Internal solitary waves in compressible fluids

DOI:
10.6100/IR327789

Document status and date:
Published: 01/01/1990

Document Version:
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

INTERNAL SOLITARY WAVES IN COMPRESSIBLE FLUIDS

PROEFSCHRIFT

ter verkrijging van de graad van doctor aan de Technische Universiteit Eindhoven, op gezag van de Rector Magnificus, prof. ir. M. Tels, voor een commissie aangewezen door het College van Dekanen in het openbaar te verdedigen op vrijdag 20 april 1990 te 14.00 uur

door

ROBERTUS HENRICUS MARGARETHA MIESEN

geboren te Sittard

Druk: Boek en Offsetdrukkerij Letru, Helmond. 04920-37797

Dit proefschrift is goedgekeurd door
de promotoren
prof. dr. F.W. Sluijter
en
prof. dr. M.P.H. Weenink
en door de copromotor dr. ir. L.P.J. Kamp

The work described in this thesis was carried out at the Physics Department of the Eindhoven University of Technology and was financially supported by "de Nederlandse organisatie voor Wetenschappelijk Onderzoek (NWO)".

Habe nun, ach! Philosophie,
Juristerei und Medizin,
Und leider auch Theologie!
Durchaus studiert, mit heißem Bemühn.
Da steh ich nur, ich armer Tor
Und bin so Klug, als wie zovor ...

(uit Goethe's Faust)

aan mijn ouders

CONTENTS

SUMMARY

In the first chapter of this thesis, several basic properties of acoustic gravity waves are discussed using the linearised equations of motion for an inviscid non–rotating compressible fluid that is stratified under gravity. After this introductory chapter, two main topics are addressed. Firstly, a critical analysis of the frequently used hydrostatic approximation and of the equally frequently used Boussinesq approximation, both applied to finite amplitude waves in stratified fluids, is given (chapter 2). Secondly, the influence of the compressibility on the properties of internal solitary waves in compressible stratified fluids is studied (chapters 4 and 5).

The hydrostatic approximation, which neglects the vertical acceleration of the fluid in the momentum equation, is shown to be valid for acoustic gravity waves with small aspect ratios (the aspect ratio being the ratio of the vertical and horizontal scales of the wave) and with frequencies much smaller than the Brunt–Väisälä frequency. The Boussinesq approximation can be used if density variations are small throughout the part of the fluid that we want to describe. For acoustic gravity waves this implies that the vertical scale of the motion must be much smaller than the scale of the stratification of the fluid, i.e. the scale height. The linear Boussinesq equations for an incompressible fluid can be used for these waves under the following additional conditions:

1) the displacement of streamlines should be small compared to the vertical scale of the wave,

2) the frequency of the wave should be smaller than the Brunt–Väisälä frequency and

3) the definition of the Brunt–Väisälä frequency for an incompressible fluid should be replaced by its compressible counterpart, i.e. the density has to be replaced by the potential density.

It is noted that if other small effects are retained, e.g. weak nonlinear effects, one should be very careful using the Boussinesq approximation even if the conditions mentioned above are met.

To study internal solitary waves in compressible fluids two integrals of the equations of motion for an inviscid compressible fluid are derived. These integrals, together with certain boundary conditions, are transformed to

equations for the displacement of a streamline and for the perturbation of the temperature, where we assume the absence of closed streamlines in order to be able to determine the two constants of integration in the integrals of motion from upstream conditions. Like for incompressible fluids, internal solitary waves in compressible fluids are described by the Korteweg–de Vries equation if the fluid is shallow, i.e. if the total depth of the fluid is much smaller than the characteristic horizontal length scale of the wave. They are described by the Benjamin–Davis–Ono equation if the fluid is deep, i.e. if the total depth of the fluid is much larger than the characteristic horizontal length scale of the wave. The corrections, due to the compressibility, to the coefficients and so to the solutions of these two equations are $\mathcal{O}(gh/c_s^2)$, where g is the gravitational acceleration, h the depth of the stratified part of the fluid and c_s the velocity of sound. For atmospheric conditions the magnitude of this parameter varies between 0.01 and 1. A type of flow for which corrections due to the compressibility are always important is a shallow isothermal shearless layer of fluid, as is discussed in this thesis.

The contents of the sections 2.4.2 and 2.4.3 have been published as: Miesen, R.H.M., Kamp, L.P.J., Sluijter, F.W. (1988). "On the application of the Boussinesq approximation for nonlinear gravity waves in the atmosphere", *Phys. Scripta* 38, 857–859.

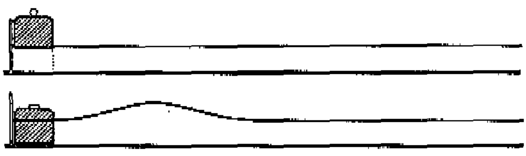
The contents of the chapters 3 and 4 will be published as: Miesen, R.H.M., Kamp, L.P.J., Sluijter, F.W. (1990). "Solitary Waves in Compressible Shallow Fluids", *Phys. Fluids A* 2 (in press).

The contents of chapter 5 have been submitted for publication as: Miesen, R.H.M., Kamp, L.P.J., Sluijter, F.W. "Solitary Waves in Compressible Deep Fluids", *Phys. Fluids A*.

HISTORICAL INTRODUCTION

> I believe I shall best introduce this phaenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water in the channel which it had put in motion; it accumulated around the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well–defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phaenomenon ....
>
> John Scott Russell (1845)

This original description by J.S. Russell, a Scottish engineer who discovered and named the *solitary wave* riding on horseback along a channel in the neighborhood of Edinburgh, shows that the history of solitary waves has been entangled with the theory of waves in fluids right from their discovery. Especially in the theory of surface waves the study of *nonlinear* wave phenomena, like solitary waves, has played an important role and vice versa [e.g. Airy 1845, Boussinesq 1872, Rayleigh 1876, Korteweg & de Vries 1895, Lamb 1932, Keller 1948, Ursell 1953, Long 1956]. After he discovered the solitary wave Russell, who estimated this phenomenon at its value, studied it extensively [Russell 1838, 1845]. In a laboratory channel he tried to produce solitary waves by either releasing an impounded elevation of water or dropping a weight at one end of a channel.



Russel's experiment.

His experiments showed that solitary waves are reproducible indeed and that a wave with amplitude (or height) a in water of depth h advances with the speed

$$c = [g(h+a)]^{1/2}, \tag{i.1}$$

where g is the gravitational acceleration. Note that this implies that a solitary wave moves faster if its amplitude is larger. He also observed that breaking occurs for $a \approx h$, that negative solitary waves, i.e. solitary waves of depression, do not exist, and that an initial elevation might, depending on the relation between its length and height, evolve into either a pure solitary wave, a solitary wave plus a wave train, or two or more solitary waves with or without a wave train.

Russell's ideas and experiments on solitary waves, as described above, seemed to contradict Airy's shallow–water theory [Airy 1845, Stokes 1891, Lamb 1932, section 252, Ursell 1953, Stoker 1957, section 10.9]. According to this theory for surface waves with horizontal wavelengths much larger than the depth of the fluid, a wave of finite amplitude cannot propagate without change of form. The Airy paradox caused considerable scientific interest at the time. The paradox was solved independently by Boussinesq [1871a,b, 1872, 1877] and Rayleigh [1876]. They showed that, if vertical acceleration and finite amplitude is accounted for correctly, water waves can propagate without change of shape. As a solitary wave solution they found

$$\eta = a\,\mathrm{sech}^2[(x-ct)/\ell], \quad \alpha := a/h \ll 1, \quad \epsilon := (h/\ell)^2 = \mathcal{O}(\alpha), \tag{i.2}$$

where η is the free–surface displacement, x the horizontal coordinate, t the time, and ℓ the characteristic length of the solitary wave. The wave speed is indeed given by (i.1) and the characteristic length ℓ is given by

$$\ell^2 = 4h^3/3a. \tag{i.3}$$

Their work was the first to contain the subtle balance between dispersion, caused by the allowance for vertical acceleration which was neglected in shallow–water theory until that time (hydrostatic approximation), and nonlinearity. It is this balance, where nonlinearity tends to steepen the wave front in consequence of the increase of wave speed with amplitude [cf. (i.1)] and dispersion tends to spread the wave front because the wave speed of any spectral component decreases with increasing wave number, that is essential to solitary waves.

A subsequent important theoretical result was the derivation by Korteweg & de Vries [1895] of an equation describing finite amplitude waves, moving in one direction in a uniform rectangular channel with an inviscid fluid. This equation is now called Korteweg–de Vries equation:

$$\eta_t + c_0[\eta_x + \tfrac{3}{2}(\eta/h)\eta_x + \tfrac{1}{6}h^2\eta_{xxx}] = 0, \tag{i.4}$$

wherein c_0 is the speed of infinitesimal long waves [cf. (i.1) with $a = 0$] and subscripts denote derivatives. A solution of this equation is given by (i.1) to (i.3). Korteweg & de Vries also obtained periodic solutions of this equation in terms of elliptic functions which they called *cnoidal* waves [see also Whitham 1974]. In fact the equivalent of (i.4) that Korteweg & de Vries derived is a somewhat more general equation in which they allow for a surface tension, which we have assumed to be zero here. A recent review on shallow–water solitary waves is the one by Miles [1980].

The interest in solitary waves waned after the resolution of the Airy paradox and no coherent research was done. Solitary waves were merely thought of as an unimportant curiosity in the theory of nonlinear surface waves. This situation did last until 1955, when computer experiments in a different field [Fermi, Pasta & Ulam 1955] gave a new impetus to nonlinear wave theory. Zabusky & Kruskal [1965] coined the soliton concept, referring to the particle like behaviour of the solitary waves they studied. It was at that time that it was discovered that the Korteweg–de Vries equation or equivalents of this equation arise in a wide variety of physical contexts [e.g Broer 1964, Van Wijngaarden 1968] and that these equations and their solutions exhibit some remarkable and far–reaching properties [see e.g. the reviews by Jeffrey & Kakutani 1972, Scott et al. 1973, Miura 1974, 1976, Kruskal 1974, 1975, Makhankov 1978].

At the same time that the theory of solitary waves and nonlinear wave equations attracted new interest, progress was made with the theory and interpretation of internal gravity waves in the atmosphere [e.g. Gossard & Munk 1954, Eckart & Ferris 1956, Eckart 1960, Hines 1960, 1963, 1965, Tolstoy 1963, Yeh & Liu 1972, 1974]. Until that time interest in these and related waves was sporadic [Väisälä 1925, Brunt 1927, Pekeris 1948, Scorer 1948, Martyn 1950]. These internal gravity waves can only exist because of the stratification of the atmosphere. Because the free surface of a fluid is a limiting form of stratification Abdullah [1949] suggested that atmospheric solitary waves might exist. The first observation of what appears to be an atmospheric solitary wave is also by Abdullah [1955]. The theoretical study of internal solitary waves, i.e. solitary waves that do not exist on the free surface of a fluid but "inside" the fluid as a consequence of internal density stratification, was initiated by Keuligan [1953] who considered a system of two superimposed fluids of different constant densities, bounded above and below by rigid surfaces. Long [1956], Abdullah [1956] and Benjamin [1966] also investigated this kind of model for internal solitary waves. A more general treatment, in which the upper boundary of the two–fluid system may be free, as well as the more difficult problem of the existence of internal solitary waves in a fluid whose density decreases exponentially with height, was given by Peters & Stoker [1960]. The problem of a continuously stratified fluid was also addressed by Ter–Krikorov [1963], Long [1965], Benjamin [1966], Benney [1966], Djordjevic & Redekopp [1978], and Weidman [1978]. Benjamin, Benney and Miles [1979] also allow for shear, as long as there are no critical layers (that is where the phase speed of the wave equals the velocity of the fluid). For the modifications necessary when critical layers occur, see Maslowe & Redekopp [1980] or Tung et al. [1982]. Miles also includes cubic nonlinearity. Gear & Grimshaw [1983] extend the theory to the second order (in α). Compressibility was accounted for by Long & Morton [1966], Shen [1966, 1967], Shen & Keller [1973], Grimshaw [1980/1981] and Miesen et al. [1990a]. Long & Morton only consider a mode that is entirely due to the compressibility and that disappears in the limit of incompressibility. The theory of Shen and Shen & Keller is very difficult to compare with solitary wave theory for incompressible fluids. Grimshaw, who uses a Lagrangian formulation of the problem, also considers compressibility of the fluid to some extent, but it does not connect well to the theory for incompressible fluids. Miesen et al. are the first to give a theory for a fully compressible fluid, generalising the theory of Gear & Grimshaw [1983] for an incompressible fluid, also allowing the background quantities to vary with the height in the fluid. For recent reviews of the theory of internal solitary waves see Grimshaw [1982] and Redekopp [1983]. A compact review of the theory of solitary waves in shallow fluids is the one by

Grimshaw [1986]. See also the books on nonlinear wave theory by Karpman [1975] and Whitham [1974].

Until now we have only discussed the "classical" shallow-water solitary wave theory. Benjamin [1967] and Davis & Acrivos [1967] were the first to discuss the problem of internal waves in a stratified layer of fluid that is contained in an infinitely deep fluid. These waves have dispersion properties that differ from the waves in shallow water, and instead of the Korteweg–de Vries equation they are therefore described by an equation which is now called the Benjamin–Davis–Ono equation [Ono, 1975]. A solitary wave solution of this equation was first given by Benjamin [1967]. This solution is called an algebraic solitary wave. The influence of compressibility on these waves has been considered only by Grimshaw [1980/1981], who also has given a second-order theory [1981], and by Miesen et al. [1990b]. Important experiments in which classical as well as algebraic solitary waves are studied have been carried out by Koop & Butler [1981] and Segur & Hammack [1982].

Observations of solitary waves in the atmosphere, though less detailed than in oceans [e.g. Gargett 1976, Farmer 1978, Haury et al. 1979, Osborne & Burch 1980, Apel et al. 1985, Liu et al. 1985, Liu 1988, Ostrovsky & Stepanyants, 1989], have been frequent since 1978 [Christie et al. 1978, 1979, 1981, Clarke et al. 1981, Shreffler & Binkowski 1981, Goncharov & Matveyev 1982, Stobie et al. 1983, Doviak & Ge 1984, Pecnik & Young 1984, Haase & Smith 1984, Noonan & Smith 1985, Bosart & Sanders 1986, Lin & Goff 1988, Shutts et al. 1988]. These observations include phenomena like "the Morning Glory" and undular bores, that can be explained in terms of solutions of nonlinear wave equations for internal waves.

In chapter 1 of this thesis an introduction is given into the linear theory of acoustic gravity waves (see also the books by Phillips [1966], Hines [1974], Gossard & Hooke [1975], Lighthill [1978], Yih [1980], Yeh & Liu [1972], Holton [1972], Kato [1980], Gill [1982], Turner [1983], and the reviews by Yeh and Liu [1974] and Jones [1976]). Frequently used approximations like the Boussinesq- and the hydrostatic approximation are discussed in chapter 2. In chapter 3 two integrals of the equations of motion for a compressible flow are derived and formulated in terms of the displacement of streamlines and the perturbation of the temperature. These two equations describe the flow exactly if it is two-dimensional and stationary in a horizontally moving frame, the background properties of the flow depending on the vertical coordinate. In the limit of incompressibility they reduce to Long's equation [1953]. In the chapters 4 and 5 the corrections, due to the compressibility, for solitary internal gravity waves in shallow and deep fluids, respectively, are discussed.

At the end of this introduction we will indicate the reasons for the interest in atmospheric solitary waves. From the academic point of view they are an interesting nonlinear wave phenomenon in a non-homogeneous medium. Furthermore they play an important role in the explanation of meteorological phenomena like the already mentioned "Morning Glory", undular bores and clear-air turbulence. These waves also cause a wind-shear hazard to airplanes, so that an accurate description of the dynamics of these waves is necessary to estimate possible risks [Christie & Muirhead 1983a,b and Doviak 1988]. Because internal solitary waves do not disperse and are dissipated very slowly they can transport energy as well as momentum over large distances. As such they are of interest to the general dynamics of the atmosphere.

REFERENCES

Apel, J.R., Holbrook, J.R., Liu, A.K., Tsai, J.J. (1985). The Sulu sea internal soliton experiment, *J. Phys. Ocean.* **15**, 1625–1651.

Abdullah, A.J. (1949). Cyclogenesis by a purely mechanical process, *J. Meteorol.* **6**, 86–97.

Abdullah, A.J. (1955). The atmospheric solitary wave, *Bull. Amer. Meteorol. Soc.* **36**, 515–518.

Abdullah, A.J. (1956). A note on the atmospheric solitary wave, *J. Meteorol.* **13**, 381–387

Airy, G.B. (1845). Tides and waves, section 392, "Encyclopedia Metropolitana" **5**, 241–396.

Benjamin, T.B. (1966). Internal waves of finite amplitude and permanent form, *J. Fluid Mech.* **25**, 241–270.

Benjamin, T.B. (1967). Internal waves of permanent form in fluids of great depth, *J. Fluid Mech.* **29**, 559–592.

Benney, D.J. (1966). Long nonlinear waves in fluids, *J. Math. Phys.* **45**, 52–63.

Bosart, L.F., Sanders, F. (1986). Mesoscale structure in the megalopolitan snowstorm of 11–12 february 1983, part III: A large amplitude gravity wave, *J. Atmos. Sci.* **43**, 924–939.

Boussinesq, M.J. (1871a). Théorie de lintumescence liquide, appelée onde solitaire ou de translation, se propageant dans un canal rectangulaire, *Acad. Sci. Paris, Comptes rendus* **72**, 755–759.

Boussinesq, M.J. (1871b). Théorie générale des mouvements qui sont propagés dans un canal rectangulaire horizontal, *Acad. Sci. Paris, Comptes rendus* **73**, 256–260.

Boussinesq, M.J. (1872). Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide continu dans ce canal des vitesses sensiblement pareilles de la surface au fond, *J. Math. Pures Appl.* 17, 55–108; translation A.C.J. Vastano, J.C.H. Mungall, Texas A&M Univ., Ref. 76–2–T (March, 1976)

Boussinesq, M.J. (1877). Essai sur la théorie des eaux courantes, Mém. présentés par divers Savants à L'Acad. Sci. Inst. France (series 2) 23, 1–680, (see also 24, 1–64)

Broer, L.J.F. (1964). Some recent advances in the theory of wave propagation, *Appl. Mech.*, Proceedings of the eleventh international congress of applied mechanics, Munich, Springer–Verlag, Berlin, 59–68.

Brunt, D. (1927). The period of simple vertical oscillations in the atmosphere, *Quart. J. Roy. Meteorol. Soc.* 53, 30–32.

Christie, D.R., Muirhead, K.J., Hales, A.L. (1978). On solitary waves in the atmosphere, *J. Atmos. Sci.* 35, 805–825

Christie, D.R., Muirhead, K.J., Hales, A.L. (1979). Intrusive density flows in the lower troposphere: A source of atmospheric solitons, *J. Geophys. Res. C* 84, 4959–4970.

Christie, D.R., Muirhead, K.J., Clarke, R.H. (1981). Solitary waves in the lower atmosphere, *Nature* 293, 46–49.

Christie, D.R., Muirhead, K.J. (1983a). Solitary waves: A hazard to aircraft operating at low altitudes, *Australian Meteor. Mag.* 31, 97–109.

Christie, D.R., Muirhead, K.J. (1983b). Solitary waves: A low–level wind shear hazard to aviation, *Int. J. Aviat. Safety* 1, 169–190.

Clarke, R.H., Smith, R.K., Reid, D.G. (1981). The morning glory of the gulf of Carpentaria, *Mon. Wea. Rev.* 109, 1733–1757.

Davis, R.E., Acrivos, A. (1967). Solitary internal waves in deep water, *J. Fluid Mech.* 29, 593–607.

Djordjevic, V.D., Redekopp L.G. (1978). The fission and disintegration of internal solitary waves moving over two–dimensional topography, *J. Phys. Oceanogr.* 8, 1016–1024.

Doviak, R.J., Ge, R.S. (1984). An atmospheric solitary gust observed with a Doppler radar, a tall tower and a surface network, *J. Atmos. Sci.* 41, 2559.

Doviak, R.J. (1988). Thunderstorm generated solitary waves: A wind shear hazard, *AIAA 26th Aerospace Sciences Meeting*, Reno, Nevada.

Eckart, C., Ferris, H.G. (1956). Equations of motion of the ocean and atmosphere, *Rev. Mod. Phys.* 28, 48–52.

Eckart, C. (1960). "Hydrodynamics of Oceans and Atmospheres." Pergamon Press, New York.

Farmer, D.M. (1987). Observations of long nonlinear internal waves in a lake, *J. Phys. Ocean.* 8, 63–73.

Fermi, E., Pasta, J., Ulam, S. (1955). Studies of nonlinear problems I, *Los Alamos Rep. LA 1940.* Reprinted in Newell (1974), 143–156.

Gargett, A.E. (1976). Generation of internal waves in the strait of Georgia, Britisch Columbia, *Deap Sea Res.* 23, 17–32.

Gear, J.A., Grimshaw, R.H.J. (1983). A second–order theory for solitary waves in a shallow fluid, *Phys. Fluids* 26, 14–29.

Gill, A.E. (1982). "Atmosphere–Ocean dynamics." Academic Press, New York.

Goncharov, V.P., Matveyev, A.K. (1982). Observations of nonlinear waves on an atmospheric inversion, *Izvestiya Atmos. Ocean. Phys.* 18, 61–64.

Gossard, E.E., Munk, W.H. (1954). On gravity waves in the atmosphere, *J. Meteorol.* 11, 259–269.

Gossard, E.E., Hooke, W.H. (1975). "Waves in the Atmosphere." Elsevier Scientific Publishing Company, Amsterdam.
Grimshaw, R.H.J. (1980/1981). Solitary waves in a compressible fluid, *Pure Appl. Geophys.* 119, 780–797.
Grimshaw, R.H.J. (1981). A second-order theory for solitary waves in deep fluids, *Phys. Fluids* 24, 1611–1618.
Grimshaw, R.H.J. (1983). Solitary waves in density stratified fluids, in "Nonlinear Deformation Waves", eds. Nigul, U., Engelbrecht, J., 431–447. IUTAM Symposium Tallinn 1982, Springer.
Grimshaw, R.H.J. (1986). Theory of solitary waves in shallow fluids, in "Encyclopedia of Fluid Mechanics". Gulf Publishing Company, Houston.
Haase, S., Smith, R.K. (1984). Morning glory wave clouds in Oklahoma: A case study, *Mon. Wea. Rev.* 112, 2078–2089.
Haury, L.R., Briscoe, M.G., Orr, M.H. (1979). Tidally generated internal wave packets in Massachusetts Bay, Nature 278, 312–317.
Hines, C.O. (1960). Internal atmospheric gravity waves at ionospheric heights, *Can. J. Phys.* 38, 1441–1481.
Hines, C.O. (1963). The upper atmosphere in motion, *Quart. J. Roy. Meteorol. Soc.* 89, 1–42.
Hines, C.O. (1965). Atmospheric gravity waves : A new toy for the wave theorist, *Radio. Sci.* 69D, 375–380.
Hines, C.O. (1974). "The Upper Atmosphere in Motion." American Geophysical Union, Washington.
Holton, J.R. (1972). "An Introduction to Dynamic Meteorology." Academic Press, London.
Jeffrey, A., Kakutani, T. (1972). Weak nonlinear dispersive waves: a discussion centered around the Korteweg–de Vries equation, *SIAM Rev.* 14, 582–643.
Jones, W.L. (1976). Linear internal gravity waves in the atmosphere, in "Encyclopedia of Physics", 177–216. Springer–Verlag, Berlin.
Karpman, V.I. (1975). "Non–linear Waves in Dispersive Media." Pergamon Press, Oxford.
Kato, S. (1980). "Dynamics of the Upper Atmosphere." D. Reidel Publishing Company, Dordrecht.
Keller, J.B. (1948). The solitary wave and periodic waves in shallow water, *Comm. Pure Appl. Math.* 1, 323–329.
Keuligan, G.H. (1953). Characteristics of internal solitary waves, *J. Res. Nat. Bur. Stnd.* 51, 133–140.
Koop, C.G., Butler, G. (1981). An investigation of internal solitary waves in a two–fluid system, *J. Fluid Mech.* 112, 225–251.
Korteweg, D.J., de Vries, G. (1895). On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary waves, *Phil. Mag* 39, 422–443.
Kruskal, M.D. (1974). The Korteweg–de Vries equation and related evolution equations, in Newell (1974), 61–84.
Kruskal, M.D. (1975). Nonlinear wave equations, "Dynamical Systems Theory and Applications", ed. J. Moser, 310–354. Springer–Verlag, New York.
Lamb, H. (1932). "Hydrodynamics." Cambridge University Press, Cambridge.
Leibovich, S., Seebass, A.R., eds. (1974). "Nonlinear Waves." Cornell University Press, Ithaca.
Lighthill, M.J. (1978). "Waves in Fluids." Cambridge University Press.
Lin, Y.L., Goff, R.C. (1988). A study of a mesoscale solitary wave in the atmosphere originating near a region of deep convection, *J. Atmos. Sci* 45, 194.

Liu, A.K., Holbrook, J.R., Apel, J.R. (1985). Nonlinear internal wave evolution in the Sulu sea, *J. Phys. Ocean.* **15**, 1613–1624.
Liu, A.K. (1988). Analysis of nonlinear internal waves in the New York bight, *J. Geophys. Res. C* **93**, 12317–12329.
Long, R.R. (1953). Some aspects of the flow of stratified flows, *Tellus* **5**, 42–58.
Long, R.R. (1956). Solitary waves in one– and two–fluid systems, *Tellus* **8**, 460–471.
Long, R.R. (1965). On the Boussinesq approximation and its role in the theory of internal waves, *Tellus* **17**, 46–52
Long, R.R., Morton, J.B. (1966). Solitary waves in compressible stratified fluids, *Tellus* **18**, 79–85.
Makhankov, V.G. (1978). Dynamics of classical solitons (in non–integrable systems), *Phys. Rep.* **35**, 1–128.
Martyn, D.F. (1950). Cellular atmospheric waves in the ionosphere and troposphere, *Proc. Roy. Soc. Ser. A* **201**, 216–234.
Maslowe, S.A., Redekopp, L.G. (1980). Long nonlinear waves in stratified shear flows, *J. Fluid Mech.* **101**, 321–348.
Miesen, R.H.M., Kamp, L.P.J., Sluijter, F.W. (1990a). Solitary waves in compressible shallow fluids, *Phys. Fluids A* **2**, in press.
Miesen, R.H.M., Kamp, L.P.J., Sluijter, F.W. (1990b). Solitary waves in compressible deep fluids, *Phys. Fluids A*, submitted for publication.
Miles, J.W. (1979). On internal solitary waves, *Tellus* **31**, 456–462
Miles, J.W. (1980). Solitary waves, *Ann. Rev. Fluid Mech.* **12**, 11–43.
Miura, R.M. (1974). The Korteweg–de Vries equation: a model equation for nonlinear dispersive waves. In Leibovich & Seebass (1974), 212–234.
Miura, R.M. (1976). The Korteweg–de Vries equation: a servey of results, *SIAM Rev.* **18**, 412–459.
Newell, A.C. ed. (1974). "Nonlinear Wave Motion." Lectures in Applied Mathematics, **15**, American Mathematical Society.
Noonan, J.A. Smith, R.K. (1985). Linear and weakly nonlinear internal wave theories applied to morning glory waves, *Geophys. Astrophys. Fluid Dyn.* **33**, 123–143.
Ono, H. (1975). Algebraic solitary waves in stratified fluids, *J. Phys. Soc. Japan* **39**, 1082–1091.
Osborne, A.R., Burch, T.L. (1980). Internal solitons in the Andaman sea, *Science* **208**, 451–460.
Ostrovsky, L.A., Stepanyants, Y.A. (1989). Do internal solitons exist in the ocean, *Rev. Geophys.* **27**, 293–310.
Pecnik, M.J., Young, J.A. (1984). Mechanics of a strong subsynoptic gravity wave deduced from satellite and surface observations, *J. Atmos. Sci.* **41**, 1850–1862.
Pekeris, C.L. (1948). The propagation of a pulse in the atmosphere, part II, *Phys. Rev.* **73**, 145–154.
Peters, A.S., Stoker, J.J., (1960). Solitary waves in liquids having non–constant density, *Comm. Pure Appl. Math.* **13**, 115–164.
Phillips, O.M. (1966). "The dynamics of the upper ocean.", Cambridge University Press.
Rayleigh, Lord (1876). On waves, *Phil. Mag.* **1**, 257–259, and, *Sci. Pap.* **1**, 251–271.
Redekopp, L.G. (1983). Nonlinear waves in geophysics: Long internal waves, in "Lectures in Applied Mathematics" **20**, 59–78. American Math. Soc.

Russell, J.S. (1838). Report of the committee on waves, *Rep. Meet. Brit. Assoc. Adv. Sci.*, 7th, Liverpool, 1837, 417–496. John Murray, London.
Russell, J.S. (1845). Report on waves, *Rep. Meet. Brit. Assoc. Adv. Sci.*, 14th, York, 1844, 311–390. John Murray, London.
Scorer, R.S. (1949). Theory of waves in the lee of mountains, *Quart. J. Roy. Meteorol. Soc.* 75, 41–56.
Scott, A.C., Chu, F.Y.F., McLaughlin, D.W. (1973). The soliton: a new concept in applied science, *Proc. IEEE* 61, 1443–1483.
Segur, H., Hammack, J. (1982). Solution models of long internal waves, *J. Fluid Mech.* 118, 285–304.
Sheffler, J.H., Binkowski, F.S. (1981). Observations of pressure jump lines in the midwest, *Mon. Wea. Rev.* 109, 1731–1725.
Shen, M.C. (1966). Solitary waves in an atmosphere with arbitrary wind and density profiles, *Phys. Fluids* 9, 1944–1950.
Shen, M.C. (1967). On long waves with cross–wind in an atmosphere, *J. Fluid Mech.* 28, 481–492.
Shen, M.C., Keller, J.B. (1973). Ray method for nonlinear wave propagation in a rotating fluid of variable depth, *Phys. Fluids* 16, 1565–1572.
Shutts, G.J., Kitchen, M., Hoare, P.H. (1988). A large amplitude gravity wave in the lower atmosphere detected by radiosonde, *Quart. J. Roy. Meteor. Soc.* 114, 579–594.
Stobie, J.G., Einaudi, F., Uccellini, L.W. (1983). A case study of gravity waves–convective storms interaction, *J. Atmos. Sci.* 40, 2804–2830.
Stoker, J.J. (1957). "Water Waves." Interscience, New York.
Stokes, G.G. (1891). Note on the theory of the solitary wave, *Phil. Mag.* 31, 314–316, and, *Math. Phys. Pap.* 5, 160–162.
Ter–Krikorov, A.M. (1963). Théory exacte des ondes longues stationnaires dans un liquide hétérogène, *J. Mécanique* 2, 351–376.
Tolstoy, I. (1963). The theory of waves in stratified fluids including the effects of gravity and rotation, *Rev. Mod. Phys.* 35, 207–230.
Tung, K.K., Chan, T.F., Kubota, T. (1982). Large amplitude internal waves of permanent form, *Studies Appl. Math.* 66, 1–44.
Turner, J.S. (1983). "Buoyancy Effects in Fluids.", Cambridge University Press, Cambridge.
Ursell, F. (1953). The long–wave paradox in the theory of gravity waves, *Proc. Camb. Phil. Soc.* 49, 685–694.
Väisälä, V. (1925). Über die Wirkung der Windschwankungen auf die Pilot–beobachtungen, *Soc. Sci. Fenn. Commentat. Phys.–Math* 2, no. 19, 19–37.
Van Wijngaarden, L. (1968). On the equations of motion for mixtures of liquid and gas bubbles, *J. Fluid Mech.* 33, 465–474.
Weidman, P.D. (1978). Internal solitary waves in a linearly stratified fluid, *Tellus* 30, 177–184. Corrigendum (1980).
Whitham, G.B. (1974). "Linear and Nonlinear Waves." Wiley, New York.
Yeh, K.C., Liu, C.H. (1972). "Theory of Ionospheric Waves." Academic Press, New York.
Yeh, K.C., Liu, C.H. (1974). Acoustic gravity waves in the upper atmosphere, *Rev. Geophys. Space Phys.* 12, 193–216.
Yih, C.–S. (1980). "Stratified Flows." Academic Press, New York.
Zabusky, N.J., Kruskal, M.D. (1965). Interaction of "solitons" in a collisionless plasma and the recurrence of initial states, *Phys. Rev. Lett.* 15, 240–243.

CHAPTER 1

LINEAR ACOUSTIC GRAVITY WAVES

1.1 INTRODUCTION

Although of the rich variety of wave phenomena physicists are familiar with, it seems that gravity dependent wave phenomena in fluids are not common property of them. Therefore we will give in this chapter an introduction into the theory of these waves. We will restrict ourselves to linear theory, because it is simple, it has been developed farthest, it describes almost all basic properties of acoustic gravity waves and because it is the point of departure of most non-linear theories. A linear theory, for any kind of waves, implies that we consider disturbances so weak that in the equations describing these waves, we can view their amplitudes as small quantities, whose products are negligible.

To understand the quite complicated physics involved with these waves we will separately consider the different features dominating acoustic gravity waves, i.e. the *compressibility* of the medium and the effect of the *gravitational acceleration*. To study the effects of the compressibility of a fluid we will describe in section 1.2 the theory of acoustic or sound waves. Section 1.2 is also devoted to the introduction of some elementary ideas concerning wave propagation, like the phase velocity, the group velocity, the dispersion equation, and the polarisation relations. The effect of gravity is analysed in section 1.3 by considering so–called internal gravity waves, first when propagating along the interface of two homogeneous fluids of different density, subsequently when propagating in a continuously stratified fluid. Also in section 1.3 the concept of dispersion and the effects of the anisotropy of the medium, due to the presence of the gravitational acceleration, are discussed. The influence of the compressibility of the fluid and the gravitational acceleration are combined in section 1.4 that is about the linear theory of acoustic gravity waves.

1.2 ACOUSTIC WAVES

The propagation of acoustic or sound waves is governed by the balance between the *inertia* and the *compressibility* of the fluid. We will neglect other properties of the fluid, like viscosity, heat conduction, inhomogeneities, and external forces including gravity. The validity of these assumptions is examined in textbooks on the subject, e.g. [Lighthill 1978].

