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A note on BRST quantization of $SU(2)$ Yang-Mills mechanics

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Abstract

The quantization of $SU(2)$ Yang-Mills theory reduced to $0 + 1$ space-time dimensions is performed in the BRST framework. We show that in the unitary gauge $A_0 = 0$ the BRST procedure has difficulties which can be solved by introduction of additional singlet ghost variables. In the Lorenz gauge $\dot{A}_0 = 0$ one has additional unphysical degrees of freedom, but the BRST quantization is free of the problems in the unitary gauge.

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1 $SU(2)$ Yang-Mills mechanics

We consider $SU(2)$ Yang-Mills mechanics obtained by the reduction of $SU(2)$ Yang-Mills field theory in $(D + 1)$ -dimensional space-time to a finite-dimensional quantum system, by taking the dynamical variables to depend on the time co-ordinate t only. The lagrangean of such a theory is

$$L_{\text{YMQM}} = \frac{1}{2}(F_{0i}^a)^2 - \frac{1}{4}(F_{ij}^a)^2, \quad (1)$$

where $i, j = (1, \dots, D)$, and

$$F_{0i}^a = \dot{A}_i^a - g\epsilon^{abc} A_0^b A_i^c, \quad F_{ij}^a = -g\epsilon^{abc} A_i^b A_j^c. \quad (2)$$

Such a system has been widely studied in the context of non-perturbative aspects of (super) Yang-Mills theories [3, 4], and as a first step in the regularized dynamics of membrane theory [5]-[10].

The lagrangean is invariant under time-dependent gauge transformations with parameters $\Lambda^a(t)$, taking the infinitesimal form

$$\delta A_0^a = \dot{\Lambda}^a - g\epsilon^{abc} A_0^b \Lambda^c, \quad \delta A_i^a = -g\epsilon^{abc} A_i^b \Lambda^c. \quad (3)$$

This invariance allows us to impose a gauge condition leaving the physical dynamics unchanged. The simplest choice is

$$A_0^a = 0. \quad (4)$$

With this condition the effective lagrangean for the remaining D -dimensional vector potentials \mathbf{A}_a becomes¹

$$L_{eff} = \frac{1}{2} \dot{\mathbf{A}}_a^2 - V[\mathbf{A}], \quad (5)$$

with the potential

$$V[\mathbf{A}] = \frac{g^2}{4} \left(\mathbf{A}_a^2 \mathbf{A}_b^2 - (\mathbf{A}_a \cdot \mathbf{A}_b)^2 \right). \quad (6)$$

In addition, we have to impose a set of (first-class) constraints corresponding to the previous equations of motion for A_0^a :

$$G^a \equiv g\epsilon^{abc} A_i^b F_{0i}^c \simeq g\epsilon_{abc} \mathbf{A}_b \cdot \dot{\mathbf{A}}_c = 0 \quad (7)$$

¹We do not distinguish between upper and lower adjoint indices (a, b, c, \dots) for $SU(2)$.

Thus, the physical trajectories in configuration space in the gauge (4) are the solutions of the Euler-Lagrange equations derived from (5) subject to the additional constraints (7).

In addition to the pure Yang-Mills theory described by the action (1), one can also construct various supersymmetric extensions, based on the reduction of supersymmetric Yang-Mills field theory in $D = 1, 3, 5, 9$. The spectra of these theories are qualitatively different [3, 8, 5, 19, 20, 21], but for the problem addressed in this note those differences are not relevant.

To keep track of the constraints, especially in the context of the Yang-Mills quantum theory, we follow the BRST procedure². Thus we introduce anti-commuting ghost degrees of freedom (b^a, c^a) as well as commuting auxiliary scalars N^a in such a way, that the total gauge-fixed action becomes invariant under a set of special ghost-dependent gauge transformations, the rigid BRST invariance. The anti-commuting BRST differentials δ_Ω are defined before gauge fixing as follows

$$\begin{aligned}\delta_\Omega A_0^a &= (D_0 c)^a = \dot{c}^a - g\epsilon^{abc} A_0^b c^c, & \delta_\Omega A_i^a &= (D_i c)^a = -g\epsilon^{abc} A_i^b c^c, \\ \delta_\Omega c^a &= \frac{g}{2} \epsilon^{abc} c^b c^c, & & \\ \delta_\Omega b^a &= iN^a, & \delta_\Omega N^a &= 0.\end{aligned}\tag{8}$$

