Large-time behavior of a two-scale semilinear reaction-diffusion system for concrete sulfatation

Document Version:
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

Large-time behavior of a two-scale semilinear
reaction-diffusion system for concrete sulfatation

by

T. Aiki, A. Muntean

Large-time behavior of a two-scale semilinear reaction-diffusion system for concrete sulfatation

Toyohiko Aiki‡ and Adrian Muntean⋆

‡Department of Mathematics, Faculty of Science, Japan Women's University
2-8-1 Mejirodai, Bunkyo-ku, Tokyo 112-8681, Japan. e-mail: aikit@fc.jwu.ac.jp
⋆CASA - Centre for Analysis, Scientific computing and Applications, Institute for
Complex Molecular Systems (ICMS), Department of Mathematics and Computer
Science, Eindhoven University of Technology, The Netherlands. e-mail:
a.muntean@tue.nl

Abstract

We study the large-time behavior of (weak) solutions to a two-scale reaction-diffusion system coupled with a nonlinear ordinary differential equations modeling the partly dissipative corrosion of concrete (/cement)-based materials with sulfates. We prove that as $t \to \infty$ the solution to the original two-scale system converges to the corresponding two-scale stationary system. To obtain the main result we make use essentially of the theory of evolution equations governed by subdifferential operators of time-dependent convex functions developed combined with a series of two-scale energy-like time-independent estimates.

1   Introduction

1.1   Organization of the paper

We study the large time behavior of a two-scale reaction-diffusion system modeling the evolution of the sulfatation reaction in concrete-based materials; see [FM13] for a rigorous derivation of the system by periodic homogenization [a direct application of two-scale convergence principles [All92, Ngu89] and multiscale analysis of PDEs posed in perforated domains [HJ91]]. To fix ideas, let us only mention here that the sulfatation reaction attacks aggressively unsaturated porous media, where the H_2S air-water transfer and bacteria interplay together in the presence of heat. This is precisely the case of most sewer pipes or of marble monuments in countries like Brazil, Japan, USA, Italy, etc.; see e.g. [BR96, ADDN04] and references cited therein. More engineering details on this scenario can be found e.g. in [GMSR09].

To show that as $t \to \infty$ the solution to the original two-scale reaction-diffusion system converges to the corresponding two-scale stationary system, we proceed as follows:

In the subsequent sections, we present the setting of the two-scale model equations (Section 1.2), give a brief outlook on the literature that inspired us to working in such framework of multiple spatial scales (Section 1.3), and finally, we delimitate the aim of the paper (Section 1.4). The main technical assumptions behind our results together with

1

the weak solvability of the problem are collected in Section 2 and Section 3, respectively. The bulk of the paper is Section 4 – the proof of our main main result - Theorem 2.6 - a characterization of the solution behavior at large times.

1.2   Two-scale model equations

Let us consider Ω and Y to be connected and bounded domains in $\mathbb{R}^3$, $\Gamma_i \subset \partial Y$, $i = 1, 2, 3$, $\partial Y = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, and Γ_1, Γ_2 and Γ_3 are disjoint. Also, $\partial\Omega = \Gamma_D \cup \Gamma_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$.

Simplifying the scenario analyzed in [FMA12], we consider here the following system of partial differential equations coupled with one ordinary differential equation for $T > 0$:

$$\partial_t w_1 - \nabla_y \cdot (d_1 \nabla_y w_1) = -\psi(w_1 - \gamma w_2) \quad \text{in } (0, T) \times \Omega \times Y, \tag{1}$$

$$\partial_t w_2 - \nabla_y \cdot (d_2 \nabla_y w_2) = \psi(w_1 - \gamma w_2) \quad \text{in } (0, T) \times \Omega \times Y, \tag{2}$$

$$\partial_t w_3 - \nabla \cdot (d_3 \nabla w_3) = -\alpha \int_{\Gamma_2} \big(h w_3 - w_2\big) d\gamma_y \quad \text{in } (0, T) \times \Omega, \tag{3}$$

$$\partial_t w_4 = \eta(w_1, w_4) \quad \text{on } (0, T) \times \Omega \times \Gamma_1. \tag{4}$$

The system is equipped with the initial conditions

$$w_i(0, \cdot, \cdot) = w_{i0} \text{ in } \Omega \times Y, i = 1, 2, w_3(0, \cdot) = w_{30} \text{ in } \Omega, w_4(0, \cdot, \cdot) = w_{40} \text{ on } \Omega \times \Gamma_1, \tag{5}$$

while the boundary conditions are

$$\begin{cases} d_1 \nabla_y w_1 \cdot \nu_y = -\eta(w_1, w_4) \text{ on } (0, T) \times \Omega \times \Gamma_1, \\ d_1 \nabla_y w_1 \cdot \nu_y = 0 \text{ on } (0, T) \times \Omega \times (\Gamma_2 \cup \Gamma_3), \\ d_2 \nabla_y w_2 \cdot \nu_y = 0 \text{ on } (0, T) \times \Omega \times (\Gamma_1 \cup \Gamma_3), \\ d_2 \nabla_y w_2 \cdot \nu_y = \alpha(h w_3 - w_2) \text{ on } (0, T) \times \Omega \times \Gamma_2, \\ d_3 \nabla w_3 \cdot \nu(x) = 0 \text{ on } (0, T) \times \Gamma_N, \quad w_3 = w_3^D \text{ on } (0, T) \times \Gamma_D, \end{cases} \tag{6}$$

where $w_1 = w_1(t, x, y)$ denotes the concentration of H_2SO_4 in $(0, T) \times \Omega \times Y$, $w_2 = w_2(t, x, y)$ the concentration of H_2S aqueous species in $(0, T) \times \Omega \times Y$, $w_3 = w_3(t, x)$ the concentration of H_2S gaseous species in $(0, T) \times \Omega$ and $w_4 = w_4(t, x, y)$ of *gypsum* concentration on $(0, T) \times \Omega \times \Gamma_1$ and η is the reaction rate of *gypsum*. ∇ without subscript denotes the differentiation w.r.t. macroscopic variable x, while ∇_y is the respective differential operators w.r.t. the micro-variable y, ν and ν_y are outward normal vectors on $\partial\Omega$ and ∂Y, respectively.

By α we denote the rate of the reaction taking place on the interface Γ_2, h is Henry's constant (see [BC66] for an extensive review on Henry's law), d_1, d_2 and d_3 are diffusion coefficients, ψ is a continuous function on $\mathbb{R}$ and γ is a positive constant. The microscale and macroscale are connected together via the right-hand side of (3) and via the *micro-macro* boundary condition $(6)_4$.

1.3   Comments on related PDE systems with multiple scales structure

Promoted initatially by G. I. Barenblatt and his co-workers (see e.g. [BZK60] and references cited therein), the research on fissured media-like equations in particular and on

double-porosity models in general has attracted an increasing interest on the mathematical analysis side especially on what pseudo-parabolic equations and nonlinear parabolic equations posed on multiple spatial scales are concerned; the focus being mostly on well-posedness aspects; see the original work of R.E Showalter and collaborators [BS85, HS90, SW91, CS95] which was a source of inspiration for later developments by M. Böhm and S. A. Meier ([MB08, Mei08, MM08, MM10]) and J. Escher and D. Treutler ([ET12, Tre12]).

Mathematical tools employed range from a fine use of strong solutions (exploiting the semigroup structure of the problem), fixed-point arguments in Bochner spaces, energy methods for parabolic equations as well as weak-convergence type methods (particularly the two-scale convergence put in the periodic homogenization context).

For a classification of two-scale PDE systems based on the used *micro-macro transmission condition*, we refer the reader to the chapter written by R. E. Showalter in U. Hornung's book [Hor97]. Furthermore, the reader will discover therein that the concept of balance equations on two-scales (or on a *distributed array of microstructures*) can be used as a stand alone modeling tool, not necessarily in the context of averaging techniques.

1.4   Aim of this paper

Very much in line with older results by Friedman, Knabner, and Tzavaras (compare [FT87, FK92]), the interest of this paper lies on the large-time asymptotics of the two-scale system (1)–(4) endowed with the initial conditions (5) and the boundary conditions (6).

In this special context of multiple spatial scales, we need to cope with two main specific difficulties:

(i) Due to the presence of the microscale (here denoted by Y) and evolution equations posed at that level, memory effects are inherently present; see e.g. [Ant93]. The question is here twofold: How strong are such memory effects and to which extent can they affect the lifespan of the coupled PDE system?

(ii) As micro-macro transmission condition we impose a nonlinear Henry's law, therefore particular care is needed while treating two-scale traces of Sobolev functions; see e.g. [MNR10].

It is worth noting that the large-time behavior is not only the most interesting mathematical question that one would pose at this stage, but also it is the most relevant one from the practical point of view – an estimate on the lifespan of the material [forced to confront, for instance, evolving free boundaries [AM10, AM13], potential clogging of the pores, and self-healing [ZACV13]] is the holy grail of the materials science.

For our problem (1)–(6), we basically show that the large-time behavior of the active concentrations is well described by the solution of the corresponding stationary system; see Theorem 2.6. In the rest of the paper, we prepare a suitable mathematical framework and then prove the large-time asymptotics for this multiscale reaction-diffusion scenario in a rigorous manner.

2 Assumptions and main results

To keep notation simple, we put

$$X := \{z \in H^1(\Omega) : z = 0 \text{ on } \Gamma_D\}, H := L^2(\Omega \times Y) \times L^2(\Omega \times Y) \times L^2(\Omega),$$

$$V := L^2(\Omega; H^1(Y)) \times L^2(\Omega; H^1(Y)) \times X, K(T) := W^{1,2}(0,T; L^2(\Omega \times \Gamma_1)) \text{ for } T > 0,$$

$$(u,v)_H := (u_1, v_1)_{L^2(\Omega \times Y)} + \gamma(u_2, v_2)_{L^2(\Omega \times Y)} + \gamma h(u_3, v_3)_{L^2(\Omega)}$$

for $u := (u_1, u_2, u_3)$, $v := (v_1, v_2, v_3) \in H$.

