Mixed finite element for swelling of cartilaginous tissues, II

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Mixed Finite Element for Swelling of Cartilaginous Tissues, II

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Hybridization of the Mixed Method

- Fraijs de Veubeke (1965, 1977) introduced a hybrid method for the mixed formulation in order to simplify the solution of the algebraic system.
- The hybridization technique reduces the number of degrees of freedom. Furthermore, in the computations we only need to compute inverses of element-wise block diagonal matrices.
- The idea behind the hybridization is to relax the continuity of the normal components of $\mathbf{q}^l$ and $\mathbf{q}^\beta_{\text{tot}}$ across the inter-element boundaries.
- This will require to enlarge the Raviart-Thomas space in which $\mathbf{q}^l$ and $\mathbf{q}^\beta_{\text{tot}}$ are sought and to introduce Lagrange multipliers to enforce the continuity of the normal components of $\mathbf{q}^l$ and $\mathbf{q}^\beta_{\text{tot}}$ across the inter-element boundaries.
- After applying the hybridization technique:

$$\mathfrak{A}(\varphi_h, c_h^+, c_h^-)\frac{d\mathbf{y}}{dt} + \mathfrak{B}(\varphi_h, c_h^+, c_h^-)\mathbf{y} = \mathfrak{F}(\varphi_h, c_h^+, c_h^-) + \frac{d\mathfrak{G}}{dt}(\varphi_h, c_h^+, c_h^-),$$

$$\mathbf{y} = \left[\tilde{\mathbf{u}}, \tilde{\mathbf{q}}^l, \tilde{\mathbf{q}}^+_{\text{tot}}, \tilde{\mathbf{q}}^-_{\text{tot}}, \tilde{\boldsymbol{\mu}}^l, \tilde{\boldsymbol{\mu}}^+, \tilde{\boldsymbol{\mu}}^-, \tilde{\boldsymbol{\lambda}}^l, \tilde{\boldsymbol{\lambda}}^+, \tilde{\boldsymbol{\lambda}}^-\right]^T$$

$$\mathfrak{A}_{ij} = \begin{cases} \mathbf{B}^T, & i = 5,\ j = 1, \\ \mathbf{0}, & i \neq 5,\ j \neq 1, \end{cases}$$

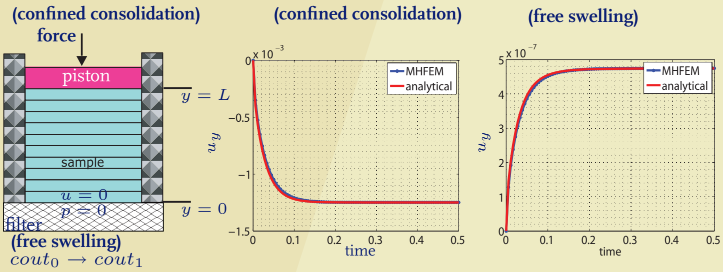
$$\mathfrak{B} = \begin{pmatrix} \mathbf{A} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^{ll}(\varphi_h/c_h^\beta) & \mathbf{C}^{l+}(\varphi_h) & \mathbf{C}^{l-}(\varphi_h) & \mathbf{D} & \mathbf{0} & \mathbf{0} & \mathbf{E} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^{l+}(\varphi_h) & \mathbf{C}^{++}(\varphi_h c_h^+) & \mathbf{0} & \mathbf{0} & \mathbf{D} & \mathbf{0} & \mathbf{0} & \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^{l-}(\varphi_h) & \mathbf{0} & \mathbf{C}^{--}(\varphi_h c_h^-) & \mathbf{0} & \mathbf{0} & \mathbf{D} & \mathbf{0} & \mathbf{0} & \mathbf{E} \\ \mathbf{0} & \mathbf{D}^T & \mathbf{0} & \mathbf{0} & & & & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{D}^T & \mathbf{0} & & & & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D}^T & & & \mathbf{O}_{3(K+L)\times 3(K+L)} & & & \\ \mathbf{0} & \mathbf{E}^T & \mathbf{0} & \mathbf{0} & & & & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{E}^T & \mathbf{0} & & & & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{E}^T & & & & & & \end{pmatrix}$$

- Finally the idea that flows are taken from $\mathcal{RT}_{-1}(\Omega)$ will reduces the above system to a smaller system:

$$\begin{pmatrix} \mathbf{A}\tilde{\mathbf{u}} \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathfrak{A}_1 & \mathfrak{A}_2 \\ \mathfrak{A}_2^T & -\mathfrak{A}_3 \end{pmatrix} \begin{pmatrix} \frac{d}{dt}\tilde{\mathbf{u}} \\ \tilde{\boldsymbol{\lambda}}^l \\ \tilde{\boldsymbol{\lambda}}^+ \\ \tilde{\boldsymbol{\lambda}}^- \end{pmatrix} = \begin{pmatrix} \mathbf{F}_1 \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathfrak{F}_1 \\ \mathfrak{F}_2 \end{pmatrix} \frac{d}{dt} \begin{pmatrix} \mathbf{0} \\ \mathbf{F}_2^+ \\ \mathbf{F}_2^- \end{pmatrix},$$