The gauge-invariance of the classical action (1) implies its invariance under the BRST transformations by construction. The BRST differential has the standard property that $\delta_\Omega^2 = 0$. The implementation of the BRST construction for the gauge $A_0 = 0$ is, to impose this gauge condition using the Nakanishi-Lautrup fields N^a as Lagrange multipliers, and complete the effective lagrangean so as to make it fully BRST invariant. For the case at hand this results in the effective lagrangean

$$L_{eff} = L_{YMQM} + N^a A_0^a + ib^a (\dot{c}^a - g\epsilon^{abc} A_0^b c^c).\tag{9}$$

We can use the gauge condition implied by the Nakanishi-Lautrup fields to eliminate A_0^a and N^a simultaneously; in a path-integral formulation, this implies integrating out a δ -functional $\delta(A_0)$. The result is

$$L_{eff} = \frac{1}{2} \dot{\mathbf{A}}_a^2 - \frac{1}{4} (F_{ij}^a)^2 + ib_a \dot{c}_a,\tag{10}$$

²For reviews, see [13] and [14].

Note that for $D = 3$ we can construct a magnetic field by $\frac{1}{2}\epsilon_{ijk}F_{jk}^a = B_i^a$, but this does not hold for a general D . The effective lagrangean (10) is invariant under the reduced form of the BRST variations (8) obtained by taking $A_0^a = 0$, and using the equation of motion for N^a :

$$\delta b^a = iN^a \simeq ig\epsilon^{abc} \left(A_i^b F_{0i}^c - ic^b b^c \right). \quad (11)$$

The BRST invariance of the effective lagrangean implies an anti-commuting conserved charge by Noether's theorem. The BRST charge takes the form

$$\Omega = c^a G^a - \frac{ig}{2} \epsilon^{abc} c^a c^b b^c. \quad (12)$$

The first-class constraints of the classical theory are summarized effectively by the statement that $\Omega = 0$; more precisely, in the phase-space formulation, all brackets of physical quantities with Ω must vanish: physical quantities must be BRST invariant; this is discussed in more detail in the next section.

2 Quantum theory

In the quantum theory the dynamical variables A_i^a and their conjugate momenta $P_i^a = \dot{A}_i^a$, as well as the Faddeev-Popov ghosts are operators satisfying (anti-)commutation relations

$$\left[A_i^a, P_j^b \right] = i \delta^{ab} \delta_{ij}, \quad \left[c^a, b^b \right]_+ = \delta^{ab} \quad (13)$$

The hamiltonian is given by

$$H_{eff} = \frac{1}{2} \mathbf{P}_a^2 + \frac{1}{4} F_{ij}^{a2}, \quad (14)$$

as for pure Yang-Mills theory, in fact. The hamiltonian determines the time-evolution of any quantity X constructed from the Yang-Mills or ghost operators by the Schrödinger equation

$$\dot{X} = i [H, X]. \quad (15)$$

Gauge transformations on $(\mathbf{A}_a, \mathbf{P}_a)$ are generated by the $SU(2)$ charges

$$G_a = g \epsilon_{abc} \mathbf{A}_b \cdot \mathbf{P}_c, \quad (16)$$

such that

$$\delta_a X = i [G_a, X], \quad \delta_a G_b = i [G_a, G_b] = -g \epsilon_{abc} G_c, \quad (17)$$

whilst more generally the BRST transformations are given by

$$\delta_\Omega X = i [\Omega, X]_\pm, \quad (18)$$

the sign depending on the fermionic parity of the quantity X : $+$ (anti-commutator) for fermionic X , and $-$ (commutator) for bosonic X . In particular, the commutation relation (17) for the gauge charges together with the ghost anti-commutator (13) implies the nilpotency of the BRST charge:

$$\Omega^2 = 0. \quad (19)$$

To complete the theory we have to define an inner product on the extended state space, such that zero-norm states decouple and physical states have positive norm. For this to happen, it is necessary that the BRST operator is self-adjoint w.r.t. this inner product. In the co-ordinate representation, with states being represented by wave functions $\Psi[\mathbf{A}, c]$, such an inner product is defined by the integral [15]

$$(\Phi, \Psi) = i \int dc^1 dc^2 dc^3 \int \prod_{i,a} dA_i^a \Phi^\dagger[\mathbf{A}, c] \Psi[\mathbf{A}, c]. \quad (20)$$

It is easily seen that with this definition the ghost operators (b^a, c^a) are self-adjoint themselves. It follows directly that, indeed,

$$(\Omega \Phi, \Psi) = (\Phi, \Omega \Psi). \quad (21)$$

3 Physical states

The physical states of Yang-Mills quantum mechanics are constructed by solving for the eigenstates and eigenvalues of the hamiltonian (14) subject to the constraint of BRST invariance.

