Platonic maps of low genus

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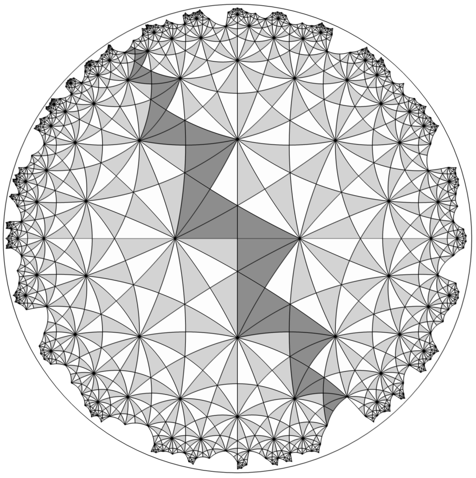
Maxim Hendriks



Eindhoven University of Technology

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Platonic Maps of Low Genus

PROEFSCHRIFT

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Preface

SYMMETRY AND BEAUTY have always been driving forces and sources of inspiration in mathematics. Symmetric shapes therefore occupy a special place in a mathematician's heart. The classical platonic solids are ancient examples of this. Plato even considered them to be possible building blocks of the universe. It is no wonder then, that mathematicians have studied those objects, and related ones, since antiquity. The first of these related objects were the archimedean solids. For them, one relinquishes some of the symmetry constraints. Almost two millennia later, Johannes Kepler and his contemporaries again enlarged the set of especially symmetric objects by allowing non-convexity. With the arrival of the study of $\mathbb{R}^n$ in the 19th century, symmetric polytopes in higher-dimensional spaces were sought. The development of topology led to the quest for symmetric tilings of even more general (metrized) spaces. The first interesting case that arises is that of a compact surface. Symmetric tilings of these surfaces are the objects of study in this thesis: *platonic maps*. The platonic solids are (part of) the special case where the underlying surface is the sphere. The combinatorial and topological viewpoint is exemplified in work of Henry Brahana and Harold Scott MacDonald Coxeter. These tilings remained somewhat of a curiosity until a firm connection between them and algebraic curves over number fields was realized by Belyĭ's theorem and the work of Alexander Grothendieck on his *dessins d'enfants*. The field then absorbed earlier results by pioneers such as Felix Klein and Walther von Dyck. In the late 20^{th} century, a 'Japanese school' constructed algebraic descriptions of platonic maps of genus 2 and 3. And starting in 2001 Marston Conder, with help from Peter Dobscányi, has been compiling complete lists of platonic maps for an ever-growing range of genera. The resulting zoo of objects provides us with a plethora of specimens to study. For a more detailed overview of the history of the subject, see e.g. the survey article by Jozef Širáň [Š2006].

Thesis roadmap / What's new? New results obtained in this thesis are: the introduction of polynomial families in Chapter 2 as a handy tool to classify some platonic maps and bundle them together; understanding the relation between triangle group inclusions and platonic maps (see Chapter 3); several classification results, most notably that of reflexive platonic maps with a prime number of vertices and of reflexive platonic maps with high density (to be found in Chapter 4); presentation of the explicit map structure on several well-known curves in Chapter 5; the computation of canonical models for reflexive platonic maps of genus $g \leq 8$ (and various others with $9 \leq g \leq 15$ in Chapter 6 and Appendix A; and finally, the construction of canonical models for the first Hurwitz triplet, the way to distinguish between the three, and the computation of Weierstraß points and their weights on these and other platonic maps. As an innovation in exposition, I propose the covering theory of platonic maps in Section 1.6, the concise but complete classification of genus 1 maps in Section 1.7 and the classification of hyperelliptic platonic maps in Section 6.3.

The use of computer algebra. I would not have been able to obtain many of the results in this thesis without the substantial use of computer algebra packages, specifically MAGMA, GAP, and MATHEMATICA. I stress that in most cases, the computer work produced a certificate at several steps during the computation, so that the results can be verified independently. The certificates and some additional MAGMA code are available online via

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Notation and terminology. Platonic maps have up to now been studied under the name "regular maps", but the author finds this terminology bland and generic. The term "platonic surface" is already in use for the Riemann surface determined by a platonic map, and of course the adjective "platonic" has been used for the platonic solids since ancient times. We therefore propose in this thesis to re-evaluate the terminology and replace "regular map" with the more evocative "platonic map". We can then differentiate these into "reflexive" and "chiral" platonic maps on an orientable surface. Logically then, the Riemann surface corresponding to a platonic map is a platonic surface. These notions are all defined and explained in Chapter 1.

All our group actions will be right actions. One reason was to avoid headaches when converting between left and right, since a lot of computation was done with the computer algebra system MAGMA, which uses right actions. Another reason is that right actions make sense in the context of left-to-right script, when interpreting a word in a map automorphism group as acting on a cell or fundamental triangle of a platonic map. So we will write RS, meaning "first apply R, then S". For commutators, it follows that we use $[x, y] = x^{-1}y^{-1}xy$ and for conjugation $x^y = y^{-1}xy$. We also abbreviate conjugation actions as $\mathrm{con}_y(x)$. If G is a group and $H \triangleleft G$, then we will denote the equivalence class of x in the quotient G/H as $[x]$, and work only with right cosets, not left ones.

Two minor words of warning: we will use the letters $e_1, \ldots, e_n$ to denote edges of platonic maps (in Chapter 4), but also as the standard basis vectors of an n-dimensional vector space (in Appendix A). Another recurring issue is the embedding of a number field into $\mathbb{C}$. We will generally work with roots of unity by letting $\zeta_n := e^{2\pi i/n}$. But whenever multiple embeddings are at stake, we might for the moment consider ζ_n to be any primitive n-th root of unity. For both ambiguities, the context will hopefully make the intention clear.

Isotypic components of representations occur throughout Appendix A, and also in

Chapter 7. They are subscripted with numbers, distinguishing different isotypics for irreducible representations of the same degree. We will mostly omit a description of the character making precise which representation is at stake, but will often give bases of the invariant subspaces to make up for this loss of precision.

Acknowledgments

No PhD candidate is an island. For one, there could be no PhD without a supervisor. Prof.dr. Arjeh M. Cohen accepted to guide my research for the past five years, and managed to regularly give me advice in spite of his busy schedule after he became dean. He taught me several important lessons, amongst which one from 1778: "Een onvermoeide arbeid komt alles te boven." I want to kindly thank you.

Of prime importance are also the members of my reading committee, prof.dr. Dan Bernstein, dr. Jan Draisma, prof.dr. Joseph Steenbrink, and prof.dr. Jaap Top. I thank them for serving on my defense committee, along with prof.dr. Marston Conder and dr. Kay Magaard. To the latter two I am especially grateful that they have taken the effort of traveling to Eindhoven over a great distance.

Thanks to all the mathematicians and other people working in the Discrete Mathematics group at the Technische Universiteit Eindhoven. Many helped me learn some new mathematics and encouraged me in one way or another. Specifically, I would like to thank dr. Jan Draisma for always keeping his door open and being so versatile as to think along about all my mathematical problems. Another thank you is due to Bas Spitters and Tyrrell McAllister for letting me join in their research on categories and logic, and Tyrrell especially for fruitful intellectual discussions about the rest of the universe.

Some people contributed very concretely to this thesis. Prof.dr. Joseph Steenbrink came to Arjeh with the problem that was the start of the work. Stephen Glasby supplied the rational representation for $\mathrm{PSL}(2, 13)$ used in Chapter 7. Prof.dr. Dan Bernstein factored various numbers for me, making me guilty of using up the department's computer resources for a month. Prof.dr. Jarke van Wijk let me use his wonderful program ORMVIEW, excellent for studying the combinatorics and geometry of many platonic maps of low genus. The program also contains stunning visualizations, and Jarke has kindly allowed me to use a multitude of illustrations in this work to bring out the beauty of the subject. Some additional images were granted by prof.dr. Carlo H. Séquin. The lists of platonic maps of low genus compiled by prof.dr. Marston Conder supplied me with all the data I needed for years of research: without his work, 223 of the pages in this book would have been blank. Finally, had Rob Eggermont not helped me, there would not be a platonic density theorem (cf. Section 4.4), only a conjecture. To all of them I owe a debt of gratitude.

My hoard of manuscript readers as well as spell, style and math checkers should

not be forgotten: Julia Collins, Agnès Denie, Annemieke Drummen, Mia Drummen, Rob Eggermont, Bernard Scharp, and Hans Sterk. All the remaining errors and melling spistakes are of course entirely mine. I am also grateful to Martin de Gier and Bernard Scharp for agreeing to be my *paranimfen*.

During my time at the Technische Universiteit Eindhoven I participated in the project Inter2Geo, part of the eContent*plus* program of the European Union. I wish to salute all collaborators on that project for their love of geometry and education, especially dr. Cyrille Desmoulins, Christian Dohrmann, dr. Santiago Egido, prof.dr. Ulrich Kortenkamp, dr. Yves Kreis, prof.dr. Colette Laborde & prof.dr. Jean-Marie Laborde, dr. Paul Libbrecht, and prof.dr. Christian Mercat.

Together with the other board members of PromoVE, the PhD organisation of the TU/e, I helped organize several events and tried to help out PhD candidates who encountered adverse conditions. For this reason I wish to acknowledge the hard labor done by Alex Juarez Cordova, Manuel Davila Delgado, Shakith Fernando, Çiçek Güven, Yulia Kiseleva, Evgeny Knutov, Elena Miloskovska and most of all Jan-Jaap Oosterwijk.

I extend my gratitude to roommates Rob Eggermont, Yael Fleischmann, Çiçek Güven, Jos in 't panhuis, and Dan Roozemond, as well as to all my other fellow PhD candidates, including Christian Eggermont, Bart Frenk, Sebastiaan de Hoogh, Relinde Jurrius, Peter van Liesdonk, Shoumin Liu, Jan-Jaap Oosterwijk, Christiane Peters, Peter Schwabe, Meilof Veeningen, and Rikko Verrijzer.

And last but not least, the unwavering support of my parents, Peter Hendriks and Esselien 't Hart, my girlfriend, Annemieke Drummen, and my other good friends have helped me battle the stress and anxiety inherent in trying to create something new.

M.H.
Eindhoven, January 2013

Contents

List of Notations

$\rightarrowtail$ injective map

$\twoheadrightarrow$ surjective map

$\subset$ and $\subseteq$ (Strict) containment as sets

$<$ and $\leq$ (Strict) containment in the appropriate category

$\lhd$ Normal subgroup containment

$\cong$ Isomorphic to

Alt_n Alternating group on n symbols

Aut Automorphism group

Dih_n Dihedral group of order n

$\mathbb{P}^n$ Complex projective n-space

Sym_n Symmetric group on n symbols

$\mathbb{Z}_n$ Cyclic group of order n

1

Introduction

PLATONIC maps stand at a crossroads of various branches of mathematics: group theory, incidence geometry, topology, Riemann surface theory, and algebraic geometry, to name a few. The possible range of background material to include is thus fairly wide, and inroads to the subject are numerous. The most intuitive way to think of a map is as a combinatorial and topological object. This will be our starting point in Section 1.1. In Section 1.2 we offer a starters of Riemann surface theory in order to whet the appetite for their connection to platonic maps in Section 1.3. Next, the group theory of platonic maps is explored in Section 1.4, to get a more computational grip on matters. After quickly highlighting the algebraic aspects of platonic maps in Section 1.5, to which we return in greater detail in later chapters, we treat their morphism / covering theory in Section 1.6. We conclude the introduction with the classification of genus 0 and genus 1 maps.

1.1 Platonic maps on surfaces

Consider a finite graph. When we say 'graph', we mean it may contain loops, or multiple edges between the same two vertices. We will think of the graph as a topological graph, which is to say realized as a compact topological space, each edge homeomorphic to $\mathbb{R}$. Our interest lies in embedding a connected graph into a surface.

Definition 1.1.1. *A* map M *is a surface (closed connected 2-manifold) Σ together with an embedded connected finite graph Γ with nonempty vertex and edge set, such that its complement is a finite disjoint union of open disks:*

$$\Sigma - \Gamma = \bigcup_{k=1}^{n} D_k, \qquad D_1 \cong \cdots \cong D_n \cong \mathrm{D}^2.$$

Example 1.1.2. The surface defined by a 2-dimensional polyhedron; the graph is formed by its vertices and edges. Or the map on a torus drawn in Figure 1.1.

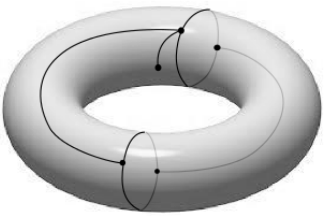


Figure 1.1: An example of a closed orientable surface with a map on it.

The embedding of the graph induces a natural decomposition of Σ into a finite number of disjoint open cells. We let $\mathrm{cells}_i(\mathbf{M})$ be the (nonempty) set of i-cells (those of dimension i), and $\mathrm{cells}(\mathbf{M}) = \cup_{i=0}^{2}\mathrm{cells}_i(\mathbf{M})$. But we will also refer to 0-cells, 1-cells and 2-cells as 'vertices', 'edges' and 'faces', respectively. The vertices and edges are of course those of the embedded graph. We say that two cells c_1, c_2 are *incident*, written $c_1 * c_2$, if as point sets $c_1 \subseteq \overline{c_2} \vee c_2 \subseteq \overline{c_1}$, the bars denoting closure. It is clear that two incident cells are either equal or have different dimension, the one of smaller dimension contained in the closure of the other. For a set of cells C, define the *residue*

$$C^* := \{d \in \mathrm{cells}(\mathbf{M}) \mid \exists c \in C, d * c\} - C.$$

Note that for a cell c, its topological boundary ∂c is equal to $c^* \cap (\cup_{i<\dim(c)}\mathrm{cells}_i(\mathbf{M}))$. A *flag* is a collection of distinct mutually incident cells. We speak of a *flag of type S* or an *S-flag* if the dimensions of its cells form the sequence S. A $(0,1,2)$-flag will also be called a *maximal flag*, since it cannot be extended to a larger flag. An *oriented flag* is a flag F with orientations on its members of positive dimension that are compatible, i.e. such that for $c, d \in F$ with $\dim(c) < \dim(d)$, the orientation on c *can* be induced on it from d as the positive orientation. Note that a vertex v determines a compatible orientation (going away from the vertex) on an incident edge e, unless the edge is a *loop*, i.e. $\partial e = \{v\}$. Similarly, an oriented edge $\overrightarrow{e}$ uniquely determines a compatible orientation on an incident face f, unless the face lying on either side of e is f. We denote the edge e with opposite orientation from $\overrightarrow{e}$ as $\overleftarrow{e}$. The *tail vertex* of $\overrightarrow{e}$ is the one that extends it to an oriented $(0,1)$-flag. Its *head vertex* is the one extending $\overleftarrow{e}$ to an oriented $(0,1)$-flag.

One more noteworthy construction is that of the *dual map* $\mathbf{M}^\vee$. To construct it, place one new vertex P_f in each 2-cell f. Next, choose a point p_e in the interior of every edge e and connect each vertex P_f to all $p_e \in \partial f$ by a simple arc. The new vertices and arcs define the cell decomposition of the dual map. If there is a cellular homeomorphism from a map to its dual, we call the map *self-dual*. We now come to the special kind of maps that are the focus of this thesis, namely those maps that have sufficiently many automorphisms.

Definition 1.1.3. *Let* $\mathbf{M}$ *be a map. A* map automorphism *of* $\mathbf{M}$ *is an equivalence* *class of cellular homeomorphisms under isotopy. We denote the group of map automorphisms of* $\mathbf{M}$ *by* $\mathrm{Aut}(\mathbf{M})$*.*

A map automorphism induces maps on the (oriented) vertex set, edge set and face set of $\mathbf{M}$, and thereby also on oriented flags.

Definition 1.1.4. *A map* $\mathbf{M}$ *is* platonic *if* $\mathrm{Aut}(\mathbf{M})$ *acts transitively on its set of oriented* $(0,1)$*-flags.*

Example 1.1.5. The map in Figure 1.1 is not platonic. One reason is the vertex with only one incident edge. Another is that the vertical edges are contained in circuits of length two, while the horizontal edges are not. There can be no automorphism of the surface mapping one to the other.

Each face of a map $\mathbf{M}$ has an edge on its boundary, and this edge is contained in precisely two oriented $(0,1)$-flags. So for a platonic map, any face can be mapped to any other by $\mathrm{Aut}(\mathbf{M})$. It follows that a traversal of the boundary of each face will encounter the same number p of incident vertices, and as many edges. Note that the same vertices and edges may occur multiple times, but we distinguish them for the count. We will come back to this multiplicity in Section 1.4. The faces are then named *p-gons*. We also see that the face is contained in exactly $2p$ oriented $(0,1,2)$-flags: there are p edges on its boundary, contained in a set of $2p$ oriented $(0,1)$-flags, and each one can be extended with the face in one compatible way.

Similarly, each vertex can be mapped to all others by $\mathrm{Aut}(\mathbf{M})$, so a traversal of a small loop around each vertex will go through the same number q of incident faces and as many edges, both counted with multiplicity. We say that the vertex has *valency q*. A vertex is contained in exactly $2q$ oriented $(0,1,2)$-flags: each of the q incident edges has a unique orientation compatible with the vertex, and each of the resulting $(0,1)$-flags can be extended in two ways to an oriented $(0,1,2)$-flag. Counting the number of oriented $(0,1,2)$-flags in three different ways, namely per vertex, per edge or per face, leads to the following fundamental *flag counting equalities* for platonic maps:

$$qv = 2e = pf = \frac{1}{2}|\{\text{oriented } (0,1,2)\text{-flags}\}|. \tag{1.1}$$

The pair (p,q) is called the *type* of a platonic map. Because of the transitivity, for each face f there is an automorphism that acts on the edges $e_1,\ldots,e_p$ encountered by a counterclockwise traversal of ∂f as the p-cycle $(e_1\cdots e_p)$. (We choose a local orientation for a neighborhood of the face to define the notion of counterclockwise.) We call this automorphism the *primitive counterclockwise rotation around the face* and denote it by R_f. Similarly, orienting a neighborhood of a vertex gives us a *primitive counterclockwise rotation around the vertex*, denoted S_v, that acts as a q-cycle $(e_1\cdots e_q)$ and $(f_1\cdots f_q)$ on the counterclockwise sequence of incident edges and faces respectively. For an edge e, there is a unique automorphism that switches its two oriented $(0,1)$-flags and also the incident faces (locally). We call this the *rotation around the edge e*, but do not use a separate symbol, because we can obtain it as follows: choose a face f and vertex v incident to e. Then also $v * f$, and the automorphism mentioned is $R_f S_v$ – first execute R_f and then S_v. This is illustrated in Figure 1.2 (left).

We leave out the indices when unnecessary. Note that this automorphism has order two, because its square fixes the oriented $(0, 1, 2)$-flags containing the edge.

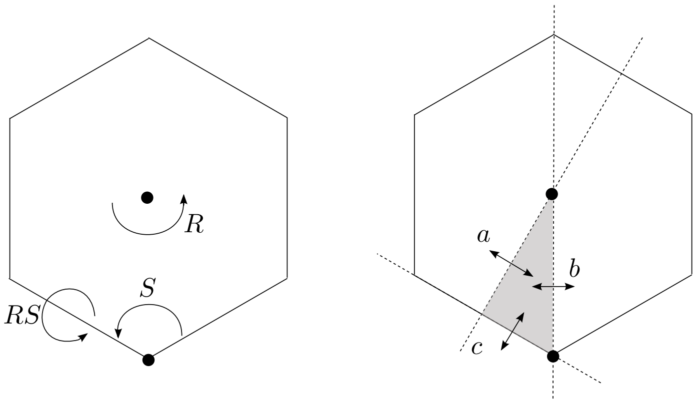


Figure 1.2: Left The primitive rotations R, S and RS around a face, vertex, and edge. Right The reflections a, b, c in the sides of a fundamental triangle of a reflexive map.

Lemma 1.1.6. *Choose a $(0, 2)$-flag and let S and R be its primitive vertex and face rotations. The subgroup $\langle R, S\rangle < \mathrm{Aut}(\mathbf{M})$ is transitive on the oriented $(0, 1)$-flags.*

Proof. We use induction on the set of oriented edges on the graph $\Gamma(\mathbf{M})$. Choose the edge $\overrightarrow{e_0}$ to be the one with edge rotation RS and tail at the rotation vertex of S. Assume that $\phi \in \langle R, S\rangle$ satisfies $\phi(\overrightarrow{e_0}) = \overrightarrow{e}$. First of all, we can then map $\overrightarrow{e_0}$ to $\overleftarrow{e}$ (same edge, the other orientation) by $RS\phi$. Second, we can then map $\overrightarrow{e_0}$ to an edge incident to the tail vertex of $\overrightarrow{e}$ by $S^k\phi$ for a suitable $k \in \mathbb{Z}/q\mathbb{Z}$. And third, we can then map $\overrightarrow{e_0}$ to an edge incident with the head vertex of $\overrightarrow{e}$ by $S^k(RS)\phi$ for a suitable $k \in \mathbb{Z}/q\mathbb{Z}$. Since $\Gamma(\mathbf{M})$ is connected, this implies we can map $\overrightarrow{e_0}$ to any edge with any orientation using $\langle R, S\rangle$. □

The lemma shows that if Σ is non-orientable, then $\mathrm{Aut}(\mathbf{M}) = \langle R, S\rangle$. We will restrict our attention to platonic maps on orientable surfaces though, called *orientable platonic maps*. This means that the underlying surface of a map is a g-holed torus Σ_g for some $g \in \mathbb{Z}_{\geq 0}$. We call g the *genus* of the map and surface. The group $\mathrm{Aut}(\mathbf{M})$ of an orientable map has a natural subdivision: there is the subgroup $\mathrm{Aut}^+(\mathbf{M})$ of orientation preserving ones and the coset $\mathrm{Aut}^-(\mathbf{M})$ of orientation reversing ones. Since an oriented $(0, 1)$-flag on an orientable surface is contained in precisely two oriented $(0, 1, 2)$-flags, and only an orientation reversing automorphism could switch them, we see that $\mathrm{Aut}^+(\mathbf{M}) = \langle R, S\rangle$ and that it makes sense to distinguish two types of orientable platonic maps.

