

## Existence of solutions to the diffusive VSC model

**Citation for published version (APA):**

Hulshof, J., Nolet, R., & Prokert, G. (2011). *Existence of solutions to the diffusive VSC model*. (CASA-report; Vol. 1154). Technische Universiteit Eindhoven.

**Document status and date:**

Published: 01/01/2011

**Document Version:**

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

**Please check the document version of this publication:**

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

**General rights**

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

[www.tue.nl/taverne](http://www.tue.nl/taverne)

**Take down policy**

If you believe that this document breaches copyright please contact us at:

[openaccess@tue.nl](mailto:openaccess@tue.nl)

providing details and we will investigate your claim.

**EINDHOVEN UNIVERSITY OF TECHNOLOGY**  
Department of Mathematics and Computer Science

CASA-Report II-54  
November 2011

Existence of solutions to the diffusive VSC model

by

J. Hulshof, R. Nolet, G. Prokert



Centre for Analysis, Scientific computing and Applications  
Department of Mathematics and Computer Science  
Eindhoven University of Technology  
P.O. Box 513  
5600 MB Eindhoven, The Netherlands  
ISSN: 0926-4507



# Existence of solutions to the Diffusive VSC model

J. Hulshof, R. Nolet VU University, G. Prokert, TU Eindhoven

November 2, 2011

## Abstract

We prove existence of classical solutions to the so-called diffusive Vesicle Supply Centre (VSC) model describing the growth of fungal hyphae. It is supposed in this model that the local expansion of the cell wall is caused by a flux of vesicles into the wall and that the cell wall particles move orthogonally to the cell surface. The vesicles are assumed to emerge from a single point inside the cell (the VSC) and to move by diffusion.

For this model, we derive a non-linear, non-local evolution equation and show the existence of solutions relevant to our application context, namely, axially symmetric surfaces of fixed shape, travelling along with the VSC at constant speed. Technically, the proof is based on the Schauder fixed point theorem applied to Hölder spaces of functions. The necessary estimates rely on comparison and regularity arguments from elliptic PDE theory.

## 1 Introduction

Describing the growth behavior of living cells is a challenging pursuit, both from the point of view of biological modelling and from the point of view of the mathematical treatment of the resulting models. Since growth of a cell proceeds primarily by incorporating new material into the cell wall and membrane, models for cell growth have to describe how the shape of a cell changes as a result of this process. In geometric models, the cell wall and the membrane are treated as a single surface without thickness. This allows one to mathematically describe the cell wall as an embedded two-dimensional manifold. In this case, the well-known “first variation of area formula” relates the local growth of cell surface area to the velocity of its particles, more precisely, to their normal velocity and the divergence of their tangential velocity.

Extreme growth behavior can be observed in fungal hyphae cells, i.e. very long, hair-shaped cells that form the mycelium of fungi. Accordingly, modelling their growth has attracted particular interest, with an emphasis on solutions given by a fixed, travelling profile. In most models for these cells, it is assumed that cell wall particles move in a direction orthogonal to the cell surface. (An exception to this is the isometric model described by Tindemans [11].) This

assumption of orthogonal growth is mostly justified by observations, with turgor pressure given as a possible physical mechanism. It will also be adopted in the present paper. Moreover, conservation of mass dictates that the surface area growth equals the local flux  $F$  of material into the cell boundary. In Section 2 we show that these assumptions determine the normal velocity as  $v_n = -F/H$  where  $H$  is the mean curvature of the manifold.

Some models express this flux solely as a function of the local geometry of the cell wall. For example, Goriely *et al.* [7] define the flux as a function of the curvature. The Vesicle Supply Centre (VSC) models, first proposed by Bartnicki-Garcia *et al.* [1], assume that material is transported towards the wall in so-called vesicles, i.e. small “sacks” bounded by a membrane. These vesicles are created at the Golgi apparatus, and transported via the cytoskeleton to the VSC, from which they are released and transported to the cell wall. On arrival at the cell wall the contents of the vesicles are used to make cell wall material while the vesicle membrane merges with the cell membrane. For modeling purposes it is not important whether the VSC acts as a distribution centre for vesicles created elsewhere, or whether it produces them itself; in both cases it can be treated as a source of vesicles. In models for tip growth, the location of the VSC often coincides with an organelle called the Spitzenkörper.

The VSC models are divided in two classes, depending on how vesicles move from the VSC to the cell wall. In the so-called ballistic model, vesicles travel in straight lines towards the cell wall. The model by Bartnicki-Garcia *et al.* [1] is of this kind, with vesicles sent in every direction isotropically. The advantage of ballistic models lies in their mathematical simplicity: The flux of vesicles arriving at a point on the cell wall can be calculated directly from its distance to the VSC and the slope of the wall. A travelling wave ansatz then yields an ordinary differential equation for the shape of the hypha. In a previous article [8] we used this to show that this model has unique, stable, travelling solutions. These solutions are tubular elongating cells growing mostly at the tip, as observed in fungal hyphae. Possible variations of the ballistic model involve including a directional preference to the release of vesicles so that more of them are focussed on the tip, or having multiple sources.

One criticism of the ballistic model, given e.g. by Koch [10], is that inside a living cell, it is highly unlikely that a vesicle will travel in a straight line to its destination. Instead it will perform a random walk and will be absorbed when it hits the cell boundary. Accordingly, the concentration of vesicles obeys a Poisson equation with a point source at the VSC and homogeneous Dirichlet boundary conditions.

Numerical calculations on this model were done by Tindemans *et al.* [12]. Possible variations of the diffusive model include further physical properties of the cell wall, e.g. elasticity or reduced absorption due to ageing [4]. A very good overview of many of the models available is given by De Keijzer *et al.* [9].

The aim of all these models is to find a travelling solution corresponding to a fungal hypha. These are solutions which are stationary in a frame of reference travelling along with the VSC (or at some fixed velocity if no VSC is present in the models). As usual, we introduce the assumption of cylindrical symmetry,

i.e. the surface can then be expressed as a curve rotated around the  $z$  axis. We seek solutions which asymptotically approximate a cylinder as  $z \rightarrow -\infty$ .

We want to stress that the diffusive VSC models are essentially nonlocal. In fact, the equations for the tip shape involve an unknown flux function which itself depends on the tip shape. Therefore this shape is determined by a condition which cannot be formulated as an ordinary differential equation.

So far, most research has focussed on numerically approximating the tip shape for these models. In this article we provide a theoretical foundation for the simplest of them by rigorously proving the existence of these travelling solutions using a Schauder fixed point argument. However, the methods described in this article should work as well for certain related models involving orthogonal growth and a flux dependent on the cell shape; on this, see also the Conclusions section.

## 1.1 Notation and conventions

In this article we will often make use of the following notation:  $\Omega$  is an open subset of  $\mathbb{R}^3$ , not necessarily compact, rotationally symmetric around the  $z$  axis with boundary  $\partial\Omega$ . Since we are working in this axially symmetric case, we will use a cylindrical coordinate system  $(r, z, \theta)^T$  in  $\mathbb{R}^3$ . The transformation to Euclidian coordinates is given by  $(x_1, x_2, x_3)^T = (r \cos \theta, r \sin \theta, z)^T$ . Often we will not mention the coordinate  $\theta$  in our calculations. The rotationally symmetric surface  $\partial\Omega$  will be parametrized by two functions  $s \mapsto r(s)$  and  $s \mapsto z(s)$ . The surface implied is the curve parametrized by these two functions at some fixed value of  $\theta$ , rotated around the  $z$  axis. When we refer to a point on the surface at pathlength  $s$  or the point  $(r(s), z(s))$ , we mean a point  $(r(s), z(s), \theta) \in \partial\Omega$  at some arbitrary, but fixed value of  $\theta$ .

For the (scalar) mean curvature  $H$  we use the conventions as found in [5]. The mean curvature is the sum (and thus not the true mean) of the principal curvatures with respect to an outward pointing normal  $\hat{n}$ . For example, the sphere of radius  $R$  has mean curvature  $-\frac{2}{R}$  at every point.

When  $u$  is a (harmonic) function defined on  $\Omega$ , the flux  $F_u$  of  $u$  will always be the negative normal derivative of  $u$  on  $\partial\Omega$ . Often  $u$  will depend on a parameter  $\xi$ , we will denote this as  $u_\xi$ . If it is clear the flux mentioned is the flux of  $u$  we will write  $F_\xi$  to denote the flux at parameter value  $\xi$ .

The proof in this article relies heavily on the use of Hölder spaces. For these spaces and their norms we will use the notation as found in [6], with  $\|\cdot\|_{k,\alpha;X}$  denoting the  $C^{k,\alpha}$  norm on the domain of definition  $X$ . The space of continuous functions from  $X$  to  $Y$  with bounded Hölder norm is denoted as  $C^{k,\alpha}(X;Y)$ . The domain  $X$  or codomain  $Y$  will be omitted if they are clear from the context.

## 2 The diffusive VSC model

### 2.1 The diffusive flux

In the diffusive VSC model, we assume the VSC is a source of vesicles which diffuse outwards toward the cell wall, where they are completely absorbed, causing growth. Each vesicle is a membrane sack containing the materials to build new cell wall. Upon absorption, the membrane of the vesicle merges with the cell membrane, while the contents build new cell wall. In the VSC model, the membrane and cell wall are treated as a single manifold, and each vesicles contributes a fixed amount of surface area to this manifold. The total amount of surface area produced by the VSC per unit of time is denoted by  $P$ . The VSC is moving in the positive  $z$  direction at speed  $c$ . We assume the motion of the cell wall is slow on the diffusion time scale of the vesicles, and so the density of vesicles is always in equilibrium. As such, it can be found by solving a Poisson equation. The assumption that vesicles are completely absorbed at the boundary yields a homogeneous Dirichlet boundary condition. Furthermore, we assume that the motion of the cell wall is slow on the diffusion time scale of the vesicles.

If at time  $t = 0$  the tip of the cell wall is at the origin, and the VSC is at distance  $\xi$  from the tip, then the density  $u_\xi$  of vesicles can be found by solving

$$\begin{aligned} \Delta u_\xi &= -P\delta(r, z + \xi - ct) && \text{in } \Omega, \\ u_\xi &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (2.1)$$

The flux of material arriving at a point is now given by  $F_\xi = -\frac{\partial u_\xi}{\partial \hat{n}}$  where  $\hat{n}$  is the outward pointing normal of  $\partial\Omega$ . This flux gives the rate per unit of area at which the surface area increases.

### 2.2 Mass balance

If one takes an arbitrary bounded region  $A \subset \partial\Omega$  of the surface of the cell, with surface area  $\|A\|$  and boundary curve  $\partial A$ , then the total flux of material absorbed in  $A$  is given by

$$\frac{d\|A\|}{dt} = \int_A F_\xi dS. \quad (2.2)$$

If one assumes  $A$  is transported by a velocity field  $v$ , then Gauss' formula for the first variation of area states that

$$\frac{d\|A\|}{dt} = - \int_A H(\hat{n} \cdot v) dS + \oint_{\partial A} (\hat{m} \cdot v) dl \quad (2.3)$$

where  $\hat{m}$  is the outward pointing normal to  $\partial A$  tangent to  $\partial\Omega$  and  $H$  is the (scalar) mean curvature. By assumption, the surface of the cell moves orthogonally to the cell surface, so  $v = v_n \hat{n}$  and the integral over  $\partial A$  vanishes. As  $A$  was chosen arbitrarily, we get from (2.2) and (2.3) that

$$v_n = -\frac{F_\xi}{H}. \quad (2.4)$$

### 2.3 Scaling

We define the typical length scale  $X$  and time-scale  $T$  of the model as follows,

$$X = \frac{P}{4\pi c} \quad T = \frac{P}{4\pi c^2}. \quad (2.5)$$

Rescaling our spacial coordinates by  $x = X\tilde{x}$  and  $t = T\tilde{t}$  we denote the rescaled domain as  $\tilde{\Omega}$ . It is now natural to rescale the dependant variables and constants as

$$\begin{aligned} u_\xi &= \frac{X}{T}\tilde{u}_{\tilde{\xi}}, & F_\xi &= \frac{1}{T}\tilde{F}_{\tilde{\xi}}, & H &= \frac{1}{X}\tilde{H}, & v_n &= \frac{X}{T}\tilde{v}_n, \\ \xi &= X\tilde{\xi}, & \tilde{c} &= 1, & \tilde{P} &= 4\pi. \end{aligned} \quad (2.6)$$

We now see that the rescaled model satisfies

$$\tilde{v}_n = -\frac{\tilde{F}_{\tilde{\xi}}}{\tilde{H}}, \quad (2.7)$$

where  $\tilde{F}_{\tilde{\xi}} = -\frac{\partial \tilde{u}_{\tilde{\xi}}}{\partial \tilde{n}}$  and  $\tilde{u}_{\tilde{\xi}}$  satisfies

$$\begin{aligned} \Delta \tilde{u}_{\tilde{\xi}} &= -4\pi\delta(\tilde{r}, \tilde{z} + \tilde{\xi} - t) && \text{in } \tilde{\Omega}, \\ \tilde{u}_{\tilde{\xi}} &= 0 && \text{on } \partial\tilde{\Omega}. \end{aligned} \quad (2.8)$$

For the remainder of this article we will drop the tildes and work with this rescaled model.

### 2.4 The unbounded travelling wave problem

We wish to find a surface satisfying the evolution equation (2.4) which moves along with the VSC, see for example Figure 1. In other words, in a coordinate system moving along with the VSC,  $\partial\Omega$  appears stationary. In this coordinate system, Gauss' formula for the variation of area on an arbitrary subsurface  $A$  states that:

$$\int_A F_\xi dS = - \int_A H(\hat{n} \cdot (v - \hat{e}_z))dS + \oint_{\partial A} (\hat{m} \cdot (v - \hat{e}_z))dl \quad (2.9)$$

For a stationary solution,  $v - \hat{e}_z$  must lie tangent to  $\partial\Omega$  and the integral over  $A$  vanishes. By assumption,  $v$  is perpendicular to  $\hat{m}$  and so

$$- \oint_{\partial A} \hat{m} \cdot \hat{e}_z dl = \int_A F_\xi dS. \quad (2.10)$$

We now choose  $A$  to be the region from the tip up to the plane located at  $z = z(s)$ , then  $\hat{m} \cdot \hat{e}_z$  is constant over  $\partial A$ . We choose cylindrical coordinates  $r$  and  $z$ , and describe  $\partial\Omega$  as the curve  $(r(s), z(s))$  rotated around the  $z$  axis. We parametrize such that  $s$  is the pathlength over  $\partial\Omega$  from the tip.



