Monotonicity conditions for multirate and partitioned explicit Runge-Kutta schemes

EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computer Science

CASA-Report 12-10
May 2012

Monotonicity conditions for multirate and partitioned explicit Runge-Kutta schemes

by

W. Hundsdorfer, A. Mozartova, V. Savcenco

Centre for Analysis, Scientific computing and Applications
Department of Mathematics and Computer Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven, The Netherlands
ISSN: 0926-4507

Monotonicity Conditions for Multirate and Partitioned Explicit Runge-Kutta Schemes

W. Hundsdorfer, A. Mozartova, V. Savcenco

Abstract Multirate schemes for conservation laws or convection-dominated problems seem to come in two flavors: schemes that are locally inconsistent, and schemes that lack mass-conservation. In this paper these two defects are discussed for one-dimensional conservation laws. Particular attention will be given to monotonicity properties of the multirate schemes, such as maximum principles and the total variation diminishing (TVD) property. The study of these properties will be done within the framework of partitioned Runge-Kutta methods. It will also be seen that the incompatibility of consistency and mass-conservation holds for 'genuine' multirate schemes, but not for general partitioned methods.

1 Introduction

Several well-known multirate schemes for conservation laws that have appeared in the literature have one of the following defects: there are schemes that are *locally inconsistent*, e.g. [1, 2, 13, 14], and schemes that are *not mass-conservative*, e.g. [19]. In this paper these two defects are discussed for one-dimensional conservation laws $u_t + f(u)_x = 0$. We will mainly concentrate on time stepping aspects for simple schemes with one level of temporal refinement. The spatial grids are assumed to be given and fixed in time. Spatial discretization of a PDE (partial differential equation) then leads to a system of ODEs (ordinary differential equations), the so-called semi-discrete system. Particular attention will be given to monotonicity properties

Willem Hundsdorfer
CWI, PO Box 94079, 1090-GB Amsterdam, The Netherlands, e-mail: willem.hundsdorfer@cwi.nl

Anna Mozartova
CWI, PO Box 94079, 1090-GB Amsterdam, The Netherlands, e-mail: a.mozartova@cwi.nl

Valeriu Savcenco
TU Eindhoven, Dept. Mathematics, PO Box 513, 5600 MB Eindhoven, The Netherlands, e-mail: v.savcenco@tue.nl

of the multirate time stepping schemes, such as maximum principles and the total variation diminishing (TVD) property. Different types of monotonicity, depending on the norm or semi-norm being used, will lead to different stepsize restrictions.

In Section 2 we will present some multirate schemes with one level of refinement, due to Osher & Sanders [14], Tang & Warnecke [19], Constantinescu & Sandu [1], and Savcenco, Hundsdorfer & Verwer [15].

For the analysis of general multirate schemes it is convenient to write them in the form of partitioned Runge-Kutta methods. Basic properties of these methods are discussed in Section 3.

In Section 4 techniques of Higueras, Ferracina and Spijker [6, 9, 10, 18] will be employed, with some suitable modifications, to obtain monotonicity results. It will be seen that the step-size restrictions for maximum-norm monotonicity and maximum principles can be more relaxed than for other norms or semi-norms.

2 Some multirate schemes of order one and two

2.1 Examples of simple schemes for the advection equation

Consider as a simple example the advection equation

$$u_t + u_x = 0 \tag{1}$$

on a one-dimensional spatial region $0 < x < 1$ with given initial value $u(x,0)$, and inflow boundary condition $u(0,t)$ or spatial periodicity. Spatial discretization is performed with the first-order upwind scheme on cells $\mathscr{C}_j = (x_j - \frac{1}{2}\Delta x_j, x_j + \frac{1}{2}\Delta x_j)$. This gives a semi-discrete system

$$u_j'(t) = \frac{1}{\Delta x_j}\big(u_{j-1}(t) - u_j(t)\big) \qquad \text{for } j \in \mathscr{I} = \{1,2,\ldots,m\}\,, \tag{2}$$

where $u_j'(t) = \frac{d}{HMV dt}u_j(t)$, and $u_j(t)$ approximates $u(x_j,t)$ or the average value over the surrounding cell $\mathscr{C}_j$.

Application of the forward Euler method with time step Δt gives the CFL stability condition $\nu_j \le 1$ for all j, where $\nu_j = \Delta t/\Delta x_j$ is the local Courant number. Suppose this stability condition is satisfied for $j \in \mathscr{I}_1$ but on $\mathscr{I}_2 = \mathscr{I} - \mathscr{I}_1$ we need to take two smaller steps with step-size $\frac{1}{2}\Delta t$ to reach $t_{n+1} = t_n + \Delta t$.

Then for this simple situation, the scheme of Osher and Sanders [14] can be written as

$$\left\{\begin{aligned} & u_j^{n+\frac{1}{2}} = \begin{cases} u_j^n & \text{for } j \in \mathscr{I}_1\,, \\ u_j^n + \frac{1}{2}\nu_j(u_{j-1}^n - u_j^n) & \text{for } j \in \mathscr{I}_2\,, \end{cases} \\ & u_j^{n+1} = u_j^n + \frac{1}{2}\nu_j(u_{j-1}^n - u_j^n) + \frac{1}{2}\nu_j(u_{j-1}^{n+\frac{1}{2}} - u_j^{n+\frac{1}{2}}) \quad \text{for } j \in \mathscr{I}\,. \end{aligned}\right. \tag{3}$$

As observed in [19], the scheme (3) is not consistent at the interface: if $i-1 \in \mathscr{I}_1$ and $i \in \mathscr{I}_2$ then

$$\frac{1}{\Delta t}\big(u_i^{n+1}-u_i^n\big) = \frac{1}{\Delta x_i}\big(u_{i-1}^n - \tfrac{1}{2}(u_i^n + u_i^{n+\frac{1}{2}})\big) = \frac{1-\frac{1}{4}\nu_i}{\Delta x_i}\big(u_{i-1}^n - u_i^n\big)\,,$$

which is consistent for fixed Courant number ν_i with the equation

$$u_t + (1-\tfrac{1}{4}\nu_i)u_x = \mathscr{O}(\Delta t) + \mathscr{O}(\Delta x_i)\,,$$

rather than the original advection equation (1).

To overcome this inconsistency, Tang and Warnecke [19] therefore proposed the modified scheme

$$\begin{cases} u_j^{n+\frac{1}{2}} = u_j^n + \frac{1}{2}\nu_j(u_{j-1}^n - u_j^n) \quad \text{for } j \in \mathscr{I}\,, \\[2ex] u_j^{n+1} = u_j^{n+\frac{1}{2}} + \begin{cases} \frac{1}{2}\nu_j(u_{j-1}^n - u_j^n) \quad \text{for } j \in \mathscr{I}_1\,, \\ \frac{1}{2}\nu_j(u_{j-1}^{n+\frac{1}{2}} - u_j^{n+\frac{1}{2}}) \quad \text{for } j \in \mathscr{I}_2\,. \end{cases} \end{cases} \tag{4}$$

This scheme, however, is not mass conserving at the interface. If $i-1 \in \mathscr{I}_1$ and $i \in \mathscr{I}_2$, then the flux at $x_{i-1/2}$ that leaves cell $\mathscr{C}_{i-1}$ over the time interval $[t_n, t_{n+1}]$ equals $\Delta t\, u_{i-1}^n$, whereas the flux that enters $\mathscr{C}_i$ over this time interval is given by $\frac{1}{2}\Delta t(u_{i-1}^n + u_{i-1}^{n+1/2})$.

