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EINDHOVEN UNIVERSITY OF TECHNOLOGY
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by

L.M.J. Florack, A. Fuster

Centre for Analysis, Scientific computing and Applications
Department of Mathematics and Computer Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven, The Netherlands
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Luc Florack Andrea Fuster

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1 Introduction

Diffusion weighted magnetic resonance imaging (dwMRI) has become a standard MRI technique for in vivo imaging of apparent water diffusion processes in fibrous tissues. Clinical use of dwMRI is hampered by the fact that dwMRI analysis requires radically new approaches, based on abstract representations, a development still in its infancy. Examples are rank-2 symmetric positive-definite tensor representations in diffusion tensor imaging (DTI), pioneered by Basser, Mattiello and Le Bihan et al. [1, 2] and explored by many others [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14], higher order symmetric positive-definite tensor representations [15, 16, 17, 18, 19, 20, 21], spherical harmonic representations in high angular resolution diffusion imaging (HARDI) [22, 23, 24, 25, 26], and SE(3) Lie group representations [27, 28, 29]. The latter type of representation, developed by Duits et al., appears to bear a particularly close relationship to the theory outlined below.

In this chapter we concentrate on a possible extension of the Riemannian paradigm used in the context of DTI in order to account explicitly for the unconstrained number of local directional degrees of freedom of general dwMRI representations. *Riemann-Finsler geometry* appears to be ideally suited for this purpose, as has already been hinted upon in previous work [16, 30, 31, 32]. However, foregoing work is either driven by heuristics or merely scratches the surface of Riemann-Finsler geometry.

Several outstanding problems remain as long as pivotal questions are left unanswered, such as the putative connection between the so-called Finsler function and anisotropic diffusion and tractography in dwMRI. We do not claim to readily resolve such fundamental issues here. However, it is clearly necessary to get the gist of the theory as a *conditio sine qua non* for clinical applicability. We therefore hope that our overview will encourage researchers to further contribute to a systematic study of Riemann-Finsler geometry in the context of dwMRI.

Riemann-Finsler geometry has its roots in Riemann's "Habilitation" [33]. Riemann focused on a special case, nowadays known as *Riemannian geometry*. Important application areas, such as Maxwell theory and Einstein's (pseudo-Riemannian) theory of general relativity, contributed to the popularity of Riemannian geometry. The general case was taken up by Finsler in his PhD thesis [34], and subsequently by Cartan [35] (who was the first to refer to it as "Finsler geometry") and others.

Although potentially much more powerful, Riemann-Finsler geometry has not been nearly as popular as its Riemannian counterpart. To some extent this may be explained by its rather mind-boggling technicalities and heavy computational demands. This should no longer withhold practitioners in our technological era, for both symbolic as well as large-scale numerical manipulations can be readily performed on state-of-the-art computers. Progress in enabling technologies, such as compressed sensing for fast imaging, also contributes to practical feasibility of dwMRI, but we believe that the major hurdle is still methodological.

2 Theory

2.1 The Finsler Function

The geometric paradigm for DTI hinges on Riemannian geometry, stipulating that the diffusion tensor is precisely the dual metric tensor $g^{ij}(x)$. It is limited to the extent that local anisotropy is captured by $\frac{1}{2}n(n-1)=6$ degrees of freedom in $n = 3$ spatial dimensions. For state-of-the-art dwMRI, in which local signal attenuations are recorded under a multitude of magnetic gradient directions, this is too restrictive. The potential power of the Riemann-Finsler paradigm lies in the fact that it removes this limitation altogether.

The pivot of Riemann-Finsler geometry is a generalised notion of length of a spatial curve C (*Hilbert's invariant integral* [36]):

$$\mathscr{L}(C) = \int_C F(x, dx) \,. \tag{1}$$

The Lagrangian function $F(x, dx)$ is known as the *Finsler function*. (An equivalent Hamiltonian formulation will be given later.) This function cannot be chosen arbitrarily. In order to interpret Eq. (1) properly, one has to require $F(x, dx) = F(x, \dot{x})dt$, so that the functional $\mathscr{L}(C)$ is well-defined and parameter invariant. More specifically, $F(x, \dot{x})$ is required to be smooth for $\dot{x} \neq 0$ and to satisfy the following properties[1]:

$$\begin{aligned} F(x, \lambda\dot{x}) &= |\lambda| F(x, \dot{x}) \qquad \text{for all } \lambda \in \mathbb{R}, & (2)\\ F(x, \dot{x}) &> 0 \qquad \text{if } \dot{x} \neq 0, & (3)\\ g_{ij}(x, \dot{x})\xi^i\xi^j &> 0 \qquad \text{if } \xi \neq 0, & (4) \end{aligned}$$

in which the *Riemann-Finsler metric tensor* is defined as

$$g_{ij}(x, \dot{x}) = \frac{1}{2} \frac{\partial^2 F^2(x, \dot{x})}{\partial \dot{x}^i \partial \dot{x}^j} \,. \tag{5}$$

In these definitions and below, $\dot{x}$ should be regarded as an a priori independent vectorial argument, not necessarily a tangent vector to an underlying parametrized curve, unless stated otherwise. (If it does denote such a tangent vector, then $\dot{x}$ is shorthand for dx/dt.)

Using Eqs. (2–5), it is not difficult to show that

$$F^2(x, \dot{x}) = g_{ij}(x, \dot{x})\dot{x}^i\dot{x}^j \,. \tag{6}$$

Riemann's "quadratic restriction" pertains to the "mildly anisotropic" case $g_{ij}(x, \dot{x}) = g_{ij}(x)$, or

$$F^2(x, \dot{x}) = g_{ij}(x)\dot{x}^i\dot{x}^j \,. \tag{7}$$

In general, the Riemann-Finsler metric tensor, Eq. (5), is homogeneous of degree zero: $g_{ij}(x, \lambda\dot{x}) = g_{ij}(x, \dot{x})$. Zero-homogeneous functions may be viewed as being defined on the *projectivized tangent bundle*, which may in turn be interpreted as a $(2n-1)$-dimensional base manifold of positions and orientations.

Since, in principle, only positions and orientations are of interest, all geometrically relevant quantities will be zero-homogeneous. Although the Finsler function itself does not qualify as such (its domain of definition is the $2n$-dimensional *slit tangent bundle* of positions and nonzero vectors), it serves as the basic generator for such quantities.