Assumption 2.1 *(A1) For $i = 1, 2$, $d_i \in L^\infty(\Omega \times Y)$ and $d_3 \in L^\infty(\Omega)$ such that $d_i(x,y) \geq d_i^0$ for a.e. $(x,y) \in \Omega \times Y$, where d_i^0 is a positive constant, and $d_3(x) \geq d_3^0$ for a.e. $x \in \Omega$, where d_3^0 is a positive constant.*

(A2) $\eta(r_1, r_2) := R(r_1)Q(r_2)$ for $r_1, r_2 \in \mathbb{R}$, where R and Q are locally Lipschitz continuous functions such that $R' \geq 0$ and $Q' \leq 0$ a.e. on $\mathbb{R}$ and

$$R(r_1) := \begin{cases} positive & if \ r_1 > 0, \\ 0 & otherwise, \end{cases} \qquad Q(r_2) := \begin{cases} positive & if \ r_2 < \beta_{max}, \\ 0 & otherwise, \end{cases}$$

where β_{max} is a positive constant.

(A3) The function ψ is increasing and locally Lipschitz continuous on $\mathbb{R}$ with $\psi(0) = 0$.

(A4) $w_3^D \in L_{loc}^2([0,\infty); H^2(\Omega)) \cap W_{loc}^{1,2}([0,\infty); L^2(\Omega)) \cap L_+^\infty((0,\infty) \times \Omega)$ with $\nabla w_3^D \cdot \nu = 0$ on $(0,\infty) \times \Gamma_N$.

(A5) $w_{i0} \in L^2(\Omega; H^1(Y)) \cap L_+^\infty(\Omega \times Y)$ for $i = 1, 2$, $w_{30} \in H^1(\Omega) \cap L_+^\infty(\Omega)$, $w_{30} - w_3^D(0, \cdot) \in X$, and $w_{40} \in L_+^\infty(\Omega \times \Gamma_1)$.

Note that in (A4) and (A5) we define $L_+^\infty(\Omega') := L^\infty(\Omega') \cap \{u : u \geq 0 \text{ on } \Omega'\}$ for a domain Ω'.

Next, we denote the two-scale problem (1)–(6) by $TP(R, Q, \psi)$ and give a definition of a solution to $TP(R, Q, \psi)$ as follows:

Definition 2.2 *We call the multiplet (w_1, w_2, w_3, w_4) a solution to $TP(R, Q, \psi)$ on $[0,T]$, $T > 0$, if (S1) $\sim$ (S5) hold:*
(S1) $w_1, w_2 \in W^{1,2}(0,T; L^2(\Omega \times Y)) \cap L^\infty(0,T; L^2(\Omega; H^1(Y))) \cap L^\infty((0,T) \times \Omega \times Y)$, $w_3 \in W^{1,2}(0,T; L^2(\Omega)) \cap L^\infty((0,T) \times \Omega)$, $w_3 - w_3^D \in L^\infty(0,T; X)$, $w_4 \in W^{1,2}(0,T; L^2(\Omega \times \Gamma_1)) \cap L^\infty((0,T) \times \Omega \times \Gamma_1)$.

$(S2) \qquad \displaystyle\int_{\Omega \times Y} \partial_t w_1 v_1 dx dy + \int_{\Omega \times Y} d_1 \nabla_y w_1 \cdot \nabla_y v_1 dx dy + \int_{\Omega \times \Gamma_1} Q(w_4) R(w_1) v_1 dx d\gamma_y$

$$= - \int_{\Omega \times Y} \psi(w_1 - \gamma w_2) v_1 dx dy \quad for \ v_1 \in L^2(\Omega; H^1(Y)) \ a.e. \ on \ [0,T].$$

$(S3) \qquad \displaystyle\int_{\Omega \times Y} \partial_t w_2 v_2 dx dy + \int_{\Omega \times Y} d_2 \nabla_y w_2 \cdot \nabla_y v_2 dx dy - \alpha \int_{\Omega \times \Gamma_2} (h w_3 - w_2) v_2 dx d\gamma_y$

4

$$= \int_{\Omega \times Y} \psi(w_1 - \gamma w_2)v_2 dx dy \quad \text{for } v_2 \in L^2(\Omega; H^1(Y)) \text{ a.e. on } [0,T].$$

(S4)
$$\int_{\Omega} \partial_t w_3 v_3 dx + \int_{\Omega} d_3 \nabla w_3 \cdot \nabla v_3 dx = -\alpha \int_{\Omega \times \Gamma_2} (hw_3 - w_2)v_3 dx d\gamma_y \text{ for } v_3 \in X \text{ a.e. on } [0,T].$$

(S5) (4) and (5) hold.

Moreover, the multiplet (w_1, w_2, w_3, w_4) is called a solution of $TP(R,Q,\psi)$ on $[0,\infty)$, if it is a solution of $TP(R,Q,\psi)$ on $[0,T]$ for any $T > 0$.

These two Theorems are concerned with the well-posedness of $TP(R,Q,\psi)$.

Theorem 2.3 *(Uniqueness) Assume (A1) $\sim$ (A5), then there exists at most one solution of $TP(R,Q,\psi)$.*

Theorem 2.4 *(Global existence of solutions to $TP(R,Q,\psi)$) Assume (A1) $\sim$ (A5), then there exists a solution (w_1, w_2, w_3, w_4) of $TP(R,Q,\psi)$ on $[0,\infty)$. Moreover, it holds that*

(i) $w_1(t), w_2(t) \geq 0$ a.e. in $\Omega \times Y$, $w_3(t) \geq 0$ a.e. in Ω and $w_4(t) \geq 0$ a.e. on $\Omega \times \Gamma_1$ for $t \geq 0$.

(ii) $w_1(t) \leq M_1$, $w_2(t) \leq M_2$ a.e. in $\Omega \times Y$, $w_3(t) \leq M_3$ a.e. in Ω and $w_4(t) \leq M_4$ a.e. on $\Omega \times \Gamma_1$ for $t \geq 0$, where $M_1 = \max\{|w_{10}|_{L^\infty(\Omega \times Y)}, \gamma|w_{20}|_{L^\infty(\Omega \times Y)}, \gamma h|w_{30}|_{L^\infty(\Omega)}, \gamma h|w_3^D|_{L^\infty((0,\infty) \times \Omega)}\}$, $M_2 = \frac{1}{\gamma} M_1$, $M_3 = \frac{1}{h} M_2$, and $M_4 = \max\{\beta_{max}, |w_{40}|_{L^\infty(\Omega \times \Gamma_1)}\}$.

Remark 2.1 *In [FMA12], we consider the system consisting in the equations (3) and (4) and prove its well-posedness (see Theorem 4.3 and Theorem 4.4 in [FMA12]):*

$$\partial_t w_1 - \nabla_y \cdot (d_1 \nabla_y w_1) = -f_1(w_1) + f_2(w_2) \quad \text{in } (0,T) \times \Omega \times Y,$$
$$\partial_t w_2 - \nabla_y \cdot (d_2 \nabla_y w_2) = f_1(w_1) - f_2(w_2) \quad \text{in } (0,T) \times \Omega \times Y,$$

where f_1 and f_2 are continuous functions on $\mathbb{R}$. It is easy to see that Theorems 2.3 and 2.4 can cover the well-posedness of $TP(R,Q,f_1 - f_2)$ in case $f_1(r) = a_1[r]^+$ and $f_2(r) = a_2[r]^+$ for $r \in \mathbb{R}$, where a_1 and a_2 are positive constants. In fact, Let $\psi(r) = a_1(r)$ for $r \in \mathbb{R}$ and $\gamma = \frac{a_2}{a_1}$. Then, by Theorem 2.4, the solution (w_1, w_2, w_3, w_4) of $TP(R,Q,\psi)$ satisfies that

- *w_1 and w_2 are nonnegative;*

- *$\psi(w_1 - \gamma w_2) = f_1(w_1) - f_2(w_2)$.*

To be able to study the large time behavior of the solution, we need the additional condition (A6) on the boundary data.

(A6) $w_3^D \in L^\infty(0,\infty; H^1(\Omega))$, $\partial_t w_3^D \in L^1((0,\infty) \times \Omega) \cap L^2((0,\infty) \times \Omega)$, $\partial_t w_3^D - \nabla d_3 \nabla w_3^D \in L^\infty(0,\infty; L^1(\Omega))$, $\partial_t(\partial_t w_3^D - \nabla d_3 \nabla w_3^D) \in L^1((0,\infty) \times \Omega)$.

Clearly, under (A6) we have: $w_3^D(t) \to w_{3\infty}^D$ in $L^2(\Omega)$ and weakly in $H^1(\Omega)$ as $t \to \infty$. Also, in order to give a statement on the large time behavior we introduce the following

5

stationary problem SP($w_{4\infty}$, $w_{3\infty}^D$) for given functions $w_{4\infty}$ and $w_{3\infty}^D$. In this problem unknown functions are $w_{1\infty}$, $w_{2\infty}$ and $w_{3\infty}$ such that

$$-\nabla_y \cdot (d_1 \nabla_y w_{1\infty}) = -\psi(w_{1\infty} - \gamma w_{2\infty}) \quad \text{in } \Omega \times Y,$$
$$-\nabla_y \cdot (d_2 \nabla_y w_{2\infty}) = \psi(w_{1\infty} - \gamma w_{2\infty}) \quad \text{in } \Omega \times Y,$$
$$-\nabla \cdot (d_3 \nabla w_{3\infty}) = -\alpha \int_{\Gamma_2} \left(hw_{3\infty} - w_{2\infty} \right) d\gamma_y \quad \text{in } \Omega,$$
$$d_1 \nabla_y w_{1\infty} \cdot \nu_y = -\eta(w_1, w_{4\infty}) \text{ on } \Omega \times \Gamma_1, \quad d_1 \nabla_y w_{1\infty} \cdot \nu_y = 0 \text{ on } \Omega \times (\Gamma_2 \cup \Gamma_3),$$
$$d_2 \nabla_y w_{2\infty} \cdot \nu_y = 0 \text{ on } \Omega \times (\Gamma_1 \cup \Gamma_3), \quad d_2 \nabla_y w_{2\infty} \cdot \nu_y = \alpha(hw_{3\infty} - w_{2\infty}) \text{ on } \Omega \times \Gamma_2,$$
$$d_3 \nabla w_{3\infty} \cdot \nu(x) = 0 \text{ on } \Gamma_N, \quad w_{3\infty} = w_{3\infty}^D \text{ on } \Gamma_D.$$