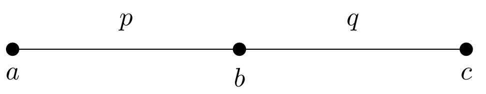
Definition 1.1.7. *An orientable platonic map $\mathbf{M}$ is called* reflexive *if $\mathrm{Aut}(\mathbf{M})$ acts transitively on its set of oriented $(0, 1, 2)$-flags. Otherwise it is called* chiral.

We will often indicate reflexive platonic maps using $\mathbf{R}$, and chiral ones using $\mathbf{C}$.

Every reflexive platonic map $\mathbf{R}$ is also simply a platonic map: we may forget the existence of reflections if we so desire. When $\mathbf{M}$ is chiral, then $\mathrm{Aut}^-(\mathbf{M}) = \emptyset$ and $\mathrm{Aut}(\mathbf{M}) = \mathrm{Aut}^+(\mathbf{M})$, whereas if $\mathbf{M}$ is reflexive, then $\mathrm{Aut}^+(\mathbf{M}) < \mathrm{Aut}(\mathbf{M})$ is a subgroup of index 2. If the map is reflexive, then there is an (orientation reversing) automorphism a switching the two oriented $(0, 1, 2)$-flags containing an edge-face pair. Such an element is called a *reflection*, as are $b := aR$ and $c := aRS$. These elements satisfy $a^2 = b^2 = c^2 = 1$. We can recover our pair (R, S) as

$$\begin{cases} R = ab \\ S = bc \end{cases}$$

Other relations that follow are $\mathrm{ord}(ab) = \mathrm{ord}(R) = p$, $\mathrm{ord}(bc) = \mathrm{ord}(S) = q$ and $\mathrm{ord}(ac) = \mathrm{ord}(abbc) = \mathrm{ord}(RS) = 2$. Since $\mathrm{Aut}(\mathbf{M}) = \langle a, b, c\rangle$, it is a quotient of the Coxeter group with diagram



and $\mathrm{Aut}(\mathbf{M}) = \mathrm{Aut}^+(\mathbf{M}) \rtimes \mathbb{Z}_2$, a semi-direct product defined by the complement $\langle a\rangle$ with conjugation action

$$\mathrm{con}_a : (R, S) \mapsto (R^{-1}, R^2S).$$

We call the closure of a connected, simply connected fundamental domain of Σ under $\mathrm{Aut}(\mathbf{R})$ that contains no point of a reflection axis a *fundamental triangle*. This is illustrated in Figure 1.2 (right), along with the three reflections in its sides a, b, c. For a chiral map, we can still define a fundamental triangle, as will be clear after the next two sections. But then a fundamental domain under $\mathrm{Aut}(\mathbf{C})$ consists of two fundamental triangles glued together along one of their sides. This will be justified in Section 1.3. A fundamental triangle corresponds uniquely to an oriented $(0, 1, 2)$-flag of $\mathbf{M}$.

Marston Conder has enumerated all reflexive and chiral platonic maps of low genus. His enumeration started with all maps of genus $g \leq 15$ in [CD2001] and has been extended steadily, now ranging up to genus 301. For the most recent update, refer to [Con2001]. Counting duals, there are 6104 reflexive platonic maps of genus $2 \leq g \leq 101$. Of these, 652 are self-dual. In this same range of genera, there are 1061 chiral platonic maps, of which 127 are self-dual. We identify the n-th reflexive (chiral) map of genus g by his numbering (for $g \geq 2$) using $\mathbf{R}_{g.n}$ ($\mathbf{C}_{g.n}$). A table of maps with $g \leq 15$ with improvements on the extra relators is found in Appendix B.

When $\mathbf{M}$ is a reflexive platonic map, $\mathrm{Aut}(\mathbf{M})$ acts simply transitively on the set of oriented $(0, 1, 2)$-flags. If the map is chiral, $\mathrm{Aut}(\mathbf{M})$ acts simply transitively on the set of oriented $(0, 1)$-flags. This suggests that the action of $\mathrm{Aut}^+(\mathbf{M})$ on the oriented $(0, 1)$-flags of $\mathbf{M}$ already contains all the information of the map, something we investigate now.

1.2 Riemann surfaces and triangle groups

In the next Section, we will see that a platonic map determines a Riemann surface. We therefore introduce the necessary concepts from Riemann surface theory, but we only have space to sketch the flow of the theory. For a more elaborate introduction, see [FK1980] and [Mir1995].

Definition 1.2.1. *Complex differentiable functions between open sets in $\mathbb{C}$ are called* holomorphic. *A Riemann surface is a connected topological space X with a maximal atlas of charts $(U_i, x_i)_{i \in I}$ such that $U_i \subseteq X$ is an open set, the U_i cover X, $x_i : U_i \to \mathbb{C}$ is a homeomorphism onto an open set, and the transition functions $x_j \circ x_i^{-1} : x_i(U_i \cap U_j) \to x_j(U_i \cap U_j)$ are holomorphic. (This last condition is vacuous when $U_i \cap U_j = \emptyset$.) We call the atlas a* complex structure *on the underlying space.*

On a Riemann surface X, one can sensibly speak of the set of complexly differentiable functions. They are called *holomorphic functions*, their collection denoted as $\mathcal{O}(X)$. More generally, the suitable morphisms for the category of Riemann surfaces are *holomorphisms*, which preserve holomorphy of functions. They also give rise to the automorphism group $\mathrm{Aut}(X)$, consisting of *biholomorphisms* $X \to X$. Holomorphic functions on domains in $\mathbb{C}$ are *conformal*: they preserve angles between curves when measured in a chart in $\mathbb{C}$. This property transfers to a Riemann surface, and equips it with a constant curvature metric. Holomorphic automorphisms thus act by isometries. Riemann surfaces are automatically oriented because $\mathbb{C}$ is, and holomorphic maps are *orientation preserving*. Also of interest to us are *anticonformal mappings* (also *antiholomorphic*). These preserve the absolute size of angles locally, but reverse orientation. The canonical example is complex conjugation $z \mapsto \overline{z}$ on $\mathbb{C}$. Geometrically, this is reflection in the real axis. We denote the set of (anti)conformal functions on X by $\mathcal{O}^*(X)$ and the group of (anti)conformal automorphisms of X, its *extended automorphism group*, by $\mathrm{Aut}^*(X)$.

The Riemann mapping theorem tells us that there are exactly three simply-connected Riemann surfaces: the Riemann sphere $\widehat{\mathbb{C}}$, the complex plane $\mathbb{C}$ and the hyperbolic plane $\mathbb{H}^2$. The curvature of the metric on the Riemann sphere is 1, whereas it is 0 for the complex plane. The hyperbolic plane has constant curvature -1. It has several models, amongst which are the upper-halfplane model $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ with metric defined by $ds^2 = dz\, d\overline{z}/y^2$ where $z = x + yi$, and the *Poincaré disk model* D^2 with metric $ds^2 = dz\, d\overline{z}/(1 - |z|)^2$. An illustration of the latter will follow shortly, the former is shown in Figure 5.3. For more on the beautiful geometry of hyperbolic spaces, see e.g. [Bea1983] and [CFKP1997].

Next, the uniformization theorem says that a Riemann surface X is isomorphic to a quotient of its universal cover $\widetilde{X}$ by a properly discontinuous fixed point free action of a subgroup $\Gamma < \mathrm{Aut}(\widetilde{X})$. If the surface is compact, this action has a compact fundamental domain in $\widetilde{X}$. The automorphism groups of the three simply-connected

Riemann surfaces all consist of Möbius transformations:

$$\begin{aligned} &\mathrm{Aut}(\widehat{\mathbb{C}}) = \mathrm{PGL}(2,\mathbb{C}) \qquad \text{(all Möbius transformations on } \widehat{\mathbb{C}}) \\ &\mathrm{Aut}(\mathbb{C}) = \{z \mapsto az + b \mid a, b \in \mathbb{C}\} = \text{Borel}(\mathrm{PGL}(2,\mathbb{C})) \\ &\mathrm{Aut}(\mathbb{H}^2) \cong \mathrm{Aut}(U^2) = \mathrm{PSL}(2,\mathbb{R}) \end{aligned}$$

Discrete subgroups of the third are called *Fuchsian groups*. A very important family of automorphism subgroups of simply-connected Riemann surfaces acting properly discontinuously are triangle groups. Their well-definedness is guaranteed by the well-known Poincaré polygon theorem [And2005].

Definition 1.2.2. *We say that a geodesic triangle with angles* $(\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r})$ *is a* (p, q, r)-triangle. *Let* U *be* $\widehat{\mathbb{C}}$, $\mathbb{C}$ *or* $\mathbb{H}^2$*, depending on whether* $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1$ *is positive, zero, or negative. Given a* (p, q, r)*-triangle* $\triangle \subset U$*, the* (full) triangle group $\Delta(p, q, r)$ *is the group of isometries generated by the reflections in the three sides of* $\triangle$*. The* triangle group $\Delta^+(p, q, r)$ *is its index* 2 *subgroup of orientation preserving maps.*

Remark 1.2.3. If we define the sides opposite the angles $(\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r})$ to be c, b, a respectively (see Figure 1.3), then $\Delta(p, q, r)$ has the presentation

$$\Delta(p, q, r) \cong \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^p = (bc)^q = (ca)^r = 1\rangle.$$

This is the Coxeter group that we saw in Section 1.1. Defining $R := ab$ and $S := bc$, we find

$$\Delta^+(p, q, r) \cong \langle R, S \mid R^p = S^q = (RS)^r = 1\rangle.$$

The group $\Delta^+(p, q, r)$ consists of products of an even number of reflections. For us, the important triangle groups will be those with $r = 2$.

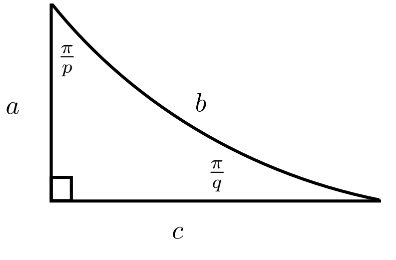
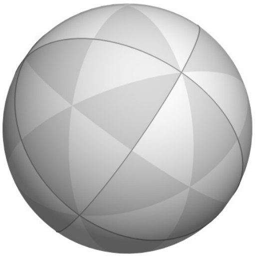


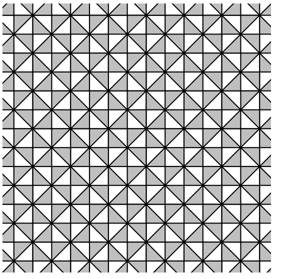
Figure 1.3: A hyperbolic $(p, q, 2)$-triangle. The letters a, b, c denote both the side and the reflections in the side they are juxtaposed to.

Proposition 1.2.4. *According to whether the sign of* $\frac{1}{p} + \frac{1}{q} - \frac{1}{2}$ *is* $+1$, 0 *or* -1*, the triangle group is a subgroup of* $\mathrm{Aut}(\widehat{\mathbb{C}})$, $\mathrm{Aut}(\mathbb{C})$ *or* $\mathrm{Aut}(\mathbb{H}^2)$.

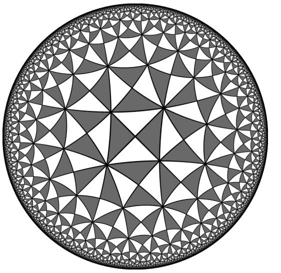
A few triangle group actions are illustrated in Figure 1.4.



(a) $\Delta(3,4,2)$



(b) $\Delta(4,4,2)$



(c) $\Delta(4,5,2)$

Figure 1.4: The three triangle groups $\Delta(3,4,2)$, $\Delta(4,4,2)$ and $\Delta(4,5,2)$ induce tilings of the respective spaces $\widehat{\mathbb{C}}$, $\mathbb{C}$ and $\mathbb{H}^2$ with (the interiors of) fundamental domains of the actions as tiles.

1.3 Platonic maps are Riemann surfaces

Let Σ_g be an orientable surface. Its mapping class group is the group of isotopy classes of homeomorphisms. The solution of the Nielsen Realization problem (in [Ker1983]) showed that every finite subgroup G of the mapping class group can be realized as a subgroup of $\mathrm{Homeo}(\Sigma_g)$. In fact, there is a Riemann surface structure on Σ_g with respect to which $G \cap \mathrm{Homeo}^+(\Sigma_g)$ acts conformally and $G \cap \mathrm{Homeo}^-(\Sigma_g)$ anticonformally. As such, the action consists of isometries with respect to the induced constant curvature metric. The curvature is either 1 (for $g = 0$, the *elliptic* case), 0 (if $g = 1$, the *parabolic* case) or -1 (if $g \geq 2$, the *hyperbolic* case).

We apply this result to $\mathrm{Aut}^+(\mathbf{M})$ of a platonic map $\mathbf{M}$. Platonicity as defined in Section 1.1 then implies that we may consider the surface underlying $\mathbf{M}$ to have a complex structure and a conformal group action transitive on the oriented $(0,1)$-flags. If we uniformize this Riemann surface $\mathbf{M}_r$, it leads us to the conclusion that $\mathbf{M}_r = \widetilde{\mathbf{M}_r}/\Gamma$ with $\Gamma < \mathrm{Aut}^+(\widetilde{\mathbf{M}_r})$ acting properly discontinuously, as we saw in the last section. As a group,

$$\Gamma \cong \pi_1(\Sigma_g) = \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \mid \pi_{i=1}^g [\alpha_i, \beta_i] \rangle.$$

The projection $\widetilde{\pi} : \widetilde{\mathbf{M}_r} \to \mathbf{M}_r$ gives us a discrete inverse image $\widetilde{\pi}^{-1}(\mathrm{Aut}^+(\mathbf{M}_r)) < \mathrm{Aut}^+(\widetilde{\mathbf{M}_r})$ acting discontinously. By cellularity and the transitivity of the group on the $(0,1)$-flags this has to be the triangle group $\Delta^+(p,q,2)$. Since $\Gamma < \ker(\widetilde{\pi}_*)$ (the action on the automorphism groups) we find $\Gamma \lhd \Delta^+(p,q,2)$ of finite index and $\widetilde{\pi}_* : \Delta^+(p,q,2) \to \Delta^+(p,q,2)/\Gamma \cong \mathrm{Aut}^+(\mathbf{M}_r)$. If the map is reflexive, then a reflection can be realized conformally and lifts to one of the universal cover. We then get $\Gamma \lhd \Delta(p,q,2)$ and $\mathrm{Aut}(\mathbf{M}_r) = \Delta(p,q,2)/\Gamma$.

The surface Σ_g has many possible complex structures on it for $g \geq 1$. These structures are parametrized by the Teichmüller space of genus g, homeomorphic to $\mathbb{R}^{6g-6}$ when $g \geq 2$. In most cases, the possible complex structures making a finite group action conformal is also far from unique. In fact, this structure is unique precisely in the case where $\mathbf{M}_r = \widetilde{\mathbf{M}_r}/\Gamma$ with $\Gamma \lhd \Delta^+(p,q,2)$. See [Woo2007].

We conclude that a platonic map $\mathbf{M}$ defines a unique complex structure. If it is reflexive, then the conformal action can be extended from $\mathrm{Aut}^+(\mathbf{M})$ to one of $\mathrm{Aut}(\mathbf{M})$ where orientation reversing automorphisms act anticonformally, so that $\mathbf{M}_r = \widetilde{\mathbf{M}_r}/\Gamma$ with $\Gamma \lhd \Delta(p,q,2)$. To summarize:

Theorem 1.3.1. *A platonic map* $\mathbf{M}$ *determines a unique Riemann surface* $\mathbf{M}_r$ *such that* $\mathrm{Aut}^+(\mathbf{M}) \leq \mathrm{Aut}(\mathbf{M}_r)$ *and* $\mathrm{Aut}(\mathbf{M}) \leq \mathrm{Aut}^*(\mathbf{M}_r)$.

We call $\mathbf{M}_r$ the *platonic surface* of the platonic map $\mathbf{M}$.

Remark 1.3.2. On $\mathbf{M}_r$ we can execute the procedure to obtain the dual cell division with the following extra constraints: we take the dual vertices to be the *centers* of the old faces (i.e. a dual vertex is the unique point of a face equidistant from the vertices on its boundary) and let the dual edges be geodesic segments. Conformality then ensures that $\mathrm{Aut}^+(\mathbf{M}_r)$ also preverves the new cell division, so that $(\mathbf{M}^\vee)_r = \mathbf{M}_r$.

Viewed the other way around, most Riemann surfaces do not have a large automorphism group that gives rise (see below) to a cell division on which this group acts platonically. This brings us to the following definition.

Definition 1.3.3. *A compact Riemann surface is* platonic *if it is of the form* U/Γ*, where* $U \in \{\widehat{\mathbb{C}}, \mathbb{C}, \mathbb{H}^2\}$ *and* $\Gamma \lhd \Delta^+(p,q,2)$ *acts fixed point freely on* U*. It is* reflexive platonic *if* $\Gamma \lhd \Delta(p,q,2)$.

When do we have $\mathbf{M}_{1,r} = \mathbf{M}_{2,r}$? It is not difficult to show that two Riemann surfaces $\mathbb{H}^2/\Gamma_1$ and $\mathbb{H}^2/\Gamma_2$ are isomorphic if and only if Γ_1 is conjugate to Γ_2 in $\mathrm{Aut}(\mathbb{H}^2)$. So we may assume $\mathrm{Aut}^+(\mathbf{M}_{i,r}) = \Delta_i/\Gamma$ for $i = 1, 2$. The Riemann surface $\mathbb{H}^2/\Gamma$ has full automorphism group Δ/Γ for some Fuchsian group Δ. But this implies that $\Delta_i \leq \Delta$ of finite index. All such inclusions have been determined by David Singerman in [Sin1972]. From his list we deduce that our two triangle groups Δ_1 and Δ_2 are not incomparable, but one can be included in the other. When $\Delta_1 = \Delta_2$, the maps are either the same or each other's dual. When $\Delta_1 \neq \Delta_2$ one map is in some sense a refinement of the other. We shall study these inclusions in more detail in Chapter 3.

1.4 Platonic maps are coset geometries

We can reconstruct the platonic map from the finite group G with a generator pair (R, S) satisfying $\mathrm{ord}(R) = p$, $\mathrm{ord}(S) = q$ and $\mathrm{ord}(RS) = 2$ as follows: by definition, such a group is a quotient of the $\Delta^+(p,q,2)$ triangle group, i.e. $G = \Delta^+(p,q,2)/\Gamma$ with $\Gamma \lhd \Delta^+(p,q,2)$ of finite index. Since the orders of R and S and RS are the same in $\Delta^+(p,q,2)$ as in the quotient G, the subgroup Γ contains no conjugate of one of these three elements, and therefore no non-trivial elements with fixed points at all.

So let U be $\widehat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{H}^2$ according to the type of triangle group. The natural projection $\pi : U \to U/\Gamma$ is apparently an unramified cover. The quotient is endowed with a platonic map $\mathbf{M}$ with $\mathrm{Aut}^+(\mathbf{M}) \cong G$ by projecting the axes of conjugates of the reflection c of a fundamental triangle in U. Thus we can abstract from the topology of $\mathbf{M}$ and just consider the group $\mathrm{Aut}^+(\mathbf{M})$.

Equivalence between platonic maps and standard map presentations. *A platonic map* $\mathbf{M}$ *is uniquely defined by a triple* (G, R, S) *where* G *is a finite group,* $R, S \in G$*,* $\mathrm{ord}(RS) = 2$ *and* $\langle R, S\rangle = G$*. A* standard map presentation *for* $\mathbf{M}$ *is a group presentation for* G *in terms of the* standard generator pair (R, S)*.*

Three relations that hold in any standard map presentation are $R^p = 1$, $S^q = 1$ and $(RS)^2 = 1$. We assume the corresponding relators as understood, and designate others as *extra relators*. The triple (R, S, RS) is a special case of what Wootton terms a *generating vector* for the group action in [Woo2007].

Remark 1.4.1. There is a similar equivalence between reflexive platonic maps and standard map group presentations by a *standard generator triple* (a, b, c) satisfying $\mathrm{ord}(a) = \mathrm{ord}(b) = \mathrm{ord}(c) = \mathrm{ord}(ca) = 2$. The reflexive structure can be deduced automatically from the platonic structure if it exists, as we saw in Section 1.1, so for uniformity we will use standard map presentations in (R, S) even for reflexive platonic maps.

In fact we could generalize our concept of a platonic map to correspond to a quotient of a general $\Delta(p, q, r)$ instead of $\Delta(p, q, 2)$. This gives the theory of hypermaps. More information can be found in [BCM2001]. We will not pursue this train of thought.

Remark 1.4.2. In case you are wondering whether $R = S$ is allowed: it is, but occurs for only one platonic map. If equality holds, then $2 = \mathrm{ord}(RS) = \mathrm{ord}(R^2)$ so $\mathrm{ord}(R) = \mathrm{ord}(S) = 4$. We find $G = \langle R\rangle \cong \mathbb{Z}_4$. This is indeed a correct map presentation, namely that of $\mathbf{R}_{1.3:1}$ which we will encounter in Section 1.7.

The equivalence allows us to define a map succinctly and to work with it computationally. All concepts determined by the topology of the map are captured by the group structure in some way. We proceed to demonstrate the usefulness of a standard map presentation concretely.

The incidence structure

Let $\mathbf{M}$ be platonic. If it is reflexive, let $W_0 := \langle b, c\rangle$, $W_1 := \langle a, c\rangle$, $W_2 := \langle a, b\rangle$, and $W_i^+ := W_i \cap \mathrm{Aut}^+(\mathbf{M})$. Otherwise, set $W_0^+ := \langle S\rangle$, $W_1^+ := \langle RS\rangle$, $W_2^+ := \langle R\rangle$. We note that $\cap_{i=0}^2 W_i$ and $\cap_{i=0}^2 W_i^+$ are both trivial, corresponding to the fact that the first action is simple on oriented $(0, 1, 2)$-flags and the second on oriented $(0, 1)$-flags.

