho_b - \frac{\partial \Psi|_E}{\partial B^2} dB^2. \quad (78)$$

From (20) the differential of internal energy is found to be

$$\begin{aligned} d\epsilon|_E &= \left(\frac{\partial \epsilon^*}{\partial T} + \frac{1}{2\rho} \int_0^{B^2} T \frac{\partial^2 M^L|_E}{\partial T^2} dB^2 \right) dT + \\ &+ \sum_{b=1}^s \left\{ \frac{\partial \epsilon^*}{\partial \rho_b} - \frac{1}{2\rho} \int_0^{B^2} \left[\frac{\partial}{\partial \rho_b} \left(M^L|_E - T \frac{\partial M^L|_E}{\partial T} \right) - \right. \right. \\ &\left. \left. - \frac{1}{\rho} \left(M^L|_E - T \frac{\partial M^L|_E}{\partial T} \right) \right] dB^2 \right\} d\rho_b - \frac{1}{2\rho} \left(M^L|_E - T \frac{\partial M^L|_E}{\partial T} \right) dB^2. \quad (79) \end{aligned}$$

Further, utilization of (15) and (46) leads to the result that

$$-\sum_{b=1}^s \frac{\partial \Psi|_E}{\partial \rho_b} d\rho_b = -\sum_{b=1}^s \frac{1}{\rho_b} (\mu_b - \Psi|_E) d\rho_b \quad (80)$$

It follows now from a substitution of (12) and (79) into (78) that a Gibbs' equation for a mixture interacting with an electromagnetic field is given here by the relationship

$$\begin{aligned} d\eta|_E = & \frac{1}{T} \frac{\partial \epsilon|_E}{\partial T} dT + \frac{1}{T} \sum_{b=1}^s \left(\frac{\partial \epsilon|_E}{\partial \rho_b} - \frac{\mu_b}{\rho} + \frac{\Psi|_E}{\rho} \right) d\rho_b + \\ & + \frac{1}{T} \left(\frac{\partial \epsilon|_E}{\partial B^2} + \frac{1}{2\rho} M|_E \right) dB^2. \end{aligned} \quad (81)$$

An expression equivalent to this one, but written in terms of the total density ρ and mass functions c_b , can be secured through the use of $c_b = \rho_b/\rho$ (cf. Sect. B.b.), their derivatives and (52): it is

$$\begin{aligned} d\eta^* = & \frac{1}{T} \frac{\partial \epsilon^*}{\partial T} dT + \frac{1}{T} \left(\frac{\partial \epsilon^*}{\partial \rho} - \frac{1}{2} p^* \right) d\rho + \\ & + \frac{1}{T} \sum_{b=1}^{s-1} \left[\frac{\partial \epsilon^*}{\partial c_b} - \left(\frac{\mu_b^*}{\rho} - \frac{\mu_s^*}{\rho} \right) \right] dc_b. \end{aligned} \quad (82)$$

Footnotes to Chapter E

- ¹ A rational mechanical definition of equilibrium is made in terms of those physically well-motivated conditions that must be satisfied in order that the entropy production density (here (D49)) equal zero. It is in particular interesting to observe that for rational mechanical theories with different degrees of generality, but which consider the same subject matter, the requirements for equilibrium need not be the same. For example, single-continuum theories of media interacting with an electromagnetic field such as that of Liu & Müller [1972, 5, p.163] and Müller [1973, 9, p.144] require, say, the non-convective current density $\tilde{\mathcal{J}}_j$ (cf. (B62)₂) to vanish. In this theory, however, such a condition need not, and indeed in general cannot, be set down. For the special case of this theory where $P_i = 0$ and $M_i = 0$ for all constituents the result that $\tilde{\mathcal{J}}_i$ is zero in equilibrium is recovered.
- ² See CFT [1960, 9, p.652, footnote 1]. Note further Liu & Müller [1972, 5, p.164] and Hirschfelder, Curtiss & Bird [1967, 9, p.515 and p.709]. Regarding experience consult, e.g., Kantrowitz & Petschek [1957, 3, p.5] and Sutton & Sherman [1965, 16, p.212]. Note also footnote F12.
- ³ With the exception of setting down the condition (55)₂ and the discussion given in Sect. F.3. I do not approach here the difficult matter of stability. It may be worthwhile to point out that the use of linearized constitutive equations is made, with physical motivation, once again for purposes of developmental simplicity; and that the absence of stability considerations here is not a consequence of that fact.

- ⁴ See footnote C35.
- ⁵ The kinematics of mixtures of magnetizable fluids here depends upon both the electric and magnetic fields. The material response thereof has, however, been taken to depend only upon the magnetic flux density. For such systems in the absence of said field, it thus follows that the constitutive relations are reduced in their descriptive content essentially to expressions appropriate to such media in the absence of an electromagnetic field.
- ⁶ Cf., for example, Müller [1973, 9, pp.174-176] and note the difference in approach there (and thus here) with that of Müller [1968, 4, p.24].
- ⁷ This point is discussed somewhat by Müller [1973, 9, pp.177-173]. From a "practical" viewpoint (cf. Slattery [1972, 8, p.499]) the generally difficult empirical determination of chemical potentials may serve as a motivation to avoid their introduction into theoretical considerations such as these. That such potentials can be measured under certain circumstances may be seen in the discussion of, say, Sage, [1965, 14, Chpt. 9]. I take the position that the usefulness in principle of chemical potentials should stimulate efforts to exploit same in practice.
- ⁸ Denbigh [1971, 1, p.215]. Note further CFT [1960, 9, pp.649-650] and footnote D3.
- ⁹ This may be seen by considering the definitions (B59), (B135), (B152)₁, (C32), (29) and (47) for the various quantities of concern.
- ¹⁰ Slattery [1972, 8, p.482].
- ¹¹ Cf. Liu & Müller [1972, 5, p.163].

- ¹² This interesting and potentially important result for plasma physics research was first obtained and discussed by Liu & Müller [1972, 5, p.169 and pp.173-174] for the special case of electrically non-conducting magnetizable fluids. It was found there as a consequence of the use of the particular definition(s) of equilibrium they employed as were dictated by the single-fluid nature of their theory. Here, by virtue of the mixture character of the model, the material response of the systems involved is such that this result is generally applicable to both electrical conductors and non-conductors.
- ¹³ Cf. Benach & Müller [1974, 1].
- ¹⁴ It may be noted that the coefficient κ_{ik}^0 in the abbreviated expression for (C30) is zero.
- ¹⁵ E.g., de Groot & Mazur [1963, 5].

F. ON SPECIFIC PLASMA SYSTEM FIELD EQUATIONS

The foregoing chapters have served to establish a particular general rational dynamic model for a multi-continuum fluid medium interacting with an electromagnetic field. The methodology of the approach involved initially the consideration of a single, physically identifiable, but otherwise arbitrary constituent; and, a treatment of its properties was seen to lead in turn to those of the class of mixtures considered here.

A total (plasma) system consists configurationally of two parts: the system interior and the system boundary. Clearly, a "complete" phenomenological description of any given material system should when necessary be able to present a treatment of each of these parts together with a physically acceptable representation of their interaction. Real media (e.g. MFD plasmas) due to their inherent physical complexity do not, however, seem in general to lend themselves to such a detailed global analysis as that indicated here. On the basis of this fact I have ¹ restricted my attention primarily to the system bulk (i.e. interior); having accommodated the system boundary by means of a discontinuity surface representation.

An adequate treatment of a given system has been assumed ² to be expressible in terms of a physically well-motivated set of field equations; and it is to these relations that I again direct attention in the next two sections. Thereafter I discuss briefly in sections F.3. and F.4. respectively some general aspects of empiricism, and boundary and initial conditions, as related to this study.