Definition 2.5 *We say that the triplet* $(w_{1\infty}, w_{2\infty}, w_{3\infty})$ *is a solution of SP($w_{4\infty}$, $w_{3\infty}^D$), if the following conditions hold:* $w_{i\infty} \in L^2(\Omega; H^1(Y)) \cap L^\infty(\Omega \times Y)$, $i = 1, 2$, $w_{3\infty} - w_{3\infty}^D \in X \cap L^\infty(\Omega)$,

$$\int_{\Omega \times Y} d_1 \nabla_y w_{1\infty} \cdot \nabla_y v_1 dx dy + \int_{\Omega \times \Gamma_1} Q(w_{4\infty}) R(w_{1\infty}) v_1 dx d\gamma_y$$
$$= -\int_{\Omega \times Y} \psi(w_{1\infty} - \gamma w_{2\infty}) v_1 dx dy \quad \text{for } v_1 \in L^2(\Omega, H^1(Y)),$$

$$\int_{\Omega \times Y} d_2 \nabla_y w_{2\infty} \cdot \nabla_y v_2 dx dy - \alpha \int_{\Omega \times \Gamma_2} (hw_{3\infty} - w_{2\infty}) v_2 dx d\gamma_y$$
$$= \int_{\Omega \times Y} \psi(w_{1\infty} - \gamma w_{2\infty}) v_2 dx dy \quad \text{for } v_2 \in L^2(\Omega, H^1(Y)),$$

$$\int_\Omega d_3 \nabla w_{3\infty} \cdot \nabla v_3 dx + \alpha \int_{\Omega \times \Gamma_2} (hw_{3\infty} - w_{2\infty}) v_3 dx d\gamma_y = 0 \quad \text{for } v_3 \in X.$$

Theorem 2.6 *(Large-time behavior) Assume (A1) $\sim$ (A6) and let* (w_1, w_2, w_3, w_4) *be a solution of TP(R, Q, ψ) on* $[0, \infty)$. *Then it holds that* $w_4(t) \to w_{4\infty}$ *in* $L^1(\Omega \times \Gamma_1)$ *as* $t \to \infty$ *for some* $w_{4\infty} \in L^\infty(\Omega \times \Gamma_1)$, *and* $\partial_t w_4 \in L^1((0, \infty) \times \Omega \times \Gamma_1)$ *and there exists a sequence* $\{t_n\}$ *with* $t_n \to \infty$ *as* $n \to \infty$ *such that*

$$w(t_n) \to w_\infty \quad \text{weakly in } H \text{ as } n \to \infty,$$

for some $w_\infty \in H$, *and* w_∞ *is a solution of SP($w_{4\infty}$, $w_{3\infty}^D$), where* $w = (w_1, w_2, w_3)$.

Moreover, if $(\psi(r) - \psi(r'))(r - r') \geq \mu|r - r'|^{p+1}$ *for* $r, r' \in \mathbb{R}$, *where* $\mu > 0$ *and* $p \geq 1$, *then SP($w_{4\infty}$, $w_{3\infty}^D$) has at most one solution and*

$$w(t) \to w_\infty \quad \text{in } H \text{ as } t \to \infty. \tag{7}$$

6

3 Well-posedness of $\mathrm{TP}(R, Q, \psi)$

The aim of this section is to show the existence and uniqueness of a solution to $\mathrm{TP}(R, Q, \psi)$ on $[0, T]$ for any $T > 0$.

First, we consider the following auxiliary problem $\mathrm{AP}(w_4)$ for given $w_4 \in K(T)$:

$$\partial_t w_1 - \nabla_y \cdot (d_1 \nabla_y w_1) = -\psi(w_1 - \gamma w_2) \quad \text{in } (0, T) \times \Omega \times Y, \tag{8}$$

$$\partial_t w_2 - \nabla_y \cdot (d_2 \nabla_y w_2) = \psi(w_1 - \gamma w_2) \quad \text{in } (0, T) \times \Omega \times Y, \tag{9}$$

$$\partial_t w_3 - \nabla \cdot (d_3 \nabla w_3) = -\alpha \int_{\Gamma_2} \left(h w_3 - w_2 \right) d\gamma_y \quad \text{in } (0, T) \times \Omega, \tag{10}$$

$$w_i(0) = w_{i0} \text{ in } \Omega \times Y \text{ for } i = 1, 2, \text{ and } w_3(0) = w_3^0 \text{ in } \Omega, \tag{11}$$

$$d_1 \nabla_y w_1 \cdot \nu_y = -\eta(w_1, w_4) \quad \text{on } (0, T) \times \Omega \times \Gamma_1, \tag{12}$$

$$d_1 \nabla_y w_1 \cdot \nu_y = 0 \quad \text{on } (0, T) \times \Omega \times (\Gamma_2 \cup \Gamma_3), \tag{13}$$

$$d_2 \nabla_y w_2 \cdot \nu_y = 0 \quad \text{on } (0, T) \times \Omega \times (\Gamma_1 \cup \Gamma_3), \tag{14}$$

$$d_2 \nabla_y w_2 \cdot \nu_y = \alpha(h w_3 - w_2) \quad \text{on } (0, T) \times \Omega \times \Gamma_2, \tag{15}$$

$$d_3 \nabla w_3 \cdot \nu(x) = 0 \text{ on } (0, T) \times \Gamma_N, w_3 = w_3^D \text{ on } (0, T) \times \Gamma_D. \tag{16}$$

From now on, we solve the above problem $\mathrm{AP}(w_4)$ by using the theory of evolution equations governed by subdifferential operators of time-dependent convex functions (see [Yam76] and [Ken81]). To apply this theory, we first define a function $\varphi^t(w_4, \cdot)$ on H for $t \in [0, T]$, w_3^D satisfying (A4) and $w_4 \in L^2(\Omega \times \Gamma_1)$ as follows:

$$\varphi^t(w_4, u)$$
$$= \begin{cases} \dfrac{1}{2} \displaystyle\int_{\Omega \times Y} d_1 |\nabla_y u_1|^2 dxdy + \int_{\Omega \times \Gamma_1} Q(w_4) \hat{R}(u_1) dxd\gamma_y + \dfrac{\gamma}{2} \int_{\Omega \times Y} d_2 |\nabla_y u_2|^2 dxdy \\ \quad + \displaystyle\int_{\Omega \times Y} \hat{\psi}(u_1 - \gamma u_2) dxdy + \dfrac{\gamma \alpha}{2} \int_{\Omega \times \Gamma_2} |h(u_3 + w_3^D(t)) - u_2|^2 dxd\gamma_y \\ \quad + \dfrac{\gamma h}{2} \displaystyle\int_{\Omega} d_3 |\nabla u_3|^2 dx \quad \text{if } u = (u_1, u_2, u_3) \in V, \\ \infty \quad \text{otherwise,} \end{cases}$$

where $\hat{R}$ and $\hat{\psi}$ are primitives of R and ψ with $\hat{R}(0) = 0$ and $\hat{\psi}(0) = 0$, respectively. Moreover, we can prove the following Lemma in a straightforward manner:

Lemma 3.1 *Let $t \in [0, T]$. If (A1) $\sim$ (A4) hold, Q is Lipschitz continuous and bounded on $\mathbb{R}$, R and ψ are Lipschitz continuous on $\mathbb{R}$ and $w_4 \in K(T)$, then $\varphi^t(w_4, \cdot)$ is proper, l.s.c. and convex on H for $t \in [0, T]$ and $D(\varphi^t(w_4, \cdot)) = V$ and $\partial \varphi^t(w_4, u)$ is single valued, where $D(\varphi^t(w_4, \cdot))$ denotes the effective domain of $\varphi^t(w_4, \cdot)$. Moreover, $u \in H$ and $u^* = \partial \varphi^t(w_4, u)$ is equivalent to $u^* = (u_1^*, u_2^*, u_3^*) \in H$ and*

$$(u_1^*, v_1)_{L^2(\Omega \times Y)} = \int_{\Omega \times Y} d_1 \nabla_y u_1 \cdot \nabla_y v_1 dxdy + \int_{\Omega \times \Gamma_1} Q(w_4) R(u_1) v_1 dxd\gamma_y$$

$$+ \int_{\Omega \times Y} \psi(u_1 - \gamma u_2) v_1 dxdy \quad \text{for } v_1 \in L^2(\Omega, H^1(Y)),$$

7

$$(u_2^*, v_2)_{L^2(\Omega \times Y)} = \int_{\Omega \times Y} d_2 \nabla_y u_2 \cdot \nabla_y v_2 dx dy - \alpha \int_{\Omega \times \Gamma_2} (h(u_3 + w_3^D(t)) - u_2) v_2 dx d\gamma_y$$

$$- \int_{\Omega \times Y} \psi(u_1 - \gamma u_2) v_2 dx dy \quad \text{for } v_2 \in L^2(\Omega, H^1(Y)),$$

$$(u_3^*, v_3)_{L^2(\Omega)} = \int_{\Omega} d_3 \nabla u_3 \cdot \nabla v_3 dx + \alpha \int_{\Omega \times \Gamma_2} (h(u_3 + w_3^D(t)) - u_2) v_3 dx d\gamma_y \text{ for } v_3 \in X.$$

The next Lemma is concerned with the continuity property of φ^t with respect to t.

Lemma 3.2 *Let $T > 0$. If (A1) $\sim$ (A5) hold, R and Q are Lipschitz continuous and bounded on $\mathbb{R}$, ψ is Lipschitz continuous on $\mathbb{R}$ and $w_4 \in K(T)$, then for any $r > 0$ there exists $b_r \in W^{1,2}(0, T)$ such that for $0 \le s \le t \le T$ and $u \in V$ with $|u|_H \le r$ it holds that*

$$\varphi^t(w_4(t), u) - \varphi^s(w_4(s), u) \le |b_r(t) - b_r(s)|(1 + |\varphi^s(w_4(s), u)|).$$

Accordingly, by the theory of evolution equations developed cf. [Ken81] and by Lemma 3.2, we can deduce the solvability of the following problem.