It should be noted that except for interface points the schemes (3) and (4) are identical. For example, if $\mathscr{I}_1 = \{j : j < i\}$ and $\mathscr{I}_2 = \{j : j \geq i\}$, then (3) and (4) give in one step the same result for $j \neq i$. Furthermore, it can be shown that, also for interface regions with a larger, but fixed, number of points, the properties of local consistency and mass conservation cannot be combined. Due to cancellation and damping effects, local inconsistencies need not show up in the global errors. In particular, the scheme (3) can be shown to be convergent in the maximum norm.

2.2 Some schemes with one refinement level for general semi-discrete problems

In this paper we will discuss monotonicity properties and temporal convergence of multirate schemes for general semi-discrete problems in $\mathbb{R}^m$,

$$u'(t) = F(u(t))\,, \qquad u(0) = u_0\,. \tag{5}$$

As applications we will consider nonlinear conservation problems with flux-limited spatial discretizations. The approximations to $u(t_n) = (u_j(t_n)) \in \mathbb{R}^m$ will be denoted by $u_n = (u_j^n) \in \mathbb{R}^m$. As above, we consider partitioning $\mathscr{I} = \mathscr{I}_1 \cup \mathscr{I}_2$. Corresponding to these sets $\mathscr{I}_k$, let I_1, I_2 be $m \times m$ diagonal matrices with diagonal entries 0 or 1, such that $(I_k)_{jj} = 1$ for $j \in \mathscr{I}_k$, $k = 1,2$. We have $I_1 + I_2 = I$, the identity matrix.

2.2.1 First-order schemes

The semi-discrete system (2) obviously fits in this form with linear function F; the general system (5) allows nonlinear problems and nonlinear discretizations. For such systems the Osher-Sanders scheme (3) becomes

$$\begin{cases} u_{n+\frac{1}{2}} = u_n + \frac{1}{2}\Delta t I_2 F(u_n)\,, \\ u_{n+1} = u_n + \frac{1}{2}\Delta t F(u_n) + \frac{1}{2}\Delta t F(u_{n+\frac{1}{2}})\,, \end{cases} \tag{6}$$

and the Tang-Warnecke scheme (4) reads

$$\begin{cases} u_{n+\frac{1}{2}} = u_n + \frac{1}{2}\Delta t F(u_n)\,, \\ u_{n+1} = u_n + \Delta t I_1 F(u_n) + \frac{1}{2}\Delta t I_2 \big(F(u_n) + F(u_{n+\frac{1}{2}})\big)\,. \end{cases} \tag{7}$$

In the following we will refer to (6) as the OS1 scheme, and to (7) as the TW1 scheme. We note that in [14] and [19] the number of sub-steps on the index set $\mathscr{I}_2$ was allowed to be larger than two for these schemes. This will be covered by the more general formulations considered in Section 3.

2.2.2 Second-order schemes

In the literature, several second-order multirate schemes for conservation laws have been derived that are based on the standard two-stage Runge-Kutta method

$$u^*_{n+1} = u_n + \Delta t F(u_n)\,, \qquad u_{n+1} = u_n + \frac{1}{2}\Delta t \big(F(u_n) + F(u^*_{n+1})\big)\,.$$

The second stage can also be written as $u_{n+1} = \frac{1}{2}(u_n + u^*_{n+1} + \Delta t F(u^*_{n+1}))$. Monotonicity properties are more clear with this form. The method is known as the explicit trapezoidal rule or the modified Euler method. In this section we consider some multirate schemes, based on this method, with one level of temporal refinement. Results on internal consistency and mass conservation are mentioned here, but a detailed discussion will only be given in Section 3.

The second-order scheme of Tang & Warnecke [19] reads

$$\begin{cases} u^*_{n+\frac{1}{2}} = u_n + \frac{1}{2}\Delta t F(u_n)\,, \\ u_{n+\frac{1}{2}} = \frac{1}{2}\big(u_n + u^*_{n+\frac{1}{2}} + \frac{1}{2}\Delta t F(u^*_{n+\frac{1}{2}})\big)\,, \\ u^*_{n+1} = I_1\big(u_n + \Delta t F(u_n)\big) + I_2\big(u_{n+\frac{1}{2}} + \frac{1}{2}\Delta t F(u_{n+\frac{1}{2}})\big)\,, \\ u_{n+1} = \frac{1}{2}I_1\big(u_n + u^*_{n+1} + \Delta t F(u^*_{n+1})\big) + \frac{1}{2}I_2\big(u_{n+\frac{1}{2}} + u^*_{n+1} + \frac{1}{2}\Delta t F(u^*_{n+1})\big)\,. \end{cases} \tag{8}$$

We will refer to this scheme as TW2. It will be shown below that this scheme is internally consistent but not mass-conserving.

Constantinescu & Sandu [1] introduced the following scheme, which will be referred to as CS2,

$$\begin{cases} u^*_{n+\frac{1}{2}} = u_n + \Delta t I_1 F(u_n) + \frac{1}{2}\Delta t I_2 F(u_n)\,, \\ u_{n+\frac{1}{2}} = u_n + \frac{1}{4}\Delta t I_2\big(F(u_n) + F(u^*_{n+\frac{1}{2}})\big)\,, \\ u^*_{n+1} = I_1\big(u_n + \Delta t F(u_{n+\frac{1}{2}})\big) + I_2\big(u_{n+\frac{1}{2}} + \frac{1}{2}\Delta t F(u_{n+\frac{1}{2}})\big)\,, \\ u_{n+1} = u_n + \frac{1}{4}\Delta t\big(F(u_n) + F(u^*_{n+\frac{1}{2}}) + F(u_{n+\frac{1}{2}}) + F(u^*_{n+1})\big)\,. \end{cases} \tag{9}$$

This scheme is mass-conserving but not internally consistent. Nevertheless, we will see that it is still convergent (with order one) in the maximum-norm due to damping and cancellation effects. Note that for non-stiff ODE systems the scheme will be consistent and convergent with order two.

The related method of Dawson and Kirby [2] is also mass-conserving but not internally consistent. However in that scheme a limiter is applied which is adapted to the outcome of previous stages, so it does not fit in the framework of this paper where the semi-discrete system is supposed to be given a priori.

In [15] a multirate scheme of order two was constructed for stiff (parabolic) problems. This is a Rosenbrock-type scheme containing a parameter γ, and setting $\gamma = 0$ yields an explicit scheme, which we will refer to as SH2. In this scheme, first a prediction v_{n+1} is computed, followed by refinement steps on $\mathscr{I}_2$ using interpolated values $v_{n+\frac{1}{2}}$ on $\mathscr{I}_1$. The scheme reads

$$\begin{cases} v^*_{n+1} = u_n + \Delta t F(u_n)\,, \\ v_{n+1} = \frac{1}{2}u_n + \frac{1}{2}v^*_{n+1} + \frac{1}{2}\Delta t F(v^*_{n+1})\,, \\ v_{n+\frac{1}{2}} = \frac{1}{2}u_n + \frac{1}{4}v^*_{n+1} + \frac{1}{4}v_{n+1}\,, \\ u^*_{n+\frac{1}{2}} = I_1 v_{n+\frac{1}{2}} + I_2\big(u_n + \frac{1}{2}\Delta t F(u_n)\big)\,, \\ u_{n+\frac{1}{2}} = I_1 v_{n+\frac{1}{2}} + \frac{1}{2}I_2\big(u_n + u^*_{n+\frac{1}{2}} + \frac{1}{2}\Delta t F(u^*_{n+\frac{1}{2}})\big)\,, \\ u^*_{n+1} = I_1 v_{n+1} + I_2\big(u_{n+\frac{1}{2}} + \frac{1}{2}\Delta t F(u_{n+\frac{1}{2}})\big)\,, \\ u_{n+1} = I_1 v_{n+1} + \frac{1}{2}I_2\big(u_{n+\frac{1}{2}} + u^*_{n+1} + \frac{1}{2}\Delta t F(u^*_{n+1})\big)\,. \end{cases} \tag{10}$$

This scheme will be seen to be internally consistent but not mass-conserving. It can be written with fewer stages; there are no function evaluations of v_{n+1} and $v_{n+1/2}$, so these vectors are just included for notational convenience. Further we note that this scheme was not intended originally as used here. Instead, the prediction values v^*_{n+1} and v_{n+1} were used in [15] to estimate local errors, and based on this estimate the partitioning $\mathscr{I} = \mathscr{I}_1 \cup \mathscr{I}_2$ was adjusted. For the schemes in the present paper the partitioning is supposed to be given, based on local Courant numbers, in which case only $I_1 v_{n+1}$ and $I_1 v_{n+1/2}$ are needed.