F.1. System interior equations

It is, of course, the specific material media and field quantities of a given physical system which determine the composition of the interior thereof. Regarding a theoretical possibility which I deem relevant to both the general problem ³ and illustrative of my purpose ⁴ here, I propose to give now a résumé of the expressions of balance and material response to be gotten from this study for the following plasma system.

Aside from the electromagnetic field, the mixture is taken to consist materially of three continua: viz., an electron, an ion (of one type), and an electromagnetically neutral gas continuum; where, in addition, the last two constituents are in their respective "ground energy states". As an example of such a system, one may think of the system as being a tertiary plasma mixture consisting of electrons and, say, neutral argon atoms and the first ionized constituent thereof.

F.1.a. Balance, conservation and constitutive equations

For the case at hand I now identify the three partial continua involved by letting the subscript a in the arbitrary constituent relations derived earlier assume only the values $a = 1, 2$ and 3 ; these denoting respectively the electron, ion and neutral gases. For the sake of convenience only, I take as the reference (i.e. the s^{th}) constituent the neutral gas continuum; and, I consider here only such mixtures as are "close to equilibrium". With their general forms having been discussed earlier, I now summarily present the relationships of concern here.

Constituent mass balances

$$\frac{\partial}{\partial t} + \frac{\rho v}{aa} j_{,j} = \frac{L}{a} \bar{c} \quad (1)$$

where

$$\frac{L}{a} \bar{c} = \sum_{\alpha=1}^3 \gamma_{aa}^{\alpha} W_{m\Lambda}^{\alpha} \frac{L}{a} \quad (2)$$

Constituent charge balances

$$\frac{\partial Q}{\partial t} + \frac{J_T}{aj,j} = \frac{L}{aa} q \bar{c} \quad (3)$$

where

$$\frac{L}{Q} = \frac{Q^F}{a} - \frac{L}{P_{a1,1}}, \quad Q^F = \frac{q\rho}{aa}, \quad \frac{J_T}{aj} = \frac{Qv}{aa} + \frac{L}{\delta_j}, \quad (4)$$

$$\frac{L}{\delta_j} = \epsilon_{jkl} \frac{L}{M_{a1,k}} + \frac{\partial P_j}{\partial t} + v_{aj} \frac{L}{P_{a1,1}} + \epsilon_{jkl} \left(\frac{L}{\vec{P}} \times \frac{L}{\vec{v}} \right)_{1,k},$$

$$\frac{L}{M_i} = \frac{L}{M(1)B_i}, \quad \frac{L}{P_i(1)} = 0 \quad \text{and for } a = 2 \text{ and } 3, \quad (5)^5$$

$$\begin{aligned} \frac{L}{P_i} &= \sum_{b=1}^3 [P_{ab}^1(1) \delta_{ik} + P_{ab}^2(1) \epsilon_{ijk} B_j + P_{ab}^3(1) B_i B_k]_{\rho, k} + \\ &+ \sum_{b=1}^2 [P_{ab}^4(1) \delta_{ik} + P_{ab}^5(1) \epsilon_{ijk} B_j + P_{ab}^6(1) B_i B_k]_{\nu, k} + \end{aligned}$$

$$\begin{aligned}
 & + [P_a^7(l)\delta_{ik} + P_a^8(l)\epsilon_{ijk}B_j + P_a^9(l)B_iB_k]T_{,k} + \\
 & + [P_a^{10}(l)\delta_{ik} + P_a^{11}(l)\epsilon_{ijk}B_j + P_a^{12}(l)B_iB_k]\delta_{,k}^3;
 \end{aligned}$$

the material coefficients of the partial polarizations still being functions of the variables l .

Linear momentum balance equations

$$\rho \dot{v}_i - t_{,a}^{ij,j} - Q\delta_i - \epsilon_{ijk}t_{,a}^{jk}B_k - \rho b_i = \bar{m}_i - \bar{c}v_i \quad (6)$$

where

$$\begin{aligned}
 t_{,a}^{ij} &= t_a^1(l)\delta_{ij} + t_a^2(l)B_iB_j + t_a^3(l)\epsilon_{ijk}B_k, \\
 \delta_{,a}^i &= E_i + \epsilon_{ijk}v_{,a}^jB_k,
 \end{aligned} \quad (7)$$

and for all three constituents

$$\begin{aligned}
 \bar{m}_i^L &= \sum_{b=1}^3 \left\{ \left[\frac{\partial P}{\partial \rho} + \rho \frac{\partial \mu^*}{\partial \rho} - \rho \left(\delta_{ab} - \frac{\rho}{\rho} \right) \frac{\partial}{\partial \rho} \left(\frac{I^M}{\rho} \right) \right] \delta_{ij} - \right. \\
 & \left. - \frac{\partial t^3}{\partial \rho} \Big|_E \epsilon_{ijk}B_k \right\}_{,b,j} + \\
 & + \sum_{b=1}^2 [m_{ab}^4(l)\delta_{ik} + m_{ab}^5(l)\epsilon_{ijk}B_j + m_{ab}^6(l)B_iB_k]v_{,b}^k + \\
 & + [m_a^7(l)\delta_{ik} + m_a^8(l)\epsilon_{ijk}B_j + m_a^9(l)B_iB_k]T_{,k}. \quad (8)
 \end{aligned}$$

Mixture angular momentum balance relationship

$${}^L t_{ij} = {}^L t_{ji} \quad (9)$$

where

$${}^L t_{ij} = \sum_{a=1}^3 ({}^L t_{ij} - \rho u_i u_j), \quad u_i = v_i - v_i \quad \text{and} \quad v_i = \frac{1}{\rho} \sum_{a=1}^3 \rho v_i \quad (10)$$

Mixture internal energy balance equation

$$\rho(\epsilon)_{,i} - {}^L t_{ij} v_{i,j} - q_{j,j} - \rho r - \delta_j \delta_j = 0 \quad (11)$$

where

$$\rho = \sum_{a=1}^3 \rho_a, \quad \epsilon = \frac{1}{\rho} \epsilon^L + \frac{1}{\rho} \sum_{a=1}^3 \frac{1}{2} \rho u_a^2, \quad (12)$$

$$\epsilon^L = \epsilon^L(1), \quad (13)$$

$$q_j = q_j^L - \sum_{a=1}^3 [{}^L t_{ij} u_i + \rho \frac{L}{a} (\epsilon_a + \frac{1}{2} u_a^2) u_j], \quad (14)$$

$$\begin{aligned} q_j^L = & \sum_{a=1}^2 [q_a^L(1) \delta_{jk} + q_a^L(1) \epsilon_{j1k} B_1 + q_a^L(1) B_j B_k] v_k + \\ & + [q^L(1) \delta_{jk} + q^L(1) \epsilon_{j1k} B_1 + q^L(1) B_j B_k] T_{,k} + \\ & + M(1) \epsilon_{j1k} B_1 \delta_k, \end{aligned}$$

$$\rho r = \sum_{a=1}^3 \rho_a (r_a + b_{ia} u_i) , \quad (15)$$

$$\begin{aligned} \delta_j^L = & \sum_{a=1}^3 [Q_a^F u_j^a + \epsilon_{jkl} M_{a,l,k}^L + \frac{\partial P_a^j}{\partial t} + \\ & + v_{ja}^L P_{a,l} + \epsilon_{jkl} (\vec{p}_a \times \vec{v}_a)_{l,k}^L] , \end{aligned} \quad (16)$$

and

$$\delta_j = E_j + \epsilon_{jkl} v_k B_l . \quad (17)$$

Mixture entropy balance equation

$$\rho(\eta)^L + \left[\frac{1}{T} (q_i^L + \sum_{a=1}^3 \epsilon_{ijk} \delta_j^L M_{a,k}^L - \sum_{a=1}^3 C_{aji}^L u_j^a) \right]_{,i} - \frac{\rho r_1}{T} \geq 0 \quad (18)$$

where

$$\eta = \eta(t) , \quad (19)$$

$$C_{aji}^L = C_a^1(t) \delta_{ji} + C_a^2(t) B_j B_i + C_a^3(t) \epsilon_{jik} B_k \quad (20)$$

and

$$\rho r_1 = \sum_{a=1}^3 \rho_a r_a . \quad (21)$$

Electric flux density equation

$$\mathcal{D}_{i,i}^L = Q^F \quad (22)$$

where

$$\mathcal{D}_i^L = D_i + P_i^L, \quad D_i = \epsilon_0 E_i, \quad (23)$$

$$P_i^L = \sum_{a=2}^3 P_a^L \quad \text{and} \quad Q^F = \sum_{a=1}^2 Q_a^F. \quad (24)$$