The role played by the n-dimensional (co)tangent spaces erected at each point x of an n-dimensional Riemannian manifold is replaced by likewise n-dimensional fibers that collectively constitute a so-called *pulled-back (co)bundle* or *Finsler (co)bundle* in Riemann-Finsler geometry. The major difference is that a pulled-back

[1] Instead of the norm condition, Eq. (2), some accounts require $F(x, \lambda\dot{x}) = \lambda F(x, \dot{x})$ instead.

(co)bundle sits over the $(2n-1)$-dimensional projectivized tangent bundle or $2n$-dimensional slit tangent bundle, rather than over the n-dimensional spatial manifold. Given x-coordinates on the spatial manifold the basis sections

$$\left.\frac{\partial}{\partial x^i}\right|_{(x,\dot{x})} \quad \text{respectively} \quad \left. dx^i \right|_{(x,\dot{x})} \tag{8}$$

for its tangent and cotangent bundles can be transplanted to the pulled-back (co)bundle. That is, $\dot{x}$ plays no role in the construction of a fiber at a fiducial point $(x, \dot{x})$.

2.2 Riemann-Finsler Geometry versus Riemannian Geometry

The nontrivial nature of the *Cartan tensor* [36, 35, 37, 38],

$$C_{ijk}(x,\dot{x}) = \frac{1}{4}\frac{\partial^3 F^2(x,\dot{x})}{\partial \dot{x}^i \partial \dot{x}^j \partial \dot{x}^k}\,, \tag{9}$$

distinguishes Riemann-Finsler geometry from its Riemannian counterpart. One can show that $C_{ijk}(x,\dot{x}) = 0$ if and only if space (the x-manifold) is Riemannian. In fact it suffices to inspect the *Cartan one-form*

$$C_i(x,\dot{x}) = g^{jk}(x,\dot{x}) C_{ijk}(x,\dot{x})\,, \tag{10}$$

in which $g^{ij}(x,\dot{x})$ denotes the inverse of $g_{ij}(x,\dot{x})$, a.k.a. the *dual Riemann-Finsler metric tensor*, i.e.

$$g^{ik}(x,\dot{x}) g_{kj}(x,\dot{x}) = \delta^i_j\,, \tag{11}$$

Indeed, space is Riemannian if and only if the Cartan one-form vanishes identically. In view of the remark made earlier on the significance of zero-homogeneous functions one often encounters the alternative definitions

$$A_{ijk}(x,\dot{x}) = F(x,\dot{x}) C_{ijk}(x,\dot{x}) \quad \text{respectively} \quad A_i(x,\dot{x}) = F(x,\dot{x}) C_i(x,\dot{x})\,. \tag{12}$$

The dual Riemann-Finsler metric may be used for index raising and lowering, e.g.

$$C^k_{ij}(x,\dot{x}) = g^{k\ell}(x,\dot{x}) C_{ij\ell}(x,\dot{x})\,. \tag{13}$$

(There is no ambiguity here by virtue of symmetry of the covariant Cartan tensor.)

Thus the Cartan one-form measures the degree in which the local structure of the Riemann-Finsler manifold deviates from Riemannian. Alternatively it tells us something about local volume distortion, viz. if B^n is the Euclidean unit ball, with volume

$$\text{vol}\, B^n = \frac{\sqrt{\pi}^{\,n}}{\Gamma(\frac{n}{2}+1)}\,, \tag{14}$$

in which Γ denotes the Gamma function (i.e. $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$), and $\mathscr{B}^n_x$ is the unit ball on the fiber at point x, i.e.

$$\mathscr{B}^n_x = \{\dot{x} \in \text{TM}_x \mid F(x,\dot{x}) < 1\}\,, \tag{15}$$

then the Cartan one-form determines the directional dependence of the *distortion*

$$\tau(x,\dot{x}) = \ln \frac{\sqrt{\det g_{ij}(x,\dot{x})}}{\sigma_F(x)}\,, \tag{16}$$

in which

$$\sigma_F(x) = \frac{\text{vol}\, B^n}{\text{vol}\, \mathscr{B}^n_x}\,, \tag{17}$$

viz.

$$C_i(x,\dot{x}) = \frac{\partial \tau(x,\dot{x})}{\partial \dot{x}^i}\,. \tag{18}$$

Cf. [39, 40] for details. In the Riemannian limit $\text{vol}\, B^n$ is proportional to $\text{vol}\, \mathscr{B}^n_x$ by a relative factor given by the square root of the product of eigenvalues of the Riemannian metric tensor $g_{ij}(x,\dot{x}) = g_{ij}(x)$, yielding $\sigma_F(x) = \sqrt{\det g_{ij}(x)}$ and $\tau(x,\dot{x}) = 0$.

2.3 Connections in Riemann-Finsler Geometry

There is no "obvious" connection (mechanism for parallel transport) on a Riemann-Finsler manifold. The Berwald, Cartan, Chern-Rund and Hashiguchi connection may all be considered "natural" extensions of the Levi-Civita connection in Riemannian geometry. For instance, the (torsion-free) Chern-Rund connection is defined by[2]

$$\Gamma^i_{jk}(x,\dot{x}) = \frac{1}{2} g^{i\ell}(x,\dot{x}) \left(\frac{\delta g_{\ell k}(x,\dot{x})}{\delta x^j} + \frac{\delta g_{j\ell}(x,\dot{x})}{\delta x^k} - \frac{\delta g_{jk}(x,\dot{x})}{\delta x^\ell} \right) . \tag{19}$$

This expression is obtained from the "classical" Christoffel symbols of Riemannian geometry by formally replacing the Riemannian metric $g_{ij}(x)$ by the Riemann-Finsler metric $g_{ij}(x,\dot{x})$, Eq. (5), and spatial derivatives by *horizontal vectors*

$$\frac{\delta}{\delta x^i} \stackrel{\text{def}}{=} \frac{\partial}{\partial x^i} - N^j_i(x,\dot{x}) \frac{\partial}{\partial \dot{x}^j} . \tag{20}$$