The action of $\mathrm{Aut}^+(\mathbf{M})$ on oriented $(0, 1)$-flags is simply transitive. Within $\mathrm{Aut}^+(\mathbf{M})$, the stabilizers of the i-cell in the fundamental $(0, 1, 2)$-flag are exactly the subgroups

W_i^+. So we can identify the i-cells with (right) cosets of W_i^+. Incidence between cells is determined by

$$W_i^+ g_1 * W_j^+ g_2 \quad :\Longleftrightarrow \quad \exists h \in \mathrm{Aut}^+(\mathbf{M}),\ g_1 h \in W_i^+ \wedge g_2 h \in W_j^+.$$

If the map is reflexive, then we have an alternative description. Now $\mathrm{Aut}(\mathbf{R})$ acts simply transitive on the oriented $(0,1,2)$-flags, and the stabilizers of the i-cell in the fundamental $(0,1,2)$-flag are the subgroups W_i. We can identify the i-cells with (right) cosets of W_i and find the simpler incidence criterion

$$W_i g_1 * W_j g_2 \quad :\Longleftrightarrow \quad g_1 g_2^{-1} \in W_i \cdot W_j.$$

Either of these criteria allows us to compute all properties and functions on a platonic map expressible in terms of the incidence relation. We have interpreted the incidence structure as a *coset geometry*.

Fixed point counting

Let $\mathrm{Fix}_i(g)$ be the set of i-cells that an element $g \in \mathrm{Aut}^+(\mathbf{M})$ fixes on the platonic map $\mathbf{M}$, and $\mathrm{Fix}(g) = \cup_{i=0}^2 \mathrm{Fix}_i(g)$.

Lemma 1.4.3 (Fixed point counting)**.** *Let* $\mathbf{M}$ *be a platonic map. For* $g \in \mathrm{Aut}^+(\mathbf{M})$*:*

$$|\mathrm{Fix}_i(g)| = |\{h \in \mathrm{Aut}^+(\mathbf{M}) \mid hgh^{-1} \in W_i^+\}|/|W_i^+|.$$

If $\mathbf{M}$ *is reflexive, then for* $g \in \mathrm{Aut}(\mathbf{M})$*:*

$$|\mathrm{Fix}_i(g)| = |\{h \in \mathrm{Aut}(\mathbf{M}) \mid hgh^{-1} \in W_i\}|/|W_i|.$$

Proof. A cell $W_i^+ h$ is fixed by g precisely when $W_i^+(hg) = (W_i^+ h)g = W_i h$, so when $hgh^{-1} \in W_i^+$. There are exactly $|W_i^+|$ (right) coset representatives wh ($w \in W_i^+$) of $W_i^+ h$. When $\mathbf{M}$ is reflexive, one has to take into account that $g \in \mathrm{Aut}^-(\mathbf{M})$ will not fix W_i^+. The cells are then identified with cosets of W_i instead. □

We note that the lemma reinforces our knowledge that $g \in \mathrm{Aut}^+(\mathbf{M})$ fixes a cell if and only if g is conjugate to an element of $W_0^+ \cup W_1^+ \cup W_2^+$. The latter is clear from the triangle group action on the universal cover.

Corollary 1.4.4. *Suppose that the Riemann surface* $\mathbf{M}_r$ *satisfies* $\mathrm{Aut}(\mathbf{M}_r) = \mathrm{Aut}^+(\mathbf{M})$*. The only points of* $\mathbf{M}_r$ *fixed by non-trivial elements of* $\mathrm{Aut}(\mathbf{M}_r)$ *are the cell centers in the cell division of* $\mathbf{M}_r$*, and only conjugates to elements of* $W_0^+ \cup W_1^+ \cup W_2^+$ *fix points. Any* $g \in \mathrm{Aut}(\mathbf{M}_r)$ *that fixes the center* x *of a cell* $W_i^+ h$ *acts as a rotation* $z \mapsto \zeta_{\mathrm{ord}(g)}^{\mathrm{rot}(g,x)} z$ *in a suitable chart. The* rotation index $\mathrm{rot}(g,x) \in \{0,\ldots,\mathrm{ord}(g)-1\}$ *is independent of the choice of chart, and equals the exponent of* $hgh^{-1} \in W_i^+$ *with respect to the generator* S*,* RS *or* R*.* □

The genus formula

The number of oriented $(0,1)$-flags on an orientable map is $|\mathrm{Aut}^+(\mathbf{M})|$. Combining the flag counting equalities

$$v = |\mathrm{Aut}^+(\mathbf{M})|/q \qquad e = |\mathrm{Aut}^+(\mathbf{M})|/2 \qquad f = |\mathrm{Aut}^+(\mathbf{M})|/p$$

with the formula $\chi(\mathbf{M}) = 2 - 2g$ relating the Euler characteristic $\chi(\mathbf{M}) := v - e + f$ of the map to its genus, we find the *genus formula* of the map:

$$g(\mathbf{M}) - 1 = \frac{|\mathrm{Aut}^+(\mathbf{M})|}{2}\left(\frac{1}{2} - \frac{1}{p} - \frac{1}{q}\right). \tag{1.2}$$

From this formula we can read off immediately that we have an *elliptic map* ($g = 0$), *parabolic map* ($g = 1$) or *hyperbolic map* ($g \geq 2$) according to whether $\frac{1}{p} + \frac{1}{q} - \frac{1}{2}$ is positive, zero, or negative. We see that the first two yield only a finite number of types. Also, it is not difficult to find the smallest positive value that this last expression can assume. This is $1/42$, attained when $(p, q) = (3, 7)$. Thus for a hyperbolic map, we find:

The Hurwitz bound. When $g(\mathbf{M}) \geq 2$, then $\mathrm{Aut}^+(\mathbf{M}) \leq 84(g(\mathbf{M}) - 1)$.

As is well known, the Hurwitz bound holds in general for the group $\mathrm{Aut}(X)$ of a Riemann surface X. A surface that lies close to the bound is automatically platonic: if $|\mathrm{Aut}(X)| > 24(g - 1)$, then the Riemann-Hurwitz formula implies the area of a fundamental domain is at most $1/12$, which implies that its uniformizing Fuchsian group must be a triangle group.

There has been a great deal of interest in Riemann surfaces attaining the Hurwitz bound, appropriately called *Hurwitz surfaces*. A group is called a *Hurwitz group* if it is the automorphism group of a Hurwitz surface. This is equivalent to the group having a generator set (g_1, g_2, g_3) of orders $(3, 7, 2)$ such that $g_1 g_2 g_3 = 1$. For an overview of known results, see [Con2010], which also has many pointers to further literature.

Theorem 1.4.5. *Of the* 26 *sporadic finite simple groups, the* 12 *groups* J_1, J_2, J_4, Fi_{22}, $\mathrm{Fi}_{24'}$, Co_3, He, Ru, HN, Ly, Th *and* M *are Hurwitz groups.*

Tuplets

The uniqueness of a conformal action for *the topological group action* on a platonic map $\mathbf{M}$ does not imply uniqueness of a conformal action for *the group*, even if we specify the genus and type of the map. The counterexamples are interesting enough to warrant special attention.

Definition 1.4.6. *A set of platonic maps of genus* g *and the same type* (p, q) *that have pairwise isomorphic groups* $\mathrm{Aut}^+(\mathbf{M})$ *is a* tuplet.

We shall also say the members of a tuplet are *siblings*.

Example 1.4.7. The first tuplet consists of the two maps $\mathbf{R}_{8.1}$ and $\mathbf{R}_{8.2}$, with $\mathrm{Aut}^+(\mathbf{R}) \cong \mathrm{PGL}(2,7)$. The second that occurs is the first Hurwitz triplet $\mathbf{R}_{14.1}$, $\mathbf{R}_{14.2}$, and $\mathbf{R}_{14.3}$, with $\mathrm{Aut}^+(\mathbf{R}) \cong \mathrm{PSL}(2,13)$. The latter tuplet is discussed extensively in Chapter 7.

The tuplet notion comes into play when constructing algebraic models for platonic maps, as done in Chapter 6. We submit the following question about their behavior. It requires reading about the canonical character (Section 6.1) first.

Question 1.4.8. Are two platonic maps members of a tuplet if and only if their canonical characters are equal?

Remark 1.4.9. A group can also act as $\mathrm{Aut}^+(\mathbf{M})$ for maps with the same genus but of different type. From the genus formula, we see that there is a restriction that such types (p_1, q_1) and (p_2, q_2) must satisfy: $p_1^{-1} + q_1^{-1} = p_2^{-1} + q_2^{-1}$. Concrete examples of this phenomenon are the maps $\mathbf{R}_{56.1}$, $\mathbf{R}_{56.2}$, $\mathbf{R}_{56.3}$ with automorphism group $\mathrm{PSL}(2,11) \rtimes \mathbb{Z}_2^2$. The first two form a tuplet of type $(3,12)$ with $\langle R, S^2\rangle \cong \mathrm{PSL}(2,11)$, the third is of type $(4,6)$ with $\langle R^2, S\rangle \cong \mathrm{PSL}(2,11)$. In all instances there is the complement $\langle a, RS\rangle$. The maps $\mathbf{R}_{92.1}$, $\mathbf{R}_{92.2}$, $\mathbf{R}_{92.3}$ exhibit exactly the same behavior with $\mathrm{PSL}(2,11)$ replaced by $\mathrm{PSL}(2,13)$. As a final example we give the maps $\mathbf{R}_{101.9}$ of type $(4,24)$ and $\mathbf{R}_{101.18}$ of type $(6,8)$, both with group $\mathbb{Z}_2 \times (\mathrm{Alt}_5 \rtimes (\mathbb{Z}_8 \rtimes_{-1} \mathbb{Z}_2))$. For $\mathbf{R}_{101.9}$, $\mathrm{Alt}_5 = \langle R^2, S^8\rangle$ with complement $\langle b, S^3, (S^{-1}R)^3\rangle$ – the reader can compute the conjugation action on Alt_5 from the standard map presentation.

Suppose we are given a finite group G and a *signature* $(g; p, q, 2)$ prescribing the type and genus of its action as a map automorphism group $G \cong \mathrm{Aut}^+(\mathbf{M})$. We can compute the number of such maps by noticing that two generator pairs (R_1, S_1) and (R_2, S_2) of G, satisfying $\mathrm{ord}(R_1S_1) = \mathrm{ord}(R_2S_2) = 2$, both describe $\mathrm{Aut}^+(\mathbf{M})$ if and only if they satisfy precisely the same relations. Hence, the maps are isomorphic if and only if there exists $\phi \in \mathrm{Aut}(G)$ such that $(R_2, S_2) = (\phi(R_1), \phi(S_1))$. The number of different platonic maps $\mathbf{M}$ with $\mathrm{Aut}^+(\mathbf{M}) \cong G$ and of type (p, q) is therefore

$$\left|\left\{(R,S) \in G^2 \mid G = \langle R, S\rangle \wedge \mathrm{ord}(R) = p \wedge \mathrm{ord}(S) = q \wedge \mathrm{ord}(RS) = 2\right\}\right| / |\mathrm{Aut}(G)|.$$

Similary, the number of reflexive platonic maps $\mathbf{R}$ of type (p, q) with $\mathrm{Aut}(\mathbf{R}) \cong G$ is:

$$\left|\left\{(a,b,c) \in G^3 \mid G = \langle a, b, c\rangle \wedge \mathrm{ord}(ab) = p \wedge \mathrm{ord}(bc) = q \wedge \mathrm{ord}(ac) = 2\right\}\right| / |\mathrm{Aut}(G)|.$$

The orientability, self-duality and chirality criteria

A standard map presentation allows us to quickly test the orientability of a platonic map. Let $\mathbf{M}$ be a platonic map. Choose a fundamental vertex and face and orient the face. Suppose (R, S) is the corresponding standard map generator pair and let e_0 be the resulting fundamental edge (see Section 1.1). If we change the orientation, then

R and S are suddenly rotations *clockwise* instead of counterclockwise, and a standard generator pair for the new map is (R^{-1}, S^{-1}). If the surface is non-orientable, then there is a path along $\Gamma(\mathbf{M})$ with a Möbius strip as tubular neighborhood. We can construct an element $g \in \langle R, S \rangle$ that transports the edge e_0 around this path back to $S(e_0)$ and reverses the local orientation of the face. This entails that the map induced by $(R, S) \mapsto (R^{-1}, S^{-1})$ is an inner automorphism of $\langle R, S \rangle = \mathrm{Aut}(\mathbf{M})$.

The orientability criterion. A platonic map is non-orientable if and only if for any standard map presentation, $(R, S) \mapsto (R^{-1}, S^{-1})$ extends to an element of $\mathrm{Inn}(\langle R, S \rangle)$. In this case, $\mathrm{Aut}(\mathbf{M}) = \langle R, S \rangle$.

Suppose $\mathbf{M}$ is orientable with $\mathrm{Aut}^+(\mathbf{M}) = \langle R, S \rangle$. The mapping inv : $(R, S) \mapsto (R^{-1}, S^{-1})$ is now not an element of $\mathrm{Inn}(\langle R, S \rangle)$. If it is still contained in the group $\mathrm{Aut}(\mathrm{Aut}^+(\mathbf{M}))$, then $\mathrm{Aut}(\mathbf{M}) > \mathrm{Aut}^+(\mathbf{M})$. Choose a fundamental flag (v_0, e_0, f_0) and identify $\overline{f_0}$ with a planar regular polygon so that the vertices of $\overline{f_0}$ are identified with vertices of the polygon. Define the mapping b on $\overline{f_0}$ to correspond to reflection of the polygon in the b-axis of the fundamental triangle, compare Figure 1.2 (right). Extend it to the whole surface with the recipe

$$b(g(x)) = \mathrm{inv}(g)(b(x)) \qquad (g \in \mathrm{Aut}^+(\mathbf{M}),\ x \in v_0 \cup e_0 \cup f_0).$$

Is b independent of the choice of g? Clearly $g_1(x) = g_2(x)$ is equivalent to $g_1 g_2^{-1}(x) = x$. There are only three possible values for the pair $(g_1 g_2^{-1}, x)$: it is either (S^i, v_0) or $(RS, \text{midpoint}(e_0))$ or $(R^i, \text{midpoint}(f_0))$. In the first case $\mathrm{inv}(S^i)(b(x)) = S^{-i}(x) = x = b(x)$. In the second, $\mathrm{inv}(RS)(b(x)) = R^{-1}S^{-1}(b(x)) = b(x)$. And in the third, $\mathrm{inv}(R^k)(b(x)) = R^{-k}(x) = b(x)$. So $\mathrm{inv}(g_1)\mathrm{inv}(g_2)^{-1}(b(x)) = \mathrm{inv}(g_1 g_2^{-1})(b(x)) = b(x)$, proving well-definedness of b. It is clear that b is bijective on cells, and we conclude it is a homeomorphism, and orientation reversing. The map is apparently reflexive.

In the other direction, an easy computation shows that for a reflexive platonic map with standard map generators (a, b, c), the conjugation action of b on $\langle R, S \rangle = \langle ab, bc \rangle$ inverts R and S. We have thus derived the following handy criterion.

The chirality criterion. An orientable platonic map $\mathbf{M}$ is reflexive if for a standard map presentation, $(R, S) \mapsto (R^{-1}, S^{-1})$ extends to an element of $\mathrm{Aut}(\mathrm{Aut}^+(\mathbf{M}))$, or chiral if this is not the case.

For a chiral map $\mathbf{C}$, the application of $(R, S) \mapsto (R^{-1}, S^{-1})$ to the relators of a standard map presentation of $\mathrm{Aut}(\mathbf{C})$ produces a standard map presentation for its mirror image.

The last criterion we treat concerns the dual map. Consider the conformal action of $\mathrm{Aut}^+(\mathbf{M})$ on $\mathbf{M}_r$ and construct the dual as in Section 1.3. Then $\mathrm{Aut}^+(\mathbf{M})$ acts simultaneously on the cell divisions corresponding to $\mathbf{M}$ and $\mathbf{M}^\vee$. A standard generator pair for $\mathbf{M}^\vee$ is given by (S, R). So the map is self-dual precisely when the mapping induced by $(R, S) \mapsto (S, R)$ is a group automorphism.

The self-duality criterion. A platonic map $\mathbf{M}$ is self-dual if for any standard map presentation, $(R, S) \mapsto (S, R)$ defines an element of $\mathrm{Aut}(\mathrm{Aut}^+(\mathbf{M}))$.

If $\mathbf{M}$ is not self-dual, then the application of $(R, S) \mapsto (S, R)$ to a standard map presentation of $\mathrm{Aut}(\mathbf{M})$ produces a standard map presentation for $\mathbf{M}^\vee$.

Self-adjacencies, multiplicities and valencies

There are various multiplicities with respect to the incidence relation that can occur for (platonic) maps. For example, the graph $\Gamma(\mathbf{M})$ need not be simple. We introduce them here to be able to distinguish degenerate kinds of maps.

Definition 1.4.10. *The platonic map* $\mathbf{M}$ *is called* loop-free *if* $|\partial e| = 2$ *for any edge* e*. It is said to be* dual-loop-free *if* $|\{f \in \mathrm{cells}_2(\mathbf{M}) \mid f * e\}| = 2$ *for any edge* e*.*

Because of platonicity, both properties hold for all edges if and only if they hold for one edge. It is easy to see that $\mathbf{M}$ is loop-free precisely when $\mathbf{M}^\vee$ is dual-loop-free. In terms of a standard map presentation, $\mathbf{M}$ contains loops if RS fixes the fundamental vertex. Because of the coset geometry description, this is when $R \in \langle S \rangle$. Dually, $\mathbf{M}$ is not dual-loop-free exactly when $S \in \langle R \rangle$. Maps with (dual-)loops are classified in Section 4.1.

Lemma 1.4.11. *Let* $\mathbf{M}_1$*,* $\mathbf{M}_2$ *be platonic maps. If* $\mathbf{M}_2$ *is loop-free and dual-loop-free, then a map morphism* $f : \mathbf{M}_1 \to \mathbf{M}_2$ *is uniquely determined by the induced mappings on cells (including orientation information).*

Proof. Under these conditions, the image of an oriented $(0, 1, 2)$-flag is uniquely determined by the cells it contains. □

We continue by defining valencies and multiplicities.

Definition 1.4.12. *The* (i, j)-degree *(or* valency*)* d_{ij} *of* $\mathbf{M}$ *is (for* $i, j \in \{0, 2\}$*) the number of different* j*-cells incident to an edge incident to a given* i*-cell:*

$$d_{ij} = |\{c_2 \in \mathrm{cells}_i(\mathbf{M}) \mid \exists e \in \mathrm{cells}_1(\mathbf{M}), c_1 * e * c_2\}| \qquad (c_1 \in \mathrm{cells}_0(\mathbf{M})).$$

The (i, j)*-multiplicity is defined by* $\mu_{ij} = q/d_{ij}$ *if* $i = 0$ *and* $\mu_{ij} = p/d_{ij}$ *if* $i = 2$*.*

Platonicity implies that d_{ij} and μ_{ij} are independent of the choice of c_1. If we choose $c_0 \in \mathrm{cells}_0(\mathbf{M})$ and an oriented $\overrightarrow{c_2} \in \mathrm{cells}_2(\mathbf{M})$ such that $c_0 * c_2$, then drawing two pictures makes it clear that

$$\mu_{20} = \left|\left\{\overrightarrow{e} \in \mathrm{cells}_1(\mathbf{M}) \mid c_0 = \mathrm{tail}(\overrightarrow{e}) \ \wedge\ e * c_2 \ \wedge\ \overrightarrow{e} \text{ compatible with } \overrightarrow{c_2}\right\}\right| = \mu_{02}.$$

We can compute the multiplicities and hence the valencies from a standard map presentation of $\mathrm{Aut}^+(\mathbf{M})$ with Proposition 1.4.13.

Proposition 1.4.13. *For a platonic map* $\mathbf{M}$ *with standard generator pair* (R, S)*, the following equalities hold:*

$$\mu_{00} = |\{g \in \langle S \rangle \mid R^{-1} g R \in \langle S \rangle\}|$$
$$\mu_{22} = |\{g \in \langle R \rangle \mid S^{-1} g S \in \langle R \rangle\}|$$
$$\mu_{02} = |\langle R \rangle \cap \langle S \rangle|$$

Proof. Take v_1 to be the fundamental vertex and e the fundamental edge, between v_1 and v_2. The group of rotations around v_2 is $R^{-1}\langle S \rangle R$ and hence the set of edges between v_1 and v_2 is the orbit of e under the group $\langle S \rangle \cap R^{-1}\langle S \rangle R$. This proves the first equality. The second follows by duality. The parameter μ_{02} is equal to the number of occurrences of the fundamental face around v_1 locally, which is the number of elements in $\langle S \rangle$ that also fix the fundamental face and hence lie in $\langle R \rangle$. □

To bring the related geometry into even clearer focus, consider the cycle $(w_1, \ldots)$ of end points of edges going out from a vertex v, ordered counterclockwise. Whenever $S_v^i(w_1) = w_1$, we also have $S_v^{i+j}(w_1) = S_v^j(w_1)$ for any j. So the cycle repeats with primitive period d_{00}. Similarly, the cycle of incident faces around vertex v repeats with period d_{02}. The cycles of incident vertices and neighboring faces (via a common edge) of a face repeat with primitive periods d_{20} and d_{22}, respectively. Also, since $S_v^{d_{02}}$ fixes v and its incident faces, and the vertices along the boundary of an incident face have period d_{20}, this rotation fixes the neighboring vertices as well. Similarly, $R_f^{d_{20}}$ fixes the neighboring faces. We have shown:

Lemma 1.4.14. *The map valencies of a platonic map satisfy* $d_{00} \mid d_{02}$ *and* $d_{22} \mid d_{20}$. □

The *reduced graph* $\overline{\Gamma}(\mathbf{M})$ of $\mathbf{M}$ is the simple graph that we get if we identify all μ_{00} edges between every pair of adjacent vertices of $\Gamma(\mathbf{M})$. It is We will use multiplicities and valencies extensively in the classification efforts of Chapter 4.