The theory of acoustic waves is based on the following (nonlinear) equations for a compressible fluid, [e.g. Gill 1982, chapter 4]:

$$\frac{D\rho}{Dt} + \rho(\nabla\cdot \mathbf{u}) = 0, \tag{1a}$$

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p, \tag{1b}$$

$$\frac{D}{Dt}(p\rho^{-\gamma}) = 0, \tag{1c}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla. \tag{1d}$$

The operator $\partial/\partial t$ deals with the local change of a quantity, while the operator $\mathbf{u}\cdot\nabla$ deals with the change of a quantity owing to its changing position in space (convection). The first equation is the well–known equation of continuity, where ρ is the mass per unit volume and $\mathbf{u} = (u,v,w)$ is the vector velocity field. This equation relates the change of the density of a volume element to the velocity field, and describes the conservation of mass. The second equation is the equation of motion and is the application of Newton's second law of motion to a small fluid element. This demands that the product of the mass per unit volume ρ and the acceleration

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} \tag{2}$$

equals the force on the element per unit volume. When viscous stresses and external forces are neglected, this force per unit volume is simply minus the gradient of the fluid pressure p. The third equation can be derived directly from the equation of state of the fluid, for isentropic motion in the absence of viscous or diffusive effects [Gill 1982, chapter 4]. Here γ is the ratio of specific heats at constant pressure and constant volume.

We assume that the fluid is at rest and homogeneous in the absence of waves. Equations (1) are then linearised, i.e. products of disturbances are neglected, according to the following scheme

$$\rho(\mathbf{x},t) = \rho_0(z) + \rho_1(\mathbf{x},t), \tag{3a}$$

$$p(\mathbf{x},t) = p_0(z) + p_1(\mathbf{x},t), \tag{3b}$$

$$\mathbf{u}(\mathbf{x},t) = 0 + \mathbf{u}_1(\mathbf{x},t), \tag{3c}$$

where ρ_0 and p_0 are the mass per unit volume and the pressure, respectively, in the absence of waves, and ρ_1, p_1, and $\mathbf{u}_1$ are disturbances of respectively the mass per unit volume, the pressure and the velocity field due to the presence of waves. Note that we have assumed that $\mathbf{u}_0 = 0$, i.e., the fluid is at rest in the absence of waves. The linearised equations (1) read

$$\frac{\partial \rho_1}{\partial t} + \rho_0(\nabla \cdot \mathbf{u}_1) = 0, \tag{4a}$$

$$\rho_0 \frac{\partial \mathbf{u}_1}{\partial t} = -\nabla p_1, \tag{4b}$$

$$\frac{\partial p_1}{\partial t} = c_s^2 \frac{\partial \rho_1}{\partial t}, \tag{4c}$$

where the definition of what will prove to be the velocity of sound is

$$c_s = (\gamma p_0/\rho_0)^{1/2}. \tag{5}$$

For perfect gases this can be replaced by [e.g. Lighthill 1978, chapter 1]

$$c_s = (\gamma R T_0)^{1/2}, \tag{6}$$

where T_0 is the value of the absolute temperature T in Kelvin in the absence of waves and R is the gas constant, which is the universal gas constant divided by the mean molecular mass of the gas.

Elimination of u_1 and ρ_1 from (4) gives the following wave equation:

$$\frac{\partial^2 p_1}{\partial t^2} = c_s^2 \nabla^2 p_1. \tag{7}$$

Similar equations apply to ρ_1 and to the components of the velocity $\mathbf{u}_1$. This equation is characteristic for phenomena involving propagation through a homogeneous medium at a single wave speed c_s, with energy conserved. A solution of (7), representing a "plane wave" traveling with velocity c_s in the positive x-direction of a cartesian coordinate system, is

$$p_1 = f(x - c_s t). \tag{8}$$

Here f(x) is the form of the wave at time $t = 0$, while the waveform at a later time t has identical shape but is shifted a distance $c_s t$ in the positive x-direction.

In order to be able to compare some basic properties of acoustic waves, internal gravity waves and acoustic gravity waves, and to introduce conceptions like group velocity, polarisation relations and so on, we will consider a special solution of the kind given by (8):

$$p_1 = A \exp(i\phi), \tag{9a}$$
$$\phi = \mathbf{k}\cdot\mathbf{x} - \omega t, \tag{9b}$$

where A is the amplitude of the pressure disturbance, ϕ is the phase-function, $\mathbf{k} = (k_x, k_y, k_z)$ is the wavevector, ω is the (radian) frequency, and $\mathbf{x}$ is the spatial position in a cartesian coordinate system. The radian frequency ω is related to the waveperiod T_w by

$$\omega = 2\pi/T_w. \tag{10}$$

The magnitude k of the wavevector $\mathbf{k}$ is related to the wavelength λ by

$$k = 2\pi/\lambda \tag{11}$$

The real part of solution (9) represents a sinusoidal wave.

By substitution of (9) into the wave equation (7) we obtain the so-called *dispersion equation*, that relates the frequency of the wave ω, to the components of the wavevector $\mathbf{k}$:

$$\omega^2 = c_s^2(k_x^2 + k_y^2 + k_z^2) = c_s^2 k^2. \tag{12}$$

The *phase velocity* $\mathbf{c}_p$ is the velocity of a surface of constant phase (e.g. the wavefront of a monochromatic wave) normal to that surface, i.e.

$$\mathbf{c}_p := (\partial\mathbf{x}/\partial t \cdot \mathbf{k})\mathbf{k}/k^2. \tag{13}$$

By differentiation of (9b) with respect to time, for constant ϕ we find

$$\mathbf{c}_p = \omega\mathbf{k}/k^2, \tag{14}$$

or with the dispersion equation (12)

$$\mathbf{c}_p = c_s\mathbf{k}/k. \tag{15}$$

Equation (15) implies that acoustic waves are dispersionless, since waves with different wavenumbers all have a phase velocity with the same magnitude, c_s.

The *group velocity* $\mathbf{c}_g$ is the velocity with which a wave packet, i.e. a superposition of solutions like (9), travels. Under, in general, mild conditions it can also be shown that wave energy is transported with the group velocity [see e.g. Lighthill, 1978, Yeh and Liu, 1974]. The group velocity is defined by

$$\mathbf{c}_g = (\partial\omega/\partial k_x, \partial\omega/\partial k_y, \partial\omega/\partial k_z). \tag{16}$$

From the dispersion equation (12) we find that the group velocity of acoustic waves is simply given by

$$\mathbf{c}_g = c_s\mathbf{k}/k, \tag{17}$$

which is equal to the phase velocity, thus demonstrating the dispersionless character of such waves.

The relations among the perturbation quantities p_1, ρ_1, u_1, v_1, and w_1 are called *polarisation relations*. They can be determined from (4) together with (9)

$$p_1/\rho_1 = c_s^2, \tag{18a}$$
$$\mathbf{u}_1/p_1 = \mathbf{k}/\rho_0. \tag{18b}$$

Equation (18b) implies that acoustic waves are longitudinal, since the fluid motion $\mathbf{u}_1$ has the same direction as the direction of propagation of the wave $\mathbf{k}$.

Thus for acoustic waves we find the simple situation depicted in figure 1.

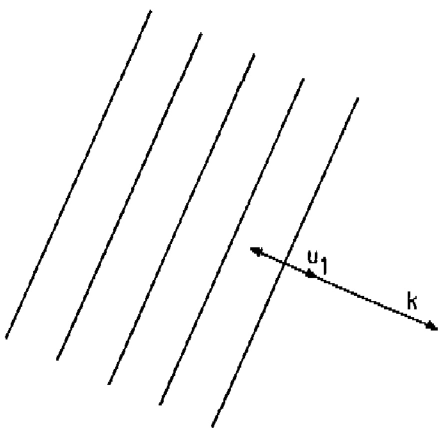


Fig. 1 For acoustic waves, the wavenumber $\mathbf{k}$*, the fluid velocity* $\mathbf{u}_1$*, the phase velocity* $\mathbf{c}_p$*, and the group velocity* $\mathbf{c}_g$ *are all normal to planes of constant phase.*

1.3 INTERNAL GRAVITY WAVES

We are all familiar with *surface gravity waves*, which are ordinary waves on the surface of, for example, a lake. These waves are controlled by the balance between a fluid's *inertia* and its tendency, under *gravity*, to return to a state of stable equilibrium with heavier fluid (water) underlying lighter (air). This tendency is due to pressure differences caused by the presence of gravity. So in contrast with the situation for acoustic waves the restoring force for a displaced fluid element is brought about by the gravitational acceleration and not by the compressibility of the fluid. In the present section we will therefore ignore this compressibility.

Suppose we have the case of a layer of fluid with a larger density ρ_1 underlying a fluid with a smaller density ρ_2 and the effect of mixing can be ignored. Along the interface of these two fluids there can exist waves that resemble surface waves in many ways and which are called *internal waves*. In fact, the theories of surface gravity waves and of internal gravity waves become identical in the limit that the density of the upper fluid becomes very small (like that of air), and if the upper fluid is assumed to extend ad infinitum. If the upper fluid does not have a density that is much smaller than the density of the lower fluid, but the upper fluid does extend ad infinitum, the two theories are identical if in the theory for surface waves the gravitational acceleration is replaced by a reduced gravitational acceleration g_r [Lighthill 1978, chapter 4.1].

$$g_r = \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}\, g. \tag{19}$$

The case of two superposed fluids of different densities is actually met, e.g. in many deep estuaries, such as Norwegian fjords, where river water moves seawards above the heavier salt water (figure 2).

If a ship enters such an estuary it may experience a substantially enhanced drag. This extra drag is due to the generation of internal waves, even though the ship may make almost no visible waves on the free surface. That the ship may generate internal waves but no surface waves is a consequence of the fact that the ship moves fast compared to the speed of internal waves and slow compared to the speed of surface waves, both with wavelengths related to the

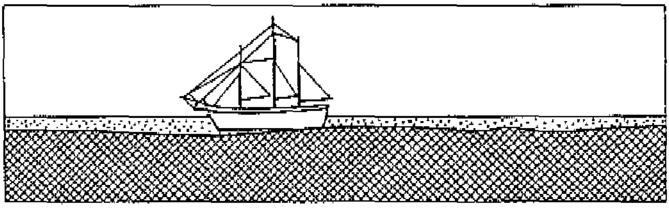


Fig.2 The generation of internal waves by a ship travelling in two superposed layers of fluid of different density.

ship's length. These lower speeds of the internal waves are due to the reduction of the effective gravitational acceleration, given by (19).

If the density change between the upper and the lower layer is not confined anymore to a region small compared with the wavelength of the wave to be considered, we have a *stratified fluid*, whose density must be considered a continuous function of the height in the fluid. As can be imagined, internal gravity waves can still exist, since there is still a balance between the inertia of the fluid and its tendency to return to a state of heavier fluid underlying lighter. But in this case the waves are no longer confined to an interface between two fluids and also propagation in vertical direction (the direction of the density gradient) will be possible. As will be seen these waves are not only dispersive (the phase velocity varies with the wavenumber) but also anisotropic: the phase velocity depends on the direction of propagation and the group velocity does not have the same direction as the phase velocity.

To gain a better (quantitative) understanding of the balance between inertia and buoyancy we consider an element of fluid at some hydrostatic equilibrium level z_0, in a fluid with density ρ_0 decreasing with height at a rate $-\,d\rho_0/dz$. The mass Δm of the fluid element at z_0 is

$$\Delta m = \rho_0(z_0)\,\Delta V, \qquad (20)$$

where ΔV is the volume of the fluid element. If we displace the fluid element quasi-static over a small vertical distance δs, it will be subjected to a buoyancy force

$$F_b = g\,[\rho_0(z_0+\delta s) - \rho_0(z_0)]\,\Delta V, \tag{21}$$

acting to return Δm to z_0 (figure 3). Together with (20) and (21), Newton's second law of motion leads to

$$g\,\frac{d\rho_0(z_0)}{dz}\,\delta s\,\Delta V = \rho_0(z_0)\,\Delta V\,\frac{d^2(\delta s)}{dt^2}. \tag{22}$$

Choosing the acceleration of gravity in the negative z-direction (cf. figure 3), $d\rho_0(z_0)/dz < 0$ for a stably stratified incompressible fluid and hence (22)

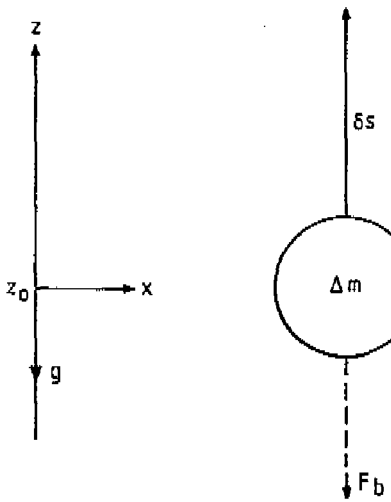


Fig. 3 The buoyancy force F_b, acting on a fluid element due to its displacement over a vertical distance δs.

describes a harmonic oscillator with a frequency ω_b, given by

$$\omega_b^2 = -\frac{g}{\rho_0}\frac{d\rho_0}{dz}. \tag{23}$$

This frequency is known as the Brunt–Väisälä frequency [Väisälä 1925, Brunt 1927]. The Brunt–Väisälä frequency is in fact the natural oscillation frequency of a stratified fluid.

If the fluid element is somehow constrained to move at an angle Θ with respect to the vertical (figure 4), the force exerted on this fluid element will be the projection of the buoyancy force now given by

$$F_b = g\,\frac{d\rho_0(z_0)}{dz}\,(\delta s\,\cos\Theta)\,\Delta V. \tag{24}$$

In this case the fluid element will therefore oscillate with a frequency ω, given by

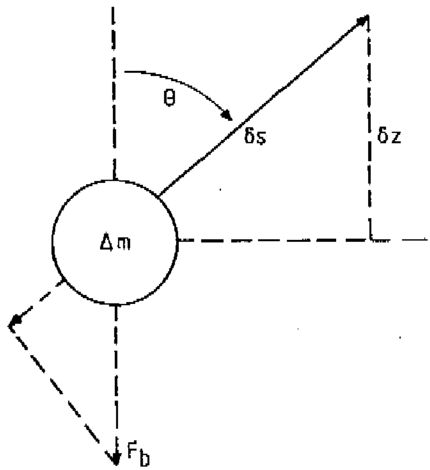


Fig. 4 The projection of the buoyancy force F_b*, acting on a fluid element due to its displacement over a vertical distance* $\delta z = \delta s\cos\Theta$*.*

$$\omega^2 = \omega_b^2 \cos^2\Theta. \tag{25}$$

Hence the Brunt–Väisälä frequency defined by (23) is the maximum frequency of oscillations in a stratified incompressible fluid.

We will now give a more formal treatment of the propagation of internal gravity waves in an incompressible non–rotating fluid. Like we did in section 1.2 for acoustic waves, we start with the continuity equation, with the equation of motion, and with an equation of state, but now for an incompressible fluid and with gravity. Heat conduction and viscosity are still neglected so that the following equations apply, [e.g. Phillips 1966, Gill 1982, chapter 4]:

$$\frac{D\rho}{Dt} = 0, \tag{26a}$$

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{g}, \tag{26b}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{26c}$$

where the vector gravitational acceleration $\mathbf{g} = (0,0,-g)$ is directed downwards (figure 3). The fact that a velocity vector field is divergence free in an incompressible fluid, as stated by (26b), expresses the solenoidal property of the velocity field [Lighthill 1986, chapter 2.4]. This property is equivalent to the assertion that the volume of a fluid element does not change by its motion as can be seen from the equation of continuity (1a) and (26a).

Again, for simplicity, we assume that the fluid velocity is zero in the absence of waves, i.e. $\mathbf{u}_0 = 0$. If $\mathbf{u}_0 = (u_0(z),0,0)$ the analysis already becomes complicated, e.g., [Booker and Bretherton 1967, Holton 1972, Kelder 1987]. Equations (26) are linearised using (3). From the equation of motion (26b) we find for the density ρ_0 and the pressure p_0 in the absence of waves

$$\frac{dp_0}{dz} = -g\rho_0, \tag{27}$$

i.e. the hydrostatic balance. Together with the law for a perfect gas

$$p = \rho RT, \tag{28}$$

we obtain

$$dp_0/p_0 = -\,dz/H, \tag{29a}$$

where H is called the scale height and is given by

$$H = RT/g. \tag{29b}$$

It may seem to be a contradiction to assume that the fluid is incompressible and to use the law for a perfect gas. However, it is to be expected that this is correct if the velocity of the fluid is small compared to the velocity of sound (this will be shown in section 3.3). Integration of (29a) from a reference height z_0, at which $p_0(z_0) = p_{00}$, to an arbitrary height z yields

$$p_0 = p_{00} \exp\left[-\int_{z_0}^{z} (dz/H)\right], \tag{30a}$$

and with (28)

$$\rho_0 = p_{00}/(RT) \exp\left[-\int_{z_0}^{z} (dz/H)\right]. \tag{30b}$$

For a constant scale height (30a,b) become

$$p_0 = p_{00} \exp[(z_0 - z)/H], \tag{31a}$$

$$\rho_0 = \rho_{00} \exp[(z_0 - z)/H], \tag{31b}$$

$$\rho_{00} = p_{00}/(RT) = p_{00}/(gH) \tag{31c}$$

Now we have evaluated the continuously stratified basic state of the fluid, we proceed with the linearised equations for the perturbations ρ_1, p_1 and $\mathbf{u}_1$, that can be obtained by substitution of (3) in (26). This gives

$$\frac{\partial \rho_1}{\partial t} + \mathbf{u}_1 \cdot \nabla \rho_0 = 0, \tag{32a}$$

$$\rho_0 \frac{\partial u_1}{\partial t} = -\nabla p_1 + \rho_1 g, \tag{32b}$$

$$\nabla \cdot u_1 = 0, \tag{32c}$$

where (32a) expresses the fact that changes of density at a fixed position are due to the bodily displacement of the mean density structure and are 90^0 out of phase with the vertical velocity. After elimination of the perturbation quantities ρ_1, p_1, u_1 and v_1 from (32), the following wave equation is found:

$$\frac{\partial^2}{\partial t^2}\left[\nabla^2 w_1 + \frac{1}{\rho_0}\frac{d\rho_0}{dz}\frac{\partial w_1}{\partial z}\right] + \omega_B^2 \nabla_\perp^2 w_1 = 0, \tag{33}$$

where

$$\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{34}$$

The Brunt–Väisälä frequency as defined by (23) generally depends on z. Using the expression for the equilibrium density (30b), (23) and (33) become respectively

$$\omega_B^2 = (g/H)(1+dH/dz) \tag{35}$$

$$\frac{\partial^2}{\partial t^2}\left[\nabla^2 w_1 - \frac{\partial w_1}{\partial z}(1 + dH/dz)H^{-1}\right] + \omega_B^2 \nabla_\perp^2 w_1 = 0. \tag{36}$$

If the scale height H is not a constant the coefficients in the partial differential equation (36) depend on z, and (36) can be solved analytically only for particular H(z), e.g. [Daniels 1975].

However, in the present analysis we will confine ourselves to the case that the scale height H is a constant and the fluid is exponentially stratified [cf. (31)]. In that case plane wave solutions, like in the analysis for acoustic waves, are possible. From (36), together with (9), we obtain the following dispersion equation for internal waves in an incompressible fluid with constant scale height:

$$\omega^2(k^2 + ik_z/H) - \omega_B^2 k_\perp^2 = 0, \tag{37}$$

here $\omega_0^2 = g/H$ is a constant, and $k_\perp^2 = k_x^2 + k_y^2$. The complex valuedness of this dispersion equation, although the medium is assumed to be completely lossless, will be discussed in the sequel.

But first (37) is examined if $|k_z| \gg 1/H$, an assumption that is almost always justified for oceanographic applications for vertical wavelengths up to hundreds of kilometers. In that case (37) becomes

$$\omega^2 = \omega_0^2 k_\perp^2/k^2, \tag{38}$$

which is equivalent to (25).

If this approximation cannot be made, one is forced to consider complex ω, k_x, k_y or k_z in order to satisfy (37). The choice of real or complex ω, k_x, k_y and k_z depends on the problem at hand and the initial or boundary conditions associated with the problem. For example, if the problem of interest is concerned with imperfect horizontal ducting, $k_\perp$ may become complex to show leakage of energy from the duct. Usually the forced oscillation case is considered for which ω and $k_\perp$ are real. For (37) to be satisfied, k_z has to be complex in that case. Let

$$k_z = k_{z,r} + ik_{z,i} \tag{39}$$

where $k_{z,r}$ and $k_{z,i}$ are real. Substitution of (39) into (37) gives

$$k_{z,i} = -(2H)^{-1}, \tag{40a}$$

$$\omega^2[k_r^2 + (2H)^{-2}] - \omega_0^2 k_\perp^2 = 0, \tag{40b}$$

where k_r^2 is defined as

$$k_r^2 = k_\perp^2 + k_{z,r}^2. \tag{41}$$

Equation (40a) implicates that the dependence on z of the field quantities ρ_1/ρ_0, p_1/ρ_0, and u_1 has the shape

$$\exp(z/2H) \cdot \exp(ik_{z,r}z), \tag{42}$$

which grows exponentially with height. Note that the averaged energy density of a wave with fixed real frequency must be independent of time at a fixed position. For an atmosphere whose properties vary only with z this means that the vertical component of the energy flux does not depend on z, i.e. the average value of $p_1 w_1 = \rho_0 (p_1/\rho_0) w_1$ is independent of z [Lighthill, 1978, chapter 4.2]. This requires that the average value of p_1/ρ_0 and w_1 vary with z as exp(z/2H), so that k_z must be complex although the medium is losless. The growth of the amplitude of internal waves shown by (42) accounts for the importance of such waves in the interpretation of atmospheric and oceanographic disturbances. It also accounts for the importance of nonlinear effects, what this thesis is about.

The phase velocity for (internal) waves is given by (14), while the group velocity can be determined from the dispersion equation (40b) with the definition (16). We find

$$c_g = \frac{\omega^3}{\omega_b^2 k_\perp^2} \left[\left[\frac{\omega_b^2}{\omega^2} - 1\right] k_x, \left[\frac{\omega_b^2}{\omega^2} - 1\right] k_y, -k_{z,r} \right]. \tag{43}$$

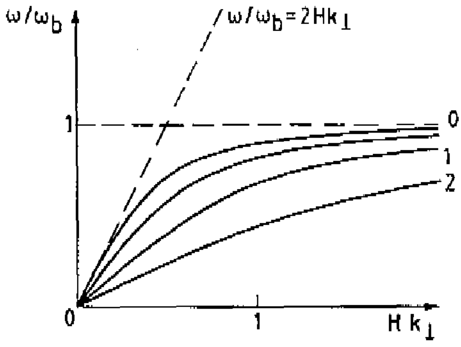


Fig. 5 Dispersion diagram for internal gravity waves in an incompressible stratified fluid with constant scale height H *and Brunt–Väisälä frequency* ω_b. *In the diagram the normalised frequency* ω/ω_b *is shown as a function of the normalised horizontal wavenumber* $Hk_\perp$, *with the vertical wavenumber as a parameter* ($Hk_{z,r} = 0, 1/2, 1, 2$).

We should note several things now. First of all, by comparing (14) and (43), and since the frequency of internal waves is always smaller than the Brunt–Väisälä frequency, which we already noted after (25) and which can easily be seen from the dispersion equation (40b), the horizontal component of the group velocity has the same sign and the same orientation as the horizontal component of the phase velocity. Furthermore it is easily seen that the sign of the vertical component of the group velocity is opposite to the sign of the vertical component of the phase velocity. The latter implicates that if an internal wave source radiates energy upward, the phase propagation of the generated waves is directed downward! Finally from (14), (40b) and (43) we deduce that the magnitudes of the vertical components of the group and the phase velocity are equal under the condition $k_{z,r} \gg 1/H$, and that the group velocity is perpendicular to the wavevector in that case.

From (32c) we immediately see that internal waves are transverse, since $\mathbf{k}\cdot\mathbf{u}_1 = 0$, i.e. the fluid motion is transverse to the wavevector. Thus for internal waves we find the situation depicted in figure 6.

From (32) together with (9) the polarisation relations are found to be given by

$$(\rho_1/\rho_0) = -\frac{i\,k_\perp^{\,2}}{\omega^2 k_z H}\,(p_1/\rho_0), \tag{44a}$$

$$u_1 = \frac{k_x}{\omega}\,(p_1/\rho_0), \tag{44b}$$

$$v_1 = \frac{k_y}{\omega}(p_1/\rho_0), \tag{44c}$$

$$w_1 = -\frac{k_\perp^{\,2}}{\omega k_z}\,(p_1/\rho_0). \tag{44d}$$

Conclusively, as we have shown in this section, the gravitational acceleration introduces an anisotropy in the fluid system. This anisotropy has some remarkable consequences, like, for example, the opposed directions of the vertical components of the group and phase velocity and the, in this case exponential, growth of the wave's amplitude with height.

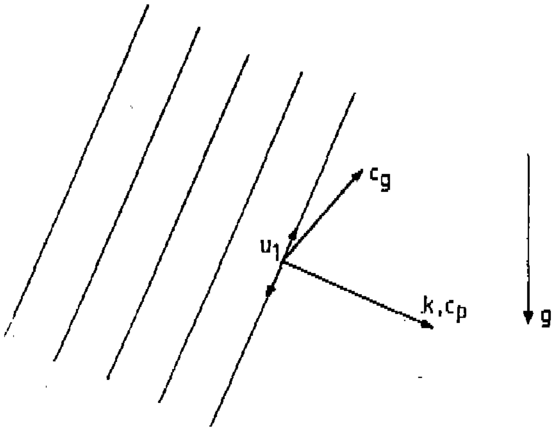


Fig. 6 For internal gravity waves in an incompressible fluid with constant scale height the wavenumber $\mathbf{k}$ *and the phase velocity* $\mathbf{c}_p$ *are perpendicular to planes of constant phase, and the fluid velocity* $\mathbf{u}_1$ *is parallel to planes of constant phase. The vertical components of the phase velocity and the group velocity have opposite directions. If the real part of the vertical wavenumber is much larger than one over the scale height, i.e. the vertical wavelength is much smaller than the scale height, then the group velocity is parallel to planes of constant phase.*

1.4 ACOUSTIC GRAVITY WAVES

As the heading of this section says, the effects of compressibility and of gravity will now be combined to study the propagation of hydrodynamic waves in a compressible stratified fluid in a gravitational field.

From what we have learned in the two preceding sections, we might anticipate some of the characteristics of such waves. Since compressibility causes acoustic waves to be longitudinal and gravity causes internal gravity waves to be transversal, we might expect waves in a compressible stratified fluid to be neither of them. Furthermore, as we have seen, internal gravity waves only exist with frequencies smaller than the Brunt–Väisälä frequency. Therefore acoustic waves are likely to be modified substantially by gravity, only if their frequency is not too large compared to the Brunt–Väisälä frequency. On the other hand, if the frequency of a wave is very small so that the fluid's velocity is much smaller than the speed of sound, compressibility will be of minor importance, and the waves are expected to be essentially internal gravity waves.

As will be shown, the definition of the Brunt–Väisälä frequency is also changed by the compressibility of the medium, since in the derivation of (23) the fluid was assumed to be incompressible and therefore the effect of adiabatic expansion or compression of a fluid element was absent.

The dispersion equation for acoustic gravity waves will of course be of fourth order in the wave frequency, indicating the existence of two modes, i.e. a "gravity" mode and an "acoustic" mode. The group velocity of the waves, which can be determined from the dispersion equation, will be even more complicated than the one for internal gravity waves, given by (43).

Now let us quantify these ideas by combining (1), describing acoustic waves, with (26), describing internal gravity waves:

$$\frac{D\rho}{Dt} + \rho(\nabla\cdot\mathbf{u}) = 0, \tag{45a}$$

$$\rho\,\frac{D\mathbf{u}}{Dt} = -\,\nabla p + \rho\mathbf{g}, \tag{45b}$$

$$\frac{D}{Dt}(p\rho^{-\gamma}) = 0. \tag{45c}$$

Using the perturbation scheme (3), the following linearised equations are obtained:

$$\frac{\partial\rho_1}{\partial t} + \mathbf{u}_1\cdot\nabla\rho_0 + \rho_0(\nabla\cdot\mathbf{u}_1) = 0, \tag{46a}$$

$$\rho_0\,\frac{\partial\mathbf{u}_1}{\partial t} = -\,\nabla p_1 + \rho_1\mathbf{g}, \tag{46b}$$

$$\frac{\partial p_1}{\partial t} + \mathbf{u}_1\cdot\nabla p_0 = c_s^2\left[\frac{\partial\rho_1}{\partial t} + \mathbf{u}_1\cdot\nabla\rho_0\right]. \tag{46c}$$

From these equations the following wave equation can be derived, cf. Appendix A, [Pekeris 1948, Lamb 1916 or Jones 1976]:

$$\frac{\partial^4 \chi_1}{\partial t^4} - c_s^2 \frac{\partial^2}{\partial t^2}\left[\nabla^2 \chi_1 - \frac{\gamma g}{c_s^2}\frac{\partial \chi_1}{\partial z} + \frac{1}{c_s^2}\frac{dc_s^2}{dz}\frac{\partial \chi_1}{\partial z}\right] - \omega_g^2 c_s^2 \nabla_\perp^2 \chi_1 = 0, \tag{47}$$

where also (27) was used and χ_1 and the Brunt–Väisälä frequency for a compressible atmosphere ω_g [cf. (23)] are defined by

$$\chi_1 = \nabla \cdot u_1, \tag{48}$$

$$\omega_g^2 = -\left[\frac{g}{\rho_0}\frac{d\rho_0}{dz} + \frac{g^2}{c_s^2}\right]. \tag{49}$$

Again, as in section 3 of this chapter, we have obtained a wave equation with coefficients depending on z that can be solved analytically for special H(z) only.

We have also found the Brunt–Väisälä frequency for a compressible atmosphere, given by (49), and which can alternatively be written as [cf. (35)]

$$\omega_g^2 = (g/H)(1 + dH/dz) - g^2/c_s^2, \tag{50}$$

where we have used (29b) and (30b). As can be seen by comparing (49) to (23), or by comparing (50) to (35), compressibility of a medium reduces the magnitude of the Brunt–Väisälä frequency. From (49) we can also see that for a compressible fluid to be stably stratified it is required that

$$-\frac{g}{\rho_0}\frac{d\rho_0}{dz} > \frac{g^2}{c_s^2}. \tag{51}$$

In order to obtain a dispersion equation in an easy way we will assume the scale height H to be constant. The coefficients of the wave equation (47) are constant in that case and plane wave solutions are possible. We then obtain the following fourth–order dispersion equation for such acoustic gravity waves in a compressible fluid [e.g. Hines 1960]

$$\omega^4 - \omega^2 c_s^2(k^2+ik_z/H) + \omega_g^2 c_s^2 k_\perp^2 = 0, \tag{52}$$

where

$$\omega_g^2 = g^2(\gamma-1)/c_s^2, \tag{53}$$

and we used that

$$c_s^2 = \gamma g H, \tag{54}$$

which follows from (6) and (29b). Now the definitions (39) and (41) are used, which imply a z–dependence of the field quantities given by (42), consistent with energy considerations given after (42). The dispersion equation (52) becomes

$$\omega^4 - \omega^2 c_s^2(k_r^2+\omega_a^2/c_s^2) + \omega_g^2 c_s^2 k_\perp^2 = 0, \tag{55}$$

where the acoustic cut–off frequency ω_a is defined by

$$\omega_a = c_s/(2H). \tag{56}$$

The name–giving of ω_a will become clear in what follows.

In the limit that the gravitational acceleration g approaches zero, the dispersion equation (55) for acoustic gravity waves reduces to the dispersion equation (12) for acoustic waves, as can be seen from (29b), (53) and (56). Both the acoustic cut–off frequency and the Brunt–Väisälä frequency become zero in this case. In the limit that the fluid becomes incompressible, i.e. the quotient of the specific heats becomes infinitely large, or, to put it differently, if the velocity of sound becomes infinitely large, the dispersion equation (55) reduces to the dispersion equation (40b) for internal waves. In this case the Brunt–Väisälä frequency for a compressible stratified fluid, given by (49), approaches the Brunt–Väisälä frequency for an incompressible stratified fluid, given by (23).

From the foregoing we can conclude that acoustic gravity waves with frequencies larger than the acoustic cut–off frequency are acoustic waves modified by gravity and that acoustic gravity waves with frequencies smaller than the Brunt–Väisälä frequency (for a compressible fluid) are internal waves modified by compressibility. The dispersion equation (55) can be visualised in several ways as shown in figures 7 to 12, where for γ the value 1.4 was used which applies to dry air.

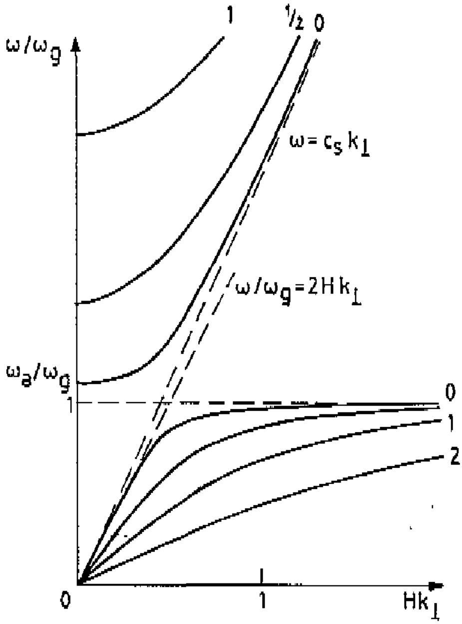


Fig. 7 Dispersion diagram for acoustic gravity waves in a compressible stratified fluid with constant scale height H *and Brunt–Väisälä frequency* ω_g*. In the diagram the normalised frequency* ω/ω_g *is shown as a function of the normalised horizontal wavenumber* $Hk_\perp$*, with the vertical wavenumber as a parameter* ($Hk_{z,r}$ = 0, 1/2, 1, 2)*. Also shown are the acoustic limit* $\omega = c_s k_\perp$*, the acoustic cut–off frequency* ω_a *and the resonance at the Brunt–Väisälä frequency. The regions of propagation of acoustic gravity waves are bounded by the curves for which* $k_{z,r} = 0$*. The value used for* γ *is 1.4.*

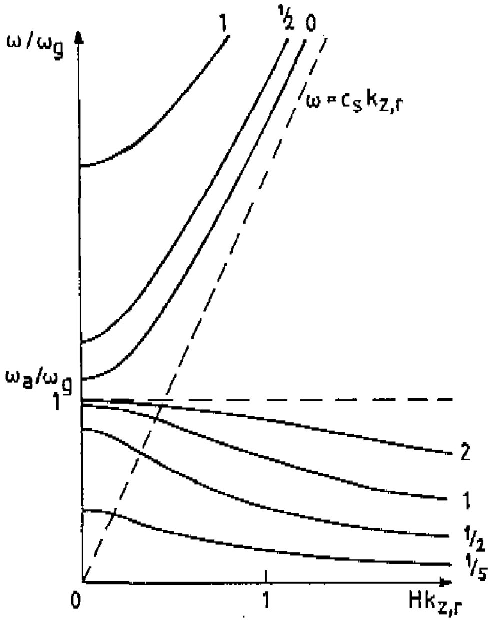


Fig. 8 Dispersion diagram for acoustic gravity waves. The normalised frequency ω/ω_g is shown as a function of the real part of the normalised vertical wavenumber $Hk_{z,r}$, *with the normalised horizontal wavenumber* $Hk_\perp$ *as a parameter.*

For the sake of completeness, we also give the group velocity of acoustic gravity waves, calculated from its definition (16), together with the dispersion equation (55):

$$c_g = \frac{\omega^3}{\omega_g^2 k_\perp^2}\left[1 - \frac{\omega^4}{c_s^2 k_\perp^2 \omega_g^2}\right]^{-1}\left[\left[\frac{\omega_g^2}{\omega^2} - 1\right]k_x,\ \left[\frac{\omega_g^2}{\omega^2} - 1\right]k_y,\ -k_{z,r}\right]. \qquad (57)$$

For $g \to 0$ this reduces to the group velocity for acoustic waves (17), and for $c_s \to \infty$ (57) reduces to the group velocity for internal waves given by (43).