Mixture current balance equation

$$\epsilon_{ijk} \mathcal{K}_{k,j}^L = Q^F v_i + J_i^F + \frac{\partial \mathcal{D}_i^L}{\partial t} \quad (25)$$

where

$$\mathcal{K}_k^L = H_k - M_k^L - \sum_{a=2}^3 (\vec{P}_a^L \times \vec{v}_a)_k, \quad (26)$$

$$H_k = \frac{1}{\mu_0} B_k \quad \text{and} \quad J_i^F = \sum_{a=1}^2 Q_a^F u_{a,i}. \quad (27)$$

Mixture charge conservation equation

$$\frac{\partial Q}{\partial t} + J_{j,j}^L = 0 \quad (28)$$

where

$$Q = \sum_{a=1}^2 Q_a^F - \sum_{a=2}^3 P_{a,1}^L \quad \text{and} \quad J_j^L = Q v_j + \mathcal{J}_j^L. \quad (29)$$

Magnetic flux density conservation equation

$$B_{i,i} = 0. \quad (30)$$

Electromagnetic induction equation

$$\epsilon_{ijk} E_{k,j} = - \frac{\partial B_i}{\partial t}. \quad (31)$$

It will be remembered⁶ that a primary aim of such a theory of mixtures as that given here is the determination of the field quantities

$$\rho_a(\vec{x},t), \quad T(\vec{x},t), \quad v_a(\vec{x},t), \quad E_i(\vec{x},t) \quad \text{and} \quad B_i(\vec{x},t); \quad (A1)$$

which, in this particular case (i.e. $a = 1,2,3$), are 19 in number. All of the expressions above are clearly not independent. However, those relationships thereof which in principle make said calculation possible and for which the system of equations and unknowns is determinate⁷ are the following: viz., (1), (6), (11), (18), (25) and (31). After setting down in Sect. F.2. the equations found here for the boundaries and interfaces of a plasma system, I shall comment somewhat further in Sect. F.3. with regard to the problem of solving these relationships.

F.1.b. On mass and charge transport

It is of considerable theoretical interest and practical importance that physical insight additional to that given by (B12) be gained here into the nature of constituent mass and charge diffusion in a multi-continuum mixture of fluids. While the mathematical investigation of this subject was begun by Fick⁸ in the mid-nineteenth century, it remains yet a problem area intensively studied. Thus, it seems that it can be fairly stated here that the general treatment of diffusion for plasmas leaves much to be desired⁹.

This section has as its foremost purpose the presentation of those expressions for mass and charge flux that result from a unified mechanical, electromagnetic and energetic non-linear theory of diffusion¹⁰. For the sake of convenience, however, I shall employ the theory above in a severely restricted form in order that a special case of particular interest here may be considered¹¹.

On magnetizable fluid mixtures

The development here proceeds from the balance equation (B85)₁ for the linear momentum of an arbitrary mixture constituent; and it may, with the use of (B15), be expressed for all s constituents in the form

$$\begin{aligned} \rho \dot{v}_i^a + u_{aj}^j v_i^a{}_{,j} - t_{aj}^j - Q \mathcal{E}_i^a - \epsilon_{ijk} \mathcal{J}_j^a B_k - \rho b_{a0}^i &= \\ &= \dot{m}_i^a - \partial_{aa} v_i^a. \end{aligned} \quad (32)$$

The result of dividing (32) by ρ and subtracting the expression secured for the s^{th} (i.e. the reference) constituent from that for any other constituent a is given by

$$\begin{aligned} (v_i^a - v_i^s) \cdot + (u_{aj}^j v_i^a{}_{,j} - u_{sj}^j v_i^s{}_{,j}) - \frac{1}{\rho} t_{aj}^j + \frac{1}{\rho} t_{sj}^j - \\ - \left(\frac{1}{\rho} Q \mathcal{E}_i^a - \frac{1}{\rho} Q \mathcal{E}_i^s \right) - \left(\frac{1}{\rho} \mathcal{J}_j^a - \frac{1}{\rho} \mathcal{J}_j^s \right) \epsilon_{ijk} B_k - \left(\frac{b_i^a}{\rho_a} - \frac{b_i^s}{\rho_s} \right) = \\ = \frac{1}{\rho} \dot{m}_i^a - \frac{1}{\rho} \dot{m}_i^s - \left(\partial_{aa} v_i^a - \partial_{ss} v_i^s \right). \end{aligned} \quad (33)$$

I now limit the class of mixtures for which (33) is to be used here to these systems for which chemical reactions, constituent polarizations and terms non-linear in $\rho_{a,i}$, $V_{b,i}$, $T_{a,i}$ and $\xi_{a,i}$ may be neglected; and also for which $\vec{M} = \vec{M}(\rho, T, B^2)$ and the constituent specific magnetizations are density-independent¹². Under these conditions use of linearized relationships for the constitutive fields is acceptable and it thus follows from (B8) and (B47)₂, (B53), (C32), (E60) and (E69) respectively that

$$Q_b = q_{bb} \rho_b, \quad \vec{D}_b = \epsilon_{jkl}^L M_{l,k}, \quad M_b = M_b B_b,$$

$$t_{ij}^L = \left(-\overset{*}{p}_b + \vec{B} \cdot \vec{M} \right) \delta_{ij} - B_j M_i + t_{ijk}^L \epsilon_{ijk} B_k$$

and

(34)

$$\begin{aligned} \bar{m}_i^L = & \sum_{d=1}^s \left[\left(\frac{\partial \overset{*}{p}_b}{\partial \rho_d} - \rho_b \frac{\partial \vec{u}}{\partial \rho_d} \right) \delta_{ij} - \frac{\partial t_{ijk}^L}{\partial \rho_d} \epsilon_{ijk} B_k \right] \rho_{d,j} + \\ & + \sum_{d=1}^{s-1} \left(m_{bd}^L \delta_{ik} + m_{bd}^L \epsilon_{ijk} B_j + m_{bd}^L B_i B_k \right) V_{d,k} + \\ & + \left(m_a^L \delta_{ik} + m_a^L \epsilon_{ijk} B_j + m_a^L B_i B_k \right) T_{,k}. \end{aligned}$$

Introduction of the relations (34) together with the use of (B10), (B48), (D37) and (E46) leads, after some calculation¹³ to the following form of (33); viz.,

$$\sum_{b=1}^{s-1} F_{cb} V_j = \dot{V}_c + \left(\frac{\mu_c^*}{c} - \frac{\mu_s^*}{s} \right) V_c + D_{ck} T_{,k} - E_{ck} B_n B_{n,k} - (q - q) \varepsilon_{cs} - \left(\frac{b_c}{0c} - \frac{b_s}{0s} \right) \quad (35)$$