One useful way to construct states is by the Fock-space approach [19], in which one starts with an oscillator basis for the dynamical degrees of freedom defined by

$$\mathbf{a}_a = \frac{1}{\sqrt{2}} (\mathbf{A}_a + i\mathbf{P}_a), \quad \mathbf{a}_a^\dagger = \frac{1}{\sqrt{2}} (\mathbf{A}_a - i\mathbf{P}_a). \quad (22)$$

These creation and annihilation operators satisfy the standard commutation relations

$$[\mathbf{a}_a, \mathbf{a}_b^\dagger] = \delta_{ab} 1_D, \quad (23)$$

where 1_D is the D -dimensional unit matrix. As implied by eq. (13) the ghost operators already behave like fermionic ladder operators. One is free to consider either c^a or b^a as creation operator; we choose c^a . Fock states are now constructed as polynomials in \mathbf{a}_a^\dagger and c^a acting on an empty state Ψ_0 defined by

$$\mathbf{a}_a \Psi_0 = b^a \Psi_0 = 0. \quad (24)$$

Such a construction differs from the standard (bosonic or fermionic) creation and annihilation operators in that c^a and b^a are self-adjoint w.r.t. the inner product (20) rather than adjoint to each other. A similar treatment of ghost ladder operators can be found in [2].

The hamiltonian can be represented as a matrix in a basis of Fock states. Subsequent diagonalization would give the spectrum of the theory³.

In the context of the co-ordinate representation this construction is realized by taking

$$\mathbf{a}_a = \frac{1}{\sqrt{2}} \left(\mathbf{A}_a + \frac{\partial}{\partial \mathbf{A}_a} \right), \quad b^a = \frac{\partial}{\partial c^a}, \quad (25)$$

and

$$\Psi_0 = N e^{-\frac{1}{2} \mathbf{A}_a \cdot \mathbf{A}_a}, \quad (26)$$

with N a normalization factor. In this representation the gauge generators are of the form

$$G_a = -ig \epsilon_{abc} \mathbf{a}_b^\dagger \cdot \mathbf{a}_c. \quad (27)$$

We analyze next the restrictions imposed by the BRST symmetry on states in order to be physical. In the BRST formalism physical states are identified with the cohomology classes of the nilpotent BRST charge Ω :

$$\mathcal{H}^{phys} \simeq \frac{\text{Ker } \Omega}{\text{Im } \Omega}. \quad (28)$$

This implies that physical states Ψ are BRST-invariant:

$$\Omega \Psi = 0, \quad (\Psi, \Psi) = 1, \quad (29)$$

³Note one can only construct a basis of finite dimension and therefore any results would be approximate (see [19]).

and state vectors differing by a BRST-exact state are identified:

$$\Psi \sim \Psi' = \Psi + \Omega\Lambda. \quad (30)$$

Therefore matrix elements of physical operators between physical states must be invariant under the BRST transformations (30):

$$(\Phi, X\Psi) = (\Phi, X\Psi'), \quad \text{if} \quad [\Omega, X]_{\pm} = 0. \quad (31)$$

These properties are guaranteed if BRST-exact states of the form $\Omega\Lambda$ decouple from the physical state space and have zero norm:

$$(\Psi, \Omega\Lambda) = (\Omega\Psi, \Lambda) = 0, \quad (\Omega\Lambda, \Omega\Lambda) = (\Lambda, \Omega^2\Lambda) = 0. \quad (32)$$

Observe that it is crucial for these results that the BRST charge is self-adjoint w.r.t. to the physical inner product.

To do any practical calculation one needs an explicit expression for the physical state vectors; this can be achieved by selecting one element from each equivalence class, using the nilpotent co-BRST operator

$$*\Omega = G^a b^a - \frac{ig}{2} \epsilon^{abc} c^a b^b b^c, \quad *\Omega^2 = 0. \quad (33)$$

Indeed, the co-BRST condition

$$*\Omega\Psi = 0, \quad (34)$$

acts as a gauge fixing condition for the BRST transformations (30), reducing the state space as required [15]. States satisfying both $\Omega\Psi = *\Omega\Psi = 0$ are called BRST harmonic. Physical states are defined as BRST harmonic states of finite norm. We build first Fock states which are BRST harmonic.