Lemma 3.3 *Let $T > 0$. If (A1) $\sim$ (A5) hold, R and Q are Lipschitz continuous and bounded on $\mathbb{R}$, ψ is Lipschitz continuous on $\mathbb{R}$ and $w_4 \in K(T)$, then for any $u_0 \in V$ there exists one and only $u \in W^{1,2}(0, T; H)$ such that $\varphi^{(\cdot)}(w_4(\cdot), u(\cdot)) \in L^\infty(0, T)$ and*

$$\frac{d}{dt}u(t) + \partial\varphi^t(w_4(t), u(t)) = f(t) \text{ in } H \text{ for } t \in [0, T] \text{ and } u(0) = u_0, \quad (17)$$

where $f(t) = (0, 0, -\partial_t w_3^D(t) + \nabla d_3 \nabla w_3^D(t))$ for $t \in [0, T]$.

Lemma 3.1 and Lemma 3.3 guarantee the existence of a solution of $AP(w_4)$.

Lemma 3.4 *Let $T > 0$. If (A1) $\sim$ (A5) hold, R and Q are Lipschitz continuous and bounded on $\mathbb{R}$, ψ is Lipschitz continuous on $\mathbb{R}$ and $w_4 \in K(T)$, then $AP(w_4)$ has a unique solution $w = (w_1, w_2, w_3)$ on $[0, T]$. Precisely speaking, $w_1, w_2 \in W^{1,2}(0, T; L^2(\Omega \times Y)) \cap L^\infty(0, T; L^2(\Omega; H^1(Y)))$, $w_3 \in W^{1,2}(0, T; L^2(\Omega))$, $w_3 - w_3^D \in L^\infty(0, T; X)$, (8) $\sim$ (16) hold in the usual sense.*

Proof. By putting $u_0 = (w_{10}, w_{20}, w_{30} - w_3^D(0))$ we see that $u_0 \in V$. Then on account of Lemma 3.3 there exists $u = (u_1, u_2, u_3)$ satisfying (17). Hence, by putting $w_1 = u_1$, $w_2 = u_2$ and $w_3 = u_3 + w_3^D$ it is clear that the assertion of this proposition is true. $\square$

Now, we show a proof of Theorem 2.4 by applying the fixed point argument in case Q, R and ψ are Lipschitz continuous.

Proposition 3.5 *Let $T > 0$. If (A1) $\sim$ (A5) hold, R and Q are Lipschitz continuous and bounded on $\mathbb{R}$, ψ is Lipschitz continuous on $\mathbb{R}$, then $TP(R, Q, \psi)$ has a unique solution on $[0, T]$.*

Proof. Let $T > 0$. By Lemma 3.4 for $w_4 \in K(T)$ these exists one and only one solution $(w_1, w_2, w_3) \in W^{1,2}(0, T; H)$ of $AP(w_4)$. Then we can define a mapping Λ_T in the following way: $\Lambda_T : K(T) \to K(T)$ is given by

$$(\Lambda_T w_4)(t) = w_{04} + \int_0^t \eta(w_1(\tau), w_4(\tau)) d\tau \quad \text{for } t \in [0, T].$$

Obviously, this mapping is well-defined. From now on we shall show that Λ_T is a contraction mapping for small $T > 0$.

Let $w_4^{(1)}, w_4^{(2)} \in K(T)$, $w^{(1)} = (w_1^{(1)}, w_2^{(1)}, w_3^{(1)})$ and $w^{(2)} = (w_1^{(2)}, w_2^{(2)}, w_3^{(2)})$ be solutions of $AP(w_4^{(1)})$ and $AP(w_4^{(2)})$, respectively, and put $w_4 = w_4^{(1)} - w_4^{(2)}$, $w = w^{(1)} - w^{(2)} = (w_1, w_2, w_3)$. It from (8) follows that

$$\partial_t w_1 - \nabla_y \cdot (d_1 \nabla_y w_1) = -\psi(w_1^{(1)} - \gamma w_2^{(1)}) + \psi(w_1^{(2)} - \gamma w_2^{(2)}) \quad \text{in } (0, T) \times \Omega \times Y.$$

By multiplying it by w_1 we have

$$\frac{1}{2}\frac{d}{dt}|w_1|^2_{L^2(\Omega \times Y)} + \int_{\Omega \times Y} d_1 |\nabla_y w_1|^2 dx dy$$
$$+ \int_{\Omega \times \Gamma_1} (Q(w_4^{(1)})R(w_1^{(1)}) - Q(w_4^{(2)})R(w_1^{(2)})) w_1 dx d\gamma_y$$
$$= - \int_{\Omega \times Y} (\psi(w_1^{(1)} - \gamma w_2^{(1)}) - \psi(w_1^{(2)} - \gamma w_2^{(2)})) w_1 dx dy \quad \text{a.e. on } [0, T]. \quad (18)$$

Similarly to (18), we see that

$$\frac{\gamma}{2}\frac{d}{dt}|w_2|^2_{L^2(\Omega \times Y)} + \gamma \int_{\Omega \times Y} d_2 |\nabla_y w_2|^2 dx dy$$
$$= \gamma \alpha \int_{\Omega \times \Gamma_2} (h w_3 - w_2) w_2 dx d\gamma_y$$
$$+ \gamma \int_{\Omega \times Y} (\psi(w_1^{(1)} - \gamma w_2^{(1)}) - \psi(w_1^{(2)} - \gamma w_2^{(2)})) w_2 dx dy \quad \text{a.e. on } [0, T],$$

and

$$\frac{\gamma h}{2}\frac{d}{dt}|w_3|^2_{L^2(\Omega)} + \gamma h \int_{\Omega} d_3 |\nabla w_3|^2 dx = -\gamma h \alpha \int_{\Omega \times \Gamma_2} (h w_3 - w_2) w_3 dx d\gamma_y \text{ a.e. on } [0, T].$$

Since (A2) implies that R is increasing on $\mathbb{R}$, we obtain

$$\int_{\Omega \times \Gamma_1} (Q(w_4^{(1)})R(w_1^{(1)}) - Q(w_4^{(2)})R(w_1^{(2)})) w_1 dx d\gamma_y$$
$$\geq - \int_{\Omega \times \Gamma_1} |Q(w_4^{(1)}) - Q(w_4^{(2)})||R(w_1^{(2)})||w_1| dx d\gamma_y \quad \text{a.e. on } [0, T].$$

Accordingly, it holds that

$$\frac{1}{2}\frac{d}{dt}|w|^2_H + d_1^0 \int_{\Omega \times Y} |\nabla_y w_1|^2 dx dy + \gamma d_2^0 \int_{\Omega \times Y} |\nabla_y w_2|^2 dx dy + \gamma h d_3^0 \int_{\Omega} |\nabla w_3|^2 dx$$

9

$$\leq \int_{\Omega \times \Gamma_1} |Q(w_4^{(1)}) - Q(w_4^{(2)})||R(w_1^{(2)})||w_1|dxd\gamma_y$$

$$- \int_{\Omega \times Y} (\psi(w_1^{(1)} - \gamma w_2^{(1)}) - \psi(w_1^{(2)} - \gamma w_2^{(2)}))(w_1 - \gamma w_2)dxdy$$

$$\leq C_Q|R|_{L^\infty(\mathbb{R})} \int_\Omega |w_1|_{L^2(\Gamma_1)}|w_4|_{L^2(\Gamma_1)}dx$$

$$\leq C_Y C_Q|R|_{L^\infty(\mathbb{R})} \int_\Omega (|\nabla_y w_1|_{L^2(Y)} + |w_1|_{L^2(Y)})|w_4|_{L^2(\Gamma_1)}dx$$

$$\leq \frac{d_1^0}{2} \int_\Omega |\nabla_y w_1|_{L^2(Y)}^2 dx + \frac{1}{2} \int_\Omega |w_1|_{L^2(Y)}^2 dx + C_0 \int_\Omega |w_4|_{L^2(\Gamma_1)}^2 dx \quad \text{a.e. on } [0,T],$$

where C_Q is a Lipschitz constant of Q, $C_0 = (\frac{1}{2d_1^0} + \frac{1}{2})(C_Y C_Q|R|_{L^\infty(\mathbb{R})})^2$ and C_Y is a positive constant such that

$$|z|_{L^2(\Gamma_1)} \leq C_Y|z|_{H^1(\Omega)} \text{ for } z \in H^1(Y). \tag{19}$$

From these inequalities, it follows that

$$\frac{1}{2}\frac{d}{dt}|w|_H^2 + \frac{d_1^0}{2} \int_{\Omega \times Y} |\nabla_y w_1|^2 dxdy + \gamma d_2^0 \int_{\Omega \times Y} |\nabla_y w_2|^2 dxdy + \gamma h d_3^0 \int_\Omega |\nabla w_3|^2 dxdy$$

$$\leq C_0 \int_\Omega |w_4|_{L^2(\Gamma_1)}^2 dx + \frac{1}{2} \int_\Omega |w_1|_{L^2(Y)}^2 dx$$

$$\leq C_1(\int_\Omega |w_4|_{L^2(\Gamma_1)}^2 dx + |w|_H^2) \quad \text{a.e. on } [0,T],$$

where $C_1 = C_0 + \frac{1}{2}$. Then by applying Gronwall's inequality we obtain

$$\frac{1}{2}|w(t)|_H^2 + \int_0^t (\frac{d_1^0}{2}|\nabla_y w_1|_{L^2(\Omega \times Y)}^2 + \gamma d_2^0|\nabla_y w_2|_{L^2(\Omega \times Y)}^2 + \gamma h d_3^0|\nabla w_3|_{L^2(\Omega)}^2)d\tau$$

$$\leq C_2 \int_0^t |w_4|_{L^2(\Omega \times \Gamma_1)}^2 d\tau \quad \text{for } t \in [0,T], \tag{20}$$

where C_2 is a positive constant. This shows that

$$|\partial_t(\Lambda w_4^{(1)}) - \partial_t(\Lambda w_4^{(2)})|_{L^2(0,T;L^2(\Omega \times \Gamma_1))}$$

$$\leq |R(w_1^{(1)})Q(w_4^{(1)}) - R(w_1^{(2)})Q(w_4^{(2)})|_{L^2(0,T;L^2(\Omega \times \Gamma_1))}$$