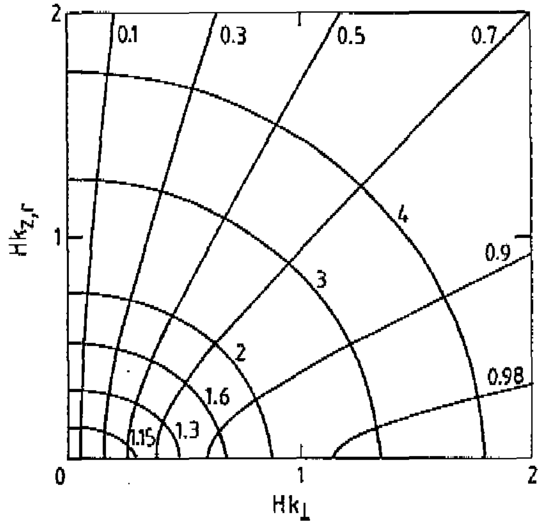


Fig. 9 Wavenumber diagram for acoustic gravity waves. The real part of the normalised vertical wavenumber $Hk_{z,r}$ is shown versus the normalised horizontal wavenumber $Hk_\perp$, with the normalised frequency of the wave ω/ω_g as a parameter. Note the difference in shape of the curves for which $\omega/\omega_g < 1$, and for which $\omega/\omega_g > \omega_a/\omega_g \simeq 1.1$.

The polarisation equations are found to be [Pekeris 1948]:

$$w_1 = (g^2k_\perp^2 - \omega^4)^{-1}\left[\omega^2 c_s^2 \frac{\partial \chi_1}{\partial z} + (gk_\perp^2 c_s^2 - \gamma g \omega^2)\chi_1\right], \tag{58a}$$

$$p_1 = (i\rho_0/\omega)(c_s^2\chi_1 - gw_1), \tag{58b}$$

$$u_1 = (k_x/\omega)(p_1/\rho_0), \tag{58c}$$

$$v_1 = (k_y/\omega)(p_1/\rho_0), \tag{58d}$$

$$\rho_1 = (i\rho_0/\omega c_s^2)\left[-c_s^2\chi_1 + \gamma g w_1 + \frac{dc_s^2}{dz}w_1\right], \tag{58e}$$

where $\partial\chi_1/\partial z = ik_z\chi_1$ and $dc_s^2/dz = 0$ if the scale height H of the fluid is a constant.

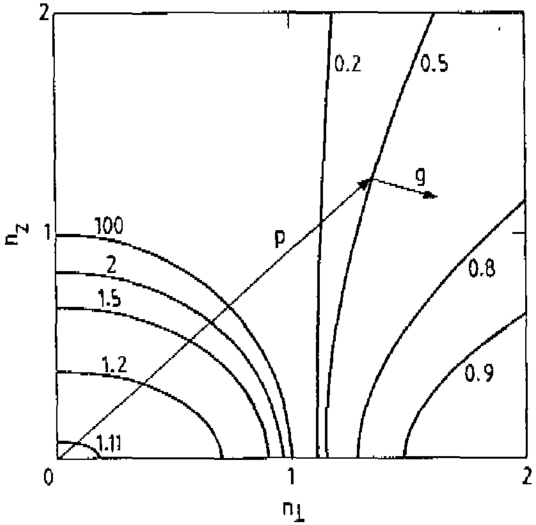


Fig. 10 Indicatrix for acoustic gravity waves. $n_z = c_s k_{z,r}/\omega$ *is shown versus* $n_\perp = c_s k_\perp/\omega$ *with the normalised frequency* ω/ω_g *as a parameter. The arrows illustrate the directions of the phase velocity* (p) *and of the group velocity* (g).

1.5 SUMMARY

In this chapter some basic properties of acoustic gravity waves have been reviewed on the basis of the linear theory of these waves. We have seen that these waves exist in a compressible fluid, stratified under gravity. Such a fluid has a natural oscillation frequency, which is called Brunt–Väisälä frequency.

Waves that are called acoustic gravity waves can in fact belong to two modes. Waves belonging to one mode are acoustic in nature, are modified by gravity and have a cut–off at a frequency that is usually slightly larger than the Brunt–Väisälä frequency. Waves belonging to the other mode are internal waves modified by the compressibility of the fluid. This mode has a resonance at the Brunt–Väisälä frequency.

Basic ideas like phase velocity, group velocity, wave equation, dispersion relation, and polarisation relation have been introduced.

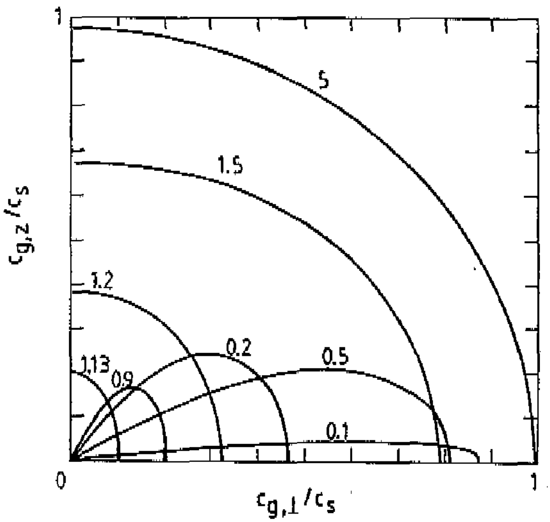


Fig. 11 Ray surface for acoustic gravity waves. The normalised vertical component of the group velocity $c_{g,z}/c_s$ *is shown versus the normalised horizontal part of the group velocity* $c_{g,\perp}/c_s$ *with the normalised frequency* ω/ω_g *as a parameter.*

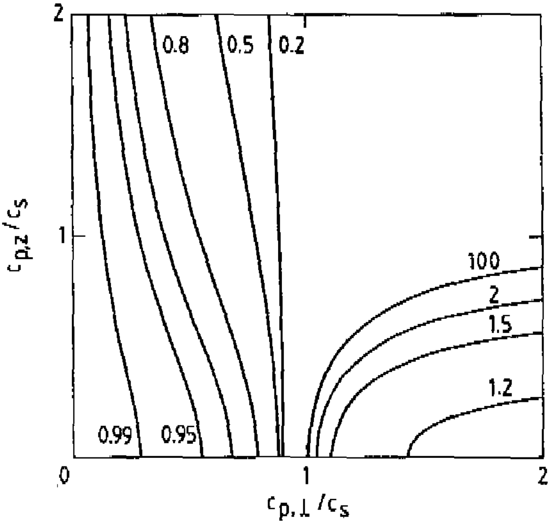


Fig. 12 *Wave normal surface for acoustic gravity waves. The normalised phase velocity in the z-direction* $c_{p,z}/c_s = \omega/(k_{z,r}c_s)$ *is shown versus* $c_{p,\perp}/c_s = \omega/(k_\perp c_s)$ *with the normalised period* ω/ω_g *as a parameter.*

REFERENCES

Booker, J.R., and Bretherton, F.P. (1967). The critical layer for internal gravity waves in a shear flow, *J. Fluid Mech.*, **27**, 513–529.

Brunt, D. (1927). The period of simple vertical oscillations in the atmosphere, *Q. J. R. Meteorolog. Soc.*, **53**, 30–32.

Daniels, G.M. (1975). The ducting of acoustic gravity waves by the thermosphere, *Geophys. Fluid Dyn.*, **6**, 359–382.

Gill, A.E. (1982). "Atmosphere–Ocean Dynamics." Academic Press, New York.

Holton, J.R. (1972). "An Introduction to Dynamic Meteorologie." Academic Press, London.

Jones, W.L. (1976). Linear internal gravity waves in the atmosphere, "Encyclopedia of Physics", 177–218. Springer–Verlag, Berlin.

Kelder, H. (1987). On waves in the upper atmosphere, Ph. D. thesis, Eindhoven University of Technology.

Hines, C.O. (1960). Internal atmospheric gravity waves at ionospheric heights, *Can. J. Phys.*, 38, 1441.

Lamb, H. (1916). "Hydrodynamics." Dover Press, New York.

Lighthill, M.J. (1978). "Waves in Fluids." Cambridge University Press, Cambridge.

Lighthill, M.J. (1986). "An informal Introduction to Fluid Dynamics." Cambridge University Press, Cambridge.

Pekeris, C.L. (1948). The propagation of a pulse in the atmosphere, part II, *Phys. Rev.*, **73**, no. 2, 145–154.

Phillips, O.M. (1966). "Dynamics of the upper ocean." Cambridge University Press.

Väisälä, V. (1925). Über die Wirkung der Windschwankungen auf die Pilot–beobachtungen, *Soc. Sci. Fenn. Commentat. Phys.–Math.*, **2**, no. 19, 19–37.

Yeh, K.C., and Liu, C.H. (1974). Acoustic gravity waves in the upper atmosphere, *Rev. Geophys. Space Phys.*, 12, 193–216.

CHAPTER 2

APPROXIMATE EQUATIONS OF MOTION

2.1 INTRODUCTION

In chapter 1 we have given an introduction into the (linear) theory of acoustic gravity waves. This was done by linearisation of the equations of motion and assuming a monochromatic plane wave solution of these linearised equations. Of course the equations of motion (1.1), (1.26) and (1.45) already suppose several approximations. In section 2 of the present chapter we will give equations of motion that are more general in the sense that they account for rotation, viscosity and heat conduction of the fluid. In section 2.2 we will also give some crude conditions under which these general equations of motion reduce to (1.45), i.e. they reduce from the Navier–Stokes equations of motion to the Euler equations of motion. In section 2.3 the hydrostatic approximation is discussed. This is an approximation in which the vertical acceleration of the fluid is neglected. The hydrostatic approximation will be shown to apply to waves with frequencies small compared to the Brunt–Väisälä frequency and with small aspect ratios; the aspect ratio being the quotient of the vertical and the horizontal scale of the wave. In section 2.4 the frequently used Boussinesq approximation and the conditions for its validity are analysed. This is done for the application of the Boussinesq approximation to linear waves, i.e. waves with infinitesimal amplitudes, as well as for its application to nonlinear waves.

2.2 EQUATIONS OF MOTION

2.2.1 Navier–Stokes equations

The continuity equation or mass conservation equation (1.1a) is derived and discussed by, e.g., [Gill 1982, section 4.2] or more generally by [Batchelor 1967, sections 2.2 and 3.1]. This equation is fundamental in all problems involving fluid motion.

The momentum equation for a viscous fluid in a coordinate system rotating with angular velocity Ω reads [Batchelor 1967, section 3.2 and section 3.3, Gill 1982, section 4.5]

$$\rho\frac{D\mathbf{u}}{Dt} + 2\rho\Omega\times\mathbf{u} + \rho\frac{D\Omega}{Dt}\times\mathbf{x} - \tfrac{1}{2}\rho\nabla|\Omega\times\mathbf{x}|^2 = -\nabla p + \rho\mathbf{g} + \mu_v[\nabla^2\mathbf{u} + \tfrac{1}{3}\nabla(\nabla\cdot\mathbf{u})], \tag{1}$$

where μ_v is the dynamic coefficient of viscosity. The second term on the left–hand side of this equation is the Coriolis force per unit volume. It is called after Coriolis [1835] who discussed it, together with the third and fourth term on the left–hand side. These two terms will be neglected because Ω can be considered constant and the potential due to the rotation is always very small on earth [Gill 1982, section 7.4] To obtain (1), the assumptions have been made that μ_v is a constant independent of temperature and that the ratio between μ_v and the second coefficient of viscosity in the non–isotropic part of the stress tensor is −2/3. This puts an upper limit to the temperature variations in the fluid for which (1) applies.

The heat, or (internal) energy equation [Gill 1982, section 4.4] for a viscous, heat conducting, perfect gas may be written as an equation for the entropy:

$$\frac{\rho^\gamma}{\gamma-1}\frac{D}{Dt}(p\rho^{-\gamma}) = -\mathscr{L}, \tag{2a}$$

where the function $\mathscr{L}$ is related to the net production of entropy. Now consider the various terms combined in this function, namely

$$\mathscr{L} = \nabla\cdot\mathbf{q} - \mathscr{H}, \tag{2b}$$

where q is the heat flux due to particle conduction and $\mathscr{H}$ represents the sum of all the heating sources. The heat flux q may be written

$$\mathbf{q} = -k\nabla T, \tag{2c}$$

where k is the thermal conduction tensor. The heating term $\mathscr{H}$ in (2b) may be written

$$\mathscr{H} = \mathscr{H}_v + \mathscr{H}_w, \tag{2d}$$

where $\mathscr{H}_v$ is the viscous dissipation rate and $\mathscr{H}_w$ is the wave heating term. The viscous heating is

$$\mathscr{H}_v = \mu_v[\tfrac{1}{2}e_{ij}e_{ij} - \tfrac{2}{3}(\nabla\cdot u)^2], \tag{2e}$$

where $e_{ij} = \partial u_i/\partial x_j + \partial u_j/\partial x_i$, is the rate of strain tensor. The wave heating contribution to $\mathscr{H}$ is not well-known. It is often assumed to be uniformly or proportional to the density.

2.2.2 Viscosity and thermal conductivity

In order to estimate the importance of viscosity and thermal conductivity in the equations of motion we introduce a characteristic mass density ρ_c, a timescale t_c and a lengthscale l, which, e.g., can be the wavelength of a wave or the scale height of the fluid. From (1) we see that viscosity can be neglected in the momentum equation if

$$l^2/t_c \gg \mu_v/\rho_c, \tag{3}$$

which is equivalent to the condition that the Reynolds number is much larger than one. The kinematic viscosity, which is equal to μ_v/ρ_c, has a value of 10^{-6} for water, of approximately 1.4×10^{-5} m^2/s for air at 1000 mbar (at the earth's surface) [List 1951, table 113] and a value of about 1.8 m^2/s for air at a height of 86 km. Using the dispersion equation (1.55) to estimate l^2/t_c, we find that viscosity can be neglected in the momentum equation for much of the acoustic gravity wave spectrum, at least in the lower atmosphere. For a more detailed study of the effects of viscosity and thermal conduction we refer to Pitteway & Hines [1963].

In the energy balance (2) the viscous dissipative terms can be neglected, as argued by Spiegel & Veronis [1960]. It can also be seen that viscous dissipation contributes negligibly to the heat equation by noting that the proportion of the viscous terms and the left-hand side of (2) scales with

$$\frac{\mu_v}{\rho_c}(c_p T t_c)^{-1} \ll 1, \tag{4}$$

where we used the law for a perfect gas (1.28), that $\gamma = c_p/c_v$, where c_p is the specific heat at constant pressure, and the fact that $Tt_c > 10^4$ for acoustic gravity waves. For water c_p is of the order 10^3, while for air it is an order one quantity.

Using (1.28) we find that heat conduction can be neglected in the heat equation (2) if

$$l^2/t_c \gg k/(c_p\rho_c), \tag{5}$$

which is equivalent to the condition that the Fourier number is much smaller than one. Since the thermal diffusivity $k/(c_p\rho)$ is 1.36 times the kinematic viscosity, according to the kinematic theory of gases, heat conduction can be neglected in the heat equation under the same conditions under which viscosity can be neglected in the momentum equation.

2.2.3 The earth's rotation

An obvious condition under which to neglect rotation of the fluid would be that the characteristic time scale of the phenomenon to be studied should be much smaller than the inverse magnitude of the angular velocity, i.e.,

$$t_c \ll |\Omega|^{-1}. \tag{6}$$

An other condition, under which rotation can be neglected, that is often used is that the Rossby number is much smaller than one [Houghton 1977].

However, in this subsection we want to examine the conditions necessary to neglect the earth's rotation in the description of gravity waves somewhat better. Therefore we will derive and study the dispersion equation for internal waves in a rotating fluid. For simplicity the compressibility of the fluid is neglected. This will not alter the conditions under which rotation can be neglected, because internal gravity waves for which the earth's rotation is of any significance have frequencies much smaller than the Brunt–Väisälä frequency and for such waves compressibility can be neglected if ω_b is replaced by ω_g

[chapter 1].

The rotation of the earth can de modeled by taking Ω in (1) in the z-direction, with magnitude f/2 [Gill 1982, section 7.4]. Here f is the Coriolis parameter, which is equal to 2sinϕ times the earth's angular velocity, where ϕ is the latitude. If this is used, the equations of motion for internal gravity in an inviscid rotating fluid become [cf. (1.26)]:

$$\frac{D\rho}{Dt} = 0, \tag{7a}$$

$$\rho\frac{Du}{Dt} - fv = -\frac{\partial p}{\partial x}, \tag{7b}$$

$$\rho\frac{Dv}{Dt} + fu = -\frac{\partial p}{\partial y}, \tag{7c}$$

$$\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} - g\rho, \tag{7d}$$

$$\nabla\cdot\mathbf{u} = 0. \tag{7e}$$

Proceeding like after (1.26) gives the wave equation [cf. (1.33)]:

$$\frac{\partial^2}{\partial t^2}\left[\nabla^2 w_1 + \frac{1}{\rho_0}\frac{d\rho_0}{dz}\frac{\partial w_1}{\partial z}\right] + f^2\left[\frac{\partial^2 w_1}{\partial z^2} + \frac{1}{\rho_0}\frac{d\rho_0}{dz}\frac{\partial w_1}{\partial z}\right] + \omega_b^2\nabla_\perp^2 w_1 = 0 \tag{8}$$

Assuming an isothermal fluid, i.e. a constant scale height, gives instead of dispersion equation (1.40b) the following dispersion equation:

$$\omega^2[k_r^2 + (2H)^{-2}] - f^2[k_{z,r}^2 + (2H)^{-2}] - \omega_b^2 k_\perp{}^2 = 0. \tag{9}$$

A diagram for this dispersion equation is given in figure 13. From this diagram and (9) it is clear that, due to the rotation of the fluid, there is a cut-off at $\omega = f$ and dispersion properties of internal gravity waves with frequencies near the Coriolis parameter are modified. By comparing (9) to (1.40b) we find the following condition for rotation of the fluid to be neglected:

$$[k_{z,r}^2 + (2H)^{-2}]/k_\perp^2 \ll \omega_b^2/f^2. \tag{10}$$

This implies that both the ratio of the vertical and the horizontal wavelength of the wave and the ratio of 4π times the scale height and the horizontal wavelength should be much larger than f/ω_b, i.e. the aspect ratio of the wave should be much larger than f/ω_b. From (9) these conditions can be seen to be equivalent to the condition $\omega \gg f$ [cf. (6)].

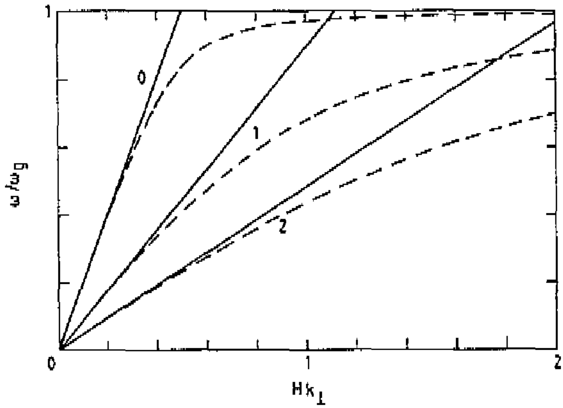


Fig. 13 Dispersion diagram for internal gravity waves in a rotating fluid (solid lines) and a non–rotating fluid (dashed lines) with constant scale height H *and Brunt–Väisälä frequency* ω_g *[replacing* ω_b *in* (9)]. *In the diagram the normalised frequency* ω/ω_g *is shown versus the normalised horizontal wavenumber* $Hk_\perp$, *with the vertical wavenumber as a parameter* ($Hk_{z,r}$ = 0, 1, 4). *To show the effect of rotation clearly we have chosen the ratio of* f/ω_g *to be* 10^{-1}, *which is correct for the abyssal part of the ocean where* $\omega_g \approx 10^{-3}$. *However, for the upper part of the ocean and the earth's atmosphere (at a latitude of approximately* 45^0) *a value of* 10^{-4} *for* f *and of* 10^{-2} *for* ω_g *applies. Note the cut–off at* $\omega/\omega_g = f/\omega_g = 0.1$.

2.2.4 Euler equations

In the previous subsections we have considered the conditions for which the effects of viscosity, thermal conductivity and rotation can be neglected. If all these effects can be neglected and no heat sources are present, the Navier–Stokes equations of motion, i.e. (1.45a), (1) and (2), become the Euler equations of motion (1.45). In the next section we will consider two further simplifications of these equations that are frequently used in studying phenomena in stratified flows: the hydrostatic approximation and the Boussinesq approximation.

2.3 THE HYDROSTATIC APPROXIMATION

In the hydrostatic approximation the vertical acceleration of the fluid is neglected. This amounts to the neglect of the left–hand side of the vertical component of the momentum equation (1.45b). In this section we will study the conditions and the implications of this approximation in a rigorous way by adopting a consistent way of bringing the equations of motion into a dimensionless form.

As we will see this approximation applies to waves with frequencies much smaller than the Brunt–Väisälä frequency and small aspect ratios. Anticipating on these conditions we will make the equations of motion (1.45) dimensionless using a kind of convective scales. However, since we are dealing with a stratified fluid a little care is needed. Because the waves to which we want to apply the hydrostatic approximation, have frequencies much smaller than the Brunt–Väisälä frequency, their velocity field multiplied by the mass density must be almost divergence free as can be seen from the continuity equation (1.45a), i.e.

$$\nabla\cdot(\rho_0 \mathbf{u}) \approx 0, \tag{11a}$$

or

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} + \frac{w}{\rho_0}\frac{d\rho_0}{dz} \approx 0, \tag{11b}$$

where assumed that $v = 0$, which simplifies the discussion a little. Note that (11) is used only to adopt a consistent scaling. From (11) we see that the derivative of the horizontal velocity u with respect to a horizontal coordinate x must be of the same order of magnitude as the derivative of the vertical velocity w with respect to the vertical coordinate z or as the vertical velocity w divided by a typical scale height, depending on the fact which of the last two quantities is largest. A balance between the second and the third term in (11b) is, for acoustic gravity waves at least, not possible, because that would require that the second term is real, i.e. that the vertical wavenumber is imaginary. Note that the horizontal and the vertical scale are not necessarily the same so that we have to introduce different horizontal and vertical scales. These considerations lead to the following dimensionless order one quantities, if the undisturbed or background velocity of the fluid is assumed to be zero,

$$\hat{x} = x/l, \qquad \hat{z} = z/h, \qquad \hat{t} = t/t_c,$$

$$\hat{u} = \left[\frac{t_c}{\alpha l}\right] u, \qquad \hat{w} = \left[\frac{t_c}{\alpha h}\right] w,$$

$$\hat{\rho}_0 = \rho_0/\rho_c, \qquad \hat{p}_0 = p_0/(c_c^2 \rho_c), \tag{12}$$

$$\hat{\rho} = (\rho - \rho_0)/(\alpha \rho_c), \qquad \hat{p} = (p - p_0)/(\alpha c_c^2 \rho_c),$$

$$\rho_0^{-1}(d\rho_0/dz) = (H_c \hat{\rho}_0)^{-1}(d\hat{\rho}_0/d\hat{z}_1),$$

where h is a characteristic vertical scale of the wave and smaller or equal to the characteristic scale height H_c, α is a dimensionless measure for the amplitude of the disturbance, c_c is a typical velocity, and z_1 is the scale on which the background density varies. Using the hydrostatic balance

$$dp_0/dz + g\rho_0 = 0, \tag{13a}$$

or

$$d\hat{p}_0/d\hat{z}_1 = -\frac{gH_c}{c_c^2}\hat{\rho}_0, \tag{13b}$$

the equations (1.45) for two-dimensional motion become

$$\frac{D\rho}{Dt} + (\rho_0+\alpha\rho)(\nabla\cdot\mathbf{u}) + \frac{h}{H_c}\frac{d\rho_0}{dz_1}w = 0, \tag{14a}$$

$$(\rho_0+\alpha\rho)\frac{Du}{Dt} = -\frac{\partial p}{\partial x}, \tag{14b}$$

$$\frac{h^2}{l^2}(\rho_0+\alpha\rho)\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} - \frac{gh}{c_c^2}\rho, \tag{14c}$$

$$\frac{c_c^2}{c_s^2}\frac{Dp}{Dt} - \frac{D\rho}{Dt} - \left[\frac{gh}{c_s^2} + \frac{h}{H_c}\frac{1}{\rho_0}\frac{d\rho_0}{dz_1}\right]\rho_0 w = 0, \tag{14d}$$

where we have dropped the hats and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \alpha(\mathbf{u}\cdot\nabla), \tag{15a}$$

$$\nabla = \left[\frac{\partial}{\partial x}, 0, \frac{\partial}{\partial z}\right], \tag{15b}$$

and we have chosen c_c as

$$c_c = l/t_c. \tag{16}$$

From (14c) we see that the vertical acceleration of the fluid can be neglected if the aspect ratio of the motion is small. The hydrostatic balance then also holds for the perturbation pressure and density. The hydrostatic approximation is therefore correct provided that

$$(h/l)^2 \ll 1, \tag{17}$$

and $\alpha \lesssim \mathcal{O}(1)$. This will prove to be a very important condition in chapters 4 and 5.

Now the dispersion equation in the hydrostatic approximation is examined. Therefore we linearise (14) by assuming that the wave's amplitude is very small, i.e. $\alpha \ll 1$. Furthermore an isothermal basic state is assumed. In that case $c_s^2 = \gamma gH$ and $H_c = H$ are constants and $\rho_0^{-1}\, d\rho_0/dz_1 = -1$ (dimensionless variables). Equations (14) become

$$\frac{\partial\rho}{\partial t} + \rho_0(\nabla\cdot u) - \frac{h}{H}\rho_0 w = 0, \tag{18a}$$

$$\rho_0\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x}, \tag{18b}$$

$$\frac{\partial p}{\partial z} + \frac{gh}{c_c^2}\rho = 0, \tag{18c}$$

$$\frac{c_c^2}{c_s^2}\frac{\partial p}{\partial t} - \frac{\partial\rho}{\partial t} + \frac{h}{H}\frac{(\gamma-1)}{\gamma}\rho_0 w = 0. \tag{18d}$$

Some algebra similar to that leading to (1.47) gives the wave equation

$$\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2 w}{\partial z^2} + \frac{h}{H}\frac{\partial w}{\partial z}\right] + \frac{gh}{c_c^2}\frac{h}{H}\frac{(\gamma-1)}{\gamma}\frac{\partial^2 w}{\partial x^2} = 0, \tag{19}$$

where one should note that

$$\partial(\hat{\rho}_0\hat{w})/\partial\hat{z} = -\frac{h}{H}\hat{\rho}_0\hat{w} + \hat{\rho}_0\,\partial\hat{w}/\partial\hat{z},$$

because we assumed ρ_0 to vary on the scale H, and w on the scale h [cf. (12)]. Returning to dimensional quantities, (19) together with (12), (16), (1.53) and (1.54) gives

$$\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2 w}{\partial z^2} + \frac{1}{H}\frac{\partial w}{\partial z}\right] + \omega_g^2\frac{\partial^2 w}{\partial x^2} = 0. \tag{20}$$

In the same way as we obtained the dispersion equations (1.40b) and (1.55) we now find from (20)

$$\omega^2[k_{z,r}^2 + (2H)^{-2}] - \omega_g^2 k_x^2 = 0. \tag{21}$$

If we compare the dispersion equations (21) and (1.55) we may note several things. First we note that for the hydrostatic approximation to be valid the condition

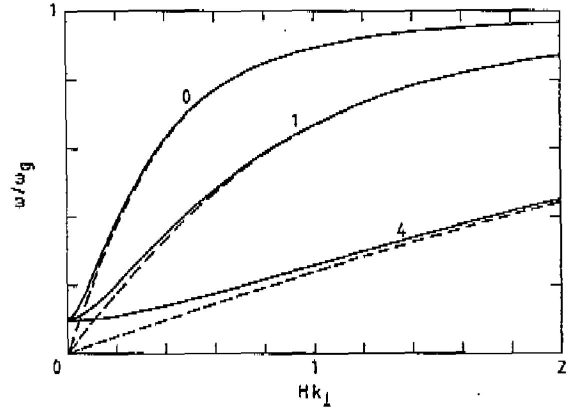


Fig. 14 Dispersion diagram for internal gravity waves in a stratified fluid (dashed lines) and for such a fluid in the hydrostatic approximation (solid lines) with constant scale height H *and Brunt–Väisälä frequency* ω_g. *In the diagram the normalised frequency* ω/ω_g *is shown versus the normalised horizontal wavenumber* $Hk_\perp$, *with the vertical wavenumber as a parameter* ($Hk_{z,r} = 0, 1, 2$).

$$k_x \ll k_{z,r} \quad \text{or} \quad k_x \ll (2H)^{-1}, \tag{22a}$$

should be satisfied. This is equivalent to (17). Furthermore we note the condition

$$\omega \ll \omega_g. \tag{22b}$$

Condition (22b) implies that the hydrostatic approximation eliminates the acoustic mode, although the important correction to the Brunt–Väisälä frequency, due to the compressibility of the fluid, is retained. This is so, since ω_g in (22) is defined by (1.53). In (1.55) ω^4 can be neglected because $\omega^2 \ll \omega_a^2$ [cf.

(22b)}. Another thing to be noted is that the horizontal group velocity in the hydrostatic approximation,

$$\frac{\partial \omega}{\partial k_x} = \omega_g[k_{z,r}^2 + (2H)^{-2}]^{-1/2}, \tag{23}$$

is determined by the value of $k_{z,r}$ only. So, if $k_{z,r}$ is determined by certain boundary conditions in such a way that it is fixed (chapters 4 and 5), the wave shows no dispersion (figure 14). This implicates that a low frequency acoustic gravity wave packet will not disperse, but will maintain its shape.

2.4 THE BOUSSINESQ APPROXIMATION

2.4.1 Historic note

In this section the Boussinesq approximation in its application to acoustic gravity waves is discussed. This approximation is attributed to Boussinesq [1903], but, as noted by Eckart & Ferris [1956], a similar approximation of the equations of motion was already introduced by Oberbeck [1879]. In fact "Boussinesq approximation" is a collective name for several different approximations used in studies of convection, gravity waves and other phenomena in fluids with density variations. To illustrate the original form of the Boussinesq approximation [Boussinesq, 1903], we rewrite (1.45b) in the form

$$\left[1 + \frac{\rho_1}{\rho_0}\right] \frac{Du_1}{Dt} = -\frac{1}{\rho_0} \nabla p_1 + \frac{\rho_1}{\rho_0} g, \tag{24}$$

where we have used (1.3), subtracted the hydrostatic balance (1.27) from (1.45b) and divided by ρ_0. In (24) the ratio ρ_1/ρ_0 appears twice: on the left–hand side in the inertia term, on the right–hand side in the buoyancy term. When disturbances are small it is clear that the density variations in the inertial term are of little importance. However if the characteristic time scale is long enough so that the inertia of the fluid is less important, the density variation in the buoyancy term will be essential to many problems. The original Boussinesq approximation now consists of neglecting variations of density in so far as they affect inertia, but retaining them in the buoyancy terms, where they occur in the combination $g\rho_1/\rho_0$. When viscosity and diffusion are included, variations of fluid properties are also neglected in this approximation. Note that the equations of motion are still nonlinear in the Boussinesq approximation.

2.4.2 The linear Boussinesq equations for incompressible fluids.

Before we consider the Boussinesq approximation for nonlinear problems (2.4.3) and for compressible flows (2.4.4) we will first study the Boussinesq approximation for an incompressible inviscid fluid, assuming that perturbations are small enough and nonlinear terms can be neglected.

The linearised Boussinesq equations for an incompressible, inviscid fluid are [cf. (1.32)]

$$\frac{\partial \rho_1}{\partial t} + w_1 \frac{d\rho_0}{dz} = 0, \tag{25a}$$

$$\frac{\partial \mathbf{u}_1}{\partial t} = -\frac{1}{\rho_0} \nabla p_1 + \frac{\rho_1}{\rho_0} \mathbf{g}, \tag{25b}$$

$$\nabla \cdot \mathbf{u}_1 = 0. \tag{25c}$$

Now the additional assumption is made, which is often included in the name "Boussinesq approximation", that density deviations from a standard density are small so that

$$\frac{1}{\rho_0} \frac{\partial f_1}{\partial z} \approx \frac{\partial}{\partial z}\left[\frac{f_1}{\rho_0}\right], \tag{26a}$$

where f_1 is a perturbation quantity $\mathbf{u}_1$, p_1 or ρ_1. This implies that

$$\left|\frac{1}{\rho_0} \frac{d\rho_0}{dz}\right| \ll \left|\frac{1}{f_1} \frac{\partial f_1}{\partial z}\right|, \tag{26b}$$

i.e. that the characteristic vertical length scale of the perturbation is much smaller than the scale height. For acoustic gravity waves this condition, which is in fact a kind of W.K.B. approximation [Fröman & Fröman 1965], is always satisfied for deep sea conditions but can be restrictive for upper sea and atmospheric applications.