Define the (total) ghost number as the operator

$$N_g = c^a b^a \quad (35)$$

Splitting the Fock space in four sectors corresponding to the eigenvalues n_g of N_g : $0, \dots, 3$, we construct states in each ghost sector as follows,

$$\begin{aligned} \Psi^{(0)}[M] &= M[\mathbf{a}^\dagger] \Psi_0, \\ \Psi_a^{(1)}[M] &= c_a M[\mathbf{a}^\dagger] \Psi_0, \\ \tilde{\Psi}_a^{(2)}[M] &= \frac{1}{2} \epsilon_{abc} c^b c^c M[\mathbf{a}^\dagger] \Psi_0, \\ \tilde{\Psi}^{(3)}[M] &= \frac{i}{3!} \epsilon_{abc} c^a c^b c^c M[\mathbf{a}^\dagger] \Psi_0, \end{aligned}$$

Here $M[\mathbf{a}^\dagger]$ is some gauge-invariant polynomial in the operators \mathbf{a}^\dagger :

$$M[\mathbf{a}^\dagger] = \sum_n \mu_{a_1 \dots a_n} \mathbf{a}_{a_1}^\dagger \dots \mathbf{a}_{a_n}^\dagger, \quad (36)$$

and the coefficients $\mu_{a_1 \dots a_n}$ are invariant $SU(2)$ tensors.

The complete set of solutions consists of two distinct classes: the states at ghost number $n_g = 0$, $\Psi^{(0)}[M]$, and those at ghost number $n_g = 3$, $\tilde{\Psi}^{(3)}[M]$. We discuss next the possibility for these states to have finite norm (see also [16, 1]). The spectrum of the hamiltonian in such a basis would be guaranteed to be physical.

4 Inner product and ghost vacuum

The existence of two classes of BRST-harmonic states at different ghost number is of crucial importance for the construction of a non-trivial physical inner product [14, 16]. Indeed, if we would only have the states at $n_g = 0$ it is quite obvious from the definition (20) that the vacuum state Ψ_0 would have zero norm:

$$(\Psi_0, \Psi_0) = 0, \quad (37)$$

whilst the BRST-invariant 3-ghost operator has a non-zero vacuum expectation value:

$$\frac{i}{3!} \epsilon^{abc} \langle c^a c^b c^c \rangle = \frac{i}{3!} (\Psi_0, \epsilon^{abc} c^a c^b c^c \Psi_0) = 1. \quad (38)$$

The problem clearly is in the definition of the ghost vacuum, in combination with the fact that the ghosts are self-conjugate. Therefore the ghost creation operators c^a do not act as annihilation operators on the conjugate (bra) vectors; if they would, the BRST charge wouldn't be self-adjoint. In particular, it is not an option to replace the space of bra states by the BRST-dual states

$$\tilde{\Psi}^\dagger[M] = \frac{i}{3!} \Psi^\dagger[M] \epsilon^{abc} c^a c^b c^c, \quad (39)$$

as proposed in [16, 17], which is equivalent to the replacement of the inner product (20) by

$$(\Phi, \Psi) \quad \rightarrow \quad \langle \Phi, \Psi \rangle = \frac{i}{3!} \epsilon^{abc} (\Phi, c^a c^b c^c \Psi). \quad (40)$$

In fact, it is clear that the ghost variables have vanishing matrix elements between *any* states (physical or unphysical):

$$\langle \Phi, c^a \Psi \rangle = 0, \quad \forall \Psi, \Phi, \quad (41)$$

i.e. the ghosts would effectively vanish as operators, and the same is true for the BRST charge Ω .

Part of the solution of this problem, also along lines suggested in [17], is to use the existence of the second set of solutions of the BRST- and co-BRST constraints with $n_g = 3$ to change the definition of the ghost vacuum. If we define a new vacuum state

$$\Psi_+ = \frac{1}{\sqrt{2}} \left(1 + \frac{i}{3!} \epsilon^{abc} c^a c^b c^c \right) \Psi_0, \quad (42)$$

with corresponding physical excited states $\Psi_+[M] = M[\mathbf{a}^\dagger] \Psi_+$, the ghost operators remain self-adjoint and the vacuum is normalizable:

$$(\Psi_+, \Psi_+) = 1. \quad (43)$$

A draw-back is, that the vacuum Ψ_+ has no well-defined ghost number, and not even a well defined Grassmann parity, being a sum of an even and odd ghost number state. Moreover, the vacuum expectation value of the ghosts (38) is changed, but still non-vanishing; actually we now have

$$\frac{i}{3!} \epsilon^{abc} \langle c^a c^b c^c \rangle_+ = \frac{i}{3!} (\Psi_+, \epsilon^{abc} c^a c^b c^c \Psi_+) = \frac{1}{2}, \quad (44)$$

and similarly

$$\frac{i}{3!} \epsilon^{abc} \langle b^a b^b b^c \rangle_+ = \frac{1}{2}. \quad (45)$$

Although these expectation values are BRST-invariant, they carry a non-zero ghost number, a manifestation of the non-invariance of both the vacuum and the inner product itself under ghost rescaling.