Here, the coefficients F_{cb} , D_{ck} and E_{ck} are given respectively by ¹⁴

$$F_{cb} = \left(\frac{L_4}{\frac{m_{cb}}{\rho_c} - \frac{L_4}{\frac{m_{sb}}{s}}} \right) \delta_{1j} + \left(\frac{L_6}{\frac{m_{cb}}{\rho_c} - \frac{L_6}{\frac{m_{sb}}{s}}} \right) B_1 B_j + \left[\frac{L_5}{\frac{m_{cb}}{\rho_c} - q \left(\delta_{ab} - \frac{\rho}{c} \right) - \frac{L_5}{\frac{m_{sb}}{s}}} - q \left(\delta_{sb} - \frac{\rho}{s} \right) \right] \varepsilon_{1kj} B_k \quad (36)$$

$$D_{ck} = \left\{ \frac{\partial}{\partial T} \left[\left(\frac{\dot{p}_c}{\rho_c} - \frac{\dot{p}_s}{s} \right) - \left(\frac{\mu_c^*}{c} - \frac{\mu_s^*}{s} \right) \right] - \left(\frac{L_7}{\frac{m_c}{\rho_c} - \frac{L_7}{\frac{m_s}{s}}} \right) \right\} \delta_{ik} - \left(\frac{L_9}{\frac{m_c}{\rho_c} - \frac{L_9}{\frac{m_s}{s}}} \right) B_i B_k - \left[\left(\frac{L_8}{\frac{m_c}{\rho_c} - \frac{L_8}{\frac{m_s}{s}}} \right) - \frac{\partial}{\partial T} \left(\frac{L_3}{\frac{c}{\rho_c} - \frac{s}{s}} \right) \right] \varepsilon_{ijk} B_j \quad (37)$$

and

$$E_{ck} = \left(\frac{L}{\frac{c}{\rho_c} - \frac{L}{\frac{s}{s}}} \right) \delta_{1k} - 2 \frac{\partial}{\partial B^2} \left(\frac{L_3}{\frac{c}{\rho_c} - \frac{L_3}{\frac{s}{s}}} \right) \varepsilon_{1jk} B_j \quad (38)$$

Now, with

$$V_j = \sum_{d=1}^{s-1} G_{bdd} u_j \quad \text{where} \quad G_{bd} = \left(\delta_{bd} - \frac{d_j}{s} \right), \quad (39)$$

the relationship secured by solving (35) for the diffusion flux of mass of an arbitrary constituent a is given here by

$$\begin{aligned} \rho_{aa} u_i = & \sum_{b=1}^{s-1} \sum_{c=1}^{s-1} \rho_{abac} (G^{-1} F^{-1})_{il} \left[\sum_{d=1}^{s-1} (G_{cdd} u_d)^* + \left(\frac{\dot{u}_c}{s} - \frac{\dot{u}_s}{s} \right)_{,l} + D_{c1k} T_{,k} - \right. \\ & \left. - E_{lk} B_n B_{n,k} - \left(\frac{q_c}{s} - q \right) \delta_l - \left(\frac{b_{c1}}{0c_1} - \frac{b_{s1}}{0s_1} \right) \right]; \quad (40) \end{aligned}$$

or, with the introduction of

$$H_{ac}^{-1} = \sum_{b=1}^{s-1} G_{abac}^{-1} F_{il}^{-1}, \quad (41)$$

by

$$\begin{aligned} \rho_{aa} u_i = & \sum_{c=1}^{s-1} \rho_{aac} H_{ii}^{-1} \left[\sum_{d=1}^{s-1} (G_{cdd} u_d)^* + \left(\frac{\dot{u}_c}{s} - \frac{\dot{u}_s}{s} \right)_{,l} + D_{c1k} T_{,k} - \right. \\ & \left. - E_{lk} B_n B_{n,k} - \left(\frac{q_c}{s} - q \right) \delta_l - \left(\frac{b_{c1}}{0c_1} - \frac{b_{s1}}{0s_1} \right) \right]. \quad (42) \end{aligned}$$

This relationship, a generalization of Fick's first equation of diffusion¹⁵, has been cast into a form permitting a rough comparison with those expressions secured from, say, "classical" particle approach considerations¹⁶; which results it also generalizes. Aside from the possibly important acceleration term $(G_{cdd} u_d)^*$ ¹⁷ and the extended physical scope of the $T_{,k}$ coefficient¹⁸, it is of particular interest here to note the presence of a new term which is related to the non-uniformity of \vec{B} ¹⁹.

Now, the non-convective electric current density is given for the special case under consideration here by (862) without the polarization terms. It follows from the introduction of (42) into that relation that the generalization here of Ohm's equation for the diffusion flux of charge is given by

$$\begin{aligned} \vec{j}_i = & \sum_{a=1}^s \sum_{c=1}^{s-1} q_a H_{aac}^{-1} \left\{ \sum_{d=1}^{s-1} \frac{1}{q_d} \left[\frac{1}{\rho_d} \delta_{cd} - \frac{1}{\rho_s} (q_d u_d) \right] \right\} + \left(\vec{u}_c - \vec{u}_s \right)_i + \\ & + \left. E_{ck} B_{nk} - \left(\begin{matrix} b_{c1} \\ 0 \end{matrix} - \begin{matrix} b_{s1} \\ 0 \end{matrix} \right) \right\} + K_{ik} T_{,k} + L_{il} \delta_l + (\vec{\nabla} \times \vec{M}|_E)_i. \quad (43) \end{aligned}$$

Here, the thermo-electric²⁰ and electrical conductivity tensors are denoted, respectively, by²¹

$$K_{ik} = \sum_{a=1}^s \sum_{c=1}^{s-1} q_a H_{aac}^{-1} D_{ic}^D \quad \text{and} \quad L_{il} = \sum_{a=1}^s \sum_{c=1}^{s-1} q_a (q_c - q_s) H_{ac}^{-1}. \quad (44)$$

F.2. System boundary and interface equations

In general, a surface of discontinuity can represent not only a mechanical, electromagnetic and/or energetic boundary or interface for the total material system in V ; but also, depending upon the physical circumstances involved, an interface between, and a boundary of regions therein.

Boundaries and interfaces of, say, a plasma (region) in V constitute in themselves physical systems the treatment of which, together with that for the (rest) of the system interior, may in certain cases²² be absolutely essential to the securing of an adequate description of the behavior of the plasma as a whole. Such a treatment could perhaps follow the methodology employed here for the study of the interior of the system²³; but, I consider this important problem too as lying beyond the scope of this initial study.

As was the case in Sect. F.1.a., the relevant jump balance equations here have been examined earlier. Hence, with $a = 1, 2$ and 3 still denoting respectively the electrons, ions and neutral gas continua, I consider it sufficient to present them now without further comment.