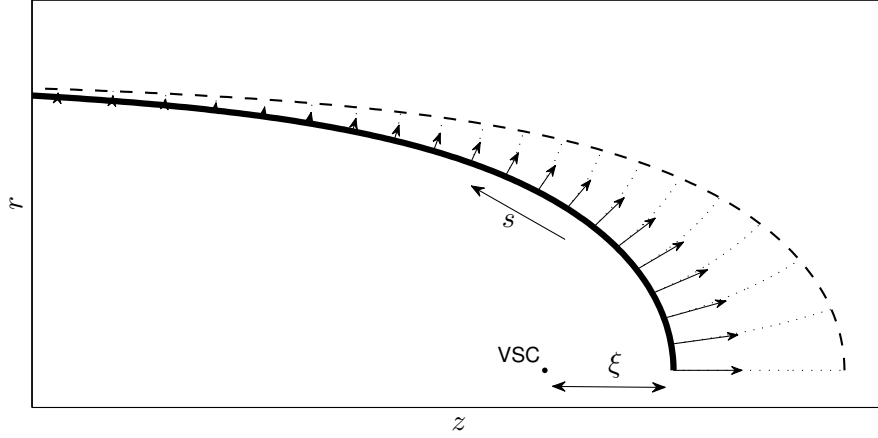


Figure 1: Travelling wave profile with model definitions. The solid curve, rotated around the  $z$  axis, is the cell boundary  $\partial\Omega$ . The arrows indicate the normal velocity and the dotted lines are the particle trajectories. The dashed curve indicates the location of the cell boundary at some later time.

In these coordinates  $\hat{m} = r'(s)\hat{e}_r + z'(s)\hat{e}_z$ , and (2.10) simplifies to

$$z'(s) = -\frac{G_\xi(s)}{r(s)}, \quad r'(s) = \sqrt{1 - \left(\frac{G_\xi(s)}{r(s)}\right)^2}, \quad (2.11)$$

where

$$G_\xi(s) = \frac{1}{2\pi} \int_A F_\xi dS = \int_0^s F_\xi(\sigma)r(\sigma)d\sigma, \quad (2.12)$$

and  $F_\xi(\sigma)$  is the flux passing through the point on the boundary at pathlength  $\sigma$ .

We wish to find functions  $r(s)$ ,  $z(s)$  and a number  $\xi^*$  such that when (2.1) is solved on the domain defined by the functions, the corresponding cumulative flux  $G_{\xi^*}$  and the functions  $r$  and  $z$  satisfy (2.11) with boundary conditions  $r(0) = z(0) = 0$ ,  $r'(0) = 1$ . We wish  $r(s)$  to remain bounded. Since by the divergence theorem  $G_\xi(s) \rightarrow 2$ , this can only be accomplished if  $r(s) \rightarrow 2$  as  $s \rightarrow \infty$ . In the rest of the article we will refer to this as the unbounded travelling wave problem.

## 2.5 The bounded travelling wave problem

The fact that the domain of the functions  $r$  and  $z$  is infinite, and therefore that the domain  $\Omega$  is unbounded makes analysis difficult. In order to handle these difficulties, we first restrict ourselves to bounded domains with a no flux condition at  $z = z(s_{max})$  for some sufficiently large  $s_{max}$ . We apply the method of reflection and define the following problem: given functions  $r(s)$  and  $z(s)$  on  $(0, s_{max})$  we define  $\partial\Omega$  to be the curve  $(r(s), z(s))$  rotated around the  $z$  axis and

reflected at the plane  $z = z(s_{max})$ . If the VSC is located at a distance  $\xi$  from the tip, the density of vesicles  $u_\xi(r, z)$  is found by solving

$$\begin{aligned} \Delta u_\xi &= -4\pi\delta(r, z + \xi) - 4\pi\delta(r, z + \eta) && \text{in } \Omega, \\ u_\xi &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{2.13}$$

where  $\eta = -2z(s_{max}) - \xi$  is the distance from the reflected VSC to the tip at  $z = 0$ . The functions  $F_\xi$  and  $G_\xi$  are still defined as above. Note that by symmetry and the divergence theorem,  $G_\xi(s_{max}) = 2$ .

Given  $s_{max}$  we wish to find functions  $r(s)$ ,  $z(s)$  and a number  $\xi^*$ , such that when (2.13) is solved on the domain defined by these functions, the corresponding cumulative flux  $G_{\xi^*}$  and the functions  $r$  and  $z$  satisfy (5.2) with boundary conditions  $r(0) = z(0) = 0$ ,  $r'(0) = 1$ , and  $r(s_{max}) = 2$ . We will refer to this problem as the bounded travelling wave problem. In Section 7 we take the limit as  $s_{max} \rightarrow \infty$  to show the existence of a solution for the unbounded travelling wave problem described in the previous section.

### 3 The Schauder map

Our approach to solve this problem relies on a Schauder fixed point argument on a subset of the product space  $C^{1,\alpha} \times C^{0,\alpha}$  containing Hölder continuous functions  $s \mapsto r(s)$  and  $s \mapsto z'(s)$ , for some  $\alpha \leq \frac{1}{2}$ . We equip this space with the  $C^{1,\beta} \times C^{0,\beta}$  topology for some  $\beta < \alpha$ . Defining  $z(s) = \int_0^s z'(\sigma) d\sigma$ , the functions  $r(s)$  and  $z(s)$  describe a boundary  $\partial\Omega$  with certain properties. (Note that since  $z \rightarrow -\infty$  if  $s_{max} \rightarrow \infty$ , we cannot claim that  $z \in C^{1,\alpha}$  is bounded uniformly in  $s_{max}$ , instead we demand this only of its derivative  $z'$ .) Since the Schauder fixed point theorem requires that we work on a closed convex subset of this product space, we cannot require that  $\partial\Omega$  is parametrized by pathlength ( $r'^2 + z'^2 = 1$  is not a convex requirement.) Instead we require that  $r'^2 + z'^2 \leq 1$  and  $r' - z' \geq 1$ , see Figure 2. Note, however, that the image of the Schauder map does satisfy the pathlength requirement  $r'^2 + z'^2 = 1$ .

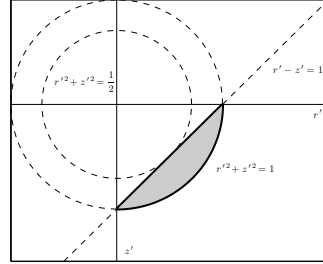


Figure 2: Convex bounds on  $r'$  and  $z'$ .

#### 3.1 The domain of the Schauder map

Given a sufficiently large  $s_{max}$  we define the domain of the Schauder map  $\Xi(M, A, C; s_{max})$  as the subset of  $C^{1,\alpha}([0, s_{max}]; \mathbb{R}) \times C^{0,\alpha}([0, s_{max}]; \mathbb{R})$  containing functions  $r$  and  $z'$  satisfying the following convex properties:

$$\|r\|_{1,\alpha} \leq M, \quad \|z'\|_{0,\alpha} \leq M, \quad (3.1)$$

$$r'(s)^2 + z'(s)^2 \leq 1, \quad r'(s) - z'(s) \geq 1, \quad (3.2)$$

$$r(s_{max}) = 2, \quad r'(s_{max}) = 0, \quad (3.3)$$

$$\frac{r'(s_2) - r'(s_1)}{s_2 - s_1} \leq A, \quad \frac{z'(s_2) - z'(s_1)}{s_2 - s_1} \leq A, \quad \text{for } s_1 < s_2, \quad (3.4)$$

$$s - \frac{1}{9}C^2s^3 \leq r(s) \leq 2 \quad \text{for } 0 \leq s \leq C^{-1}, \quad (3.5)$$

In Section 4 we will show that one can solve the Dirichlet problem (2.13) for every  $\xi$ ,  $\xi_{min} \leq \xi \leq \xi_{max}$ , with  $\xi_{min}$  and  $\xi_{max}$  to be determined later, and obtain a family  $G_\xi$  of cumulative fluxes, parametrized by  $\xi$ , with certain properties. This defines a map  $\Psi_1 : \Xi(M, A, C; s_{max}) \rightarrow C^1([\xi_{min}, \xi_{max}]; C^{1,\alpha}([0, s_{max}]; \mathbb{R}))$ . Given  $G_\xi \in \text{Im}(\Psi_1)$  and a value of the parameter  $\xi$ , (2.11) can be seen as an ordinary differential equation, which can be solved to obtain functions  $r_\xi(s)$  and  $z_\xi(s)$ . In Section 5 we will show that one can find a unique value  $\xi^*$  such that  $r_{\xi^*}(s_{max}) = 2$ . This defines a map  $\Psi_2 : \text{Im}(\Psi_1) \rightarrow \Xi(\tilde{M}, \tilde{A}, \tilde{C})$ . In Section 6 we will choose  $M$ ,  $A$  and  $C$  such that the composition  $\Psi = \Psi_2 \circ \Psi_1$  maps from  $\Xi(M, A, C; s_{max})$  to itself. We then use Schauder's fixed point theorem to show that the map  $\Psi$ , which we will refer to as the Schauder map, has a fixed

point. Since solutions to (2.11) satisfy  $r'^2 + z'^2 = 1$ , this fixed point describes a surface parametrized by pathlength and solves the bounded travelling wave problem defined in Section 2.5.

**Lemma 3.1** *The set  $\Xi(M, A, C; s_{max})$  is closed in the  $C^{1,\beta} \times C^{0,\beta}$  topology.*

**Proof** Let  $(r_n, z'_n)$  be a sequence in  $\Xi(M, A, C; s_{max})$  which converges to  $(r, z')$  in  $C^{1,\beta} \times C^{0,\beta}$ . We need to prove that  $(r, z')$  satisfies (3.1) to (3.5). We can clearly take the limit to see that  $(r, z)$  satisfies (3.2) to (3.5) so we need only concern ourselves with the Hölder norm established in (3.1). Now since  $r_n$  is bounded in the  $C^{1,\alpha}$  norm it has a convergent subsequence in the  $C^{1,\beta}$  norm, clearly the limit of this subsequence is  $r$  and thus  $\|r\|_{1,\alpha} \leq M$ . Similarly  $\|z'\|_{0,\alpha} \leq M$ . ■

### 3.2 Estimates on $r(s)$ , $z(s)$ and distances

The definition of the set  $\Xi(M, A, C; s_{max})$  yields several estimates on  $r(s)$ ,  $z(s)$  and distances between points on the boundary that will be used throughout this article.

First of all, (3.2) gives  $0 \leq r' \leq 1$ ,  $-1 \leq z' \leq 0$ ,  $r'^2 + z'^2 \geq \frac{1}{2}$ ,  $r(0) = 0$ ,  $r'(0) = 1$  and  $z'(0) = 0$ . The estimate given by (3.5) gives an asymptotic approximation for  $r(s)$  in the tip, at small  $s$ . The monotonicity of  $r$  then gives a lower bound away from the tip,

$$\frac{8}{9}C^{-1} \leq r(s) \leq 2 \quad \text{for } C^{-1} \leq s \leq s_{max}. \quad (3.6)$$

Using this we can establish an asymptotic estimate for  $z(s)$  at small  $s$ ,

$$z(s) \geq -\sqrt{s^2 - r(s)^2} \geq -\frac{1}{2}Cs^2 \quad \text{for } 0 \leq s \leq C^{-1}. \quad (3.7)$$

while the requirement that  $r' - z' \geq 1$  implies that

$$-s \leq z(s) \leq 2 - s. \quad (3.8)$$

for all  $s$ .

The choice of parametrization of the curve  $s \mapsto (r(s), z(s))$  yield various useful bounds on the distances between points on the curve.

**Lemma 3.2** *The distance between points  $(r(s_2), z(s_2))$  and  $(r(s_1), z(s_1))$  is bounded from above and below by the difference in parameter values  $s_2 - s_1$ .*

$$\frac{1}{2}(s_2 - s_1)^2 \leq (r(s_2) - r(s_1))^2 + (z(s_2) - z(s_1))^2 \leq (s_2 - s_1)^2. \quad (3.9)$$

**Proof** Let  $d(s)$  be the distance from  $(r(s), z(s))$  to  $r(s_1), z(s_1)$ ,

$$d(s) = \sqrt{(r(s) - r(s_1))^2 + (z(s) - z(s_1))^2}. \quad (3.10)$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} \left(\frac{d}{ds}d(s)\right)^2 &= \left(\frac{r(s) - r(s_1)}{d(s)}r'(s) + \frac{z(s) - z(s_1)}{d(s)}z'(s)\right)^2 \\ &\leq r'(s)^2 + z'(s)^2 \leq 1 \end{aligned} \quad (3.11)$$

so  $d(s_2) \leq (s_2 - s_1)$ . For the lower bound we see that

$$\begin{aligned} d(s_2)^2 &= \frac{1}{2}((r(s_2) - r(s_1)) - (z(s_2) - z(s_1)))^2 + \frac{1}{2}((r(s_2) - r(s_1)) + (z(s_2) - z(s_1)))^2 \\ &\geq \frac{1}{2}(s_2 - s_1)^2, \end{aligned} \quad (3.12)$$

since  $r'(s) - z'(s) \geq 1$ .

Geometrically, the upper bound is achieved when the path between the points at pathlength  $s_1$  and  $s_2$  consists of a straight line. The lower bound is achieved when the path consists solely of vertical and horizontal segments. ■

Furthermore, the estimates for  $r(s)$  and  $z(s)$  established in the previous section allow us to calculate a lower bound for the distance between the VSC and the boundary of the cell, an important ingredient for bounding the flux.

**Lemma 3.3** *If  $\frac{1}{2}C^{-1} < \xi_{min} \leq \xi$  then the distance  $d_\xi(s)$  from the point  $(r(s), z(s))$  to the VSC is bounded from below by a nonzero constant  $d_{min}$  depending only on  $C$  and  $\xi_{min}$ .*

**Proof** Using the asymptotics for  $r$  and  $z$  at small  $s$ , (3.5) and (3.7),

$$d_\xi(s) = \sqrt{r(s)^2 + (\xi + z(s))^2} \geq \xi + z(s) \geq \xi_{min} - \frac{1}{2}C^{-1} \quad \text{for} \quad 0 \leq s \leq C^{-1},$$

while for large  $s$ ,

$$d_\xi(s) \geq r(s) \geq \frac{8}{9}C^{-1} \quad \text{for} \quad C^{-1} \leq s \leq s_{max},$$

by (3.6). The minimum of these two estimates gives a lower bound for the distance. ■

### 3.3 Exterior spheres

For bounds on the flux in the next section we require that it is possible to be able to touch a sphere of fixed radius to every point of the boundary, in such a way that the interior of the sphere does not intersect  $\Omega$ . If the second derivatives of  $r$  and  $z$  were bounded from above, this would be a relatively straightforward task involving the calculation of the first principle curvature. The upper bound on the difference quotient given by (3.4) is in fact sufficient for this task.