The coefficients $N^j_i(x,\dot{x})$ define the so-called *nonlinear connection* [36]:

$$N^j_i(x,\dot{x}) = \gamma^j_{ik}(x,\dot{x})\dot{x}^k - C^j_{ik}(x,\dot{x})\gamma^k_{\ell m}(x,\dot{x})\dot{x}^\ell \dot{x}^m , \tag{21}$$

in which the *formal Christoffel symbols of the second kind* are introduced as

$$\gamma^i_{jk}(x,\dot{x}) = \frac{1}{2} g^{i\ell}(x,\dot{x}) \left(\frac{\partial g_{\ell k}(x,\dot{x})}{\partial x^j} + \frac{\partial g_{j\ell}(x,\dot{x})}{\partial x^k} - \frac{\partial g_{jk}(x,\dot{x})}{\partial x^\ell} \right) . \tag{22}$$

Note that in the Riemannian limit, both Eq. (19) as well as Eq. (22) simplify to

$$\Gamma^i_{jk}(x) = \frac{1}{2} g^{i\ell}(x) \left(\frac{\partial g_{\ell k}(x)}{\partial x^j} + \frac{\partial g_{j\ell}(x)}{\partial x^k} - \frac{\partial g_{jk}(x)}{\partial x^\ell} \right) . \tag{23}$$

These are the standard *Christoffel symbols of the second kind* defining the (torsion-free) *Levi-Civita connection* in Riemannian geometry. A computation reveals that[3]

$$\begin{aligned} &\Gamma_{ijk}(x,\dot{x}) = \\ &\quad \gamma_{ijk}(x,\dot{x}) - C_{hjk}(x,\dot{x})G^h_{\dot{x}^i}(x,\dot{x}) - C_{hji}(x,\dot{x})G^h_{\dot{x}^j}(x,\dot{x}) + C_{hik}(x,\dot{x})G^h_{\dot{x}^j}(x,\dot{x}) , \end{aligned} \tag{24}$$

in which indices have been lowered with the help of the Riemann-Finsler metric tensor:

$$\Gamma_{ijk}(x,\dot{x}) = g_{j\ell}(x,\dot{x})\Gamma^\ell_{ik}(x,\dot{x}) \quad \text{resp.} \quad \gamma_{ijk}(x,\dot{x}) = g_{j\ell}(x,\dot{x})\gamma^\ell_{ik}(x,\dot{x}) , \tag{25}$$

and in which the *geodesic coefficients* are defined as[4]

$$G^i_{\dot{x}^j}(x,\dot{x}) = \frac{\partial G^i(x,\dot{x})}{\partial \dot{x}^j} \quad \text{with} \quad G^i(x,\dot{x}) = \frac{1}{2}\gamma^i_{jk}(x,\dot{x})\dot{x}^j\dot{x}^k . \tag{26}$$

In fact we have

$$G^i_{\dot{x}^j}(x,\dot{x}) = N^i_j(x,\dot{x}) \tag{27}$$

recall Eq. (21).

2.4 Horizontal-Vertical Splitting

The heuristic coupling of position and orientation is formalized in terms of the so-called *horizontal* and *vertical basis vectors*, recall Eq. (20),

$$\frac{\delta}{\delta x^i} \stackrel{\text{def}}{=} \frac{\partial}{\partial x^i} - N^\ell_i(x,\dot{x}) \frac{\partial}{\partial \dot{x}^\ell} \quad \text{and} \quad \frac{\partial}{\partial \dot{x}^i} . \tag{28}$$

[2]Caveat: In [37] Rund defines these symbols as $\Gamma^{*i}_{jk}(x,\dot{x})$.
[3]Caveat: In [37] Rund defines these symbols as $\Gamma^*_{ijk}(x,\dot{x})$.
[4]Caveat: In [36] Bao et al. write $G^i(x,\dot{x}) = \gamma^i_{jk}(x,\dot{x})\dot{x}^j\dot{x}^k$.

These constitute a basis for the horizontal and vertical tangent bundles over the slit tangent bundle:

$$\mathscr{H}_{(x,\dot{x})}\mathrm{TM} = \text{span}\left\{ \left.\frac{\delta}{\delta x^i}\right|_{(x,\dot{x})} \right\} \quad \text{and} \quad \mathscr{V}_{(x,\dot{x})}\mathrm{TM} = \text{span}\left\{ \left.\frac{\partial}{\partial \dot{x}^i}\right|_{(x,\dot{x})} \right\} . \tag{29}$$

Their direct sum yields the complete tangent bundle

$$\mathrm{TTM}\backslash\{0\} = \mathscr{H}\mathrm{TM} \oplus \mathscr{V}\mathrm{TM} \tag{30}$$

pointwise. By the same token one considers the *horizontal* and *vertical basis covectors*,

$$dx^i \quad \text{and} \quad \delta\dot{x}^i \stackrel{\text{def}}{=} d\dot{x}^i + N^i_\ell(x,\dot{x})dx^\ell , \tag{31}$$

yielding the corresponding horizontal and vertical cotangent bundles:

$$\mathscr{H}^*_{(x,\dot{x})}\mathrm{TM} = \text{span}\left\{ \left. dx^i\right|_{(x,\dot{x})} \right\} \quad \text{and} \quad \mathscr{V}^*_{(x,\dot{x})}\mathrm{TM} = \text{span}\left\{ \left. \delta\dot{x}^i\right|_{(x,\dot{x})} \right\} , \tag{32}$$

such that

$$\mathrm{T^*TM}\backslash\{0\} = \mathscr{H}^*\mathrm{TM} \oplus \mathscr{V}^*\mathrm{TM} \tag{33}$$

pointwise.

The above vectors and covectors satisfy the following duality relations:

$$dx^i\left(\frac{\delta}{\delta x^j}\right) = \delta\dot{x}^i\left(\frac{\partial}{\partial \dot{x}^j}\right) = \delta^i_j \quad \text{and} \quad dx^i\left(\frac{\partial}{\partial \dot{x}^j}\right) = \delta\dot{x}^i\left(\frac{\delta}{\delta x^j}\right) = 0 . \tag{34}$$

Incorporating a natural scaling so as to ensure zero-homogeneity with respect to $\dot{x}$ (so that it indeed represents orientation rather than "velocity") we conclude that

$$\mathrm{TTM}\backslash\{0\} = \text{span}\left\{ \frac{\delta}{\delta x^i} \, , \; F(x,\dot{x})\frac{\partial}{\partial \dot{x}^i} \right\} , \tag{35}$$

and similarly

$$\mathrm{T^*TM}\backslash\{0\} = \text{span}\left\{ dx^i \, , \; \frac{\delta\dot{x}^i}{F(x,\dot{x})} \right\} . \tag{36}$$

The so-called *Sasaki metric* furnishes the slit tangent bundle with a natural Riemannian metric:

$$g(x,\dot{x}) = g_{ij}(x,\dot{x})\, dx^i \otimes dx^j + g_{ij}(x,\dot{x})\, \frac{\delta\dot{x}^i}{F(x,\dot{x})} \otimes \frac{\delta\dot{x}^j}{F(x,\dot{x})} . \tag{37}$$

The horizontal and vertical tangent bundles, Eq. (29), are orthogonal relative to this metric.