$$\leq C_R|Q|_{L^\infty(\mathbb{R})}|w_1|_{L^2(0,T;L^2(\Omega \times \Gamma_1))} + C_Q|R|_{L^\infty(\mathbb{R})}|w_4|_{L^2(0,T;L^2(\Omega \times \Gamma_1))})$$

$$\leq (1 + C_Y)C_R|Q|_{L^\infty(\mathbb{R})}|w_1|_{L^2(0,T;L^2(\Omega;H^1(Y)))} + C_Q|R|_{L^\infty(\mathbb{R})}|w_4|_{L^2(0,T;L^2(\Omega \times \Gamma_1))})$$

$$\leq C_3|w_4|_{L^2(0,T;L^2(\Omega \times \Gamma_1))}$$

$$\leq C_3|\int_0^t \partial_\tau w_4 d\tau|_{L^2(0,T;L^2(\Omega \times \Gamma_1))}$$

$$\leq C_3 T|\partial_t w_4|_{L^2(0,T;L^2(\Omega \times \Gamma_1))}, \tag{21}$$

where C_R is a Lipschitz constant of R and C_3 is a positive constant depending only on C_2. Moreover, by using (21) we have

$$|\Lambda w_4^{(1)} - \Lambda w_4^{(2)}|_{L^2(0,T;L^2(\Omega \times \Gamma_1))} \leq |\int_0^t \partial_t(\Lambda w_4^{(1)} - \Lambda w_4^{(2)})d\tau|_{L^2(0,T;L^2(\Omega \times \Gamma_1))}$$

$$\leq \quad T|\partial_t(\Lambda w_4^{(1)}) - \partial_t(\Lambda w_4^{(2)})|_{L^2(0,T;L^2(\Omega\times\Gamma_1))}$$
$$\leq \quad C_4 T^2 |\partial_t w_4|_{L^2(0,T;L^2(\Omega\times\Gamma_1))},$$

where C_4 is a positive constant.

Hence, if T_1 is sufficiently small, then Λ_{T_1} is a contraction mapping. Namely, Banach's fixed point theorem shows that $TP(R,Q,\psi)$ has a solution on $[0,T_1]$. Furthermore, the choice of T_1 is independent of initial values so that we conclude that $TP(R,Q,\psi)$ has a solution on $[0,T]$.

The uniqueness of a solution is a direct consequence of the Banach's fixed point theorem. □

Lemma 3.6 *Let $T > 0$ and assume that (A1) $\sim$ (A5) hold, R and Q are Lipschitz continuous and bounded on $\mathbb{R}$, ψ is Lipschitz continuous on $\mathbb{R}$. If (w_1, w_2, w_3, w_4) is a solution of $TP(R,Q,\psi)$ on $[0,T]$, then it holds that*

$$0 \leq w_i \leq M_i \ a.e. \ on \ (0,T) \times \Omega \times Y \ for \ i = 1, 2,$$

$$0 \leq w_3 \leq M_3 \ a.e. \ on \ (0,T) \times \Omega, \ and \ 0 \leq w_4 \leq M_4 \ a.e. \ on \ (0,T) \times \Omega \times \Gamma_1,$$

where M_i is the positive constant defined in Theorem 2.4 for each $i = 1, 2, 3, 4$.

The proof of this lemma is quite similar to that of [FMA12, Theorem 4.4] so that we omit a proof of Lemma 3.6.

By using Proposition 3.5 and Lemma 3.6, we can now prove Theorem 2.3 and Theorem 2.4.

Sketch of the proofs of Theorem 2.3 and Theorem 2.4. First, for $m > 0$ we put

$$R_m(r) = \left\{ \begin{array}{ll} R(m) & \text{for } r > m, \\ R(r) & \text{otherwise,} \end{array} \right. \quad Q_m(r) = \left\{ \begin{array}{ll} Q(m) & \text{for } r > m, \\ Q(r) & \text{for } 0 \leq r \leq m, \\ Q(0) & \text{otherwise,} \end{array} \right.$$

and

$$\psi_m(r) = \left\{ \begin{array}{ll} \psi(m) & \text{for } r > m, \\ \psi(r) & \text{for } 0 \leq r \leq m, \\ \psi(-m) & \text{otherwise.} \end{array} \right.$$

Then, Proposition 3.5 implies that $TP(R_m, Q_m, \psi_m)$ has a solution $(w_{1m}, w_{2m}, w_{3m}, w_{4m})$ on $[0,T]$ for any $T > 0$. Moreover, on account of Lemma 3.6 we can choose a positive constant $m > 0$ such that $R_m(w_{1m}) = R(w_{1m})$, $Q_m(w_{1m}) = Q(w_{1m})$ and $\psi_m(w_{1m} - \gamma w_{2m}) = \psi(w_{1m} - \gamma w_{2m})$, since M_i is independent of m for each $i = 1, 2, 3, 4$. This shows the conclusion of Theorem 2.4. Moreover, the uniqueness of a solution of $TP(R,Q,\psi)$ is a direct consequence of Proposition 3.5. □

4 The quest of the large-time behavior

To prove Theorem 2.6, throughout this section we always assume (A1) $\sim$ (A5) and (A6), and denote a solution of $TP(R,Q,\psi)$ on $[0,\infty)$ by (w_1, w_2, w_3, w_4). Clearly, by putting

$f := \partial_t w_3^D - \nabla d_3 \nabla w_3^D$ there exists $f_\infty \in L^\infty(\Omega)$ such that $f(t) \to f_\infty$ in $L^1(\Omega)$ as $t \to \infty$. Also, let $m_0 = \max\{M_1, M_2, M_3, M_4\}$. Then for some $m_1 \geq m_0$ it holds that $R_{m_1}(z_1) = R(z_1), \psi_{m_1}(z_1 - \gamma z_2) = \psi(z_1 - \gamma z_2)$ for $z = (z_1, z_2, z_3) \in U(m_0)$, where $U(m_0) = \{z = (z_1, z_2, z_3) \in H | 0 \leq z_1 \leq m_0, 0 \leq z_2 \leq m_0$ a.e. on $\Omega \times Y, 0 \leq z_3 \leq m_0$ a.e. on $\Omega\}$. Moreover, for simplicity, we write R, Q and ψ as R_{m_1}, Q_{m_1} and ψ_{m_1}, respectively, in this section, and set

$$\ell_1 = R(M_1), \ell_2 = |Q(w_4)|_{L^\infty((0,\infty) \times \Omega \times \Gamma_1)}, \ell_3 = \hat{R}(M_1), \ell_4 = \sup\{|Q'(r)| : |r| \leq m_0|\}.$$

First, we show the convergence of $w_4(t)$ as $t \to \infty$.

Lemma 4.1 $w_4(t)$ converges to some function $w_{4\infty} \in L^\infty(\Omega \times \Gamma_1)$ in $L^1(\Omega \times \Gamma_1)$ as $t \to \infty$, $0 \leq w_{4\infty} \leq m_0$ a.e. on $\Omega \times \Gamma_1$, and

$$\partial_t w_4 \in L^1(0, \infty; L^1(\Omega \times \Gamma_1)) \cap L^2(0, \infty; L^2(\Omega \times \Gamma_1)). \tag{22}$$

Proof. By (A2) it is obvious that $\partial_t w_4 = \eta(w_1, w_4) \geq 0$ a.e. on $(0, \infty) \times \Omega \times \Gamma_1$. Then Lemma 3.6 and the Lebesgue monotone convergence theorem imply that $w_4(t) \to w_{4\infty}$ in $L^1(\Omega \times \Gamma_1)$ as $t \to \infty$, where $w_{4\infty} \in L^\infty(\Omega \times \Gamma_1)$.

It is easy to see that

$$\int_0^T \int_{\Omega \times \Gamma_1} |\partial_t w_4| dx d\gamma_y dt = \int_0^T \int_{\Omega \times \Gamma_1} \partial_t w_4 dx d\gamma_y dt$$
$$= \int_{\Omega \times \Gamma_1} (w_4(T) - w_{04}) dx d\gamma_y \quad \text{for } T > 0,$$

and

$$\int_0^T \int_{\Omega \times \Gamma_1} |\partial_t w_4|^2 dx d\gamma_y dt \leq \ell_1 \ell_2 \int_0^T \int_{\Omega \times \Gamma_1} |\partial_t w_4| dx d\gamma_y dt \text{ for } T > 0.$$

Then these estimates lead to (22). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now, we provide uniform estimates on the time derivative of solutions.

Lemma 4.2 *It holds that*

$$\partial_t w_1, \partial_t w_2 \in L^2(0, \infty; L^2(\Omega \times Y)), \partial_t \tilde{w}_3 \in L^2(0, \infty; L^2(\Omega)),$$

$$w_1, w_2 \in L^\infty(0, \infty; L^2(\Omega; H^1(Y))), \tilde{w}_3 \in L^\infty(0, \infty; X),$$

where $\tilde{w}_3 = w_3 - w_3^D$.