Both problems can be solved by introducing a fourth ghost θ , with conjugate anti-ghost ζ :

$$[\theta, \zeta]_+ = 1. \quad (46)$$

The new ghost θ is taken to be a BRST singlet and has ghost number $n_g(\theta) = -3$; thus it has the same quantum numbers as the invariant anti-ghost operator, whilst ζ has the quantum numbers of the corresponding ghost operator:

$$\theta \sim \frac{i}{3!} \epsilon^{abc} b^a b^b b^c, \quad \zeta \sim \frac{i}{3!} \epsilon^{abc} c^a c^b c^c. \quad (47)$$

We then define the physical vacuum state

$$\Phi_0 = \frac{1}{\sqrt{2}} \left(1 + \frac{1}{3!} \theta \epsilon^{abc} c^a c^b c^c \right) \Psi_0, \quad (48)$$

and the physical excited states

$$\Phi[M] = M[\mathbf{a}^\dagger] \Phi_0. \quad (49)$$

These physical states have a well-defined ghost number $n_g(\Phi[M]) = 0$ and Grassmann parity (even). This is especially important in the supersymmetric extensions of the theory, as the action of the gaugino operators would otherwise cause problems with sign-changes for odd ghost number terms.

Simultaneously we also redefine the inner product (20) in the co-ordinate representation on the full state space to

$$(\Phi, \Psi) = \int d\theta \int dc^1 dc^2 dc^3 \int \prod_{i,a} dA_i^a \Phi^\dagger[\mathbf{A}, c] \Psi[\mathbf{A}, c]. \quad (50)$$

W.r.t. this inner product all ghosts, including the new singlet ghost, are self-adjoint, and so is the BRST charge Ω . Observe, that the ghost integration measure now has vanishing ghost number as well. Finally, the 3-ghost operator vacuum expectation value vanishes trivially:

$$\frac{i}{3!} \epsilon^{abc} \langle c^a c^b c^c \rangle_\theta = \frac{i}{3!} (\Phi_0, \epsilon^{abc} c^a c^b c^c \Phi_0) = 0. \quad (51)$$

Of course, there arise new vacuum expectation values

$$\frac{1}{3!} \langle \theta \epsilon^{abc} c^a c^b c^c \rangle_\theta = \frac{1}{3!} \langle \zeta \epsilon^{abc} b^a b^b b^c \rangle_\theta = \frac{1}{2}, \quad (52)$$

but these expectation values are both BRST invariant and have vanishing ghost number.

In passing, let us point out a further result of some interest: it is possible to define new anti-ghost operators β^a and η by

$$\beta^a = b^a + \frac{1}{2} \epsilon^{abc} \theta c^b c^c, \quad \eta = \zeta - \frac{1}{3!} \epsilon^{abc} c^a c^b c^c. \quad (53)$$

These redefinitions preserve the ghost number. Moreover, one easily establishes the anti-commutation relations

$$[c^a, \beta^b]_+ = \delta^{ab}, \quad [\theta, \eta]_+ = 1, \quad [\eta, \beta^a]_+ = [\theta, c^a]_+ = 0, \quad (54)$$

with all other anti-commutators vanishing as well. In addition

$$\beta^a \Phi_0 = \eta \Phi_0 = 0, \quad (55)$$

suggesting that Φ_0 is the actual Fock vacuum for the new anti-ghosts (η, β^a) . Unfortunately, it is to be noted that these antighosts are no longer self-adjoint w.r.t. the inner product (50):

$$\beta_a^\dagger = b^a - \frac{1}{2} \epsilon^{abc} \theta c^b c^c, \quad \eta^\dagger = \zeta + \frac{1}{3!} \epsilon^{abc} c^a c^b c^c. \quad (56)$$

Hence these operators do not annihilate the conjugate vacuum: for a general state vector Ψ

$$(\Phi_0, \beta^a \Psi) = (\beta_a^\dagger \Phi_0, \Psi) \neq 0. \quad (57)$$

Moreover, the conjugate ghosts have non-trivial anti-commutation relations with the original anti-ghosts, e.g.:

$$[\beta_a^\dagger, \eta]_+ = -\epsilon^{abc} c^b c^c, \quad [\beta_a^\dagger, \beta^b]_+ = -\epsilon^{abc} \theta c^c. \quad (58)$$

Therefore the ghost variables (β^a, η) are not of much use in the construction of states. Nevertheless, they do provide a good way to characterize the ghost dependence of the physical states by the conditions (55).