Geodesic walls and Petrie paths

The cell division that a platonic map $\mathbf{M}$ induces on its underlying surface has a refinement that is of great use. To construct it we consider $\mathbf{M}_r$. The group $\Delta(p, q, 2)$ has a $(p, q, 2)$-triangle of $\widetilde{\mathbf{M}_r}$ as fundamental domain, which induces a cell division of $\widetilde{\mathbf{M}_r}$ like the ones seen in Figure 1.4. We know that $\mathbf{M}_r = \widetilde{\mathbf{M}_r}/\Gamma$ for some $\Gamma \lhd \Delta^+(p, q, 2)$, and even $\Gamma \lhd \Delta(p, q, 2)$ if $\mathbf{M}$ is reflexive. The cell division is therefore Γ-invariant and projects down to an $\mathrm{Aut}(\mathbf{M}_r)$-invariant cell division of $\mathbf{M}$. We call it the *barycentric subdivision* of $\mathbf{M}$. Its vertex set splits naturally into three types: the vertices, edge centers and face centers of the map $\mathbf{M}_r$. In the barycentric graph, these have degrees $2q$, 4, and $2p$, respectively. The barycentric faces are the fundamental triangles of $\mathbf{M}$. An extra property is now also clear: in $\widetilde{\mathbf{M}_r}$, any edge of a fundamental triangle lies on a unique geodesic. This geodesic is a reflection axis for an element of $\Delta(p, q, 2)$, and is contained within the graph of the tiling of $\widetilde{\mathbf{M}_r}$. It must project down to a geodesic loop on $\mathbf{M}_r$ contained within the graph of the barycentric subdivision. Platonic maps can be viewed as chamber complexes. In that comparison, the fundamental triangles are apartments and these geodesic loops are walls. Hence, we name them *geodesic walls*. The geodesic walls will help us in Chapter 7.

We could have specified the incidence structure of the barycentric subdivision topologically, by putting a new vertex in each edge and face, and connecting them up in the obvious way. In this setting, the geodesic walls are paths within the graph of the barycentric subdivision for which one continues 'straight ahead' at each vertex, i.e. two consecutive edges of the path divide the edges at their common vertex into two sets, those to the left and those to the right, and the two sets are of equal size. We have the following simple but important lemma.

Lemma 1.4.15. *The geodesic walls of a reflexive platonic map do not self-intersect.*

Proof. For any geodesic wall γ on $\mathbf{R}_r$ there is a reflection of $\mathbf{R}_r$ in γ. This reflection is an isometry of $\mathbf{M}_r$. Let p be any point on γ and restrict the reflection to a disk around p that is reflected to itself. A reflection on a disk has exactly one axis, so there can be no self-intersection at p. □

One should contrast this with a generic geodesic on a compact Riemann surface, which need neither be a closed loop nor simple.

Depending on the parity of p and q, the barycentric subdivision has different types of geodesic walls. Each wall γ has a periodic pattern of barycentric vertex types, that we denote v, e, and f according to the cell type of $\mathbf{M}$ they are the center of. There is an automorphism $T_\gamma \in \mathrm{Aut}^+(\mathbf{M})$ that acts as a translation of γ along itself by one primitive period of this vertex pattern. Table 1.1 lists a word of $\mathrm{Aut}^+(\mathbf{M})$ that accomplishes the primitive translation in each case. The number of barycentric

Parity of (p,q)	Cell center pattern	Primitive translation word
(0,0)	ef	$R^{p/2+1}S$
	ev	$RS^{q/2+1}$
	vf	$R^{p/2}S^{q/2}$
(0,1)	ef	$RSR^{p/2}$
	evfv	$RS^{(q+1)/2}R^{p/2}S^{(q+1)/2}$
(1,0)	ev	$SRS^{q/2}$
	efvf	$SR^{(p+1)/2}S^{q/2}R^{(p+1)/2}$
(1,1)	vevfef	$R^{(p+1)/2}SR^{(p+1)/2}S^{(q-1)/2}R^{-1}S^{(q-1)/2}$

Table 1.1: Translations along the different types of geodesic walls that border a fundamental triangle, in terms of the rotations R and S around the incident vertex and face center (of the original map).

vertices (or equivalently, edges) on each wall type can be counted by computing the order of its primitive translation word and multiplying by the number of occurrences within each period.

Now let $\mathbf{R}$ be a reflexive platonic map. Look back at Figure 1.3 and mark each barycentric edge as an a-, b- or c-edge according to the type of cell centers it connects. Every barycentric edge can be expressed uniquely as $g(x)$ for x an edge of a chosen fundamental triangle and $g \in \mathrm{Aut}^+(\mathbf{R})$. This allows us to compute, for $w \in \{a, b, c\}$, the set $\mathrm{Fix}_w^{\mathrm{bar}}(r)$ of barycentric w-edges that a reflection $r \in \mathrm{Aut}^-(\mathbf{R})$ fixes. We are mainly interested in its size:

$$|\mathrm{Fix}_w^{\mathrm{bar}}(r)| = |\{(g, x) \in \mathrm{Aut}^+(\mathbf{R}) \times \{a, b, c\} \mid r = g^{-1}xg\}|.$$

From this we infer that the number of w-walls (walls containing a w-edge) fixed by a reflection r is $|\mathrm{Fix}_w^{\mathrm{bar}}(r)|$/length($w$-wall). One reflection can fix multiple walls, even of different type, e.g. on $\mathbf{R}_{3.8}$, where the reflection a fixes one a-wall and one c-wall.

In Chapter 7, we will also have occasion to use a different kind of path from a geodesic one on the barycentric graph.

Definition 1.4.16. *A* Petrie path *on the barycentric graph is a path formed by alternatingly choosing the first edge left and the first edge right at each successive vertex.*

Unless $q = 2$, there are two different Petrie paths through a (directed) edge, depending on whether the first turn after it is left or right. For any two succesive edges $\vec{e}_1$, $\vec{e}_2$ the map automorphism that maps $\vec{e}_1 \mapsto \vec{e}_2$ moves the Petrie path to *the other* Petrie path through $\vec{e}_2$. Because we also know that $\mathrm{Aut}(\mathbf{M})$ is transitive on oriented $(0, 1)$-flags, it is transitive on Petrie paths, and so the *Petrie path length* is uniquely defined. From a picture one sees that applying $(abc)^2$ to a barycentric triangle moves it along two edges of a Petrie path, whence $\mathrm{ord}(abc) = 2\mathrm{ord}(R^2S^2) = 2\mathrm{ord}([R, S])$ is the Petrie path length.

Example 1.4.17. In [CM1980], Coxeter and Moser define the map $\{p, q\}_r$ as having type (p, q) with $[R, S]^r$ as only extra relator. This extra relator is not sufficient to define a *finite* group if p, q, r are all big. If the group is finite, then $\{p, q\}_r$ defines a platonic map, it will be reflexive automatically and have Petrie path length r by decree. The following six polynomial families are definable as a $\{p, q\}_r$-map: $\mathbf{Fer}(n)$, $\mathcal{F}^{(4,2n)}_{(n-1)^2}$, $\mathcal{F}_{1.4:n}$, $\mathcal{F}^\vee_{1.4:n}$, $\mathcal{F}_{1.2:n}$, and $\mathcal{F}^\vee_{1.2:n}$. Confer Chapter 2 for the definition of these families. We conjecture that the only other solutions (p, q, r) are $(3, 7, 4)$ ($\mathbf{R}_{3.1}$), $(3, 7, 6)$ ($\mathbf{R}_{14.1}$), $(3, 7, 7)$ ($\mathbf{R}_{14.3}$), $(3, 7, 8)$, $(3, 8, 4)$ ($\mathbf{R}_{8.1}$), $(3, 8, 5)$, $(3, 9, 5)$, $(3, 10, 4)$, $(3, 11, 4)$, $(3, 12, 2)$ (same map as $(3, 6, 2)$), $(4, 5, 3)$ ($\mathbf{R}_{4.2}$), $(4, 5, 4)$, $(4, 7, 3)$, $(5, 5, 2)$, $(5, 6, 2)$ ($\mathbf{R}_{9.16}$), $(5, 8, 2)$, $(5, 9, 2)$, and $(6, 7, 2)$.

Example 1.4.18. The sporadic finite simple group J_1 has seven inequivalent presentations as a Hurwitz group. These cannot all be distinguished by their Petrie path lengths. We can determine them by using two word orders, e.g. $(\mathrm{ord}([R, S]), \mathrm{ord}(R^2SR^2S^2R^2S^3))$. The seven curves belong to the values $(10, 6)$, $(10, 10)$, $(11, 5)$, $(11, 10)$, $(15, 5)$, $(15, 6)$, and $(19, 10)$. For some of these one needs still more relators to define the group.

1.5 Platonic maps are algebraic curves

That platonic maps have a unique Riemann surface structure is surprising. But the surprise gets bigger: our topological objects define algebraic curves! By Jean-Pierre Serre's GAGA theorem [Ser1955], the category of Riemann surfaces with holomorphic maps is equivalent to the category of smooth complex projective algebraic curves with rational mappings as morphisms. And there is more. Let us take a platonic map $\mathbf{M}$ and consider the cover $\pi : \mathbf{M}_r \to \mathbf{M}_r/\mathrm{Aut}^+(\mathbf{M}_r) \cong \widetilde{\mathbf{M}_r}/\Delta^+(p,q,2) \cong \widehat{\mathbb{C}}$. By cellularity and the transitivity of the group on the $(0,1)$-flags, all vertices are identified, as are the edge centers and the face centers. These are also the only points at which π ramifies, since any elements with fixed points in $\widetilde{\mathbf{M}_r}$ is conjugate to either R, S or RS. Thus, the ramification points yield only three branch points on $\widehat{\mathbb{C}}$, which we can take to be $\{0, 1, \infty\}$. The edges of a fundamental triangle of $\mathbf{M}_r$ are mapped to edges between these three vertices, dividing $\widehat{\mathbb{C}}$ into two triangles. This is shown in Figure 1.5.

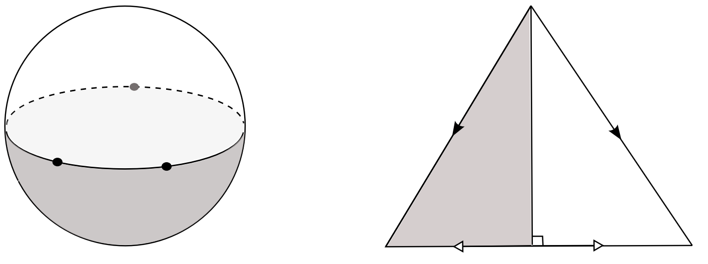


Figure 1.5: Left. The Riemann sphere $\mathbf{M}_r/\mathrm{Aut}^+(\mathbf{M}_r) \cong \widehat{\mathbb{C}}$. Right. The two triangles on $\widehat{\mathbb{C}}$ result from glueing the edges of a fundamental triangle and an adjacent mirror image.

Perhaps the simplest algebraic curve is $\mathbb{P}^1$, the *complex projective line*. It is equivalent to $\widehat{\mathbb{C}}$ under GAGA. In 1979, Russian mathematician Gennadii Vladimirovich Belyĭ published the astonishing result (cf. [Bel1979]) that a complex projective algebraic curve is definable over $\overline{\mathbb{Q}}$ precisely when it admits a rational mapping to $\mathbb{P}^1$ that branches over at most three points. Such a map is called a *Belyĭ function*. We have exhibited just such a mapping by our holomorphic $\mathbf{M}_r \to \widehat{\mathbb{C}}$. The conclusion is inevitable.

Theorem 1.5.1. *A platonic map* $\mathbf{M}$ *defines a unique algebraic curve* $\mathbf{M}_a$ *up to isomorphism. The curve* $\mathbf{M}_a$ *is definable over a number field* $K \subseteq \overline{\mathbb{Q}}$ *and admits a canonical Belyĭ function* $\pi : \mathbf{M}_a \to \mathbf{M}_a/\mathrm{Aut}(\mathbf{M}_a) \cong \mathbb{P}^1$.

The theorem leads to a number of research projects. Can we prove that a certain algebraic curve C supports a platonic map? According to the previous discussion, this is precisely the case when $C/\mathrm{Aut}(C) \cong \mathbb{P}^1$ and the projection $C \to C/\mathrm{Aut}(C)$ branches over three points. Therefore, a positive solution is obtained by finding an

$\mathrm{Aut}(C)$-invariant Belyĭ function. This is a route taken in Chapter 5. Another problem is to find an algebraic model for a given platonic map. One construction strategy is described in Chapter 6 and worked out for all reflexive platonic maps of genus $g \leq 8$ and various other cases. This work makes the GAGA correspondence effective for the platonic surfaces under consideration, which it is not in general.

1.6 Map morphisms and platonic covers

With a little extra effort, the concept of map automorphism introduced in Section 1.1 can be broadened to that of a morphism between different maps.

Definition 1.6.1. *A* map morphism *or* map cover *is a branched cellular cover* $\pi : \mathbf{M}_1 \to \mathbf{M}_2$. *The map* $\mathbf{M}_2$ *is a* quotient map *of* $\mathbf{M}_1$. *We call two map morphisms equivalent if there is a homotopy* $H : \mathbf{M}_1 \times [0,1] \to \mathbf{M}_2$ *between them, for which every* $H(\cdot, t)$ *is a cellular branched cover.*

We remark that every map morphism is surjective, being a cover. One can proceed to define the notion of automorphism as a morphism with an inverse. The new definition for automorphism coincides with the earlier one, since an automorphism in the new sense is forced to be a homeomorphism; this will follow from the Riemann-Hurwitz formula discussed below.

A map cover $\pi : \mathbf{M}_1 \to \mathbf{M}_2$ has a *ramification locus* $\mathcal{R}_\pi \subset \mathbf{M}_1$. This is the set of points in $\mathbf{M}_1$ at which π is not a local homeomorphism. The *branch locus* $\mathcal{B}_\pi \subset \mathbf{M}_2$ is the set of points in $\mathbf{M}_2$ at which the covering property fails. In other words, $\mathcal{B}_\pi = \pi(\mathcal{R}_\pi)$. Both $\mathcal{R}_\pi$ and $\mathcal{B}_\pi$ are discrete subspaces and hence finite.

Because a branched cover is a local homeomorphism away from the finitely many ramification points, $\dim(\pi(c)) = \dim(c)$ for all $c \in \mathrm{cells}(\mathbf{M}_1)$. Thus, π also induces a mapping on (oriented) flags, preserving their type. Continuity of a map morphism guarantees that $\pi(\overline{c}) = \overline{\pi(c)}$ and hence $\forall c_1, c_2 \in \mathrm{cells}(\mathbf{M}_1),\ \big(c_1 * c_2 \implies \pi(c_1) * \pi(c_2)\big)$. Because of cellularity and surjectivity, we also see that $\forall c \in \mathrm{cells}(\mathbf{M}_1),\ \pi(c^*) = \pi(c)^*$. But we can determine the behavior of a map morphism even more accurately.

Lemma 1.6.2. *Let* $\pi : \mathbf{M}_1 \to \mathbf{M}_2$ *be a map morphism. An edge of* $\mathbf{M}_2$ *contains no branch point, and an edge of* $\mathbf{M}_1$ *no ramification point.*

Proof. If the edge $e \in \mathrm{cells}_1(\mathbf{M}_2)$ contained a branch point q, then for any $p \in \pi^{-1}(q)$, the inverse image $\pi^{-1}(e) \ni p$ would not be contained in the edge of $\mathbf{M}_1$ containing p, so π would not be cellular. $\square$

Inside a 2-cell of $\mathbf{M}_1$ a map cover can have multiple branch points. This is exemplified by covers $\overline{D}^2 \to \overline{D}^2$ given by so-called finite Blaschke products, holomorphic functions of the form

$$f(z) = \frac{z - z_1}{1 - \overline{z}_1 z} \cdots \frac{z - z_n}{1 - \overline{z}_n z}$$

for a sequence of points $z_1, \ldots, z_n \in D^2$ that may contain repetitions. If a point z_i occurs d_i times, then locally around z_i there is d_i-fold ramification. We note, however, that we can let $z_i \to 0$ continuously by a homotopy of f. Doing this for all points of the sequence, we 'move all difficult behavior to one point'. We prove a topological lemma extending this trick to general maps.

Lemma 1.6.3. *Let $\pi : \mathbf{M}_1 \to \mathbf{M}_2$ be a map morphism. There exists an equivalent morphism that has at most one ramification point in any face of $\mathbf{M}_1$ and one branch point in any face of $\mathbf{M}_2$.*

Proof. Take $c \in \mathrm{cells}_2(\mathbf{M}_1)$ and suppose $\pi(c) = d$. First, we treat exceptional cases. If ∂c contains no embedded S^1, then ∂c is a tree. Contracting this tree to a point shows that $g(\mathbf{M}_1) = 0$ and that c is its only 2-cell. The same must then hold for $\mathbf{M}_2$. A loop close to ∂c then gets mapped to a loop close to ∂d with a certain winding number n. This shows that π is homotopic to $z \mapsto z^n$ on $\widehat{\mathbb{C}}$ with at most one of its two ramification points in c. So now suppose ∂c does contain an embedded S^1. We identify $\overline{c}$ with $\overline{D}^2$ and simply forget any degree 1 vertices and 'internal edges' that happen to lie in its interior D^2. The other exception that can still occur is that ∂d is a tree. In that case, $g(\mathbf{M}_2) = 0$ and we choose a projection $\mathrm{pr} : \overline{D}^2 \to S^2 \cong \Sigma(\mathbf{M}_2)$ mapping $D^2 \xrightarrow{\sim} d$. Lift the cellular structure on $\overline{d}$ and replace $\pi|_{\overline{c}}$ by a pr-lift. If we prove the statement for this map, it is also true for the original. Thus, we identify $\overline{d}$ with $\overline{D}^2$, and forget about possible internal edges and vertices of d.

We may thus assume the cover on the cell c is equivalent to $\pi|_{\overline{c}} : \overline{D}^2 \twoheadrightarrow \overline{D}^2$. Without loss of generality $\pi|_{\overline{c}}$ preserves orientation. We now orient ∂c counterclockwise and label its cells as $(v_{c,0}, \overrightarrow{e_{c,0}}, v_{c,1}, \ldots, \overrightarrow{e_{c,m-1}}, v_{c,0})$. We label ∂d similarly with indices from 0 to $n-1$, and may suppose $\pi(v_{c,0}) = v_{d,0}$ and $\pi(\overrightarrow{e_{c,0}}) = \overrightarrow{e_{d,0}}$. By induction we prove that $\pi(\overrightarrow{e_{c,i}}) = \overrightarrow{e_{d,i \bmod n}}$ for all i. Assume it is true for $i-1$. Either $\pi(\overrightarrow{e_{c,i}}) = \overrightarrow{e_{d,i}}$ or $\pi(\overrightarrow{e_{c,i}}) = \overleftarrow{e_{d,i-1}}$ because π is cellular. If the second option is true, take distinct points $p_1 \in e_{c,i-1}$, $p_2 \in e_{c,i}$ with $\pi(p_1) = \pi(p_2) \in e_{d,i-1}$: an arc from p_1 to p_2 inside c maps to a loop winding around $v_{d,i}$, implying $\pi(c) \nsubseteq d$. From this fact we learn that $\pi|_{\partial c}$ is homotopic to $z \mapsto z^n$. Extend this homotopy to a homotopy of $\overline{D}^2$ with the Alexander trick:

$$H(x,t) = \begin{cases} t\,\pi|_{\overline{c}}(x/t) & \text{if } 0 \leq ||x|| < t \\ \pi|_{\overline{c}}(x/||x||) & \text{if } t \leq ||x|| \leq 1 \end{cases}.$$

Apply this procedure to each 2-cell of $\mathbf{M}_1$ in turn. This vindicates the first claim. The second follows by composing $\pi_{\overline{c}}$ with a suitable Möbius transformation for each $c \in \mathrm{cells}_2(\mathbf{M}_1)$ to ensure that for any $d \in \mathrm{cells}_2(\mathbf{M}_2)$, the ramification points of the cells in $\pi^{-1}(d)$ are all sent to the same point of d. □

Up to equivalence, map morphisms exhibit a strong rigidity. From Lemma 1.6.3 we readily deduce that two map morphisms that agree on all oriented $(0,1,2)$-flags are equivalent. What is more, the behavior on a single oriented $(0,1,2)$-flag already determines a map morphism's equivalence class.

Proposition 1.6.4 (Rigidity of map morphisms)**.** *Two map morphisms* $\pi_1, \pi_2 : \mathbf{M}_1 \to \mathbf{M}_2$ *that agree on one oriented* $(0,1,2)$*-flag are equivalent.*

Proof. We prove that the morphisms agree on all oriented $(0,1,2)$-flags by induction over the graph $\Gamma(\mathbf{M}_1)$. Assume that $\pi_1 = \pi_2$ on the oriented flag $(c_0, \overrightarrow{c_1}, \overrightarrow{c_2})$. There are two induction steps. First, take a small disk D around c_0 and consider the subdivision induced by the cells of $\mathbf{M}_1$. Label these regions as $e_0, f_0, e_1, f_1, \ldots$, where e_0 is the part containing the 'tail' of c_1 and f_0 the region of c_2 to the left of that tail. Similarly, we label the cells of $\pi_1(c_0)^* = \pi_2(c_0)^*$ as $e_0', f_0', e_1', f_1', \ldots$ where $e_0' = \pi_1(e_0)$. Note that multiple regions can belong to the same cell of $\mathbf{M}_1$ (cf. Section 1.4, map multiplicities). But because π_1 and π_2 agree on the *oriented* $(0,1,2)$-flag $(c_0, \overrightarrow{c_1}, \overrightarrow{c_2})$, also $\pi_2(e_0) = e_0'$. The two morphisms send the region f_0 to 'the same' region as well (the two images have non-empty intersection), and hence also the regions e_1 and f_1. This means they agree on the oriented $(0,1,2)$-flag $(c_0, \overrightarrow{e_1}, \overrightarrow{e_2})$. By induction they agree on all oriented $(0,1,2)$-flags containing c_0.