Cf. Appendix A for further motivation.

2.5 Horizontal Curves and Geodesics

The notion of horizontality expresses the fact that spatial trajectories have a "natural" manifestation in the orientation subdomain, viz. through identification of the trajectory's tangent vector (or rather equivalence class of local tangent vectors in the zero-homogeneous case) with the vector $\dot{x}$ (whence its suggestive notation). In other words, interpreted as a curve along the Finsler manifold a parametrized spatial curve, $x = \xi(t)$, say, has a natural parametrization $(x,\dot{x}) = (\xi(t),\dot{\xi}(t))$. A tangent vector of this curve is given by

$$\boldsymbol{T}(t) = \dot{\xi}^i(t)\frac{\partial}{\partial x^i} + \ddot{\xi}^i(t)\frac{\partial}{\partial \dot{x}^i} . \tag{38}$$

The aforementioned splitting suggests that, using Eq. (28), we rather decompose this vector as follows:

$$\boldsymbol{T}(t)=\dot{\xi}^i(t)\frac{\delta}{\delta x^i}+\left(\ddot{\xi}^i(t)+N^i_j(\xi(t),\dot{\xi}(t))\dot{\xi}^j(t)\right)\frac{\partial}{\partial \dot{x}^i}\,. \tag{39}$$

The requirement of horizontality then entails that the vertical component vanishes in the sense that

$$\delta\dot{x}^i\left(\boldsymbol{T}(t)\right)=0\,. \tag{40}$$

Using the basic duality relations, Eqs. (34), this means that

$$\ddot{\xi}^i(t)+N^i_j(\xi(t),\dot{\xi}(t))\dot{\xi}^j(t)=0\,. \tag{41}$$

By virtue of Eq. (21), using the fact that $C^j_{ik}(\xi,\dot{\xi})\dot{\xi}^k=0$ (a trivial consequence of the homogeneity property of the Cartan tensor with respect to $\dot{x}$), this can be simplified to

$$\ddot{\xi}^i(t)+\gamma^i_{jk}(\xi(t),\dot{\xi}(t))\dot{\xi}^j(t)\dot{\xi}^k(t)=0\,. \tag{42}$$

This *geodesic equation* has the same form as in the Riemannian case, except for the fact that Christoffel symbols have been replaced by their Finslerian counterparts, Eq. (22).

2.6 Lagrangian versus Hamiltonian Frameworks

The non-singular Riemann-Finsler metric enables the same kind of index gymnastics in Riemann-Finsler geometry as the Riemann metric does in Riemannian geometry. In particular we have the "velocity"–"momentum" (i.e. $\dot{x}$–y) duality expressed by the equations

$$y_i=g_{ij}(x,\dot{x})\dot{x}^j\quad\text{and}\quad\dot{x}^i=g^{ij}(x,y)y_j\,, \tag{43}$$

in which the *dual Riemann-Finsler metric* has now been redefined such that

$$g^{ik}(x,y)g_{kj}(x,\dot{x})=\delta^i_j\,, \tag{44}$$

assuming the aforementioned relationship between $\dot{x}$ and y, recall Eq. (11). Note that, unlike before, the dual metric tensor has been expressed as a function of momentum, not velocity.

The foregoing formulation of the theory, in which geometric quantities are considered as functions of position and velocity, is known as the *Lagrangian framework*. The alternative formulation, in which the independent variables are position and momentum, is known as the *Hamiltonian framework*. The connection between the Lagrangian and corresponding Hamiltonian frameworks is particularly elegant in Riemann-Finsler geometry, in which the *Hamiltonian function* (or *dual Finsler function*) is given by

$$H(x,y)=F(x,\dot{x})\,, \tag{45}$$

again assuming Eq. (43) to hold. As a consequence, the contravariant or dual Riemann-Finsler metric tensor plays a similar role in the Hamiltonian framework as the covariant Riemann-Finsler metric tensor does in the Lagrangian framework.

The physical interpretations of the two formulations depend on context and typically differ. The Lagrangian formalism emphasizes the role of *geodesic congruences*, i.e. families of geodesics viewed as "particle trajectories", for which the variable $\dot{x}$ serves as particle velocity. In the Hamiltonian formalism one considers the "wave phenomena" induced by such geodesic congruences, in which case one tends to think of the variable y as wave momentum, which, by definition, is the covector along which the wave fronts propagate. Recall that in anisotropic media wave fronts induced by the interference of the disturbances caused by individual particles do not travel in the same direction as the particles themselves. This is expressed by Eq. (43), as the (dual) metric is not necessarily diagonal.

2.7 Relation to Diffusion Weighted MRI

Because of its physical interpretation the Hamiltonian framework appears to be most naturally related to the dwMRI acquisition protocol. This is clear in the Riemannian limit of DTI, since it is the Hamiltonian that governs signal attenuation via the Bloch-Torrey equations under the Stejskal-Tanner assumption [41, 42, 43, 44, 45]:

$$S(x,y) = S(x,0)\exp\left(-\tau H^2(x,y)\right) . \tag{46}$$

Here τ denotes a time constant related to the time Δ between a pair of balanced diffusion-sensitizing gradients G_i and pulse duration δ (according to Stejskal-Tanner's scheme [44] we have, $\tau = \Delta - \delta/3$), and

$$H^2(x,y) = D^{ij}(x) y_i y_j , \tag{47}$$

with covector y_i denoting the normalized diffusion-sensitizing gradient,

$$y_i = \gamma \delta G_i , \tag{48}$$

in which γ is the gyromagnetic ratio of hydrogen. Indeed, the Riemann geometric rationale of DTI stipulates that $D^{ij}(x)$ is independent of y—an assertion that is inconsistent in significant parts of the brain—and coincides up to a constant proportionality factor with the dual metric tensor $g^{ij}(x)$. If we adopt Eq. (46) *without* the quadratic restriction, Eq. (47), we can invoke the powerful machinery of Riemann-Finsler geometry instead, recall Eq. (6) and Eq. (45):

$$H^2(x,y) = g^{ij}(x,y) y_i y_j , \tag{49}$$

with $\tau D^{ij}(x,y) = g^{ij}(x,y)$. Clearly this is still at best an approximation of reality due to the mono-exponential decay stipulated by the Stejskal-Tanner equation, Eq. (46), and the axiomatic homogeneity constraint on the Finsler function and thus on the physical scaling behaviour of the Hamiltonian function, recall Eq. (2):

$$H(x, \lambda y) = |\lambda| H(x,y) \qquad \text{for all } \lambda \in \mathbb{R}. \tag{50}$$