Proof. By multiplying (1) by $\partial_t w_1$, we can obtain

$$\int_{\Omega \times Y} |\partial_t w_1|^2 dx dy + \frac{1}{2} \frac{d}{dt} \int_{\Omega \times Y} d_1 |\nabla_y w_1|^2 dx dy + \frac{d}{dt} \int_{\Omega \times \Gamma_1} Q(w_4) \hat{R}(w_1) dx d\gamma_y$$
$$= -\int_{\Omega \times Y} \psi(w_1 - \gamma w_2) \partial_t w_1 dx dy + \int_{\Omega \times \Gamma_1} Q'(w_4) \partial_t w_4 \hat{R}(w_1) dx d\gamma_y \quad \text{a.e. on } [0, \infty).$$

12

Next, we multiply (2) by $\gamma\partial_t w_2$. Then we see that

$$\gamma\int_{\Omega\times Y}|\partial_t w_2|^2 dxdy + \frac{\gamma}{2}\frac{d}{dt}\int_{\Omega\times Y}d_2|\nabla_y w_2|^2 dxdy$$

$$+\gamma\alpha\int_{\Omega\times\Gamma_2}(w_2 - h(\tilde{w}_3 + w_3^D))\partial_t w_2 dxd\gamma_y$$

$$= \gamma\int_{\Omega\times Y}\psi(w_1 - \gamma w_2)\partial_t w_2 dxdy \quad \text{a.e. on } [0,\infty).$$

Easily, we have

$$\partial_t\tilde{w}_3 - \nabla d_3\nabla\tilde{w}_3 = -\alpha\int_{\Gamma_2}\big(h(\tilde{w}_3 + w_3^D) - w_2\big)d\gamma_y - f \quad \text{a.e. on } (0,\infty)\times\Omega,$$

where $f = \partial_t w_3^D - \nabla d_3\nabla w_3^D$. Here, we multiply it by $\gamma h\partial_t\tilde{w}_3$ and observe that

$$\gamma h\int_\Omega|\partial_t\tilde{w}_3|^2 dx + \frac{\gamma h}{2}\frac{d}{dt}\int_\Omega d_3|\nabla\tilde{w}_3|^2 dx$$

$$= -\alpha\gamma h\int_{\Omega\times\Gamma_2}(h(\tilde{w}_3 + w_3^D) - w_2)\partial_t\tilde{w}_3 dxd\gamma_y - \gamma h\int_\Omega f\partial_t\tilde{w}_3 dx$$

$$= -\alpha\gamma h\int_{\Omega\times\Gamma_2}(h(\tilde{w}_3 + w_3^D) - w_2)(\partial_t\tilde{w}_3 + \partial_t w_3^D)dxd\gamma_y$$

$$+\alpha\gamma h\int_{\Omega\times\Gamma_2}(h(\tilde{w}_3 + w_3^D) - w_2)\partial_t w_3^D dxd\gamma_y - \gamma h\frac{d}{dt}\int_\Omega f\tilde{w}_3 dx + \gamma h\int_\Omega\partial_t f\tilde{w}_3 dx$$

a.e. on $[0,\infty)$. By adding these equations we obtain

$$\int_{\Omega\times Y}|\partial_t w_1|^2 dxdy + \gamma\int_{\Omega\times Y}|\partial_t w_2|^2 dxdy + \gamma h\int_\Omega|\partial_t\tilde{w}_3|^2 dx$$

$$+\frac{1}{2}\frac{d}{dt}\left(\int_{\Omega\times Y}d_1|\nabla_y w_1|^2 dxdy + \gamma\int_{\Omega\times Y}d_2|\nabla_y w_3|^2 dxdy + \gamma h\int_\Omega d_3|\nabla\tilde{w}_3|^2 dx\right)$$

$$+\frac{d}{dt}\int_{\Omega\times\Gamma_1}Q(w_4)\hat{R}(w_1)dxd\gamma_y + \gamma\alpha\frac{d}{dt}\int_{\Omega\times\Gamma_2}|w_2 - h(\tilde{w}_3 + w_3^D)|^2 dxd\gamma_y$$

$$+\frac{d}{dt}\int_{\Omega\times Y}\hat{\psi}(w_1 - \gamma w_2)dxdy$$

$$= \int_{\Omega\times\Gamma_1}Q'(w_4)\partial_t w_4\hat{R}(w_1)dxd\gamma_y + \alpha\gamma h\int_{\Omega\times\Gamma_2}(h(\tilde{w}_3 + w_3^D) - w_2)\partial w_3^D dxd\gamma_y \quad (23)$$

$$-\gamma h\frac{d}{dt}\int_\Omega f\tilde{w}_3 dx + \gamma h\int_\Omega\partial_t f\tilde{w}_3 dx \quad \left(=: \sum_{i=1}^4 I_i(\cdot)\right) \quad \text{a.e. on } [0,\infty).$$

Since

$$I_1(t) \le \ell_3\ell_4\int_{\Omega\times\Gamma_1}|\partial_t w_4(t)|dxd\gamma_y \text{ for a.e. } t \ge 0,$$

it holds that $I_1 \in L^1(0, \infty)$. It is clear that

$$
\begin{aligned}
I_2 &= \alpha\gamma h \int_{\Omega\times\Gamma_2} (hw_3 - w_2)\partial_t w_3^D dx d\gamma_y \\
&\leq \alpha\gamma h(hM_3 + M_2) \int_{\Omega\times\Gamma_2} |\partial_t w_3^D| dx d\gamma_y \quad \text{a.e. on } [0, \infty)
\end{aligned}
$$

so that $I_2 \in L^1(0, \infty)$. Also, we see that

$$
\int_0^t I_3(\tau)d\tau = \gamma h(-\int_\Omega f(t)\tilde{w}_3(t)dx + \int_\Omega f(0)(w_{30} + w_3^D(0))dx) \leq B_1 \text{ for } t \geq 0,
$$

where B_1 is a positive constant, and

$$
I_4(t) \leq \gamma h(M_3 + |w_3^D|_{L^\infty((0,\infty)\times\Omega)}) \int_\Omega |\partial_t f(t)| dx \quad \text{for } t \geq 0
$$

so that $I_4 \in L^\infty(0, \infty)$. Hence, we have proved this lemma. $\qquad\square$

To describe the large-time behavior of the solution, we introduce the following notations: We put $w(t) := (w_1(t), w_2(t), w_3(t)) \in H$ for $t \geq 0$,

$$
\omega(w_0) = \left\{ z \in H \,\middle|\, \begin{array}{l} w(t_n) \to z \text{ weakly in } H \text{ as } n \to \infty \text{ for some sequence } \{t_n\} \\ \text{with } t_n \to \infty \text{ as } n \to \infty \end{array} \right\},
$$

where $w_0 = (w_{10}, w_{20}, w_{30})$,

$$
\varphi_1^t(u) = \varphi^t(w_4(t), u) - \gamma h \int_\Omega f(t)u_3 dx \text{ for } u = (u_1, u_2, u_3) \in H,
$$

$$
\varphi_1^\infty(u)
$$
$$
= \begin{cases} \dfrac{1}{2}\displaystyle\int_{\Omega\times Y} d_1|\nabla_y u_1|^2 dxdy + \int_{\Omega\times\Gamma_1} Q(w_{4\infty})\hat{R}(u_1)dxd\gamma_y + \dfrac{\gamma}{2}\int_{\Omega\times Y} d_2|\nabla_y u_2|^2 dxdy \\ + \displaystyle\int_{\Omega\times Y} \hat{\psi}(u_1 - \gamma u_2)dxdy + \dfrac{\gamma\alpha}{2}\int_{\Omega\times\Gamma_2} |h(u_3 + w_{3\infty}^D) - u_2|^2 dxd\gamma_y \\ + \dfrac{\gamma h}{2}\displaystyle\int_\Omega d_3|\nabla u_3|^2 dx - \gamma h \int_\Omega f_\infty u_3 dx \quad \text{if } u = (u_1, u_2, u_3) \in V, \\ \infty \quad \text{otherwise}, \end{cases}
$$

and

$$
F(\varphi_1^\infty) = \{z \in H | \varphi_1^\infty(z) = \min_{u\in H} \varphi_1^\infty(u)\}.
$$

Clearly, the similar results to Lemma 3.1 hold for φ_1^∞. Here, we note that w satisfies (S2), (S3) and (S4) if and only if $\tilde{w} = (w_1, w_2, \tilde{w}_3)$ with $\tilde{w}_3 = w_3 - w_3^D$ is a solution of the following evolution equation:

$$
\tilde{w}_t(t) + \partial\varphi_1^t(\tilde{w}(t)) = 0 \text{ in } H \text{ for a.e. } t \geq 0. \tag{24}
$$

14

Moreover, since by Theorem 2.4 $\{w(t)\}_{t \geq 0}$ is bounded in H, there exist a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ and $w_\infty = (w_{1\infty}, w_{2\infty}, w_{3\infty}) \in U(m_0)$ such that

$$w(t_n) \to w_\infty \text{ weakly in } H \text{ as } n \to \infty,$$

that is, $w_\infty \in \omega(w_0)$.

The next Lemma guarantees the existence of a solution to $SP(w_{4\infty}, w_{3\infty}^D)$.

Lemma 4.3 $w_\infty \in F(\varphi_1^\infty)$ *and* w_∞ *is a solution of* $SP(w_{4\infty}, w_{3\infty}^D)$.

Proof. By integrating (23) over $[s, t]$ with $0 \leq s \leq t$ we have

$$\varphi_1^t(\tilde{w}(t)) - \varphi_1^s(\tilde{w}(s)) + \int_s^t |\partial_t \tilde{w}|_H^2 d\tau \leq \sum_{i=1}^4 \int_s^t I_i d\tau \quad \text{for } 0 \leq s \leq t,$$

where $\tilde{w} = (w_1, w_2, \tilde{w}_3)$. It is obvious that the function $t \to \varphi_1^t(\tilde{w}(t)) + \int_0^t |\partial_t \tilde{w}|_H^2 d\tau - \sum_{i=1}^4 \int_0^t I_i d\tau$ is non-increasing on $[0, \infty)$ and $\varphi_1^t(\tilde{w}(t))) \geq 0$ for $t \geq 0$. Then by Lemma 4.2 we see that $\lim_{t \to \infty} \varphi_1^t(\tilde{w}(t))$ exists so that we can put $q_0 = \lim_{t \to \infty} \varphi_1^t(\tilde{w}(t))) \geq 0$.