Constituent mass equation

$$\llbracket \rho_a (v_i - u_i) \hat{n}_i \rrbracket = \hat{c}_a^S. \quad (45)$$

Constituent charge equation

$$\llbracket Q_a (v_i - u_i) \hat{n}_i \rrbracket + \llbracket \hat{J}_i \hat{n}_i \rrbracket = \hat{c}_a^{SC}. \quad (46)$$

Mixture-electromagnetic field linear momentum equation

$$\llbracket v_i \rrbracket \rho_a (v_j - u_j) \hat{n}_j - \llbracket G_i \rrbracket u_n - \llbracket (t_{ij} + t_{ij}^E) \hat{n}_j \rrbracket = \hat{m}_i^{SM} \quad (47)$$

where

$$\hat{m}_i^{SM} = \sum_{a=1}^3 \hat{m}_i^S - \hat{m}_i^{SE}.$$

Mixture-electromagnetic field energy equation

$$\llbracket \epsilon + \frac{1}{2} v^2 \rrbracket \rho_a (v_i - u_i) \hat{n}_i - \llbracket \epsilon \rrbracket u_n + \llbracket (q_i - t_{ji} v_j - S_i^E) \hat{n}_i \rrbracket = \hat{e}^{SM} \quad (48)$$

where

$$\hat{e}^{SM} = \sum_{a=1}^3 \hat{e}_a^{SM}.$$

Mixture entropy equation

$$T[\dot{h}] \rho (v_i - u_i) \hat{n}_i + [\dot{q}_i \hat{n}_i] + \left[\sum_{a=1}^3 \epsilon_{ijk} \delta_{ja} M_a^k \hat{n}_i \right] - \left[\sum_{a=1}^3 C_{aji} u_j \hat{n}_i \right] \geq 0. \quad (49)$$

Electric flux density equation

$$[\dot{D}_i] = \epsilon_{ijk} h_j \hat{n}_k + Q^{SF} \hat{n}_i. \quad (50)$$

where

$$Q^{SF} = \sum_{a=1}^2 Q_a^{SF}. \quad (51)$$

Mixture current equation

$$[\dot{C}_k] = g \hat{n}_k + u_n h_k. \quad (52)$$

Mixture charge equation

$$[Q(v_i - u_i) \hat{n}_i] + [\dot{\rho}_i \hat{n}_i] = 0. \quad (53)$$

Magnetic flux density equation

$$[\dot{B}_i] = \epsilon_{ijk} k_j \hat{n}_k. \quad (54)$$

Electromagnetic induction equation

$$[\dot{E}_i] = f \hat{n}_i - u_n k_i. \quad (55)$$

F.3. On solutions and applications

The mathematical description of the behavior of a given medium under a particular set of circumstances is given by the solution to the set of governing equations of that system appropriate to those conditions. At least in principle, with the exception of boundary and/or initial conditions discussed below, all of the ingredients necessary here to calculate the fields (A1) as functions of position and time for a given (but not arbitrary) physical region V are represented by the set of equations noted earlier in Sect. F.1.

Before examining this point further, there is an important observation that I wish to make regarding the constitutive equations employed in that set of equations. The material response coefficients here are continuous functions²⁴ of $l = \{\rho_a (a = 1,2,3), T, B^2\}$ and not constants²⁵. Further, under the requirement that they satisfy the reduced entropy inequality²⁶, the constitutive coefficients are themselves, either individually or in combination, also subject to certain (in)equality conditions²⁷. Hence, the coefficients of the plasma system equations here are neither arbitrary in the variables upon which they depend, nor entirely unrestricted and independent of one another in terms of their range of variability.

Now, with regard to such equations as these and the matter of their solution, rational mechanics appears²⁸ to view as physically meaningful (but not exclusively²⁹) problems thereto related that are "properly posed". Problems of physical mathematics are called properly posed³⁰ if they satisfy the three-part postulate of Hadamard: viz., a solution for the problem should exist that is unique and stable. This concept rests upon the not

necessarily correct belief ³¹ that a high-fidelity mathematical description of the physical nature of a material system is securable from its governing equations together with the "data" (e.g. ³² the boundary and/or initial conditions ³³, constitutive coefficients and configurational aspects of the system) indicated therein. Although this approach seems adequate in most cases ³⁴, there are ³⁵ an increasing number of physically meaningful problems which in one or more respects do not satisfy the requirements above; i.e., the so-called "improperly posed" problems ³⁶.

On the existence of solutions

While the physical phenomena related to a given material system exist in their own right, it is desirable to ensure that solutions of the descriptive equations of that system also exist. Unfortunately, investigation of here relevant problems such as, e.g., domains of existence ³⁷, the relationship of solution existence to problem data ³⁸, and the determination of necessary and sufficient conditions for solution existence in general has hardly begun ³⁹. With the exception of certain important results (e.g. the Cauchy-Kovalevsky theorem), it seems at the present time that the difficult matter of demonstrating the existence of a solution is in all but relatively few situations unsettled.

On the uniqueness of solutions

The issue of descriptive completeness for a given problem is generally taken ⁴⁰ to be resolved with the establishment of a solution thereof that is uniquely determined by the data of that problem. Although the question of uniqueness is supposedly less difficult than that of existence ⁴¹, here

too relatively little study has been devoted to the problem of uniqueness in general ⁴² and to the relationship of arbitrary (e.g. non-analytic) data to solution uniqueness in particular ⁴³.

Further, aside from the fact that there are cases where existence theorems are invalid but for which uniqueness can be proved ⁴⁴, there are more importantly problems for which this condition is as such considered an inappropriate one for a solution to fulfil ⁴⁵. There are, however, cases for which this aspect of a properly-posed problem can be salvaged; but then, by means of additional a priori assumptions which restrict either the constitutive equations or the boundary conditions ⁴⁶ (i.e. the allowed solutions ⁴⁷) of that problem. From a practical viewpoint, i.e. one of application, it is clear ⁴⁸ in any case that the criterion of uniqueness is a desirable one to be met by any given solution.

On the stability of solutions

This last and very significant condition stipulates ⁴⁹ that the solution to a given problem must depend in a continuous manner upon the data of that problem. The fulfilment of the stability requirement thus necessitates that for "sufficiently small" ⁵⁰ changes in said data, the solution thereto related must in turn experience an arbitrarily small change.

There are ⁵¹, however, meaningful and here relevant problems for which this condition cannot be satisfied in an arbitrary yet total physical region V ⁵². Generally speaking, the content of this requirement appears ⁵³ to also need a great deal more investigation; and this with particular regard to the determination of criteria to be met by both data and solutions in order to ensure satisfaction of this postulate ⁵⁴.

Now, it will be remembered (cf. Sect. A.2.b.) that the calculation of the fields (A1) was taken to be the main aim of this type of mixture theory. And further, that (cf. Sect. D...) it is in particular those solutions for a region V which are physically admissible that are of concern in this study.

By virtue of the structure of the system equations above it can readily be seen that at this level of developmental generality very little can be said concerning the desired solutions. From what has been said above it should be clear that it is not always possible, or even necessary, that a physical problem be properly posed. Although such problems do appear in practice to be dominant ⁵⁵, the generally held contention ⁵⁶ that only such problems are physically meaningful is known by a small but growing number of researchers to be incorrect ⁵⁷. The essential difference between problems that are properly posed and those that are not lies ⁵⁸ in the role played by the data of a given physical problem with regard to the mathematical model thereof.

It is thus important to note that in general the matter regarding the approach(es) by which the class of admissible thermodynamic processes in V for the fields (A1) can be established remains an open one here. This does not mean that further systematic progress towards the goal set is no longer possible. On the contrary, from a comprehensive comparative study of these and other better understood relations, additional insight into the mathematical character of said solutions and the system data could be gained. Such an undertaking does not, however, fall within the scope of this endeavor.

On applications

The approach from this point onwards to the system of general partial differential equations given in Sect. F.1.a. is essentially the same as that elsewhere in mathematical physics. Their applicability to the investigation of any given detail problem obviously requires that they be made (more) tractable; the degree of simplification depending upon the circumstances of the particular problem involved.

Steps in this direction can be made in two ways. The first entails the introduction of physical approximations such as, e.g., dimensional analysis arguments⁵⁹ which make possible further treatment of a reduced class of dynamically homologous systems⁶⁰. Not unimportant here is the possibility that a rational mechanical development such as that here can, at least in some special and already "well-studied" cases, suggest new experiments.