5 Lorenz gauge

We will now show, that the problems with the definition of physical states and inner products sketched in sect. 4 do not exist in the Lorenz gauge quantization. The starting point for our analysis is again the classical theory defined in eqs. (1)-(3), and the representation of the nilpotent BRST algebra defined in eq. (8). In the $(0+1)$ -dimensional reduction of the Yang-Mills theory, the Lorenz gauge takes the form

$$\dot{A}_0 = 0. \quad (59)$$

A convenient BRST-invariant extension of the classical lagrangean for this gauge is

$$\begin{aligned} L_{Lorenz} &= L_{YMQM} + N^a \dot{A}_0^a - \frac{1}{2} N_a^2 - i\dot{b}^a (D_0 c)^a \\ &\simeq \frac{1}{2} (D_0 \mathbf{A}^a)^2 + \frac{1}{2} (\dot{A}_0^a)^2 - \frac{1}{4} (F_{ij}^a)^2 - i\dot{b}^a (D_0 c)^a, \end{aligned} \quad (60)$$

where the last line results from elimination of the auxiliary fields N^a . The corresponding hamiltonian is

$$\begin{aligned}
H_{Lorenz} &= \frac{1}{2} \left(\mathbf{P}^a + g\epsilon^{abc} A_0^b \mathbf{A}^c \right)^2 + \frac{1}{2} (P_0^a)^2 + \frac{1}{4} (F_{ij}^a)^2 - \frac{g^2}{2} \left(\epsilon^{abc} A_0^b \mathbf{A}^c \right)^2 \\
&\quad + i \left(u^a - ig\epsilon^{abc} A_0^b c^c \right) v^a,
\end{aligned} \tag{61}$$

where the canonical momenta are defined by

$$\begin{aligned}
\mathbf{P}_a &= (D_0 \mathbf{A})^a, \quad P_0^a = \dot{A}_0^a, \\
u^a &= -(D_0 c)^a, \quad v^a = \dot{b}^a.
\end{aligned} \tag{62}$$

The conserved BRST charge takes the form

$$\Omega = G^a c^a - \frac{ig}{2} \epsilon^{abc} c^a c^b v^c + P_0^a u^a, \tag{63}$$

with the gauge charges G^a as in eq. (16). As neither of the expressions (61) and (63) suffer from ordering ambiguities, they can be interpreted directly as quantum operators, with the fundamental commutation relations given by

$$\begin{aligned}
[\mathbf{A}_a, \mathbf{P}_b] &= i\delta_{ab} 1_D, \quad [A_0^a, P_0^b] = i\delta^{ab}, \\
[c^a, v^b]_+ &= \delta^{ab}, \quad [b^a, u^b]_+ = \delta^{ab}.
\end{aligned} \tag{64}$$

The quantum equations of motion and the BRST transformations then again take the form (15), (18). In the co-ordinate representation, the BRST-invariant inner product of two wave functions in the full ghost-extended Hilbert space now takes the form

$$(\Phi, \Psi) = i \int \prod_a db^a dc^a \int \prod_a dA_0^a \int \prod_{i,a} dA_i^a \Phi^\dagger[\mathbf{A}, A_0, c, b] \Psi[\mathbf{A}, A_0, c, b]. \tag{65}$$

To fix the BRST gauge, we introduce the co-BRST operator

$${}^* \Omega = G^a v^a + \frac{ig}{2} \epsilon^{abc} c^a v^b v^c + P_0^a b^a. \tag{66}$$