Second, we consider the head vertex c_0' of $\overrightarrow{c_1}$ and suppose $c_0' \neq c_0$. Take a neighborhood D of c_1 homeomorphic to a disk and subdivide it into regions as above. We have $\pi_1(c_0') = \pi_2(c_0')$ and we can orient D such that c_2, $\pi_1(c_2)$, and $\pi_2(c_2)$ have points on the left of c_1, $\pi_1(c_1)$, and $\pi_2(c_1)$ when inducing the positive orientation on these respective edges. This implies that $(c_0', \overleftarrow{c_1}, \overleftarrow{c_2})$ has the same image under π_1 and π_2. This finishes the second induction step. Because $|\text{cells}(\mathbf{M})| < \infty$ and $\Gamma(\mathbf{M}_1)$ is connected, we conclude from the two steps that π_1 and π_2 agree on all $(0,1,2)$-flags. □

Remark 1.6.5. The necessity of *oriented* $(0,1,2)$-flags in Proposition 1.6.4 stems from the fact that either $\mathbf{M}_1$ or $\mathbf{M}_2$ may contains loops or dual-loops (see Section 1.4). A map cover of $\mathbf{R}_{1.3:1}$ (the torus map with one face, see Section 1.7) for example, cannot be defined by only specifying the images of cells. If both maps are loop-free and dual-loop-free, all mention of orientations in the proof can be dispensed with.

To correctly define the subcategory of platonic maps, we want map covers that combine with $\mathrm{Aut}(\mathbf{M})$. The definition also involves the orientation.

Definition 1.6.6. *A map cover* $\pi : \mathbf{M}_1 \to \mathbf{M}_2$ *between platonic maps is* platonic *if it preserves positive orientation and*

$$\forall g \in \mathrm{Aut}(\mathbf{M}_1)\ \forall c_1, c_2 \in \text{cells}(\mathbf{M}_1), (\pi(c_1) = \pi(c_2) \implies \pi \circ g(c_1) = \pi \circ g(c_2)).$$

From general covering theory we know that fiber size is constant for any cover, when counted with multiplicity. This size is the *degree* of the cover. For a map cover π we can also speak of the set of inverse cells $\pi^{-1}(c) \subseteq \text{cells}(\mathbf{M}_1)$ of a cell $c \in \text{cells}(\mathbf{M}_2)$. If π is platonic, the number of inverse images is constant per cell type. This follows by noting that if we take two inverse images c_1, c_2 of $d_1, d_2 \in \text{cells}_i(\mathbf{M}_2)$, there is a $g \in \mathrm{Aut}(\mathbf{M}_1)$ for which $g(c_1) = c_2$. Since $\pi(c) = d_1 = \pi(c_1)$ implies $\pi(g(c)) = \pi(g(c_1)) = \pi(c_2) = d_2$, we obtain a (non-canonical) bijection

$$\begin{aligned} \pi^{-1}(d_1) &\xrightarrow{\sim} \pi^{-1}(d_2) \\ c &\mapsto g(c) \end{aligned} .$$

One can realize a map cover $\pi : \mathbf{M}_1 \to \mathbf{M}_2$ as a cellular Riemann surface cover $(\mathbf{M}_1)_r \to (\mathbf{M}_2)_r$: a Riemann surface structure on $\mathbf{M}_2$ lifts via π to a unique Riemann surface structure on $\mathbf{M}_1$ such that π is holomorphic [For1977, Theorem 4.6]. Without loss of generality we can choose an equivalent map cover that has at most one ramification point in each cell of $\mathbf{M}_1$. For a ramification point $x \in \Sigma(\mathbf{M}_1)$, there are then suitable charts around x and $\pi(x)$ such that $\pi(z) = z^n$ locally. If the map cover is platonic, then the exponent n is equal to p_1/p_2 if x lies inside a 2-cell of $\mathbf{M}_1$, and equal to q_1/q_2 if x is a vertex of $\mathbf{M}_1$. Hence, the ramification and branching orders only depend on the *type* of cell in the respective maps, and we have the two ramification indices $\mathrm{mult}_0(\pi) := q_1/q_2$ and $\mathrm{mult}_2(\pi) := p_1/p_2$. If both indices are 1, the cover is unbranched. If only $\mathrm{mult}_2(\pi) = 1$ we say the cover "branches/ramifies (only) over vertices". If only $\mathrm{mult}_0(\pi) = 1$ we say the cover "branches/ramifies (only) over faces". A platonic map cover π is a normal cover (also called Galois or regular).

The Riemann-Hurwitz formula for platonic covers. The famous Riemann-Hurwitz formula for ramified covers (see e.g. [Mir1995]) takes on a special form for platonic covers. Let $\pi : \mathbf{M}_1 \to \mathbf{M}_2$ be a map cover. We have

$$\begin{aligned}\chi(\mathbf{M}_1) &= \deg(\pi)\chi(\mathbf{M}_2) - \sum_{p\in\mathbf{M}_1} (\mathrm{mult}(\pi, p) - 1) \\ &= \deg(\pi)\chi(\mathbf{M}_2) + f_1\left(1 - \frac{p_1}{p_2}\right) + v_1\left(1 - \frac{q_1}{q_2}\right).\end{aligned}$$

Remark 1.6.7. If $g(\mathbf{M}_2) \geq 1$, then $\chi(\mathbf{M}_2) \leq 0$ and since $\deg(\pi) \geq 2$ unless $\mathbf{M}_2 = \mathbf{M}_1$, we find that $\chi(\mathbf{M}_1) \leq 2\chi(\mathbf{M}_2) \leq 0$, so that $g(\mathbf{M}_1) \geq 2g(\mathbf{M}_2) + 1$.

The group theory behind platonic covers is what we wanted to investigate next. We derived the following results.

Proposition 1.6.8. *A map cover* $\pi : \mathbf{M}_1 \to \mathbf{M}_2$ *induces an epimorphism* $\pi_{\mathrm{Aut}} : \mathrm{Aut}^+(\mathbf{M}_1) \twoheadrightarrow \mathrm{Aut}^+(\mathbf{M}_2)$.

Proof. Let $g \in \mathrm{Aut}^+(\mathbf{M}_1)$. We define $\pi_{\mathrm{Aut}}(g)(x) := \pi\circ g\circ\pi^{-1}(x)$. This is well-defined as a function $\mathbf{M}_2 \to \mathbf{M}_2$, since two different points $y_1, y_2 \in \pi^{-1}(x)$ satisfy $\pi(y_1) = \pi(y_2)$ and hence $\pi(g(y_1)) = \pi(g(y_1))$ by the definition of platonic map covers. It is continuous because around $y \in \pi^{-1}(x)$ we can find a neighborhood in which π is a homeomorphism. Furthermore, take $c \in \mathrm{cells}(\mathbf{M}_2)$ and $x_1, x_2 \in c$. Connect the two points by a path γ within c. Since π is cellular, any π-lift of γ will lie within one cell of $\mathbf{M}_1$, hence so will $\pi_{\mathrm{Aut}}(g)(\gamma)$. This shows that $\pi_{\mathrm{Aut}}(g)$ is cellular as well. Finally, because $\pi \circ g^{-1} \circ \pi^{-1}$ is clearly an inverse of our mapping, we conclude $\pi_{\mathrm{Aut}}(g) \in \mathrm{Aut}(\mathbf{M}_2)$.

The mapping π_{Aut} is a homomorphism because $\pi_{\mathrm{Aut}}(g)\pi_{\mathrm{Aut}}(h)$ satisfies the defining equation for $\pi_{\mathrm{Aut}}(gh)$ and this defines the map uniquely. The remaining task is to prove surjectivity. Take $g \in \mathrm{Aut}(\mathbf{M}_2)$ and let F be an oriented $(0, 1)$-flag of $\mathbf{M}_2$. The flags F and $g(F)$ have lifts to oriented $(0, 1)$-flags $\tilde{F}_1, \tilde{F}_2$ on $\mathbf{M}_1$. By platonicity

$\widetilde{F}_2 = \widetilde{g}(\widetilde{F}_1)$ for some $\widetilde{g} \in \mathrm{Aut}(\mathbf{M}_1)$, and so

$$\pi_{\mathrm{Aut}}(\widetilde{g})(F) = \pi \circ \widetilde{g}(\widetilde{F}_1) = \pi(\widetilde{F}_2) = g(F).$$

This implies $\pi_{\mathrm{Aut}}(\widetilde{g}) = g$. □

Proposition 1.6.9. *For a platonic cover $\pi : \mathbf{M}_1 \to \mathbf{M}_2$, we have*

$$\ker \pi_{\mathrm{Aut}} = \{g \in \mathrm{Aut}(\mathbf{M}_1) \mid \forall c \in \mathrm{cells}(\mathbf{M}_1), \pi(g(c)) = \pi(c)\},$$

This normal subgroup of $\mathrm{Aut}^+(\mathbf{M}_1)$ does not contain a rotation around an edge of $\mathbf{M}_1$. For the converse, let $\mathbf{M}$ be a platonic map. A normal subgroup $N \lhd \mathrm{Aut}^+(\mathbf{M})$ containing no edge rotation gives rise to a platonic quotient map $\mathbf{M}/N$, and the natural projection $\pi : \mathbf{M} \to \mathbf{M}/N$ is a platonic cover of degree $|N|$ with $\ker \pi_{\mathrm{Aut}} = N$.

Proof. ($\Longrightarrow$) If $g \in \ker \pi_{\mathrm{Aut}}$, then for any $c \in \mathrm{cells}(\mathbf{M}_1)$ one has $\pi \circ g(c) = \pi_{\mathrm{Aut}}(g)(\pi(c)) = \pi(c)$. Vice versa, if $g \in \mathrm{Aut}^+(\mathbf{M}_1)$ and for all cells $c \in \mathrm{cells}(\mathbf{M}_1)$ we have $\pi(c) = \pi \circ g(c) = \pi_{\mathrm{Aut}}(g)(\pi(c))$, then $\pi_{\mathrm{Aut}}(g)$ fixes all oriented $(0,1)$-flags and so it must be the identity. Suppose $g \in \ker \pi_{\mathrm{Aut}}$ is an edge rotation around $e \subset \mathbf{M}_1$. From Lemma 1.6.2 we know e contains no ramification points. Choose an open neighborhood U of e containing no ramification points and such that $g(U) = U$. Then $\pi_{\mathrm{Aut}}(g) = \pi \circ g \circ \pi^{-1}$ sends $\pi(U)$ to itself and flips $\pi(e)$ around. Hence $\pi_{\mathrm{Aut}}(g) \neq 1$.

($\Longleftarrow$) If $N \lhd \mathrm{Aut}^+(\mathbf{M})$, consider $\mathbf{M}_r = \mathbb{H}^2/\Gamma$, with $\Gamma \lhd \Delta^+(p,q,2)$ and natural projection $\nu : \mathbb{H}^2 \to \mathbf{M}_r$. We have an induced homomorphism ν_* on the automorphism groups and $\Gamma' := (\nu_*)^{-1}(N) \lhd \Delta^+(p,q,2)$. Certainly $\Gamma \leq \Gamma'$. We therefore have the platonic surface $\mathbf{M}'_r = \mathbb{H}^2/\Gamma'$ with natural projection $\nu' : \mathbb{H}^2 \to \mathbf{M}'_r$, and this yields a unique branched cover $\pi : \mathbf{M}_r \to \mathbf{M}'_r$ completing the factorization $\nu' = \pi \circ \nu$. This π is the map cover we are after. If N contains no edge rotations, neither does Γ'. The map ν' thus has no ramification on (ν pullbacks of) edges. Therefore, π has the same property. Since $\mathbf{M}'_r$ is divided into disks by the images of all the edges, it follows that π is cellular. That π is platonic follows from the fact that for any $x_1, x_2 \in \mathbb{H}^2$,

$$\begin{aligned}
\pi(x_1\Gamma) = \pi(x_2\Gamma) &\Longleftrightarrow \nu'(x_1) = \pi \circ \nu(x_1) = \pi \circ \nu(x_2) = \nu'(x_2) \\
&\Longrightarrow \forall g \in \Delta^+(p,q,2),\ \nu'(g(x_1)) = \nu'(g(x_2)) \\
&\Longrightarrow \forall g \in \mathrm{Aut}(\mathbf{M}_r),\ \pi(g(x_1)\Gamma) = \pi(g(x_2)\Gamma).
\end{aligned}$$

The second step is valid because $\Gamma' \lhd \Delta^+(p,q,2)$. The kernel of π_{Aut} is N, since two points $x_1\Gamma, x_2\Gamma \in \mathbf{M}_r$ are N-related precisely when x_1, x_2 are Γ'-related. Such points have the same image under π. This also proves that the degree of π is $|N|$. □

The proof of 1.6.9 shows us that the Riemann surface structures on $\mathbf{M}_1$ and $\mathbf{M}_2$ of a platonic map cover for which $\mathrm{Aut}(\mathbf{M}_1)$ and $\mathrm{Aut}(\mathbf{M}_2)$ act holomorphically are those for which the map cover that has at most one ramification point in each 2-cell of $\mathbf{M}_1$ and at most one branch point in each 2-cell of $\mathbf{M}_2$ is holomorphic. Even more important is that Proposition 1.6.9 enables us to work wholly group-theoretically with platonic covers, something of great use. As a side note, a platonic cover $\pi : \mathbf{M}_1 \to \mathbf{M}_2$ has an associated dual platonic cover $\pi^\vee : \mathbf{M}_1^\vee \to \mathbf{M}_2^\vee$. This is easily understood by considering the dual maps as cell divisions on the same Riemann surfaces.

Corollary 1.6.10. *A platonic map cover π is unbranched if and only if* $\ker(\pi_{\mathrm{Aut}}) \cap (\langle S\rangle \cup \langle R\rangle) = 1$. □

Remark 1.6.11. Reflexive maps can have chiral quotients, but these are rare in low genus. The first example is $\mathbf{R}_{19.14}/\langle R^3S^2RS^{-1}\rangle = \mathbf{C}_{7.1}$.

Remark 1.6.12. A platonic map that does not platonically cover another platonic map is a basic building block of the category. We call such a map *irreducible*. Platonic maps with $\mathrm{Aut}^+(\mathbf{R})$ a simple group are certainly irreducible, but there are others. Appendix E lists the irreducible reflexive platonic maps of genus $2 \leq g \leq 101$.

Remark 1.6.13. After having seen map covers, it is an opportune moment to introduce the *gonality* $\mathrm{gon}(X)$ of a Riemann surface X. This is the smallest degree of a branched cover $X \to \widehat{\mathbb{C}}$. A Riemann surface of gonality 2 is called *hyperelliptic* (see also Chapter 6), and if the gonality is 3 it is called *trigonal*. We transport gonality to platonic maps in the obvious way. A known upper bound for $\mathrm{gon}(X)$ is $\lfloor (g+3)/2 \rfloor$ (cf. [KL1972]). The degree $\mathrm{gon}(\mathbf{M})$ is not always realized by a platonic cover. An example is $\mathbf{R}_{3.3}$ of type $(3, 12)$, which is trigonal. However, the only platonic cover to the Riemann sphere it admits is a 4-cover $\mathbf{R}_{3.3}/\langle S^3\rangle = \mathbf{Tet}$. Platonic covers can improve on the above-mentioned generic upper bound. The resulting possible range of gonalities is listed in Appendix C. That table incorporates knowledge of platonic covers, the canonical models discussed in Chapter 6, and the diagonal map constructions from Chapter 3.

Example 1.6.14. In Section 1.7 we will encounter the hosohedra $\mathbf{Hos}(n)$ ($n \in \mathbb{Z}_{\geq 1}$). There is a platonic cover $\mathbf{Hos}(n) \to \mathbf{Hos}(m)$ exactly when $m \mid n$, equivalent to the branched cover $z \mapsto z^{n/m}$ on $\widehat{\mathbb{C}}$.

There are also examples of platonic covers between whole polynomial families of platonic maps (introduced in Chapter 2). One is given in the following proposition.

Proposition 1.6.15. *There is a platonic 2-cover* $\mathcal{F}_n^{(2n+2,2n+2)}(2k) \to \mathcal{F}_n^{(2n+1,4n+2)}(k)$ *branched over faces.*

Proof. Let $\mathbf{R}_1$ be the map of genus $2k$ in the statement. From the standard map presentation

$$\mathrm{Aut}^+(\mathbf{R}_1) = \left\langle R, S \,\middle|\, R^{4k+2}, S^{4k+2}, (RS)^2, [R,S] \right\rangle$$

we see that $\langle R^{2k+1}\rangle \lhd \mathrm{Aut}^+(\mathbf{R}_1)$. Form the quotient map $\mathbf{M}_2 = \mathbf{R}_1/\langle R^{2k+1}\rangle$. It has the standard map presentation

$$\mathrm{Aut}^+(\mathbf{R}_2) = \left\langle \overline{R}, \overline{S} \,\middle|\, \overline{R}^{2k+1}, \overline{S}^{4k+2}, (\overline{RS})^2, [\overline{R}, \overline{S}] \right\rangle.$$

Since $[R, S] = 1$, we find $1 = (RS)^2 = R^2S^2$, which implies $R^{2k+1} = S^{-2k}R$ in $\mathrm{Aut}(\mathbf{R}_1)$. This yields the relation $\overline{R} = \overline{S}^{2k}$ for $\mathbf{M}_2$, which is the extra relator for $\mathcal{F}_n^{(2n+1,4n+2)}$. This shows that the quotient map is the purported one. In particular it is reflexive again. □

Multiplicity quotients

One obtains several particularly important platonic covers and quotient maps by trying to identify all edges that have the same set of incident vertices, or satisfy a similar condition.

Proposition 1.6.16. *Let* M *be a platonic map with standard generator pair* (R, S)*. The subgroups* $\langle S^{d_{00}}\rangle$*,* $\langle S^{d_{02}}\rangle = \langle R^{d_{20}}\rangle$*, and* $\langle R^{d_{22}}\rangle$ *are normal in* $\mathrm{Aut}(\mathbf{M})$*.*

Proof. Let S_v be the primitive counterclockwise rotation around v. The subgroup $\langle S_v^{d_{00}}\rangle$ also fixes all neighboring vertices of v. Hence, for a neighbor w we have $S_v^{d_{00}} \in \langle S_w\rangle$. Comparing orders, we see that $S_v^{d_{00}}$ generates $\langle S_w^{d_{00}}\rangle$, so $S_w^{d_{00}} \in \langle S_v^{d_{00}}\rangle$. By induction (or appealing to platonicity), we find that the subgroup $\langle S_v^{d_{00}}\rangle$ is the set of automorphisms in $\mathrm{Aut}^+(\mathbf{M})$ that fixes all vertices. Any conjugate by an element of $\mathrm{Aut}(\mathbf{M})$ will lie in this set as well. This proves $\langle S^{d_{00}}\rangle \lhd \mathrm{Aut}(\mathbf{M})$. Similar reasoning holds for the other two subgroups. □

The proposition shows that multiplicity quotients even preserve reflexivity. We call the quotient maps (and the map covers) by the type of valency / multiplicity present in their description: the μ_{00}-quotient, the μ_{02}-quotient and the μ_{22}-quotient.

Proposition 1.6.17. *The* μ_{00}*-quotient is a map of type* $(p/\mu_{20}, q/\mu_{00})$ *with the reduced graph* $\overline{\Gamma}(\mathbf{M})$ *as graph. The* μ_{22}*-quotient is a map of type* $(p/\mu_{22}, q/\mu_{02})$ *with dual graph the reduced graph* $\overline{\Gamma^\vee}(\mathbf{M})$*.*

Proof. Since $\langle S^{d_{00}}\rangle \cap \langle R\rangle = \langle R^{d_{20}}\rangle$, the μ_{00}-cover has ramification index $q/d_{00} = \mu_{00}$ over vertices and $p/d_{20} = \mu_{20}$ over faces. From this the claim about the map type follows. The subgroup fixes all vertices, so $\mathbf{M}/\langle S^{d_{00}}\rangle$ has the same number of vertices as $\mathbf{M}$, but between two vertices there is only one edge. The proof for the other quotient is similar. □

1.7 Platonic maps of genus 0 or 1

Platonic maps of genus 0 satisfy the inequality $\frac{1}{p} + \frac{1}{q} > \frac{1}{2}$ by the genus formula. The only solutions (p, q) are found to be $(2, n)$, $(3, 3)$, $(3, 4)$, and $(3, 5)$ up to duality. If $p = 2$ the above equation yields $v = 2$ and $e = f$, resulting in the family of *hosohedra* $\mathbf{Hos}(n)$ defined by the map automorphism group

$$\mathrm{Aut}(\mathbf{Hos}(n))^+ = \left\langle R, S \,\middle|\, R^n, S^2, (RS)^2 \right\rangle .$$

Such a map is easily visualized (as a beach ball), and so is its dual *dihedron* $\mathbf{Dih}(n)$: The others solution give platonic maps with the combinatorial structure of the classical platonic solids. We abbreviate their names to **Tet** for the tetrahedron; **Oct** for the octahedron and **Cub** for its dual, the cube; **Ico** for the icosahedron and **Dod** for

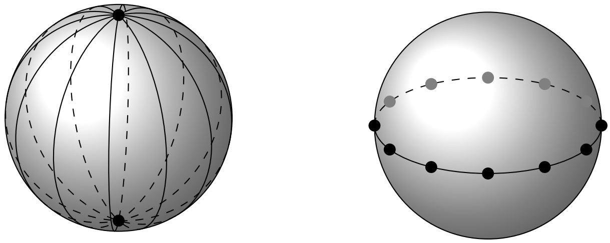


Figure 1.6: The hosohedron $\mathbf{Hos}(12)$ and the dihedron $\mathbf{Dih}(12)$.

its dual, the dodecahedron. They are defined by the map presentations

$$\begin{aligned}\mathrm{Aut}^+(\mathbf{Tet}) &= \left\langle R, S \,\middle|\, R^3, S^3, (RS)^2 \right\rangle \\ \mathrm{Aut}^+(\mathbf{Oct}) &= \left\langle R, S \,\middle|\, R^3, S^4, (RS)^2 \right\rangle \\ \mathrm{Aut}^+(\mathbf{Ico}) &= \left\langle R, S \,\middle|\, R^3, S^5, (RS)^2 \right\rangle\end{aligned}$$

Combinatorial data of these objects, which is standard fare, is listed in Appendix B. The genus formula implies that platonic maps of genus 1 obey $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. The

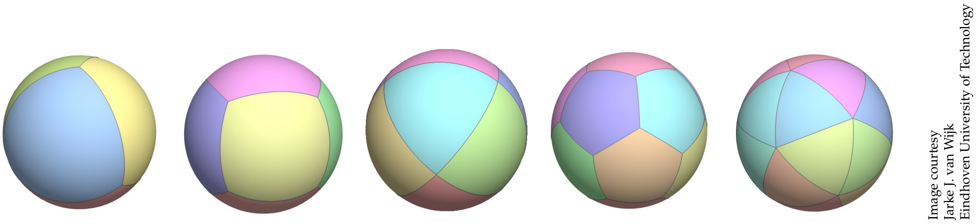


Figure 1.7: The platonic solids as platonic maps: **Tet**, **Cub**, **Oct**, **Dod**, and **Ico**.

only solutions for (p, q) are $(4, 4)$, $(3, 6)$, and $(6, 3)$. We leave the last case out of consideration by taking the dual. The group $\mathrm{Aut}^+(\mathbf{R})$ is thus a quotient of $\Delta^+(3, 6, 2)$ or $\Delta^+(4, 4, 2)$, both of which are realized as discrete translation subgroups of the euclidean plane $\mathbb{E}^2$ (subgroups of affine Coxeter groups):

$$\begin{aligned}\Delta^+(3, 6, 2) &= \left\langle (1, 0), \left(\tfrac{1}{2}, \tfrac{1}{2}\sqrt{3}\right) \right\rangle = A_2 \\ \Delta^+(4, 4, 2) &= \langle (1, 0), (0, 1) \rangle = A_1^2\end{aligned}$$

Our map is thus a quotient $\pi : \mathbb{E}^2 \to \mathbb{E}^2/\Lambda$ by a sublattice Λ of A_1^2 or A_2. This situation is illustrated in Figure 1.8. A subgroup Λ that yields a torus $\mathbb{E}^2/\Lambda$ is a lattice generated by two independent integral linear combinations of the lattice basis $\langle b_1, b_2 \rangle$. Without loss of generality, we may assume that $\pi((0, 0))$ is a vertex of the map. We analyze the two lattices separately.