This boils down to the physical constraint that the apparent diffusion coefficient scales quadratically with the diffusion-sensitizing gradient, which is not invariably true. The conditions under which this is a reasonable approximation will need to be made explicit.

2.8 Indicatrix and Figuratrix

The *indicatrix* at fixed point x is the level set, or "glyph", of the Riemann-Finsler unit sphere, $F(x,\dot{x}) = 1$, or, by virtue of Eq. (6),

$$g_{ij}(x,\dot{x})\dot{x}^i \dot{x}^j = 1 . \tag{51}$$

The *figuratrix* at fixed point x is the Hamiltonian counterpart, i.e. the level set defined by $H(x,y) = 1$, recall Eq. (49):

$$g^{ij}(x,y) y_i y_j = 1 . \tag{52}$$

One can show that, as a result of zero-homogeneity of the Riemann-Finsler (dual) metric tensor, both indicatrix as well as figuratrix represent *convex glyphs*.

A convenient interpretation of these structures is obtained by "freezing" the (co)vector argument of the Riemann-Finsler (dual) metric tensor in Eqs. (51–52), so that one ends up with quadratic forms. These are known as the *osculating indicatrix* and *osculating figuratrix*, respectively:

$$\begin{aligned} g_{ij}(x,\dot{x}_0)\dot{x}^i \dot{x}^j &= 1 , && (53) \\ g^{ij}(x,y_0) y_i y_j &= 1 . && (54) \end{aligned}$$

One could think of these as gauge figures of a parametrized family of inner products on the tangent, respectively cotangent space of the spatial domain, each direction (specified by $\dot{x}_0$ or y_0) having its own unique instance. Of course, in the DTI/Riemannian case the coefficients are orientation independent, so that each point in space has a single and unambiguously defined ellipsoidal shape representing the entire family. In general it is clear that the Cartan tensor, Eq. (9), must play a pivotal role in geometric tractography methods extending the DTI rationale, as it relates the individual members of these families.

2.9 Covariant Derivatives

The horizontal and vertical one-forms given by Eq. (36) can be used as a basis for decomposing the *covariant differential* of an arbitrary tensor field on the slit tangent bundle. For simplicity consider

$$\boldsymbol{T}(x,\dot{x}) = T^i_j(x,\dot{x}) \frac{\partial}{\partial x^j} \otimes dx^i \,, \tag{55}$$

and

$$\nabla\boldsymbol{T}(x,\dot{x}) = \left(\nabla\boldsymbol{T}\right)^i_j(x,\dot{x}) \frac{\partial}{\partial x^j} \otimes dx^i \,. \tag{56}$$

Then each component on the r.h.s. is a one-form, and can thus be written as a sum of horizontal and vertical one-forms. By definition,

$$\left(\nabla\boldsymbol{T}\right)^i_j(x,\dot{x}) = T^i_{j|k}(x,\dot{x})\, dx^k + T^i_{j;k}(x,\dot{x})\, \frac{\delta \dot{x}^k}{F(x,\dot{x})} \,. \tag{57}$$

By evaluation on the corresponding dual basis, Eq. (35), one obtains the components

$$T^i_{j|k}(x,\dot{x}) = \frac{\delta T^i_j(x,\dot{x})}{\delta x^k} + T^\ell_j(x,\dot{x})\Gamma^i_{\ell k}(x,\dot{x}) - T^i_\ell(x,\dot{x})\Gamma^\ell_{jk}(x,\dot{x}) \,, \tag{58}$$

$$T^i_{j;k}(x,\dot{x}) = F(x,\dot{x})\, \frac{\partial T^i_j(x,\dot{x})}{\partial \dot{x}^k} \,. \tag{59}$$

Eqs. (58-59) are the components of the *horizontal covariant derivative* and the *vertical covariant derivative* of the tensor field, respectively. Higher order tensors are treated similarly. Their horizontal covariant derivatives will contain as many "correction terms" involving the Riemann-Finsler Γ-symbols of Eq. (19) as indicated by their order. Note the elegant similarity with the Riemannian case.

Some cases are particularly important, e.g. those involving the Riemann-Finsler metric tensor or its dual. We have

$$g_{ij|k}(x,\dot{x}) = 0 \,, \tag{60}$$

$$g_{ij;k}(x,\dot{x}) = 2F(x,\dot{x})\, C_{ijk}(x,\dot{x}) \,, \tag{61}$$

$$g^{ij}{}_{|k}(x,\dot{x}) = 0 \,, \tag{62}$$

$$g^{ij}{}_{;k}(x,\dot{x}) = -2F(x,\dot{x})\, C^{ij}_k(x,\dot{x}) \,. \tag{63}$$

The Kronecker tensor is covariantly constant both horizontally as well as vertically:

$$\delta^i_{j|k} = 0 \,, \tag{64}$$

$$\delta^i_{j;k} = 0 \,. \tag{65}$$

Thus, unlike in the Riemannian case, the Riemann-Finsler metric tensor is covariantly constant only along horizontal directions, whereas its behaviour in vertical direction is governed by the Cartan tensor (the covariant derivative is said to be "almost metric compatible").