Requiring states to be simultaneously BRST and co-BRST invariant leads to the conditions

$$G^a \Psi = 0, \quad \Sigma^a \Psi = 0, \quad F_0^a \Psi = 0, \quad (67)$$

where

$$\Sigma^a = ig\epsilon^{abc} c^b v^c, \quad (68)$$

is the generator of the rigid $SU(2)$ transformations, which is still an invariance of the theory, on the conjugate ghosts variables (c^a, v^a) . In contrast to the unitary gauge $A_0^a = 0$, in the Lorenz gauge the BRST conditions do not fix the physical states completely. We can still impose a further constraint fixing the dependence of physical states on the anti-ghost variables (b^a, u^a) , by requiring states to be rigid $SU(2)$ singlets w.r.t. all variables:

$$\tilde{\Sigma}^a \Psi = 0, \quad \tilde{\Sigma}^a = ig\epsilon^{abc} b^b u^c. \quad (69)$$

Indeed, it is easily checked that $\tilde{\Sigma}^a$ is a BRST- and co-BRST invariant operator; therefore the constraint can be imposed consistently on all physical states.

The full set of solutions of conditions (67) and (69) are wave functions which are $SU(2)$ singlets (i.e., gauge invariant), which do not depend on A_0^a , and whose ghost dependence is constrained to the form

$$\begin{aligned} \Psi_{phys}[\mathbf{A}, c, b] &= \Psi_1[\mathbf{A}] + \frac{i}{3!} \epsilon^{abc} c^a c^b c^c \Psi_2[\mathbf{A}] + \frac{i}{3!} \epsilon^{abc} b^a b^b b^c \Psi_3[\mathbf{A}] \\ &+ \frac{i}{(3!)^2} (\epsilon^{abc} c^a c^b c^c) (\epsilon^{def} b^d b^e b^f) \Psi_4[\mathbf{A}]. \end{aligned} \quad (70)$$

With the standard assignment of the ghost number $+1$ for c^a and -1 for b^a , requiring the states to have vanishing ghost number and definite Grassmann parity imposes the further constraint

$$\Psi_2[\mathbf{A}] = \Psi_3[\mathbf{A}] = 0. \quad (71)$$

Finally, requiring the inner product (65) to be positive definite in the subspace of physical states, we have to fix the space of physical states to be represented by factorized wave functions

$$\begin{aligned} \Psi_{phys} &= \frac{1}{\sqrt{2}} \left[1 + \frac{i}{(3!)^2} (\epsilon^{abc} c^a c^b c^c) (\epsilon^{def} b^d b^e b^f) \right] \Psi_M, \\ &= \frac{1}{\sqrt{2}} \left(1 - i \prod_a (c^a b^a) \right) \Psi_M, \end{aligned} \quad (72)$$

where Ψ_M can be taken as a physical Fock state of the form (36). Observe that the operator $(i/3!) \epsilon^{abc} b^a b^b b^c$ plays the same role here as the extra ghost θ in our construction of the states in the unitary gauge. Obviously, as in the unitary gauge, we can define a ghost operator with non-zero vacuum expectation value

$$\frac{i}{(3!)^2} \langle \prod_a (c^a b^a) \rangle = \frac{1}{2}, \quad (73)$$

but like (52) it is BRST invariant and has vanishing ghost number. Finally, defining the vacuum state of the physical subspace as

$$\Phi_0 = \frac{1}{\sqrt{2}} \left(1 - i \prod_a (c^a b^a) \right) \Psi_0, \quad (74)$$

where Ψ_0 is the Fock vacuum of the Yang-Mills system, one can again define ghost operators annihilating Φ_0 by taking

$$\gamma^a = c^a + \frac{i}{2 \cdot 3!} \epsilon^{abc} v^b v^c \epsilon^{def} u^d u^d u^f, \quad \beta^a = b^a - \frac{i}{2 \cdot 3!} \epsilon^{abc} u^b u^c \epsilon^{def} v^d v^d v^f. \quad (75)$$

As might be expected from our previous analysis, these operators are not self-adjoint and do not define a good basis for a complete Fock-space construction in the ghost sector. Nevertheless, the conditions

$$\gamma^a \Phi_0 = \beta^a \Phi_0 = 0 \quad (76)$$

provide a convenient way to characterize the physical ghost vacuum.

Finally we should remark, that in the physical subspace the integration over A_0^a is of course divergent in the absence of damping, as the physical wave functions are A_0 -independent. This divergence can be absorbed in a wave-function renormalization factor

$$N = \frac{1}{\sqrt{\int \prod_a dA_0^a}}. \quad (77)$$

Knowing this, we can remove the A_0^a from the physical inner product and effectively set $N = 1$; we observe, that N is BRST-invariant, and the procedure does not jeopardize the BRST-invariance of the integration measure.