Next, we show that

$$\varphi_1^\infty(\tilde{w}_\infty) \leq \liminf_{n \to \infty} \varphi_1^{t_n}(\tilde{w}(t_n)), \tag{25}$$

where $\tilde{w}_\infty = (w_{1\infty}, w_{2\infty}, w_{3\infty} - w_{3\infty}^D)$. In fact, we observe that for each n

$$|\varphi_1^\infty(\tilde{w}(t_n)) - \varphi_1^{t_n}(\tilde{w}(t_n))|$$
$$\leq \int_{\Omega \times \Gamma_1} |Q(w_{4\infty})\hat{R}(w_1(t_n)) - Q(w_4(t_n))\hat{R}(w_1(t_n))| dx d\gamma_y$$
$$+ \frac{\gamma \alpha}{2} \int_{\Omega \times \Gamma_2} ||h(\tilde{w}_3(t_n) + w_{3\infty}^D) - w_2(t_n)|^2 - |h(\tilde{w}_3(t_n) + w_3^D(t_n)) - w_2(t_n))|^2| dx d\gamma_y$$
$$+ h \int_\Omega |(f_\infty - f(t_n))\tilde{w}_3(t_n)| dx$$
$$\leq \ell_3 \ell_4 \int_{\Omega \times \Gamma_1} |w_{4\infty} - w_4(t_n)| dx d\gamma_y + h \int_\Omega |f_\infty - f(t_n)||\tilde{w}_3(t_n)| dx$$
$$+ \frac{\gamma \alpha}{2} \int_{\Omega \times \Gamma_2} |h(w_{3\infty}^D - w_3^D(t_n))(2h\tilde{w}_3(t_n) + h(w_3^D(t_n) + w_{3\infty}^D) - 2w_2(t_n))| dx d\gamma_y.$$

Then, for some positive constant C_5, we obtain:

$$|\varphi_1^\infty(\tilde{w}(t_n)) - \varphi_1^{t_n}(\tilde{w}(t_n))|$$
$$\leq C_5 \int_{t_n}^\infty (\int_{\Omega \times \Gamma_1} |\partial_t w_4| dx d\gamma_y + \int_{\Omega \times \Gamma_2} |\partial_t w_3^D| dx d\gamma_y + \int_\Omega |\partial_t f| dx) dt \text{ for } n$$

so that

$$\lim_{n \to \infty} (\varphi_1^\infty(\tilde{w}(t_n)) - \varphi_1^{t_n}(\tilde{w}(t_n))) = 0. \tag{26}$$

Hence, we see that

$$\liminf_{n \to \infty} \varphi_1^{t_n}(\tilde{w}(t_n)) \geq \liminf_{n \to \infty}(\varphi_1^{t_n}(\tilde{w}(t_n)) - \varphi_1^\infty(\tilde{w}(t_n)) + \liminf_{n \to \infty} \varphi_1^\infty(\tilde{w}(t_n)) \geq \varphi_1^\infty(\tilde{w}_\infty).$$

15

Thus (25) is true. Moreover, it is clear that $\varphi_1^\infty(\tilde{w}_\infty) \le q_0$ so that $\tilde{w}_\infty \in V$.

Based on Lemma 4.2, we can take a subsequence $\{t'_n\}$ with $t'_n \to \infty$ as $n \to \infty$ such that

$$n \le t'_n \le n+1 \text{ for each } n \text{ and } \partial_t \tilde{w}(t'_n) \to 0 \text{ in } H \text{ as } n \to \infty. \tag{27}$$

Let $z \in V$. Similarly to (26), we can prove that $\varphi_1^{t'_n}(z) \to \varphi_1^\infty(z)$ as $n \to \infty$. Also, by (24) we have

$$
\begin{aligned}
(-\tilde{w}_t(t'_n), z - w(t'_n))_H &= (\partial\varphi_1^{t'_n}(w(t'_n)), z - \tilde{w}(t'_n))_H \\
&\le \varphi_1^{t'_n}(z) - \varphi_1^{t'_n}(\tilde{w}(t'_n)) \quad \text{for each } n.
\end{aligned}
$$

By letting $n \to \infty$ in the above inequality, we obtain $0 \le \varphi_1^\infty(z) - q_0$ for any $z \in V$. This shows that $q_0 = \min \varphi^\infty$, that is, and $\tilde{w}_\infty \in F(\varphi_1^\infty)$. Moreover, we see that $0 = \partial\varphi^\infty(\tilde{w}_\infty)$. Hence, Lemma 3.1 together with $w_\infty \in U(m_0)$ implies the conclusion of this Lemma. $\square$

The next Lemma guarantees the uniqueness of solutions to $SP(w_{4\infty}, w_{3\infty}^D)$.

Lemma 4.4 *If $(\psi(r) - \psi(r'))(r - r') \ge \mu|r - r'|^{p+1}$ for $r, r' \in \mathbb{R}$, where $\mu > 0$ and $p \ge 1$, then $SP(w_{4\infty}, w_{3\infty}^D)$ has at most one solution.*

Proof. Let $w_\infty^{(1)}(:= (w_1^{(1)}, w_2^{(1)}, w_3^{(1)}))$ and $w_\infty^{(2)}(:= (w_1^{(2)}, w_2^{(2)}, w_3^{(2)}))$ be solutions of $SP(w_{4\infty}, w_{3\infty}^D)$ and $w_\infty^{(1)} - w_\infty^{(2)} = (w_{1\infty}, w_{2\infty}, w_{3\infty})$.

By Definition 2.5 we see that

$$
\begin{aligned}
0 &= \int_{\Omega \times Y} d_1 |\nabla_y w_{1\infty}|^2 dxdy + \int_{\Omega \times \Gamma_1} Q(w_{4\infty})(R(w_1^{(1)}) - R(w_1^{(2)})) w_{1\infty} dxd\gamma_y \\
&\quad + \int_{\Omega \times Y} (\psi(w_1^{(1)} - \gamma w_2^{(1)}) - \psi(w_1^{(2)} - \gamma w_2^{(2)}))(w_{1\infty} - \gamma w_{2\infty}) dxdy \\
&\quad + \gamma \int_{\Omega \times Y} d_2 |\nabla_y w_{2\infty}|^2 dxdy + \alpha\gamma \int_{\Omega \times \Gamma_2} |hw_{3\infty} - w_{2\infty}|^2 dxd\gamma_y + \alpha\gamma \int_\Omega d_3 |\nabla w_{3\infty}|^2 dx.
\end{aligned}
$$

Using of the monotonicity of R and ψ, we obtain $\nabla w_{3\infty} = 0$ a.e. on Ω. Since $w_{3\infty} = 0$ a.e. on Γ_D, we obtain $w_{3\infty} = 0$ a.e. on Ω. Accordingly, we have $w_2 = 0$ a.e. on $\Omega \times \Gamma_2$. Immediately, we infer that $w_2 = 0$ a.e. on $\Omega \times Y$. Moreover, it follows that $\int_{\Omega \times Y} d_1 |\nabla_y w_{1\infty}|^2 dxdy + \mu \int_{\Omega \times Y} |w_{1\infty} - \gamma w_{2\infty}|^{p+1} dxdy = 0$. This implies the uniqueness of a solution of the stationary problem. $\square$

Now, we accomplish the proof of Theorem 2.6.

Proof of Theorem 2.6.

Considering the above arguments, it is sufficient to show (7). Let $\{t'_n\}$ be a sequence satisfying (27). First, we observe that

$$
\begin{aligned}
&(\partial\varphi_1^{t'_n}(\tilde{w}(t'_n)) - \partial\varphi_1^\infty(\tilde{w}_\infty), \tilde{w}(t'_n) - \tilde{w}_\infty)_H \\
&\ge \int_{\Omega \times Y} d_1 |\nabla_y(w_1(t'_n) - w_{1\infty})|^2 dxdy \\
&\quad + \int_{\Omega \times \Gamma_1} (Q(w_4(t'_n))R(w_1(t'_n)) - Q(w_{4\infty})R(w_{1\infty}))(w_1(t'_n) - w_{1\infty}) dxd\gamma_y
\end{aligned}
$$

16

$$+\mu \int_{\Omega \times Y} |(w_1(t'_n) - w_{1\infty}) - \gamma(w_2(t'_n) - w_{2\infty})|^{p+1} dxdy$$

$$+\gamma \int_{\Omega \times Y} d_2 |\nabla_y(w_2(t'_n) - w_{2\infty})|^2 dxdy$$

$$+\alpha\gamma \int_{\Omega \times \Gamma_2} |h(\tilde{w}_3(t'_n) - \tilde{w}_{3\infty}) - (w_2(t'_n) - w_{2\infty})|^2 dxd\gamma_y$$

$$+\alpha\gamma h \int_{\Omega \times \Gamma_2} (w_3^D(t'_n) - w_{3\infty}^D)\{(w_2(t'_n) - w_{2\infty}) - h(\tilde{w}_3(t'_n) - \tilde{w}_{3\infty})\}dxd\gamma_y$$

$$+\gamma h \int_{\Omega} d_3 |\nabla(\tilde{w}_3(t'_n) - \tilde{w}_{3\infty})|^2 dx + \gamma h \int_{\Omega} (f(t'_n) - f_\infty)(\tilde{w}_3(t'_n) - \tilde{w}_{3\infty})dx$$

$$=: \sum_{i=1}^{8} J_{in} \quad \text{for each } n.$$

For each n, it is easy to see that

$$J_{1n} + J_{4n} + J_{7n}$$
$$\geq d_1^0 \int_{\Omega \times Y} |\nabla_y(w_1(t'_n) - w_\infty)|^2 dxdy + \gamma d_2^0 \int_{\Omega \times Y} |\nabla_y(w_2(t'_n) - w_{2\infty})|^2 dxdy$$
$$+\gamma h d_3^0 \int_{\Omega} |\nabla(\tilde{w}_3(t'_n) - \tilde{w}_{3\infty})|^2 dx,$$

and

$$J_{2n} \geq \int_{\Omega \times \Gamma_1} (Q(w_4(t'_n)) - Q(w_{4\infty}))R(w_{1\infty})(w_1(t'_n) - w_{1\infty})dxd\gamma_y.$$

On the other hand, we have

$$(\partial\varphi_1^{t'_n}(\tilde{w}(t'_n)) - \partial\varphi_1^{\infty}(\tilde{w}_\infty), \tilde{w}(t'_n) - \tilde{w}_\infty)_H = (-\tilde{w}_t(t'_n), \tilde{w}(t'_n) - \tilde{w}_\infty)_H \quad \text{for each } n. \quad (28)$$