2.10 The Unit Vector ℓ and the Hilbert Form ω

Given any base point $(x, \dot{x})$ on the slit tangent bundle, there exists a distinguished direction on the pulled back bundle, viz. the one pointed out by the vector $\dot{x}$. The Finsler function allows us to normalize it to unit Finslerian length:

$$\boldsymbol{\ell}(x, \dot{x}) = \ell^i(x, \dot{x}) \frac{\partial}{\partial x^i} , \tag{66}$$

with

$$\ell^i(x, \dot{x}) = \frac{\dot{x}^i}{F(x, \dot{x})} . \tag{67}$$

Its dual is the so-called *Hilbert form*:

$$\boldsymbol{\omega}(x, \dot{x}) = \ell_i(x, \dot{x}) dx^i , \tag{68}$$

with

$$\ell_i(x, \dot{x}) = g_{ij}(x, \dot{x}) \ell^j(x, \dot{x}) . \tag{69}$$

By construction we have

$$\boldsymbol{\omega}(\boldsymbol{\ell}) = 1 . \tag{70}$$

With the definitions of the previous section it readily follows that $\boldsymbol{\ell}$ and $\boldsymbol{\omega}$ are covariantly constant along horizontal directions, but not along vertical directions:

$$\ell^i_{|k}(x, \dot{x}) = 0 , \tag{71}$$
$$\ell^i_{;k}(x, \dot{x}) = \delta^i_k - \ell^i(x, \dot{x}) \ell_k(x, \dot{x}) , \tag{72}$$
$$\ell_{i|k}(x, \dot{x}) = 0 , \tag{73}$$
$$\ell_{i;k}(x, \dot{x}) = g_{ik}(x, \dot{x}) - \ell_i(x, \dot{x}) \ell_k(x, \dot{x}) . \tag{74}$$

The distinguished pair $(\boldsymbol{\ell}(x, \dot{x}), \boldsymbol{\omega}(x, \dot{x}))$ suggests that it may be beneficial to adjust the basis for the pulled back bundle so as to (i) make it orthonormal relative to the Riemann-Finsler metric, and (ii) align one of the basis vectors with the a priori preferred direction. Likewise for the dual bundle. This can be achieved through suitable choice of so-called n-*beins* $u^i_a(x, \dot{x})$ and $v^a_i(x, \dot{x})$ $(i, a = 1, \ldots, n)$:

$$\boldsymbol{e}_a(x, \dot{x}) = u^i_a(x, \dot{x}) \frac{\partial}{\partial x^i} , \tag{75}$$
$$\boldsymbol{\omega}^a(x, \dot{x}) = v^a_i(x, \dot{x}) dx^i , \tag{76}$$

with inverse

$$\frac{\partial}{\partial x^i} = v^a_i(x, \dot{x}) \boldsymbol{e}_a(x, \dot{x}) , \tag{77}$$
$$dx^i = u^i_a(x, \dot{x}) \boldsymbol{\omega}^a(x, \dot{x}) , \tag{78}$$

such that

$$\boldsymbol{e}_n(x, \dot{x}) \stackrel{\text{def}}{=} \boldsymbol{\ell}(x, \dot{x}) \quad \text{and} \quad \boldsymbol{\omega}^n(x, \dot{x}) \stackrel{\text{def}}{=} \boldsymbol{\omega}(x, \dot{x}) , \tag{79}$$

recall Eqs. (66–70). These n-beins simultaneously induce ($2n$-dimensional) Sasaki-orthonormal bases for the tangent and cotangent bundle of the slit tangent bundle, viz. (with some ambiguity in the name of keeping notation simple)

$$\boldsymbol{e}_a(x, \dot{x}) = u^i_a(x, \dot{x}) \frac{\delta}{\delta x^i} , \tag{80}$$
$$\boldsymbol{e}_{n+a}(x, \dot{x}) = u^i_a(x, \dot{x}) F(x, \dot{x}) \frac{\partial}{\partial \dot{x}^i} , \tag{81}$$
$$\boldsymbol{\omega}^a(x, \dot{x}) = v^a_i(x, \dot{x}) dx^i , \tag{82}$$
$$\boldsymbol{\omega}^{n+a}(x, \dot{x}) = v^a_i(x, \dot{x}) \frac{\delta \dot{x}^i}{F(x, \dot{x})} . \tag{83}$$

with inverse

$$
\begin{aligned}
\frac{\delta}{\delta x^i} &= v_i^a(x,\dot{x})\boldsymbol{e}_a(x,\dot{x})\,, & (84)\\
\frac{\partial}{\partial \dot{x}^i} &= v_i^a(x,\dot{x})\boldsymbol{e}_{n+a}(x,\dot{x})\,, & (85)\\
dx^i &= u_a^i(x,\dot{x})\boldsymbol{\omega}^a(x,\dot{x})\,, & (86)\\
\frac{\delta \dot{x}^i}{F(x,\dot{x})} &= u_a^i(x,\dot{x})\boldsymbol{\omega}^{n+a}(x,\dot{x})\,. & (87)
\end{aligned}
$$

Note that, by construction,

$$
\begin{aligned}
u_a^i(x,\dot{x})g_{ij}(x,\dot{x})u_b^j(x,\dot{x}) &= \eta_{ab}\,, & (88)\\
v_i^a(x,\dot{x})g^{ij}(x,\dot{x})v_j^b(x,\dot{x}) &= \eta^{ab}\,, & (89)
\end{aligned}
$$

in which η_{ab} and η^{ab} are the components of the identity matrix (1 if $a=b$, 0 otherwise).

3 Conclusion and Discussion

Riemann-Finsler geometry naturally extends the Riemannian rationale used in the context of DTI to general dwMRI representations, such as HARDI. It can be equivalently approached from a Lagrangian or Hamiltonian perspective, although the latter appears to be most closely related to the physics of dwMRI acquisition and its underlying model in terms of an appropriately generalized mono-exponential Stejskal-Tanner equation. We have illustrated its potential application by deriving the corresponding geodesic equations for dwMRI tractography from the apparent diffusion coefficient, obtained as a function of position and orientation, without the quadratic restriction inherent to the DTI model, yet retaining quadratic scaling in the *magnitude* of the gradient magnetic field. Although this does not cover the general (multi-exponential) case, the conditions for and limitations of this conjecture, and in particular the added value relative to DTI, remain to be investigated.

Despite intriguing heuristic analogies it remains an open problem how to exactly relate the present Riemann-Finsler framework to the Cartan geometric approach by Duits et al. [27, 28, 29]. A suitable construction of the n-beins of Section 2.10 might provide the necessary insight.