Assuming that differentiations with respect to y are zero, which is no loss of generality because the equations are linear and three–dimensional solutions can be obtained from superposition of two–dimensional solutions [Yih 1980, section 2.4], a streamfunction ψ is introduced as

$$u = -\frac{\partial\psi}{\partial z}, \tag{27a}$$

$$w = \frac{\partial\psi}{\partial x}, \tag{27b}$$

so that (25c), requiring that the velocity field is divergence free, is identically satisfied. Using (26), elimination of p_1 from (25) gives

$$\frac{\partial^3\psi}{\partial t\,\partial x^2} + \frac{\partial^3\psi}{\partial t\,\partial z^2} + g\,\frac{\partial\tilde{\rho}}{\partial x} = 0, \tag{28a}$$

$$\frac{\partial\tilde{\rho}}{\partial t} - \frac{\omega_b^2}{g}\,\frac{\partial\psi}{\partial x} = 0, \tag{28b}$$

where

$$\tilde{\rho} = \rho_1/\rho_0, \tag{29}$$

and ω_b^2 is defined by (1.35). From (28) we find the following wave equation

$$\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2}\right] + \omega_b^2\,\frac{\partial^2\psi}{\partial x^2} = 0. \tag{30}$$

Assuming an isothermal atmosphere and a plane wave solution (1.9) the following dispersion equation is found

$$\omega^2 k^2 = \omega_b^2 k_\perp^2. \tag{31}$$

Comparison of this dispersion equation with (1.37) and (1.40) shows two consequences of the approximation (26). Firstly, for $\omega < \omega_b$ the vertical wavenumber is real now because ω and $k_\perp$ have been assumed real, so that the exponential growth of the field quantities, given by (1.42), is neglected. This consequence can be corrected for afterwards by scaling the perturbation quantities with $\rho_0^{-1/2}$ [Lighthill 1978, section 4.2, Turner 1973, section 1.3], giving an improved Boussinesq approximation. This can be understood in terms of the energy considerations given after (1.42). Secondly, the dispersion properties of the waves are changed. This consequence is of no concern provided that

$$(2H)^{-2} \ll k^2, \tag{32}$$

which is somewhat more general than (26b).

2.4.3 The Boussinesq approximation for nonlinear problems

We will now study the Boussinesq approximation for incompressible inviscid fluids in a more rigorous way. Although the Boussinesq approximation sometimes works surprisingly well in fluids with large density variations its validity depends very much on the type of flow considered. In the theory of internal solitary waves, for example, certain nonlinear terms are retained which are of the same order as those neglected in the Boussinesq approximation, and the whole phenomenon depends on the consistent inclusion of all terms to this order [Long 1965, Benjamin 1966, Miesen et al. 1989].

If we consider incompressible two-dimensional flow it is profitable to introduce again a streamfunction, defined by (27). Elimination of p from the momentum equation (1.26b) and using (27) gives:

$$\rho\Big\{\psi_{xxt} + \psi_{zzt} + \psi_x(\psi_{xxz}+\psi_{zzz}) - \psi_z(\psi_{xxx}+\psi_{xzz})\Big\} + \rho_x\Big\{\psi_{xt} - \psi_z\psi_{xx} + \psi_x\psi_{xz}\Big\} + \rho_z\Big\{\psi_{zt} - \psi_z\psi_{xz} + \psi_x\psi_{zz}\Big\} + g\rho_x = 0, \tag{33a}$$

where we have denoted differentiations as subscripts. Equation (1.26a), together with (27), gives

$$\rho_z\psi_x - \rho_x\psi_z + \rho_t = 0. \tag{33b}$$

If we use again that

$$\rho = \rho_0 + \rho_1,$$
$$\tilde{\rho} = \rho_1/\rho_0,$$

where $\rho_0 = \rho_0(z)$, we find from (33)

$$(1+\tilde{\rho})\Big\{\psi_{xxt} + \psi_{zzt} + \psi_x(\psi_{xxz}+\psi_{zzz}) - \psi_z(\psi_{xxx}+\psi_{xzz})\Big\} + \tilde{\rho}_x\Big\{\psi_{xt} - \psi_z\psi_{xx} + \psi_x\psi_{xz}\Big\} + \Big\{\tilde{\rho}_z - (1+\tilde{\rho})\omega_b^2/g\Big\}\Big\{\psi_{zt} - \psi_z\psi_{xz} + \psi_x\psi_{zz}\Big\} + g\tilde{\rho}_x = 0, \tag{34a}$$

$$\{\tilde{\rho}_z - (1+\tilde{\rho})\omega_b^2/g\}\psi_x - \tilde{\rho}_x\psi_z + \tilde{\rho}_t = 0, \tag{34b}$$

where ω_b is defined by (1.23). Note that there is as yet no restriction on the magnitude of $\tilde{\rho}$.

The variables are now made dimensionless and of order one in magnitude:

$$\hat{x} = \frac{x}{l}, \quad \hat{z} = \frac{z}{h}, \quad \hat{\rho} = \frac{\tilde{\rho}}{\alpha}, \quad \beta = \frac{h}{H_c}, \quad \epsilon = \frac{h}{l}, \tag{35}$$

where l and h are the characteristic horizontal and vertical lengthscales of the wave phenomenon, α is a measure for the amplitude of the wave, H_c is again a characteristic value for $\rho_0(\partial\rho_0/\partial z)^{-1}$, β and ϵ are parameters and $\hat{x}$, $\hat{z}$ and $\hat{\rho}$ are of order one in magnitude. How to make t dimensionless can be seen from (31) and the definition of ω_b (1.23):

$$\hat{t} = \frac{h}{l}\,(g/H_c)^{1/2}\, t = \epsilon\,(g/H_c)^{1/2}\, t, \tag{36}$$

where we assumed that $h \lesssim \mathcal{O}(l)$ and that the dispersion equation (31) is still valid to some approximation. To make ψ dimensionless and $\mathcal{O}(1)$ we note that (28b), together with (35) and (36), gives

$$\epsilon^2 \frac{\partial^3 \psi}{\partial \hat{t} \partial \hat{x}^2} + \frac{\partial^3 \psi}{\partial \hat{t} \partial \hat{z}^2} + \alpha\beta^{-1} h^2 \frac{\partial \hat{\rho}}{\partial \hat{x}} = 0 \tag{37}$$

So we define

$$\hat{\psi} = \frac{\beta\psi}{\alpha h^2 (g/H_c)^{1/2}}, \tag{38}$$

again assuming that $h \leq \mathcal{O}(l)$. Using (35), (36) and (38) we find from (34)

$$(1+\alpha\rho)\left\{\epsilon^2\psi_{xxt} + \psi_{zzt} + \frac{\alpha}{\beta}(\epsilon^2\psi_{xxz} + \psi_{zzz})\psi_x - \frac{\alpha}{\beta}(\epsilon^2\psi_{xxx} + \psi_{xzz})\psi_z\right\} +$$

$$\alpha\epsilon^2\rho_x\left\{\psi_{xt} - \frac{\alpha}{\beta}\psi_z\psi_{xx} + \frac{\alpha}{\beta}\psi_x\psi_{xz}\right\} + \beta\left\{\frac{\alpha}{\beta}\rho_z - (1+\alpha\rho)\omega_b^2 H_c/g\right\}$$

$$\times \left\{\psi_{zt} - \frac{\alpha}{\beta}\psi_z\psi_{xz} + \frac{\alpha}{\beta}\psi_x\psi_{zz}\right\} + \rho_x = 0, \tag{39a}$$

$$\left\{\frac{\alpha}{\beta}\rho_z - (1+\alpha\rho)\omega_b^2 H_c/g\right\}\psi_x - \frac{\alpha}{\beta}\rho_{xzz} + \rho_t = 0, \tag{39b}$$

where we have dropped the hats. Note that $\omega_b^2 H_c/g \equiv 1$ for an isothermal atmosphere and otherwise of magnitude one.

By comparing (39) with (28) we find the following conditions for the Boussinesq approximation for atmospheric gravity waves to be valid

$$\alpha \ll \beta, \;\; \beta \ll 1, \;\; \epsilon = \mathcal{O}(1). \tag{40}$$

The condition $\beta \ll 1$ is equivalent to (26b), but the condition $\alpha \ll \beta$ is more severe than the normally used condition $\alpha \ll 1$. So for example if H = 10km, gravity waves for which $h = 1$ km and $\alpha = 0.1$ cannot be treated in this Boussinesq approximation because the nonlinear terms in the equations cannot be neglected. We note that the condition $\alpha \ll \beta$ is consistent with the condition $a/h \ll 1$, given by Long [1965] for a linearly stratified incompressible fluid, where a is the order of the amplitude of the displacement η of a streamline. This can be seen by noting that for an incompressible fluid

$$\alpha\bar{\rho} = \frac{\rho_1}{\rho_0} = \frac{\eta}{\rho_0}\frac{d\rho_0}{dz} = \mathcal{O}\left[\frac{\eta}{H_c}\right] = \mathcal{O}\left[\frac{h}{H_c}\frac{a}{h}\frac{\eta}{a}\right] = \mathcal{O}\left[\beta\frac{a}{h}\frac{\eta}{a}\right], \tag{41}$$

so that $\alpha = \mathcal{O}\left[\beta\frac{a}{h}\right]$ and therefore $\alpha \ll \beta$ provided that $\frac{a}{h} \ll 1$.

Another implication of the use of the Boussinesq approximation is, that to obtain a solution of (39) we have to expand ψ and ρ not only in α but also in β (and if $\epsilon \ll 1$ also in ϵ) [Long 1965], e.g.

$$\rho = \rho_{00}(z) + \alpha\rho_{10} + \beta\rho_{01} + \alpha^2\rho_{20} + \alpha\beta\rho_{11} + \beta^2\rho_{02} + \dots, \tag{42}$$

where the first index gives the order in α and the second index gives the order in β. We then choose α as

$$\alpha = \beta^{n+1} \tag{43a}$$

where n = 1,2,... so that indeed $\alpha \ll \beta$ when $\beta \ll 1$. Then the following expansion is possible

$$\rho = \rho_0 + \beta\rho_1 + \beta^2\rho_2 + \dots, \tag{43b}$$

For an isothermal fluid the equations for terms of the first order in β become

$$\psi_{1xxt} + \psi_{1zzt} + \rho_{1x} = 0, \tag{44a}$$

$$-\psi_{1x} + \rho_{1t} = 0, \tag{44b}$$

which is of course equivalent with (28). The second order equations are

$$\psi_{2xxt} + \psi_{2zzt} + \rho_{2x} - \psi_{1zt} = 0 \tag{45a}$$

$$-\psi_{2x} + \rho_{2t} = 0 \tag{45b}$$

The form of higher order equations depends on the value of n. The results of this paragraph are especially important in the study and interpretation of nonlinear phenomena associated with internal gravity waves.

2.4.4 The Boussinesq approximation for a compressible fluid.

In order that the Boussinesq approximation is valid for compressible fluids, additional conditions are necessary. These conditions amount to the assumption that the fluid can be considered incompressible except in the definition of the Brunt–Väisälä frequency and they can be inferred by writing the equations of motion for a compressible inviscid fluid (1.45) as

$$\nabla\cdot\mathbf{u} = -\frac{1}{\rho}\frac{D\rho}{Dt}, \tag{46a}$$

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \rho\mathbf{g}, \tag{46b}$$

$$\frac{D\rho}{Dt} = \frac{1}{c_s^2}\frac{Dp}{Dt}, \tag{46c}$$

and comparing these equations with the equations of motion for an incompressible fluid (1.26). Under the same condition that nonlinear terms can be neglected for an incompressible fluid, they can be neglected for a compressible fluid as can be seen by introducing the scaled variables (35), (36) and (38) into (46) and using (33). We then obtain the following linear equations for a compressible fluid:

$$\nabla\cdot\mathbf{u}_1 = -\frac{1}{\rho_0}\frac{\partial\rho_1}{\partial t} - \frac{w_1}{\rho_0}\frac{d\rho_0}{dz}, \tag{47a}$$

$$\rho_0\frac{\partial\mathbf{u}_1}{\partial t} = -\nabla p_1 + \rho_1\mathbf{g}, \tag{47b}$$

$$\frac{\partial\rho_1}{\partial t} + w_1\frac{d\rho_0}{dz} = \frac{1}{c_s^2}\left[\frac{\partial p_1}{\partial t} + w_1\frac{dp_0}{dz}\right]. \tag{47c}$$

If the right–hand side of (47a,c) can be neglected, these equations are equivalent to the linearised equations of motion (25) for an incompressible fluid. The conditions for which this is correct are found by introducing the scaled variables (35), (36) and (38) into (47). We find that the two terms on the right–hand side of (47a) are $\mathcal{O}[\alpha\epsilon(g/H_c)^{1/2}]$, whereas the left–hand side of this equation is $\mathcal{O}[\alpha\epsilon(g/H_c)^{1/2}/\beta]$. This means that the right–hand side of (47a) can be neglected if $\beta \ll 1$, which is the same condition as the condition for the Boussinesq approximation for an incompressible fluid. The neglect of the first term on the right–hand side of (47a) means that changes of perturbations take place on such a long time scale that the rate of change of the perturbation density negligibly affects the equation of continuity. Neglecting the second term is just equivalent to the assumption (26) that leads to the Boussinesq approximation.

The first term on the right–hand side of (47c) is $\mathcal{O}[\alpha\beta\epsilon(g/H_c)^{1/2}\rho_0]$, where we used (47b) to find the correct scaling for p_1 and we used that $c_s^2 = \mathcal{O}(gH_c)$. Again, by comparing it to the first term on the left–hand side, this implies that this term can be neglected, under the same condition under which the Boussinesq approximation is valid for an incompressible fluid. Neglecting this term means that the density changes resulting from pressure changes at a fixed level, allowed for by the compressibility of the fluid, are much smaller than the density changes due to the vertical displacement of the fluid. However, the second term on the right–hand side of (47c) is of the same order as the second term on the left–hand side of (47c), as can be seen from the hydrostatic balance, the expression for the velocity of sound (1.54) and the definition of the scale height. Therefore the second term on the right–hand side of (47c) cannot be neglected. As we will see, retaining this term only changes the Brunt–Väisälä frequency from its incompressible value (1.23) to its compressible value (1.49). This correction, due to the compressibility, is by no means small. For an isothermal atmosphere, for example, it changes the value of the Brunt–Väisälä frequency by a factor $[(\gamma-1)/\gamma]^{1/2} \approx 0.53$. Another way to put it is that we account for the compressibility of the fluid by using the potential density instead of the density ρ_0 in the equations (25) for an incompressible fluid [Turner 1973, section 1.2 and 1.3, Spiegel & Veronis 1960].

Thus in the Boussinesq approximation for a compressible fluid the linearised equations of motion become

$$\nabla \cdot \mathbf{u}_1 = 0, \tag{48a}$$

$$\rho_0 \frac{\partial \mathbf{u}_1}{\partial t} = -\nabla p_1 + \rho_1 \mathbf{g}, \tag{48b}$$

$$\frac{\partial \rho_1}{\partial t} - \frac{\omega_g^2}{g} \rho_0 w_1 = 0. \tag{48c}$$

This leads to the wave equation (30) with ω_g replacing ω_b, and so to the dispersion equation (figure 15)

$$\omega^2 k^2 = \omega_g^2 k_\perp^2. \tag{49}$$

Comparing (49) to (1.55), we do not only find condition (32) but also the condition

$$\omega^2 \ll c_s^2 k^2, \tag{50}$$

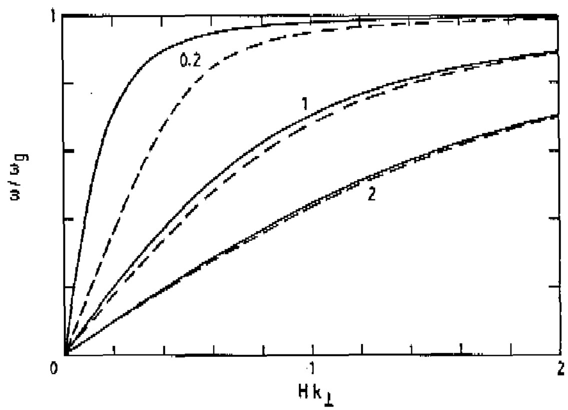


Fig. 15 Dispersion diagram for internal gravity waves in a stratified fluid (dashed lines) and for such a fluid in the Boussinesq approximation (solid lines) with constant scale height H *and Brunt–Väisälä frequency* ω_g. *In the diagram the normalised frequency* ω/ω_g *is shown versus the normalised horizontal wavenumber* $Hk_\perp$, *with the vertical wavenumber as a parameter* ($Hk_{z,r} = 0.2, 1, 2$).

which can be seen to be satisfied using (49) and (32), and the fact that $\omega_g H/c_s$ is of order one.

As we can see from the dispersion equation (49) sound waves have been eliminated from the equations of motion by the Boussinesq approximation. This is obvious since from the discussion after (47) we know that the two main effects governing the propagation of sound, i.e. the local density change associated with the compressibility and the effect that this density change affects the divergence of the velocity field, are neglected in the Boussinesq approximation.

2.5 SUMMARY

In this chapter we have introduced the (Navier–Stokes) equations of motion for fluids. The conditions for which viscosity and thermal conductivity can be neglected have been given and these conditions apply for most atmospheric and oceanographic applications with respect to acoustic gravity waves .

The influence of the rotation of the earth on the dispersion of internal gravity waves has been studied briefly. From this study it is clear that the rotation of the earth can be neglected if the aspect ratio of the wave is much larger than $f/\omega_b \approx 10^{-2}$ and that this condition is equivalent to the condition that $\omega \gg f$ ($f \approx 10^{-4}$).

The hydrostatic approximation, in which the vertical acceleration of the fluid in the momentum equation is neglected, is shown to be valid for internal gravity waves with small aspect ratios having frequencies much smaller than the Brunt–Väisälä frequency

The last approximation considered in this chapter is the Boussinesq approximation. After a historic note, in which the origin and the ambiguity of the name of this approximation is pointed out, the conditions for its validity for an incompressible fluid are considered. These conditions require that the ratio of the characteristic vertical scale and the scale height is very small, and that the relative density perturbations are much smaller than this ratio. This last condition is equivalent to the condition that the relative displacement a/h of a streamline in a fluid of depth h is much smaller than one, and that the relative density difference throughout the part of the fluid we want to describe the wave in is also much smaller than one. If the Boussinesq equations are to be applied to a compressible fluid the additional condition that the wave's frequency is smaller than the Brunt–Väisälä frequency is imposed. This implies that the acoustic waves are not described by the Boussinesq equations. In general one should be careful using the Boussinesq approximation, especially if other small effects are retained, since its validity may depend strongly on the specific problem to be studied.

REFERENCES

Batchelor, G.K. (1967). "An Introduction to Fluid Dynamics." Cambridge University Press, London and New York.

Benjamin, T.B. (1966). Internal waves of finite amplitude and permanent form, *J. Fluid Mech.* 25, 241–270.

Boussinesq, J. (1903). "Théorie Analytique de la Chaleur." Vol. 2, Gauthier–Villars, Paris

Coriolis, G. (1835). Mémoire sur les équations du mouvement rélatifs des systèmes de corps, *J. Ec. Polytech. (Paris)* 15, 142.

Eckart, C., Ferris, H.G. (1956). Equations of motion of the ocean and atmosphere, *Rev. Mod. Phys.* 28, 48–52.

Fröman, N., Fröman, P.O. (1965). "JWKB approximation." North–Holland Publishing Company, Amsterdam.

Gill, A.E. (1982). "Atmosphere–Ocean dynamics." Academic Press, New York.

Houghton, J.T. (1977). "The physics of atmospheres." Cambridge University Press.

Lighthill, M.J. (1978). "Waves in Fluids." Cambridge University Press, Cambridge.

List, R.J. (1951). "Smithsonian Meteorological Tables," 6th ed. Smithson. Misc. Collect. No. 114, Smithson. Inst. Washington, D.C.

Long, R.R. (1965). On the Boussinesq approximation and its role in the theory of internal waves, *Tellus* 17, 46–52.

Miesen, R.H.M., Kamp, L.P.J., Sluijter, F.W. (1988). On the application of the Boussinesq approximation for nonlinear gravity waves in the atmosphere, *Phys. Scripta* 38, 857–859.

Miesen, R.H.M., Kamp, L.P.J., Sluijter, F.W. (1990). Internal solitary waves in compressible shallow fluids, *Phys. Fluids A* 2 (in press).

Oberbeck, A. (1879). Über die Wärmeleitung der Flüssigkeiten bei berück–sichtigung der Strömungen infolge von Temperaturdifferenzen, *Ann. Physik* 7, 271–292.

Pitteway, M.L.V., Hines, C.O. (1963). The viscous damping of atmospheric gravity waves, *Can. J. Phys.* 41, 1935–1948.

Spiegel, E.A., Veronis, G. (1960). On the Boussinesq approximation for a compressible fluid, *Astrophys. J.* 131, 442–447.

Turner, J.S. (1983). "Buoyancy Effects in Fluids.", Cambridge University Press, Cambridge.

Yih, C.–S. (1980). "Stratified Flows." Academic Press, New York.

CHAPTER 3

INTEGRALS OF MOTION

3.1 INTRODUCTION

In this chapter we will adopt an approach by Long & Morton [1965] to obtain two integrals of the equations of motion for a compressible fluid. Together with certain boundary conditions these integrals are transformed to equations for the vertical displacement of a streamline and for the perturbation of the temperature, in a quasi–Lagrangian coordinate system. In the limit of incompressibility these equations are shown to be equivalent to equation (8) of Gear & Grimshaw [1983], which is in fact Long's equation [Long 1953].

3.2 INTEGRALS OF MOTION

The geometry we consider is shown in Fig. 16. There is a steady flow bounded below by a rigid boundary at $z = 0$, and above by either a free surface or a rigid boundary at $z = h$. The horizontal coordinate x is the coordinate in a frame moving horizontally with the phase speed c of the wave. At $|x| = \infty$ the flow is horizontal with speed $-\bar{c} := u_0(z) - c$ and has a density profile $\rho_0(z)$. Finally, in what follows we will adopt a quasi–Lagrangian description of the fluid motion. Accordingly, all quantities are considered to be functions of z, i.e. the height of a streamline far upstream, and x. The vertical displacement $\eta = \eta(x,z)$ of a streamline is determined by $z^* = z + \eta$, where z^* is the Eulerian vertical coordinate.

The equations describing the motion of an inviscid compressible fluid are discussed in chapter 2 of this thesis [e.g. Gill 1982, or cf. (1.45)]:

$$\frac{D\rho}{Dt} + \rho(\nabla\cdot u) = 0, \tag{1a}$$

$$\rho\frac{Du}{Dt} = -\nabla p + \rho g, \tag{1b}$$

$$\frac{D}{Dt}(p\rho^{-\gamma}) = 0, \tag{1c}$$

where ρ is the density, p is the pressure, $\mathbf{u} = (u,v,w)$ is the velocity, g is the gravitational acceleration pointing in the negative z^*–direction, γ is the ratio of specific heats, $\nabla = (\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z^*})$ and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{u}\cdot\nabla). \tag{1d}$$

Here x, y, and z^* are the Cartesian (Eulerian) coordinates. We now define the potential density $\bar{\rho}$ by

$$\frac{\bar{p}}{\bar{\rho}^\gamma} := \frac{p}{\rho^\gamma}. \tag{2}$$

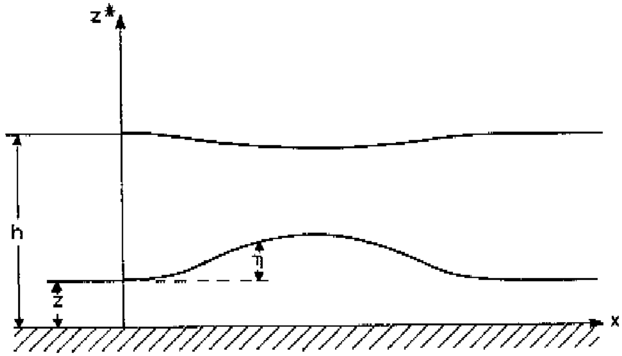


Fig. 16 The geometry: a steady flow bounded below by a rigid boundary at $z = 0$*, and above by either a free surface or a rigid boundary at* $z = h$*. The horizontal coordinate* x *is the coordinate in a frame moving horizontally with the phase speed c of the wave. At* $|x| = \infty$ *the flow is horizontal with speed* $-\bar{c} := u_0(z) - c$ *and has a density profile* $\rho_0(z)$.

The potential density is the density of a fluid parcel if its pressure is increased (or decreased) adiabatically from p to $\bar{p}$. In case of adiabatic motion we can see from (1c) that $\bar{\rho}$ is a conserved quantity; $\bar{p}$ is a constant which we may assign arbitrarily. With the assumption of *steady, two–dimensional* motion, and with the definitions

$$P' := \frac{\gamma}{\gamma-1}\,\bar{p}^{1/\gamma}\,p^{1-1/\gamma}, \tag{3}$$

$$\sigma := \frac{1}{\gamma-1}, \tag{4}$$

equations (1) become

$$\nabla\cdot(P'^{\sigma}\mathbf{u}) = 0, \tag{5a}$$

$$\bar{\rho}\,\frac{d\mathbf{u}}{dt} = -\nabla P' + \bar{\rho}\mathbf{g}, \tag{5b}$$

$$\frac{d\bar{\rho}}{dt} = 0, \tag{5c}$$

where $\mathbf{u} = (u,0,w)$, $\nabla = \left[\frac{\partial}{\partial x},0,\frac{\partial}{\partial z^*}\right]$, and

$$\frac{d}{dt} := u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z^*}. \tag{5d}$$

Let us define

$$P^* = \frac{P'}{\rho_c g h_c}, \tag{6}$$

where ρ_c is a given value of ρ to be specified later and h_c is a characteristic length. Then (5a) implies the existence of a stream function ψ, such that

$$uP^{*\sigma} = -\frac{\partial\psi}{\partial z^*}, \tag{7a}$$

$$wP^{*\sigma} = \frac{\partial\psi}{\partial x}. \tag{7b}$$

In the limit that the fluid is incompressible, i.e. $c_s \to \infty$ or $\gamma \to \infty$, $\sigma \to 0$, the stream function defined by (7) becomes equivalent to the stream function for an incompressible flow defined by (2.27). That ψ is indeed a stream function follows from the fact that **u** is tangent to the line ψ = constant, as can be seen using (7). Notice from (5c) that $\bar{\rho} = \bar{\rho}(\psi)$.

From (5) to (7) we can obtain the following two integrals of the equations of motion of an inviscid compressible fluid [Appendix B]:

$$\frac{\zeta}{P^{*\sigma}} + \frac{1}{\bar{\rho}}\frac{d\bar{\rho}}{d\psi}\left[\frac{q^2}{2} + gz^*\right] = H(\psi), \tag{8a}$$

$$P^* + \frac{\bar{\rho}}{\rho_c g h_c}\left[\frac{q^2}{2} + gz^*\right] = L(\psi), \tag{8b}$$

where

$$\begin{aligned}\zeta &= \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z^*}\\ &= \frac{1}{P^{*\sigma}}\left[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^{*2}}\right] - \frac{1}{P^{*2\sigma}}\left[\frac{\partial P^{*\sigma}}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial P^{*\sigma}}{\partial z^*}\frac{\partial\psi}{\partial z^*}\right],\end{aligned} \tag{8c}$$

is the vorticity,

$$\begin{aligned}q^2 &= u^2 + w^2\\ &= \frac{1}{P^{*2\sigma}}\left[\left[\frac{\partial\psi}{\partial x}\right]^2 + \left[\frac{\partial\psi}{\partial z^*}\right]^2\right],\end{aligned} \tag{8d}$$

is two times the kinetic energy per unit mass and $H(\psi)$ and $L(\psi)$ are arbitrary functions, determined by the conditions far upstream. The equations (8a) and (8b) are in fact special forms of Bernouilli's equation and Helmholtz' vorticity equation for a two–dimensional compressible stratified fluid.

If the upper boundary is a free surface, the boundary condition can be found by considering the streamline coinciding with this boundary. Along this streamline the following quantity is constant [Yih 1980]

$$gz^* + \tfrac{1}{2}q^2 = \text{constant, for } z = h. \tag{9}$$

This in fact means that the perturbation pressure is zero at a free surface, as can be seen from (8b).

The equations (8) and (9) for $\psi(x,z^*)$ and $P^*(x,z^*)$ can be transformed to equations for

$$\eta(x,z) = z^* - z, \tag{10a}$$

$$\delta(x,z) = P^*/P(z) - 1, \tag{10b}$$

where z is the Lagrangian coordinate and P(z) is the value of P^* far upstream.From (2), (3), (6), (10b), (1.6) and (1.28) we find

$$\begin{aligned}\delta(x,z) &= \frac{\gamma p}{\rho c_s{}^2} - 1,\\ &= T/T_0 - 1,\end{aligned}$$

where the velocity of sound c_s is the value of $(\gamma p/\rho)^{1/2}$ far upstream, i.e. $(\gamma p_0/\rho_0)^{1/2}$, and T_0 is the temperature far upstream. The equations for $\eta(x,z)$ and $\delta(x,z)$ are [Appendix C]:

$$\begin{aligned}&\bar{\rho}\bar{c}^2\left[\frac{\eta_x}{1+\eta_z}\right]_x + \left[\bar{\rho}\bar{c}^2\,\frac{\eta_z+\frac{1}{2}(\eta_z^2-\eta_x^2)}{(1+\eta_z)^2}\right]_z - g\bar{\rho}_z\eta(1+\delta)^{2\sigma}\\ &\quad + \frac{\sigma\bar{\rho}\bar{c}^2}{(1+\delta)(1+\eta_z)^2}\left[(1+\eta_x^2)\delta_z - (1+\eta_z)\delta_x\eta_x\right]\\ &\quad + \tfrac{1}{2}(\bar{\rho}\bar{c}^2)_z\,[(1+\delta)^{2\sigma}-1] = 0,\end{aligned} \tag{11a}$$

$$\delta + (\sigma c_s^2)^{-1}\left[\tfrac{1}{2}\bar{c}^2(1+\delta)^{-2\sigma}\frac{(1+\eta_x^2)}{(1+\eta_z)^2} + g\eta - \tfrac{1}{2}\bar{c}^2\right] = 0, \tag{11b}$$

with the boundary conditions

$$\eta = 0, \text{ for } z = 0, \tag{11c}$$

$$g\eta + \frac{1}{2}\left[\frac{\bar{c}^2(1+\eta_x^2)}{(1+\eta_z)^2}(1+\delta)^{-2\sigma} - \bar{c}^2\right] = 0, \text{ for } z = h, \tag{11d}$$

where j = 0 or 1 for a rigid or a free upper boundary, respectively. Note that from the definition of the potential density (2), the hydrostatic balance (1.27), the definition of the velocity of sound and the fact that the potential density is a conserved quantity [see (5c)], we can see that the Brunt–Väisälä frequency for a compressible fluid, defined by (1.49), is given by

$$\omega_g^2 = -g\bar{\rho}_z/\bar{\rho}. \tag{12}$$

Equations (11) describe the inviscid *steady two–dimensional compressible* flow exactly. As long as all the streamlines originate upstream the constants of integration [Appendix B] $H(\psi)$ and $L(\psi)$ can be determined from upstream conditions [Appendix C]. Equation (11) is the equivalent for a compressible flow of (18) of [Gear and Grimshaw, 1983] for an incompressible flow. This can be seen by letting $c_s \to \infty$, i.e. $\gamma \to \infty$ and $\sigma \to 0$.

3.3 THE HYDROSTATIC AND THE BOUSSINESQ APPROXIMATION

In the present section, once again, the hydrostatic and the Boussinesq approximation are discussed. Note that as yet we did not use any approximation to derive equations (11a,b), which describe two–dimensional steady motion, from (1). Since we are now dealing with only two equations, (11a,b) respectively, instead of four [cf. (2.14), (2.25) and (2.46)], it is easier to make a thorough scale analysis of the two–dimensional steady compressible flow. Therefore the following dimensionless quantities are introduced [cf. (2.35)]:

$$\hat{x} = \frac{x}{l}, \quad \hat{z} = \frac{z}{h}, \quad \hat{\eta} = \frac{\eta}{\alpha_1 h}, \quad \hat{\delta} = \frac{\delta}{\alpha_2}, \quad \hat{\rho} = \frac{\bar{\rho}}{\rho_c}, \quad \hat{c} = \frac{\bar{c}}{c_c},$$

$$\hat{\rho}_z = H_{c1}\rho_c^{-1}\bar{\rho}_{z1}, \quad (\hat{c}^2)_z = H_{c2}c_c^{-2}(\bar{c}^2)_{z2}, \quad \alpha_1 = \frac{a}{h}, \quad \beta_{1,2} = \frac{h}{H_{c1,2}}, \quad \epsilon = \frac{h}{l}, \tag{13}$$

where $\hat{x}$, $\hat{z}$, $\hat{\eta}$, $\hat{\delta}$, $\hat{\rho}$, $\hat{c}$, $\hat{\rho}_{z1}$ $(\hat{c}^2)_{z2}$ are dimensionless and of order one and α_1, α_2, β_1, β_2 and ϵ are parameters. Equations (11a,b,d) become

$$\epsilon^2\rho c^2\left[\frac{\eta_x}{1+\alpha_1\eta_z}\right]_x + \rho c^2\left[\frac{\eta_z+\frac{1}{2}\alpha_1(\eta_z^2-\epsilon^2\eta_x^2)}{(1+\alpha_1\eta_z)^2}\right]_z - \beta_1 R\rho_{z1}\eta(1+\alpha_2\delta)^{2\sigma}$$

$$+ \frac{\sigma\rho c^2}{(1+\alpha_2\delta)(1+\alpha_1\eta_z)^2}\left[(1+\epsilon^2\alpha_1^2\eta_x^2)\frac{\alpha_2}{\alpha_1}\delta_z - (1+\alpha_1\eta_z)\epsilon^2\alpha_2\delta_x\eta_x\right]$$

$$+ [\beta_1\rho_{z1}c^2+\beta_2\rho(c^2)_{z2}]\left\{\frac{\eta_z+\frac{1}{2}\alpha_1(\eta_z^2-\epsilon^2\eta_x^2)}{(1+\alpha_1\eta_z)^2} + \tfrac{1}{2}\alpha_1^{-1}[(1+\alpha_2\delta)^{2\sigma}-1]\right\} = 0, \tag{14a}$$

$$\delta + \frac{c_c^2}{\sigma c_s^2}\left\{\tfrac{1}{2}\alpha_2^{-1}c^2[(1+\alpha_2\delta)^{-2\sigma}-1]\frac{(1+\epsilon^2\alpha_1^2\eta_x^2)}{(1+\alpha_1\eta_z)^2} + \frac{\alpha_1 R}{\alpha_2}\eta\right\} = 0, \tag{14b}$$

$$\eta + \frac{j c^2}{2\alpha_1 R}[(1+\alpha_2\delta)^{-2\sigma}-1]\frac{(1+\epsilon^2\alpha_1^2\eta_x^2)}{(1+\alpha_1\eta_z)^2} = 0, \text{ for } z = 1, \tag{14c}$$

where we have dropped the hats and defined a Richardson number $R = gh/c_c^2$. Assuming that disturbances are small, i.e.

$$\alpha_1 \ll 1, \qquad \alpha_2 \ll 1, \tag{15}$$

these equations become, correct to the lowest order in an expansion with respect to α_1 and α_2,

$$\rho c^2\left[\epsilon^2\eta_{xx} + \eta_{zz} + \sigma\frac{\alpha_2}{\alpha_1}\delta_z\right] - \beta_1 R\rho_{z1}\eta + [\beta_1\rho_{z1}c^2+\beta_2\rho(c^2)_{z2}]\left[\eta_z+\sigma\frac{\alpha_2}{\alpha_1}\delta\right] = 0 \tag{16a}$$

$$\delta + \frac{c_c^2}{\sigma c_s^2}\left[-\sigma c^2\delta + \frac{\alpha_1 R}{\alpha_2}\eta\right] = 0, \tag{16b}$$

$$\eta - \frac{j\alpha_2\sigma c^2}{\alpha_1 R}\delta = 0, \text{ for } h = 1. \tag{16c}$$

Making the hydrostatic approximation implies that we assume that the aspect ratio ϵ of the disturbance is small, so that the first term in (16a) can be neglected. Note that in that case all differentiations with respect to x have disappeared from the equations and therefore the x–dependence of the solutions

η and δ is not determined by these equations. This also means that the waves show no dispersion with respect to the horizontal wavenumber [cf. (2.23)].