For the lattice A_2, the supposed existence of a rotation S around $\pi((0, 0))$ over $\pi/3$ forces the sublattice to be of the form $\langle mb_1 + nb_2, -nb_1 + (m + n)b_2 \rangle$ (with $(m, n) \in$

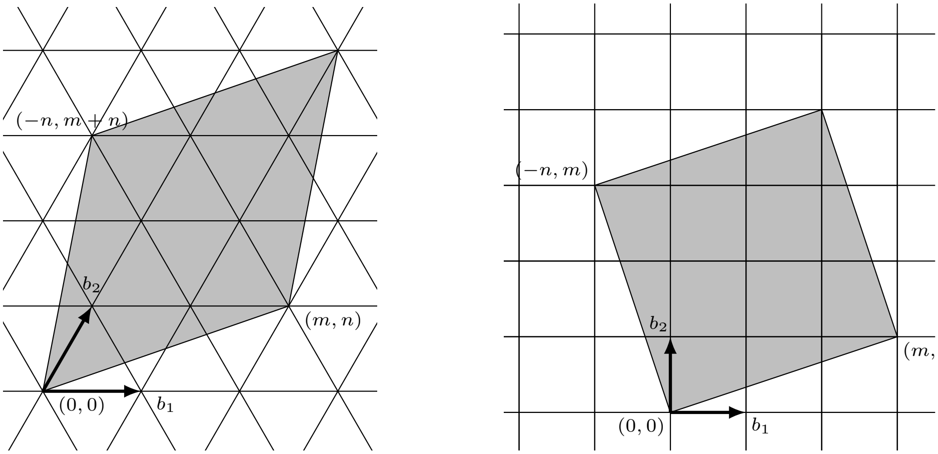


Figure 1.8: The lattices A_1^2 and A_2 in $\mathbb{E}^2$, yielding platonic maps of type $(3,6)$ (left) and $(4,4)$ (right) with a fundamental domain indicated (grey).

$\mathbb{Z}^2_{\geq 0} - \{(0,0)\}$). Calling the rotation over $\pi/3$ around (the image under π of) the triangle wedged in by the basis vectors R, we find that the translations $S^{-2}R : v \mapsto v + b_1$ and $S^2R^{-1} : v \mapsto v + b_2$ generate the whole translation group, leading to a collection of platonic maps we call $\mathbf{M}_{1.1:(m,n)}$ with standard map presentation

$$\mathrm{Aut}^+(\mathbf{M}_{1.1}(m,n)) = \left\langle R,S \,\middle|\, R^3, S^6, (RS)^2, (S^{-2}R)^m(S^2R^{-1})^n, (S^{-2}R)^{-n}(S^2R^{-1})^{m+n} \right\rangle$$

To write down its isomorphism type explicitly, we study the translation subgroup. Its order equals m^2+mn+n^2, the area of a fundamental domain. Let $d = \gcd(m,n)$. We show that $\mathrm{ord}(\pi((1,0))) = (m^2+mn+n^2)/d$. For if $i(m,n)+j(-n,m+n) = (k,0)$, then $in/(m+n) = -j \in \mathbb{Z}$, so $(m+n)/d$ divides i. Writing $k = im - jn = (m^2+mn+n^2)i/(m+n)$, we see that k must be an integral multiple of $(m^2+mn+n^2)/d \in \mathbb{Z}$. Since $(m+n)/d(m,n) - n/d(-n,m+n) = ((m^2+mn+n^2)/d, 0)$, we have proved the claim. The quotient group is

$$(\mathbb{Z}^2/\Lambda)/\langle \pi((1,0))\rangle \cong \mathbb{Z}^2/\langle (1,0),(m,n),(-n,m+n)\rangle \cong \mathbb{Z}/\langle n, m+n\rangle \cong \mathbb{Z}_d$$

The quotient $\mathbb{Z}_d$ can actually be found as a complementary subgroup, and so we find

$$\mathrm{Aut}^+(\mathbf{M}(m,n)) \cong (\mathbb{Z}_{(m^2+mn+n^2)/d} \times \mathbb{Z}_d) \rtimes \mathbb{Z}_6.$$

Some extra combinatorial data is readily computed: $v = (m^2+mn+n^2)$, $e = 3(m^2+mn+n^2)$, $f = 2(m^2+mn+n^2)$. A map of this family is reflexive if and only if the reflection of $\mathbb{E}^2$ switching b_1 and b_2 leaves Λ invariant. This is equivalent to $(n,m) \in \Lambda$, and assuming without loss of generality $m \leq n$, we see that either $(n,m) = (m,n)$ or $(n,m) = (-n, m+n)$. This leads to the cases $m = n$ and $n = 0$ respectively. We can simplify the relators under one of these assumptions, and arrive at the two 1-parameter families $\mathbf{R}_{1.1:n}$, and $\mathbf{R}_{1.2:n}$ (illustrated in Figure 1.9) defined by:

$$\mathrm{Aut}^+(\mathbf{R}_{1.1:n}) = \left\langle R,S \,\middle|\, R^3, S^6, (RS)^2, [R,S^2]^n \right\rangle,$$
$$\mathrm{Aut}^+(\mathbf{R}_{1.2:n}) = \left\langle R,S \,\middle|\, R^3, S^6, (RS)^2, [R,S]^n \right\rangle.$$

For the lattice A_1^2, the supposed existence of a rotation S around $\pi((0,0))$ over $\pi/2$ forces the sublattice to be of the form $\langle mb_1 + nb_2, -nb_1 + mb_2\rangle$ (with $(m,n) \in \mathbb{Z}^2_{\geq 0} - \{(0,0)\}$). Calling the rotation over $\pi/2$ around (the image under π of) the square wedged in by the basis vectors R, we find that the translations $S^{-1}R : v \mapsto v + b_1$ and $SR^{-1} : v \mapsto v + b_2$ generate the whole translation group, leading to a collection of platonic maps we call $\mathbf{M}_{1.2:(m,n)}$ with standard map presentation

$$\mathrm{Aut}^+(\mathbf{M}_{1.2}(m,n)) = \left\langle R, S \,\middle|\, R^4, S^4, (RS)^2, (S^{-1}R)^m(SR^{-1})^n, (S^{-1}R)^{-n}(SR^{-1})^m \right\rangle$$

To write down its isomorphism type explicitly, we study the translation subgroup. Its order equals $m^2 + n^2$, the area of a fundamental domain. Let $d = \gcd(m,n)$. We show that $\mathrm{ord}(\pi((1,0))) = (m^2+n^2)/d$. For if $i(m,n) + j(-n,m) = (k,0)$, then $in/m = -j \in \mathbb{Z}$, so m/d divides i. Writing $k = im - jn = (m^2+n^2)i/m$, we see that k must be an integral multiple of $(m^2+n^2)/d \in \mathbb{Z}$. Since $m/d(m,n) - n/d(-n,m) = ((m^2+n^2)/d, 0)$, we have proved the claim. The quotient group is

$$(\mathbb{Z}^2/\Lambda)/\langle \pi((1,0))\rangle \cong \mathbb{Z}^2/\langle (1,0),(m,n),(-n,m)\rangle \cong \mathbb{Z}/\langle n,m\rangle \cong \mathbb{Z}_d$$

The quotient $\mathbb{Z}_d$ can actually be found as a complementary subgroup, and so we find

$$\mathrm{Aut}^+(\mathbf{M}(m,n)) \cong (\mathbb{Z}_{(m^2+n^2)/d} \times \mathbb{Z}_d) \rtimes \mathbb{Z}_4.$$

Again, extra combinatorial data is readily computed: $v = (m^2+n^2)$, $e = 2(m^2+n^2)$, $f = (m^2+n^2)$. Like we saw before, a map of this family is reflexive if and only if the reflection of $\mathbb{E}^2$ switching b_1 and b_2 leaves Λ invariant. This is equivalent to $(n,m) \in \Lambda$. We assume again without loss of generality that $m \leq n$, but this time we see that either $(n,m) = (-n,m)$ or $(n,m) = (m,n)$. This leads to the cases $n = 0$ and $m = n$ respectively. We can simplify the relators under one of these assumptions, and arrive at the two 1-parameter families $\mathbf{R}_{1.3:n}$, and $\mathbf{R}_{1.4:n}$ (illustrated in Figure 1.9) defined by:

$$\mathrm{Aut}^+(\mathbf{R}_{1.3:n}) = \left\langle R, S \,\middle|\, R^4, S^4, (RS)^2, (RS^{-1})^n \right\rangle,$$
$$\mathrm{Aut}^+(\mathbf{R}_{1.4:n}) = \left\langle R, S \,\middle|\, R^4, S^4, (RS)^2, [R,S]^n \right\rangle.$$

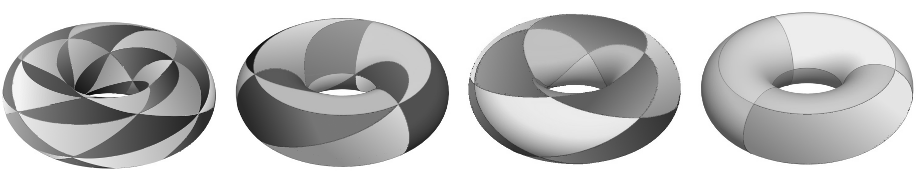


Image courtesy
Jarke J. van Wijk
Eindhoven University of Technology

Figure 1.9: The torus maps $\mathbf{R}_{1.1:3}$, $\mathbf{R}_{1.2:3}$, $\mathbf{R}_{1.3:3}$, and $\mathbf{R}_{1.4:3}$.

2

Polynomial families of platonic maps

THIS chapter contains the defining building blocks for the rest of the thesis: polynomial families of platonic maps. These recur again and again in attempts at classification, examples of which can be found in Chapter 4 or Section 6.3, but to our knowledge, the concept has not been formalized yet. We do so here. In the rest of the chapter we list, in order of increasing complexity, polynomial families of maps that we discovered, and ones that were already known.

The starting point is: can one find any pattern in the somewhat bewildering set of (reflexive) platonic maps? Coxeter and Moser [CM1980] already indicated a few general group-theoretic recipes for constructing whole families of maps. The Wiman type I/II maps, and later on the Accola-Maclachlan maps and Kulkarni maps (see Chapter 5) were discovered as forming sequences, each admitting of one coherent description. From a more geometrical viewpoint, Sequin [Seq2010] recently constructed topological realizations in $\mathbb{R}^3$ of the Wiman type I/II and Accola-Maclachlan maps that not only bring out their combinatorial patterns, but are also beautiful to look at.

Inspired by these results, we stared long and hard at Marston Conder's list of platonic maps of genus at most 101, and found additional patterns. Most importantly, we want to make their existence crystal clear and describe the pattern in an exact way. To this end, we formalize these patterns group-theoretically with the notion of a *polynomial family of platonic maps*. One advantage gained by defining such a family is that it furnishes us with succinct standard map presentations for all members at once. The discovery of a new family also naturally leads to the follow-up questions "Can we visualize this family in an understandable way?" and "Can we find (parametrized) algebraic models for all members of this family?". We have not yet gotten around to exploring those questions.

Definition 2.0.1. *A* (t-parameter) polynomial family of platonic maps $\mathcal{F}_{g(\overline{n})}^{(p(\overline{n}),q(\overline{n}))}$ *is* *a collection of platonic maps $\mathbf{M}(\overline{n})$, indexed by $\overline{n} = (n_1, \ldots, n_t) \in \mathbb{Z}^t_{\geq 0}$ and defined by map automorphism groups*

$$\mathrm{Aut}^+(\mathbf{M}(\overline{n})) = \left\langle R, S \;\middle|\; R^{p(\overline{n})}, S^{q(\overline{n})}, (RS)^2, w_1(\overline{n}), \ldots, w_k(\overline{n}) \right\rangle$$

such that the following conditions hold:

1. *$p(\overline{n}), q(\overline{n}) \in \mathbb{Q}[\overline{n}]$ are integer-valued polynomials;*
2. *every relator $w_i(\overline{n})$ is a finite product of subwords $w_i(\overline{n}) = w_{i1}^{e_{i1}} \cdots w_{im(i)}^{e_{im(i)}}$ with $m(i) \in \mathbb{Z}_{\geq 0}$, the w_{ij} fixed words in the free group on R and S, and $e_{ij} \in \mathbb{Q}[\overline{n}]$ integer-valued polynomials;*
3. *each member $\mathcal{F}^{(p(\overline{n}),q(\overline{n}))}_{g(\overline{n})}(\overline{n})$ is a platonic map of type $(p(\overline{n}), q(\overline{n}))$.*

When there are multiple families with the same parameters, we affix numbers (1), (2) etcetera to the family name, e.g. $\mathcal{F}^{(4,16n)(1)}_{32n-7}$.

If $p(\overline{n})$, $q(\overline{n})$, and all $e_{ij}(\overline{n})$ are constant, we get the same map for all parameter values. This degenerate case shall not be used. The group size $|\mathrm{Aut}^+(\mathbf{M}(\overline{n}))|$ need not be a polynomial. An example occurs in Section 2.10. Given the group size, the genus formula allows one to compute the genera $g(\overline{n})$. The Hurwitz bound implies that $g(\overline{n})$ can only be constant if it is 0 or 1. In Section 1.7 we have already met the polynomial families living within these genera: the maps $\mathbf{Hos}(n)$ and $\mathbf{R}_{1.k:n}$.

To define a polynomial family of maps, it is not enough to simply write down a random set of words $w_i(\overline{n})$. The group

$$\Gamma(\overline{n}) := \langle w_1(\overline{n}), \ldots, w_k(\overline{n}) \rangle^{\Delta^+(p(\overline{n}),q(\overline{n}),2)} \lhd \Delta^+(p(\overline{n}), q(\overline{n}), 2)$$

must act torsion freely for all $\overline{n} \in \mathbb{Z}^t_{\geq 0}$. We also wish to get a good handle on our polynomial families. Specifically, we want to know their group orders, from which we can compute their genera. To prove that all members of a polynomial family have a certain group order is a separate problem in each case; one has to analyze the group structure of $\mathrm{Aut}^+(\mathbf{M}(\overline{n}))$. One way to do this is to look for a normal subgroup $H(\overline{n})$ with quotient $Q(\overline{n}) = \mathrm{Aut}^+(\mathbf{M}(\overline{n}))/H(\overline{n})$, such that both subgroup and quotient are amenable to easy description. The group $\mathrm{Aut}^+(\mathbf{M}(\overline{n}))$ is then an extension of $Q(\overline{n})$. To define this extension, we have several tools at our disposal. If the subgroup is abelian, we can define it with the conjugation action of $Q(\overline{n})$ on $H(\overline{n})$ and a 2-cocycle in $\mathrm{H}^2(Q(\overline{n}), H(\overline{n}))$. For completeness, we introduce these notions in the next section. In most cases, we can even find $Q(\overline{n})$ as a complement inside $\mathrm{Aut}^+(\mathbf{M}(\overline{n}))$ and can use a semi-direct product for description. For some polynomial families of reflexive platonic maps we prove the structural claims of $\mathrm{Aut}^+(\mathbf{M}(\overline{n}))$ by taking recourse to $\mathrm{Aut}(\mathbf{M}(\overline{n}))$. Identities that are useful in all of the proofs are $RS = S^{-1}R^{-1}$ and others derived from the three standard relators. Besides these, the use of the extra relator number one will be indicated in a formula manipulation step by $e1$, etcetera.

Remark 2.0.2. For some polynomial families, one can add or leave out a relator and get another polynomial family. In this way, a whole *family tree* may be constructed.

Our definition might be too general in the following way. The only polynomial families we have found can be described with one or two parameters. Whether any family exists that can not be described with fewer than three parameters is doubtful: varying the type (p, q) requires at most two parameters and it seems unlikely that $\Delta^+(p, q, 2)$ contains a sequence of normal subgroups parametrizable by some words with polynomial exponents in the two generators. But who knows?

We could have made our definition more general in the following way. The relators we allow for a polynomial family are built up as a finite product of words whose base is a constant word and exponent a polynomial. These expressions have "depth" 1. We could also have allowed finite products of powers with polynomial exponent but base a depth 1 expression, forming depth 2 expressions, and so on. The general class of polynomial words of depth n is then defined recursively starting from $\{R, S\}$ and building up with either a product operation or taking a polynomial power. But since we have not discovered families of higher depth, we have no need for this generality.

2.1 Group extensions

The group structure of a polynomial family may involve a non-split group extension. We list the concepts relevant to us here. See [Rob1982, Ch.11] for a thorough study.

Definition 2.1.1. *A group G is called a* group extension *of the group Q by the group N if there is an exact sequence*

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{\pi} Q \longrightarrow 1$$

In this situation $N \leq G$. Let us identify N with its image $i(N)$ for simplicity. Because $N = \ker(\pi)$ it is normal in G. A set-theoretic function $\sigma : Q \to G$ for which $\sigma\pi = 1$ is called a *transversal function*. If one can find a transversal function that is a homomorphism, or in other words if σ is a section of π, then we call the sequence *split*. We can then write G as a semi-direct product $G \cong N \rtimes_c Q$ with the conjugation action c on N defined by $c(g) : n \mapsto \sigma(g)^{-1} n \sigma(g)$.

If the sequence is not split, but N is abelian, then any transversal function still gives us a homomorphism $G \to \mathrm{Aut}(N)$ because the conjugation action of an element $g \in G$ on N is well-defined and depends only on the coset Ng. But we need the extra information of a 2-*cocycle* to specify the extension. Because we shall need more terminology from this field in Chapter 6, we give a slightly broader setup. We write the operation of N as addition.

Definition 2.1.2. *Let N, Q be groups, N abelian, with a homomorphism $\sigma : Q \to \mathrm{Aut}(N)$. The* cochain group *$C^k(Q, N)$ is the set of all maps $Q^k \to N$ with pointwise* *addition. The* coboundary operator $\partial = \partial_q : C^k(Q, N) \to C^{k+1}(Q, N)$ *is defined by*

$$\partial(c)(q_1, \ldots, q_{k+1}) := c(q_2, \ldots, q_{k+1})^{q_1} + \sum_{i=1}^{i=k} (-1)^i c(q_1, \ldots, q_i q_{i+1}, \ldots, q_{k+1}) + (-1)^{k+1} c(q_1, \ldots, q_k).$$

We define the k-cocycle group $Z^k(Q, N)$ *to be* $\ker(\partial_k) \le C^k(Q, N)$. *The* k-coboundary group $B^k(Q, N)$ *is* $\mathrm{im}(\partial_{k-1}) \le C^k(Q, N)$.

The coboundary operator is a chain map, i.e. $\partial \circ \partial = 0$. Hence $B^k(Q, N) \le Z^k(Q, N)$ and we can define the k-th *cohomology group*

$$H^k(Q, N) := Z^k(Q, N)/B^k(Q, N).$$

In the special case $k = 2$, we find the following condition on a 2-chain c for it to be a 2-cocycle:

$$c(q_2, q_3)^{q_1} - c(q_1 q_2, q_3) + c(q_1, q_2 q_3) - c(q_1, q_2) = 0 \qquad (\forall q_1, q_2, q_3 \in Q).$$

Fact 2.1.3. *A* 2*-cocycle* $c \in Z^2(Q, N)$ *determines a group extension by the multiplication law*

$$(q_1, n_1) \cdot (q_2, n_2) = (q_1 q_2, c(q_1, q_2) + n_1^{q_2} + n_2) \qquad (q_i \in Q, n_i \in N).$$

Two 2-cocycles determine the same group extension if they differ by a 2-coboundary, and we have a bijection between the extensions of Q by an abelian group N with fixed conjugation action (Q-module structure) and the cohomology group $H^2(Q, N)$.

Fact 2.1.4. *The group extension determined by a* 2*-cocycle is split if and only if the* 2*-cocycle is a coboundary.*

Below, we shall write down a 2-cocycle $C = (c(q_1, q_2))_{q_1, q_2 \in Q}$ as an array of group elements.

2.2 Polynomial families with $v = 1$

2.2.1 — The polynomial family Wi1$(n) = \mathcal{F}_n^{(2n+1,4n+2)}$ (Wiman type I maps)

Proposition 2.2.1. *The group*

$$G(n) = \langle R, S \mid R^{2n+1}, S^{4n+2}, (RS)^2, RS^{-2n} \rangle$$

is isomorphic to $\mathbb{Z}_{4n+2}$*. These groups define a polynomial family of reflexive platonic maps with* $v = 1$*,* $e = 2n + 1$*, and* $f = 2$*.*

Proof. The extra relator shows that $R \in \langle S \rangle$. Moreover, it implies $R^{2n+1} = S^{2n(2n+1)} = 1$ and $(RS)^2 = S^{4n+2} = 1$, so the presentation is equivalent to $\langle S \mid S^{4n+2} \rangle$. To show reflexivity we compute that $R^{-1}S^{2n} \overset{e1}{=} S^{-2n}S^{2n} = 1$ and apply the chirality criterion. ☐

Remark 2.2.2. In fact, $\mathrm{Aut}(\mathcal{F}_n^{(2n+1,4n+2)}(n)) \cong \mathrm{Dih}_{8n+4}$. A few members of this family from the Conder list are: $\mathbf{Dih}(1)$, $\mathbf{R}_{1.2:1}$, $\mathbf{R}_{2.4}$, $\mathbf{R}_{3.9}$.