**Lemma 3.4** *Let  $B_R$  be a ball of radius  $R \leq \frac{1}{4A}$  touching  $\partial\Omega$  at the point  $(r(s_1), z(s_1))$ . Then the distance from any point  $(r(s_2), z(s_2)) \in \partial\Omega$  to the centre  $(r_c, z_c)$  of  $B_R$  is always greater than  $R$ .*

**Proof** Let  $\frac{1}{2}\pi \leq \theta \leq \pi$  be such that

$$\cos \theta = \frac{z'(s_1)}{\sqrt{r'(s_1)^2 + z'(s_1)^2}}, \quad \sin \theta = \frac{r'(s_1)}{\sqrt{r'(s_1)^2 + z'(s_1)^2}}. \quad (3.13)$$

The centre of the ball  $B_R$  is then given by

$$r_c = r(s_1) - R \cos \theta, \quad z_c = z(s_1) + R \sin \theta. \quad (3.14)$$

The distance  $d_c$  between the point  $(r(s_2), z(s_2))$  and the centre of this sphere is given by

$$\begin{aligned} d_c^2 &= (r(s_2) - r(s_1))^2 + (z(s_2) - z(s_1))^2 + R^2 \\ &\quad + 2R((r(s_2) - r(s_1)) \cos \theta - (z(s_2) - z(s_1)) \sin \theta). \end{aligned} \quad (3.15)$$

Integrating the difference quotients (3.4) we can estimate

$$\begin{aligned} r(s_2) - r(s_1) &\leq r'(s_1)(s_2 - s_1) + \frac{1}{2}A(s_2 - s_1)^2, \\ z(s_2) - z(s_1) &\leq z'(s_1)(s_2 - s_1) + \frac{1}{2}A(s_2 - s_1)^2. \end{aligned} \quad (3.16)$$

Substituting this, the linear terms in  $(s_2 - s_1)$  drop out and

$$d_c^2 \geq (r(s_2) - r(s_1))^2 + (z(s_2) - z(s_1))^2 + R^2 + AR(s_2 - s_1)^2(\cos \theta - \sin \theta). \quad (3.17)$$

The distance between points at parameter values  $s_2$  and  $s_1$  can be estimated using Lemma (3.2) and so,

$$d_c^2 \geq \frac{1}{2}(s_2 - s_1)^2 + R^2 - 2AR(s_2 - s_1)^2, \quad (3.18)$$

and so if we choose  $R \leq \frac{1}{4A}$  this distance will always be greater than  $R$ . ■

## 4 The Dirichlet problem

Given a domain  $\Omega$  given by the functions  $r(s)$  and  $z(s)$  as described in the previous section, we wish to find a solution  $u_\xi$  to (2.13). We then wish to find various estimates for the flux  $F_\xi(s)$  passing through the point  $(r(s), z(s))$  and the cumulative flux  $G_\xi(s)$ , defined as

$$G_\xi(s) = \int_0^s F_\xi(\sigma) r(\sigma) \sqrt{r'(\sigma)^2 + z'(\sigma)^2} d\sigma. \quad (4.1)$$

Note that if  $\partial\Omega$  is parametrized by pathlength, which for example is the case in the fixed point of the Schauder map, then this definition is equivalent to (2.12).

## 4.1 The domain $\Omega$

**Lemma 4.1** *If the boundary  $\partial\Omega$  is given by  $C^{1,\alpha}$  Hölder continuous functions  $r(s)$  and  $z(s)$  as described previously, then the enclosed domain  $\Omega$  is of class  $C^{1,\alpha}$ .*

**Proof** For the purposes of this proof, we extend the functions  $r$  and  $z$  to the interval  $[0, 2s_{max}]$  by reflection, so for  $s > s_{max}$ ,  $r(s) = r(2s_{max} - s)$  and  $z(s) = 2z(s_{max}) - z(2s_{max} - s)$ . Now  $r'(s_{max}) = 0$  and  $z'(s_{max}) = -1$  so the derivatives are continuous. For  $s_1 < s_{max} < s_2$ ,

$$\begin{aligned} |r'(s_2) - r'(s_1)| &\leq |r'(s_2) - r'(s_{max})| + |r'(s_{max}) - r'(s_1)| \\ &\leq \|r\|_{1,\alpha} |s_2 - s_{max}|^\alpha + \|r\|_{1,\alpha} |s_{max} - s_1|^\alpha \\ &\leq 2 \|r\|_{1,\alpha} |s_2 - s_1|^\alpha, \end{aligned} \quad (4.2)$$

and similar for the Hölder quotient of  $z'$ , therefore these functions are Hölder continuous. We now need to prove that each point of  $\partial\Omega$  has a neighbourhood which can be described as the graph of a  $C^{1,\alpha}$  function. We examine the point  $x^*$  at pathlength  $s^*$  and angle  $\theta^*$ . Without loss of generality we can assume that  $\theta^* = 0$  due to the rotational symmetry. A point  $x$  given by the parameters  $s$  and  $\theta$  in the neighbourhood of  $x^*$  has Euclidian coordinates  $(x_1, x_2, x_3)^T = (r(s) \cos \theta, r(s) \sin \theta, z(s))^T$ . We now introduce new coordinates  $\xi$  such that the origin lies on  $x^*$ , rotated such that the direction  $\xi_1$  lies tangent to the curve  $r(s), z(s)$  and the direction  $\xi_2$  lies in the direction of rotation by  $\theta$ . Then

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{bmatrix} -z'(s^*) & 0 & r'(s^*) \\ 0 & 1 & 0 \\ -r'(s^*) & 0 & -z'(s^*) \end{bmatrix} \begin{pmatrix} r(s) \cos \theta - r(s^*) \\ r(s) \sin \theta \\ z(s) \end{pmatrix}. \quad (4.3)$$

Now at  $(s, \theta) = (s^*, 0)$  we have  $\frac{\partial \xi}{\partial s} = (0, 0, -1)^T$  and  $\frac{\partial \xi}{\partial \theta} = (0, r(s^*), 0)^T$ . Therefore, if  $s^* \neq 0$  then  $r(s^*) \neq 0$  and by the implicit function theorem, there is a neighbourhood around  $x^*$  where we can write  $\xi_3$  (and  $s$  and  $\theta$ ) as a  $C^{1,\alpha}$  function of  $\xi_1$  and  $\xi_2$ .

If, on the other hand,  $s^* = 0$  then since  $r'(0) = 1$ ,  $r(s)$  is invertible in a neighbourhood of zero, and its inverse is  $C^{1,\alpha}$ . The tip can now be described as the graph  $x_3 = z(r^{-1}(\sqrt{x_1^2 + x_2^2}))$ . ■

This lemma implies that the domain  $\Omega$  is of class  $C^{1,\alpha}$ , this ensures us (see for example [6] Theorem 8.34) that there is a unique solution to the Dirichlet problem (2.13) which is  $C^{1,\alpha}$ . Since  $\partial\Omega$  is not  $C^2$ , it does not satisfy an interior sphere condition everywhere, and we cannot use the boundary point lemma to conclude that  $F_\xi > 0$  everywhere. This motivates the following Lemma for less smooth domains.

**Lemma 4.2** *Let  $\Omega$  be a (not necessarily rotationally symmetric) domain sufficiently smooth that the maximum principle and divergence theorem hold and a*

normal direction  $\hat{n}$  to the boundary can be defined almost everywhere. Let  $u > 0$  be a weak solution of

$$\begin{aligned} \Delta u &= f(x) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{4.4}$$

where  $f$  has compact support away from the boundary. Then  $F_u = -\frac{\partial u}{\partial \hat{n}} > 0$  almost everywhere on  $\partial\Omega$ .

**Proof** Assume there is an area  $A \subset \partial\Omega$  of positive measure such that  $F_u = 0$  on  $A$ . Let  $B_R$  be a ball of radius  $R$  centred on a point in the interior of  $A$ . Then  $R$  can be chosen sufficiently small such that  $(\partial\Omega \cap B_R) \subset A$  and  $B_R \cap \text{supp } f = \emptyset$ . Let  $v$  solve

$$\begin{aligned} \Delta v &= 0 && \text{in } B_R, \\ v &= u && \text{on } \partial B_R \cap \Omega, \\ v &= 0 && \text{on } \partial B_R \setminus \Omega, \end{aligned} \tag{4.5}$$

then by the maximum principle  $v > 0$  on  $B_R$  and  $u \leq v$  on  $B_R \cap \Omega$ . Since  $u = v$  on  $\partial B_R \cap \Omega$ ,  $F_u \leq F_v$  on  $\partial B_R \cap \Omega$ . The ball  $B_R$  satisfies an interior sphere condition, so by the boundary point lemma,  $F_v > 0$  on  $\partial B_R \setminus \Omega$ . By the divergence theorem, the total flux of  $v$  over  $\partial B_R$  must be zero, so  $\int_{\partial B_R \cap \Omega} F_v dS < 0$ . This implies that  $\int_{\partial B_R \cap \Omega} F_u dS < 0$ . By the divergence theorem, the total flux of  $u$  over  $\partial(B_R \cap \Omega)$  must be zero, so  $\int_{\partial\Omega \cap B_R} F_u dS > 0$ . This is in contradiction with our assumption that  $F_u = 0$  on  $A$ . ■

This Lemma implies that  $G_\xi(s)$  is strictly monotone in  $s$ , even though its derivative might occasionally be zero.

## 4.2 Bounds on $F_\xi(s)$

The uniform upper bound on the curvature allows us to touch a sphere of radius  $R = \frac{1}{4A}$  to any point on  $\partial\Omega$  such that this sphere lies outside of  $\Omega$ . This together with the bounds for the distances to the VSC allows us to establish an uniform upper bound for the flux.

**Lemma 4.3** *For sufficiently large  $s_{max}$ , the flux  $F_\xi(s) = -\frac{\partial u}{\partial \hat{n}}$  passing through the point  $(r(s), z(s))$  on the boundary is bounded,*

$$F_\xi(s) \leq F_{max}(\xi_{min}, \xi_{max}, A, C). \tag{4.6}$$

This implies that we can estimate  $G'_\xi(s) \leq F_{max}s$  and  $G'_\xi(s) \leq 2F_{max}$ .

**Proof** Let  $B_R$  be a sphere of radius  $R = \frac{1}{4A}$  touching  $\partial\Omega$  at the point  $(r(s), z(s))$ , we now solve

$$\begin{aligned} \Delta v &= -4\pi\delta(r, z + \xi) - 4\pi\delta(r, z + \eta) && \text{outside of } B_R, \\ v &= 0 && \text{on } \partial B_R. \end{aligned} \tag{4.7}$$



Since  $B_R$  lies outside of  $\Omega$ ,  $u \leq v$  and the boundaries touch at  $(r(s), z(s))$ , the flux of  $v$  at this point gives us an upper bound for  $F_\xi(s)$ . We can determine  $v$  using reflection techniques. For simplicity we consider the sources at  $z = -\xi$  and  $z = -\eta$  separately and write  $v = v_1 + v_2$  with  $v_1$  and  $v_2$  the individual contributions from these two sources. For  $\frac{1}{2}\pi \leq \theta \leq \pi$  let,

$$\cos \theta = \frac{z'(s)}{\sqrt{r'(s)^2 + z'(s)^2}}, \quad \sin \theta = \frac{r'(s)}{\sqrt{r'(s)^2 + z'(s)^2}}. \quad (4.8)$$

Let  $\rho$  be the distance from the VSC to the centre of  $B_R$ ,

$$\rho^2 = d_\xi(s)^2 + R^2 + 2R((z(s) + \xi) \sin \theta - r(s) \cos \theta), \quad (4.9)$$

where  $d_\xi(s)$  is the distance from the VSC to the point  $(r(s), z(s))$ . Let  $(\tilde{r}, \tilde{z})$  be the point, on the line from the VSC to the centre of  $B_R$ , at a distance  $\tilde{\rho} = \frac{R^2}{\rho}$  from the centre the sphere. Then

$$\begin{aligned} r(s) - \tilde{r} &= \frac{R^2}{\rho^2} r(s) + \left(1 - \frac{R^2}{\rho^2}\right) R \cos \theta, \\ z(s) - \tilde{z} &= \frac{R^2}{\rho^2} (z(s) + \xi) - \left(1 - \frac{R^2}{\rho^2}\right) R \sin \theta. \end{aligned} \quad (4.10)$$

This point acts as a reflected source of strength  $-4\pi \frac{R}{\rho}$ . The contribution of the source at the VSC to  $v$  is given by

$$v_1(r, z) = \frac{1}{d_1(r, z)} - \frac{R}{\rho} \frac{1}{\tilde{d}_1(r, z)} \quad (4.11)$$

where  $d_1(r, z)$  is the distance from the point  $(r, z)$  to the VSC, and  $\tilde{d}_1(r, z)$  is the distance from  $(r, z)$  to the reflected point  $(\tilde{r}, \tilde{z})$  inside  $B_R$ . Note that  $d_1(r(s), z(s)) = d_\xi(s)$ . If we denote the VSC as the point  $O$ ,  $(\tilde{r}, \tilde{z})$  as  $P$ ,  $(r(s), z(s))$  as  $X$  and the centre of the sphere  $B_R$  as  $C$ , then the triangles  $OXC$  and  $XPC$  are similar. Thus  $\frac{d_\xi(s)}{\tilde{d}_1(s)} = \frac{R}{\tilde{\rho}} = \frac{\rho}{R}$ . The contribution to the flux at  $(r(s), z(s))$  is then given by

$$\begin{aligned} F_1(s) &= \frac{(z(s) + \xi) \sin \theta - r(s) \cos \theta}{d_\xi(s)^3} - \frac{R}{\rho} \frac{(z(s) - \tilde{z}) \sin \theta - (r(s) - \tilde{r}) \cos \theta}{\tilde{d}_1(r(s), z(s))^3} \\ &= \frac{\rho^2 - R^2}{R d_\xi(s)^3} = 2 \frac{(z(s) + \xi) \sin \theta - r(s) \cos \theta}{d_\xi(s)^3} + \frac{1}{R d_\xi(s)} \\ &\leq 2 \frac{\xi_{max} + 2}{d_\xi^3} + \frac{1}{R d_\xi}. \end{aligned} \quad (4.12)$$

By Lemma 3.3 we can estimate  $d_\xi$  in terms of  $C$  and  $\xi_{min}$  while  $R$  can be expressed in terms of  $A$ . We treat the source at  $z = -\eta$  similarly to obtain,

$$v_2(r, z) = \frac{1}{d_2(r, z)} - \frac{R}{\rho} \frac{1}{\tilde{d}_2(r, z)}, \quad (4.13)$$

and,

$$\begin{aligned}
F_2(s) &= 2 \frac{(z(s) + \eta) \sin \theta - r(s) \cos \theta}{d_\eta(s)^3} + \frac{1}{Rd_\eta(s)} \\
&\leq 6 \frac{s_{max}}{d_\eta(s)} + \frac{1}{Rd_\eta(s)}.
\end{aligned} \tag{4.14}$$

where  $d_2(r, z)$  and  $\tilde{d}_2(r, z)$  are the distances from the point  $(r, z)$  to the source at  $z = -\eta$  respectively the reflection of this source inside  $B_R$  and  $d_\eta(s) = d_2(r(s), z(s))$ . Note that  $d_\eta(s) \geq s_{max} - 2 - \xi_{max}$ . Combining these contributions yields that  $F_\xi(s) \leq F_1(s) + F_2(s) \leq F_{max}$ . Since  $\frac{s_{max}}{d_\eta(s)} \rightarrow 1$  and  $\frac{1}{d_\eta(s)} \rightarrow 0$  as  $s_{max} \rightarrow \infty$  this term can be bounded independently of  $s_{max}$  assuming  $s_{max}$  is sufficiently large. Thus  $F_{max}$  depends on  $A, C, \xi_{min}$  and  $\xi_{max}$ . ■

### 4.3 Bounds on $G_\xi(s)$

By using the upper bound on the flux derived in the previous section and then integrating we can obtain estimates for the cumulative flux  $G_\xi$ . However, it will be important to have estimates on  $G_\xi$  which do not depend on the parameter  $A$ . In order to do this we will use the following comparison principle.