Using the Boussinesq approximation for a compressible fluid means in the first place that the compressibility can be neglected except in the definition of the Brunt–Väisälä frequency. Since $\sigma = 0$ for an incompressible fluid but order one for a compressible fluid, neglecting the compressibility in (16a) requires

$$\alpha_2/\alpha_1 \ll 1. \tag{17a}$$

In the second place, in the Boussinesq approximation the basic density and velocity are assumed to vary on a much larger scale than the vertical scale of the wave. This implies that

$$\beta_1 \ll 1, \qquad \beta_2 \ll 1. \tag{17b}$$

Because $R \geq \frac{1}{4}$ in all stable cases, the free boundary condition (16c) or (11d) becomes a rigid boundary condition in the Boussinesq approximation.

From (16b) and the definition of R we find that the condition (17a) is satisfied if $gh/c_s^2 \ll 1$. With the definition of c_s this can be seen to be equivalent to the condition $\beta_1 \ll 1$. The conditions for the Boussinesq approximation are therefore $\alpha_1 \ll 1$, $\beta_1 \ll 1$ and $\beta_2 \ll 1$ [cf. (2.40) and (2.41)].

3.4 SUMMARY

In this chapter we have derived two equations, i.e. (11a,b), from the (nonlinear) equations of motion without using any approximation. These two equations describe a compressible steady two–dimensional flow, if there are no closed streamlines, in terms of the displacement of a streamline $\eta(x,z)$ and in terms of the relative temperature perturbation $\delta(x,z)$. These equations also have been used to discuss once again the hydrostatic and the Boussinesq approximation.

REFERENCES

Gear, J.A., Grimshaw, R.R. (1983). A second–order theory for solitary waves in a shallow fluid, *Phys. Fluids* 26, 14–29.

Gill, A.E. (1982). "Atmosphere–Ocean dynamics." Academic Press, New York.

Long, R.R. (1953). Some aspects of the flow of stratified flows, *Tellus* 5, 42–58.

Long, R.R., Morton, J.B. (1966). Solitary waves in compressible stratified fluids, *Tellus* 18, 79–85.

Yih, C.–S. (1980). "Stratified Flows." Academic Press, New York.

CHAPTER 4

INTERNAL SOLITARY WAVES IN COMPRESSIBLE SHALLOW FLUIDS

4.1 INTRODUCTION

Internal solitary waves in incompressible stratified fluids have been studied extensively [e.g, Benney 1966, Benjamin 1966, 1967, Grimshaw 1981, Gear & Grimshaw 1983]. If the fluid can be considered shallow, i.e. if the depth of the fluid is much smaller than the wavelength, the solitary waves are described, to the first order in the wave amplitude, by the Korteweg–de Vries equation. In that case they have the characteristic "sech^2" profile, phase speeds that vary linearly with the wave amplitude and wavelengths that vary inversely with the square root of the wave amplitude.

The effects of the compressibility of the fluid are discussed to some extent by only a few authors [Grimshaw 1980/1981, Long & Morton 1966, Shen 1966, 1967, Shen & Keller 1973]. The parameter that measures the effect of the compressibility is given by $h/(\gamma H_c)$ or gh/c_s^2, where h is a vertical scale typical for the vertical structure of the wave, H_c is a vertical scale typical for the stratification of the fluid, γ is the ratio of specific heats, g is the magnitude of the gravitational acceleration, and c_s is the velocity of sound. For oceanographic observations [e.g. Liu, Holbrook & Apel 1985] this parameter is always smaller than 10^{-3}, and compressibility effects can be neglected. However, in the atmospheric case [e.g., Lin & Goff 1988, Pecnick & Young 1984, Stobie, Einaudi & Uccellini 1983], where h can be the total height of the troposphere, the parameter that measures the effect of the compressibility is an order one quantity. Even if this parameter is small, but γ is of order of magnitude one (γ for air is 1.4), the compressibility of the fluid can have important consequences [Grimshaw 1980/1981].

In this chapter a theory for a fully compressible fluid is presented, that represents a generalisation of the theory for solitary waves in incompressible shallow fluids as presented by Gear & Grimshaw [1983]. In the limit of incompressibility, i.e. $c_s \to \infty$, our results reduce to the results of their paper.

Before we proceed with compressible fluids, the theory for solitary waves in incompressible fluids, correct to the first order in the wave amplitude, is reviewed briefly. For that purpose we consider an inviscid, incompressible fluid with a background density profile $\rho_0(z)$ and a background velocity profile $u_0(z)$, bounded below by a rigid boundary $z = 0$, and above either by a free surface whose equilibrium position is at $z = h$, or by a rigid boundary at $z = h$. Here, z is a (Lagrangian) vertical coordinate, and x shall be a horizontal coordinate in a frame moving with the phase speed c of the wave. Furthermore, the basic state is assumed to be stable, i.e. stably stratified (chapter 1) and with a Richardson number larger than $\frac{1}{4}$. Weakly nonlinear long waves in shallow fluids are characterised by the equality of two small parameters α en $\epsilon^2 = (h/\ell)^2$, where α is a measure of the amplitude of the vertical displacement η_1 of a streamline due to the solitary wave and ℓ is the horizontal scale of the wave. The equality of α, which is a measure for the nonlinearity of the problem, and ϵ^2, which is a measure for the strength of the dispersion of the wave, represents the balance between nonlinearity and dispersion that is characteristic for solitary waves. Thus, we let $\eta_1 = \alpha A(X)\varphi(z)$, where $X = \epsilon x$, and $\varphi(z)$ is the modal function describing the vertical structure of the wave, normalised to 1 at its maximum. It satisfies the following linear eigenvalue problem correct to $\mathcal{O}(\epsilon^2)$ [Gear & Grimshaw 1983]:

$$(\rho_0\bar{c}_0^2\varphi_z)_z + \rho_0\omega_b^2\varphi = 0, \text{ for } 0 < z < h, \tag{1a}$$
$$\varphi = 0, \text{ for } z = 0, \tag{1b}$$
$$\varphi - j\bar{c}_0^2\varphi_z/g = 0, \text{ for } z = h, \tag{1c}$$

where $\bar{c}_0 = c_0 - u_0(z)$, $\omega_b = (-g\rho_{0z}/\rho_0)^{1/2}$ is the Brunt–Väisälä frequency for an incompressible stratified fluid and the subscripts z denote derivatives. The variable j takes the value 0 or 1 according to the upper boundary being rigid or free, respectively. The eigenvalue to be found is c_0, i.e. the *linear* long wave phase speed. We shall assume for simplicity that there are no critical layers and so $\bar{c}_0$ is not zero for any value of z. From the equations correct to the first order in the wave amplitude an integrated form of the Korteweg–de Vries equation is found that determines the amplitude $A(X)$ and the correction $\epsilon^2 c_1$ to the linear long wave phase speed c_0 [Benney 1966, Gear & Grimshaw 1983]:

$$-c_1\lambda A + \tfrac{1}{2}\mu A^2 + \nu A_{XX} = 0, \tag{2a}$$

$$A(X) = a\,\mathrm{sech}^2(X/\ell), \tag{2b}$$

$$c_1 = \mu a/(3\lambda),\ \ell^2 a = 12\nu/\mu. \tag{2c}$$

The coefficients λ, μ and ν are known in terms of the modal function $\varphi(z)$ and are given by [Gear & Grimshaw 1983]

$$\lambda = 2\int_0^h \rho_0\bar{c}_0\varphi_z^2\,dz, \tag{3a}$$

$$\mu = 3\int_0^h \rho_0\bar{c}_0^2\varphi_z^3\,dz, \tag{3b}$$

$$\nu = \int_0^h \rho_0\bar{c}_0^2\varphi^2\,dz. \tag{3c}$$

In the next section (section 4.2) we will give the analysis that leads to the analogy of equations (1) to (3) for a compressible fluid. In section 4.3 we discuss three special cases: (I) there is no background shear flow and the Brunt–Väisälä frequency is constant; (II) there is no background shear flow and the Brunt–Väisälä frequency is constant for $0 < z < d$ and zero for $d < z < h$, forming a simple model of an inversion layer; (III) the background shear flow is linear and the Brunt–Väisälä frequency is constant. For the last case only the eigenvalue problem is discussed. In (II) the Boussinesq approximation is made, which also implies that the upper boundary can be considered rigid [section 3.3, Gear & Grimshaw 1983]. The validity of this approximation, also if it is used for (I) and (III), is discussed in connection with the effects of the compressibility. Since the analytical expressions obtained for the special cases are complicated, some numerical results are also presented.

4.2 ANALYSIS

4.2.1 The linear problem

We will start our analysis for a compressible fluid from the equations (3.11) and seek solutions of these equations with the following asymptotic expansions:

$$\bar{c} = \bar{c}_0(z) + \epsilon^2 c_1 + \epsilon^4 c_2 + \ldots, \tag{4a}$$
$$\eta = \epsilon^2 \eta_1(X,z) + \epsilon^4 \eta_2(X,z) + \ldots, \tag{4b}$$
$$\delta = \epsilon^2 \delta_1(X,z) + \epsilon^4 \delta_2(X,z) + \ldots, \tag{4c}$$

where

$$\eta_1 = A(X)\varphi(z), \tag{4d}$$
$$X = \epsilon x. \tag{4e}$$

At leading order we find from (3.11a,b):

$$(\bar{\rho}\bar{c}_0^2 \eta_{1z})_z + \bar{\rho}\omega_g^2 \eta_1 + (\sigma\bar{\rho}\bar{c}_0^2 \delta_1)_z = 0, \tag{5a}$$
$$\delta_1 = \sigma^{-1}(c_s^2 - \bar{c}_0^2)^{-1}(\bar{c}_0^2 \eta_{1z} - g\eta_1). \tag{5b}$$

Equations (4d) and (5), together with the boundary conditions, give at leading order, the following eigenvalue problem:

$$\left[\bar{\rho}\bar{c}_0^2 \frac{c_s^2 \varphi_z - g\varphi}{c_s^2 - \bar{c}_0^2}\right]_z + \bar{\rho}\omega_g^2 \varphi = 0, \text{ for } 0 < z < h, \tag{6a}$$
$$\varphi = 0, \text{ for } z = 0, \tag{6b}$$
$$\varphi - j\bar{c}_0^2 \varphi_z / g = 0, \text{ for } z = h, \tag{6c}$$

where [cf. (3.12) and (1.46)]

$$\omega_g = (-g\bar{\rho}_z/\bar{\rho})^{1/2} = (-g\rho_{0z}/\rho_0 - g^2/c_s^2)^{1/2}, \tag{7}$$

is the Brunt–Väisälä frequency for a compressible stratified fluid. In the limit of incompressibility ($c_s \to \infty$), (6) reduces to (1), i.e. the eigenvalue problem for the incompressible fluid.

In terms of $\rho_0(z)$, (6a) can be written as

$$(\rho_0 \bar{c}_0^2 \varphi_z)_z + \rho_0 \omega_g^2 \varphi + \rho_0 \frac{c_s^2(\bar{c}_0^2)_z - \bar{c}_0^2(c_s^2)_z}{c_s^2(c_s^2 - \bar{c}_0^2)}(\bar{c}_0^2 \varphi_z - g\varphi) = 0, \tag{8}$$

which, if $\bar{c}_0^2 \ll c_s^2$ and $(c_s^2)_z/(\bar{c}_0^2)_z \ll c_s^2/\bar{c}_0^2$, is (2.15a) of Grimshaw [1980/1981]:

$$(\rho_0\bar{c}_0^2\varphi_z)_z + \rho_0[\omega_g^2 - g(\bar{c}_0^2)_z/c_s^2]\varphi = 0. \tag{9}$$

Before we continue to the next order in ϵ^2, we shall discuss the differential equation (8) briefly. If the fluid is almost incompressible the last term of (8) is smaller than the first two terms. From the first two terms of this equation, i.e. the differential equation if the fluid is incompressible, we see that $\bar{c}_0^2/c_s^2$ is of $\mathcal{O}(h^2\omega_g^2/c_s^2) = \mathcal{O}[h^2/(\gamma H^2)]$, where h is a typical vertical scale of the modal function φ. If $\gamma = \mathcal{O}(1)$, equation (8) can, correct to $\mathcal{O}(h/H) = \mathcal{O}(\beta)$, be written as

$$(\rho_0\bar{c}_0^2\varphi_z)_z + \rho_0\omega_g^2\varphi - \rho_0\frac{\bar{c}_0^2}{c_s^2}(c_s^2)_z\left[\frac{\bar{c}_0^2}{c_s^2}\varphi_z - \frac{g}{c_s^2}\varphi\right]$$
$$- \rho_0(\bar{c}_0^2)_z\left[\frac{g}{c_s^2}\varphi\right] + \mathcal{O}(\beta^2) = 0. \tag{10}$$

If furthermore $|(c_s^2)_z/c_s^2| \lesssim H^{-1}$, which is true for most practical circumstances, (8) becomes

$$(\rho_0\bar{c}_0^2\varphi_z)_z + \rho_0\omega_g^2\varphi - g\rho_0(\bar{c}_0^2)_z\varphi/c_s^2 + \mathcal{O}(\beta^2) = 0. \tag{11}$$

If also $|(\bar{c}_0^2)_z/\bar{c}_0^2| \lesssim H^{-1}$, e.g. $u_0(z) = 0$, then

$$(\rho_0\bar{c}_0^2\varphi_z)_z + \rho_0\omega_g^2\varphi + \mathcal{O}(\beta^2) = 0, \tag{12}$$

and the differential equation for the modal function for a compressible fluid is the same, correct to the first order in the Boussinesq parameter h/H, as the differential equation for an incompressible fluid if the "compressible definition" for the Brunt–Väisälä frequency (7) is used.

4.2.2 The Korteweg–De Vries equation

At the next order in ϵ^2 we obtain (Appendix D):

$$\left[\bar{\rho}\bar{c}_0^2\frac{c_s^2\eta_{2z} - g\eta_2}{c_s^2 - \bar{c}_0^2}\right]_z + \bar{\rho}\omega_g^2\eta_2 + f_2 = 0, \text{ for } 0 < z < h, \tag{13a}$$

$$\eta_2 = 0, \text{ for } z = 0, \tag{13b}$$

$$\eta_2 - j\bar{c}_0^2\eta_{2z}/g + jg_2 = 0, \text{ for } z = h, \tag{13c}$$

and the inhomogeneous terms f_2, g_2 are given by

$$
\begin{aligned}
f_2 = \bar\rho\bar c_0^2\eta_{1xx} &+ [2\bar\rho\bar c_0 c_1 c_s^2(c_s^2-\bar c_0^2)^{-2}(c_s^2\eta_{1z}-g\eta_1)]_z \\
&- \Big[\bar\rho\bar c_0^2\Big[\tfrac{3}{2}\eta_{1z}^2 + (c_s^2-\bar c_0^2)^{-3}[\bar c_0^2(\tfrac{7}{2}c_s^4 + (\tfrac{1}{2}\sigma^{-1}-4)c_s^2\bar c_0^2 + \tfrac{3}{2}\bar c_0^4)\eta_{1z}^2 \\
&- gc_s^2(2c_s^2 + \bar c_0^2/\sigma)\eta_1\eta_{1z} + g^2c_s^2(1 + \tfrac{1}{2}\sigma^{-1})\eta_1^2]\Big]\Big]_z,
\end{aligned}
\tag{13d}
$$

and

$$
\begin{aligned}
g_2 = -2\bar c_0 c_1(c_s^2-\bar c_0^2)^{-1}(c_s^2\eta_{1z}/g-\eta_1) &+ \bar c_0^2\Big[\tfrac{3}{2}c_s^{-2}(c_s^2-\bar c_0^2)\eta_{1z}^2/g \\
&+ (c_s^2-\bar c_0^2)^{-2}[\bar c_0^2(\tfrac{7}{2}c_s^2 + (\tfrac{1}{2}\sigma^{-1}-4)\bar c_0^2 + \tfrac{3}{2}\bar c_0^4/c_s^2)\eta_{1z}^2/g \\
&- (2c_s^2+\bar c_0^2/\sigma)\eta_1\eta_{1z} + g(1+\tfrac{1}{2}\sigma^{-1})\eta_1^2]\Big].
\end{aligned}
\tag{13e}
$$

The solution of (13a,b) can be found by the method of variation of parameters. In order that this solution satisfies the second boundary condition (13c) it is necessary and sufficient that (Appendix E)

$$
\begin{aligned}
&\int_0^h\Big\{\varphi f_2\exp[-\int_0^z(g/c_s^2)dz]\Big\}\,dz \\
&\qquad+ \Big\{j\bar\rho\bar c_0^2c_s^2(c_s^2-\bar c_0^2)^{-1}\,\varphi_z g_2\exp[-\int_0^z(g/c_s^2)dz]\Big\}\Big|_{z=h} = 0.
\end{aligned}
\tag{14}
$$

Equation (14), together with (13d,e), (6c) and after partial integration, yields the following integrated form of the Korteweg–de Vries equation for $A(X)$:

$$-c_1\lambda A + \tfrac{1}{2}\mu A^2 + \nu A_{xx} = 0, \tag{15a}$$

$$\lambda = \int_0^h 2\rho_0\bar c_0(c_s^2-\bar c_0^2)^{-2}\,(c_s^2\varphi_z-g\varphi)^2dz, \tag{15b}$$

$$
\begin{aligned}
\mu = \int_0^h\rho_0\bar c_0^2\{3\varphi_z^2 &+ (c_s^2-\bar c_0^2)^{-3}[\bar c_0^2(7c_s^4 + (1/\sigma-8)\bar c_0^2c_s^2 + 3\bar c_0^4)\varphi_z^2 \\
&- 2gc_s^2(2c_s^2+\bar c_0^2/\sigma)\varphi\varphi_z + g^2c_s^2(2+1/\sigma)\varphi^2]\}(\varphi_z-g\varphi/c_s^2)dz,
\end{aligned}
\tag{15c}
$$

$$\nu = \int_0^h\rho_0\bar c_0^2\varphi^2dz. \tag{15d}$$

Here we used that (Appendix F):

$$\rho_0(z) = \bar{\rho}(z)\exp\left[-\int_0^z (g/c_s^2)dz\right]. \tag{16}$$

If $\bar{c}_0^2 \ll c_s^2$, an assumption that is almost always correct for atmospheric applications, λ, μ and ν can be approximated by

$$\lambda = \int_0^h 2\rho_0\bar{c}_0(\varphi_z^2 + 2\bar{c}_0^2\varphi_z^2/c_s^2 - 2g\varphi\varphi_z/c_s^2 + g^2\varphi^2/c_s^4)dz, \tag{17a}$$

$$\mu = \int_0^h \rho_0\bar{c}_0^2[3\varphi_z^3 + 7\bar{c}_0^2\varphi_z^3/c_s^2 - 7g\varphi\varphi_z^2/c_s^2 + g^2(\gamma+5)\varphi^2\varphi_z/c_s^4 - g^3(\gamma+1)\varphi^3/c_s^6]dz, \tag{17b}$$

$$\nu = \int_0^h \rho_0\bar{c}_0^2\varphi^2 dz. \tag{17c}$$

If the velocity of sound becomes infinite, which means that the fluid is incompressible, the coefficients of the Korteweg–de Vries equation, given by (15) or (17), become equal to the coefficients for the incompressible case as given by (3) [cf. Benney 1966, or Gear & Grimshaw 1983]. Furthermore we note that the results (15) for the coefficients λ, μ and ν differ from those of Grimshaw's [1980/1981] treatment of the compressible case. This is due to the fact that Grimshaw does include compressibility only partially and furthermore applies the Boussinesq approximation, both of which we do not adopt (see Appendix G).

The importance of the "compressibility correction terms" in (17) depends on the magnitude of the two parameters $\bar{c}_0^2/c_s^2 = \mathcal{O}[h^2/(\gamma H^2)]$ [see the discussion before (10)], and $gh/c_s^2 = h/(\gamma H)$, where h and H are the scales for the vertical structure of the wave and for the stratification of the fluid, respectively. If $\gamma = \mathcal{O}(1)$ and $h/H \ll 1$ the last and the second term in (17a) and the last two terms and the second term in (17b) can be neglected:

$$\lambda = \int_0^h 2\rho_0\bar{c}_0(\varphi_z^2 - 2g\varphi\varphi_z/c_s^2)dz + \mathcal{O}(\beta^2), \tag{18a}$$

$$\mu = \int_0^h \rho_0\bar{c}_0^2(3\varphi_z^3 - 7g\varphi\varphi_z^2/c_s^2)dz + \mathcal{O}(\beta^2), \tag{18b}$$

$$\nu = \int_0^h \rho_0\bar{c}_0^2\varphi^2 dz. \tag{18c}$$

4.3 DISCUSSION

4.3.1 CASE I: Uniform stratification: $u_0 = 0$, $\omega_g^2 = N^2$

Here N is a constant, which means that the scaleheight of the atmosphere is a constant. This also means that the atmosphere is isothermal and the velocity of sound is a constant. For an isothermal atmosphere H is constant and the density is given by [cf. (1.30)]

$$\rho_0(z) = \rho_c \exp(-z/H). \tag{19}$$

If we anticipate that $c_0^2 \ll c_s^2$, (6) becomes [see also (9)]:

$$(\rho_0\varphi_z)_z + \rho_0 N^2\varphi/c_0^2 = 0, \tag{20a}$$

$$\varphi = 0, \text{ for } z = 0, \tag{20b}$$

$$\varphi = jc_0^2\varphi_z/g, \text{ for } z = h. \tag{20c}$$

This eigenvalue problem is the same as that for solitary waves in an incompressible shallow fluid, if our definition of the Brunt–Väisälä frequency, i.e. (7), is replaced by its incompressible counterpart [cf. case (II) of Gear & Grimshaw 1983]. That the consequences of this are by no means trivial is due to the fact that H in (19) cannot be replaced by g/N^2, as for an incompressible fluid.

The solution of (20) is

$$\varphi = M\rho_0^{-1/2}\sin(r_s z), \tag{21a}$$

$$c_0^2 = N^2/[r_s^2 + (4H^2)^{-1}], \tag{21b}$$

where

$$\tan(r_s h) = jr_s N^2\{g[r_s^2 + (4H^2)^{-1}] - N^2/(2H)\}^{-1},\ s = 0,1,2,\dots\ . \tag{21c}$$

For a rigid upper boundary condition $j = 0$, so $r_s h = s\pi$, and the $s = 0$ mode is excluded. For a free upper boundary condition

$$c_0^2 = \frac{N^2h^2}{s^2\pi^2}\left[1 - \frac{2N^2h}{g s^2\pi^2} + \mathcal{O}[(\beta/2s\pi)^2]\right],\ s = 1,2,\dots\ , \tag{22a}$$

and

$$c_0^2 = gh\left[1 - \frac{h}{2H} + \frac{N^2h}{3g} + \mathcal{O}(\beta^2)\right],\ s = 0, \tag{22b}$$

The $s = 0$ mode is the free surface mode, while the other modes are internal. Note that the decrease of the phase speed, due to the stratification of the fluid, of the free surface mode compared to the phase speed of long surface waves on a shallow homogeneous fluid $(gh)^{1/2}$ is approximately twice as large if the compressibility is taken into account. The eigenvalue r_s and so c_0 can be obtained numerically very easily by means of the iterative procedure

$$F(r_s^{(n+1)}) = h^{-1}\arctan\left[N^2 r_s^{(n)}\left[g[r_s^{(n)2}+1/(4H^2)] - N^2/(2H)\right]^{-1}\right] + s\pi/h,$$
$$n = 0,1,2,\dots, \tag{23a}$$
$$r_s^{(0)} = 0,\ s = 1,2,3,\dots, \tag{23b}$$
$$r_0^{(0)} = [N^2/(gh) - (4H^2)^{-1}]^{1/2}. \tag{23c}$$

The iteration is stopped when $r_s^{(n)} \approx r_s^{(n-1)}$, whereafter c_0 is calculated from (21b). The constant M in (21a) is chosen so that the maximum value of $|\varphi(z)|$ is 1. This value is attained at z_m; we find that

$$(h-z_m)r_s = \arctan\left[\frac{2Hr_s}{1-2jHN^2/g}\right],\ s = 1,2,\dots, \tag{24a}$$

and

$$M\rho_c^{1/2} = [\sin(r_s z_m)]^{-1}\exp[-z_m/(2H)]$$
$$= (-1)^{s+1}\left[1 + \left[\frac{1-2jHN^2/g}{2Hr_s}\right]^2\right]^{1/2}$$
$$\times\left[\frac{r_s^2H^2+\frac{1}{4}}{r_s^2H^2+\frac{1}{4}+j(N^2H/g)(N^2H/g-1)}\right]^{1/2}\exp[-z_m/(2H)]. \tag{24b}$$

Notice that for an incompressible fluid $N^2H/g = 1$ but for a compressible fluid (with $\gamma < 2$) $N^2H/g < \frac{1}{2}$. Taking account of this by adding $j\pi$ to the right hand side of (24a) if the fluid is incompressible, (24) reduces to (48) of Gear & Grimshaw [1983]. For the free surface mode ($s = 0$) and sufficiently large H, the maximum is attained at $z_m = h$, and M is chosen so that $\varphi(h) = 1$.

Next we will evaluate λ, μ and ν from (17) to find the correction to the phase velocity (c_1) and the wavelength (ℓ) of the solitary wave from (2). The integrals that appear are basic and can be found, e.g., in Gradshteyn & Ryzhik [1965]. Further evaluation of these expressions for λ, μ and ν yields rather complicated expressions that will not be given here. Instead some numerical results are presented. But first we remark that for $h < (2-j)sH$ we can approximate λ and μ within a few percent by [see (18) and (21a)]

$$\lambda/M^2 \approx 2c_0 r_s^2 \int_0^h \cos^2(r_s z)dz = c_0 r_s[hr_s + \tfrac{1}{2}\sin(2hr_s)], \tag{25a}$$

$$\mu/M^2 \approx c_0^2 M \rho_c^{-1/2}\Big[3r_s^3 \int_0^h \exp(z/2H)\cos^3(r_s z)dz + r_s^2\Big[\frac{9}{2H} - \frac{7g}{c_s^2}\Big]\int_0^h \exp(z/2H)\cos^2(r_s z)\sin(r_s z)dz\Big], \tag{25b}$$

and ν is given by

$$\nu/M^2 = c_0^2 \int_0^h \sin^2(r_s z)dz = \tfrac{1}{2}c_0^2\Big[h - \frac{1}{2r_s}\sin(2hr_s)\Big]. \tag{25c}$$

If $j = 0$ evaluation of (25) gives with (2c)

$$\frac{hc_1}{ac_0} \approx \frac{4MHr_s[(-1)^s e^{h/2H} - 1]}{\rho_c^{1/2}(36H^2r_s^2 + 1)}\left[\frac{7gH}{c_s^2}\frac{(H^2r_s^2+1/12)}{(H^2r_s^2+1/4)} - 1\right], \tag{26a}$$

$$\frac{a\ell^2}{h^3} \approx \frac{\rho_c^{1/2}(36H^2r_s^2 + 1)}{2MHh^2r_s^3\,[(-1)^s e^{h/2H} - 1]}\left[\frac{7gH}{c_s^2}\frac{(H^2r_s^2+1/12)}{(H^2r_s^2+1/4)} - 1\right]^{-1}, \tag{26b}$$

which, again for $h < 2sH$, can be approximated as

$$\frac{hc_1}{ac_0} \approx \frac{4Mh[(-1)^s e^{h/2H} - 1]}{9s\pi\rho_c^{1/2}H}, \tag{27a}$$

$$\frac{a\ell^2}{h^3} \approx \frac{9\rho_c^{1/2}H}{2s\pi Mh[(-1)^s e^{h/2H} - 1]}, \tag{27b}$$

where we used that $gH/c_s^2 = 1/1.4$ for air. In that case the "correction" due to the compressibility is about five times ($7gH/c_s^2$) larger than the term for the incompressible case and of the opposite sign. This means that compared to a solitary wave in an incompressible fluid, a solitary wave in a compressible fluid has a wavelength that is twice as small and the correction to the long wave phase speed for this wave is four times larger. Furthermore, the sign of the amplitude a of the wave changes so that a wave of elevation (e.g., j = 0, s = 1) becomes a wave of depression. So a solitary wave in a compressible fluid can even be qualitatively different from the same wave in an incompressible fluid. However, before we discuss how such qualitatively different behavior is possible, some numerical results are presented.

We have calculated the linear long wave phase velocity c_0, the correction c_1 to it, and the length ℓ of the well-known 1-soliton solution of the Korteweg- de Vries equation (Table I). We did so for the first two modes (s = 1,2) and for free (j = 1) and rigid (j = 0) upper boundary conditions, using (17) and the formulae in this section. The results are compared with those for an incompressible fluid, calculated from (48) and (49) of Gear & Grimshaw [1983]. If the velocity of sound is assumed to be infinite in (17), and the definition of the Brunt–Väisälä frequency for an incompressible isothermal fluid $N = (g/H)^{1/2}$ is used, the results are identical to those obtained from the formulae of Gear & Grimshaw. If the Brunt–Väisälä frequency is not defined in this way (c_s is still infinite), (17) already gives different results for j = 1. This is due to the fact that in order to simplify their results for j = 1, Gear & Grimshaw explicitly use that the scaleheight of the fluid $H = N^2/g$ [see their eq. (45)], which is only valid for an incompressible isothermal fluid. Therefore (49) of Gear & Grimshaw cannot, without any further consideration, be used as an approximation for a compressible fluid. The fact that for s = 1 and j = 1 there is reasonably good agreement between the results is a coincidence, since the last three terms in (17b), i.e. the terms due to the compressibility, give a much larger contribution to μ than the first incompressible term does. This can be checked by numerical evaluation.

The reason why, at least in this example, the results are influenced strongly by the compressibility of the fluid is that for a shearless isothermal fluid the coefficient in front of the nonlinear term in the Korteweg–de Vries equation is of $\mathcal{O}(h/H)$, because it is a well-known fact that in the Boussinesq

	h/H	$s = 1$ $-10^2(hc_1/ac_0)$		$-10(a\ell^2/h^3)$		$s = 2$ $-10^3(hc_1/ac_0)$		$-(a\ell^2/h^3)$	
j=0	0.1	2.83	(–0.707)	71.6	(–286)	0.349	(–0.0873)	145	(–580)
	0.2	5.65	(–1.41)	35.8	(–143)	1.38	(–0.345)	36.7	(–147)
	0.5	14.0	(–3.53)	14.3	(–57.1)	8.30	(–2.08)	6.09	(–24.4)
	1	27.4	(–7.01)	7.06	(–28.2)	31.0	(–7.80)	1.61	(–6.54)
	2	50.5	(–13.6)	3.39	(–13.5)	108	(–27.6)	0.449	(–1.79)
j=1	0.1	3.74	(2.51)	53.5	(77.7)	4.95	(15.9)	10.2	(3.14)
	0.2	7.48	(5.07)	26.4	(36.9)	10.7	(32.0)	4.72	(1.55)
	0.5	18.6	(13.2)	10.2	(12.7)	32.2	(80.6)	1.55	(0.597)
	1	36.8	(28.0)	4.74	(5.04)	80.9	(164)	0.602	(0.278)
	2	70.6	(64.0)	2.02	(1.67)	215	(346)	0.214	(0.119)

TABLE I $-hc_1/ac_0$ *and* $-a\ell^2/h^3$ *as a function of* h/H *for the modes* $s = 1,2$ *and for free surface* ($j = 1$) *and rigid* ($j = 0$) *upper boundary conditions in case I. The numbers between parenthesis are the values for an incompressible fluid. For the ratio of specific heats we used* $\gamma = 1.4$.

approximation (where it is assumed that $H \gg h$) this coefficient becomes zero for an incompressible fluid [Long 1965]. However, the terms due to the compressibility of the medium are of the order gh/c_s^2, which for a compressible fluid, i.e. γ of order of magnitude one, is the same order of magnitude as h/H.

Finally in Fig. 17 we display graphs of the streamlines for $s = 1$, $|a|/h = 0.1$, $h/H = 0.2$ and $j = 0, 1$. Note that for an incompressible fluid the solitary wave is a wave of elevation for $j = 0$ (rigid upper lid) and a wave of depression for $j = 1$ (free upper surface). However, for a compressible fluid the solitary wave is a wave of the depression for both $j = 0$ and $j = 1$. Also note the difference in wavelength for compressible and incompressible solitary waves, which, for the values of the parameters used, is quite small if $j = 1$ but is a factor 2 if $j = 0$.