6 Discussion

In this paper we have shown, that although the physical content of the (0+1)-dimensional Yang-Mills theory is clearest in the unitary gauge $A_0^a = 0$, the BRST quantization works in a more straightforward way in the Lorenz gauge $\dot{A}_0^a = 0$. An important part of the discussion and analysis was based on the construction of a BRST-invariant inner product w.r.t. which the BRST charge Ω is self-adjoint.

To get a little more algebraic and geometric insight into the constructions, consider again the unitary gauge, in which a general state is represented by a wave function

$$\Psi[c] = \psi + c^a \psi_a + \frac{i}{2!} c^a c^b \psi_{ab} + \frac{i}{3!} c^a c^b c^c \psi_{abc}. \quad (78)$$

Defining the dual wave function

$$\tilde{\Psi}[c] = \tilde{\psi} + c^a \tilde{\psi}_a + \frac{i}{2!} c^a c^b \tilde{\psi}_{ab} + \frac{i}{3!} c^a c^b c^c \tilde{\psi}_{abc}. \quad (79)$$

with components

$$\begin{aligned} \tilde{\psi} &= \frac{1}{3!} \epsilon_{abc} \psi_{abc}, & \tilde{\psi}_a &= \frac{1}{2!} \epsilon_{abc} \psi_{bc}, \\ \tilde{\psi}_{ab} &= \epsilon_{abc} \psi_c, & \tilde{\psi}_{abc} &= \epsilon_{abc} \psi, \end{aligned} \quad (80)$$

we recognize that the physical states (42) are characterized as the self-dual states $\tilde{\Psi} = \Psi$, such that the inner product (20) becomes

$$\begin{aligned} i \int dc^1 dc^2 dc^3 \Psi^\dagger \Psi &= \frac{1}{3!} \epsilon_{abc} \left(\psi_{abc}^\dagger \psi + \psi^\dagger \psi_{abc} + 3\psi_a^\dagger \psi_{bc} + 3\psi_{ab}^\dagger \psi_c \right) \\ &= 2\psi^\dagger \psi + 2\psi_a^\dagger \psi_a. \end{aligned} \quad (81)$$

In particular, with $\tilde{\psi} = \psi = \sqrt{2} \psi_M$ and $\psi_a = \tilde{\psi}_a = 0$, this reduces to

$$i \int dc^1 dc^2 dc^3 \Psi^\dagger[M] \Psi[M] = \psi_M^\dagger \psi_M. \quad (82)$$

Hence this inner product is positive definite for physical states. Of course, one can also consider the anti-self dual states $\tilde{\Psi} = -\Psi$, which then have a

negative definite norm. This should not surprise us, as the existence of a self-adjoint nilpotent BRST operator $\Omega^2 = 0$ is possible only in a space with indefinite norm. The important point is, that the space of physical states should have positive norm, and that is realized in the subspace of self-dual states.

Generalization of this discussion to the Lorenz gauge is simple. Each component in the wave-function expansion (78) now is a function of the additional ghost variables b_a , and we can again distinguish between components which are self-dual or anti-self-dual w.r.t. the expansion in b_a . In this formulation the physical states are then identified with the wave functions for which the components of zero ghost-number are completely self dual, i.e. self-dual both with respect to the c -ghost duality and with respect to the b -ghost duality.

We have discussed in particular the case of $SU(2)$ Yang-Mills theory. The generalization to $SU(N)$ is straightforward; with $r = N^2 - 1$ generators, and the same number of ghost and anti-ghost variables, the self-dual physical states in the unitary gauge are of the form

$$\Psi[c] = \frac{1}{\sqrt{2}} \left(1 + \frac{i^{[r/2]}}{r!} \epsilon^{a_1 \dots a_r} c^{a_1} \dots c^{a_r} \right) \psi_M. \quad (83)$$

For odd r (even N), both ghost number and Grassmann parity of the wave functions are ill-defined; for even r (odd N), it is only the ghost number which is violated. In both case, introduction of a singlet ghost θ with ghost number $n_g(\theta) = -r$ solves the problems. On the other hand, in the Lorenz gauge this is taken care of automatically by the anti-ghost variables, as the operator

$$\frac{i^{[r/2]}}{r!} \epsilon^{a_1 \dots a_r} b^{a_1} \dots b^{a_r} \quad (84)$$

has the same quantum numbers and plays the same role.

Finally we note, that as we have constructed precisely one BRST-invariant wave function for each physical state, in the supersymmetric extension the computation of the Witten index [18]-[21] is not affected by including the ghost degrees of freedom in the appropriate way.

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