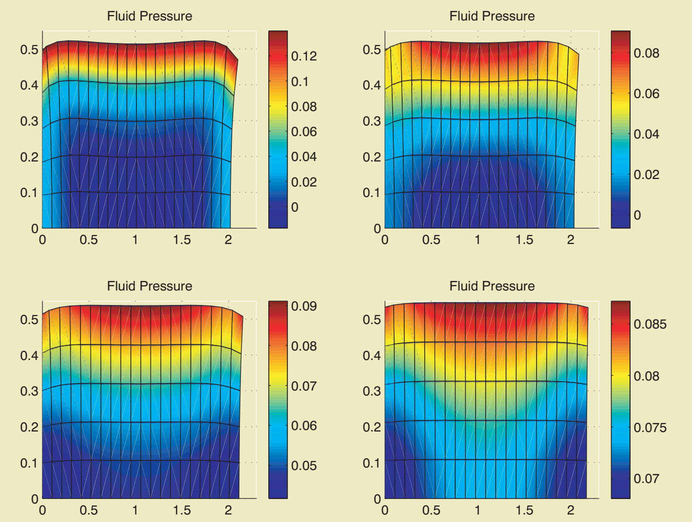
- $\mathfrak{A}_1$ and $\mathfrak{A}_3$ are symmetric positive definite matrices.

Confined Consolidation and Free Swelling



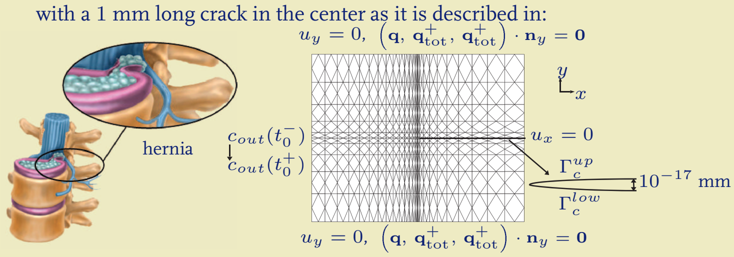
Two-dimensional Free Swelling

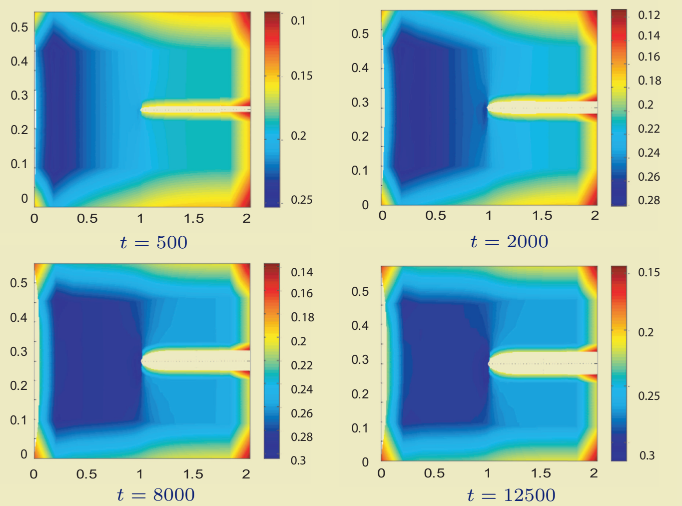
- We consider a swelling cylinder to test the two-dimensional hybridization method.
- A cylinder of height 0.5 mm and radius 1 mm is in equilibrium with an external salt concentration.
- The external salt concentration is reduced from 4.6×10^2 mol m^{-3} to 4×10^2 mol m^{-3}.
- A change of salt concentration of this solution generates a change at the boundary of ion concentration and electro-chemical potentials as well as pressure and voltage.



Opening Cracks in the Intervertebral Discs

- Low back pain is common today's society. 75 % of all people will experience back pain at some time of their lives.
- The intervertebral disc serves as a shock absorber, load distributer.
- The disc loses its ability to hold wanter, resulting in decreased ability to absorb shock and narrowing of the nerve openings in the sides of the spine, which may pinch the nerves.
- We consider the a two-dimensional rectangular geometry 2×0.5 mm with a 1 mm long crack in the center as it is described in:





- The loss of proteoglycan in degenerate discs has a major effect on the disc load-bearing behaviour. With loss of proteoglycans (and therefore fixed charges), the osmotic pressure of the disc falls and the dis is less able to maintain hydration under load.

Reference

- Malakpoor et al., Mathematical modelling and numerical solution of swelling of cartilaginous tissues. Part II. ESAIM: Mathematical Modelling and Numerical Analysis, 2006.