**Theorem 4.4** Comparison principle. *Let  $\Omega$  and  $\tilde{\Omega}$  be domains with  $0 \in \Omega \cap \tilde{\Omega}$ , with boundaries sufficiently smooth that the divergence theorem and the strong maximum principle hold. Let  $u$  and  $\tilde{u}$  solve*

$$\begin{aligned}
\Delta u &= -\delta(x) & \text{in } \Omega, & & u &= 0 & \text{on } \partial\Omega, \\
\Delta \tilde{u} &= -\delta(x) & \text{in } \tilde{\Omega}, & & \tilde{u} &= 0 & \text{on } \partial\tilde{\Omega}.
\end{aligned}$$

Then,

$$\int_{\partial\Omega \setminus \tilde{\Omega}} F_u dA \leq \int_{\partial\tilde{\Omega} \cap \Omega} F_{\tilde{u}} dA, \quad \int_{\partial\tilde{\Omega} \setminus \Omega} F_{\tilde{u}} dA \leq \int_{\partial\Omega \cap \tilde{\Omega}} F_u dA.$$

If  $\partial\Omega$  and  $\partial\tilde{\Omega}$  satisfy an interior sphere condition, then the inequalities are strict.

**Proof** Let  $v$  solve

$$\Delta v = -\delta(x) \quad \text{in } \Omega \cap \tilde{\Omega}, \quad v = 0 \quad \text{on } \partial(\Omega \cap \tilde{\Omega}).$$

Then by the maximum principle,  $0 \leq v \leq u$  and  $0 \leq v \leq \tilde{u}$  on  $\Omega \cap \tilde{\Omega}$ . Moreover, since  $v = u$  on  $\partial\Omega \cap \tilde{\Omega}$ ,  $F_v \leq F_u$  on  $\partial\Omega \cap \tilde{\Omega}$ . Similarly,  $F_v \leq F_{\tilde{u}}$  on  $\partial\tilde{\Omega} \cap \Omega$ . By the divergence theorem,

$$\int_{\partial\Omega \cap \tilde{\Omega}} F_u dA + \int_{\partial\Omega \setminus \tilde{\Omega}} F_u dA = \int_{\partial\Omega \cap \tilde{\Omega}} F_v dA + \int_{\partial\tilde{\Omega} \cap \Omega} F_v dA.$$

Substituting the inequalities for  $F_v$  yields the first inequality. The second inequality follows by symmetry. If the boundaries satisfy an interior sphere condition, then by the boundary point lemma,  $F_v < F_u$  on  $\partial\Omega \cap \tilde{\Omega}$  and  $F_v < F_{\tilde{u}}$  on  $\partial\tilde{\Omega} \cap \Omega$  yielding strict inequalities. ■

**Corollary 4.5** *The result also holds if the source  $\delta(x)$  is replaced by several sources  $\sum_i w_i \delta(x - x_i)$  of different positive weights  $w_i$  for  $x_i \in \Omega \cap \tilde{\Omega}$ . Similarly we can replace the point source  $\delta(x)$  by a positive function  $f(x)$  with compact support within  $\Omega \cap \tilde{\Omega}$ .*

This Lemma allows us to compare the integrated flux on  $\partial\Omega$  to that of domains for which the solution of the Dirichlet problem is exactly known, for example spheres and half planes. In this way we establish the following bounds on  $G_\xi(s)$ .

**Lemma 4.6** *If  $\xi \leq \frac{3}{40}$  then there exists a  $s^*$  such that  $G(s^*) \geq s^*$ .*

**Proof** The reflected source at  $z = -\eta$  contributes positively to the flux. Since we are interested in a lower bound, we can safely ignore it. For some  $R$ ,  $\xi < R < 2$  let  $B_R$  be a ball of radius  $R$  centred around the VSC. The surface of the ball may intersect  $\partial\Omega$  in multiple points, let  $s^*$  identify the coordinate of the point furthest from the tip where  $\partial\Omega$  intersects the surface of this ball,

$$s^* = \max\{s | (r(s), z(s)) \in \partial\Omega \cap \partial B_R\}. \quad (4.15)$$

Let  $\partial B_R^+$  be that part of  $\partial B_R$  which lies in the positive  $z$  halfspace. Note that since  $z(s) \leq 0$  and  $R > \xi$ ,  $\partial B_R^+$  is non empty and lies outside of  $\Omega$ . The comparison principle, Theorem 4.4, now states that

$$\begin{aligned} 2\pi G_\xi(s^*) &\geq \int_{\partial\Omega \cap B_R} F_u dA \geq \int_{\partial B_R \setminus \Omega} \frac{1}{R^2} dA \geq \int_{\partial B_R^+} \frac{1}{R^2} dA \\ &\geq \pi \frac{R^2 - (R - \xi)^2}{R^2}. \end{aligned} \quad (4.16)$$

We now set  $R = 2\xi$  to obtain that  $G(s^*) \geq \frac{3}{8}$ . The point  $(r(s^*), z(s^*))$  lies on  $\partial B_R$ , so  $r(s^*) \leq 2\xi$  and  $z(s^*) \geq -3\xi$ . By integrating (3.2),  $s^* \leq r(s^*) - z(s^*) \leq 5\xi$  and so if  $\xi \leq \frac{3}{40}$  then  $s^* \leq \frac{3}{8} \leq G_\xi(s^*)$ . ■

**Lemma 4.7** *If  $s_{max}$  is chosen large enough that  $s_{max} \geq \xi_{max} + 2$  then*

$$G_\xi(s) \leq 8 \left(\frac{s}{\xi}\right)^2 \quad \text{for} \quad 0 \leq s \leq \min\left(C^{-1}, \sqrt{\xi_{min} C^{-1}}\right). \quad (4.17)$$

**Proof** We wish to use the estimates at the tip derived in Section 3.2 so we must require that  $s \leq C^{-1}$ . Furthermore if  $s \leq \sqrt{\xi_{min} C^{-1}}$ , then by (3.7),  $z(s) \geq -\frac{1}{2}\xi$ . Also, if  $s_{max} \geq \xi_{max} + 2$ , then by (3.8),  $z(s_{max}) \leq -\xi_{max}$ , so

$\eta = -2z(s_{max}) - \xi \geq 2\xi_{max} - \xi$ . This enables us to estimate the difference in the  $z$  coordinate between the point at pathlength  $s$ , the source at the VSC and the reflected source:

$$z(s) + \xi \geq \frac{1}{2}\xi \quad \text{and,} \quad z(s) + \eta \geq \frac{1}{2}\xi. \quad (4.18)$$

We define  $\tilde{\Omega}$  to be the half space  $\{(r, z) | z \leq z(s)\}$ . Let

$$\begin{aligned} \tilde{u}(r, z) = & (r^2 + (z + \xi)^2)^{-\frac{1}{2}} - (r^2 + (z - 2z(s) - \xi)^2)^{-\frac{1}{2}} \\ & + (r^2 + (z + \eta)^2)^{-\frac{1}{2}} - (r^2 + (z - 2z(s) - \eta)^2)^{-\frac{1}{2}}. \end{aligned} \quad (4.19)$$

Then  $\tilde{u}$  solves the Dirichlet problem on  $\tilde{\Omega}$  with sources at  $z = -\xi$  and  $z = -\eta$ . The comparison principle for integrated fluxes now states that

$$\begin{aligned} 2\pi G_\xi(s) &= \int_{\partial\Omega \setminus \tilde{\Omega}} F_u dA \leq \int_{\partial\tilde{\Omega} \cap \Omega} F_{\tilde{u}} dA \\ &= 4\pi \left[ 1 - \left( 1 + \left( \frac{r(s)}{\xi + z(s)} \right)^2 \right)^{-\frac{1}{2}} + 1 - \left( 1 + \left( \frac{r(s)}{\eta + z(s)} \right)^2 \right)^{-\frac{1}{2}} \right] \end{aligned} \quad (4.20)$$

(4.21)

now using the inequality  $1 - 1/\sqrt{1+x} \leq \frac{1}{2}x$ , the upper bound  $r(s) \leq s$ , and the estimates (4.18), we obtain

$$G_\xi(s) \leq \left( \frac{r(s)}{\xi + z(s)} \right)^2 + \left( \frac{r(s)}{\eta + z(s)} \right)^2 \leq 8 \left( \frac{s}{\xi} \right)^2. \quad (4.22)$$

■

#### 4.4 Monotonicity of $G_\xi$ in $\xi$

We wish to know how the cumulative flux  $G_\xi$  changes as the distance  $\xi$  between the tip and the VSC is varied while the domain  $\Omega$  remains fixed. We can write the Dirichlet problem as follows, let  $\tilde{u}_\xi(r, z)$  solve

$$\begin{aligned} \Delta \tilde{u}_\xi &= 0 && \text{in } \Omega, \\ \tilde{u}_\xi &= -\frac{1}{(r^2 + (z + \xi)^2)^{\frac{1}{2}}} - \frac{1}{(r^2 + (z + \eta)^2)^{\frac{1}{2}}} && \text{on } \partial\Omega \end{aligned} \quad (4.23)$$

then

$$u_\xi(r, z) = \tilde{u}_\xi(r, z) + \frac{1}{(r^2 + (z + \xi)^2)^{\frac{1}{2}}} + \frac{1}{(r^2 + (z + \eta)^2)^{\frac{1}{2}}} \quad (4.24)$$

solves (2.13). We now differentiate with respect to  $\xi$ ; note that  $\frac{d\eta}{d\xi} = -1$ . Let  $\tilde{v}_\xi(r, z)$  solve

$$\begin{aligned} \Delta \tilde{v}_\xi &= 0 && \text{in } \Omega, \\ \tilde{v}_\xi &= \frac{z + \xi}{(r^2 + (z + \xi)^2)^{\frac{3}{2}}} - \frac{z + \eta}{(r^2 + (z + \eta)^2)^{\frac{3}{2}}} && \text{on } \partial\Omega \end{aligned} \quad (4.25)$$

By Lemma 3.3,  $\tilde{v}_\xi$  is bounded on  $\partial\Omega$ , and so by the maximum principle it is bounded in  $\Omega$ . We define  $v_\xi(r, z)$  as

$$v_\xi(r, z) = \tilde{v}_\xi(r, z) - \frac{z + \xi}{(r^2 + (z + \xi)^2)^{\frac{3}{2}}} + \frac{z + \eta}{(r^2 + (z + \eta)^2)^{\frac{3}{2}}}, \quad (4.26)$$

then  $v_\xi = \frac{\partial}{\partial \xi} u_\xi$ , essentially  $v_\xi$  solves a Dirichlet problem with zero boundary condition and two dipoles of opposite orientation at  $z = -\xi$  and  $z = -\eta$  as sources. We will first show that  $\Omega$  can be divided into two connected subsets, where  $v_\xi$  is positive or negative. We will then show that this also divides the boundary  $\partial\Omega$  into two connected subsets, bordering the respective subsets of  $\Omega$ . Lastly we will examine the cumulative flux of  $v_\xi$  and prove a monotonicity result for  $G_\xi(s)$ .

Near the dipole source,  $\Omega$  is divided into two regions where  $v_\xi$  is positive or negative with a surface separating these two. The following Lemma states this division can be extended to the whole domain.

**Lemma 4.8** *Let  $v_\xi$  be defined as in (4.26) on a domain  $\Omega$  sufficiently smooth that the maximum principle holds and such that the points  $(0, -\xi)$  and  $(0, -\eta)$  lie in the interior of  $\Omega$ . Let*

$$\begin{aligned} \Omega^+ &= \{(r, z, \theta) \in \Omega | v(r, z) > 0 \text{ and } z > z(s_{max})\}, \\ \Omega^- &= \{(r, z, \theta) \in \Omega | v(r, z) < 0 \text{ and } z > z(s_{max})\}, \end{aligned} \quad (4.27)$$

then for sufficiently large  $s_{max}$ ,  $\Omega^-$  and  $\Omega^+$  are connected sets.