A Horizontal and Vertical Vector and Covector Transformation

We may consider the partial derivatives with respect to x^i and $\dot{x}^i$ as coordinate vector fields on the tangent bundle TM , and consider the effect of a coordinate transformation induced by a change of coordinates on the base manifold M, $x = x(\xi)$ say. Since $\dot{x}$ is a vector, this induces the following *vector transformation law* for its components $\dot{x}^i$:

$$\dot{x}^i = \frac{\partial x^i}{\partial \xi^p}\dot{\xi}^p\,, \tag{90}$$

or, equivalently,

$$\frac{\partial}{\partial \dot{\xi}^p} = \frac{\partial x^i}{\partial \xi^p}\frac{\partial}{\partial \dot{x}^i}\,, \tag{91}$$

so that, by construction,

$$\dot{x}^i\frac{\partial}{\partial x^i} = \dot{\xi}^p\frac{\partial}{\partial \xi^p}\,. \tag{92}$$

As a result,

$$\frac{\partial}{\partial \xi^p} = \frac{\partial x^i}{\partial \xi^p}\frac{\partial}{\partial x^i} + \frac{\partial^2 x^i}{\partial \xi^p \partial \xi^q}\dot{\xi}^q \frac{\partial}{\partial \dot{x}^i}\,. \tag{93}$$

Given the definition of the horizontal vectors, Eq. (20), and of the nonlinear connection, Eq. (21), it is then a tedious but straightforward exercise to deduce that

$$\frac{\delta}{\delta \xi^p} = \frac{\partial x^i}{\partial \xi^p}\frac{\delta}{\delta x^i}\,, \tag{94}$$

similar to the vector transformation law for the vertical components, recall Eq. (91).

Likewise one has the *covector transformation law* for the basic horizontal and vertical one-forms, recall Eq. (31):

$$dx^i = \frac{\partial x^i}{\partial \xi^p} d\xi^p\,, \tag{95}$$

respectively

$$\delta \dot{x}^i = \frac{\partial x^i}{\partial \xi^p} \delta \dot{\xi}^i\,. \tag{96}$$

It is the "natural" transformation behaviour expressed by Eqs. (91,94–96) that motivates the stated definitions of the basic horizontal and vertical vectors and covectors.

References

[1] Basser, P.J., Mattiello, J., Le Bihan, D.: Estimation of the effective self-diffusion tensor from the NMR spin echo. Journal of Magnetic Resonance **103** (1994) 247–254

[2] Basser, P.J., Mattiello, J., Le Bihan, D.: MR diffusion tensor spectroscopy and imaging. Biophysics Journal **66**(1) (1994) 259–267

[3] Arsigny, V., Fillard, P., Pennec, X., Ayache, N.: Log-Euclidean metrics for fast and simple calculus on diffusion tensors. Magnetic Resonance in Medicine **56**(2) (2006) 411–421

[4] Astola, L., Florack, L., ter Haar Romeny, B.: Measures for pathway analysis in brain white matter using diffusion tensor images. In Karssemeijer, N., Lelieveldt, B., eds.: Proceedings of the Twentieth International Conference on Information Processing in Medical Imaging–IPMI 2007 (Kerkrade, The Netherlands). Volume 4584 of Lecture Notes in Computer Science., Berlin, Springer-Verlag (2007) 642–649

[5] Astola, L., Florack, L.: Sticky vector fields and other geometric measures on diffusion tensor images. In: Proceedings of the 9th IEEE Computer Society Workshop on Mathematical Methods in Biomedical Image Analysis, held in conjuction with the IEEE Computer Society Conference on Computer Vision and Pattern Recognition (Anchorage, Alaska, USA, June 23–28, 2008), IEEE Computer Society Press

[6] Astola, L., Fuster, A., Florack, L.: A Riemannian scalar measure for diffusion tensor images. Pattern Recognition **44**(9) (2011) 1885–1891 doi:10.1016/j.patcog.2010.09.009.

[7] Deriche, R., Calder, J., Descoteaux, M.: Optimal real-time Q-ball imaging using regularized Kalman filtering with incremental orientation sets. Medical Image Analysis **13**(4) (August 2009) 564–579

[8] Fillard, P., Pennec, X., Arsigny, V., Ayache, N.: Clinical DT-MRI estimation, smoothing, and fiber tracking with log-Euclidean metrics. IEEE Transactions on Medical Imaging **26**(11) (November 2007)

[9] Florack, L.M.J., Astola, L.J.: A multi-resolution framework for diffusion tensor images. [46] Digital proceedings.

[10] Lenglet, C., Deriche, R., Faugeras, O.: Inferring white matter geometry from diffusion tensor MRI: Application to connectivity mapping. In Pajdla, T., Matas, J., eds.: Proceedings of the Eighth European Conference on Computer Vision (Prague, Czech Republic, May 2004). Volume 3021–3024 of Lecture Notes in Computer Science., Berlin, Springer-Verlag (May 2004) 127–140

[11] Lenglet, C., Rousson, M., Deriche, R., Faugeras, O.: Statistics on the manifold of multivariate normal distributions: Theory and application to diffusion tensor MRI processing. Journal of Mathematical Imaging and Vision **25**(3) (October 2006) 423–444

[12] Lenglet, C., Prados, E., Pons, J.P.: Brain connectivity mapping using Riemannian geometry, control theory and PDEs. SIAM Journal on Imaging Sciences **2**(2) (2009) 285–322

[13] Pennec, X., Fillard, P., Ayache, N.: A Riemannian framework for tensor computing. International Journal of Computer Vision **66**(1) (January 2006) 41–66

[14] Prados, E., Soatto, S., Lenglet, C., Pons, J.P., Wotawa, N., Deriche, R., Faugeras, O.: Control theory and fast marching techniques for brain connectivity mapping. In: Proceedings of the IEEE Computer Society Conference on Computer Vision and Pattern Recognition (New York, USA, June 2006). Volume 1., IEEE Computer Society Press (2006) 1076–1083

[15] Florack, L., Balmashnova, E.: Two canonical representations for regularized high angular resolution diffusion imaging. In Alexander, D., Gee, J., Whitaker, R., eds.: MICCAI Workshop on Computational Diffusion MRI (September 10 2008, New York, USA). (2008) 85–96

[16] Florack, L., Balmashnova, E., Astola, L., Brunenberg, E.: A new tensorial framework for single-shell high angular resolution diffusion imaging. Journal of Mathematical Imaging and Vision **3**(38) (August 13 2010) 171–181 Published online: DOI 10.1007/s10851-010-0217-3.