2.2.3 Monotonicity assumptions

We will consider monotonicity properties of the numerical methods with suitable norms or semi-norms $\|v\|$ for $v = (v_j) \in \mathbb{R}^m$. More general sublinear functionals can be included in the theory as well, and also these will be denoted by $\|v\|$.

The basic monotonicity assumption on the semi-discrete system is given by

$$\|v + \tau_1 I_1 F(v) + \tfrac{1}{2}\tau_2 I_2 F(v)\| \leq \|v\| \qquad \text{for all } v \in \mathbb{R}^m \text{ and } 0 \leq \tau_1, \tau_2 \leq \tau_0\,, \tag{11}$$

where $\tau_0 > 0$ is a problem dependent parameter. A related assumption, used for instance in [10, 18], is

$$\|v + \tfrac{1}{k}\tau_k I_k F(v)\| \leq \|v\| \qquad \text{for all } v \in \mathbb{R}^m \text{ and } 0 \leq \tau_k \leq \tau_0,\ k = 1, 2\,, \tag{12}$$

It is easily seen that (12) implies (11).

For the multirate schemes we shall determine step-size coefficients C such that we have the monotonicity property

$$\|u_{n+1}\| \leq \|u_n\| \quad \text{whenever } \Delta t \leq C\tau_0\,. \tag{13}$$

For a given scheme, the optimal step-size coefficient C will be called the threshold factor for monotonicity. In general, such monotonicity properties are intended to ensure that unwanted overshoots or numerical oscillations will not arise. Following [16, 17] we will call a scheme total variation diminishing (TVD) if (13) holds with the semi-norm $\|v\|_{\mathrm{TV}}$. If the (semi-)norm is not specified, methods that have a positive threshold C can be called strong stability preserving (SSP), as in [3, 4] for standard, single-rate methods.

The optimal values C may depend on the norm. As we will see, under assumption (11), the thresholds C are in general larger for the maximum norm than for the total variation semi-norm.

Example. Well-known examples are the maximum norm $\|v\|_\infty = \max_{1\leq j\leq m} |v_j|$ and the total variation semi-norm $\|v\|_{\mathrm{TV}} = \sum_{j=1}^m |v_{j-1} - v_j|$ with $v_0 = v_m$, the latter arising from one-dimensional scalar PDEs with spatial periodicity.

Apart from such (semi-)norms, we can also consider sublinear functionals. For example, following [18], define $\|v\|_+ = \max_{1\leq j\leq m} v_j$ and $\|v\|_- = -\min_{1\leq j\leq m} v_j$.

Then, having (13) for both these sublinear functionals amounts to the maximum principle $\min_{1\le i\le m} u_i^0 \le u_j^n le \max_{1\le i\le m} u_i^0$ for all $n \ge 1$ and $1 \le j \le m$. In general, this is of course somewhat stronger than having monotonicity in the maximum-norm, $\|u_{n+1}\|_\infty \le \|u_n\|_\infty$, but for the schemes considered in this paper the associated threshold values C will be the same. ◇

Example. Consider a scalar conservation law $u_t + f(u)_x = 0$ with a periodic boundary condition, and with $0 \le f'(u) \le \alpha$. Spatial discretization in conservation form gives semi-discrete systems (5) with

$$F_j(v) = \frac{1}{\Delta x_j}\big(f(v_{j-\frac{1}{2}}) - f(v_{j+\frac{1}{2}})\big)$$

where $v_{j\pm 1/2}$ are the values at the cell boundaries, determined from the components of $v = (v_i) \in \mathbb{R}^m$. Using limiters in the discretization it can be guaranteed that

$$0 \;\le\; \frac{v_{j-\frac{1}{2}} - v_{j+\frac{1}{2}}}{v_{j-1} - v_j} \;\le\; 1+\mu$$

with a constant $\mu \ge 0$ determined by the limiter; see also formula (8) in [2]. This holds trivially for the first-order upwind discretization with $\mu = 0$; for higher-order schemes with limiting we get $\mu = 1$. It follows that $F_j(v)$ can then be written as

$$F_j(v) = \frac{a_j(v)}{\Delta x_j}\big(v_{j-1} - v_j\big)\,, \qquad j = 1,\dots,m\,,\;\; v_0 = v_m\,,$$

where

$$0 \le a_j(v) \le \alpha(1+\mu) \qquad \text{for all } j \text{ and } v \in \mathbb{R}^m.$$

Suppose that $\Delta x_j = h$ for $j \in \mathscr{I}_1$ and $\Delta x_j = \frac{1}{2}h$ for $j \in \mathscr{I}_2$. Then a well-known lemma of Harten [7, Lemma 2.2] shows that the monotonicity assumptions (11) and (12) will be valid for the total variation semi-norm $\|\cdot\|_{\mathrm{TV}}$ provided that

$$\frac{\alpha\tau_0}{h} \;\le\; \frac{1}{1+\mu}\,.$$

Moreover, it is easy to see that the assumptions (11) and (12) will also hold under the same CFL restriction with the maximum-norm and the functionals $\|\cdot\|_\pm$ of the previous example. ◇

2.2.4 Example: monotonicity for the TW1 scheme

General results on monotonicity will be presented in Section 4 in a more general setting, but it is illustrative to first show the derivation of monotonicity results for the simple TW1 scheme under assumption (11) with the maximum norm or a general semi-norm to see how the different step-size restrictions arise.

In the first stage of the TW1 scheme (7) we have of course

$$\|u_{n+\frac{1}{2}}\| \le \|u_n\| \qquad \text{whenever } \Delta t \le \tau_0 \,.$$

The second stage can be written in the form

$$u_{n+1} = (1-\theta)u_n + \theta\big(u_{n+\frac{1}{2}} - \tfrac{1}{2}\Delta t F(u_n)\big) + \Delta t I_1 F(u_n) + \tfrac{1}{2}\Delta t I_2\big(F(u_n) + F(u_{n+\frac{1}{2}})\big)\,,$$

with arbitrary $\theta \in [0,1]$. This leads to

$$\begin{aligned} u_{n+1} \,=\, &(1-\theta)\Big(u_n + \frac{2-\theta}{2(1-\theta)}\Delta t I_1 F(u_n) + \tfrac{1}{2}\Delta t I_2 F(u_n)\Big) \\ &+ \theta\Big(u_{n+\frac{1}{2}} + \frac{1}{2\theta}\Delta t I_2 F(u_{n+\frac{1}{2}})\Big)\,. \end{aligned} \tag{14}$$

Under assumption (11) this gives the monotonicity property (13) with

$$C \,=\, \max_{0\le\theta\le1} \min\Big(1\,,\, \frac{2(1-\theta)}{2-\theta}\,,\,\theta\Big) \,=\, 2-\sqrt{2}\,. \tag{15}$$

This value $C \approx 0.58$ is valid for general semi-norms. So, in particular, it provides a TVD result for schemes with limiters.