**Proof** Let  $B$  be a ball centred around the point  $(0, -\xi)$ , such that  $d = \text{dist}(\partial B, \partial\Omega) > 0$ . The boundary condition imposed on  $w_\xi$  is bounded and continuous, and thus by the maximum principle,  $w_\xi$  is bounded in  $\Omega$ . By [6] Theorem 2.10 the derivatives of  $w_\xi$  are bounded in  $B$ ,  $\sup_B \left| \frac{\partial w_\xi}{\partial z} \right| \leq \frac{3}{d} \sup_\Omega |w|$ . We now examine  $v_\xi(r, z)$  on the cylinder defined by  $r \leq R$  and  $|z + \xi| \leq Z$  for some sufficiently small  $R$  and  $Z = \frac{1}{\sqrt{3}}R$ . Since  $\eta = O(s_{max})$ , the contribution of the reflected source to  $v_\xi$  in this cylinder is  $O(s_{max}^{-2})$  and its contribution to  $\frac{\partial v_\xi}{\partial z} = O(s_{max}^{-3})$ , so we can choose  $s_{max}$  sufficiently large that

$$\left| \frac{z + \eta}{(r^2 + (z + \eta)^2)^{\frac{3}{2}}} \right| \leq 1, \quad \text{and} \quad \left| \frac{\partial}{\partial z} \left( \frac{z + \eta}{(r^2 + (z + \eta)^2)^{\frac{3}{2}}} \right) \right| \leq 1. \quad (4.28)$$

We now estimate  $v_\xi$  on the caps of the cylinder,

$$\begin{aligned}
v_\xi(r, Z - \xi) &\leq \sup_{\Omega} |w_\xi| + 1 - \frac{Z}{(R^2 + Z^2)^{\frac{3}{2}}} \\
&\leq \sup_{\Omega} |w_\xi| + 1 - \frac{9}{8} \frac{1}{R^2}, \\
v_\xi(r, -Z - \xi) &\geq -(\sup_{\Omega} |w_\xi| + 1) + \frac{Z}{(R^2 + Z^2)^{\frac{3}{2}}} \\
&\geq -(\sup_{\Omega} |w_\xi| + 1) + \frac{9}{8} \frac{1}{R^2},
\end{aligned} \tag{4.29}$$

while on the cylinder,

$$\frac{\partial v_\xi}{\partial z}(R, z) \leq \sup_B \left| \frac{\partial w_\xi}{\partial z} \right| + 1 - \frac{R^2 - 2Z^2}{(R^2 + Z^2)^{\frac{5}{2}}} \leq \sup_B \left| \frac{\partial w_\xi}{\partial z} \right| + 1 - \frac{3\sqrt{3}}{32} \frac{1}{R^3}. \tag{4.30}$$

Therefore we can choose an  $R_{max}$  such that for all  $R < R_{max}$ ,  $v_\xi(R, Z - \xi) < 0$ ,  $v_\xi(R, -Z - \xi) > 0$  and  $\frac{\partial v_\xi}{\partial z}(R, z) < 0$ . Thus  $z \rightarrow v_\xi(R, z)$  has a unique zero  $z_0$ . In other words, there exists a function  $z_0(r)$  defined on the interval  $[0, R_{max}]$  such that  $|z_0(r) + \xi| \leq \frac{1}{\sqrt{3}}r$  and  $v_\xi(r, z_0(r)) = 0$ . By the implicit function theorem,  $z_0$  is continuous and so this function defines a surface  $S$  which is part of the interface between  $\Omega^+$  and  $\Omega^-$ . By continuity,  $\partial\Omega^+ \cap S$  and  $\partial\Omega^- \cap S$  are both non empty and both boundaries contain the dipole at  $z = -\xi$ . Let  $\Omega_1$  and  $\Omega_2$  be two connected components of  $\Omega^+$  or  $\Omega^-$ . Assume  $(0, -\xi) \notin \partial\Omega_{1,2}$ , then  $v_\xi$  is harmonic in  $\Omega_{1,2}$  (note that we excluded the singularity in  $(0, -\eta)$  by demanding that  $z > z(s_{max})$ ), and  $v_\xi = 0$  on  $\partial\Omega_{1,2}$ . The maximum principle then implies that  $v_\xi = 0$  on  $\Omega_{1,2}$  which is a contradiction. However, since all points in the neighbourhood of  $(0, -\xi)$  for which  $v_\xi = 0$  are contained in a subset  $A$  of  $S$ ,  $A \subset \partial\Omega_{1,2}$  so  $\Omega_1$  and  $\Omega_2$  are connected and must be equal to one another.  $\blacksquare$

The division of  $\Omega$  into two regions of positive and negative  $v_\xi$  similarly divides the boundary into two parts.

**Lemma 4.9** *Let  $\Omega$  be a domain sufficiently smooth such that the maximum principle holds, let  $\Omega^+$ ,  $\Omega^-$  and  $v_\xi$  be defined as in the previous lemma. Then  $\partial\Omega^+ \cap \partial\Omega$  and  $\partial\Omega^- \cap \partial\Omega$  are closed connected sets.*

**Proof** This is essentially a 2D argument, since we assume radial symmetry we restrict ourselves to some plane at  $\theta = 0$ . We examine the region on the boundary with positive or negative flux. Let  $I^\pm = \{s \in [0, s_{max}] | (r(s), z(s)) \in \partial\Omega^\pm \cap \partial\Omega\}$  be the set of parameter values whose respective points on the boundary border  $\Omega^+$  respectively  $\Omega^-$ . Clearly these are closed sets. Let  $R_{max}$  be as in the proof of Lemma 4.8 and let  $z^\pm = -\xi \pm \frac{1}{\sqrt{3}}R_{max} - \xi$ . By the arguments of the previous Lemma,  $(0, z^+) \in \tilde{\Omega}^-$  and  $(0, z^-) \in \tilde{\Omega}^+$ .

Let  $s^- \in I^-$  and  $s^+ \in I^+$ , we will first show that  $s^- \leq s^+$ . Assume  $s^- > s^+$ , since  $\Omega^+$  and  $\Omega^-$  are connected, there exist paths connecting  $(r(s^-), z(s^-))$  to  $(0, z^+)$  and  $(r(s^+), z(s^+))$  to  $(0, z^-)$ , such that these paths lie completely inside  $\Omega^-$  respectively  $\Omega^+$ . Since  $z(s^+) > z(s^-)$  by the monotonicity of  $z$ , clearly these paths must intersect, which is a contradiction. Thus for  $s^- \in I^-$ , all points  $s < s^-$  are also in  $I^-$ , similarly for  $s^+ \in I^+$  all points  $s > s^+$  are in  $I^+$ . Thus  $I^+$  and  $I^-$  are closed intervals. ■

We now have enough information on  $v_\xi$  near the boundary to prove the following monotonicity result.

**Lemma 4.10** *For sufficiently large  $s_{max}$  the cumulative flux  $G_\xi$  is strictly monotone and differentiable in  $\xi$ ,*

$$\frac{\partial G_\xi}{\partial \xi}(s) < 0, \quad (4.31)$$

furthermore, this derivative is  $C^{1,\alpha}$  Hölder continuous.

**Proof** We wish to examine the cumulative flux  $H_\xi(s)$  of  $v_\xi$ ,

$$H_\xi(s) = \frac{\partial}{\partial \xi} G_\xi(s) = \int_0^s F_v(\sigma) r(\sigma) d\sigma, \quad (4.32)$$

where  $F_v(\sigma) = -\frac{\partial v_\xi}{\partial \bar{n}}(r(\sigma), z(\sigma))$ . By Lemmas 4.2 and 4.9 there are two closed intervals  $I^+$  and  $I^-$  such that the flux  $F_v$  is positive, respectively negative, everywhere on these intervals and cannot be zero on an open subinterval. If  $s^* \in [0, s_{max}] \setminus (I^+ \cup I^-)$  then there would exist a  $R > 0$  such that a ball of radius  $R$  around  $(r(s^*), z(s^*))$  lies neither in  $\Omega^+$  nor in  $\Omega^-$  (where  $\Omega^\pm$  is defined as in Lemma 4.8.) This is a contradiction since  $v_\xi$  cannot be zero on an open subset of  $\Omega$ . Therefore the union of  $I^+$  and  $I^-$  is the whole interval  $[0, s_{max}]$ . The flux must be zero on the intersection of these two intervals, and so this intersection must be either empty or be equal to a singleton  $\{s_0\}$ . It cannot be empty since the union of two disjoint closed intervals cannot be an interval. Therefore  $I^- = [0, s_0]$  and  $I^+ = [s_0, s_{max}]$ . The cumulative flux  $H_\xi(s)$  is strictly decreasing for  $s \in I^-$  and strictly increasing for  $s \in I^+$ , by the divergence theorem  $H_\xi(s_{max}) = 0$  so  $H_\xi(s) < 0$  for  $s \in (0, s_{max})$ . ■

## 5 The travelling wave ODE

In this section we will assume we are given a family of functions  $G_\xi(s)$  parametrized by  $\xi$  with

$$\begin{aligned} 0 < G_\xi(s) < 2 & \quad \text{for } 0 < s < s_{max}, \\ G_\xi(s_{max}) &= 2, \\ G'_\xi(s) &\geq 0, \\ G_\xi(s) &\leq \frac{1}{2} \tilde{C}(\xi) s^2 \quad \text{for } 0 \leq s \leq \tilde{C}(\xi)^{-1}, \end{aligned} \quad (5.1)$$

where the constant  $\tilde{C}(\xi)$  is given in Lemma 4.7. Using this we will solve the travelling wave ODE for each  $\xi$ ,

$$r'_\xi(s) = \sqrt{1 - \left(\frac{G_\xi(s)}{r_\xi(s)}\right)^2}, \quad r_\xi(0) = 0, \quad r'_\xi(0) = 1 \quad (5.2)$$

and show that there exists a  $\xi^*$  such that  $r_{\xi^*}(s_{max}) = 2$ . Since this differential equation is not Lipschitz, we cannot use standard arguments for existence and uniqueness of solutions. In fact, there are many solutions satisfying  $r_\xi(0) = 0$ , in subsection 5.1 we will use a contraction argument to show that there is a unique solution  $r_{f,\xi}$  which also satisfies  $r'_{f,\xi}(0) = 1$ . In subsection 5.3 we then show there is a unique solution  $r_{b,\xi}$  satisfying  $r_{b,\xi}(s_{max}) = 2$ . Finally in subsection 5.4 we show that for  $\xi = \xi^*$  these two solutions match, giving the desired solution.

## 5.1 The forward solution

In this section we show that, for each  $\xi$  there exists a solution starting at  $s = 0$ . We substitute  $r_\xi(s) = s - s^3x(s)$ , a function  $x(s)$  solving the ODE must then be a fixed point of the integral operator  $\Phi$ .

$$\Phi[x](s) = \frac{1}{s^3} \int_0^s 1 - \sqrt{1 - \left(\frac{G_\xi(\sigma)}{\sigma - \sigma^3x(\sigma)}\right)^2} d\sigma \quad (5.3)$$

We examine  $\Phi$  on the ball  $B$  of radius  $\frac{1}{9}\tilde{C}(\xi)^2$  in the space of continuous bounded functions on the interval  $[0, \tilde{C}(\xi)^{-1}]$  equipped with the supremum norm.

**Lemma 5.1** *The integral operator  $\Phi$  has a unique fixed point on  $B$ .*

**Proof** First of all  $\Phi : B \rightarrow B$ . To see this, assume  $x(s) \leq a\tilde{C}(\xi)^2$  for some value of  $a$ . If  $a \leq \frac{1}{2}$  then for  $s \leq \tilde{C}(\xi)^{-1}$ ,  $\frac{G_\xi(s)}{s - s^3x(s)} \leq \frac{1}{2} \frac{\tilde{C}(\xi)}{1-a} s \leq \frac{1}{2(1-a)} \leq 1$ . Now for  $u \leq 1$ ,  $1 - \sqrt{1-u} \leq u$  and so  $\Phi[x](s) \leq \frac{1}{12} \frac{\tilde{C}(\xi)^2}{(1-a)^2}$ . If we set  $a = \frac{1}{9}$  then  $\Phi[x](s) \leq a\tilde{C}(\xi)^2$ .

Furthermore,  $\Phi$  is a contraction on  $B$ . Let  $\|\cdot\|$  be the supremum norm on  $B$ ,

$$\|\Phi[x_2] - \Phi[x_1]\| \leq \frac{1}{s^3} \int_0^s \left| \frac{\partial}{\partial x} \left( 1 - \sqrt{1 - \left(\frac{G_\xi(\sigma)}{\sigma - \sigma^3x}\right)^2} \right) \right| \cdot |x_2(\sigma) - x_1(\sigma)| d\sigma. \quad (5.4)$$

Now,

$$\begin{aligned} \frac{\partial}{\partial x} \left( 1 - \sqrt{1 - \left(\frac{G_\xi(s)}{s - s^3x}\right)^2} \right) &= \frac{G_\xi(s)^2}{(1 - s^2x)^3 \sqrt{1 - \left(\frac{G_\xi(s)}{s - s^3x}\right)^2}} \\ &\leq \frac{1}{4(1-a)^3 \sqrt{1 - \frac{1}{4(1-a)^2}}} s^2 \leq \frac{1}{2} s^2 \quad \text{for } a = \frac{1}{9}. \end{aligned} \quad (5.5)$$



Therefore,  $\|\Phi[x_2] - \Phi[x_1]\| \leq \frac{1}{6} \|x_2 - x_1\|$ . By the Banach fixed point theorem,  $\Phi$  must have a fixed point. ■

We will denote this fixed point as  $r_{f,\xi}$ .

## 5.2 Determination of $\xi_{min}$ and $\xi_{max}$ .

We wish to find a value  $\xi^*$  such that  $r_{f,\xi^*}$  can be continued to the interval  $(0, s_{max}]$  with  $r_{f,\xi^*}(s_{max}) = 2$ . Clearly, since  $r_{f,\xi}(s) \leq s$ , if at some point  $s^*$ ,  $G_\xi(s^*) \geq s^*$  then we cannot continue the solution at this value of  $\xi$  since  $r'_{f,\xi}(s^*)$  would not be defined. By Lemma 4.6 such a value of  $s^*$  exists, and so  $\xi^* \geq \xi_{min} = \frac{3}{40}$ .

If there exists an  $s^* \leq s_{max}$  such that  $r_{f,\xi}(s^*) \geq 2$ , the solution can be continued till infinity, however, due to the monotonicity of  $r_{f,\xi}$  it will be impossible to meet the requirement that  $r_{f,\xi}(s_{max}) = 2$ . By Lemma 5.1,  $r_{f,\xi}(\tilde{C}(\xi)^{-1}) \geq \frac{8}{9} \tilde{C}(\xi)^{-1}$ , so if  $\tilde{C}(\xi) \leq \frac{4}{9}$  and  $\tilde{C}(\xi)^{-1} < s_{max}$  then the continuation of  $r_\xi$  will grow too large. By Lemma 4.7,  $\tilde{C}(\xi) = \frac{16}{\xi^2}$  and so  $\xi^* \leq \xi_{max} = 6$ .

## 5.3 The backwards solution

In this section we will show that there exists a unique solution from  $s = s_{max}$  with  $r_\xi(s_{max}) = 2$  extending backwards till  $s = 0$ .