[17] Jensen, J.H., Helpern, J.A., Ramani, A., Lu, H., Kaczynski, K.: Diffusional kurtosis imaging: The quantification of non-Gaussian water diffusion by means of magnetic resonance imaging. Magnetic Resonance in Medicine **53**(6) (2005) 1432–1440

[18] Jian, B., Vemuri, B.C., Özarslan, E., Carney, P.R., Mareci, T.H.: A novel tensor distribution model for the diffusion-weighted MR signal. NeuroImage **37** (2007) 164–176

[19] Liu, C., Bammer, R., Acar, B., Moseley, M.E.: Characterizing non-gaussian diffusion by using generalized diffusion tensors. Magnetic Resonance in Medicine **51**(5) (2004) 924–937

[20] Özarslan, E., Mareci, T.H.: Generalized diffusion tensor imaging and analytical relationships between diffusion tensor imaging and high angular resolution imaging. Magnetic Resonance in Medicine **50**(5) (2003) 955–965

[21] Özarslan, E., Shepherd, T.M., Vemuri, B.C., Blackband, S.J., Mareci, T.H.: Resolution of complex tissue microarchitecture using the diffusion orientation transform (DOT). NeuroImage **31** (2006) 1086–1103

[22] Descoteaux, M., Angelino, E., Fitzgibbons, S., Deriche, R.: Apparent diffusion coefficients from high angular resolution diffusion imaging: Estimation and applications. Magnetic Resonance in Medicine **56**(2) (2006) 395–410

[23] Descoteaux, M., Angelino, E., Fitzgibbons, S., Deriche, R.: Regularized, fast, and robust analytical Q-ball imaging. Magnetic Resonance in Medicine **58**(3) (2007) 497–510

[24] Florack, L.M.J.: Codomain scale space and regularization for high angular resolution diffusion imaging. [46] Digital proceedings.

[25] Hess, C.P., Mukherjee, P., Tan, E.T., Xu, D., Vigneron, D.B.: Q-ball reconstruction of multimodal fiber orientations using the spherical harmonic basis. Magnetic Resonance in Medicine **56** (2006) 104–117

[26] Tuch, D.S.: Q-ball imaging. Magnetic Resonance in Medicine **52** (2004) 1358–1372

[27] Duits, R., Franken, E.M.: Left invariant parabolic evolution equations on $SE(2)$ and contour enhancement via invertible orientation scores, part I: Linear left-invariant diffusion equations on $SE(2)$. Quarterly of Applied Mathematics **68**(2) (June 2010) 255–292

[28] Duits, R., Franken, E.M.: Left invariant parabolic evolution equations on $SE(2)$ and contour enhancement via invertible orientation scores, part II: Nonlinear left-invariant diffusion equations on invertible orientation scores. Quarterly of Applied Mathematics **68**(2) (June 2010) 293–331

[29] Duits, R., Franken, E.: Left-invariant diffusions on the space of positions and orientations and their application to crossing-preserving smoothing of HARDI images. International Journal of Computer Vision **12**(3) (2011) 231–264

[30] Astola, L.J., Florack, L.M.J.: Finsler geometry on higher order tensor fields and applications to high angular resolution diffusion imaging. International Journal of Computer Vision **92**(3) (2011) 325–336

[31] Astola, L.J., Jalba, A.C., Balmashnova, E.G., Florack, L.M.J.: Finsler streamline tracking with single tensor orientation distribution function for high angular resolution diffusion imaging. Journal of Mathematical Imaging and Vision **41**(3) (2011) 170–181

[32] Melonakos, J., Pichon, E., Angenent, S., Tannenbaum, A.: Finsler active contours. IEEE Transactions on Pattern Analysis and Machine Intelligence **30**(3) (2008) 412–423

[33] Riemann, B.: Über die Hypothesen, welche der Geometrie zu Grunde liegen. In Weber, H., ed.: Gesammelte Mathematische Werke. Teubner, Leibzig (1892) 272–287

[34] Finsler, P.: Ueber Kurven und Flächen in allgemeinen Räumen. PhD thesis, University of Göttingen,, Göttingen, Germany (1918)

[35] Cartan, E.: Les Espaces de Finsler. Hermann, Paris (1934)

[36] Bao, D., Chern, S.S., Shen, Z.: An Introduction to Riemann-Finsler Geometry. Volume 2000 of Graduate Texts in Mathematics. Springer-Verlag, New York (2000)

[37] Rund, H.: The Differential Geometry of Finsler Spaces. Springer-Verlag, Berlin (1959)

[38] Rund, H.: The Hamilton-Jacobi Theory in the Calculus of Variations. Robert E. Krieger Publishing Company, Huntington, N.Y. (1973)

[39] Shen, Z.: Lectures on Finsler Geometry. World Scientific, Singapore (2001)

[40] Mo, X.: An Introduction to Finsler Geometry. Volume 1 of Peking University Series in Mathematics. World Scientific (2006)

[41] Bloch, F.: Nuclear induction. Physical Review **70** (1946) 460–473

[42] Haacke, E.M., Brown, R.W., Thompson, M.R., Venkatesan, R.: Magnetic Resonance Imaging: Physical Principles and Sequence Design. John Wiley & Sons, New York (1999)

[43] Stejskal, E.O.: Use of spin echoes in a pulsed magnetic-field gradient to study anisotropic, restricted diffusion and flow. Journal of Computational Physics **43**(10) (1965) 3597–3603

[44] Stejskal, E.O., Tanner, J.E.: Spin diffusion measurements: Spin echoes in the presence of a time-dependent field gradient. Journal of Computational Physics **42**(1) (1965) 288–292

[45] Torrey, H.C.: Bloch equations with diffusion terms. Physical Review D **104** (1956) 563–565

[46] Aja Fernández, S., de Luis Garcia, R., eds.: CVPR Workshop on Tensors in Image Processing and Computer Vision, Anchorage, Alaska, USA, June 24–26, 2008. In Aja Fernández, S., de Luis Garcia, R., eds.: CVPR Workshop on Tensors in Image Processing and Computer Vision, Anchorage, Alaska, USA, June 24–26, 2008, IEEE (2008) Digital proceedings.

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