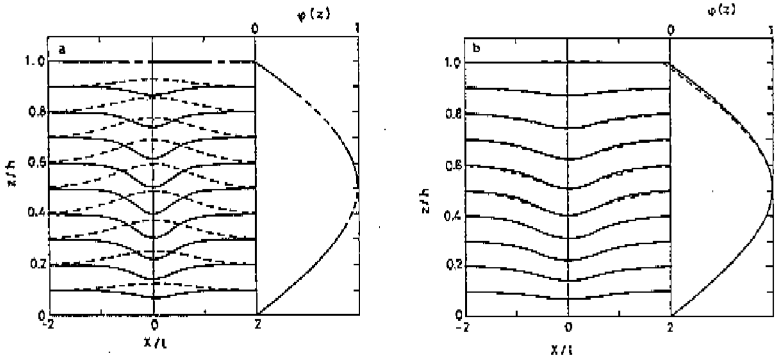


Fig. 17. The streamlines as a function of X/ℓ *and* z/h *for the mode* s = 1, $|a|/h = 0.1$, $h/H = 0.2$ *when* (a) j = 1 *and* (b) j = 0 *for case* I. *Here* ℓ *is the wavelength of the solitary wave in the incompressible fluid.* – – –, *incompressible fluid;* ———, *compressible fluid.*

4.3.2 CASE II: Inversion layer: $u_0 = 0$, $\omega_g^2 = N^2\, H(d-z)$

Here H is the Heaviside function, so that ω_g takes the constant value N below the inversion level z = d and is zero above this level. The lower layer is therefore an isothermal uniformly stratified layer with constant velocity of sound. In the upper layer the Brunt–Väisälä frequency is zero, implying constant density. Of course, such a model is an idealisation of a real physical system like the atmosphere. The idea behind this idealisation is that in the upper layer ($z > d$) the temperature, and so the scale height $H = c_s^2/\gamma g$ of the fluid, increases suddenly to a much larger value than in the lower layer ($z < d$). A much larger value of the scaleheight implies that the density of the fluid is almost constant. However, for a compressible fluid like air, γ is of order of magnitude one, and to be consistent with the assumption of constant density, terms proportional to g/c_s^2 have to be neglected in the upper layer.

First the eigenvalue problem is solved, again anticipating that $\bar{c}_0^2 \ll c_s^2$. For the lower layer (20a,b) applies, while for the upper layer the boundary condition (20c) is used and

$$\varphi_{zz} = 0, \text{ for } d < z < h, \tag{28}$$

since $\bar{c}_0 = c_0$ and ρ_0 are constants and ω_g is zero in (9). Requiring continuity of the solution and its first derivative (dynamic boundary condition) gives

$$\varphi(z) = \begin{cases} (-1)^{s-1}\, M \exp(z/2H)\ \sin(r_s z), & \text{for } 0 \le z \le d, \\ (-1)^{s-1}\, M\, \dfrac{(h - jc_0^2/g - z)}{(h - jc_0^2/g - d)} \exp(d/2H)\ \sin(r_s d), & \text{for } d \le z \le h, \end{cases} \tag{29a}$$

where

$$c_0^2 = N^2/[r_s^2 + (4H^2)^{-1}], \tag{29b}$$

and

$$\tan(r_s d) = -\, r_s(h - jc_0^2/g - d)\left[1 + \frac{h - j\,c_0^2/g - d}{2H}\right]^{-1},\ s = 1,2,3,\ldots\,. \tag{29c}$$

Note that as for case I the compressibility of the fluid only changes the definition of the Brunt–Väisälä frequency in the eigenvalue problem.

In order to compare our results for a compressible fluid with those of Gear & Grimshaw [1983] for an incompressible fluid and to estimate in that way the importance of the compressibility when studying solitary wave propagation, especially in the atmosphere, we shall use the Boussinesq approximation. This is done although corrections due to non–Boussinesq terms may be of the same order of magnitude as corrections due to the compressibility of the fluid. In the Boussinesq approximation (29) becomes [cf. Gear & Grimshaw 1983]

$$\varphi(z) = \begin{cases} (-1)^{s-1} \sin(r_s z), & \text{for } 0 \le z \le d, \\ (-1)^{s-1} \dfrac{(h-z)}{(h-d)} \sin(r_s d), & \text{for } d \le z \le h, \end{cases} \tag{30a}$$

where

$$c_0^2 = N^2 r_s^{-2}, \tag{30b}$$

and

$$\tan(r_s d) = -\, r_s(h-d),\ s = 1,2,3,\ldots\,. \tag{30c}$$

Next, from (17) and (30) we can evaluate λ, μ and ν to find c_1 and ℓ from (2c) to study the correction due to the compressibility to them. Because we already use the Boussinesq approximation, which is correct only if h/H is much less than one, and to obtain expressions of reasonable size for c_1 and ℓ, we neglect terms of $\mathcal{O}(c_0^2/c_s^2)$, $\mathcal{O}[(gh/c_s^2)^2]$ and $\mathcal{O}[(gh/c_s^2)^3]$. Equation (17) then becomes (18) with $\varphi(z)$ given by (30a). We find that

$$\frac{hc_1}{ac_0} = \frac{2(-1)^{s-1}r_s^2(h-d)^2\{\sin(r_s d)-(7g/6r_s c_s)[1+\cos^2(r_s d)/2\cos^2(r_s d/2)]\}}{3[1+r_s^2(h-d)^2 d/h-(2g/hc_s^2)(h-d)^2]}, \tag{31a}$$

$$\frac{a\ell^2}{h^3} = \frac{3+r_s^2(h-d)^2(2+d/h)}{(-1)^{s-1}r_s^4 h^2(h-d)^2\{\sin(r_s d)-(7g/6r_s c_s^2)[1+\cos^2(r_s d)/2\cos^2(r_s d/2)]\}}, \tag{31b}$$

which for $c_s \to \infty$ reduces to (56) of Gear & Grimshaw [1983]. With these expressions we have calculated hc_1/ac_0 and $a\ell^2/h^3$ for several values of d/h for the modes $s = 1,2$ and for $h/H = 0.2$ and $h/H = 0.5$ (Table II). Comparison with the results for an incompressible fluid shows that differences, although less important than in case I, are appreciable. Corrections due to the compressibility are more important for smaller s and larger h/H and d/H. For an incompressible fluid the solitary waves are always waves of elevation since the amplitude a is always positive, as can be seen from (31) with $c_s \to \infty$ and from Table II. The inversion layer $z = d$ is displaced upwards by these waves. However, for a compressible fluid the waves are waves of depression if $s = 1$ and $d/h \to 1$. This is due to the fact that for an incompressible isothermal layer of fluid $\mu \to 0$ if the Boussinesq approximation is used [see case I]. So, when $d \to h$ the inversion layer in this case becomes just an isothermal uniformly stratified layer of fluid and the only terms contributing to μ are the ones due to the compressibility.

We can conclude this case with a remark on the two–layer fluid [case IV in the paper by Gear & Grimshaw 1983]. The two–layer fluid is a typical incompressible model since both layers of fluid are assumed to have constant densities, being equivalent with the assumption that $H = c_s^2/\gamma g \to \infty$ for both layers. If the fluid was assumed to be compressible, i.e. $\gamma = \mathcal{O}(1)$, this would imply $g/c_s^2 \to 0$ and the coefficients in the Korteweg–de Vries equation are the same as for an incompressible fluid.

	d/h	s = 1: $10^2(hc_1/ac_0)$	s = 1: $10^2(a\ell^2/h^3)$	s = 2: $10^2(hc_1/ac_0)$	s = 2: $10^3(a\ell^2/h^3)$
$\frac{h}{H}=0.2$	0.1	632 (636)	0.797 (0.789)	666 (663)	0.934 (0.937)
	0.3	178 (182)	7.24 (7.02)	218 (216)	9.01 (9.11)
	0.5	76.1 (80.5)	23.3 (21.9)	123 (121)	27.3 (27.7)
	0.7	23.0 (28.1)	86.7 (70.5)	69.0 (67.0)	65.8 (67.6)
	0.9	–4.82 (1.83)	–421 (1110)	11.3 (10.6)	447 (478)
$\frac{h}{H}=0.5$	0.1	625 (636)	0.810 (0.789)	669 (663)	0.929 (0.937)
	0.3	171 (182)	7.60 (7.02)	222 (216)	8.88 (9.11)
	0.5	69.4 (80.5)	25.9 (21.9)	127 (121)	26.6 (27.7)
	0.7	15.2 (28.1)	133 (70.5)	72.0 (67.0)	63.2 (67.6)
	0.9	–14.8 (1.83)	–137 (1110)	12.4 (10.6)	408 (478)

TABLE II hc_1/ac_0 *and* $a\ell^2/h^3$ *as a function of* d/h *for the modes* s = *1,2 and for* h/H = *0.2 and* h/H = *0.5 in case II. The numbers between parenthesis are for an incompressible fluid. For the ratio of specific heats we used* $\gamma = 1.4$.

4.3.3 CASE III: Isothermal shear layer: $u_0 = -kz$, $\omega_g^2 = N^2$

Here k and N are constants, so the background state is that of a linear shear flow in an isothermal stratified fluid. We will assume that the Richardson number N^2/k^2 is greater that $\frac{1}{4}$ so that the background state is stable. We will also assume that there are no critical layers so that $\bar{c}_0 \neq 0$ for $0 \leq z \leq h$. In the first two examples the compressibility of the fluid did not, aside from a redefinition of the Brunt–Väisälä frequency, alter the eigenvalue problem materially. After redefining this frequency the linear problems for compressible and incompressible fluids were equivalent. Significant consequences of the compressibility were found in the next order, i.e. when evaluating the coefficients of the Korteweg–de Vries equation. However, in the present example we will show that also the eigenvalue problem, and thus the eigenvalues c_0 and eigenfunction $\varphi(z)$, can change considerably due to the compressibility of the fluid. Note that we do not use the Boussinesq approximation and do not assume $\bar{c}_0^2$ to be much smaller than c_s^2. Since the fluid is isothermal c_s is a constant and the eigenvalue problem (8) becomes

$$(\rho_0 \bar{c}_0^2 \varphi_z)_z + \rho_0 N^2 \varphi + \rho_0 \frac{(\bar{c}_0^2)_z}{c_s^2 - \bar{c}_0^2} (\bar{c}_0^2 \varphi_z - g\varphi) = 0, \text{ for } 0 < z < h, \tag{32a}$$

with boundary conditions

$$\varphi = 0, \text{ for } z = 0 \tag{32b}$$
$$\varphi = j\bar{c}_0^2 \varphi_z / g, \text{ for } z = h. \tag{32c}$$

We can write (32a) like

$$\tilde{\varphi}_{yy} + [q_0(y) + q_1(y)]\tilde{\varphi} = 0, \text{ for } 0 < z < h, \tag{33a}$$

where

$$y = 1 + kz/c_0, \tag{33b}$$
$$\tilde{\varphi}(y) = y(1-a_4y^2)^{-1/2} \exp(-a_1y/2)\, \varphi(y), \tag{33c}$$
$$q_0(y) = a_2/y^2, \tag{33d}$$
$$q_1(y) = a_1y^{-1}(1-a_1y/4) + y^{-1}(1-a_4y^2)^{-1}(a_1a_4y^2-a_3) - 3a_4(1-a_4y^2)^{-2}, \tag{33e}$$
$$a_1 = c_0/(Hk), \tag{33f}$$
$$a_2 = N^2/k^2, \tag{33g}$$
$$a_3 = 2gc_0/(kc_s^2), \tag{33h}$$
$$a_4 = c_0^2/c_s^2. \tag{33i}$$

If $\bar{c}_0^2 \ll c_s^2$, $a_4 \to 0$ and solutions of (33a) can be found in terms of Whittaker functions [see e.g. Kamke 1956]. In the Boussinesq approximation $a_1 = 0$, and in the incompressible limit both a_3 and a_4 are zero. If a_1, a_3, and a_4 are zero, $q_1(y) \equiv 0$ and the solution of (33a) satisfying boundary condition (32b), i.e. $\tilde{\varphi}(1) = 0$, is given by

$$\tilde{\varphi}(y) = M\, y^{1/2} \sin[r \ln(y)], \tag{34}$$

or

$$\varphi(y) = M\, y^{-1/2} \sin[r \ln(y)], \tag{35a}$$

where

$$r^2 = a_2 - \tfrac{1}{4} = N^2/k^2 - \tfrac{1}{4}, \tag{35b}$$

and the upper boundary condition (32c) determines the eigenvalues c_0. An approximate solution of (33a), if $a_2 \gg 1$ and a_1/a_2, a_3/a_2, and $a_4/a_2 \ll 1$, is [Nayfeh 1973]

$$\tilde{\varphi}(y) = L\, q_0^{-1/4} \{\sin[\tilde{\beta}(y)] + P \cos[\tilde{\beta}(y)]\}, \tag{36a}$$

where

$$\begin{aligned}\tilde{\beta}(y) &= \int^y q_0^{1/2}(1+q_1/2q_0)dy \\ &= a_2^{1/2}\ln(y) + \tfrac{1}{2}a_2^{-1/2}\Big\{-a_1^2y^2/8 - \tfrac{3}{2}a_4y^2(1-a_4y^2)^{-1} \\ &\qquad + \tfrac{1}{2}a_4^{-1/2}(a_1-a_3)\ln\Big[\Big(\frac{1+a_4^{1/2}y}{1-a_4^{1/2}y}\Big)\Big]\Big\}.\end{aligned} \tag{36b}$$

The lower boundary condition (32b) yields

$$P = -\tan[\tilde{\beta}(1)]. \tag{36c}$$

By comparing (36) with (35a) or by substitution of (36) in (33), one finds that an even better approximation is obtained with

$$\varphi(y) = M\, y^{-1/2} \exp(a_1y/2)\, \{\sin[\beta(y)] - \tan[\beta(1)]\cos[\beta(y)]\}, \tag{37a}$$

$$\beta(y) = r\ln(y) + \tfrac{1}{2}a_2^{-1/2}\Big\{-a_1^2y^2/8 - \tfrac{3}{2}a_4y^2(1-a_4y^2)^{-1} + \tfrac{1}{2}a_4^{-1/2}(a_1-a_3)\ln\Big[\Big(\frac{1+a_4^{1/2}y}{1-a_4^{1/2}y}\Big)\Big]\Big\}. \tag{37b}$$

The condition $a_2 \gg 1$ can then be replaced by $a_2 \geq 1$, while the conditions a_1/a_2, a_3/a_2, and $a_4/a_2 \ll 1$ still apply.

The eigenvalues c_0 are determined by the upper boundary condition

(32c). If the fluid is incompressible and the Boussinesq approximation is used $a_1 = a_3 = a_4 = j = 0$ and we find [Gear & Grimshaw 1983, equation (38b)]:

$$c_0 = kh[\exp(s\pi/r)-1]^{-1}, \ |s| = 1,2,3,\dots \,. \tag{38}$$

From (34) we see that $\frac{a_1}{a_2}$ and $\frac{a_3}{a_2}$ are $\mathcal{O}(\frac{c_0 k}{g})$ and that $\frac{a_4}{a_2}$ is $\mathcal{O}[(\frac{c_0 k}{g})^2]$. From (38) we find that for $|s|/r \geq 1$ the phase velocity c_0 is of the order $\mathcal{O}(kh)$ or smaller. Therefore the conditions $a_1/a_2 \ll 1$, $a_3/a_2 \ll 1$, and $a_4/a_2 \ll 1$ for the approximate solution (36a) to be valid require $k^2 \ll g/h$, a condition that is always satisfied under practical circumstances.

If a_1, a_3, and a_4 are non–zero, the eigenvalues c_0 can be determined using an iterative procedure. From (32c), (33b), and (37) we find

$$\breve{h} = G(\breve{h}), \tag{39a}$$

where

$$\breve{h} = 1 + kh/c_0, \tag{39b}$$

$$\begin{aligned} G(\breve{h}) = \exp\Bigg\{ & \frac{1}{r}\arctan\Bigg\{\tan[\beta(1)] + \frac{jkc_0}{g}\breve{h}\left[1+\frac{kc_0}{2g}\breve{h}(1-a_1\breve{h})\right]^{-1} \\ & \times \left[r + \tfrac{1}{4}a_2^{-1/2}\Big[-a_1\breve{h}^2/2-6a_4\breve{h}^2(1-a_4\breve{h}^2)^{-1}-6a_4^2\breve{h}^4(1-a_4\breve{h}^2)^{-2} \right. \\ & \left. +2\breve{h}(a_1-a_3)(1-a_4\breve{h}^2)^{-1}\Big]\right] \Big[1+\tan[\beta(1)]\tan[\beta(\breve{h})]\Big]\Bigg\} \\ & + (4ra_2^{1/2})^{-1}\left\{a_1^2\breve{h}^2/4+3a_4\breve{h}^2(1-a_4\breve{h}^2)^{-1}+a_4^{-1/2}(a_1-a_3)\ln\left[\frac{(1-a_4^{1/2}\breve{h})}{(1-a^{1/2}\breve{h})}\right]\right\} \\ & + s\pi/r\Bigg\}. \end{aligned} \tag{39c}$$

With a simple iterative procedure like the one discussed in case I we find the eigenvalues c_0. We have calculated c_0/hk if $r = 1$ for several values of h/H for the modes $s = 1,2$ and $j = 0,1$ (Table III). This was also done if $a_3 = 0$ (Boussinesq approximation) and if $a_1 = a_3 = a_4 = 0$ (incompressible fluid in the Boussinesq approximation).

h/H	s = 1		s = 2	
	j = 0	j = 1	j = 0	j = 1
0.25	425 (452) [502]	398 (424) [471]	17.6 (18.7) [20.7]	16.6 (17.6) [19.5]
0.5	389 (452) [546]	343 (396) [478]	16.3 (18.7) [22.5]	14.5 (16.6) [20.0]
1	297 (452) [598]	262 (340) [461]	12.7 (18.7) [24.9]	11.1 (14.4) [19.7]
1.5	183 (452) [556]	321 (285) [484]	7.99 (18.7) [24.3]	12.9 (12.3) [19.9]

TABLE III *The dimensionless phase velocity* $10^4(c_0h/k)$ *for* $r = 1$, $s = 1,2$, $j = 0,1$ *and* $h/H = 0.25$, *0.5, 1, 1.5, calculated from (39). The numbers between parenthesis are for* $a_1 = a_3 = a_4 = 0$, *i.e., for an incompressible fluid in the Boussinesq approximation. The numbers between brackets are for* $a_3 = 0$, *i.e., for a compressible fluid in the Boussinesq approximation. For the ratio of specific heats we used 1.4.*

From Table III we see that corrections for the compressibility tend to make the phase velocity larger, while *not* making the Boussinesq approximation makes the phase velocity smaller. The magnitude of the corrections entirely due to the compressibility can be inferred from comparison of the numbers between parenthesis and brackets.

4.4 SUMMARY AND CONCLUSIONS

In this chapter we have studied the effects of the compressibility of a density–stratified shallow fluid on nonlinear long solitary gravity waves that are described by the Korteweg–de Vries equation and we compared our results with those for an incompressible fluid. To that end, starting from two integrals of motion for the compressible inviscid fluid, we performed a rigorous derivation of the eigenvalue problem for the modal function which describes the vertical structure of the wave phenomenon.

To first order in the wave amplitude we found, just as in the incompressible case, the Korteweg–de Vries equation of which the coefficients are given in terms of integrals of the modal function. In the limit of incompressibility these coefficients reduce to the values that have been given previously in the literature for an incompressible fluid [Benney 1966, Benjamin 1966, Grimshaw 1983]. However, they differ from those found by Grimshaw [1980/1981] who also treated the compressible case. The reason for this is that our approach is more general in that we do include compressibility fully and furthermore we do not apply the Boussinesq approximation.

The modification of the coefficients in the Korteweg–de Vries equation have been studied in some detail for three special cases.

For an isothermal fluid with a constant Brunt–Väisälä frequency and without shear these modifications are most pronounced. Assuming a rigid upper boundary of the fluid, not only the wavelength of the solitary wave becomes approximately twice as small as in the incompressible case but also the correction to the long wave phase speed becomes approximately four times larger. Furthermore it turned out, again assuming a rigid upper boundary, that the inclusion of compressibility results in a change of sign of the coefficient in front of the nonlinear term in the Korteweg–de Vries equation thus causing a solitary wave of elevation to become one of depression. The importance of the effects of the compressibility in this example is due to the fact that the coefficient in front of the nonlinear term in the Korteweg–de Vries equation becomes zero in the Boussinesq approximation if the fluid is incompressible.

For the case of an inversion layer the corrections due to the compressibility, though less pronounced than in the preceding case, are still appreciable. The most significant effects are found for the lowest modal modes, that is for the modal modes with the smallest vertical wavenumber, and for larger inversion heights.

Finally, for the case of an isothermal shear layer important changes are found in the phase speeds and modal functions, i.e., the eigenvalues and eigenfunctions, respectively, of the eigenvalue problem, thus giving rise to considerable changes in the coefficients of the Korteweg–de Vries equation.

REFERENCES

Benjamin, T.B. (1966). Internal waves of finite amplitude and permanent form, *J. Fluid Mech.* **25**, 241–270.

Benjamin, T.B. (1967). Internal waves of permanent form in fluids of great depth, *J. Fluid Mech.* **29**, 559–592.

Benney, D.J. (1966). Long nonlinear waves in fluids, *J. Math. Phys.* **45**, 52–63. a shallow fluid, *Phys. Fluids* **26**, 14–29.

Gear, J.A., Grimshaw, R.H.J. (1983). A second order theory for solitary waves in a shallow fluid, *Phys. Fluids* **26**, 14–29.

Gill, A.E. (1982). "Atmosphere–Ocean dynamics." Academic Press, New York.

Gradshteyn, I.S., Ryzhik, I.M. (1965). "Table of integrals, series, and products." Academic Press, New York, 129.

Grimshaw, R.H.J. (1980/1981). Solitary waves in a compressible fluid, *Pure Appl. Geophys.* **119**, 780–797.

Grimshaw, R.H.J. (1981). A second–order theory for solitary waves in deep fluids, *Phys. Fluids* **24**, 1611–1618.

Ince, E.L. (1956). "Ordinary differential equations." Dover Publications, New York, 122.

Kamke, E. (1956). "Differentialgleichungen, Lösungsmethoden und Lösungen." Akademische Verlagsgesellschaft, Leipzig, 451.

Lin, Y.L., Goff, R.C. (1988). A study of a mesoscale solitary wave in the atmosphere originating near a region of deep convection, *J. Atmos. Sci* **45**, 194–205.

Liu, A.K., Holbrook, J.R., Apel, J.R. (1985). Nonlinear internal wave evolution in the Sulu sea, *J. Phys. Ocean.* **15**, 1613–1624.

Long, R.R. (1965). On the Boussinesq approximation and its role in the theory of internal waves, *Tellus* **17**, 46–52

Long, R.R., Morton, J.B. (1966). Solitary waves in compressible stratified fluids, *Tellus* **18**, 79–85.

Nayfeh A., (1973). "Perturbation Methods." John Wiley and Sons, New York, 314.

Pecnik, M.J., Young, J.A. (1984). Mechanics of a strong subsynoptic gravity wave deduced from satellite and surface observations, *J. Atmos. Sci.* **41**, 1850–1862.

Shen, M.C. (1966). Solitary waves in an atmosphere with arbitrary wind and density profiles, *Phys. Fluids* **9**, 1944–1950.

Shen, M.C. (1967). On long waves with cross–wind in an atmosphere, *J. Fluid Mech.* **28**, 481–492.

Shen, M.C., Keller, J.B. (1973). Ray method for nonlinear wave propagation in a rotating fluid of variable depth, *Phys. Fluids* **16**, 1565–1572.

Stobie, J.G., Einaudi, F., Uccellini, L.W. (1983). A case study of gravity waves–convective storms interaction, *J. Atmos. Sci.* **40**, 2804–2830. permanent form, *Studies Appl. Math.* **66**, 1–44.

Yih, C.–S. (1980). "Stratified Flows." Academic Press, New York.

CHAPTER 5

INTERNAL SOLITARY WAVES IN COMPRESSIBLE DEEP FLUIDS

5.1 INTRODUCTION

Long internal solitary waves in shallow stratified fluids have been studied comprehensively [Benney 1966, Benjamin 1966, Gear & Grimshaw 1983, Miesen et al. 1990, chapter 4]. To the first order in the wave amplitude these waves are described by the Korteweg–de Vries equation. A well–known solution of this equation has the characteristic "sech^2" profile, phase speeds that vary linearly with the wave amplitude and wavelengths that vary inversely with the square root of the wave amplitude. However, in deep fluids, where the total depth of the fluid is much larger than the horizontal scale of the waves, solitary waves are algebraic [Benjamin 1967, Davis & Acrivos 1967]. These algebraic solitary waves have phase speeds that also vary linearly with the wave amplitude but their wavelengths vary inversely with the wave amplitude (and not with the square root of the wave amplitude like for solitary waves in shallow fluids). The algebraic solitary wave is a solution of the Benjamin–Davis–Ono equation, which describes the balance between dispersion and wave steepening due to weak nonlinear effects, for long internal waves in a density stratified layer of fluid that is confined in an infinitely deep fluid. The waves are long compared to the typical scale of the stratification.

As far as the author knows, the changes of the deep fluid solitary waves due to the compressibility have only been discussed by Grimshaw [1980/1981]. He distinguishes two dimensionless parameters measuring the effects of the Boussinesq approximation and the compressibility and studies solitary waves in a compressible fluid, correct to first order in these parameters, using the equations of motion in Lagrangian coordinates. Furthermore an approximation concerning the profile of the velocity of sound is used [Grimshaw 1980/1981, Miesen et al. 1990].

In this chapter a theory for solitary waves in a fully compressible fluid is presented, related closely to the theory for solitary waves in incompressible deep fluids as presented by Grimshaw [1981] and to the theory for solitary waves in both incompressible [Gear & Grimshaw 1983] and compressible [Miesen et al. 1990] shallow fluids. In the limit of incompressibility our results coincide with the results of Grimshaw [1981] and Christie [1989] for incompressible fluids. As we will see, for a compressible fluid with a ratio of specific heats $\gamma = \mathcal{O}(1)$, the dimensionless parameter h/H_c measures both the effects of the Boussinesq approximation and the compressibility. Here h is a scale typical for the vertical structure of the wave and H_c is a scale typical for the stratification of the fluid. For the lower atmosphere H_c lies between 5 km and 9 km. In some cases, e.g. Christie, Muirhead & Hales [1978] where $h/H_c < 1/10$, compressibility can be neglected and the Boussinesq approximation can be made with small errors. In other cases, e.g. Noonan & Smith [1985] where $0.2 < h/H_c < 0.4$, important changes due to the compressibility and the inadequacy of the Boussinesq approximation must be expected. In the three special cases in the present paper we will concentrate on the effects of compressibility since the validity of the Boussinesq approximation can also be studied from incompressible theory. Note, however, that significance of the compressibility and the invalidity of the Boussinesq approximation go together. Unfortunately, not using the Boussinesq approximation while incorporating compressibility would require a numerical approach for all but the simplest cases.

Before we proceed with compressible fluids, the theory for solitary waves in incompressible deep fluids, correct to the first order in the wave amplitude, is reviewed briefly. An inviscid, incompressible fluid is considered for which there is a background density profile $\rho_0(z)$ and a background velocity profile $u_0(z)$, bounded below by the rigid boundary $z = 0$, and such that $\rho_0(z) \to$ constant and $u_0(z) \to 0$ as $z \to \infty$. Here, z is a (Lagrangian) coordinate and x will be a horizontal coordinate in a frame moving with the phase speed c of the wave. Furthermore, it is assumed that the basic state of the fluid is stable, i.e. the fluid is stably stratified and the Richardson number is larger than $\frac{1}{4}$. Weakly nonlinear long waves in deep fluids are characterised by the equality of two small parameters α and $\epsilon = h/\ell$, where α is a measure of the amplitude of the vertical displacement η_1 of a streamline due to the solitary wave (Fig. 18) and ℓ is the horizontal scale of the wave. Again, like in chapter 4, the equality of these parameters represents the balance between nonlinearity and dispersion, measured by α and ϵ, respectively. Let $\eta_1(X,z) = \alpha A(X)\varphi(z)$, where $X = \epsilon x$, and

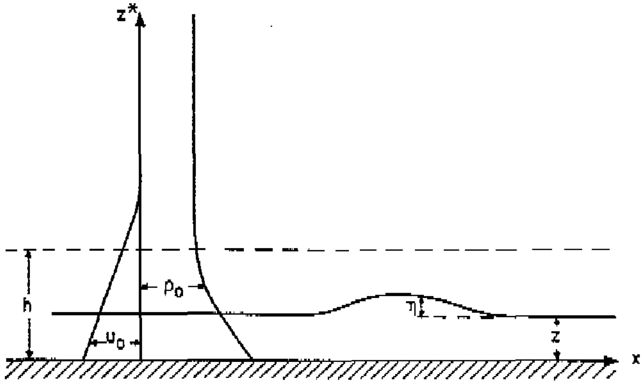


Fig. 18 The geometry of the problem: a compressible flow, bounded from below by a rigid boundary at z = 0, *is steady in a frame moving horizontally , i.e. in the* x*−direction, with the phase velocity* c *of the wave. The fluid is stratified predominantly in a layer of depth* h *and has a background horizontal velocity profile* $u_0(z)$ *and a density profile* $\rho_0(z)$. *The vertical displacement* $\eta_1(x,z)$ *of a streamline is given by the difference between the Eulerian vertical coordinate* z* *and the Lagrangian vertical coordinate* z.

$\varphi(z)$ is the modal function describing the vertical structure of the wave, which is normalised to 1 at its maximum value. It satisfies the following eigenvalue problem [Grimshaw 1981]:

$$(\rho_0 \bar{c}_0^2 \varphi_z)_z + \rho_0 \omega_b^2 \varphi = 0, \ \text{for } 0 < z < \infty, \quad (1a)$$

$$\varphi = 0, \ \text{for } z = 0, \quad (1b)$$

$$\varphi_z \to 0, \text{ as } z \to \infty, \quad (1c)$$

where $\bar{c}_0 = c_0 - u_0(z)$, $\omega_b = (-g\rho_{0z}/\rho_0)^{1/2}$ is the Brunt–Väisälä frequency for an incompressible stratified fluid, and the subscripts z denote derivatives. The eigenvalues c_0 of the problem are the linear long wave phase speeds. We will assume that there are no critical levels so that $\bar{c}_0$ is not zero for $0 < z < \infty$.

From the integrated equations of motion, i.e. Long's equation [Long 1953], one finds correct to the first order in the wave amplitude the so-called Benjamin–Davis–Ono equation [Benjamin 1967, Davis & Acrivos 1967, Ono 1975]:

$$-c_1\lambda A + \tfrac{1}{2}\mu A^2 - \kappa \mathscr{B}(A) = 0, \tag{2a}$$

where

$$\mathscr{B}(A) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\tau| \exp(i\tau X)\mathscr{A}(\tau)d\tau, \tag{2b}$$

and

$$\mathscr{A}(\tau) = \int_{-\infty}^{\infty} \exp(-i\tau X)A(X)dX. \tag{2c}$$

This equation determines A(X) and the correction ϵc_1 to the linear long wave phase speed c_0. The algebraic solitary wave solution of this equation is [Benjamin 1967]

$$A(X) = a\ell^2/(X^2 + \ell^2), \tag{3a}$$

and its phase speed $c \approx c_0 + \epsilon c_1$, where

$$c_1 = \tfrac{1}{4}\mu a/\lambda, \;\; \ell a = 4\kappa/\mu\,. \tag{3b}$$

The coefficients λ, μ and κ are known in terms of the modal function $\varphi(z)$ and are given by [e.g. Grimshaw 1981]

$$\lambda = 2\int_0^{\infty}\rho_0\bar{c}_0\varphi_z^2 dz\,, \tag{4a}$$

$$\mu = 3\int_0^{\infty}\rho_0\bar{c}_0^2\varphi_z^3 dz, \tag{4b}$$

$$\kappa = (\rho_0 c_0^2\varphi^2)_\infty\,, \tag{4c}$$

where the subscript "∞" denotes a quantity evaluated as $z \to \infty$. As already noted, and as can be seen from (3b), the algebraic solitary wave (3a) has a phase speed which is linear in the amplitude a, and has a wavelength ℓ which is inversely proportional to a.

In the next section (section 5.2), starting from the equations (3.11a,b,c), the analysis is given that leads to the analogon of equations (1) to (4) for a compressible fluid. In section 5.3 we discuss three special cases that can all be regarded as models of the nocturnal inversion in the atmosphere: (I) there is no background shear flow and the Brunt–Väisälä frequency is constant throughout a layer of depth h (i.e., $0 \leq z < h$), and is zero above this layer (i.e., $z > h$); (II) there is a linear background shear flow and the Brunt–Väisälä frequency is constant throughout a layer of depth h, and is zero above this layer; (III) there is no background shear flow and the Brunt–Väisälä frequency has a "sech^2" profile. In case (II) and (III) the Boussinesq approximation is used, although, as already noted, its effects are of the same order of magnitude as the effects of the compressibility. This is done in order to estimate analytically the importance of the compressibility in these cases. Correct to the first order in the parameter h/H_c, effects, due the invalidity of the Boussinesq approximation, can just be added to the effects of the compressibility.

5.2 ANALYSIS

5.2.1 Inner expansion

First we consider the part of the fluid that is clearly stratified (figure 18) and where we can identify a characteristic vertical scale, compared to which the waves are long. The ratio of the characteristic vertical scale and a characteristic horizontal lenght scale of the wave is ϵ. We seek solutions of the equations (3.11a,b,c), which govern finite amplitude waves in inviscid compressible stratified shear flows, having the following expansions

$$\eta = \epsilon\eta_1(X,z) + \epsilon^2\eta_2(X,z) + \ldots, \quad (5a)$$
$$\delta = \epsilon\delta_1(X,z) + \epsilon^2\delta_2(X,z) + \ldots, \quad (5b)$$
$$\bar{c} = \bar{c}_0(z) + \epsilon c_1 + \epsilon^2 c_2 + \ \ldots, \quad (5c)$$

where

$$\eta_1(X,z) = A(X)\varphi(z), \quad (5d)$$
$$X = \epsilon x. \quad (5e)$$

Substitution of (5) into (3.11a,b) gives [Miesen et al. 1990, see also chapter 4.2] at leading order in ϵ

$$\left[\bar{\rho}\bar{c}_0^2 \frac{c_s^2\varphi_z - g\varphi}{c_s^2 - \bar{c}_0^2}\right]_z + \bar{\rho}\omega_g^2\varphi = 0, \tag{6}$$

or

$$(\rho_0\bar{c}_0^2\varphi_z)_z + \rho_0\omega_g^2\varphi + \rho_0 \frac{c_s^2(\bar{c}_0^2)_z - \bar{c}_0^2(c_s^2)_z}{c_s^2(c_s^2-\bar{c}_0^2)} (\bar{c}_0^2\varphi_z - g\varphi) = 0, \tag{7a}$$

with boundary condition

$$\varphi = 0, \text{ for } z = 0. \tag{7b}$$

Equations (7a,b) reduce to (1a,b) in the limit of incompressibility.