Next, consider the maximum-norm. The second stage can also be written as

$$u_{n+1} = I_1\big(u_n + \Delta t I_1 F(u_n)\big) + I_2\big(u_{n+\frac{1}{2}} + \tfrac{1}{2}\Delta t I_2 F(u_{n+\frac{1}{2}})\big)\,.$$

It follows that the monotonicity property (13) is valid for the maximum norm with step-size coefficient

$$C = 1\,, \tag{16}$$

and the same holds for maximum principles; cf. [19, Lemma 2.1]).

Note that this result (16) has been obtained by using the inequality $\|I_1 v + I_2 w\| \le \max(\|v\|, \|w\|)$, which holds for the maximum-norm and for the convex functionals $\|\cdot\|_\pm$ from the previous example, but not for general norms or semi-norms.

3 Partitioned Runge-Kutta methods

3.1 General properties

In the multirate examples considered thus far, only one level of refinement was used to keep the notation simple. Generalizations can be formulated in terms of partitioned Runge-Kutta methods by which the schemes are presented in a compact fashion; see also [1, 5]. Explicit methods are in general preferred for applications to conservation laws, but in the analysis below diagonally implicit methods could also be easily included.

As in (5), the semi-discrete system in $\mathbb{R}^m$ is written as $u'(t) = F(u(t))$, $u(0) = u_0$. Let $\mathscr{I} = \mathscr{I}_1 \cup \cdots \cup \mathscr{I}_r$ be an index partitioning with corresponding diagonal matrices $I = I_1 + \cdots + I_r$, where I is the identity matrix, the entries of the I_k are zero or one and $I_k I_l$ is the zero matrix if $k \neq l$.

For a time step from t_n to $t_{n+1} = t_n + \Delta t$, we consider partitioned Runge-Kutta methods

$$\begin{cases} v_{n,i} = u_n + \Delta t \sum_{k=1}^{r} \sum_{j=1}^{i-1} a_{ij}^{(k)} I_k F(v_{n,j}), \qquad i = 1, \ldots, s, \\ u_{n+1} = u_n + \Delta t \sum_{k=1}^{r} \sum_{j=1}^{s} b_j^{(k)} I_k F(v_{n,j}). \end{cases} \tag{17}$$

The internal stage vectors $v_{n,i}$, $i = 1, \ldots, s$, give approximations at intermediate time levels. The multirate schemes of the previous sections all fit in this form with $r = 2$. With $r > 2$ more levels of temporal refinement are allowed.

3.1.1 Internal consistency and conservation

Let $c_i^{(k)} = \sum_{j=1}^{i-1} a_{ij}^{(k)}$, $i = 1, \ldots, s$. If we have

$$c_i^{(k)} = c_i^{(l)} \qquad \text{for all } 1 \leq k, l \leq r \text{ and } 1 \leq i \leq s, \tag{18}$$

then the internal vectors $v_{n,i}$ will be consistent approximations to $u(t_n + c_i \Delta t)$, and the method will be called *internally consistent*. This is an important property for the accuracy of the method when applied to ODEs obtained by semi-discretization.

Apart from consistency, we will also regard global *conservation*, for example mass conservation. Suppose that $h^T = (h_1, \ldots, h_m)$ is such that $h^T u(t) = \sum_j h_j u_j(t)$ is a conserved quantity for the ODE system (5). This will hold for arbitrary initial value u_0 provided that

$$h^T F(v) = 0 \qquad \text{for all } v \in \mathbb{R}^m. \tag{19}$$

For the partitioned Runge-Kutta scheme we have

$$\begin{aligned} h^T u_{n+1} &= h^T u_n + \Delta t \sum_{k=1}^{r} \sum_{j=1}^{s} b_j^{(k)} h^T I_k F(v_{n,j}) \\ &= h^T u_n + \Delta t \sum_{k \neq l} \sum_{j=1}^{s} \big(b_j^{(k)} - b_j^{(l)}\big) h^T I_k F(v_{n,j}), \end{aligned}$$

for any $1 \leq l \leq r$. Therefore, as noted in [1], the conservation property $h^T u_{n+1} = h^T u_n$ will be valid provided that

$$b_j^{(k)} = b_j^{(l)} \qquad \text{for all } 1 \leq k, l \leq r \text{ and } 1 \leq j \leq s. \tag{20}$$

3.1.2 Order conditions for non-stiff problems

The order conditions for partitioned Runge-Kutta methods applied to non-stiff problems can be found found in [8, Thm. I.15.9] for $r = 2$. The order p for non-stiff problems may not correspond to the order of convergence for semi-discrete systems arising from PDEs with boundary conditions or interface conditions, and therefore p is often referred to as the *classical order*.

To write the order conditions for $r > 2$ in a compact way, let $A_k = (a_{ij}^{(k)}) \in \mathbb{R}^{s\times s}$ and $b_k = (b_i^{(k)}) \in \mathbb{R}^s$ contain the coefficients of the method, and set $e = (1,\ldots,1)^T \in \mathbb{R}^s$. Then the conditions for order $p = 1$ are just

$$b_k^T e = 1 \qquad \text{for } \; k = 1,\ldots,r\,, \tag{21}$$

that is $\sum_{j=1}^{s} b_j^{(k)} = 1$ for all k. To have order $p = 2$ the coefficients should satisfy

$$b_k^T A_l\, e = \frac{1}{2} \qquad \text{for } \; k,l = 1,\ldots,r\,. \tag{22}$$

The number of conditions quickly increase for higher orders; for $p = 3$ we get

$$b_k^T C_{l_1} A_{l_2} e = \frac{1}{3}\,, \quad b_k^T A_{l_1} A_{l_2} e = \frac{1}{6} \qquad \text{for } \; k,l_1,l_2 = 1,\ldots,r\,, \tag{23}$$

where $C_l = \mathrm{diag}(A_l e)$.

3.1.3 Formulation for non-autonomous systems

For non-autonomous systems

$$u'(t) = F(t,u(t))\,, \qquad u(0) = u_0\,, \tag{24}$$

we will use the partitioned method (17) with the stage function values $F(v_{n,j})$ replaced by $F(t_n + c_j\Delta t, v_{n,j})$. If (18) is valid, the abscissa are naturally taken as $c_i = c_i^{(k)}$, which is independent of k.

If (18) does not hold, then a proper choice of the abscissa is less obvious. For the OS1 and CS2 multirate schemes with $r = 2$ it is natural to take $c_i = c_i^{(2)}$. As generalization of the autonomous case we will therefore use

$$c_i = c_i^{(r)}\,, \qquad i = 1,\ldots,s\,. \tag{25}$$

Note that if $h^T F(t,v) = 0$ for all $t \in \mathbb{R}$, $v \in \mathbb{R}^m$, then we still have the conservation property $h^T u_{n+1} = h^T u_n$ if the scheme satisfies (20).

The alternative of replacing $I_k F(v_{n,j})$ in (17) by $I_k F(t_n + c_j^{(k)}\Delta t, v_{n,j})$ will destroy this conservation property. If the non-autonomous form originates from a source term in the PDE, this loss of conservation may be of little concern, but for the advection equation $u_t + \big(a(x,t)u\big)_x = 0$ with time-dependent velocity it is still a very desirable property.