2.2.2 — The polynomial family $\mathcal{F}_n^{(4n,4n)}$ (Diagonal Wiman type II maps)

Proposition 2.2.3. *The group*

$$G(n) = \left\langle R, S \;\middle|\; R^{4n}, S^{4n}, (RS)^2, RS^{-(2n-1)} \right\rangle$$

is isomorphic to $\mathbb{Z}_{4n}$. *These groups define a polynomial family of reflexive platonic maps with* $v = 1$, $e = 2n$ *and* $f = 1$.

Proof. The extra relator shows that $R \in \langle S \rangle$. Moreover, it implies $R^{4n} = S^{4n(2n-1)} = 1$ and $(RS)^2 = S^{4n} = 1$, so the presentation is equivalent to $\langle S \mid S^{4n} \rangle$. ☐

Remark 2.2.4. In fact, $\mathrm{Aut}(\mathbf{R}) \cong \mathrm{Dih}_{8n}$. A few members of this family are: $\mathbf{R}_{1.3:1}$, $\mathbf{R}_{2.6}$, $\mathbf{R}_{3.12}$. For $n = 0$ we get the cell decomposition of the sphere without 1-cells, which we have excluded as a map because it has no edges. These maps are the D_1-maps (cf. Chapter 3) of the polynomial family $\mathbf{Wi2}(n) = \mathcal{F}_n^{(4,4n)}$ called the "Wiman type II maps". That name will be clarified in Chapter 5. The present family thus consists of "diagonal Wiman type II maps". The Wiman type II maps satisfy $v = 2$. A visualization of the first few is shown in Figure 2.1. Such visualizations bring out how a pattern in group structure translates into a pattern in topological structure.

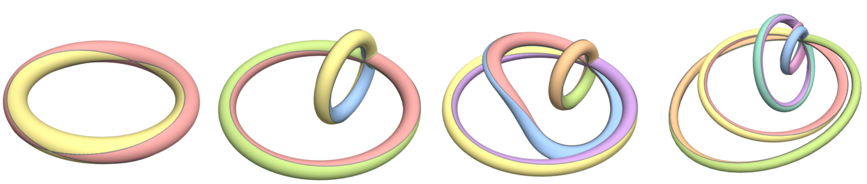


Figure 2.1: The maps $\mathbf{R}_{1.4:1}$, $\mathbf{R}_{2.3}$, $\mathbf{R}_{3.7}$, and $\mathbf{R}_{4.5}$ of the polynomial family $\mathcal{F}_n^{(4,4n)}$ of Wiman type II maps.

2.3 Polynomial families with $v = 2$

All maps with two vertices are determined by two parameters (q, k), as derived in Section 4.2. Any 2-vertex map $\mathbf{R}$ has a standard map presentation of the form

$$\mathrm{Aut}^+(\mathbf{R}) = \left\langle R, S \,\middle|\, R^{p(q)}, S^q, (RS)^2, R^{-1}SRS^{-k} \right\rangle.$$

The number of polynomial families within this set is most likely infinite. The reason to indicate some of them is that these polynomial families can appear in the solution to other classification problems. Because of Theorem 4.2.1, their polynomiality is clear once the relations $k^2 \equiv 1 \bmod q$ and $p = 2q/\gcd(k+1, q)$ have been verified. In Table 2.1 we list the families we found by the parameters (p, q, k) and the genus g for each family.

2.4 Polynomial families with $v = 4$

2.4.1 — The polynomial family $\mathrm{AM}(n) = \mathcal{F}_{n-1}^{(4,2n)}$ (Accola-Maclachlan maps)

This family will be encountered again in Chapters 5 and 6. Its Riemann surface realizations are the Accola-Maclachlan curves. In this chapter we only present a group-theoretical analysis. We also note that this is a subfamily of Coxeter's family $\{p, q \mid 2\}$, presented later as family 2.10.3.

Proposition 2.4.1. *The groups*

$$G(n) = \left\langle R, S \,\middle|\, R^4, S^{2n}, (RS)^2, (RS^{-1})^2 \right\rangle$$

have order $8n$. They define a polynomial family of reflexive platonic maps with $v = 4$, $e = 4n$, $f = 2n$.

Proof. The subgroup $H(n) := \langle R^2, S\rangle$ is normal in $G(n)$ because clearly $S^{-1}R^2S \in H(n)$ and $R^{-1}SR = R^{-2}S^{-1} \in H(n)$. It has the complement $\langle RS\rangle$. Moreover,

$$[R^2, S] = R^{-2}S^{-1}R^2S = R(RS^{-1}R)RS \overset{e1}{=} RSRS = 1$$

and we conclude that $G(n) = (\langle R^2\rangle \times \langle S\rangle) \rtimes \langle RS\rangle \cong (\mathbb{Z}_2 \times \mathbb{Z}_{2n}) \rtimes \mathbb{Z}_2$ of order $8n$. The conjugation action that specifies the semi-direct product is

$$\mathrm{con}_{RS} : \; (R^2, S) \mapsto (R^2, R^2S^{-1}) \qquad \square$$

The proof shows that the subgroup $\langle S, R^{-1}SR\rangle = \langle S, R^{-2}S^{-1}\rangle = H(n)$ has index 2 in $G(n)$. Therefore, each member has a diagonal map. The resulting diagonal family of type $(2n, 2n)$ satisfies $v = 2$ and $[R, S] = 1$, see Table 2.1. Some members from the

$p(n)$	$q(n)$	$k(n)$	$g(n)$	Conder list maps
2	n	-1	0	$\mathbf{Hos}(n)$
4	$4n$	$2n-1$	n	$\mathbf{R}_{1.4:1}, \mathbf{R}_{2.3}, \mathbf{R}_{3.7}$
6	$9n-6$	$6n-5$	$3n-2$	$\mathbf{R}^{\vee}_{1.2:1}, \mathbf{R}_{4.9}, \mathbf{R}_{7.8}$
6	$9n-3$	$3n-2$	$3n-1$	$\mathbf{R}_{2.5}, \mathbf{R}_{5.11}, \mathbf{R}_{8.6}$
10	$25n-20$	$10n-9$	$10n-8$	$\mathbf{R}^{\vee}_{2.4}, \mathbf{R}_{12.6}, \mathbf{R}_{22.14}$
10	$25n-15$	$5n-4$	$10n-6$	$\mathbf{R}_{4.11}, \mathbf{R}_{14.9}, \mathbf{R}_{24.8}$
10	$25n-10$	$20n-9$	$10n-4$	$\mathbf{R}_{6.10}, \mathbf{R}_{16.13}, \mathbf{R}_{26.11}$
10	$25n-5$	$15n-4$	$10n-2$	$\mathbf{R}_{8.8}, \mathbf{R}_{18.6}, \mathbf{R}_{28.29}$
$2n$	$2n$	1	$n-1$	$\mathbf{Hos}(2), \mathbf{R}_{1.4:1}, \mathbf{R}_{2.5}$
$4n-2$	$8n-4$	$4n-1$	$4n-4$	$\mathbf{Hos}(4), \mathbf{R}_{4.9}, \mathbf{R}_{8.8}$
$6n-4$	$18n-12$	$12n-7$	$9n-9$	$\mathbf{Hos}(6), \mathbf{R}_{9.25}, \mathbf{R}_{18.8}$
$6n-2$	$18n-6$	$6n-1$	$9n-6$	$\mathbf{R}_{3.7}, \mathbf{R}_{12.6}, \mathbf{R}_{21.31}$
$8n-6$	$32n-24$	$24n-31$	$16n-16$	$\mathbf{Hos}(8), \mathbf{R}_{16.13}, \mathbf{R}_{32.9}$
$8n-2$	$32n-8$	$8n-1$	$16n-8$	$\mathbf{R}_{8.6}, \mathbf{R}_{24.10}, \mathbf{R}_{40.17}$
$8n$	$8n$	$4n+1$	$4n-1$	$\mathbf{R}_{3.10}, \mathbf{R}_{7.11}, \mathbf{R}_{11.13}$
$12n-10$	$18n-15$	$6n-4$	$9n-9$	$\mathbf{Hos}(3), \mathbf{R}_{9.29}, \mathbf{R}_{18.11}$
$12n-2$	$18n-3$	$12n-1$	$9n-3$	$\mathbf{R}_{6.10}, \mathbf{R}_{15.19}, \mathbf{R}_{24.14}$
$16n-4$	$32n-8$	$24n-13$	$16n-6$	$\mathbf{R}_{10.21}, \mathbf{R}_{26.13}$
$16n$	$32n$	$8n-9$	$16n-14$	$\mathbf{R}_{2.3}, \mathbf{R}_{18.9}$
$20n-18$	$50n-45$	$30n-26$	$25n-25$	$\mathbf{Hos}(5), \mathbf{R}_{25.40}, \mathbf{R}_{50.14}$
$20n-14$	$50n-35$	$10n-6$	$25n-20$	$\mathbf{R}_{5.11}, \mathbf{R}_{30.9}, \mathbf{R}_{55.53}$
$20n-6$	$50n-15$	$40n-11$	$25n-10$	$\mathbf{R}_{15.17}, \mathbf{R}_{40.20}, \mathbf{R}_{65.138}$
$20n-2$	$50n-5$	$20n-1$	$25n-5$	$\mathbf{R}_{20.9}, \mathbf{R}_{45.34}, \mathbf{R}_{70,14}$
$24n-16$	$72n-48$	$12n-7$	$36n-27$	$\mathbf{R}_{9.24}, \mathbf{R}_{45.32}, \mathbf{R}_{81.173}$
$24n-8$	$72n-24$	$60n-19$	$36n-15$	$\mathbf{R}_{21.30}, \mathbf{R}_{57.66}, \mathbf{R}_{93.25}$
$2n-2$	n^2-1	n	$\frac{1}{2}(n+1)(n-2)$	[CM1980]
$2n+2$	n^2-1	$-n$	$\frac{1}{2}n(n-1)$	[CM1980]

Table 2.1: Several polynomial families of (reflexive) platonic maps with $v = 2$.

Conder list are: $\mathbf{Dih}(4)$, $\mathbf{R}_{1.3:2}$,$\mathbf{R}_{2.2}$, $\mathbf{R}_{3.6}$, $\mathbf{R}_{4.4}$; four of them are shown in Figure 2.2.

2.4.2 — The polynomial family $\mathbf{Kul}(n) = \mathcal{F}^{(4,8n)}_{4n-1}$ (Kulkarni maps)

The maps of this family correspond to the Kulkarni curves, also discussed in Chapters 5 and 6.

Proposition 2.4.2. *The groups*

$$G(n) = \left\langle R, S \,\middle|\, R^4, S^{8n}, (RS)^2, R^{-2}SR^2S^{-(4n+1)} \right\rangle$$

have order $32n$. They define a polynomial family of reflexive platonic maps with $v = 4$, $e = 16n$, $f = 8n$.

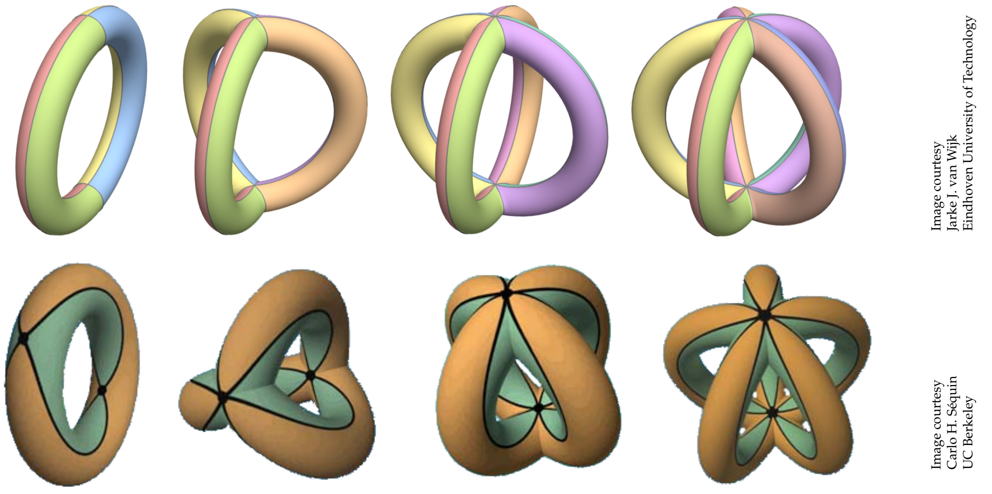


Figure 2.2: The maps $\mathbf{R}_{1.3:2}$, $\mathbf{R}_{2.2}$, $\mathbf{R}_{3.6}$, and $\mathbf{R}_{4.4}$ of the polynomial family $\mathcal{F}^{(4,2n)}_{n-1}$ of Accola-Maclachlan maps (top row) and their D_1-maps (bottom row).

Proof. Just like for the Accola-Maclachlan maps, the subgroup $H(n) := \langle R^2, S\rangle$ is normal in $G(n)$ because clearly $S^{-1}R^2S \in H(n)$ and $R^{-1}SR = R^{-2}S^{-1} \in H(n)$. It has the complement $\langle RS\rangle$, which acts by the following conjugations:

$$RS(R^2)RS = RSR^{-1}S = S^{-1}R^{-2}S,$$
$$RS(S)RS = RSR^{-1} = S^{-1}R^{-2}.$$

Mimicking this structure, we can instantiate $H(n)$ as a semi-direct product $\mathbb{Z}_{8n} \rtimes \mathbb{Z}_2$ with cyclic generators x, y respectively and conjugation action $y^{-1}xy = x^{-(4n+1)}$. This group can be extended by $\mathbb{Z}_2 = \langle z\rangle$ with conjugation action $z^{-1}xz = x^{-1}y^{-1}$ and $z^{-1}yz = x^{-1}y^{-1}x$. Setting $R := zx^{-1}$ and $S := x$, we get back our original relators, whence

$$G(n) = (\langle S\rangle \rtimes \langle R^2\rangle) \rtimes \langle RS\rangle \cong (\mathbb{Z}_{8n} \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2.$$

The parameters are thus proved correct. □

Some members from the Conder list are: $\mathbf{R}_{3.5}$, $\mathbf{R}_{7.3}$, $\mathbf{R}_{11.2}$, $\mathbf{R}_{15.6}$.

2.5 Polynomial families with $v = 6$

2.5.1 — The polynomial family $\mathcal{F}^{(6,2n)}_{2n-2}$ of type $(6, g+2)$

This family is also a subfamily of the family 2.10.3.

Proposition 2.5.1. *The groups*

$$G(n) := \left\langle a, b, c \,\middle|\, a^2, b^2, c^2, (ab)^6, (bc)^{2n}, (ca)^2, (abcb)^2 \right\rangle$$

have order $24n$. *They define a polynomial family of reflexive platonic maps of genus* $2n-2$ *and type* $(6, 2n)$, *with parameters* $v = 6$, $e = 6n$, *and* $f = 2n$. *Furthermore,*

$$\mathrm{Aut}^+(\mathbf{R}(n)) = \left\langle R, S \,\middle|\, R^6, S^{2n}, (RS)^2, (RS^{-1})^2 \right\rangle .$$

Proof. The subgroup $\langle (ab)^3 c \rangle$ of $G(n)$ is normal by the following computations:

$$\begin{aligned}
a((ab)^3c)a &= bababac = (ab)^{-3}c = (ab)^3c, \\
b((ab)^3c)b &= baba(babcb) \overset{e1}{=} babacba \overset{e1}{=} cbababa = ((ab)^3c)^{-1}, \\
c((ab)^3c)c &= c(ab)^3 = c(ab)^{-3} = ((ab)^3c)^{-1}.
\end{aligned}$$

It has the complement $\langle a, b \rangle \cong \mathrm{Dih}_{12}$, as can readily be checked by computing a presentation for the quotient $G(n)/H(n)$. The conjugation action was just computed.

Taking $\mathbb{Z}_{2n} = \langle x \rangle$ and $\mathrm{Dih}_{12} = \left\langle a, b \,\middle|\, a^2, b^2, (ab)^6 \right\rangle$ with conjugation action of the latter on x analogous to the one above, we get a semi-direct product with the presentation of $G(n)$ by defining $c := (ab)^{-3}x$:

$$\begin{aligned}
&c^2 = (ab)^{-3}x(ab)^3x = x^{-1}x = 1, \\
&ac = a(ab)^3x = (ba)^3ax = (ab)^3xa = ca, \\
&(bc)^2 = (b(ab)^3x)^2 = (ab)^3bxb(ab)^3x = (ab)^3x^{-1}(ab)^3x = x^2 \implies (bc)^{2n} = 1, \\
&(abcb)^2 = (ab)^4xb(ab)^4xb = (ab)^4x(ba)^4bxb = xbxb = xx^{-1} = 1.
\end{aligned}$$

This proves that

$$G(n) = \langle (ab)^3c \rangle \rtimes \langle a, b \rangle \cong \mathbb{Z}_{2n} \rtimes \mathrm{Dih}_{12}$$

is of order is $24n$. The other claims are now straightforward. □

A few members from the Conder list are: $\mathbf{Dih}(6)$, $\mathbf{R}^{\vee}_{2.2}$, $\mathbf{R}_{4.7}$, $\mathbf{R}_{6.7}$.

2.5.2 — The polynomial family $\mathcal{F}^{(6,3n)}_{3n-2}$ of type $(6, g+2)$

Proposition 2.5.2. *The groups*

$$G(n) = \left\langle R, S \,\middle|\, R^6, S^{3n}, (RS)^2, [R^2, S], (RS^{-2})^2 \right\rangle$$

have order $18n$. *They define a polynomial family of reflexive platonic maps of genus* $3n-2$ *and type* $(6, 3n)$. *The maps have parameters* $v = 6$, $e = 9n$, *and* $f = 3n$.

Proof. Let $H(n) := \langle R^2, S\rangle$, which is abelian by relator (e1). Some deliberation tells one that $H(n) \lhd G(n)$, the most difficult part of which is

$$R^{-1}SR = R^{-2}S^{-1} \in H(n).$$

Clearly, $H(n)$ has the complement $\langle RS\rangle$. We now set $\mathbb{Z}_3 = \langle x\rangle$, $\mathbb{Z}_{3n} = \langle y\rangle$, and $\mathbb{Z}_2 = \langle z\rangle$, we form $(\mathbb{Z}_3\times\mathbb{Z}_{3n})\rtimes\mathbb{Z}_2$ with conjugation action $\mathrm{con}_z : (x, y) \mapsto (x, y^{-1}x^{-1})$. Then (think $zy^{-1} = R$ and $y = S$)

$$\begin{aligned}
&(zy^{-1})^2 = (zy^{-1}z)y^{-1} = yxy^{-1} = x \implies (zy^{-1})^6 = 1,\\
&(zy^{-1})^2 = xy = yx = y(zy^{-1})^2 \text{ and}\\
&(zy^{-3})^2 = (zy^{-3}z)y^{-3} = (xy)^3y^{-3} = x^3y^3y^{-3} = 1
\end{aligned}$$

show that $\langle zy^{-1}, y\rangle$ has our original presentation for $\langle R, S\rangle$. We conclude that

$$G(n) = (\langle R^2\rangle \times \langle S\rangle) \rtimes \langle RS\rangle \cong (\mathbb{Z}_3 \times \mathbb{Z}_{3n}) \rtimes \mathbb{Z}_2.$$

The resulting platonic map is reflexive, since $(R^{-1}S^2)^2 = (R^{-1}(RS^{-2})^{-1}R)^2 = 1$ and obviously $[R^{-2}, S^{-1}] = 1$. The other claims are straightforward. □

A few members from the Conder list are: $\mathbf{R}^{\vee}_{1.1:1}$, $\mathbf{R}_{4.8}$, $\mathbf{R}_{7.6}$, $\mathbf{R}_{10.18}$, $\mathbf{R}_{13.12}$.

2.6 Polynomial families with $v = 8$

2.6.1 — The polynomial family $\mathcal{F}^{(6,9n-3)}_{12n-7}$ of type $(6, \frac{3}{4}(g+3))$

Proposition 2.6.1. *The groups*

$$G = \left\langle R, S \,\middle|\, R^6, S^{9n-3}, (RS)^2, (RS^{-2})^2, R^{-3}SR^3S^{-(3n-2)} \right\rangle$$

have order $24(3n-1)$*. They define a polynomial family of reflexive platonic maps of genus* $12n-7$ *and type* $(6, 9n-3)$*. The maps have parameters* $v = 8$*,* $e = 12(3n-1)$*, and* $f = 4(3n-1)$*.*

Proof. Note that

$$R^{-1}S^3R = R^{-1}S^2SR \overset{e1}{=} S^{-2}RSR = S^{-3},$$

so $\langle S^3\rangle \lhd G(n)$. The quotient has presentation

$$G(n)/\langle S^3\rangle = \left\langle \overline{R}, \overline{S} \,\middle|\, \overline{R}^6, \overline{S}^3, (\overline{RS})^2, (\overline{RS}^{-2})^2, [\overline{R}^3, \overline{S}] \right\rangle.$$

The quotient group turns out to be isomorphic to $\mathrm{Alt}_4 \times \mathbb{Z}_2$, the center being $\langle \overline{R}^3\rangle$. Some more manipulation shows that this quotient group can be realized as the complement $\langle R^2, RS\rangle$, and so

$$G(n) = \langle S^3\rangle \rtimes \langle R^2, RS\rangle \cong \mathbb{Z}_{3n-1} \rtimes (\mathrm{Alt}_4 \times \mathbb{Z}_2)$$

with conjugation action defined by $\mathrm{con}_{R^2} : S^3 \mapsto S^3$ and $\mathrm{con}_{RS} : S^3 \mapsto S^{-3}$. The fact that the maps are reflexive follows from $(R^{-1}S^2)^2 = (R^{-1}(RS^{-2})^{-1}R)^2 = 1$ and taking the inverse of the second extra relator. The other claims are straightforward. □

A few members from the Conder list are: $\mathbf{R}_{5.10}$, $\mathbf{R}_{17.23}$, $\mathbf{R}_{29.13}$, $\mathbf{R}_{41.34}$.