Then we obtain

$$d_1^0 \int_{\Omega \times Y} |\nabla_y(w_1(t'_n) - w_\infty)|^2 dxdy + \gamma d_2^0 \int_{\Omega \times Y} |\nabla_y(w_2(t'_n) - w_{2\infty})|^2 dxdy$$

$$+\alpha\gamma d_3^0 \int_{\Omega} |\nabla(\tilde{w}_3(t'_n) - \tilde{w}_{3\infty})|^2 dx$$

$$+\mu \int_{\Omega \times Y} |(w_1(t'_n) - w_{1\infty}) - \gamma(w_2(t'_n) - w_{2\infty})|^{p+1} dxdy$$

$$+\alpha\gamma \int_{\Omega \times \Gamma_2} |h(\tilde{w}_3(t'_n) - \tilde{w}_{3\infty}) - (w_2(t'_n) - w_{2\infty})|^2 dxd\gamma_y$$

$$\leq E(t'_n) + |\tilde{w}_t(t'_n)|_H |\tilde{w}(t'_n) - \tilde{w}_\infty|_H \quad \text{for each } n, \quad (29)$$

where

$$E(t) = \int_{\Omega \times \Gamma_1} |Q(w_4(t)) - Q(w_{4\infty})||R(w_{1\infty})||w_1(t) - w_{1\infty}|dxd\gamma_y$$

17

$$+\alpha\gamma h\int_{\Omega\times\Gamma_2}(w_3^D(t'_n)-w_{3\infty}^D)\{(w_2(t'_n)-w_{2\infty})-h(\tilde{w}_3(t'_n)-\tilde{w}_{3\infty})\}dxd\gamma_y$$

$$+\gamma h\int_\Omega|f(t)-f_\infty||\tilde{w}_3(t)-\tilde{w}_{3\infty}|dx\quad\text{for }t\geq 0.$$

By letting $n\to\infty$ in the above inequality, we infer that $\tilde{w}_3(t'_n)\to\tilde{w}_{3\infty}$ in X, $w_2(t'_n)\to w_{2\infty}$ and $w_1(t'_n)\to w_{1\infty}$ in $L^2(\Omega;H^1(Y))$ as $n\to\infty$. In particular, $\tilde{w}(t'_n)\to\tilde{w}_\infty$ in H as $n\to\infty$.

From (28), it follows that

$$-(\partial\varphi_1^t(\tilde{w}(t))-\partial\varphi_1^\infty(\tilde{w}_\infty),\tilde{w}(t)-\tilde{w}_\infty)_H=\frac{1}{2}\frac{d}{dt}|\tilde{w}(t)-\tilde{w}_\infty|_H^2\quad\text{for a.e. }t\geq 0.$$

Then, similarly to (29), we can show that

$$\frac{1}{2}\frac{d}{dt}|\tilde{w}(t)-\tilde{w}_\infty|_H^2\leq E(t)\quad\text{for a.e. }t\geq 0.$$

By integrating it over $[t'_n,t]$ with $t'_n\leq t\leq t'_n+2$ for n, we have

$$\frac{1}{2}|\tilde{w}(t)-\tilde{w}_\infty|_H^2\leq\frac{1}{2}|\tilde{w}(t'_n)-\tilde{w}_\infty|_H^2+\int_{t'_n}^{t'_n+2}E(\tau)d\tau.\tag{30}$$

By the assumption (A6) and Lemma 4.1 for any $\varepsilon>0$ we can take a positive integer N_1 such that

$$\frac{1}{2}|\tilde{w}(t'_n)-\tilde{w}_\infty|_H^2<\frac{1}{2}\varepsilon\text{ for }n\geq N_1\text{ and }E(t)<\frac{1}{4}\varepsilon\text{ for }t\geq N_1.$$

Then for $t\geq N_1+1$ there exists a positive integer $n\geq N_1$ such that $n+1\leq t\leq n+2$. In this case we see that $n\leq t'_n\leq n+1\leq t\leq n+2\leq t'_n+2$. Hence, on account of (30) we infer that $\frac{1}{2}|\tilde{w}(t)-\tilde{w}_\infty|_H^2<\varepsilon$ for $t\geq N_1+1$. This is the conclusion of this theorem. Thus we have proved Theorem 2.6. □

References

[ADDN04] D. Aregba-Driollet, F. Diele, and R. Natalini. A mathematical model for the sulphur dioxide aggression to calcium carbonate stones: Numerical approximation and asymptotic analysis. *SIAM J. Appl. Math.*, 64(5):1636–1667, 2004.

[All92] G. Allaire. Homogenization and two-scale convergence. *SIAM J. Math. Anal.*, 23(6):1482–1518, 1992.

[AM10] T. Aiki and A. Muntean. Large time behavior of solutions to a moving-interface problem modeling concrete carbonation. *Communications on Pure and Applied Analysis*, 9(5):1117–1129, 2010.

[AM13] T. Aiki and A. Muntean. A free-boundary problem for concrete carbonation: Rigorous justification of the $\sqrt{t}$-law of propagation. *Interfaces and Free Boundaries*, 2013. to appear.

[Ant93]   N. Antonic. Memory effects in homogenisation: Linear second-order equations. *Archive for Rational Mechanics and Analysis*, 125(1):1–24, 1993.

[BC66]   R. Battino and H. L. Clever. The solubility of gases in liquids. *Chemical Reviews*, 66(4):395–463, 1966.

[BR96]   M. Böhm and I. G. Rosen. Global weak solutions and uniqueness for a moving boundary problem for a coupled system of quasilinear diffusion-reaction equations arising as a model of chemical corrosion of concrete surfaces - (part 3). Preprints aus dem Institut für Mathematik. Institut für Mathematik, 1996.

[BS85]   M. Böhm and R. E. Showalter. Diffusion in fissured media. *SIAM J. Math. Anal.*, 16(3):500–509, 1985.

[BZK60]   G.I. Barenblatt, I. P. Zheltov, and I. N. Kochina. Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks. *Journal of Applied Mathematics and Mechanics*, 24(5):1286–1303, 1960.

[CS95]   J.D. Cook and R.E. Showalter. Microstructure diffusion models with secondary flux. *J. Math. Anal. Appl.*, 189(3):731 – 756, 1995.

[ET12]   J. Escher and D. Treutler. Strong solutions of semilinear matched microstruture models. *Journal of Evolution Equations*, 12(2):459–480, 2012.

[FK92]   A. Friedman and P. Knabner. A transport model with micro- and macrostructure. *Journal of Differential Equations*, 98(2):328 – 354, 1992.

[FM13]   T. Fatima and A. Muntean. Sulfate attack in sewer pipes: Derivation of a concrete corrosion model via two-scale convergence. *Nonlinear Analysis: Real World Applications*, 2013. to appear.

[FMA12]   T. Fatima, A. Muntean, and T. Aiki. Distributed space scales in a semilinear reaction-diffusion system including a parabolic variational inequality: A well-posedness study. *Adv. Math. Sci. Appl.*, 22:295–318, 2012.

[FT87]   A. Friedman and A. E. Tzavaras. A quasilinear parabolic system arising in modelling of catalytic reactors. *Journal of Differential Equations*, 70(2):167 – 196, 1987.

[GMSR09]   P.J.P. Gleize, E.V. Motta, D.A. Silva, and H.R. Roman. Characterization of historical mortars from Santa Catarina (Brazil). *Cement and Concrete Composites*, 31(5):342 – 346, 2009.

[HJ91]   U. Hornung and W. Jäger. Diffusion, convection, adsorption, and reaction of chemicals in porous media. *Journal of Differential Equations*, 92(2):199–225, 1991.

[Hor97]   U. Hornung, editor. *Homogenization and Porous Media*, volume 6 of *Interdisciplinary Applied Mathematics*. Springer-Verlag, New York, 1997.

[HS90]   U. Hornung and R. E. Showalter. Diffusion models for fractured media. *J. Math. Anal. Appl.*, 147(1):69 – 80, 1990.

[Ken81]   N. Kenmochi. Solvability of nonlinear evolution equations with time-dependent constraints and applications. *Bull. Fac. Education, Chiba Univ.*, 30:1–87, 1981.

[MB08]   S. A. Meier and M. Böhm. A note on the construction of function spaces for distributed-microstructure models with spatially varying cell geometry. *Int. J. Numer. Anal. Model.*, 5(5):109 – 125, 2008.

[Mei08]   S. A. Meier. *Two-scale models for reactive transport and evolving microstructure.* PhD thesis, University of Bremen, Germany, 2008.

[MM08]   S. A. Meier and A. Muntean. A two-scale reaction-diffusion system with micro-cell reaction concentrated on a free boundary. *Comptes Rendus Mécanique*, 336(6):481–486, 2008.

[MM10]   S. A. Meier and A. Muntean. A two-scale reaction-diffusion system: Homogenization and fast-reaction limits. *GAKUTO Int. Ser. Math. Sci. Appl.*, 32:441–459, 2010.

[MNR10]   A. Muntean and M. Neuss-Radu. A multiscale Galerkin approach for a class of nonlinear coupled reaction-diffusion systems in complex media. *J. Math. Anal. Appl.*, 371(2):705–718, 2010.

[Ngu89]   G. Nguestseng. A general convergence result for a functional related to the theory of homogenization. *SIAM J. Math. Anal*, 20:608–623, 1989.

[SW91]   R. E. Showalter and J. Walkington. Micro-structure models of diffusion in fissured media. *J. Math. Anal. Appl.*, 155:1–20, 1991.

[Tre12]   D. Treutler. Well-posedness for a quasilinear generalization of the matched microstructure model. *Nonlinear Analysis: Real World Applications*, 13(6):2622 – 2632, 2012.

[Yam76]   Y. Yamada. On evolution equations generated by subdifferential operators. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 23(3):491–515, 1976.

[ZACV13] S. V. Zemskov, B. Ahmad, O. Copuroglu, and F. J. Vermolen. Modeling of a self-healing process in blast furnace slag cement exposed to accelerated carbonation. *Journal of Physics: Conference Series*, 410(1):012088, 2013.

PREVIOUS PUBLICATIONS IN THIS SERIES:

Number	Author(s)	Title	Month
13-10	G. Kakuba M.J.H. Anthonissen	Local defect correction for boundary integral equation methods	May '13
13-11	M. Oppeneer S.W. Rienstra P. Sijtsma	Efficient mode-matching based on closed form integrals of Pridmore-Brown modes	May '13
13-12	D.K. Singh S.W. Rienstra	A systematic impedance model for non-linear Helmholtz resonator liner	May '13
13-13	M.H. Duong	Finite-time blow-up and variational approximation scheme for a Wigner-Fokker-Planck equation with a nonlocal perturbation	May '13
13-14	T. Aiki A. Muntean	Large-time behavior of a two-scale semilinear reaction-diffusion system for concrete sulfatation	May '13