Example. The OS1 scheme (6) leads to the partitioned method (17) with $r = 2$ and coefficients given by the tableau

$$\begin{array}{|c|c} a_{ij}^{(1)} & a_{ij}^{(2)} \\ \hline b_j^{(1)} & b_j^{(2)} \end{array} \; = \; \begin{array}{|cc|cc} 0 & & 0 & \\ 0 & 0 & 1/2 & 0 \\ \hline 1/2 & 1/2 & 1/2 & 1/2 \end{array} \quad .$$

For non-autonomous systems $u'(t) = F(t, u(t))$ the scheme with (25) reads

$$\begin{cases} u_{n+\frac{1}{2}} = u_n + \frac{1}{2}\Delta t I_2 F(t_n, u_n)\,, \\ u_{n+1} = u_n + \frac{1}{2}\Delta t F(t_n, u_n) + \frac{1}{2}\Delta t F(t_{n+\frac{1}{2}}, u_{n+\frac{1}{2}})\,. \end{cases}$$

The use of $I_k F(t_n + c_j^{(k)}\Delta t, v_{n,j})$ instead of $I_k F(t_n + c_j \Delta t, v_{n,j})$, $c_j = c_j^{(2)}$, would lead to the same formula for $u_{n+1/2}$ in the first stage, but then

$$u_{n+1} = u_n + \frac{1}{2}\Delta t F(t_n, u_n) + \frac{1}{2}\Delta t I_1 F(t_n, u_{n+\frac{1}{2}}) + \frac{1}{2}\Delta t I_2 F(t_{n+\frac{1}{2}}, u_{n+\frac{1}{2}})\,,$$

which is no longer conservative. $\diamond$

The above order conditions have been derived for autonomous systems, but with (25) they are also valid for non-autonomous systems. This follows from the fact that $u'(t) = F(t, u(t))$ can be written as an equivalent, augmented autonomous system $u'(t) = F(\vartheta(t), u(t))$, $\vartheta'(t) = 1$, with $\vartheta(0) = 0$, and application of the partitioned method to this augmented system gives the same result as to the original, non-autonomous system provided the additional equation $\vartheta'(t) = 1$ is included in the index set $\mathscr{I}_r$.

3.1.4 Conservation versus internal consistency

For the multirate schemes that have been considered in this paper, the conditions for internal consistency (18) and conservation (20) did not match. This incompatibility is valid for all 'genuine' multirate schemes that are based on one single method $\mathscr{M}_{\mathrm{RK}}$, that is, for schemes (17) that reduce to m_k applications (with step-size $\Delta t/m_k$) of this base method $\mathscr{M}_{\mathrm{RK}}$ to cover $[t_n, t_{n+1}]$ in case that $\mathscr{I}_k = \mathscr{I}$ and the other $\mathscr{I}_l$ are empty.

Consider, as simple example, a quadrature problem $u'(t) = g(t) \in \mathbb{R}^m$, which is just a special case of (24). (In a PDE context, this can be viewed as a degenerate case of advection with a source term where the advective velocity happens to be zero.) Suppose (18) and (20) are valid with $c_j^{(k)} = c_j$, $b_j^{(k)} = b_j$ for all k, and let $\mathscr{J} = \{i \in \mathscr{I} : b_i \neq 0\}$. Then for the quadrature problem we simply get

$$u_{n+1} = u_n + \Delta t \sum_{i \in \mathscr{J}} b_i \, g(t_n + c_i \Delta t)\,,$$

which is independent of the partitioning. However, if this is the result of a base method $\mathscr{M}_{\rm RK}$ with $m_1 = 1$, $\mathscr{I}_1 = \mathscr{I}$, then the result for $m_2 = 2$, $\mathscr{I}_2 = \mathscr{I}$ should be

$$u_{n+1} = u_n + \tfrac{1}{2}\Delta t \sum_{i \in \mathscr{J}} b_i \Big(g\big(t_n + \tfrac{1}{2} c_i \Delta t\big) + g\big(t_n + \tfrac{1}{2}(1 + c_i)\Delta t\big) \Big)\,,$$

which is not the same for arbitrary source terms g.

Note that for general partitioned Runge-Kutta methods there is no conflict between (18) and (20). Given a scheme with the same $c_i^{(k)} = c_i^{(l)}$ (for all i,k,l), but different weights $b_i^{(k)} \neq b_i^{(l)}$ (for some i,k,l), we can add an extra stage with new weights b_i^* that are independent of k, to make it mass-conserving. Of course, this will increase the computational work per step, and for the TW1, TW2 and SH2 schemes such a modification does not seem to lead to efficient new schemes.

4 Monotonicity and convex Euler combinations

We are in particular interested in the case where the partitioned Runge-Kutta method (17) stands for a multirate scheme that takes m_k substeps of size $\Delta t / m_k$ on $\mathscr{I}_k$ to cover $[t_n, t_{n+1}]$, $k = 1, \ldots, r$, with $m_1 = 1 < m_2 < \cdots < m_r$. The corresponding monotonicity assumption is

$$\Big\| v + \sum_{k=1}^{r} \frac{\tau_k}{m_k} I_k F(v) \Big\| \;\leq\; \|v\| \qquad \text{for all } v \in \mathbb{R}^m \text{ and } \tau_k \leq \tau_0,\ k = 1, \ldots, r\,, \tag{26}$$

where $\|\cdot\|$ is a sublinear functional or (semi-)norm. We will also consider

$$\big\| v + \frac{\tau_0}{m_k} I_k F(v) \big\| \;\leq\; \|v\| \qquad \text{for all } v \in \mathbb{R}^m \text{ and } k = 1, \ldots, r\,, \tag{27}$$

which generalizes the assumptions made in [10] and [18]. Of course, (26) implies (27). On the other hand, if (27) is valid, then the inequality in (26) will hold under the step-size restriction $\tau_1 + \cdots + \tau_m \leq \tau_0$. If we are dealing with the maximum-norm, then (26) and (27) are equivalent.

In the following we denote for $l = 1, \ldots, r$,

$$\left\{ \begin{array}{rcll} \kappa_{ij}^{(l)} & = & m_l a_{ij}^{(l)}\,, & \quad 1 \leq i, j \leq s\,, \\ \kappa_{s+1,j}^{(l)} & = & m_l b_j^{(l)}\,, & \quad 1 \leq j \leq s\,, \\ \kappa_{i,s+1}^{(l)} & = & 0\,, & \quad 1 \leq i \leq s+1\,. \end{array} \right. \tag{28}$$

These coefficients will be grouped in the $(s+1)\times(s+1)$ matrix $\mathscr{K}_l = (\kappa_{ij}^{(l)})$. It is convenient to add $v_{n,s+1} = u_{n+1}$ to the internal vectors. Then (17) can be written as

$$v_{n,i} \;=\; u_n + \sum_{l=1}^{r}\sum_{j=1}^{i-1} \kappa_{ij}^{(l)} \frac{\Delta t}{m_l} I_l F(v_{n,j}), \qquad i = 1,\ldots,s+1. \tag{29}$$

Depending on the monotonicity assumption, we can consider various ways to represent this partitioned scheme in terms of convex Euler combinations. For this we will introduce new method coefficients $\alpha_{ij}^{(k)}$, $\beta_{ij}^{(k)}$ with corresponding lower triangular matrices $\mathscr{A}_k = (\alpha_{ij}^{(k)})$ and $\mathscr{B}_k = (\beta_{ij}^{(k)})$. Such convex Euler forms are also called Shu-Osher forms, after [17] where such representations were used originally to demonstrate the TVD property of certain Runge-Kutta methods.