**Lemma 5.2** *Given  $\xi \in (\xi_{min}, \xi_{Max})$  and  $s^* \in (0, s_{max})$ . Then any two functions  $r_1$  and  $r_2$  solving (5.2) on an interval  $(s_0, s^*]$  having equal endpoints,  $r_1(s^*) = r_2(s^*)$ , must be equal over the entire interval.*

**Proof** We write the ODE as  $r' = f(r(s), s)$  and examine the derivative to  $r$ ,

$$\frac{\partial f}{\partial r} = \frac{1}{f(r, s)} \frac{G_\xi(s)^2}{r^3} > 0. \quad (5.6)$$

Now assume  $r_1$  and  $r_2$  are two different solutions to (5.2) on an interval  $I = (s^* - \delta, s^*)$  such that  $r_1(s^*) = r_2(s^*)$ . We will show that there is a contradiction. If  $r_1$  and  $r_2$  are different, there exists an  $s_0 \in I$  such that  $r_2(s_0) \neq r_1(s_0)$ . Without loss of generality, assume that  $r_2(s_0) > r_1(s_0)$ . The difference between  $r_2$  and  $r_1$  satisfies the differential equation

$$\frac{d}{ds} (r_2(s) - r_1(s)) = \int_{r_1(s)}^{r_2(s)} \frac{\partial f}{\partial r} dr > 0. \quad (5.7)$$

and so  $r_2(s) - r_1(s) > r_2(s_0) - r_1(s_0) > 0$  for all  $s > s_0$ , specifically at  $s = s^*$ . ■

**Lemma 5.3** *Given  $\xi \in (\xi_{min}, \xi_{max})$  and  $s^* \in (0, s_{max}]$  then there exists a unique solution  $r : (0, s^*] \rightarrow [0, 2]$  to the differential equation (5.2) satisfying  $r(s^*) = G_\xi(s^*)$ .*

**Proof** We examine the differential equation

$$r'(s) = f(s, r(s)), \quad (5.8)$$

where

$$f(s, r) = \begin{cases} \sqrt{1 - \left(\frac{G_\xi(s)}{r}\right)^2} & \text{if } r \geq G_\xi(s) \text{ and } s \leq s_{max}, \\ 0 & \text{if } r \leq G_\xi(s) \text{ and } s \leq s_{max}, \\ \sqrt{1 - \left(\frac{2}{r}\right)} & \text{if } r \geq 2 \text{ and } s \geq s_{max}, \\ 0 & \text{if } r \leq 2 \text{ and } s \geq s_{max}. \end{cases} \quad (5.9)$$

The function  $f$  is continuous, so by Peano's existence theorem there exist solutions  $s \mapsto r(s)$  satisfying  $r(s^*) = G_\xi(s^*)$  defined in a neighbourhood of  $s^*$ . Assume there is an  $s_0 < s^*$  in this neighbourhood such that  $r(s_0) < G_\xi(s_0)$ . Let  $s_1 = \min\{s \in (s_0, s^*) \mid r(s) \geq G_\xi(s)\}$ , this definition makes sense since  $r$  and  $G_\xi$  are continuous and  $r(s^*) = G_\xi(s^*)$ . By the mean value theorem there exists an  $s_2 \in (s_0, s_1)$  such that

$$r'(s_2) = \frac{r(s_1) - r(s_0)}{s_1 - s_0} > \frac{G_\xi(s_1) - G_\xi(s_0)}{s_1 - s_0} > 0, \quad (5.10)$$

since  $G_\xi(s)$  is strictly monotone. Thus by (5.8),  $r(s_2) \geq G_\xi(s_2)$  which is in contradiction with our definition of  $s_1$ . Therefore  $r(s) \geq G_\xi(s)$  for  $s \leq s^*$  and the restriction of our solution to  $s \leq s_{max}$  solves (5.2), repeating this argument enables us to extend this solution until  $s = 0$ . Uniqueness then follows from Lemma 5.2.  $\blacksquare$

Setting  $s^* = s_{max}$  in Lemma 5.3 we obtain a unique solution on  $(0, s_{max}]$ .

## 5.4 Matching

In the previous sections we have constructed solutions to the travelling wave ODE. The forward solution, which we denote as  $r_{f,\xi}$ , exists on an interval  $[0, \tilde{C}(\xi)]$  while the backward solution  $r_{b,\xi}$ , exists on the interval  $(0, s_{max}]$ . We will examine both solutions at the point  $s = \bar{C}^{-1}$ , where

$$\bar{C}^{-1} = \min_{\xi \in [\xi_{min}, \xi_{max}]} \{\tilde{C}(\xi)^{-1}\}. \quad (5.11)$$

If we examine the solutions at  $\xi = \xi_{min}$  we see that at some point  $s^*$ ,  $r_{f,\xi_{min}}(s^*) = G_{\xi_{min}}(s^*) \leq r_{b,\xi_{min}}(s^*)$ . Since solutions to the same ODE cannot intersect,

$$r_{f,\xi_{min}}(\bar{C}^{-1}) \leq r_{b,\xi_{min}}(\bar{C}^{-1}). \quad (5.12)$$

Similarly, examining the solution at  $\xi = \xi_{max}$  we see that  $r_{f,\xi_{max}}(\bar{C}(\xi_{max})^{-1}) \geq 2 \geq r_{b,\xi_{max}}(\bar{C}(\xi_{max})^{-1})$  and so

$$r_{f,\xi_{max}}(\bar{C}^{-1}) \geq r_{b,\xi_{max}}(\bar{C}^{-1}). \quad (5.13)$$

**Lemma 5.4** *The forwards and backwards solutions  $r_{f,\xi}$  and  $r_{b,\xi}$  are strict monotone in  $\xi$  for  $s > 0$ ,*

$$\frac{\partial r_{f,\xi}}{\partial \xi} > 0, \quad \frac{\partial r_{b,\xi}}{\partial \xi} < 0. \quad (5.14)$$

**Proof** If we differentiate the ODE (5.2) to  $\xi$  we see that  $u = \frac{\partial r_\xi}{\partial \xi}$  satisfies the differential equation

$$\frac{du}{ds}(s) = f(s)u(s) + g(s), \quad (5.15)$$

where

$$f(s) = \frac{1}{\sqrt{1 - \left(\frac{G_\xi(s)}{r_\xi(s)}\right)^2}} \frac{G_\xi(s)^2}{r_\xi(s)^3}, \quad g(s) = -\frac{1}{\sqrt{1 - \left(\frac{G_\xi(s)}{r_\xi(s)}\right)^2}} \frac{G_\xi(s)}{r_\xi(s)^2} \frac{\partial G_\xi}{\partial \xi}. \quad (5.16)$$

Since  $r_\xi$  and  $G_\xi$  are positive,  $f(s) > 0$  for  $s > 0$  and by Lemma 4.10,  $g(s) > 0$  for  $s > 0$ .

To study the forward solution, we examine the solution of this ODE with initial condition  $u(0) = 0$ . By (3.5) and Lemma 4.7 its clear that  $f$  and  $g$  remain bounded as  $s \rightarrow 0$ , so by the variation of constants formula

$$\frac{\partial r_{f,\xi}}{\partial \xi}(s) = e^{-\mu(s)} \int_0^s e^{\mu(\sigma)} g(\sigma) d\sigma > 0 \quad (5.17)$$

where

$$\mu(s) = \int_0^s f(\sigma) d\sigma. \quad (5.18)$$

For the backwards solution,  $r_{b,\xi}(s_{max}) = G_\xi(s_{max}) = 2$  for all  $\xi$ . So a similar variations of constants arguments, with  $u(s_{max}) = 0$  yields that  $\frac{\partial r_{b,\xi}}{\partial \xi} < 0$ . ■

The inequalities for the forward and backward solutions at  $s = \tilde{C}^{-1}$ , together with the strict monotonicity established in the above Lemma yield that there must exist an unique  $\xi^* \in [\xi_{min}, \xi_{max}]$  such that

$$r_{f,\xi^*}(\tilde{C}^{-1}) = r_{b,\xi^*}(\tilde{C}^{-1}). \quad (5.19)$$

We now define  $r_{\xi^*}(s)$  to be equal to  $r_{f,\xi^*}$  if  $s \leq \tilde{C}^{-1}$  and equal to  $r_{b,\xi^*}$  otherwise. Thus  $r_{\xi^*}$  solves (5.2) with the desired boundary conditions at  $s = 0$  and  $s = s_{max}$ .

## 6 Fixed point of the Schauder map

In Sections 4 and 5 we defined a map from  $\Xi(M, A, C; s_{max})$  to  $C^{1,\alpha} \times C^{0,\alpha}$ . In this section we will choose the constants  $C$ ,  $A$  and  $M$  such that the image of this map is a subset of the original domain. We will then show that both domain and image are compact and the map is continuous in the  $C^{1,\beta}$  topology for any  $\beta < \alpha$ , and thereby satisfies all the requirements of Schauder's fixed point theorem.

## 6.1 Determination of $C$

From Lemma 4.7 we have that

$$G_{\xi^*}(s) \leq \frac{1}{2} \tilde{C} s^2 \quad \text{for } 0 \leq s \leq \tilde{C}^{-1}, \quad (6.1)$$

where  $\tilde{C} = \max(\frac{16}{\xi_{min}^2}, \sqrt{\frac{C}{\xi_{min}}}, C)$ . From Lemma 5.1 and the monotonicity of  $r_{\xi^*}$  we then have that

$$\begin{aligned} s - \frac{1}{9} \tilde{C}^2 s^3 \leq r_{\xi^*}(s) \leq 2 & \quad \text{for } 0 \leq s \leq \tilde{C}^{-1}, \\ \frac{8}{9} \tilde{C}^{-1} \leq r_{\xi^*}(s) \leq 2 & \quad \text{for } \tilde{C}^{-1} \leq s \leq s_{max}, \end{aligned} \quad (6.2)$$

If we choose  $C \geq \frac{16}{\xi_{min}^2} = \frac{25600}{9}$  then  $\tilde{C} = C$ .

## 6.2 Determination of $A$

If one were to differentiate (5.2) to  $s$  we would obtain

$$r_{\xi^*}'' = \frac{G_{\xi^*}^2}{r_{\xi^*}^3} - \frac{G_{\xi^*}}{r_{\xi^*}^2} \frac{G'_{\xi^*}}{r'_{\xi^*}} \quad z_{\xi^*}'' = \frac{G_{\xi^*}}{r_{\xi^*}^2} r'_{\xi^*} - \frac{G'_{\xi^*}}{r_{\xi^*}}. \quad (6.3)$$

Since  $r'_{\xi^*}$  might be equal to zero, the negative contribution to  $r_{\xi^*}''$  may be infinite and we cannot say that  $r$  is twice differentiable. The difference quotients of the first derivatives are however always defined, and equal to

$$r'_{\xi^*}(s_2) - r'_{\xi^*}(s_1) = \int_{s_1}^{s_2} \frac{G_{\xi^*}^2}{r_{\xi^*}^3} - \frac{G_{\xi^*}}{r_{\xi^*}^2} \frac{G'_{\xi^*}}{r'_{\xi^*}} ds \leq \int_{s_1}^{s_2} \frac{G_{\xi^*}^2}{r_{\xi^*}^3} ds, \quad (6.4)$$

where without loss of generality we can assume that  $s_2 \geq s_1$ . Now using (6.1), (6.2) and the fact that by the divergence theorem,  $G_{\xi^*}(s) \leq 2$  one can estimate

$$\begin{aligned} \frac{G_{\xi^*}^2}{r_{\xi^*}^3} &\leq \frac{1}{4} \frac{C^2}{(1 - \frac{1}{9} C^2 s^2)^3} s \leq \frac{1}{4} \left(\frac{9}{8}\right)^3 C \quad \text{for } 0 \leq s \leq C^{-1}, \\ \frac{G_{\xi^*}^2}{r_{\xi^*}^3} &\leq 4 \left(\frac{9}{8}\right)^3 C^3 \quad \text{for } C^{-1} \leq s \leq s_{max}, \\ \frac{G_{\xi^*}}{r_{\xi^*}^2} &\leq \frac{1}{2} \frac{C}{(1 - \frac{1}{9} C^2 s^2)^2} \leq \frac{1}{4} \left(\frac{9}{8}\right)^2 C \quad \text{for } 0 \leq s \leq C^{-1}, \\ \frac{G_{\xi^*}}{r_{\xi^*}^2} &\leq 2 \left(\frac{9}{8}\right)^2 C^2 \quad \text{for } C^{-1} \leq s \leq s_{max}. \end{aligned} \quad (6.5)$$

All four of these estimates are bounded and only depend on the constant  $C$  determined previously, and so we can estimate

$$r'_{\xi^*}(s_2) - r'_{\xi^*}(s_1) \leq A(s_2 - s_1). \quad (6.6)$$

Similar arguments yield the estimate for the difference quotient on  $z'_{\xi^*}$ .

### 6.3 Hölder continuity

We have now established enough bounds on  $r_{\xi^*}$  and  $G_{\xi^*}$  to establish an uniform  $C^{1,\alpha}$  Hölder norm. Since  $r'_{\xi^*}$  is the square root of a  $C^1$  function, we expect it to be bounded for Hölder exponent  $\alpha \leq \frac{1}{2}$ .

**Lemma 6.1** *There is an  $M$ , independent of  $s_{max}$ , such that the solutions  $r_{\xi^*}$  and  $z'_{\xi^*}$  have the following Hölder norms for Hölder exponent  $\alpha \leq \frac{1}{2}$ :*

$$\begin{aligned} \|r_{\xi^*}\|_{1,\alpha} &\leq M, \\ \|z'_{\xi^*}\|_{0,\alpha} &\leq M. \end{aligned} \tag{6.7}$$

**Proof** We estimate the difference in first derivatives at points  $s_1$  and  $s_2$ ,

$$\begin{aligned} |r'_{\xi^*}(s_2) - r'_{\xi^*}(s_1)| &= \left| \sqrt{1 - \left(\frac{G_{\xi^*}(s_2)}{r_{\xi^*}(s_2)}\right)^2} - \sqrt{1 - \left(\frac{G_{\xi^*}(s_1)}{r_{\xi^*}(s_1)}\right)^2} \right| \\ &\leq \sqrt{\left| \left(\frac{G_{\xi^*}(s_2)}{r_{\xi^*}(s_2)}\right)^2 - \left(\frac{G_{\xi^*}(s_1)}{r_{\xi^*}(s_1)}\right)^2 \right|} \\ &\leq \left| \max \left( \frac{d}{ds} \left( \frac{G_{\xi^*}^2}{r_{\xi^*}^2} \right) \right) \right|^{\frac{1}{2}} \cdot |s_2 - s_1|^{\frac{1}{2}}. \end{aligned} \tag{6.8}$$

Now,

$$\frac{d}{ds} \left( \frac{G_{\xi^*}(s)^2}{r_{\xi^*}(s)^2} \right) = \frac{2G_{\xi^*}G'_{\xi^*}}{r_{\xi^*}^2} - \frac{2G_{\xi^*}^2 r'_{\xi^*}}{r_{\xi^*}^3}, \tag{6.9}$$

and using Lemma 4.3 and estimates (6.5) we see all these terms are uniformly bounded by constants depending only on  $C$ ,  $A$ ,  $\xi_{min}$ , and  $\xi_{max}$ . Similarly, by (6.3) and (6.5),  $z''$  can be bounded from above by  $A$ , and below by a constant depending only on  $C$ ,  $A$  and  $F_{max}$ , and so the Hölder norm of  $z'$  can be bound similarly. The constants  $C$ ,  $A$ ,  $\xi_{max}$  and  $\xi_{min}$  have been determined in the previous sections,  $F_{max}$  depends only on these constants, thus  $M$  can be expressed in terms of these known constants. Specifically,  $M$  does not depend on  $s_{max}$ . ■

### 6.4 Continuity of $(r, z) \rightarrow G_\xi$

We wish to show that, for a sequence  $(r_n, z'_n) \in C^{1,\alpha} \times C^{0,\alpha}$  uniformly, describing boundaries  $\partial\Omega_n$  such that  $(r_n, z_n) \rightarrow (\bar{r}, \bar{z})$  in the  $C^{1,\beta}$  topology, the associated fluxes  $F_n \rightarrow \bar{F}$  in the  $C^{0,\beta}$  topology. In order to do this we need to create bijections between the domains  $\Omega_n$  and  $\bar{\Omega}$ .