The next step involves the use of mathematical approximations (e.g. perturbation techniques, etc.). Here too, rational mechanics could aid in the process of determining how its results, the system equations, could be utilized most fully with the aid of modern computing methods. It must be emphasized, however, that the primary mathematical concern at this point is one of determining applicable solutions to said equations and not the formulation thereof. Indeed, while the former matter involves approximations, the latter is "exact"⁶¹.

F.4. On empirical considerations

Generally speaking there seem to be three types of empirical data. First are those data necessary for the physical determination of the constitutive equations required for a given system; such data being possibly of a micro- and/or macroscopic nature. Then there are two kinds of data which, assuming existence etc., relate directly to the solutions of the complete set of macroscopic equations for (regions of) V . One of these is concerned with the determination of the usually dynamic system boundary and initial conditions; while, the other data are those representing the empirical results meant to be compared with some specific theoretical prediction based upon the above mentioned system of equations.

Regarding the microscopic empiricism related to this study I wish to say the following. The investigation of what may be termed the quantum dynamic properties of matter (e.g. interparticle forces) does not fall within the realm of plasma physics as such; and certainly not within that of rational mechanics. Rather, it is the subject matter of disciplines concerned most directly with, say, atomic and molecular physics and physical chemistry. Although such properties undoubtedly determine (to some degree) the macro-nature of a plasma system, a detailed knowledge of them does not follow from plasma considerations.

At present empirical plasma physics concerns itself essentially with the measuremental determination of the parameters entering into the macroscopic system equations (e.g. constituent number densities, temperature(s), etc.); and, the macro-process characteristics of plasmas (e.g. flow properties). The former quantities are of considerable importance in connection with the study of plasmas in general while the latter relate more to the detailed understanding of a particular given plasma system.

There is some understanding in plasma physics of the difficulties involved with regard to the physical determination⁶² and mathematical specification⁶³ of dynamic boundary and/or initial conditions. Unfortunately, however, it appears that much less appreciation is shown there with regard to the significant role played by constitutive equations in the determination of plasma behavior in general.

In this regard it is most important to realize the following fact; viz., knowledge of a micro-physical nature which is indispensable to particle theory approaches to the description of plasma behavior is, generally spoken, at present often either inadequate⁶⁴ or more important inaccessible⁶⁵. This point has great significance with regard to degree of validity, or equivalently the range of applicability, of the results of such approaches.

Rational mechanics, on the other hand, has been shown here to lead to results which are, although simpler in conceptual development, more general and physically relevant than those in common usage. Not only the theoretical but also the practical advantages of this approach can be noted as follows. In general, rational mechanical constitutive equations such as those proposed here (cf. Chpt. C.) have at least the following properties: viz., they are properly invariant and take account of the symmetry properties of the medium in question; they permit the description of non-linear material response; they are applicable to the study of non-homogeneous processes in non-equilibrium systems; they have coefficients that are not constants but functions of macroscopic field variables; they exhibit numerous known and even more unknown independent (i.e. uncoupled) and dependent (i.e. coupled) effects; and, they are susceptible, at least in principle, to determination physically within a program of rational empiricism⁶⁶.

Footnotes to Chapter F

¹ See page 44.

² Here, on page 19; note also Benach & Müller [1974, 1, Chpt. 2].

³ See page 14.

⁴ This point is discussed on page 19.

⁵ As is well-known, electrons possess by virtue of their spins an intrinsic (spin) magnetic moment. It is in terms of this property that a (induced) magnetization density may be assumed to exist for the electron gas continuum; which mixture constituent consists, indeed, wholly of free (i.e. "conduction") electrons.

Now, the electron gas kinematics leading to a diamagnetic condition for the system, the electron cyclotron motions, are described by the balance, conservation and jump balance equations for that constituent. On the other hand, the alignment parallel (and to a lesser degree anti-parallel) to \vec{B} of the electron spins relates to the dynamics of the electron continuum.

It is in particular the condition of paramagnetism resulting from such alignments which is of concern here. The description of this property is, as it should be, a matter for constitutive theory. The spin-dependent character of this latter condition requires a quantum mechanical explanation; while the former does not. A discussion of (aspects of) this matter may be found, e.g., in Bates [1951, 1, pp. 52-53 and p.154], Van Vleck [1952, 2, pp.349-353 and pp.359-360] and Bozorth [1961, 1, p.467].

The retention or neglect of either one or both of these properties in a given problem should depend upon the demonstrated relative (in)-

significance of their respective contributions to each other as well as to the other possible terms which must also be considered in the problem of concern. Regarding the somewhat cavalier attitude taken in plasma physics with regard to this matter, see Grad [1971, 2].

⁶ Cf. page 20.

⁷ The importance of determinacy (i.e. that n equations are needed to "determine" n quantities) is here or elsewhere a relative one; and, the demand for its satisfaction essentially a blind one. Although it is usually considered a "self-evident" requirement to be met in the treatment of a system of equations, there are (cf. Courant & Hilbert [1965, 7, p.231]), for example, over-determined (improperly posed (cf. Sect. F.3.)) problems that are physically meaningful.

In connection with a system of equations and the matter of its determinedness it is worthwhile to note CFT [1960, 9, p.701, footnote 2] and the caution pointed out by Truesdell [1966, 14, pp.117-118].

⁸ Fick [1855, 1].

⁹ Cf. Monroe [1973, 8] and Grad [1971, 3]. It may be further noted here that there exists considerable discrepancy in the literature of plasma physics regarding the acceptability of theoretical results for the problem of diffusion. For example, McDaniel [1964, 5, pp.489-490] expresses the view that the results of Chapman-Enskog theory considerations are "of wide applicability and give good results". In contrast hereto it has been stated by Sutton & Sherman [1965, 17, p.126] that "the Chapman-Enskog method is incorrect in principle when applied to ionized gases". See also footnote A39.

- ¹⁰ The general history of diffusion research that is relevant to this study may be found in CFT [1960, 9, Sect. 295], Truesdell [1962, 10] and Truesdell & Noll [1965, 19, Sect. 130].
- ¹¹ See Benach & Müller [1974, 1, Sect. 5] for a treatment of the somewhat more general case which involves consideration also of the constituent polarizations.
- ¹² Most particularly, a desire once again for developmental simplicity motivates the linearizations assumed here. At least two essential consequences of such linearizations are, first, that the descriptive scope of the theory is restricted to media sufficiently dilute that higher-order terms are justifiably neglected; and second, such systems are "close to equilibrium" (cf. footnote E2). It is perhaps necessary to remark that a non-linear theory that is linearized remains just that: a non-linear and not a linear theory. Lastly, the assumed density-independence of the specific magnetizations does not mean that the magnetizations are density independent.
- ¹³ This treatment parallels that given by Müller [1973, 9, pp.182-185] for the simpler case of non-electromagnetic fluids. Note also in this regard Truesdell [1962, 10].
- ¹⁴ By inspection it is clear, that $F_{cb}{}^{ij}$, $D_c{}^{lk}$ and $E_c{}^{lk}$ all possess the same general form; viz., that of a tensor component $A_{ij} = A_1(1)\delta_{ij} + A_2(1)B_i B_j + A_3(1)\epsilon_{ijk} B_k$.
- ¹⁵ Fick [1855, 1, p.66]. While Fick did not identify his expression thusly, more recent workers in this field have: cf., for example, McDaniel [1964, 5, p.49 and p.489] and Slattery [1972, 8, p.478]. The so-called second equation of Fick (Fick [ibid, eqns. (1) and (2)]) is

noted as such by, e.g., Hirschfelder, Curtiss & Bird [1967, 9, pp. 518-519] and Slattery [1972, 8, p.479]; which relationship is the actual equation for diffusion.