Inequalities for matrices or vectors in this section are to be understood componentwise, that is, $P = (p_{ij}) \geq 0$ means that all p_{ij} are non-negative. Furthermore, if $P \in \mathbb{R}^{(s+1)\times q_1}$ and $Q \in \mathbb{R}^{(s+1)\times q_2}$, then $[P\ Q]$ stands for the matrix whose first q_1 columns equal those of P and the other columns equal those of Q. In this section we let $e = (1,1,\ldots,1)^T \in \mathbb{R}^{s+1}$, and we use the convention $\alpha/\beta = +\infty$ if $\alpha \geq 0$, $\beta = 0$.

4.1 *Maximum-norm monotonicity*

A suitable form of (29) to obtain results on monotonicity in the maximum-norm is

$$v_{n,i} \;=\; \sum_{k=1}^{r} I_k\Big(\big(1-\alpha_i^{(k)}\big)u_n + \sum_{j=1}^{i-1}\big(\alpha_{ij}^{(k)} v_{n,j} + \beta_{ij}^{(k)}\frac{\Delta t}{m_k}F(v_{n,j})\big)\Big), \tag{30}$$

where $\alpha_i^{(k)} = \sum_{j=1}^{i-1}\alpha_{ij}^{(k)}$ and $i = 1,\ldots,s+1$. To have correspondence between (29) and (30) the coefficients should satisfy

$$\mathscr{K}_k = \big(I - \mathscr{A}_k\big)^{-1}\mathscr{B}_k, \qquad k = 1,\ldots,r. \tag{31}$$

Further we want the coefficients to be such that

$$\alpha_i^{(k)} \leq 1, \quad \alpha_{ij}^{(k)}, \beta_{ij}^{(k)} \geq 0 \qquad \text{for } 1 \leq j < i \leq s+1,\ 1 \leq k \leq r. \tag{32}$$

For such coefficients, let

$$C \;=\; \min_{i,j,k} \alpha_{ij}^{(k)}/\beta_{ij}^{(k)}. \tag{33}$$

If there are no coefficients such that (31) and (32) are satisfied, we set $C = 0$.

Theorem 1. *Consider (30) with (32) and let C be given by (33). Assume (26) is valid in the maximum-norm. Then $\|u_{n+1}\|_\infty \leq \|u_n\|_\infty$ whenever $\Delta t \leq C\tau_0$.*

Proof. The form (30) is equivalent to

$$I_k v_{n,i} \;=\; I_k\Big(\big(1-\alpha_i^{(k)}\big)u_n + \sum_{j=1}^{i-1}\big(\alpha_{ij}^{(k)} v_{n,j} + \beta_{ij}^{(k)}\frac{\Delta t}{m_k} I_k F(v_{n,j})\big)\Big)\,, \qquad k=1,\ldots,r\,.$$

We have $v_{n,1}=u_n$. Suppose (induction assumption) that $\|v_{n,j}\|_\infty \le \|u_n\|_\infty$ for $j = 1,\ldots,i-1$. Since

$$\alpha_{ij}^{(k)} v_{n,j} + \beta_{ij}^{(k)}\frac{\Delta t}{m_k} I_k F(v_{n,j}) = \big(\alpha_{ij}^{(k)} - C\beta_{ij}^{(k)}\big)v_{n,j} + C\beta_{ij}^{(k)}\big(v_{n,j} + \frac{\Delta t}{Cm_k} I_k F(v_{n,j})\big)\,,$$

we then have

$$\|\alpha_{ij}^{(k)} v_{n,j} + \beta_{ij}^{(k)}\frac{\Delta t}{m_k} I_k F(v_{n,j})\|_\infty \le \alpha_{ij}^{(k)}\|v_{n,j}\|_\infty \le \alpha_{ij}^{(k)}\|u_n\|_\infty\,.$$

It follows that $\|I_k v_{n,i}\|_\infty \le \|u_n\|_\infty$ for $k=1,\ldots,r$, and hence $\|v_{n,i}\|_\infty \le \|u_n\|_\infty$. Using induction with respect to $i=1,\ldots,s+1$ the proof thus follows. □

It is obvious that we are in particular interested in the optimal value of C in (33) for a given method (29). To obtain a suitable expression for this optimal value, we can follow the construction of Ferracina & Spijker [6] and Higueras [9] for the individual Runge-Kutta methods given by the coefficients $\mathscr{K}_k$.

Theorem 2. *The optimal value for $C \ge 0$ in (33), under the constraints (31) and (32), equals the largest $\gamma \ge 0$ such that*

$$(I+\gamma\mathscr{K}_k)^{-1}[e \;\; \gamma\mathscr{K}_k] \ge 0\,, \qquad k=1,\ldots,r\,. \tag{34}$$

Proof. Suppose $\gamma \ge 0$ is such that (34) holds. We take $\mathscr{B}_k = (I+\gamma\mathscr{K}_k)^{-1}\mathscr{K}_k$ and $\mathscr{A}_k = \gamma\mathscr{B}_k$. With this choice it is easily seen that (31) and (32) are valid and that (33) holds with $C=\gamma$.

On the other hand, suppose that we have (31), (32) and (33) with $C \ge 0$, and set $\gamma = C$. Then

$$\big(I+\gamma\mathscr{K}_k\big)^{-1}[e \;\; \gamma\mathscr{K}_k] = \big(I-\mathscr{M}_k\big)^{-1}[(I-\mathscr{A}_k)e \;\; \gamma\mathscr{B}_k]\,,$$

where $\mathscr{M}_k = \mathscr{A}_k - \gamma\mathscr{B}_k$. From (33) we know that $\mathscr{M}_k \ge 0$, and since it is a strictly lower triangular matrix we also have

$$(I-\mathscr{M}_k)^{-1} = I + \mathscr{M}_k + \mathscr{M}_k^2 + \ldots + \mathscr{M}_k^s \;\ge\; 0\,.$$

It follows that (34) is valid. □

4.2 Monotonicity under assumption (27)

If we assume (27) for a general (semi-)norm or sublinear functional, then a suitable form for (29) is

$$v_{n,i} \;=\; \big(1-\underline{\alpha}_i^{(0)}\big)u_n + \sum_{k=1}^{r}\sum_{j=1}^{i-1}\big(\underline{\alpha}_{ij}^{(k)} v_{n,j} + \underline{\beta}_{ij}^{(k)}\frac{\Delta t}{m_k} I_k F(v_{n,j})\big)\,, \tag{35}$$

where $\underline{\alpha}_i^{(0)} = \sum_{j=1}^{i-1}\big(\underline{\alpha}_{ij}^{(1)}+\cdots+\underline{\alpha}_{ij}^{(r)}\big)$, $i=1,\ldots,s+1$, and

$$\mathscr{K}_k = \Big(I - \sum_{l=1}^{r}\underline{\mathscr{A}}_l\Big)^{-1}\underline{\mathscr{B}}_k\,, \qquad k=1,\ldots,r\,. \tag{36}$$

We want

$$\underline{\alpha}_i^{(0)} \le 1\,, \quad \underline{\alpha}_{ij}^{(k)}, \underline{\beta}_{ij}^{(k)} \ge 0 \qquad \text{for } 1\le j<i\le s+1\,,\ 1\le k\le r\,, \tag{37}$$

with an optimal

$$\underline{C} = \min_{i,j,k}\underline{\alpha}_{ij}^{(k)}/\underline{\beta}_{ij}^{(k)}\,. \tag{38}$$

Theorem 3. *Assume (27) is valid.*
(i) Consider (35) with (37) and let $\underline{C}$ *be given by (38). Then* $\|u_{n+1}\|\le\|u_n\|$ *whenever* $\Delta t \le \underline{C}\tau_0$.
(ii) The optimal $\underline{C}\ge 0$ *in (38), under the constraints (36) and (37), equals the largest* $\gamma \ge 0$ *such that*

$$\Big(I+\sum_{l=1}^{r}\gamma\mathscr{K}_l\Big)^{-1}[e\ \ \gamma\mathscr{K}_k] \ge 0\,, \qquad k=1,\ldots,r\,. \tag{39}$$

The proof of this result is similar to that of the Theorems 1 and 2. In fact, the result for $r=2$ can be obtained directly from Higueras [10] and Spijker [18]. Further we note that the coefficient matrices $\underline{\mathscr{A}}_k$ and $\underline{\mathscr{B}}_k$ which lead to an optimal value $\underline{C}$ are in this case given by $\underline{\mathscr{B}}_k = (I+\sum_l \gamma\mathscr{K}_l)^{-1}\mathscr{K}_k$ and $\underline{\mathscr{A}}_k = \gamma\underline{\mathscr{B}}_k$.