At the next order in ϵ we find [cf. Miesen et al. 1990, or chapter 4.2]

$$\left[\bar{\rho}\bar{c}_0^2 \frac{c_s^2\eta_{2z} - g\eta_2}{c_s^2 - \bar{c}_0^2}\right]_z + \bar{\rho}\omega_g^2\eta_2 + f_2 = 0, \tag{8a}$$

$$\eta_2 = 0, \text{ for } z = 0, \tag{8b}$$

and the inhomogeneous term f_2 is given by

$$\begin{aligned} f_2 = {} & [2\bar{\rho}\bar{c}_0 c_1 c_s^2(c_s^2-\bar{c}_0^2)^{-2}(c_s^2\eta_{1z}-g\eta_1)]_z \\ & - \Big\{\bar{\rho}\bar{c}_0^2\Big[\tfrac{3}{2}\eta_{1z}^2 + (c_s^2-\bar{c}_0^2)^{-3}\Big[\bar{c}_0^2[\tfrac{1}{2}c_s^4 + (\tfrac{1}{2}\sigma^{-1}-4)c_s^2\bar{c}_0^2 + \tfrac{3}{2}\bar{c}_0^4]\eta_{1z}^2 \\ & - gc_s^2(2c_s^2+\bar{c}_0^2/\sigma)\eta_1\eta_{1z} + g^2c_s^2(1+\tfrac{1}{2}\sigma^{-1})\eta_1^2\Big]\Big]\Big\}_z. \end{aligned} \tag{8c}$$

A solution of (8) can be found by the method of variation of parameters

$$\eta_2 = \varphi\int_0^z \frac{\chi f_2}{\Delta a_0}\,dz - \chi\int_0^z \frac{\varphi f_2}{\Delta a_0}\,dz \tag{9}$$

Here $\chi(z)$, like $\varphi(z)$, is a solution of (6), linearly independent of $\varphi(z)$. The

Wronskian of the solutions φ and χ is defined as

$$\Delta = \varphi\chi_z - \varphi_z\chi, \tag{10a}$$

and can be calculated using Abel's identity [Ince 1956]

$$\Delta = \Delta_0 \exp\left[-\int_0^z (a_1/a_0)dz\right]. \tag{10b}$$

where a_0, a_1 are the coefficients of the second and first derivative in (8a), respectively. From (10) we find

$$\varphi\chi_z - \varphi_z\chi = [\bar{\rho}\bar{c}_0^2 c_s^2(c_s^2 - \bar{c}_0^2)^{-1}]^{-1} \exp\left[\int_0^z (g/c_s^2)dz\right]. \tag{11}$$

Until now the correction c_1 to the linear phase velocity c_0 and the amplitude $A(X)$ are undetermined. They are determined by constructing an outer expansion, valid in the region $z \gg h$ where z scales, like x, with ϵ^{-1}, and matching this outer expansion with the inner expansion obtained above. This matching will also yield a second boundary condition for (7a) and (18). It is useful to anticipate that matching at leading order in ϵ will give condition (1c) as the second boundary condition for (7a). The equations (7a,b) and (1c) are an eigenvalue problem with eigenvalues c_0 and eigenfunctions or modes $\varphi(z)$. One of these modes with eigenvalue c_0 is considered. To achieve the matching, the properties of φ and χ as $z \to \infty$ must be determined. From (11) together with (1c) it follows that

$$\chi \approx z/(\rho_0 c_0^2 \varphi)_\infty + B_1, \text{ as } z \to \infty, \tag{12}$$

where B_1 is an arbitrary constant and we used that [Appendix F]

$$\rho_0(z) = \bar{\rho}(z) \exp\left[-\int_0^z (g/c_s^2)dz\right]. \tag{13}$$

The subscript ∞ denotes evaluation of the quantity as $z \to \infty$. Substitution of (12) into (9) gives

$$\eta_2 \approx -\frac{z}{(\bar{\rho}c_\delta^2\phi)_\infty}\int_0^\infty \frac{\varphi f_2}{\Delta a_0}\,dz + B_2, \tag{14a}$$

where the constant B_2 is given by

$$B_2 = \varphi(\infty)\int_0^\infty \frac{\chi f_2}{\Delta a_0}\,dz - B_1\int_0^\infty \frac{\varphi f_2}{\Delta a_0}\,dz. \tag{14b}$$

5.2.1 Outer expansion and matching

In the region for $z \to \infty$ we use the following scaling of variables

$$X = \epsilon x, \tag{15a}$$
$$Z = \epsilon z, \tag{15b}$$
$$\eta = \epsilon\hat{\eta}(X,Z). \tag{15c}$$

The scaling of x and η is consistent with the fact that the waves in the lower and the upper part of the fluid are coupled. Since the upper part of the fluid is homogeneous and of infinite thickness there is no characteristic scale for z, so that spatial scales in the vertical will be roughly equal to those in the horizontal [Doviak & Chen 1988]. Therefore z is scaled like x.

From (3.11b) we find that $\delta \to 0$ as $z \to \infty$, since $c_s \to \infty$ as $z \to \infty$. This is so because $\rho_0 \to$ constant as $z \to \infty$ which implies that the scaleheight of the fluid becomes infinitely large; for a compressible fluid that means that $c_s \to \infty$. Equations (3.11a) and (15) then give

$$\hat{\eta}_{XX} + \hat{\eta}_{ZZ} + \mathcal{O}(\epsilon^2) = 0, \tag{16}$$

where we used that $\omega_g \to 0$ and $\bar{c}_0 \to$ constant as $z \to \infty$. The solution of this equation that satisfies boundary condition (1c), i.e. $\hat{\eta} \to 0$ as $z \to \infty$, reads

$$\hat{\eta} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(i\tau X - |\tau|Z)\,\hat{\mathcal{A}}(\tau)\,d\tau + \mathcal{O}(\epsilon^2), \tag{17a}$$

where $\hat{\mathscr{A}}(\tau)$ is the Fourier transform of $\hat{A}$ [see (2c)] and

$$\hat{A}(X) := \hat{\eta}(X,0). \tag{17b}$$

Substitution of $Z = \epsilon z$ and expansion of (17a) in powers of ϵ gives with (15c)

$$\eta = \epsilon\hat{\eta}(X,\epsilon z) = \epsilon\hat{A} - \epsilon^2\mathscr{B}(\hat{A}) + \ldots, \tag{18a}$$

where

$$\mathscr{B}(\hat{A}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}|\tau|\exp(i\tau X)\,\hat{\mathscr{A}}(\tau)d\tau. \tag{18b}$$

Matching (18a) with the inner solution (5a,d) as $z \to \infty$, where η_2 is given by (14a), gives

$$\varphi \to \text{constant, as } z \to \infty, \tag{19a}$$

which is equivalent to (1c), and

$$\hat{A} = A\varphi(\infty), \tag{19b}$$

$$\mathscr{B}(\hat{A}) = \frac{1}{(\rho_0 c_0^2\varphi)_\infty}\int_0^{\infty}\frac{\phi f_2}{\Delta a_0}\,dz. \tag{19c}$$

This, together with (8a), (10a) and (11) gives

$$(\rho_0 c_0^2\varphi^2)_\infty\mathscr{B}(A) = \int_0^{\infty}\exp\left[-\int_0^{z}(g/c_0^2)dz\right]\varphi f_2\,dz. \tag{20}$$

Partial integration of the right-hand side of (20) gives, together with the boundary conditions (7b) and (19a) and equation (13),

$$-\lambda c_1 A + \tfrac{1}{2}\mu A^2 - \kappa\mathscr{B}(A) = 0, \tag{21}$$

where in this case λ, μ, and κ are given by

$$\lambda = \int_0^\infty 2\rho_0\bar{c}_0(c_s^2-\bar{c}_0^2)^{-2}(c_s^2\varphi_z-g\varphi)^2\, dz, \tag{22a}$$

$$\mu = \int_0^\infty \rho_0\bar{c}_0^2\Big\{3\varphi_z^2 + (c_s^2-\bar{c}_0^2)^{-3}\Big[\bar{c}_0^2[7c_s^4 + (1/\sigma-8)\bar{c}_0^2c_s^2 + 3\bar{c}_0^4]\varphi_z^2 - 2gc_s^2(2c_s^2+\bar{c}_0^2/\sigma)\varphi\varphi_z + g^2c_s^2(2+1/\sigma)\varphi^2\Big]\Big\}(\varphi_z-g\varphi/c_s^2)\, dz \tag{22b}$$

$$\kappa = (\rho_0c_0^2\varphi^2)_\infty. \tag{22c}$$

Since $\bar{c}_0^2/c_s^2 = \mathcal{O}(h^2/H_c^2)$ [see section 4.2] and $gh/c_s^2 = \mathcal{O}(h/H_c)$, (22) gives correct to $\mathcal{O}(h/H_c)$

$$\lambda = \int_0^\infty 2\rho_0\bar{c}_0(\varphi_z^2 - 2g\varphi\varphi_z/c_s^2)\, dz + \mathcal{O}(h^2/H_c^2), \tag{23a}$$

$$\mu = \int_0^\infty \rho_0\bar{c}_0^2(3\varphi_z^3 - 7g\varphi\varphi_z^2/c_s^2)dz + \mathcal{O}(h^2/H_c^2), \tag{23b}$$

$$\kappa = (\rho_0c_0^2\varphi^2)_\infty, \tag{23c}$$

where we used $1/\sigma = \gamma - 1$. For an incompressible fluid (i.e., $c_s \to \infty$ or $\gamma \to \infty$) (23a,b) reduce to (4a,b). A solution of (21) is given by (3a), as shown by Benjamin [1967], and c_1 and ℓ are given by (3b).

5.3 DISCUSSION

5.3.1 Case I: Constant ω_g layer, $u_0 = 0$.

For this case we assume that the background velocity $u_0 = 0$ and that the Brunt–Väisälä frequency ω_g is constant throughout a layer of depth h (i.e. $0 \le z \le h$), and is zero above this layer (i.e. $z > h$): $\omega_g^2 = N^2H(h-z)$. Here, H is the Heaviside function. If ω_g is constant, and the fluid is an ideal gas, the fluid must be isothermal and c_s is constant. If ω_g is zero the fluid must be homogeneous, i.e. the scaleheight of the fluid is infinite and c_s is infinite too. For this fluid configuration the density is given by

$$\rho_0(z) = \begin{cases} \rho_c \exp(-z/H), \text{ for } 0 \le z \le h, \\ \rho_c \exp(-h/H), \text{ for } z > h, \end{cases} \tag{24}$$

where H is the scaleheight of an isothermal fluid. For such an isothermal fluid the Brunt–Väisälä frequency and the velocity of sound, in terms of H, are given by

$$N^2 = \frac{g(\gamma-1)}{\gamma H}, \tag{25a}$$

$$c_s^2 = \gamma g H. \tag{25b}$$

The eigenvalue problem (7a,b), (19a) then becomes

$$(\rho_0 c_0^2 \varphi_z)_z + \rho_0 N^2 H(h-z)\varphi = 0, \text{ for } 0 < z < \infty, \tag{26a}$$
$$\varphi = 0, \text{ for } z = 0, \tag{26b}$$
$$\varphi_z \to 0, \text{ as } z \to \infty, \tag{26c}$$

which is equivalent to the eigenvalue problem for an incompressible fluid [Grimshaw 1981], with only the Brunt–Väisälä frequency defined differently.

Equations (26a,b,c) have the solution

$$\varphi(z) = \begin{cases} M\, \rho_0^{-1/2} \sin(\gamma_s z), \text{ for } 0 \le z \le h,\ s = 0,1,2,\ldots, \\ 1, \qquad\qquad \text{for } z > h, \end{cases} \tag{27a}$$

$$c_0^2 = N^2/[\gamma_s^2 + (2H)^{-2}]. \tag{27b}$$

The dynamic boundary condition [e.g. Benjamin 1966] requires continuity of the first derivative of φ at $z = h$, which gives

$$\gamma_s h = \pi(s+\tfrac{1}{2}) + \arctan\left[\frac{1}{2H\gamma_s}\right],\ s = 0,1,2,\ldots . \tag{27c}$$

M is a constant chosen so that the maximum value of $\varphi(z)$, attained at $z = h$, is 1:

$$M = \rho_c^{1/2} \exp[-h/(2H)] \sin(\gamma_s h). \tag{27d}$$

If $\beta = h/H$ is small, (27c) can be approximated by

$$\gamma_s h = r_s \left[1 + \frac{h}{2Hr_s^2} + \mathcal{O}(\beta^2)\right], \tag{28}$$

where

$$r_s = \pi(s+\tfrac{1}{2}). \tag{29}$$

Now λ, μ and κ will be evaluated. Correct to $\mathcal{O}(\beta)$ we find from (23a,b), (24) and (27a,b,c) that

$$\frac{\lambda h}{2c_0 M^2} = \tfrac{1}{2}r_s^2 + \frac{3h}{4H} - \frac{gh}{c_s^2} + \mathcal{O}(\beta^2), \tag{30a}$$

$$\frac{(-1)^s \mu h^2 \rho_c^{1/2}}{c_s^2 M^3} = 2r_s^2 + \frac{h}{H}\left[2 + r_s^2 + (-1)^s \frac{r_s}{3}\right] - \tfrac{2}{3}(-1)^s \left[\frac{gh}{c_s^2}\right] r_s + \mathcal{O}(\beta^2). \tag{30b}$$

From (3b), (27d) and (28) we find that

$$\frac{c_1}{c_0} = \frac{a}{2h}\left\{1 + \frac{h}{2Hr_s}[-r_s^{-1} + \tfrac{1}{3}(-1)^s] + \left[\frac{gh}{c_s^2}\right] r_s^{-1}[\tfrac{7}{6}(-1)^{s+1} + 2r_s^{-1}]\right\} + \mathcal{O}(\beta^2), \tag{31a}$$

$$\frac{\ell}{h} = \frac{2h}{a r_s^2}\left\{1 - \frac{h}{2Hr_s}[2r_s^{-1} + \tfrac{1}{3}(-1)^s] + \tfrac{7}{6}(-1)^s \left[\frac{gh}{c_s^2}\right] r_s^{-1}\right\} + \mathcal{O}(\beta^2). \tag{31b}$$

For an incompressible fluid in the Boussinesq approximation [Grimshaw 1981], c_1/c_0 and ℓ/h are given by the first term in (31a,b). If we use that for the present case $c_s^2 = \gamma g H$ and take for γ the value for air, i.e. $\gamma = 1.4$, we find for $s = 0$

$$\frac{c_1}{c_0} = \frac{a}{2h}\left[1 - 0.0481\,\frac{h}{H}\right] \qquad \frac{\ell}{h} = \frac{8h}{a\pi^2}\left[1 + 0.0191\,\frac{h}{H}\right]. \tag{32}$$

For $s = 1$ we find

$$\frac{c_1}{c_0} = \frac{a}{2h}\left[1 + 0.183\,\frac{h}{H}\right], \qquad \frac{\ell}{h} = \frac{8h}{9a\pi^2}\left[1 - 0.167\,\frac{h}{H}\right]. \tag{33}$$

Note that for $s = 0$ the corrections due to the compressibility and due to non–Boussinesq terms are both important [they are both $\mathcal{O}(\beta)$], but they cancel one another almost completely. In fact for this case and for $s = 0$, terms of $\mathcal{O}(\beta^2)$ should be included. The corrections are larger if $s = 1$, although the vertical scale of this mode is smaller. Note that the canceling of the compressibility correction terms and the non–Boussinesq ones may very well be caused by the special choice of a layer with constant ω_g. On the other hand, however, we note a more general tendency that corrections for the compressibility and due to not making the Boussinesq approximation, have opposite signs [chapter 4].

5.3.2 Case II: Constant ω_g shear layer.

In this example we will again consider an inversion kind of model. A uniformly stratified layer of depth h, with constant Brunt–Väisälä frequency N and linear wind profile, $u_0(z) = -u_c - kz$, lies beneath a neutrally stable layer (i.e. $\omega_g = 0$) with constant wind component u_∞. Consistent with the assumption of a neutrally stable layer for $z > h$ is, as we have discussed in case I, the assumption that $c_s \to \infty$ for $z > h$. In order to simplify the algebra we will, unlike Christie [1989], assume that there is no density discontinuity across the interface $z = h$: this algebra is not necessary to illustrate our point.

The differential equation (7a), i.e. (4.8), for φ is approximated by (4.12). This requires that $|k/\bar{c}_0| \le H^{-1}$, i.e. the shear should not be too strong. Solutions of the eigenvalue problem for an incompressible fluid for this configuration have been found by Maslowe and Redekopp [1979, 1980] and Clarke et al. [1981], if the Boussinesq approximation is used. Since we want to obtain corrections for the compressibility of the fluid and these corrections are of the same order of magnitude as the terms neglected in the Boussinesq approximation, we should not use the Boussinesq approximation. In that case the eigenvalue problem cannot be solved in terms of elementary functions. Therefore we will use the Boussinesq approximation anyway, but the results, i.e. the corrections due to the compressibility, can only be used to estimate the importance of the compressibility for this model. To be completely, all terms of the first order in β should have been incorporated.

In this case the eigenvalue problem has the following solutions

$$\varphi(z) = \begin{cases} M[1+kz/(c_0+u_c)]^{-1/2}\sin\{r\ln[1+kz/(c_0+u_c)]\}, & \text{for } 0 \le z \le h, \\ 1, & \text{for } z > h, \end{cases} \tag{34a}$$

where

$$M = [1+kh/(c_0+u_c)]^{1/2}\left[\sin\{r\ln[1+kh/(c_0+u_c)]\}\right]^{-1}, \tag{34b}$$

$$r = (N^2/k^2 - 1/4)^{1/2}. \tag{34c}$$

The Richardson number N^2/k^2 is assumed to be larger than 1/4 so that the basic state is stable. The eigenvalues are obtained from the dynamic boundary condition:

$$c_0+u_c = kh\left\{\exp[r^{-1}\arctan(2r)+s\pi/r] - 1\right\}^{-1}, \; s = 0,\pm1,\pm2,\ldots\,. \tag{34d}$$

Using this, (34b) can be written as

$$M = (-1)^s\,[1+kh/(c_0+u_c)]^{1/2}\,(1+\tfrac{1}{4}r^{-2})^{1/2}. \tag{34e}$$

For a positive mode number ($s = 0,1,2,\ldots$), $(c_0+u_c)/kh > 0$, the wave is propagating in the opposite sense to the background flow and the maximum of $|\varphi|$ is attained close to $z = 0$ at $z = z_m$. It can easily be shown that

$$1 + kz_m/(c_0+u_c) = \exp[r^{-1}\arctan(2r)], \tag{35a}$$

$$\varphi(z_m) = (-1)^s\left[\frac{1+kh/(c_0+u_c)}{1+kz_m/(c_0+u_c)}\right]^{1/2}. \tag{35b}$$

For a negative mode number s, $(c_0+u_c)/kh < 0$, the wave is propagating in the same sense as the background flow and the maximum of $|\varphi|$ is attained at $z = h$, i.e. $\varphi(h) = 1$.

For large Richardson N^2/k^2, $r \to \infty$, and we find from (34d)

$$c_0+u_c = \mathrm{sign}(k)\,\frac{hN}{(s+\frac{1}{2})\pi}\left\{1 + \tfrac{1}{2}r^{-1}[\pi^{-1}(s+\tfrac{1}{2})^{-1} - \pi(s+\tfrac{1}{2})] + \mathcal{O}\left[\frac{s^2}{r^2}\right]\right\}. \tag{36}$$

If we compare this result with its equivalent for a shallow fluid [Gear & Grimshaw 1983 or chapter 4] we see that, correct to order $\mathcal{O}(s/r)$, s is replaced by $s+\frac{1}{2}$, due to the different condition at $z = h$. If we compare (36) with the eigenvalues c_0 for case I, given by (27b) and (28), we see that for $r \to \infty$, i.e., $k \to 0$, they become identical as should be expected.

Next we will evaluate λ and μ. Like in the foregoing example λ and μ will be approximated by

$$\lambda = \int_0^\infty 2\rho_0 \bar{c}_0(\varphi_z^2 - 2g\varphi\varphi_z/c_s^2)dz + \mathcal{O}(\beta^2), \tag{37a}$$

$$\mu = \int_0^\infty \rho_0 \bar{c}_0^2(3\varphi_z^2 - 7g\varphi\varphi_z^2/c_s^2)dz + \mathcal{O}(\beta^2). \tag{37b}$$

After straightforward but cumbersome algebra we find from (34a,d) and (37a,b)

$$\frac{\lambda}{2\rho_c M^2 k} = \tfrac{1}{2}r^2(1-x_h^{-1}) + \left[\frac{g(c_0+u_c)}{2k\,c_s^2}\right][\ln(x_h) - 2], \tag{38a}$$

$$\frac{\mu}{\rho_c M^3 k^2} = 2r^3(r^2+\tfrac{9}{4})^{-1}\Big\{\Big[1 + (-1)^s x_h^{-3/2}(r^2+\tfrac{1}{4})^{1/2}\Big]$$
$$- \left[\frac{7g(c_0+u_c)}{18k\,c_s^2}\right]\Big[3 - 4(-1)^s x_h^{-1/2}(r^2+\tfrac{1}{4})^{-1/2}\Big]\Big\}, \tag{38b}$$

where

$$x_h = 1 + kh/(c_0+u_c) = \exp[r^{-1}\arctan(2r) + s\pi/r], \tag{39}$$

and we used that $c_s \to \infty$ and $\varphi_z = 0$ for $z > h$. If we compare (38a,b) for μ and λ with the results of Christie [1989, Appendix A of that paper], we notice that in our expression for μ there is an extra factor $[1 + (-1)^s x_h^{-3/2}(r^2+\frac{1}{4})^{1/2}]$ in the limit of incompressibility. However, if we consider the limiting case $k \to 0$ and compare the results with Christie's and ours (Case I) in that limit, we see that this factor is correct. The difference must therefore be caused by a misprint in the paper by Christie [1989]. From (3b), (22d), (34e), (38a,b) and (39) we finally find

$$\frac{c_1}{c_0+u_c} = \frac{a}{2h}\, x_h^{3/2}\frac{(r^2+1/4)^{1/2}}{(r^2+9/4)} \left\{\left[(-1)^s + x_h^{-3/2}(r^2+\tfrac{1}{4})^{1/2}\right] - \left[\frac{7gh}{18c_s^2(x_h-1)}\right]\right.$$

$$\left.\times \left[3(-1)^s - 4x_h^{-1/2}(r^2+\tfrac{1}{4})^{-1/2}\right]\right\} \left\{1 + \left[\frac{ghx_h}{r^2c_s^2(x_h-1)^2}\right][\ln(x_h) - 2]\right\}^{-1}, \tag{40a}$$

$$\frac{\ell}{h} = \frac{2(c_0-u_\infty)^2}{ahk^2x_h^{3/2}}\,\frac{(r^2+9/4)}{(r^2+1/4)^{3/2}} \left\{\left[(-1)^s + x_h^{-3/2}(r^2+\tfrac{1}{4})^{1/2}\right] - \left[\frac{7gh}{18c_s^2(x_h-1)}\right]\right.$$

$$\left.\times \left[3(-1)^s - 4x_h^{-1/2}(r^2+\tfrac{1}{4})^{-1/2}\right]\right\}^{-1}. \tag{40b}$$

These expressions for $c_1/(c_0+u_c)$ and ℓ/h can be obtained from their incompressible counterparts by multiplication by $[1+(h/H)F_c]$ and by $[1+(h/H)F_\ell]$, respectively, (Fig. 19)

$$1+(h/H)F_c := \left\{1 - \left[\frac{7gh}{18c_s^2(x_h-1)}\right]\left[\frac{3(-1)^s-4x_h^{-1/2}(r^2+\frac{1}{4})^{-1/2}}{(-1)^s+4x_h^{-3/2}(r^2+\frac{1}{4})^{1/2}}\right]\right\}$$

$$\times \left\{1 + \left[\frac{ghx_h}{r^2c_s^2(x_h-1)^2}\right][\ln(x_h) - 2]\right\}^{-1}, \tag{41a}$$

$$1+(h/H)F_\ell := \left\{1 - \left[\frac{7gh}{18c_s^2(x_h-1)}\right]\left[\frac{3(-1)^s-4x_h^{-1/2}(r^2+\frac{1}{4})^{-1/2}}{(-1)^s+4x_h^{-3/2}(r^2+\frac{1}{4})^{1/2}}\right]\right\}^{-1}. \tag{41b}$$

Note that for $s = 1,3,5,\ldots$, μ, given by (38b), can be zero for an incompressible fluid, and $c_1 \to 0$ and $\ell \to \infty$ [see (3b)], if

$$(r^2 + \tfrac{1}{4}) = x_h^3, \tag{42a}$$

or

$$\tfrac{1}{2}r\ln(r^2+\tfrac{1}{4}) = \arctan(2r) + s\pi,\ s = 1,3,5,\ldots \tag{42b}$$

For example, if $s = 1$, $r = 4.543$. In such cases the corrections due to the compressibility (and due to non–Boussinesq terms) are very important.

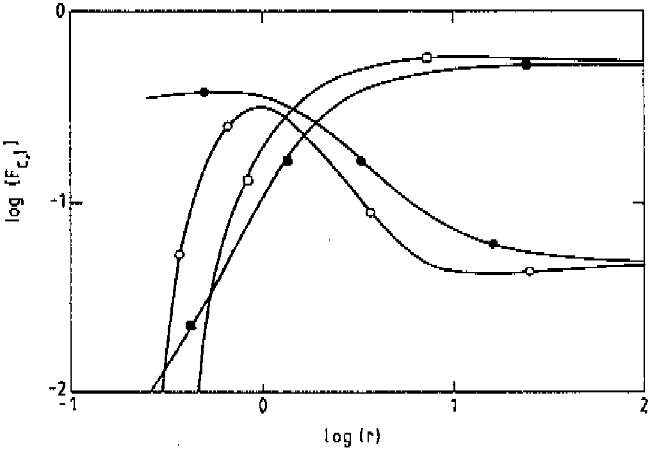


Fig. 19 The corrections to c_1 *(*F_c*, circles) and* ℓ *(*F_ℓ*, squares) due to the compressibility of the fluid for the lowest modes propagating upstream (i.e.* $s = 0$*, solid circles and squares) and downstream (i.e.* $s = -1$*, open circles and squares).*

5.3.3 **Case III: $\omega_g^2 = N^2\mathrm{sech}^2(z/h)$, $u_0 = 0$.**

Again we will assume no background velocity, $u_0 = 0$, and use the Boussinesq approximation. The profile for the Brunt–Väisälä frequency is given by

$$\omega_g^2 = N^2\mathrm{sech}^2(z/h). \tag{43}$$

For a compressible fluid there is an associated velocity of sound profile. This, of course, means that the solutions of the eigenvalue problem for a compressible fluid differ from the solutions of the eigenvalue problem for an incompressible fluid [see (6)]. However, we shall assume that the eigenvalue problem is not changed substantially by the compressibility for the single purpose to estimate the importance of the compressibility in the computation of the coefficients of the Benjamin–Davis–Ono equation in a crude way. This is correct to $\mathcal{O}(\beta)$ as is clear from the discussion in 4.2.1. Note that in this way a lower limit for the importance of the compressibility is obtained, since changes in the eigenvalue problem and so in the eigenfunctions, i.e. the modal functions also change the coefficients of the Benjamin–Davis–Ono equation. We will now evaluate (37a,b) using the first mode solution of the incompressible eigenvalue problem in the Boussinesq approximation, ω_g^2 given by (43) [Grimshaw 1981]:

$$\varphi(z) = \tanh(z/h), \tag{44a}$$
$$c_0 = Nh/\sqrt{2}. \tag{44b}$$

Substitution of (44a) into (37a,b) gives

$$\frac{c_1}{c_0} = \frac{3a}{10h}\left(1 + \tfrac{11}{32} gh/c_s^2\right) + \mathcal{O}(\beta^2), \tag{45a}$$

$$\frac{\ell}{h} = \frac{5h}{2a}\left(1 + \tfrac{45}{32} gh/c_s^2\right) + \mathcal{O}(\beta^2). \tag{45b}$$

Again, as in foregoing examples, we see that the importance of the compressibility is measured by the parameter gh/c_s^2.

5.4 SUMMARY AND CONCLUSIONS

Long solitary waves in inviscid compressible stratified deep fluids are described by the Benjamin–Davis–Ono equation [Benjamin 1967, Davis & Acrivos 1967, Ono 1975]. The coefficients of this equation are given in terms of the modal function φ, which is the solution of the eigenvalue problem with eigenvalues c_0, i.e. the long wave phase speed of these waves. A solution of the Benjamin–Davis–Ono equation is the algebraic solitary wave, first given by Benjamin [1967]. Differences between the wavelength and the correction to the long wave phase speed of this algebraic solitary wave in a compressible fluid and in an incompressible fluid are typically $\mathcal{O}(gh/c_s^2)$, where h is the vertical scale of the wave and c_s is the velocity of sound. This has been illustrated by the study of three special cases.

REFERENCES

Benjamin, T.B. (1966). Internal waves of finite amplitude and permanent form, *J. Fluid Mech.* 25, 241–270.

Benjamin, T.B. (1967). Internal waves of permanent form in fluids of great depth, *J. Fluid Mech.* 29, 559–592.

Benney, D.J. (1966). Long nonlinear waves in fluids, *J. Math. Phys.* 45, 52–63. a shallow fluid, *Phys. Fluids* 26, 14–29.

Christie, D.R. (1989). *J. Atmos. Sci.* 46, 1462.

Christie, D.R., Muirhead, K.J., Hales, A.L. (1978). On solitary waves in the atmosphere, *J. Atmos. Sci.* 35, 805–825.

Clarke, R.H., Smith, R.K., Reid, D.G. (1981). The morning glory of the gulf of Carpentaria, *Mon. Wea. Rev.* 109, 1733–1757.

Davis, R.E., Acrivos, A. (1967). Solitary internal waves in deep water, *J. Fluid Mech.* 29, 593–607.

Doviak, R.J., Chen, S. (1988). Observations of a thunderstorm generated gust compared with solitary wave theory, Technical report DOT/FAA/SA-88/1 National Technical Information Service, Springfield, Verginia 22161.

Gear, J.A., Grimshaw, R.H.J. (1983). A second order theory for solitary waves in a shallow fluid, *Phys. Fluids* 26, 14–29.

Grimshaw, R.H.J. (1980/1981). Solitary waves in a compressible fluid, *Pure Appl. Geophys.* 119, 780–797.

Grimshaw, R.H.J. (1981). A second-order theory for solitary waves in deep fluids, *Phys. Fluids* 24, 1611–1618.

Ince, E.L. (1956). "Ordinary differential equations." Dover Publications, New York, 122.

Long, R.R. (1953). Some aspects of the flow of stratified flows, *Tellus* 5, 42–58.

Maslowe, S.A., Redekopp, L.G. (1979). *Geophys. Astrophys. Fluid Dyn.* 13, 185.

Maslowe, S.A., Redekopp, L.G. (1980). Long nonlinear waves in stratified shear flows, *J. Fluid Mech.* 101, 321–348.

Miesen, R.H.M., Kamp, L.P.J., Sluijter, F.W. (1990). Internal solitary waves in compressible shallow fluids, *Phys. Fluids A* 2, in press.

Noonan, J.A., Smith, R.K. (1985). Linear and weakly nonlinear internal wave theories applied to morning glory waves, *Geophys. Astrophys. Fluid Dyn.* 33, 123–143.

Ono, H. (1975). Algebraic solitary waves in stratified fluids, *J. Phys. Soc. Japan* 39, 1082–1091.

APPENDICES

APPENDIX A

Equations (46a,b,c) are written like

$$\frac{\partial \rho_1}{\partial t} + w_1 \frac{d\rho_0}{dz} = -\rho_0 \chi_1, \tag{A1a}$$

$$\rho_0 \frac{\partial u_1}{\partial t} = -\frac{\partial p_1}{\partial x}, \tag{A1b}$$

$$\rho_0 \frac{\partial v_1}{\partial t} = -\frac{\partial p_1}{\partial y}, \tag{A1c}$$

$$\rho_0 \frac{\partial w_1}{\partial t} = -\frac{\partial p_1}{\partial z} - g\rho_1, \tag{A1d}$$

$$\frac{\partial p_1}{\partial t} + w_1 \frac{dp_0}{dz} = -\rho_0 c_s^2 \chi_1, \tag{A1e}$$

where χ_1 is defined by (48). Differentiation of (A1b,c,d) with respect to t, using (A1a,e) to eliminate ρ_1 and p_1 respectively, and substitution of these three equations into the definition (48) of χ_1 yields

$$\frac{\partial^2 \chi_1}{\partial t^2} = \nabla_1^2 (c_s^2 \chi_1 - g w_1) + \frac{\partial}{\partial z}\left\{\rho_0^{-1} \frac{\partial}{\partial z}[\rho_0 (c_s^2 \chi_1 - g w_1)]\right\} + g\frac{\partial}{\partial z}\left[\chi_1 + \frac{w_1}{\rho_0}\frac{d\rho_0}{dz}\right],$$

where we used (27) to eleminate p_0. This equation can be written as

$$g\nabla^2 w_1 = -\frac{\partial^2 \chi_1}{\partial t^2} + \nabla^2 (c_s^2 \chi_1) + \frac{\partial}{\partial z}\left[\frac{c_s^2}{\rho_0}\frac{d\rho_0}{dz}\chi_1 + g\chi_1\right]. \tag{A2a}$$

Differentiation of (A1d) with respect to t and using (A1a,e) to eliminate ρ_1 and p_1 respectively, gives

$$g\frac{\partial^2 w_1}{\partial t^2} = \frac{g}{\rho_0}\frac{\partial}{\partial z}(\rho_0 c_s^2 \chi_1) - g^2 \frac{\partial w_1}{\partial z} + g^2 \chi_1. \tag{A2b}$$

Substitution of (A2a) into (A2b) to eliminate w_1 gives after some straightforward algebra equation (47), where ω_g is defined by (49).