**Lemma 6.2** *There exist surjective mappings  $\phi_n : \bar{\Omega} \rightarrow \Omega_n \subset \mathbb{R}^3$  such that  $\phi_n \in C^{1,\alpha}$  uniformly, and  $\phi_n \rightarrow \text{Id}$  in the  $C^{1,\beta}$  topology.*

**Proof** We first define  $\phi_n$  on the boundary  $\partial\bar{\Omega}$  by mapping the point at pathlength  $s$  on  $\partial\bar{\Omega}$  to the point at the same pathlength on  $\partial\Omega_n$ . For  $x = (x_1, x_2, x_3) = (\bar{r}(s) \cos \theta, \bar{r}(s) \sin \theta, \bar{z}(s)) \in \partial\bar{\Omega}$ ,

$$\begin{aligned}\phi_n^{(1)}(x) &= r_n(s) \cos \theta, \\ \phi_n^{(2)}(x) &= r_n(s) \sin \theta, \\ \phi_n^{(3)}(x) &= z_n(s).\end{aligned}\tag{6.10}$$

We treat these as boundary conditions and extend  $\phi_n^{(i)}$  to the interior by solving  $\Delta\phi_n^{(i)} = 0$  for  $i = 1, 2, 3$ . By Schauder's boundary estimates for  $C^{1,\alpha}$  domains ([6], Thm 8.33)  $\|\phi_n^{(i)}\|_{1,\alpha} \leq C(\|r_n\|_{1,\alpha} + \|z_n\|_{1,\alpha})$  where the constant  $C$  depends only on the boundary  $\partial\bar{\Omega}$ . As such, the functions  $\phi_n^{(i)}$  are uniformly bounded in the  $C^{1,\alpha}$  Hölder norm. Furthermore,  $\phi_n(\bar{\Omega}) = \Omega_n$ , if there were a point in  $\Omega_n$  which was not mapped to, the image would have a different topological class than  $\bar{\Omega}$  which is impossible for a continuous map. If we were to apply the same procedure to construct a map  $\bar{\phi} : \bar{\Omega} \rightarrow \bar{\Omega}$  then on the boundary  $\bar{\phi}^{(1)}(x)|_{\partial\bar{\Omega}} = \bar{r}(s) \cos \theta = x_1$ . The harmonic extension of this boundary would be  $\bar{\phi}^{(1)}(x) = x_1$  and similar for the  $x_2$  and  $x_3$  coordinates, so  $\bar{\phi}$  is the identity mapping  $Id$ . We now examine  $\phi_n^{(1)} - x_1$ , this function is harmonic and assumes values  $(r_n(s) - \bar{r}(s)) \cos \theta$  on the boundary. We apply Schauder's boundary estimates again to see that  $\|\phi_n - Id\|_{1,\beta} \leq C(\|r_n - \bar{r}\|_{1,\beta} + \|r_n - \bar{z}\|_{1,\beta})$  with the constant  $C$  depending only on  $\partial\bar{\Omega}$ . Since  $r_n \rightarrow \bar{r}$  and  $z_n \rightarrow \bar{z}$  in  $C^{1,\beta}$ ,  $\phi_n \rightarrow Id$  in  $C^{1,\beta}$ . ■

We now examine the Dirichlet problem on  $\Omega_n$  and  $\bar{\Omega}$ . For simplicity we place the source at the origin, the tip of each domain lies at  $(r, z) = (0, z_n(0))$ . We write  $u_n(x) = \frac{1}{|x|} - w_n$ ,  $\bar{u} = \frac{1}{|x|} - \bar{w}$  where  $w_n$  and  $\bar{w}$  are the solutions of

$$\begin{aligned}\Delta w_n &= 0 \quad \text{in } \Omega_n, & w_n &= \frac{1}{|x|} \quad \text{for } x \in \partial\Omega_n, \\ \Delta \bar{w} &= 0 \quad \text{in } \bar{\Omega}, & \bar{w} &= \frac{1}{|x|} \quad \text{for } x \in \partial\bar{\Omega}.\end{aligned}\tag{6.11}$$

Now  $\frac{1}{|x|}$  is smooth away from the origin, and all its derivatives can be bounded by a function of  $d_{min}$ , the minimal distance between  $\partial\Omega_n$  and the origin for all  $n$ . So  $\left\|\frac{1}{|x|}\right\|_{1,\alpha;\partial\Omega_n} \leq C(\|r_n\|_{1,\alpha} + \|z_n\|_{1,\alpha})$  where the constant  $C$  depends only on  $d_{min}$ . By the boundary estimates,  $\|w_n\|_{1,\alpha;\Omega_n} \leq C\left(\left\|\frac{1}{|x|}\right\|_{1,\alpha;\partial\Omega_n}\right)$  where the constant  $C$  depends on the  $C^{1,\alpha}$  norms of  $r_n$  and  $z_n$ . Since these are uniformly bounded, there is a  $C$  independent of  $n$  such that  $\|w_n\|_{1,\alpha;\Omega_n} \leq C$ . We now examine the compositions  $w_n \circ \phi_n : \bar{\Omega} \rightarrow \mathbb{R}$ . Since, by the above Lemma, the maps  $\phi_n$  are uniformly bounded in  $C^{1,\alpha}$ , these compositions are uniformly bounded in  $C^{1,\alpha}$ . On the boundary  $w_{n_j} \circ \phi_{n_j}|_{\partial\bar{\Omega}} = \frac{1}{|\phi_{n_j}|}$  and clearly converges to  $\bar{w}|_{\partial\bar{\Omega}}$ .

**Lemma 6.3** *Let  $\phi_n : \bar{\Omega} \rightarrow \Omega_n \subset \mathbb{R}^3$  be a sequence of mappings,  $w_n : \Omega_n \rightarrow \mathbb{R}$  be a sequence of harmonic functions, and let  $\bar{w} : \bar{\Omega} \rightarrow \mathbb{R}$  be a harmonic function, such that:*

- *The mappings  $\phi_n \rightarrow Id$  in the  $C^{1,\beta}(\bar{\Omega}; \mathbb{R}^3)$  norm.*
- *The compositions  $w_n \circ \phi_n$  are uniformly bounded in the  $C^{1,\alpha}(\bar{\Omega}; \mathbb{R})$  norm.*
- *On the boundary,  $w_n \circ \phi_n|_{\partial\bar{\Omega}} \rightarrow \bar{w}|_{\partial\bar{\Omega}}$  in the  $C^{1,\beta}(\partial\bar{\Omega}; \mathbb{R})$  norm.*

*Then we have convergence on the full set  $\bar{\Omega}$ , the compositions  $w_n \circ \phi_n \rightarrow \bar{w}$  in the  $C^{1,\beta}(\bar{\Omega}; \mathbb{R})$  norm.*

**Proof** Since  $w_n \circ \phi_n$  is uniformly bounded in  $C^{1,\alpha}$  it has a convergent subsequence in  $C^{1,\beta}$ . Let us examine such a convergent subsequence, which we will also denote as  $w_{n_j} \circ \phi_{n_j}$  and let the limit be denoted as  $\tilde{w}$ . Each  $w_{n_j}$  is a weak solution of the Dirichlet problem, so for all  $v_{n_j} \in C_0^1(\Omega_{n_j}; \mathbb{R})$

$$\int_{\Omega_{n_j}} \nabla w_{n_j} \cdot \nabla v_{n_j} dx = 0 \quad (6.12)$$

Now  $\Omega_{n_j} = \phi_{n_j}(\bar{\Omega})$  and so we can perform a coordinate transform. Furthermore, for sufficiently large  $n$  the Jacobian  $D\phi_{n_j}$  is sufficiently close to the identity matrix that it can be inverted, and  $\phi_{n_j}$  is a bijection. So writing  $v_{n_j} = v \circ \phi_{n_j}^{-1}$ , the above statement holds for all  $v \in C_0^1(\bar{\Omega}; \mathbb{R})$ . Now

$$\begin{aligned} \int_{\phi_{n_j}(\bar{\Omega})} \nabla w_{n_j} \cdot \nabla v_{n_j} &= \int_{\bar{\Omega}} ((\nabla w_{n_j}) \circ \phi_{n_j}) \cdot ((\nabla v_{n_j}) \circ \phi_{n_j}) |\det D\phi_{n_j}| dx \\ &= \int_{\bar{\Omega}} (D\phi_{n_j})^{-1} \nabla(w_{n_j} \circ \phi_{n_j}) \cdot (D\phi_{n_j})^{-1} \nabla v |\det D\phi_{n_j}| dx \\ &= 0. \end{aligned} \quad (6.13)$$

We now take the limit as  $n \rightarrow \infty$ , and see that for all  $v$

$$\int_{\bar{\Omega}} \nabla \tilde{w} \cdot \nabla v dx = 0, \quad (6.14)$$

and so  $\tilde{w}$  is weakly harmonic. Since  $w_n \circ \phi_n|_{\partial\bar{\Omega}} \rightarrow \bar{w}|_{\partial\bar{\Omega}}$  and there is only one harmonic function with a given boundary value,  $\tilde{w} = \bar{w}$ .

Now assume that  $w_n \circ \phi_n$  does not converge to  $\bar{w}$ , then there exist  $\epsilon$  and  $N$  such that for all  $n \geq N$ ,  $\|w_n - \bar{w}\|_{1,\beta} \geq \epsilon$ . This is in contradiction with the fact that there is a subsequence such that  $w_{n_j} \rightarrow \bar{w}$ . ■

We can now prove the convergence of the fluxes  $F_n$ . We consider each flux to be a function on  $\partial\bar{\Omega}$  parametrized by  $s$ . Then

$$F_n = |(\nabla u_n) \circ \phi_n| = \left| (D\phi_n)^{-1} \left( \nabla \left( \frac{1}{|\phi_n|} \right) - \nabla(w_n \circ \phi_n) \right) \right|, \quad (6.15)$$

and clearly

$$F_n \rightarrow \left| \nabla \left( \frac{1}{|x|} \right) - \nabla \bar{w} \right| = \bar{F}, \quad (6.16)$$

in  $C^{0,\beta}$ .

### 6.5 Continuity of $G_\xi \rightarrow (\xi^*, r_{\xi^*}, z'_{\xi^*})$

We wish to show that for a sequence,  $G_{n,\xi}$  of families of cumulative fluxes parametrized by  $\xi$ , uniformly bounded in  $C^{1,\alpha}$  and converging to  $\bar{G}_\xi$  in  $C^{1,\beta}$  for each value of  $\xi$ , the associated solutions  $(r_n, z'_n, \xi_n^*)$  converge to  $(\bar{r}, \bar{z}', \bar{\xi}^*)$ . Each  $r_n$  satisfies

$$\begin{aligned} \int_0^s \sqrt{r_n(\sigma)^2 - G_{n,\xi_n^*}(\sigma)^2} d\sigma &= \frac{1}{2} r_n(s)^2, \\ \int_0^{s_{max}} \sqrt{r_n(\sigma)^2 - G_{n,\xi_n^*}(\sigma)^2} d\sigma &= 2. \end{aligned} \quad (6.17)$$

Since  $r_n$  is uniformly bounded in  $C^{1,\alpha}$  and  $\xi_n \in [\xi_{min}, \xi_{max}]$  we examine a subsequence of solutions  $r_{n_j}$  and  $\xi_{n_j}$ , such that  $r_{n_j} \rightarrow \tilde{r}$  and  $\xi_{n_j}^* \rightarrow \tilde{\xi}^*$ . Now

$$\begin{aligned} \left\| G_{n_j, \xi_{n_j}^*} - \bar{G}_{\tilde{\xi}^*} \right\|_{1,\beta} &\leq \left\| G_{n_j, \xi_{n_j}^*} - G_{n_j, \tilde{\xi}^*} \right\|_{1,\beta} + \left\| G_{n_j, \tilde{\xi}^*} - \bar{G}_{\tilde{\xi}^*} \right\|_{1,\beta}, \\ &\leq \left\| \frac{\partial G_{n_j, \xi}}{\partial \xi} \right\|_{1,\beta} \left| \xi_{n_j} - \tilde{\xi} \right| + \left\| G_{n_j, \tilde{\xi}^*} - \bar{G}_{\tilde{\xi}^*} \right\|_{1,\beta}. \end{aligned} \quad (6.18)$$

By Lemma 4.10 the derivative of the flux to  $\xi$  is uniformly bounded in  $C^{1,\alpha}$ , so  $G_{n_j, \xi_{n_j}^*} \rightarrow \bar{G}_{\tilde{\xi}^*}$  in  $C^{1,\beta}$ . Therefore we can pass through the limit and see that  $\tilde{r}$ ,  $\tilde{\xi}^*$  and  $\bar{G}_{\tilde{\xi}^*}$  satisfy (6.17). Since, given a family of cumulative fluxes, solutions to the travelling wave ODE are unique,  $\tilde{r} = \bar{r}$  and  $\tilde{\xi}^* = \bar{\xi}^*$ . Now assume that  $r_n$  and  $\xi_n^*$  do not converge to  $\bar{r}$  and  $\bar{\xi}$ , then there exist  $\epsilon$  and  $N$  such that for all  $n \geq N$ ,  $\|r_n - \bar{r}\|_{1,\beta} \geq \epsilon$  or  $|\xi_n^* - \bar{\xi}^*| \geq \epsilon$ . This is in contradiction with the fact that there is a subsequence such that  $r_{n_j} \rightarrow \bar{r}$  and  $\xi_{n_j}^* \rightarrow \bar{\xi}^*$ .

The quotients  $z'_n = \frac{G_{n,\xi_n^*}}{r_{n,\xi_n^*}}$  are uniformly bounded in  $C^{1,\alpha}$  and thus have a convergent subsequence  $z'_{n_j}$  which converges to  $\bar{z}'$ . Now clearly  $\bar{z}'(0) = \bar{z}'(0)$  and for  $s > 0$ ,

$$|\bar{z}'(s) - z'(s)| \leq \left| \bar{z}'(s) - z'_{n_j}(s) \right| + \frac{1}{r_{n_j}(s)} \left| G_{n_j, \xi_{n_j}^*}(s) - \bar{G}_{\bar{\xi}^*}(s) \right| + \frac{\bar{z}'(s)}{r_{n_j}(s)} \left| \bar{r}(s) - r_{n_j}(s) \right|. \quad (6.19)$$

Since the right hand side converges to zero,  $\bar{z}'(s) = z'(s)$  for all  $s$ . A similar argument as used for the convergence of  $r_n$  and  $\xi_n$  now yields that  $z'_n \rightarrow \bar{z}'$  in  $C^{0,\beta}$ .



## 6.6 Fixed point

We now have all the ingredients to use Schauder's fixed point theorem. The necessary constants  $C$ ,  $A$ ,  $M$ ,  $\xi_{min}$  and  $\xi_{max}$  have been determined in Sections 6.1, 6.2, 6.3, and Section 5.2. Since  $\Xi(M, A, C; s_{max})$  is bounded in the  $C^{1,\alpha} \times C^{0,\alpha}$  norm, its compact in the  $C^{1,\beta} \times C^{0,\beta}$  norm. In Sections 4 and 5 we define a map from  $\Xi(M, A, C; s_{max})$  to itself, in 6.4 and 6.5 we prove this map is continuous in the  $C^{1,\beta} \times C^{0,\beta}$  norm. Therefore this map has a fixed point, this fixed point solves the bounded travelling wave problem as defined in Section 2.5.