4.3 Monotonicity under assumption (26)

Finally, if (26) is assumed for a general (semi-)norm or sublinear functional, then we consider

$$v_{n,i} \;=\; \big(1-\overline{\alpha}_i^{(0)}\big)u_n + \sum_{j=1}^{i-1}\Big(\overline{\alpha}_{ij}^{(0)} v_{n,j} + \sum_{k=1}^{r}\overline{\beta}_{ij}^{(k)}\frac{\Delta t}{m_k} I_k F(v_{n,j})\Big)\,, \tag{40}$$

where $\overline{\alpha}_i^{(0)} = \sum_{j=1}^{i-1}\overline{\alpha}_{ij}^{(0)}$, $i=1,\ldots,s+1$, and

$$\mathscr{K}_k = (I-\overline{\mathscr{A}}_0)^{-1}\overline{\mathscr{B}}_k\,, \qquad k=1,\ldots,r\,. \tag{41}$$

Here we want

$$\overline{\alpha}_i^{(0)} \le 1\,, \quad \overline{\alpha}_{ij}^{(0)}, \overline{\beta}_{ij}^{(k)} \ge 0 \qquad \text{for } 1 \le j < i \le s+1\,,\ 1 \le k \le r\,. \tag{42}$$

such that

$$\overline{C} = \min_{i,j,k} \overline{\alpha}_{ij}^{(0)} / \overline{\beta}_{ij}^{(k)} \tag{43}$$

is optimal.

Theorem 4. *Consider (40) with (42) and let* $\overline{C}$ *be given by (43). Assume (26) is valid. Then* $\|u_{n+1}\| \le \|u_n\|$ *whenever* $\Delta t \le \overline{C}\tau_0$.

The proof is similar to that of Theorem 1. For this case there is no convenient representation (comparable to (34) and (39)) of the optimal step-size coefficient $\overline{C}$. An optimization code can be used to determine this optimal value. However, from the previous results we do obtain useful upper and lower bounds for $\overline{C}$.

Theorem 5. *The optimal values* C, $\underline{C}$, $\overline{C}$ *in (33), (38) and (43) satisfy*

$$\frac{1}{r}\overline{C} \;\le\; \underline{C} \;\le\; \overline{C} \;\le\; C\,.$$

Consequently, if $\underline{C} = 0$ *then* $\overline{C} = 0$.

Proof. Given an optimal $\overline{C}$ with corresponding coefficient matrices $\overline{\mathscr{A}}_0$, $\overline{\mathscr{B}}_k$, we can take $\mathscr{A}_k = \overline{\mathscr{A}}_0$, $\mathscr{B}_k = \overline{\mathscr{B}}_k$. Then (31) and (32) hold and $\min_{i,j,k} \alpha_{ij}^{(k)} / \beta_{ij}^{(k)} \ge \overline{C}$. Consequently we have $C \ge \overline{C}$ for the optimal value C.

Likewise, for a given optimal $\underline{C}$ with corresponding $\underline{\mathscr{A}}_k$, $\underline{\mathscr{B}}_k$, we can choose $\overline{\mathscr{B}}_k = \underline{\mathscr{B}}_k$, $\overline{\mathscr{A}}_0 = \sum_{l=1}^r \underline{\mathscr{A}}_l$. Then (41) and (42) hold and we have $\min_{i,j,k} \overline{\alpha}_{ij}^{(0)} / \overline{\beta}_{ij}^{(k)} \ge \underline{C}$, showing that $\overline{C} \ge \underline{C}$.

On the other hand, for given optimal $\overline{C}$ with corresponding $\overline{\mathscr{A}}_0$, $\overline{\mathscr{B}}_k$, we can take $\underline{\mathscr{B}}_k = \overline{\mathscr{B}}_k$, $\underline{\mathscr{A}}_k = \frac{1}{r}\overline{\mathscr{A}}_0$. It follows that $\underline{C} \ge \frac{1}{r}\overline{C}$. □

4.4 *Application: multirate schemes with one level of refinement*

The monotonicity results for the multirate schemes of Section 2.2 are presented in Table 1. The table gives the optimal step-size coefficients C, $\overline{C}$ and $\underline{C}$ for the various cases:

C = step-size coefficient for maximum-norm monotonicity;
$\overline{C}$ = step-size coefficient for monotonicity under (27) ;
$\underline{C}$ = step-size coefficient for monotonicity under (26) .

It was seen in Section 2.2 that for scalar conservation laws $u_t + f(u)_x = 0$ with flux-limited spatial discretizations, the monotonicity assumptions will hold for these three cases with the same τ_0.

Table 1 Optimal step-size coefficients for the multirate schemes with one level of refinement. (The entry $\overline{C}$ for the scheme SH2 is a lower bound.)

	C	$\overline{C}$	$\underline{C}$
OS1	1	0.667	0.423
TW1	1	0.580	0.423
TW2	1	0	0
CS2	1	0	0
SH2	0.5	0.284	0.284

The results for the first-order schemes OS1 and TW1 can be quite easily derived analytically as in Subsection 2.2.4; we get $C = 1$, $\overline{C} = 2/3$, $\underline{C} = 1 - 1/\sqrt{3}$ for OS1, and $C = 1$, $\overline{C} = 2 - \sqrt{2}$, $\underline{C} = 1 - 1/\sqrt{3}$ for TW1.

The optimal values C, $\overline{C}$ for the second-order schemes have been found numerically, using (34) and (39). For the TW2 and CS2 schemes we have $\underline{C} = 0$ and therefore also $\overline{C} = 0$. The fact that $\underline{C} = 0$ for these two schemes can also be shown analytically, similar to [10], by considering (39) for small $\gamma > 0$. The value of $\overline{C}$ for SH2 was obtained with the MATLAB optimization code FMINIMAX. This does not provide a guarantee that the solution is a global optimum, and therefore this $\overline{C}$ is to be considered as a lower bound. The fact that we merely have $C = 1/2$ for the SH2 scheme is due to the first stage.

The result $C = 1$ for the OS1 and TW1 scheme was already given in [11, 14, 19] in terms of maximum principles. For the CS2 scheme the same result has been proved in [1].

The optimal values C are such that we will have monotonicity in the maximum-norm, as well as maximum principles, provided that $\Delta t \leq C\tau_0$. Likewise, for spatial discretization with limiting the TVD property will hold if $\Delta t \leq \overline{C}\tau_0$. All this under corresponding assumptions (11) for the semi-discrete system.