- ¹⁶ Cf. Hirschfelder, Curtiss & Bird [1965, 9, Sects. 11.1 and 11.2] and Sutton & Sherman [1965, 17, p.176].
- ¹⁷ See Müller [1968, 4, p.36] and Müller [1973, 9, pp.185-186].
- ¹⁸ The prediction by Chapman and experimental verification by Dootson (cf. Chapman & Dootson [1917, 1]) of thermal diffusion in gases is, according to Hirschfelder, Curtiss & Bird [1967, 9, p.479], "one of a number of historically interesting instances of the prediction of experimentally observable phenomena by rigorous theoretical analysis"; while Clarke & McChesney [1964, 1, p.149] consider it "one of the triumphs of the theory" (of Chapman and Enskog). I consider it no less noteworthy that such a result emerges naturally here as elsewhere (cf., for example, Müller [1968, 4, Sect. 10]) from the considerations of rational mechanics.
- ¹⁹ Cf. Benach & Müller [1974, 1, (5.6)] which result evidences yet another new term; one arising from the inclusion there of the constituent polarizations in the model treated.
- ²⁰ Thermo-electric effect relationships have been secured and discussed for the case of scalar material coefficients by Spitzer [1962, 8, pp. 143-146]; and, for the more difficult case involving tensor coefficients by Shkarofsky, Johnston & Bachynski [1966, 12, Sect. 8-7 and p.433]. The result presented here generalizes the results of these authors; among others. It may be said in passing that the knowledge (cf. Hall [1964, 2]) of the possibly important role played by the thermo-electric effect in laboratory, say MFD, plasmas appears to

have been neglected by "experimentalists" in this field.

- 21 From (36) and (41) it follows that L_{ik} , similar for example to $F_{cb\ ij}$, also possesses the general form $L_{ik} = L_1(I)\delta_{ik} + L_2(I)B_i B_k + L_3(I)\epsilon_{ijk}B_j$. Here (cf. Alfven & Fälthammar [1963, 1, p.181]), $L_1(I)$, $L_2(I)$ and $L_3(I)$ relate, respectively, to the "Pedersen (i.e. cross-) conductivity", the usual scalar conductivity, and the "Hall conductivity". Note further the absence of mixture rules in these considerations.
- 22 For example, those considerations involving mechanical, electromagnetic and/or energetic interfaces and "boundary layers".
- 23 Cf. the investigation of Fisher & Leitman [1970, 3] concerning a single-continuum approach to this problem.
- 24 See page 108 with regard to this point; noting, that said coefficients are implicit functions of both position and time.
- 25 The development of a rational mechanical theory such as that given here permits the material response coefficients of concern to depend in a physically meaningful and mathematically responsible manner upon, essentially, any number of assumed independent field quantities. On the other hand, the traditional approaches to the description of plasma behavior do not seem to consider the possibility of, say, a \vec{B} -field dependence of said coefficients. This, together with the fact that numerous studies of plasma phenomena utilize arguments of dimensional analysis in which the various non-dimensional parameters are evaluated in terms of these coefficients under conditions of parameter constancy, leads to the following observation: viz., in practice these approaches are severely limited in the theoretical independence of their transport properties on external fields; said properties thus "unwillingly" being forced to act as constants with respect to those fields.

- ²⁶ See the discussion of Chpt. D regarding admissible thermodynamic processes in V.
- ²⁷ Under less general conditions than those taken here the requirement that constitutive equations must satisfy the reduced entropy inequality can be used (cf. Müller [1968, 4, p.23]) to establish explicit conditions of (in)equality to be met by the various material coefficients involved in a given treatment. Due, however, to the excessive number of such coefficients related to this study, I have not undertaken to determine these conditions.
- ²⁸ CFT [1960, 9, p.701], Jaunzemis [1967, 12, p.348] and Bressan [1972, 2, p.5].
- ²⁹ Note, e.g., CFT [ibid, footnote 4].
- ³⁰ Courant & Hilbert [1965, 7, p.227], Lavrentiev [1967, 14, p.1] and Payne [1973, 10, p.1]. Other identifications for this property include the following: proper and correctly set (John [1955, 1, p.591]); correct (Lavrentiev [1967, 14, p.1]); just set (CFT [1960, 9, p.701]); well set ([ibid, footnote 4]); and, well posed (Payne [1973, 10, p.2]).
- ³¹ See Courant [1952, 1, p.277], Finn & Noll [1957, 1], Lavrentiev [1967, 14, p.2] and Payne [1973, 10, p.2].
- ³² Cf., for example, Payne [1973, 10, p.1] and Ishimaru [1963, 6, p.600]. For the solution of a particular detail problem the general equations of, say, Sect. F.1.a. may be considered as they stand; or, they may be reduced to other expressions in common usage. Regardless of the choice made, at least the following information remains to be given. First, a determination must be made for which media and under exactly what specific physical conditions (e.g. for what ranges of independent

parameter variation) the above cited totality of equations are physically valid, and if desired, applicable.

And next, the system related interface and boundary conditions (cf. Sect. F.2.) and/or the parameter initial values must be supplied. "What can be said by way of answer", note Courant & Friedrichs [1963, 4, p.367] with particular regard to the problem of boundary conditions, "is still tentative and far from a clear-cut mathematical statement". See also Lieberstein [1972, 3, p.252].

It goes almost without saying that physical data are subject to "observational error"; which error is always hoped, and apparently most often taken, to be "small".

- 33 It should be borne in mind that, similar to constitutive equations, these conditions are postulated; their motivation in any given problem lying in the relevant physical insight of the investigator of the system in question. Note also in this regard footnote A64.
- 34 Cf. Ames [1965, 1, p.475] and Payne [1973, 10, p.2]. It is interesting to note that indeed some problems must be considered under the condition of being properly posed if they are not to lead to aphysical conclusions. An example of such a situation results in a "thermo-dynamic paradox"; which matter is discussed by Ishimaru [1963, 6].
- 35 Payne [1973, 10, p.2] and Lavrentiev [1967, 14, p.2].
- 36 Similar to the situation pointed out in footnote 30 above, there exist various names for this condition. Some of them are: improperly posed (John [1955, 1], Courant & Hilbert [1965, 7, p.280] and Payne [1973, 10, p.1]); ill-posed and non-well-posed (Payne [1973, 10, p.1]); not well-posed (Ames [1965, 1, p.477]); and, incorrect (Lavrentiev [1967, 14, p.1]).

- ³⁷ Domains of existence in V are defined by the regions "bounded" therein by such dynamic surfaces of discontinuity (e.g. interfaces) as may be present. See again in this regard the relationships of Sect. F.2. and note the discussion of Courant & Hilbert [1965, 7, pp.147-153]; and, say, that of Shercliff [1965, 15, p.121].
- ³⁸ For example, the matter of non-analytic data is raised by Courant & Hilbert [1965, 7, p.237]; while Payne [1973, 10, p.3 and 7]) comments in general upon this point.
- ³⁹ Lieberstein [1972, 3, p.74] and Payne [1973, 10, p.3]. The present attitude from a point of view of application regarding the existence of solutions seems to be the following (Jordan & Eringen [1964, 3, p.113]): "In the absence of an existence theorem, demonstration of the phenomena under consideration will have to suffice."
- ⁴⁰ Courant & Hilbert [1965, 7, p.227].
- ⁴¹ Payne [1973, 10, p.3]. Cf. also Lieberstein [1972, 3, p.74].
- ⁴² Mathematically, an example of this is given in the study of Finn & Noll [1957, 1]. Slattery [1972, 8, p.69], in reflecting the approach taken by most physical researchers, states that one only asks for "a" solution; leaving it, similar to the case of existence cited above, to experiment to suggest the uniqueness thereof. In principle at least, it is clear that such an attitude is an unfortunate one with particular regard to meaningful physical research and application.
- ⁴³ See Courant [1952, 1, p.278].
- ⁴⁴ Courant & Hilbert [1965, 7, p.237].
- ⁴⁵ Courant & Hilbert [1965, 7, p.227].
- ⁴⁶ John [1955, 1, p.592] and CFT [1960, 9, p.701, footnote 4].