APPENDIX B

P′ is eliminated from (5b) by cross–differentiation of its x– and z–component. With use of (7) and the fact that $\bar{\rho} = \bar{\rho}(\psi)$ we find

$$\frac{d\bar{\rho}}{d\psi}\left[uP^{*\sigma}\frac{du}{dt} + wP^{*\sigma}\frac{dw}{dt} + gwP^{*\sigma}\right] + \bar{\rho}\,\frac{d}{dt}\left[\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z^*}\right]$$

$$+ \bar{\rho}\left[\frac{\partial u}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z^*} - \frac{\partial u}{\partial x}\frac{\partial u}{\partial z^*} - \frac{\partial u}{\partial z^*}\frac{\partial w}{\partial z^*}\right] = 0. \qquad \text{(B1)}$$

With ζ and q defined by (8c,d) respectively (B1) gives

$$P^{*\sigma}\,\frac{d\bar{\rho}}{d\psi}\frac{d}{dt}\left[\frac{q^2}{2} + gz^*\right] + \bar{\rho}\,\frac{d\zeta}{dt} + \bar{\rho}\left[\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z^*}\right]\left[\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z^*}\right] = 0, \qquad \text{(B2)}$$

or, using (5a) and (8c)

$$P^{*\sigma}\,\frac{d\bar{\rho}}{d\psi}\frac{d}{dt}\left[\frac{q^2}{2} + gz^*\right] + \bar{\rho}\,\frac{d\zeta}{dt} - \bar{\rho}\,\frac{1}{P^{*\sigma}}\frac{dP^{*\sigma}}{dt}\,\zeta = 0. \qquad \text{(B3)}$$

Using (5c), equation (B3) gives

$$\bar{\rho}\;P^{*\sigma}\,\frac{d}{dt}\left[\frac{\zeta}{P^{*\sigma}} + \frac{1}{\bar{\rho}}\frac{d\bar{\rho}}{d\psi}\left[\frac{q^2}{2} + gz^*\right]\right] = 0, \qquad \text{(B4)}$$

which after integration yields (8a).

The second integral of the equations of motion may be obtained by multiplying the x–component of (5b) by u, multiplying the z*–component by w and adding. The result is

$$u\frac{\partial P'}{\partial x} + w\frac{\partial P'}{\partial z^*} + g\bar{\rho}w + \frac{1}{2}\bar{\rho}\left[u\frac{\partial u^2}{\partial x} + w\frac{\partial u^2}{\partial z^*} + u\frac{\partial w^2}{\partial x} + w\frac{\partial w^2}{\partial z^*}\right] = 0. \quad \text{(B5)}$$

With (5c) and the definition of q^2 (8d), (B5) gives

$$\frac{d}{dt}\left[P' + g\rho z^* + \bar{\rho}\frac{q^2}{2}\right] = 0. \quad \text{(B6)}$$

After integration and with the definition of P^* (6) this yields the second integral of motion (8b).

APPENDIX C

From (7a) and the conditions far upstream we see that by definition

$$\left.\frac{\partial\psi}{\partial z}\right|_x = \bar{c}P^\sigma, \quad \text{(C1)}$$

so

$$\frac{d\bar{\rho}}{d\psi} = \frac{d\bar{\rho}}{dz}\left[\left.\frac{\partial\psi}{\partial z}\right|_x\right]^{-1} = (\bar{c}P^\sigma)^{-1}\frac{d\bar{\rho}}{dz}. \quad \text{(C2)}$$

We will also use

$$0 = \frac{dz^*}{dx} = \left.\frac{\partial z^*}{\partial x}\right|_z + \left.\frac{\partial z^*}{\partial z}\right|_x \left.\frac{\partial z}{\partial x}\right|_{z^*},$$

from which we find that

$$\left.\frac{\partial z}{\partial x}\right|_{z^*} = -\left.\frac{\partial z^*}{\partial x}\right|_z \left[\left.\frac{\partial z^*}{\partial z}\right|_x\right]^{-1} = -\eta_x(1+\eta_z)^{-1}, \quad \text{(C3)}$$

where subscripts denote derivatives. Furthermore we find that

$$\left.\frac{\partial\psi}{\partial x}\right|_{z^*} = \left.\frac{\partial\psi}{\partial x}\right|_z + \left.\frac{\partial\psi}{\partial z}\right|_x \left.\frac{\partial z}{\partial x}\right|_{z^*} = -\bar{c}P^\sigma\eta_x(1+\eta_z)^{-1}, \quad \text{(C4a)}$$

and in the same way that

$$\frac{\partial^2\psi}{\partial x^{*2}}\bigg|_{z^*} = -\left[\frac{\bar{c}P^\sigma\eta_x}{1+\eta_z}\right]_x + (2\bar{c}P^\sigma)^{-1}\left[\frac{\bar{c}^2\eta_x^2P^{2\sigma}}{(1+\eta_z)^2}\right]_z, \tag{C4b}$$

$$\frac{\partial\psi}{\partial z^*}\bigg|_x = \bar{c}P^\sigma(1+\eta_z)^{-1}, \tag{C4c}$$

$$\frac{\partial^2\psi}{\partial z^{*2}}\bigg|_x = (2\bar{c}P^\sigma)^{-1}\left[\frac{\bar{c}^2P^{2\sigma}}{(1+\eta_z)^2}\right]_z, \tag{C4d}$$

$$\frac{\partial P^{*\sigma}}{\partial x}\bigg|_{z^*} = \sigma(P^*)^{\sigma-1}\left[P\delta_x - (P_z+P\delta_z+P_z\delta)\frac{\eta_x}{1+\eta_z}\right], \tag{C4e}$$

$$\frac{\partial P^{*\sigma}}{\partial z^*}\bigg|_x = \sigma(P^*)^{\sigma-1}\left[\frac{P_z+P\delta_z+P_z\delta}{1+\eta_z}\right]. \tag{C4f}$$

Substitution of (C4) into (8a,b) and (9) gives

$$\bar{\rho}\bar{c}^2P^{2\sigma}\left[\frac{\eta_x}{1+\eta_z}\right]_x - \tfrac{1}{2}\bar{\rho}\left[\bar{c}^2P^{2\sigma}\frac{(1+\eta_x^2)}{(1+\eta_z)^2}\right]_z$$

$$+\frac{\sigma\bar{c}^2P^{2\sigma}\bar{\rho}}{P(1+\delta)(1+\eta_z)}\left[-P\delta_x\eta_x + (P_z+P\delta_z+P_z\delta)\left\{\frac{1+\eta_x^2}{1+\eta_z}\right\}\right]$$

$$-\tfrac{1}{2}\bar{\rho}_z\left[\frac{\bar{c}^2P^{2\sigma}(\eta_x^2+1)}{(1+\eta_z)^2}\right] - g\bar{\rho}_z(z+\eta)(1+\delta)^{2\sigma}P^{2\sigma} = \bar{\rho}\bar{c}P^\sigma(P^*)^{2\sigma}\,H(\psi), \tag{C5a}$$

$$P + P\delta + \frac{\bar{\rho}}{\rho_c g h_c}\left[\tfrac{1}{2}\bar{c}^2(P/P^*)^{2\sigma}\frac{(1+\eta_x^2)}{(1+\eta_z)^2} + g(z+\eta)\right] = L(\psi), \tag{C5b}$$

$$g(z+\eta) + \tfrac{1}{2}\bar{c}^2(P/P^*)^{2\sigma}\frac{(1+\eta_x^2)}{(1+\eta_z)^2} = \text{constant, for } z = h. \tag{C5c}$$

Evaluation of (C5) for $|x| \to \infty$ gives the right–hand sides in terms of $\bar{\rho}$, $\bar{c}$, P and z. Using these expressions and that $\rho_c g h_c P/\bar{\rho} = \sigma c_s^2$ [equation (2) to (4) and (6)] gives after some manipulations (11).

APPENDIX D

At order ϵ^4 (2.11) yields:

$$(\bar{\rho}\bar{c}_0^2\eta_{2z})_z + \bar{\rho}\omega_g^2\eta_2 + (\sigma\bar{\rho}\bar{c}_0^2\delta_2)_z + \bar{\rho}\bar{c}_0^2\eta_{1xx} + (2\bar{\rho}\bar{c}_0c_1\eta_{1z})_z - \tfrac{3}{2}(\bar{\rho}\bar{c}_0^2\eta_{1z}^2)_z$$
$$+ 2\sigma\bar{\rho}\omega_g^2\eta_1\delta_1 + (2\sigma\bar{\rho}\bar{c}_0c_1\delta_1)_z - (\sigma\bar{\rho}\bar{c}_0^2\delta_{1z})(2\eta_{1z}+\delta_1)$$
$$+ \sigma(\sigma-1/2)(\bar{\rho}\bar{c}_0^2)_z\delta_1^2 = 0, \tag{D1a}$$

$$\delta_2 + \sigma^{-1}(c_s^2-\bar{c}_0^2)^{-1}\Big\{(g\eta_2 - \bar{c}_0^2\eta_{2z}) + \bar{c}_0^2[\tfrac{3}{2}\eta_{1z}^2 + 2\sigma\delta_1\eta_{1z} + \sigma(\sigma+1/2)\delta_1^2]$$
$$-2\bar{c}_0c_1(\eta_{1z} + \sigma\delta_1)\Big\} = 0. \tag{D1b}$$

Equation (D1a) can be written as

$$(\bar{\rho}\bar{c}_0^2\eta_{2z})_z + \bar{\rho}\omega_g^2\eta_2 + (\sigma\bar{\rho}\bar{c}_0^2\delta_2)_z + \bar{\rho}\bar{c}_0^2\eta_{1xx} + [2\bar{\rho}\bar{c}_0c_1(\eta_{1z}+\sigma\delta_1)]_z$$
$$- \tfrac{3}{2}(\bar{\rho}\bar{c}_0^2\eta_{1z}^2)_z - [(\sigma\bar{\rho}\bar{c}_0^2\delta_1)(2\eta_{1z}+\sigma\delta_1+\tfrac{1}{2}\delta_1)]_z = 0, \tag{D2}$$

where we used (5a). Substitution of (D1b) into (D2) gives:

$$\left[\bar{\rho}\bar{c}_0^2\,\frac{c_s^2\eta_{2z} - g\eta_2}{c_s^2 - \bar{c}_0^2}\right]_z + \bar{\rho}\omega_g^2\eta_2 + \bar{\rho}\bar{c}_0^2\eta_{1xx} + \left[2\bar{\rho}\bar{c}_0c_1c_s^2(c_s^2-c_0^2)^{-2}(c_s^2\eta_{1z}-g\eta_1)\right]_z$$
$$- \tfrac{3}{2}(\bar{\rho}\bar{c}_0^2\eta_{1z}^2)_z - \left\{\bar{\rho}\bar{c}_0^2(c_s^2-\bar{c}_0^2)^{-1}[2\sigma c_s^2\delta_1\eta_{1z} + \sigma(\sigma+1/2)c_s^2\delta_1^2 + \tfrac{3}{2}\bar{c}_0^2\eta_{1z}^2]\right\}_z = 0 \tag{B3}$$

Substitution of δ_1 from (5b) gives (13a) with f_2 given by (13d).

At order ϵ^4 the boundary condition (2.11d) becomes

$$g\eta_2 - j\bar{c}_0^2(\eta_{2z}+\sigma\delta_2) - 2j\bar{c}_0c_1(\eta_{1z}+\sigma\delta_1) + j\bar{c}_0^2[2\sigma\eta_{1z}\delta_1 + \tfrac{3}{2}\eta_{1z}^2$$
$$+ \sigma(\sigma+1/2)\delta_1^2] = 0, \quad \text{for } z = h. \tag{D4}$$

Substitution of δ_1 and δ_2, from (5b) and (D2), respectively, gives after some rearrangements

$$g\eta_2 - j\bar{c}_0^2\eta_{2z} - 2j\bar{c}_0c_1(c_s^2-\bar{c}_0^2)^{-1}(c_s^2\eta_{1z}-g\eta_1) + j\bar{c}_0^2(c_s^2-\bar{c}_0^2)^{-2}$$
$$\times\left[2\eta_{1z}(c_s^2-\bar{c}_0^2)(\bar{c}_0^2\eta_{1z}-g\eta_1) + \tfrac{3}{2}\eta_{1z}^2(c_s^2-\bar{c}_0^2)^2 + (1+\tfrac{1}{2}\sigma^{-1})(\bar{c}_0^2\eta_{1z}-g\eta_1)^2\right] = 0,$$
$$\text{for } z = h. \qquad \text{(D5)}$$

From (D5) we can obtain (13c), with g_2 given by (13e).

APPENDIX E

We want to solve the differential equation (13a)

$$a_0(z)\eta_{2zz} + a_1(z)\eta_{2z} + a_2(z)\eta_2 + f_2 = 0, \text{ for } 0 < z < h, \qquad \text{(E1a)}$$

where

$$a_0(z) = \bar{\rho}\bar{c}_0^2c_s^2(c_s^2-\bar{c}_0^2)^{-1}, \qquad \text{(E1b)}$$
$$a_1(z) = [\bar{\rho}\bar{c}_0^2c_s^2(c_s^2-\bar{c}_0^2)^{-1}]_z - g\bar{\rho}\bar{c}_0^2(c_s^2-\bar{c}_0^2)^{-1}, \qquad \text{(E1c)}$$
$$a_2(z) = -[g\bar{\rho}\bar{c}_0^2(c_s^2-\bar{c}_0^2)^{-1}]_z + \bar{\rho}\omega_g^2. \qquad \text{(E1d)}$$

The solution $\eta_2(z)$ is subjected to the boundary conditions (13a,b). A solution of the homogeneous problem [f_2 and g_2 are zero in (13)] is $\varphi(z)$. If a solution of the homogeneous problem, linearly independent of $\varphi(z)$, is taken to be $\chi(z)$, a solution of (E1) is given by

$$\eta_2 = \varphi\int_0^h \frac{\chi f_2}{\Delta a_0}\,dz - \chi\int_0^h \frac{\varphi f_2}{\Delta a_0}\,dz, \qquad \text{(E2)}$$

as can be verified by substitution. The Wronskian Δ of φ and χ is defined as [Ince 1956]

$$\Delta = \begin{vmatrix} \varphi & \chi \\ \varphi_z & \chi_z \end{vmatrix} = \varphi\chi_z - \varphi_z\chi, \qquad \text{(E3)}$$

and can be calculated using Abel's identity [Ince 1956]:

$$\Delta = \Delta_0 \exp\left[-\int_0^h (a_1/a_0)dz\right]. \tag{E4}$$

The solution for η_2 given by (E2) satisfies the boundary condition (13b) identically. The second boundary condition gives

$$\int_0^h \frac{\varphi f_2}{\Delta a_0} dz + \frac{i\varphi_z g_2}{\Delta}\bigg|_{z=h} = 0, \tag{E5}$$

where we used (6c) and (E3). Calculating Δ from Abel's identity (E4) gives (14).

APPENDIX F

The equilibrium pressure $p_0(z)$, determined from the hydrostatic balance, i.e. (3.1b) with $u = 0$ [cf. (1.27)],

$$\frac{\partial p_0}{\partial z} + g\rho_0 = 0, \tag{F1}$$

and from the law for a perfect gas [cf. (1.28)]

$$p_0 = \rho_0 RT, \tag{F2}$$

where R is the gas constant and T is the absolute temperature, reads

$$p_0 = p_{00} \exp\left[-\int_0^z (\frac{g}{RT})dz\right], \tag{F3}$$

where p_{00} is the pressure at $z = 0$. The velocity of sound, defined by $c_s^2 = \gamma p_0/\rho_0$, can, with the law for a perfect gas, be written as

$$c_s = (\gamma RT)^{1/2}. \tag{F4}$$

Therefore (F3) can be written as

$$p_0 = p_{00} \exp\left[-\int_0^z (\gamma g/c_s^2)dz\right], \tag{F5}$$

Evaluation of (3.2) far upstream, and assigning $\bar{p}$ the value p_{00} yields

$$\rho_0 = \bar{\rho}(p_0/p_{00})^{1/\gamma}, \tag{F6}$$

which is equivalent to (16).

APPENDIX G

If terms of $\mathcal{O}(\bar{c}_0^2/c_s^2) = \mathcal{O}[(h/H)^2]$, of $\mathcal{O}[(gh/c_s^2)^2] = \mathcal{O}[(h/H)^2]$ and of $\mathcal{O}[(gh/c_s^2)^3]$ are neglected, (17a,b) become correct to $\mathcal{O}(h/H)$

$$\lambda = \int_0^h 2\rho_0\bar{c}_0(\varphi_z^2 - 2g\varphi\varphi_z/c_s^2)dz, \tag{G1a}$$

$$\mu = \int_0^h \rho_0\bar{c}_0^2(3\varphi_z^3 - 7g\varphi\varphi_z^2/c_s^2)dz. \tag{G1b}$$

We multiply (8) with φ and integrate which gives:

$$\int_0^h [\varphi(\rho_0\bar{c}_0^2\varphi_z)_z + \rho_0\omega_g^2\varphi^2]\, dz + \mathcal{O}(\bar{c}_0^2/c_s^2) + \mathcal{O}(gh/c_s^2) = 0\,. \tag{G2}$$

If the upper boundary is rigid, or is approximated to be rigid (Boussinesq approximation), partial integration of (G2) gives

$$\int_0^h \rho_0\bar{c}_0^2\varphi_z^2dz = \int_0^h \rho_0\omega_g^2\varphi^2dz + \mathcal{O}(\bar{c}_0^2/c_s^2) + \mathcal{O}(gh/c_s^2). \tag{G3a}$$

In the same way multiplication of (8) by φ^2 and integration gives

$$\int_0^h 2\rho_0\bar{c}_0^2\varphi\varphi_z^2dz = \int_0^h \rho_0\omega_g^2\varphi^3dz + \mathcal{O}(\bar{c}_0^2/c_s^2) + \mathcal{O}(gh/c_s^2). \tag{G3b}$$

Inserting (G3) into (G4) shows that (2.19a,b) of Grimshaw is equivalent with (17), correct to $\mathcal{O}(gh/c_s^2)$ and if the Boussinesq approximation can be used.

LIST OF SYMBOLS

(page of definition enclosed within brackets)

a: amplitude of a solitary wave [4, 76, 100]

A: x–dependent part of η_1 [76, 78, 98, 101]

$\mathcal{A}$: Fourier transform of A(X) [100]

$\mathcal{B}$: [100]

c: phase speed of a solitary wave[4] and velocity of the moving frame [65, 76, 98]

$\tilde{c}(z)$: phase speed c minus background velocity $u_0(z)$ [65]

c_0: linear long wave phase speed [5, 76, 99]

$\bar{c}_0(z)$: linear long wave phase speed c_0 minus background velocity $u_0(z)$ [76, 98]

c_1: correction to c_0 [76, 77, 100, 101]

c_c: characteristic velocity [48]

$c_s(z)$: velocity of sound [15, 32]

c_g: group velocity [17]

c_p: phase speed [17]

c_p: specific heat at constant pressure [44]

c_v: specific heat at constant volume [44]

d: inversion height [87]

f: Coriolis parameter [45]

F_b: buoyancy force [21]

g: gravitational acceleration [4]

h: depth of a fluid [4], characteristic vertical scale [48]

h: depth of a fluid, characteristic vertical scale [75, 98]

H(z): scale height [24]

H_c: characteristic value of the scale height [48]

$\mathcal{H}(z)$: Heaviside function [87, 106]

j: parameter that is 0 for a rigid and 1 for a free boundary [70]

k: wavevector [16]

k: magnitude of **k** (chapters 1, 2); background velocity gradient [90, 109]

k_x: x–component of the wavevector [16]

k_y: y–component of the wavevector [16]

k_z: z–component of the wavevector [16]

k_r: magnitude of the real part of **k** [26]

$k_{z,i}$: imaginary part of k_z [26]

$k_{z,r}$: real part of k_z [26]

$k_\perp$: magnitude of the component of $\mathbf{k}$ that is perpendicular to $\mathbf{g}$ [25]

ℓ: characteristic length of a solitary wave [4, 76, 100]

l: characteristic lengthscale [43]

M: normalisation factor for φ [83, 88, 106]

N: Brunt–Väsälä frequency for a compressible isothermal fluid [83, 87, 90, 106, 109, 113]

p: pressure [14]

$p_0(z)$: background pressure [15]

p_1: perturbation pressure [15]

$\bar{p}$: reference pressure [66]

p^*: normalised pressure proportional to the temperature [67, 69]

R: gas constant [16]

r_s: eigenvalue for mode s [83, 88]

s: mode number [83, 88, 106, 110]

t: time [4]

t_c: characteristic timescale

T: absolute temperature [16]

T_0: absolute temperature in the absence of waves [16]

$\mathbf{u}$: vector velocity field [14]

$\mathbf{u}_1$: perturbation velocity field [15]

u: x–component of the velocity [14]

$u_0(z)$: background velocity [65, 76, 98]

v: y–component of the velocity [14]

w: z–component of the velocity [14]

$\mathbf{x}$: vector in a right–handed Cartesian coordinate system [15]

x: Cartesian coordinate (parallel to the lower boundary of the fluid) [4]

X: $:= \epsilon x$ [76, 78, 98, 101]

y: Cartesian coordinate (parallel to the lower boundary of the fluid) [25]

z: Cartesian coordinate in the direction opposed to $\mathbf{g}$ (chapters 1, 2) [15, 21]; Lagrangian coordinate in the direction opposed to $\mathbf{g}$ (chapters 3, 4, 5) [65, 76, 98]

z^*: Eulerian (Cartesian) coordinate in the direction opposed to $\mathbf{g}$ [65]

z_m: height at which the modal function $\varphi(z)$ attaines its maximum [83]

Z: $:= \epsilon z$ [104]
α: measure for the amplitude of a (solitary) wave [4, 48, 56, 76, 98]
β: ratio of characteristic vertical length (h, h) and characteristic scale height (H_c, H) [56, 79]
γ: ratio of specific heats [14]
δ: perturbation of the temperature [69]
η: vertical displacement of a streamline [4, 65]
η_1: vertical displacement of a streamline correct to $\mathcal{O}(\alpha)$ [77, 101]
ϵ: ratio of a characteristic vertical and horizontal length [4]
ϕ: phase function [16]
$\varphi(z)$: modal function [76, 78, 98, 101]
κ: coefficient in the BDO equation [100, 106]
λ: coefficient in the KdV equation and the BDO equation [76, 81, 100, 106]
μ: coefficient in the KdV equation and the BDO equation [76, 81, 100, 106]
ν: coefficient in the KdV equation [76, 81]
μ_v: dynamic coefficient of viscosity [42]
ρ: mass density [14]
$\tilde{\rho}$: $= \rho_1/\rho_0$ [54]
$\rho_0(z)$: background mass density [15, 76, 98]
ρ_1: perturbation of the mass density [15]
ρ_c: characteristic density [43]
$\bar{\rho}$: potential density [66]
σ: $:= (\gamma-1)^{-1}$ [67]
$\chi(z)$: solution of the eigenvalue problem (chapters 4 and 5), independent of $\varphi(z)$ [102, 122]
χ_1: divergence of u_1 [31]
ψ: streamfunction [54]
ω: radian frequency [16]
ω_a: acoustic cut–off frequency [32]
ω_b: Brunt–Väisälä frequency for an incompressible fluid [22]
ω_g: Brunt–Väisälä frequency for a compressible fluid [31, 70]
Ω: angular velocity of a rotating fluid [42]

SAMENVATTING

In het eerste hoofdstuk van dit proefschrift worden enkele fundamentele eigenschappen van akoestische zwaartegolven besproken aan de hand van de gelineariseerde bewegingsvergelijkingen voor een niet–viskeuze, niet–roterende, samendrukbare, gelaagde vloeistof. Na dit inleidende hoofdstuk worden twee onderwerpen bestudeerd. Ten eerste, de veelvuldig gebruikte hydrostatische– en Boussinesq–benaderingen in samenhang met een zorvuldige analyse van de voorwaarden voor de geldigheid van deze benaderingen bij toepassing op golven met een amplitude van eindige grootte in een samendrukbare, gelaagde vloeistof. Ten tweede, de invloed van de samendrukbaarheid op de eigenschappen van interne solitaire golven in zo'n vloeistof.

De hydrostatische benadering, waarbij de versnelling in vertikale richting verwaarloosd wordt in de impulsbalans, blijkt geldig te zijn voor akoestische zwaartegolven met een vertikale lengteschaal die veel kleiner is dan de horizontale lengteschaal, d.w.z. voor akoestische zwaartegolven met een kleine aspektverhouding, en frequenties veel kleiner dan de Brunt–Väisälä–frequentie. De Boussinesq–benadering kan worden gebruikt wanneer de dichtheidsvariaties, in het gedeelte van de stroming dat we willen beschrijven, klein zijn. Voor akoestische zwaartegolven betekent dit dat de vertikale schaal van de golven klein moet zijn ten opzichte van de schaal van de gelaagdheid van de vloeistof, i.e. de schaalhoogte. De gelineariseerde Boussinesq–vergelijkingen voor een onsamendrukbare vloeistof kunnen gebruikt worden ter beschrijving van deze golven als aan de volgende voorwaarden is voldaan:

1) de uitwijking van stroomlijnen moet klein zijn ten opzichte van de vertikale schaal van de stroming,

2) de frequentie van de golven moet kleiner zijn dan de Brunt–Väisälä –frequentie en

3) voor de definitie van deze frequentie moet de definitie van de Brunt–Väisälä–frequentie voor een samendrukbare vloeistof worden gebruikt.

De laatste voorwaarde is gelijkwaardig aan het gebruik van de massadichtheid onder genormaliseerde omstandigheden in plaats van de werkelijke massadichtheid. We merken op dat men zeer voorzichtig moet zijn met het gebruik van de Boussinesq–benadering als andere kleine effecten, zoals zwak niet–lineaire effecten, in de beschouwingen worden meegenomen, zelfs indien aan de overige bovengenoemde voorwaarden is voldaan.

Om interne solitaire golven in samendrukbare vloeistoffen te bestuderen worden twee integralen van de bewegingsvergelijkingen voor een niet–viskeuze samendrukbare vloeistof afgeleid. Deze integralen worden, evenals bepaalde randvoorwaarden, getransformeerd tot vergelijkingen voor de uitwijking van stroomlijnen en voor de verstoring van de temperatuur. Hierbij nemen we aan dat er geen gesloten stroomlijnen zijn, zodat de twee integratieconstanten in de bewegingsintegralen bepaald kunnen worden uit de condities stroomopwaarts. Interne solitaire golven in samendrukbare vloeistoffen worden, evenals interne solitaire golven in onsamendrukbare vloeistoffen, beschreven door de Korteweg–de Vries–vergelijking als de vloeistof ondiep is, d.w.z. als de totale diepte van de vloeistof veel kleiner is dan de karakteristieke horizontale lengteschaal van de golf. Zij worden door de Benjamin–Davis–Ono–vergelijking beschreven als de vloeistof diep is, d.w.z. als de totale diepte van de vloeistof veel groter is dan karakteristieke horizontale lengteschaal van de golf. De correcties ten gevolge van de samendrukbaarheid van de vloeistof op de coëfficiënten en dus op de oplossingen van deze vergelijkingen zijn $\mathcal{O}(gh/c_s^2)$, waarin g de grootte van de zwaartekrachtsversnelling is, h de diepte van het gelaagde deel van de vloeistof en c_s de geluidssnelheid. Voor omstandigheden zoals die in de atmosfeer voorkomen ligt de grootte van deze parameter tussen 0.01 en 1. Het soort stroming waarvoor deze correcties altijd van belang zijn is de ondiepe isotherme uniforme stroming, zoals in dit proefschrift wordt uitgelegd.

De inhoud van de paragrafen 2.4.2 en 2.4.3 zijn gepubliceerd als: Miesen, R.H.M., Kamp, L.P.J., Sluijter, F.W. (1988). "On the application of the Boussinesq approximation for nonlinear gravity waves in the atmosphere", *Phys. Scripta* **38**, 857–859.

De inhoud van de hoofdstukken 3 en 4 wordt gepubliceerd als: Miesen, R.H.M., Kamp, L.P.J., Sluijter, F.W. (1990). "Solitary Waves in Compressible Shallow Fluids", *Phys. Fluids A* **2**.

De inhoud van hoofdstuk 5 is aangeboden voor publikatie aan *Phys. Fluids A* als: Miesen, R.H.M., Kamp, L.P.J., Sluijter, F.W. "Solitary Waves in Compressible Deep Fluids".

CURRICULUM VITAE

19 juni 1965	:	Geboren te Sittard
3 juni 1983	:	Diploma Gymnasium–β, Henric van Veldeke–College Maastricht
8 juli 1987	:	Doctoraal examen Technische Natuurkunde (met lof)
1 augustus 1987	:	In dienst van de Nederlandse organisatie voor Wetenschappelijk Onderzoek (N.W.O.). Promotie–onderzoek bij Prof. dr. F.W. Sluijter, Prof. dr. M.P.H. Weenink en dr. ir. L.P.J. Kamp naar verschillende aspecten van gravito–akoestische golven.

STELLINGEN

behorende bij het proefschrift van

R.H.M. Miesen

Eindhoven, 20 april 1990

I

Voor de verklaring van de eigenschappen van interne solitaire golven in een ondiepe compressibele vloeistof zijn de correcties ten gevolge van de compressibiliteit van de vloeistof van essentieel belang.

[1] R.H.M. Miesen, L.P.J. Kamp and F.W. Sluijter, Phys. Fluis A 2, in press (1990).
[2] Dit proefschrift.

II

Demping van gravito–akoestische golven als gevolg van de eindige geleiding van de ionosfeer vindt plaats met een karakteristieke demping die tussen 0 en $\omega_p/2$ ligt, waarbij ω_p de Pedersen frequentie is.

[1] C.O. Hines and W.H. Hooke, J. Geophys. Res. 75, 2563 (1970).
[2] R.H.M. Miesen, P.C. de Jagher, L.P.J. Kamp and F.W. Sluijter, J. Geophys. Res. D **94**, 16269 (1989).

III

Anders dan bij demping die het gevolg is van warmtegeleiding of viscositeit hangt de sterkte van de demping die het gevolg is van de eindige geleiding van de ionosfeer niet alleen van de frequentie van de golf af, maar ook van de voortplantingsrichting. Dit kan een verklaring zijn voor de waargenomen voorkeursrichting wat betreft de voortplanting van zogenaamde "Travelling Ionosferic Disturbances".

[1] R.H.M. Miesen, P.C. de Jagher, L.P.J. Kamp and F.W. Sluijter, J. Geophys. Res. D **94**, 16269 (1989).
[2] H. Kelder, T.A.Th. Spoelstra., J. Atmos. Terr. Phys. **49**, 7 (1987).

IV

Ten gevolge van chemische reakties onder invloed van zonnestraling bestaat er een instabiliteit die mede verklaring is voor de excitatie van gravito akoestische golven.

[1] P.C. de Jagher, J. Geophys. Res. D, accepted for publication (1990).

V

De benadering die gebruikt wordt in de studie van golven in gelaagde vloeistoffen en die met de naam Boussinesq–benadering wordt aangeduid, is in feite los komen te staan van de door Boussinesq ingevoerde benadering.

[1] J. Boussinesq, Théorie Analytique de la Chaleur 2, Gauthier Vilars, Paris (1903).
[2] J.S. Turner, Buoyancy Effects in Fluids, Cambridge (1973).
[3] R.H.M. Miesen, L.P.J. Kamp and F.W. Sluijter, Phys. Scripta **38**, 857 (1988).

VI

Zelfs onder strenge condities kan de zogenaamde Boussinesq benadering leiden tot onjuiste resultaten.

[1] T.B. Benjamin, J. Fluid Mech. 25, 241 (1966).
[2] R.H.M. Miesen, L.P.J. Kamp and F.W. Sluijter, Phys. Fluids A 2, in press (1990).

VII

Ondanks het feit dat de totale hoeveelheid ozon in de atmosfeer afneemt, neemt ook de hoeveelheid ultra–violette straling op het aardoppervlak af.

[1] C. Brühl and P.J. Crutzen, Geophys. Res. Letters 16, 703 (1989).
[2] S.A. Penkett, Nature 341, 28 september, 283 (1989)

VIII

De potentiaal, in de impulsbalans voor een vloeistof, die het gevolg is van rotatie van de vloeistof kan niet, zoals door Gill gedaan wordt, geschreven worden als $\frac{1}{2}\Omega^2 x^2$, maar wel als $\frac{1}{2}|\Omega \times x|^2$.

[1] A.E. Gill, Atmosphere–Ocean Dynamics, section 4.5.1, International Geophysics Series 30, Academic Press (1982).

IX

De naamgeving van vergelijkingen en methoden gebeurt vaak onzorgvuldig. Een recent voorbeeld daarvan is de vergelijking die in de literatuur Benjamin(–Davis)–Ono–vergelijking wordt genoemd. Deze vergelijking zou Benjamin–Davis–Acrivos–of anders Benjamin–Davis–Acrivos–Ono–vergelijking moeten heten.

[1] T.B. Benjamin, J. Fluid Mech. 29, 559 (1967).
[2] R.E. Davis and A. Acrivos, J. Fluid Mech 29, 593 (1967).
[3] H. Ono, J. Phys. Soc. Japan 39, 1082 (1975).

X

Omdat interne solitaire zwaartegolven kunnen leiden tot over–correctie tijdens het landen of opstijgen van vliegtuigen vormen zij een reëel gevaar voor de lucht–vaart.

[1] D.R. Christie and K.J. Muirhead, Intern. J. Aviation Safety 1, 169 (1983).
[2] R.J. Doviak and D.R. Christie, J. Aircraft 26, 423 (1989).

XI

De positie zoals die door het Christen Democratisch Appel de laatste 10 jaar in de Nederlandse politiek is ingenomen, heeft afbreuk gedaan aan de waarde van het parlement als instrument van het volk en dus aan het democratisch gehalte van het parlement.

XII

De ontwikkeling en de verspreiding van geluidsapparatuur en geluidsdragers heeft de beleving van live uitgevoerde klassieke muziek sterk beinvloed: doordat muzikale lijnen reeds bij de luisteraar bekend zijn, is de technische perfectie van het spel een belangrijke rol in de luisterervaring gaan spelen. In dit kader moet ook het ontstaan van vele gespecialiseerde orkesten gezien worden.

XIII

De technische ontwikkeling van blaasinstrumenten heeft het mogelijk gemaakt dat muziek, met name muziek geschreven in de 18^e en begin 19^e eeuw, in een te snel tempo wordt uitgevoerd. Goede voorbeelden hiervan zijn Mozart's hoornconcerten (K. 412, 417, 447, 495). Het tempo waarin met name de rondo's uit deze concerten in de moderne concertpraktijk gespeeld worden, is op natuurhoorn, het instrument waarvoor deze concerten geschreven zijn, nauwelijks uitvoerbaar en doet afbreuk aan het originele (dans)karakter van deze delen.

XIV

Bij de ondergang van het Romeinse rijk kan het bestaan van interne zwaartegolven een rol hebben gespeeld

[1] O. Pettersson, Svenska Hydrografisk–Biologiska Kommisionens Skrifter 5, 1 (1912).