Comparison of these theoretical values with experiments for the Burgers equation $u_t + (\frac{1}{2}u)_x = 0$ with solution values $u \in [-1, 1]$ and flux-limited spatial discretizations did not show a clear correspondence. As was noted before, we then have $\tau_0 = \frac{1}{2}\Delta x$ for both the maximum-norm and the total variation semi-norm. Therefore, with $\nu = \Delta t/\Delta x$, the TVD property is guaranteed by the above results for $\nu \leq \frac{1}{2}\overline{C}$ and the maximum principle for $\nu \leq \frac{1}{2}C$. For the Burgers' experiment with a moving shock it was observed that for the schemes TW2, CS2 and SH2 there was no overshoots for $\nu \leq 1$, whereas the TVD property was valid for $\nu \leq 0.8$ approximately. Therefore, for that test, the theoretical optimal values $\overline{C} = 0$ for the TW2 and CS2 schemes in Table 1 are much too pessimistic. The same seems to hold for the small value $C = \frac{1}{2}$ of the SH2 scheme compared to the value $C = 1$ for TW2 and CS2. This may be caused by the fact that spatial discretizations with flux-limiting (or of WENO type) do add some local diffusion near very steep gradients, which may counteract an overshoot or increase of total variation of the time stepping scheme.

5 Concluding remarks

In this paper some multirate schemes based on the forward Euler method and the two-stage explicit trapezoidal rule have been analyzed. All these methods can be written as partitioned Runge-Kutta methods.

For the analysis of the monotonicity properties of the schemes we followed the TVD/SSP framework of [3, 4, 17], assuming monotonicity of one forward Euler step with suitable local time steps. The monotonicity assumptions in this paper consist of generalizations of the assumptions made in [10] and [18], together with more relaxed assumptions which are still valid for 1D scalar conservation laws with flux-limited spatial discretizations.

Different monotonicity thresholds were found for maximum-norm monotonicity and maximum principles on the one hand, and the TVD property on the other hand. However, these theoretical differences did not reveal themselves in numerical tests. In practical situations, the threshold C found for maximum-norm monotonicity seems the most relevant.

Many multirate schemes are not internally consistent. This may lead to low accuracy at interface points. An analysis of the local discretization errors even suggests lack of convergence, but this is too pessimistic. Also for the other schemes, that are internally consistent, propagation of the leading local error terms has to be studied to understand the proper convergence behaviour.

The use of a high-order Runge-Kutta methods as basis for a multirate scheme or a partitioned scheme will not directly lead to a high order of accuracy at interface points. The discretization errors have to be considered within the PDE context. Regarding the semi-discrete as a fixed (non-stiff) ODE will in general lead to a too optimistic estimate of the rate of convergence. Such an accuracy analysis is part of our current research.

The partitioning considered in this paper was grid point based, that is, componentwise in the semi-discrete system, with $F = I_1 F + I_2 F$. For a conservative spatial discretization of a conservation law, splittings of F could also be based on the fluxes, leading to a splitting $F = F^1 + F^2$ with F^1, F^2 containing fluxes and $h^T F^k(v) = 0$ for all v, instead of (19), and this automatically guarantees mass conservation. However, monotonicity assumptions such as (11) will not be valid in the maximum-norm with this decomposition. This can be seen already quite easily for the first-order upwind advection discretization (2). Moreover, such a decomposition of F can easily lead to inconsistencies, since we do not have $F^k(u(t)) = \mathscr{O}(1)$, no matter how smooth the solution is. For example, for the first-order upwind system (2) such a decomposition gives a completely inconsistent result.

Acknowledgements The work of W. Hundsdorfer is supported by Award No. FIC/2010/05 from King Abdullah University of Science and Technology (KAUST). The work of A. Mozartova has been supported by a grant from the Netherlands Organisation for Scientific Research NWO.

References

1. E.M. Constantinescu, A. Sandu, *Multirate timestepping methods for hyperbolic conservation laws*. J. Sci. Comput. 33 (2007), 239-278.
2. C. Dawson, R. Kirby, *High resolution schemes for conservation laws with locally varying time steps*. SIAM J. Sci. Comput. 22 (2000), 2256–2281.
3. S. Gottlieb, C.-W. Shu, E. Tadmor, *Strong stability preserving high-order time discretization methods*. SIAM Review 42 (2001), 89–112.
4. S. Gottlieb, D. Ketcheson, C.-W. Shu, *Strong Stability Preserving Runge-Kutta and Multistep Time Discretizations*, World Scientific, 2011.
5. M. Günther, A. Kværnø, P. Rentrop, *Multirate partitioned Runge-Kutta methods*. BIT 41 (2001), 504–514.
6. L. Ferracina, M.N. Spijker, *An extension and analysis of the Shu-Osher representation of Runge-Kutta methods*. Math. Comp. 74 (2005), 201–219.
7. A. Harten, *High resolution schemes for hyperbolic conservation laws*. J. Comput. Phys. 49 (1983), 357–393.
8. E. Hairer, S.P. Nørsett, G. Wanner, *Solving Ordinary Differential Equations I – Nonstiff Problems*. Second edition, Springer Series Comput. Math. 8, Springer, 1993.
9. I. Higueras, *Representations of Runge-Kutta methods and strong stability preserving methods*. SIAM J. Numer. Anal. 43 (2005), 924–948.
10. I. Higueras, *Strong stability for additive Runge-Kutta methods*. SIAM J. Numer. Anal. 44 (2006), 1735–1758.
11. R. Kirby, *On the convergence of of high resolution methods with multiple time scales for hyperbolic conservation laws*. Math. Comp. 72 (2003), 1239–1250.
12. R.J. LeVeque, *Finite Volume Methods for Hyperbolic Problems*. Cambridge Texts in Appl. Math., Cambridge Univ. Press, 2002.
13. N.M. Maurits, H. van der Ven, A.E.P. Veldman, *Explicit multi-time stepping methods for convection dominated flow problems*. Comput. Meth. Appl. Mech. Engrg. 157 (1998), 133–150.
14. S. Osher, R. Sanders, *Numerical approximations to nonlinear conservation laws with locally varying time and space grids*. Math. Comp. 41 (1983), 321–336.
15. V. Savcenco, W. Hundsdorfer, J.G. Verwer, *A multirate time stepping strategy for stiff ordinary differential equations.* BIT 47 (2007), 137–155.
16. C.-W. Shu, *High order ENO and WENO schemes for computational fluid dynamics.* In: *High-Order Methods for Computational Physics*, Eds. T.J. Barth, H. Deconinck, Lect. Notes Comp. Sc. Eng. 9, Springer, 1999, 439–582.
17. C.-W. Shu, S. Osher, *Efficient implementation of essentially non-oscillatory shock-capturing schemes*. J. Comput. Phys. 77 (1988), 439–471.
18. M.N. Spijker, *Stepsize restrictions for general monotonicity in numerical initial value problems*. SIAM J. Numer. Anal. 45 (2007), 1226–1245.
19. H.-Z. Tang, G. Warnecke, *High resolution schemes for conservation laws and convection-diffusion equations with varying time and space grids*. J. Comput. Math. 24 (2006), 121–140.

PREVIOUS PUBLICATIONS IN THIS SERIES:

Number	Author(s)	Title	Month
12-06	E.N.M. Cirillo A. Muntean	Dynamics of pedestrians in regions with no visibility – a lattice model without exclusion	March '12
12-07	J.M.L. Maubach W.H.A. Schilders	Micro- and macro-block factorizations for regularized saddle point systems	Apr. '12
12-08	K. Kumar I.S. Pop F.A. Radu	Convergence analysis for a conformal discretization of a model for precipitation and dissolution in porous media	Apr. '12
12-09	C.R. Prins J.H.M. ten Thije Boonkkamp T.W. Tukker W.L. IJzerman	An inverse method for the design of TIR collimators to achieve a uniform color light beam	Apr. '12
12-10	W. Hundsdorfer A. Mozartova V. Savcenco	Monotonicity conditions for multirate and partitioned explicit Runge-Kutta schemes	May '12

Ontwerp: de Tantes,
Tobias Baanders, CWI