- ⁴⁷ The term "allowed" reflects the role played by the (here unspecified) mathematical conditions to be met in order that a set of equations be considered solvable. The fulfilment of said conditions establishes the actual class of admissible thermodynamic process in V available to this theoretical model.
- ⁴⁸ Courant [1952, 1, p.278].
- ⁴⁹ Lieberstein [1972, 3, pp.74-75 and p.81], Courant & Hilbert [1965, 7, p.127] and Courant [1952, 1, p.277].
- ⁵⁰ Cf. John [1955, 1, p.591].
- ⁵¹ Courant [1952, 1, pp.277-278], Courant & Hilbert [1965, 7, p.278] and Lavrentiev [1967, 14, p.2].
- ⁵² A particular example hereof is that of a mixture of electromagnetic fluids moving in V and in which there are propagating singular surfaces. If the given data become discontinuous in V , the solution to the relationships involving said data will become discontinuous (cf. Courant & Hilbert [1965, 7, p.486] and Payne [1973, 10, p.7]).
- ⁵³ Not unlike the situation noted for the properties of existence and uniqueness, it may be observed (Payne [1973, 10, pp.6-7]) that here, in the case of improperly posed problems, "One usually tries to deal with ... (such) ... problems by measuring an over-abundance of data and then trying to compensate for the lack of knowledge."
- From a rational mechanical point of view this situation, while understandable, is unacceptable. One is here indeed compelled to inquire into the possibility of bringing method into this problem.
- ⁵⁴ This point has considerable import with regard to the use of mathematical approximation methods (e.g. numerical techniques) in the treatment of physical problems. See, e.g., Courant & Hilbert [1965, 7, p.227].

55 Payne [1973, 10, p.2].

56 Note, e.g., Petrowski [1955, 2, pp.72-73] and Sommerfield [1964, 6, p. 236].

57 Courant & Hilbert [1965, 7, p.230] and Payne [1973, 10, p.2].

58 John [1955, 1, p.592].

59 The significance of the kinematic properties related to a given medium with regard to the determination of the (degree of) applicability of the constitutive equations proposed for that system is briefly noted by Jaunzemis [1967, 12, pp.297-298]. Concerning dimensional analysis see Ericksen [1960, 9, p.797, footnote 4].

60 The matter of similarity transformations, scaling laws (i.e. axioms!) and soforth appear to play an important role in the area of physical applications.

61 The theory given here is mathematically exact in the sense that the development thereof, essentially fulfilled with the procuring of the non-linear constitutive equations, was free of mathematical approximations. The development of the theory beyond that point could, in principle at least, have been carried through without introducing the linearizations of said relationships used later; which introduction was, once again, motivated by the desire for "concrete" results.

Physically, the theory is exact to the extent that the postulated applicability of Euclidean (or Galilean) transformations holds for a mixture of electromagnetic fluids. According to Toupin ([1963, 10, p. 107]), however, for the conditions under which the media of concern in this theory are taken to exist, I may conclude that there is support for this assumption.

- 62 See, e.g., Thom [1965, 18, p.383].
- 63 Note again the remarks of footnote 32 above.
- 64 Regarding inadequacy see page 11 and footnotes A32-35.
- 65 It is sufficient here to note the remarks of Chapman & Cowling [1970, 2, p.7].
- 66 With a rational mechanical theory serving as a departure point, explicit criteria deriving therefrom (e.g. restrictions on the constitutive equations) can conceivably be used in the design of detail experiments which yield phenomena possessing a low, if not vanishing, degree of degeneracy (i.e. phenomenon coupling).

In strong contrast to numerous experiments conducted, this approach, taking as full advantage as possible of all available intra- and interdisciplinary knowledge relevant to the system in question, makes possible at least in principle a unique empirical determination of the macroscopic structure of the constitutive equations of that system.

With the knowledge thus gained it becomes possible, depending upon the meaningfulness of said results (i.e. their physical reproducibility) to move further in a number of directions: viz., new experiments can be prepared which will serve to improve the foregoing results, or which have the purpose of investigating particular system related phenomena of interest (e.g. flow behavior). The end result of a succession of such experiments would not inconceivably be physically applicable systems which have equally well-understood design, control and optimization properties.

G. CONCLUSIONS AND CLOSURE

The theory developed here demonstrates clearly the feasibility of employing rational mechanics as a serious new alternative to the problem of describing the dynamic behavior of a general class of plasma systems. Indeed, this particular non-linear constitutive theory has been seen to yield not only known, but also new results of a rather general character.

While for some problems of interest this theory is readily observed to be either too broad or too restricted with regard to its generality, a truly meaningful delimiting of its range of applicability must await the outcome of further careful study of both its physical contents as well as its mathematical character. Nonetheless, by virtue of the fact that applicable results of less generality can be recovered from this model with the aid of additional simplifying assumptions, a certain measure of confidence in said theory may be assumed to exist.

Closure

Even more essential to recognize here than the aforementioned results is the matter of the approach itself and the spirit underlying it. The attitude taken by rational mechanics with regard to its subject matter is such that it, at least, suggests not only the possibility of, but also a means for, the systematic establishment of a "total" macroscopic theory of plasmas.

The general theoretical framework set down here can, to the best of my knowledge, serve as an initial basis for the first comprehensive assimilation, organization, interpretation and reduction of the large body of

knowledge regarding plasmas; both theoretical and empirical as well as intra- and interdisciplinary. Due to the directness, logical simplicity, mathematical rigor and physical clarity of this approach, I contend that rational mechanics has much to offer to the study of plasmas that is both necessary and worthwhile.

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VITA

As the first of four children I was born in Cleveland, Ohio (U.S.A.) on 20 July 1940. With the conclusion of my primary and secondary "education" I entered military service in March, 1959 for a period of six years. Shortly thereafter in February, 1960 I matriculated to the University of Miami at Coral Gables, Florida.

In June, 1963 I received the Bachelor of Science degree; and, two years later with the submission and defense of a theoretical thesis on magneto-fluid-dynamic vorticity, I was granted the degree of Master of Science. Having been honorably discharged from my military responsibilities three months earlier, I departed the subtropical clime of southern Florida for the continent of Europe in July, 1965.

After a period of orientation, my association with the "Technische Hogeschool Eindhoven" began in February, 1966. At the present time I am married and the father of a wonderful son.

TOWARDS A RATIONAL DYNAMICS OF PLASMAS

STELLINGEN

1. The education of enlightened and responsible technologists would be enhanced by the explicit introduction into their studies of a more adequate historical perspective of, and a relevant contemporary motivation for the subject matter to be considered.
2. The support of foundations research studies would be a meaningful step towards the securing of a comprehensive and mature leadership in technology.

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3. In a society with an increasing degree of relative ignorance the need to (re)consider the function of mass communication media therein appears to be under-estimated.
4. The process of democratization would be well-served with the securing of more rational criteria to be satisfied by persons occupying positions of authority and decision.
5. The introduction of a layer of color into automobile tires could serve as a worthwhile measure of the wear, and thus the safeness, thereof.
6. Happiness is being understood.