## 7 The limit as $s_{max} \rightarrow \infty$

In the previous sections we have shown that, given a sufficiently large value of  $s_{max}$  one can find a solution to the travelling wave problem with a bounded domain, as described in Section 2.5. In this section we will show that as  $s_{max} \rightarrow \infty$  the solution profiles approach a limit profile which solves the travelling wave problem on an unbounded domain.

### 7.1 The limit profile $r_\infty, z_\infty$

We examine a sequence  $s_{max,n}$  such that  $s_{max,n} \rightarrow \infty$  as  $n \rightarrow \infty$  and such that  $s_{max,n+1} - s_{max,n} > 2$ . The estimates established in (3.8) then imply that  $|z(s_{max})|$  increases monotonically to infinity. Previously we showed that for each value of  $s_{max,n}$  one can find a distance  $\xi_n^*$  and two functions  $r_n : [0, s_{max,n}] \rightarrow [0, 2]$  and  $z_n : [0, s_{max,n}] \rightarrow \mathbb{R}^-$  solving the travelling wave problem on a bounded domain. Since  $\xi_n^* \in [\xi_{min}, \xi_{max}]$ , it is possible to choose our sequence  $s_{max,n}$  such that  $\xi_n^*$  converges to some value  $\xi_\infty^*$  in the same interval.

Since the functions  $r_{\xi_n^*}, z'_{\xi_n^*}$  are uniformly bound in  $C^{1,\alpha} \times C^{0,\alpha}$ , by the Arzela-Ascoli theorem it is always possible to find a subsequence such that the restrictions to a compact interval converge in  $C^{1,\beta} \times C^{0,\beta}$ . We use this to construct  $r_\infty$  and  $z'_\infty$ .

Let  $n_j^{(0)}$  be the sequence  $1, 2, 3, \dots$ . For each  $i \geq 1$  we define the subsequence  $n_j^{(i)}$  of  $n_j^{(i-1)}$  such that the restrictions of  $r_{n_j^{(i)}}$  and  $z'_{n_j^{(i)}}$  to the interval  $[0, s_{max,i}]$  converge. We now examine the diagonal sequence  $n_j^{(j)}$ . Let  $I \subset \mathbb{R}^+$  be an arbitrary compact interval. For sufficiently large  $j$ ,  $r_{n_j^{(j)}}$  and  $z'_{n_j^{(j)}}$  are defined on this interval and their restrictions converge to functions  $r_\infty$  and  $z'_\infty$ . Since  $I$  was arbitrary, the functions  $r_\infty$  and  $z'_\infty$  are defined on  $\mathbb{R}^+$ . Integrating  $z'_\infty$  with respect to  $s$  yields the function  $z_\infty$ . To simplify notation, for the remainder of Section 7 we will denote the sequences of functions  $r_{n_j^{(j)}}$  and  $z_{n_j^{(j)}}$  as  $r_n$  and  $z_n$ .

## 7.2 The cumulative flux $G_\infty$

The functions  $r_\infty$  and  $z_\infty$  define a boundary  $\partial\Omega_\infty$  of a domain  $\Omega_\infty$  in the same manner as described previously. In this case, the domain is unbounded and there is no zero flux boundary condition at  $z = z(s_{max})$ . We place a source at a distance  $\xi_\infty$  from the tip and solve the Dirichlet problem (2.1) to obtain a vesicle density  $u_\infty$  and the associated flux  $F_\infty$  and cumulative flux  $G_\infty$ . By Lemma 4.1,  $\Omega_\infty$  is of class  $C^{1,\alpha}$  and so  $G_\infty \in C^{1,\alpha}(\mathbb{R})$ . By the divergence theorem,  $G_\infty(s) \rightarrow 2$  as  $s \rightarrow \infty$ .

The curvature of the limit profile  $\partial\Omega_\infty$  has the same upper bound as for the bounded profiles. This again gives an outer sphere condition which enables us to give an upper bound for the flux  $F_\infty$ .

**Lemma 7.1** *The flux  $F_\infty$  passing through the point  $(r_\infty(s), z_\infty(s))$  on the boundary is bounded,*

$$F_\infty \leq F_{max}, \quad (7.1)$$

where  $F_{max}$  is the same as in Lemma 4.3. This implies we can estimate  $G'_\infty(s) \leq sF_{max}$  and  $G'_\infty(s) \leq 2F_{max}$ .

**Proof** The proof is essentially the same as the first part of the proof of Lemma 4.3. In that proof, the reflected source gives a positive contribution and thus can be neglected. ■

The comparison principle established in Theorem 4.4 allows us to calculate the asymptotics of  $G_\infty$  as  $s \rightarrow \infty$ .

**Lemma 7.2** *The cumulative flux  $G_\infty$  approaches its limit value of 2 algebraically. For sufficiently large  $s$ ,*

$$0 < 2 - G_\infty(s) \leq \frac{4}{(s - 2 - \xi_\infty)^2}. \quad (7.2)$$

**Proof** We assume  $s > \xi_\infty + 2$ , then  $z(s) < -\xi$ . We use similar arguments as in Lemma 4.7 except we compare with a halfspace at the other side of the VSC. Let  $\tilde{\Omega}$  be the halfspace  $\{(r, z) | z \geq z(s)\}$ . Let

$$\tilde{u}(r, z) = (r^2 + (z + \xi)^2)^{-\frac{1}{2}} - (r^2 + (z - 2z(s) - \xi)^2)^{-\frac{1}{2}}, \quad (7.3)$$

then  $\tilde{u}$  solves the Dirichlet problem on  $\tilde{\Omega}$  with a source at  $z = -\xi_\infty$ . The comparison principle now states that

$$\begin{aligned} 2\pi(2 - G_\infty(s)) &= \int_{\partial\Omega \setminus \tilde{\Omega}} F_u dA \leq \int_{\partial\tilde{\Omega} \cap \Omega} F_{\tilde{u}} dA \\ &\leq 4\pi \left[ 1 - \left( 1 + \left( \frac{r(s)}{z(s) + \xi_\infty} \right)^2 \right)^{-\frac{1}{2}} \right], \\ &\leq 2\pi \left( \frac{r(s)}{z(s) + \xi_\infty} \right)^2 \leq 2\pi \left( \frac{2}{s - 2 - \xi_\infty} \right)^2, \end{aligned} \quad (7.4)$$

where we used the inequalities  $r(s) < 2$  and  $-s < z(s) < -s + 2$ . ■

In fact it is possible, by comparing with a capped cylinder instead of a half plane, to show that  $G_\infty$  approaches 2 exponentially fast, however the above result is sufficient for our purposes.

### 7.3 The limit of $G_n$

Let  $I \subset \mathbb{R}^+$  be an arbitrary compact interval. For sufficiently large  $n$ ,  $G_n$  is defined on  $I$ . In this section we wish to show that, restricted to  $I$ ,  $G_n \rightarrow G_\infty$  in the  $C^{1,\beta}(I)$  topology. It is sufficient to show this in the case that  $I = [0, s^*]$  for some arbitrary, but sufficiently large value of  $s^*$ . This can be shown using arguments similar to those used in Section 6.4 when showing the continuity of the Schauder map.

**Lemma 7.3** *Let  $I = [0, s^*]$  for some arbitrary but sufficiently large value of  $s^*$ . Then,*

$$G_n|_I \rightarrow G_\infty|_I \quad (7.5)$$

*in the  $C^{1,\beta}(I)$  norm.*

**Proof** We cut off  $\Omega_\infty$  at the plane  $z = z_\infty(s^*)$ . Let  $\bar{\Omega}_\infty = \{(r, z) \in \Omega_\infty | z > z_\infty(s^*)\}$ . Similarly we cut off  $\Omega_n$  at the plane  $z = z_n(s^*)$ , so  $\bar{\Omega}_n = \{(r, z) \in \Omega_n | z > z_n(s^*)\}$ . Since the restrictions of  $r_n$  and  $z_n$  to  $I$  approach  $r_\infty$  and  $z_\infty$  in  $C^{1,\beta}$  and are uniformly bounded in  $C^{1,\alpha}$ , by Lemma 6.2 there exist mappings  $\phi_n : \bar{\Omega} \rightarrow \bar{\Omega}_n \subset \mathbb{R}^3$  which are uniformly bounded in  $C^{1,\alpha}$  and converge to the identity in the  $C^{1,\beta}$  norm. Let  $u_n$  be the solution of the Dirichlet problem (with a zero flux condition at the plane  $z = z_n(s_{max,n})$ ) on  $\Omega_n$ , restricted to  $\bar{\Omega}_n$ . Similarly, let  $u_\infty$  be the solution to the (unbounded) Dirichlet problem, restricted to  $\bar{\Omega}_\infty$ . On  $\bar{\Omega}_n$  respectively  $\bar{\Omega}_\infty$  we define

$$\begin{aligned} w_n(r, z) &= \frac{1}{\sqrt{r^2 + (z + \xi_n)^2}} - u_n, \\ w_\infty(r, z) &= \frac{1}{\sqrt{r^2 + (z + \xi_\infty)^2}} - u_\infty. \end{aligned} \quad (7.6)$$

We examine the compositions  $w_n \circ \phi_n$  viewed as functions of  $s$  and restricted to  $I$ . So  $w_n \circ \phi_n|_I(s) = w_n(r_n(s), z_n(s))$ . By Lemma 3.3 the terms in the square roots are bounded from below. Since  $r_n \rightarrow r_\infty$ ,  $z_n \rightarrow z_\infty$  in  $C^{1,\beta}$  and  $\xi_n \rightarrow \xi_\infty$ , the compositions  $w_n \circ \phi_n|_I \rightarrow w_\infty(r_\infty, z_\infty)|_I$  in the  $C^{1,\beta}$  norm. By Lemma 6.2,  $w_n \circ \phi_n \rightarrow w_\infty$  on  $\bar{\Omega}$ . The same arguments as used at the end of section ?? now yield that  $G_n \rightarrow G_\infty$  in  $C^{1,\beta}$ . ■

### 7.4 Solution to the unbounded travelling wave problem

For sufficiently large  $n$ ,  $r_n$  is defined on the interval  $I$ , and for  $s \in I$ ,

$$r_n(s)^2 = 2 \int_0^s \sqrt{r_n(\sigma)^2 - G_n(\sigma)^2} d\sigma. \quad (7.7)$$

Since  $r_n \rightarrow r_\infty$  and  $G_n \rightarrow G_\infty$ , we pass through the limit to see that  $r_\infty$  and  $G_\infty$  satisfy the same equation. Differentiating to  $s$  and dividing by  $r_\infty(s)$  shows that  $r_\infty$  satisfies the travelling wave ODE on any arbitrary interval  $I$  and thus it satisfies it for all  $s$ . Hence  $r_\infty$ ,  $z'_\infty$  and  $\xi_\infty^*$  solve the unbounded travelling wave problem.

## 8 Conclusions

Using a Schauder fixed point argument, we have shown the existence of travelling solutions to the diffusive Vesicle Supply Centre model. Note that Schauder's fixed point theorem does not guarantee uniqueness, and we have not been able to show that this solution is unique by other means. Possibly one can use the comparison principle (Theorem 4.4) along with the equation for travelling solutions (2.11). This is a direction for future research.

Since the equations in Section 2.4 hold for any flux, not only for the diffusive case, the Schauder map  $\Psi$  defined in section 3.1 will have the same form for many models. Therefore the method described in this article should yield an existence proof for various related models, provided similar estimates as those in Sections 4 through 6 can be derived.

## References

- [1] S. BARTNICKI-GARCIA, F. HERGERT, G. GIERZ : Computer simulation of fungal morphogenesis and the mathematical basis for hyphal (tip) growth, *Protoplasma* 153, 46-57 (1989)
- [2] S. BARTNICKI-GARCIA, C.E. BRACKER, G. GIERZ, R. LOPEZ-FRANCO, H. LU : Mapping the growth of fungal hyphae: orthogonal cell wall expansion during tip growth and the role of turgor. *Biophys. J.* 79, 2382-2390 (2000)
- [3] S. BARTNICKI-GARCIA, G. GIERZ : A Three-Dimensional Model of Fungal Morphogenesis Based on the Vesicle Supply Center Concept, *J.Theor.Biol.* 208, 151-164 (2001)
- [4] E. EGGEN : Self-regulation in tip-growth: The role of cell wall ageing, *J. Theor. Biol.* 283, 113-121 (2011)
- [5] T. FRANKEL *The Geometry of Physics : an introduction*, Cambridge University Press, 1997.
- [6] D. GILBARG, N.S. TRUDINGER *Elliptic Partial Differential Equations of Second Order, 2nd Edition*, Springer-Verlag, (1983)
- [7] A. GORIELY, G. KÁROLYI, M. TABOR Growth induced curve dynamics for filamentary micro-organisms, *J. Math. Biol.* 51, 355-366 (2005)

- [8] J. HULSHOF, R. NOLET, G. PROKERT: Existence and linear stability of solutions of the ballistic VSC model, submitted, preprint version: CASA report 09/33, TU Eindhoven, <http://www.win.tue.nl/analysis/reports/rana09-33.pdf>
- [9] M.N. DE KEIJZER, A.M.C. EMONS, B.M. MULDER Modeling Tip Growth: Pushing Ahead, Plant Cell Monographs 12, 103-122 (2009)
- [10] A. KOCH : The problem of hyphal growth in streptomycetes and fungi, J.Theor.Biol. 171, 137-150 (1994)
- [11] S. TINDEMANS : Master's Thesis, Modeling Tip Growth in Fungal Hyphae (2004)
- [12] S, TINDEMANS, N. KERN, B. MULDER : The diffusive vesicle supply center model for tip growth in fungal hyphae, J. Theor. Biol. 238, 937-948 (2006)

## PREVIOUS PUBLICATIONS IN THIS SERIES:

Number	Author(s)	Title	Month
II-50	M.A.T. van Hinsberg J.H.M. ten Thijsse Boonkkamp F. Toschi H.J.H. Clercx	On the efficiency and accuracy of interpolation methods for spectral codes	Sept. '11
II-51	A. Bergant Q. Hou A. Keramat A.S. Tijsseling	Experimental and numerical analysis of water hammer in a large-scale PVC pipeline apparatus	Oct. '11
II-52	A. Keramat A.S. Tijsseling Q. Hou A. Ahmadi	Fluid-structure interaction with pipe-wall viscoelasticity during water hammer	Oct. '11
II-53	Q. Hou L.X. Zhang A.S. Tijsseling A.C.H. Kruisbrink	Rapid filling of pipelines with the SPH particle method	Oct. '11
II-54	J. Hulshof R. Nolet G. Prokert	Existence of solutions to the diffusive VSC model	Nov. '11