2.6.2 — The polynomial family $\mathcal{F}^{(4,3n)}_{3n-3}$ of type $(4, g+3)$

Proposition 2.6.2. *The groups*

$$G(n) = \left\langle R, S \,\middle|\, R^4, S^{3n}, (RS)^2, (RS^{-2})^2 \right\rangle$$

have order $24n$*. They define a polynomial family of reflexive platonic maps of genus* $3n-3$ *and type* $(4, 3n)$*. These maps have parameters* $v = 8$*,* $e = 12n$*, and* $f = 6n$*.*

Proof. We derive

$$R^{-1}S^3R = (R^{-1}S^2)SR \overset{e1}{=} S^{-2}RSR = S^{-2}S^{-1} = S^{-3}.$$

This shows that $\langle S^3 \rangle \lhd G(n)$. Computing a presentation for the quotient yields quickly that $G(n)/\langle S^3 \rangle \cong \mathrm{Sym}_4$. We have shown the existence of an exact sequence

$$1 \longrightarrow \langle S^3 \rangle \longrightarrow G(n) \longrightarrow \mathrm{Sym}_4 \longrightarrow 1.$$

When $3 \nmid n$, then $\langle S^3 \rangle$ has $\langle R, S^n \rangle$ as a complement: the computations

$$(RS^n)^2 = RS^{n-1}SRSS^{n-1} = RS^{n-1}R^{-1}S^{n-1} = S^{-(n-1)}S^{n-1} = 1 \qquad \text{if } n \equiv 1 \bmod 3$$
$$(RS^{-n})^2 = RS^{-n-1}SRSS^{-n-1} = RS^{-n-1}R^{-1}S^{-n-1} = S^{n+1}S^{-n-1} = 1 \qquad \text{if } n \equiv 2 \bmod 3$$

together with $R^4 = (S^n)^3 = 1$ show that $|\langle R, S^n \rangle| \leq 24$, but all cosets of $\langle S^3 \rangle$ are readily checked to contain a representative of $\langle R, S^n \rangle$. Therefore $\langle R, S^n \rangle \cong \mathrm{Sym}_4$ and the group $\mathbb{Z}_n \rtimes \mathrm{Sym}_4$ are seen to have the presentation of $G(n)$ when taking the conjugation action following from the above. Hence, whenever $3 \nmid n$, the exact sequence splits and

$$G(n) = \langle S^3 \rangle \rtimes \langle R, S^n \rangle \cong \mathbb{Z}_n \rtimes \mathrm{Sym}_4.$$

All claims then follow. But the case $3 \mid n$ needs special treatment. We deduce

$$R^{-2}S^3R^2 = R^{-1}S^{-3}R = S^3 \qquad \Longrightarrow \qquad [R^2, S^3] = 1,$$
$$S^{-3}[R^2, S]S^3 = [R^2, S]S^{-3}S^3 = [R^2, S] \qquad \Longrightarrow \qquad [[R^2, S], S^3],$$
$$[S, R^2] = S^{-1}R^{-2}SR^2 = S^{-1}RS^{-1}R \overset{e1}{=} SR^{-1}SR = SR^{-2}S^{-1} = SR^2S^{-1} =$$
$$= R^{-1}S^{-1}RS^{-1} \overset{e1}{=} R^{-1}SR^{-1}S = R^{-2}S^{-1}R^2S = [R^2, S],$$
$$R^2[R^2, S]R^2 = S^{-1}R^2SR^2 = [S, R^2] = [R^2, S].$$

It follows that $H(n) = \langle R^2, [R^2, S], S^3\rangle$ is abelian. We find the quotient $G(n)/H(n) \cong \mathrm{Sym}_3$ and with some more work the conjugation action

$$\mathrm{con}_R : (R^2, [R^2, S], S^3) \mapsto (R^2, [S, R^2], S^{-3})$$
$$\mathrm{con}_S : (R^2, [R^2, S], S^3) \mapsto ([S, R^2], R^2, S^3)$$

and 2-cocycle (rows and columns indexed by $([1], [S^{-1}], [R], [S], [RS^{-1}], [RS])$):

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & S^{-3} & RSR^{-1}S & \cdot & RSR^{-1}S & RSR^{-1}S^{-2} \\ \cdot & \cdot & R^2 & \cdot & R^2 & R^2 \\ \cdot & \cdot & R^2 & S^3 & R^2S^3 & R^2 \\ \cdot & S^3 & RSR^{-1}S & \cdot & RSR^{-1}S & RS^{-1}RS^2 \\ \cdot & \cdot & \cdot & S^{-3} & S^{-3} & \cdot \end{bmatrix}$$

Taking the group $\mathbb{Z}_2^2 \times \mathbb{Z}_n$ and constructing an extension by the above cocycle and conjugation action yields a group with our presentation, so $|G(n)| = 24n$ and the other parameters follow. Admittedly, this description also works when $3 \nmid n$, but is more involved than the analysis as a direct product.

Reflexivity of all resulting platonic maps is a consequence of the fact that $(R^{-1}S^2)^2 = (R^{-1}(RS^{-2})^{-1}R)^2 = 1$. □

Some members from the Conder list are: **Cub**, $\mathbf{R}_{3.4}$, $\mathbf{R}_{6.3}$, $\mathbf{R}_{9.11}$, $\mathbf{R}_{12.1}$, $\mathbf{R}_{15.5}$.

Each family member $\mathbf{R}(n)$ is a platonic cover of $\mathcal{F}_{3n-3}^{(4,3n)}(1) = \mathbf{Cub}$: form the quotient map $\mathbf{R}(n)/\langle S^3\rangle$. Computing a presentation for the quotient $G(n)/\langle S, R^{-1}SR\rangle^{G(n)}$ shows quickly that $[G(n) : \langle S, R^{-1}SR\rangle] = 2$. By Theorem 3.1.1, every family member $\mathbf{R}(n)$ has a diagonal map $D(\mathbf{R}(n))$ of type $(3n, 3n)$. This gives rise to a new reflexive polynomial family $\mathcal{F}_{3n-3}^{(3n,3n)}$ with standard map presentation

$$\mathrm{Aut}^+(D(\mathbf{R}(n))) = \left\langle R, S \;\middle|\; R^{3n}, S^{3n}, (RS)^2, R^3S^3 \right\rangle$$

of order $12n$ and parameters $v = 4$, $e = 6n$, $f = 4$. Some members from the Conder list are: **Tet**,$\mathbf{R}_{3.8}$, $\mathbf{R}_{6.9}$, $\mathbf{R}_{9.27}$, $\mathbf{R}_{12.8}$, $\mathbf{R}_{15.18}$. A map $\mathbf{R}(n)$ of this new family is an n-cover of $\mathcal{F}_{3n-3}^{(3n,3n)}(1) = \mathbf{Tet}$: form the quotient map $\mathbf{R}(n)/\langle R^3, S^3\rangle = \mathbf{R}(n)/\langle S^3\rangle$.

2.6.3 — The polynomial family $\mathcal{F}_{4n-3}^{(4,4n)}$ of type $(4, g + 3)$

Proposition 2.6.3. *The groups*

$$G(n) = \left\langle R, S \;\middle|\; R^4, S^{4n}, (RS)^2, [R^2, S^2] \right\rangle$$

are of order $32n$. They define a polynomial family of reflexive platonic maps of genus $4n-3$ and type $(4, 4n)$. The maps have parameters $v = 8$, $e = 16n = 4g + 12$, $f = 8n = 2g + 6$.

Proof. Note first that $RS^{-2}R = R^{-1}R^2S^{-2}R = R^{-1}S^{-2}R^2R = R^{-1}S^{-1}S^{-1}R^{-1} = [R,S]$. From this we deduce four identities:

$$\begin{aligned}
R^{-1}S^2R &= R^{-1}S^2R^{-1}R^2 = [R,S]^{-1}R^2\\
R^{-1}[R,S]R &= R^{-2}S^{-1}(RSR) = R^{-2}S^{-2}\\
S^{-1}R^2S &= S^{-1}R^{-1}R^{-1}S = S^{-1}R^{-1}SRS^2 = [R,S]^{-1}S^2\\
S^{-1}[R,S]S &= S^{-1}R^{-1}S^{-1}RS^2 = RSS^{-1}RS^2 = R^2S^2
\end{aligned}$$

Together, these prove that $H(n) := \langle R^2, S^2, [R,S]\rangle \lhd G(n)$. The quotient is quickly seen to be $\mathbb{Z}_2^2$. The structure of $H(n)$ is analyzed by computing

$$\begin{aligned}
R^{-2}[R,S]R^2 &= R^{-2}R^{-1}S^{-1}RSR^2 = RS^{-1}S^{-1}R^{-1}R^2 = RS^{-2}R = [R,S]\\
S^{-2}[R,S]S^2 &= S^{-2}(SR^2S)S^2 = S^{-2}S^2(SR^2S) = [R,S]
\end{aligned}$$

This shows that $H(n)$ is abelian. Now $[R,S]^2 = SR^2SSR^2S = SS^2R^2R^2S = S^4$ tells us that $([R,S]S^{-2})^2 = 1$ and hence $H(n)$ is a quotient of $\langle R^2\rangle \times \langle [R,S]S^{-2}\rangle \times \langle S^2\rangle$. The extension

$$1 \longrightarrow H(n) \longrightarrow G(n) \longrightarrow \mathbb{Z}_2^2 \longrightarrow 1$$

is described by the following conjugation action (as just proven) and 2-cocycle (by some additional effort, rows/columns indexed by $([1],[R],[S],[RS])$):

$$\begin{array}{l}
\mathrm{con}_R : (R^2,[R,S]S^{-2},S^2) \mapsto (R^2,[R,S]S^{-2},R^2S^{-2})\\
\mathrm{con}_S : (R^2,[R,S]S^{-2},S^2) \mapsto ([R,S]^{-1}S^2,R^2,S^2)
\end{array}
\qquad
\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot\\
\cdot & R^2 & \cdot & R^2\\
\cdot & RS^{-2}R & S^2 & R^2\\
\cdot & S^{-2} & RS^2R^{-1} & \cdot
\end{bmatrix}$$

Defining the extension of $\mathbb{Z}_2^2 \times \mathbb{Z}_{2n}$ by the analogous conjugation and 2-cocycle gives it the presentation above and this completes our analysis: $|G(n)| = 32n$, $\mathrm{ord}(R) = 4$, $\mathrm{ord}(S) = 4n$ and the other parameters follow. It is immediate from the presentation that the chirality criterion is satisfied. □

Members from the Conder list include: $\mathbf{R}_{1.4:2}$, $\mathbf{R}_{5.6}$, $\mathbf{R}_{9.10}$, $\mathbf{R}_{13.4}$.

A computation of a presentation for $G(n)/\langle S, R^{-1}SR\rangle^{G(n)}$ yields that the index $[G(n) : \langle S, R^{-1}SR\rangle] = 2$, so each family member $\mathbf{R}(n)$ has a diagonal map $D(\mathbf{R}(n))$. This gives rise to a diagonal family $\mathcal{F}_{4n-3}^{(4n,4n)}$ with standard map presentation

$$\mathrm{Aut}^+(D(\mathbf{R}(n))) = \left\langle R,S \,\middle|\, R^{4n}, S^{4n}, (RS)^2, [R,S^2], [R^2,S], R^4S^4 \right\rangle$$

of order $16n$ and parameters $v = 4$, $e = 8n = 2g+6$, $f = 4$. (We could leave out the third extra relator.) Some members from the Conder list are: $\mathbf{R}_{1.3:2}$, $\mathbf{R}_{5.13}$, $\mathbf{R}_{9.28}$, $\mathbf{R}_{13.19}$.

2.7 Polynomial families with $v = 16$

2.7.1 — The polynomial family $\mathcal{F}_{8n-7}^{(4,4n)}$ of type $(4, \frac{1}{2}(g+7))$

Proposition 2.7.1. *The groups*

$$G(n) = \langle R, S \mid R^4, S^{4n}, (RS)^2, (RS^{-1})^4, (RS^{-3})^2 \rangle$$

have order $64n$*. They define a polynomial family of reflexive platonic maps of genus* $8n-7$ *and type* $(4, 4n)$*. These maps have parameters* $v = 16$*,* $e = 32n = 4(g+7)$*,* $f = 16n = 2(g+7)$*.*

Proof. We deduce

$$R^{-1} = SRS \overset{e2}{=} S(S^3R^{-1}S^3)S = S^4R^{-1}S^4 \quad \Longrightarrow \quad R^{-1}S^4R = S^{-4},$$

and so $\langle S^4 \rangle \lhd G(n)$. The quotient group $Q(n) = G(n)/\langle S^4 \rangle$ has presentation

$$Q(n) = \left\langle \overline{R}, \overline{S} \,\middle|\, \overline{R}^4, \overline{S}^4, (\overline{RS})^2, (\overline{RS}^{-1})^4, (\overline{RS}^{-3})^2 \right\rangle.$$

We can delete the last relator because $\overline{RS}^{-3} = \overline{RS}$. Either by coset enumeration (using a computer algebra system), or by noticing that this is $\mathrm{Aut}^+(\mathbf{R}_{1.3:4})$ and looking back at Section 1.7, we gain the insight that $Q(n)$ is of order 64 and isomorphic to a semi-direct product $\mathbb{Z}_4^2 \rtimes \mathbb{Z}_4$. To finish a complete description of the structure of $G(n)$ and thereby a proof that its order is as claimed, we need to describe the extension of $Q(n)$ by $\langle S^4 \rangle$. Note that we have already deduced the conjugation action of $\overline{R}$ and $\overline{S}$ on $\langle S^4 \rangle$, so it suffices to find the 2-cocycle describing this extension. This can be done, but the result is a bit unwieldy. An alternative proof is furnished as follows. Consider the group

$$G_\infty = \langle R, S, T \mid R^4, (RS)^2, (RS^{-1})^4, (RS^{-3})^2, TS^{-4} \rangle.$$

We know that $G(n) = G_\infty/\langle T^n \rangle$ and $\langle T \rangle \lhd G_\infty$ is of index 64. Set

$$t(n) := \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}.$$

We can extend the assignment $T \mapsto t(1)$ to a linear representation of $\langle T \rangle$. This representation can be induced up to a representation ρ of G_∞ having degree 128. We have $t(1)^n = t(n)$ for any $n \in \mathbb{Z}$, so reducing mod n yields a representation over $\mathbb{Z}/n\mathbb{Z}$ with $\mathrm{ord}(\rho(T)) = n$. This implies $|G(n)| = 64n$. Reflexivity of the maps follows from $(R^{-1}S)^4 = (R^{-1}(RS^{-1})^{-1}R)^2 = 1$ and $(R^{-1}S^3)^2 = (R^{-1}(RS^{-3})^{-1}R)^2 = 1$. □

Some members from the Conder list are: $\mathbf{R}_{1.3:4}$, $\mathbf{R}_{9.7}$, $\mathbf{R}_{17.12}$.

Computation of a presentation for the quotient group $G(n)/\langle R^{-1}SR, S\rangle^{G(n)}$ quickly yields that $[G(n) : \langle R^{-1}SR, S\rangle] = 2$. So each family member $\mathbf{R}(n)$ has a diagonal map $D(\mathbf{R}(n))$, giving rise to a polynomial family $\mathcal{F}_{8n-7}^{(4n,4n)}$ of type $(4n, 4n)$. It has the standard map presentation

$$\mathrm{Aut}^+(D(\mathbf{R}(n))) = \langle R, S \mid R^{4n}, S^{4n}, (RS)^2, [RS, SR], R^4S^4 \rangle$$

of order $32n$ with parameters $v = 8$, $e = 16n = 2(g + 7)$ and $f = 8$. Some members from the Conder list are: $\mathbf{R}_{1.4:2}$, $\mathbf{R}_{9.22}$, $\mathbf{R}_{17.36}$.

2.7.2 — The polynomial family $\mathcal{F}_{12n-7}^{(4,6n)}$ of type $(4, \frac{1}{2}(g + 7))$

Proposition 2.7.2. *The groups*

$$G(n) = \langle R, S \mid R^4, S^{6n}, (RS)^2, R^{-1}S^6RS^6, [R^2, S^3] \rangle$$

have order $96n$. They form a polynomial family of reflexive platonic maps of genus $12n - 7$ and type $(4, 6n)$. These maps have parameters $v = 16$, $e = 48n = 4(g + 7)$, and $f = 24n = 2(g + 7)$.

Proof. The relator $(e1)$ immediately shows us that $\langle S^6 \rangle \lhd G(n)$. The quotient $Q(n) := G(n)/\langle S^6 \rangle \cong G(1)$ has presentation

$$Q(n) = \left\langle \overline{R}, \overline{S} \,\middle|\, \overline{R}^4, \overline{S}^6, (\overline{RS})^2, [\overline{R}^2, \overline{S}^3] \right\rangle.$$

Coset enumeration and some extra computations show that this is isomorphic to a semi-direct product $\langle \overline{R}^2, \overline{S}^3, \overline{S}^{-1}\overline{R}^2\overline{S}, \overline{RS}^2\overline{R}^{-1}\overline{S}^2 \rangle \rtimes \langle \overline{RS}, \overline{R}^2\overline{S}^2\overline{R}^2 \rangle \cong \mathbb{Z}_2^4 \rtimes \mathrm{Sym}_3$ of order 96, but we will step over the details. In the same way as in Proposition 2.7.1 one proves that $|G(n)| \geq 96n$. This completes the proof that the groups have the order as claimed. The map parameters follow easily. Reflexivity of the maps is made clear by the computations $RS^{-6}R^{-1}S^{-6} \overset{e1}{=} RR^{-1}S^6RR^{-1}S^{-6} = 1$ and $[R^{-2}, S^{-3}] \overset{e2}{=} 1$. □

Some members from the Conder list are: $\mathbf{R}_{5.4}$, $\mathbf{R}_{17.10}$, $\mathbf{R}_{29.2}$.

Computation of a presentation for the quotient group $G(n)/\langle R^{-1}SR, S \rangle^{G(n)}$ quickly yields that $[G(n) : \langle R^{-1}SR, S \rangle] = 2$. So each family member $\mathbf{R}(n)$ has a diagonal map $D(\mathbf{R}(n))$, giving rise to a polynomial family of reflexive platonic maps $\mathcal{F}_{12n-7}^{(6n,6n)}$ of type $(6n, 6n)$. Its members have the standard map presentation

$$\mathrm{Aut}^+(D(\mathbf{R}(n))) = \langle R, S \mid R^{6n}, S^{6n}, (RS)^2, [RS, SR], SR^4SR^{-2} \rangle$$

of order $48n$ with parameters $v = 8$, $e = 24n = 2(g + 7)$, $f = 8$. Some members of the Conder list are: $\mathbf{R}_{5.10}$, $\mathbf{R}_{17.34}$, $\mathbf{R}_{29.26}$.

2.7.3 — The polynomial family $\mathcal{F}_{16n-7}^{(4,8n)}$ of type $(4, \frac{1}{2}(g + 7))$

Proposition 2.7.3. *The groups*

$$G(n) = \langle R, S \mid R^4, S^{8n}, (RS)^2, (RS^{-3})^2, (RS^{-1})^4S^{4n} \rangle$$

have order $128n$*. They define a polynomial family of reflexive platonic maps of genus* $16n-7$ *and type* $(4,8n)$*. These maps have parameters* $v=16$*,* $e=64n=4(g+7)$*,* $f=32n=2(g+7)$*.*

Proof. Exactly like in Proposition 2.7.1 we derive $R^{-1}S^4R=S^{-4}$ and thereby that $\langle S^4\rangle \lhd G(n)$. The quotient $Q(n)=G(n)/\langle S^4\rangle$ has presentation

$$Q(n)=\left\langle \overline{R},\overline{S}\,\middle|\,\overline{R}^4,\overline{S}^4,(\overline{RS})^2,(\overline{RS}^{-3})^2,(\overline{RS}^{-1})^4\right\rangle,$$

which is isomorphic to $\mathrm{Aut}^+(\mathbf{R}_{1.3:4})\cong\mathbb{Z}_4^2\rtimes\mathbb{Z}_4$, as in aforementioned proposition. To complete the proof that the extension $G(n)$ of $Q(n)$ by $\langle S^4\rangle$ has order $128n$, we construct a matrix representation for it by viewing the group as the equivalent

$$\widetilde{G}(n)=\left\langle R,S,T\,\middle|\,R^4,TS^{-4},T^{2n},(RS)^2,(RS^{-3})^2,(RS^{-1})^4T^n\right\rangle.$$

We have shown above that $\langle T\rangle\lhd\widetilde{G}(n)$ is of index 64. Now again set

$$t(k):=\begin{pmatrix}1&0\\k&1\end{pmatrix}\in\mathrm{GL}_2(\mathbb{Z}/2n\mathbb{Z})$$

and consider the linear representation $T^k\mapsto t(1)^k=t(k)$. We induce it up to a faithful representation ρ of $\widetilde{G}(n)$ having degree 128. The representation satisfies $\mathrm{ord}(\rho(T))=2n$ and by inspection it becomes clear that the image $\rho(G(n))$ fullfills all group relations. This finishes the proof that $|G(n)|=128n$ and defines a platonic map of the claimed type. The other parameters follow readily. Reflexivity of the maps follows from $(R^{-1}S^3)^2=(R^{-1}(RS^{-3})^{-1}R)^2=1$ and

$$(R^{-1}S)^4=(R^{-1}(RS^{-1})^{-1}R)^4=R^{-1}(RS^{-1})^4R\overset{e2}{=}R^{-1}S^{-4n}R=S^{4n}.\qquad\square$$

Some members from the Conder list are: $\mathbf{R}_{9.8}$, $\mathbf{R}_{25.12}$, $\mathbf{R}_{41.14}$.

Computation of a presentation for the quotient group $G(n)/\langle R^{-1}SR,S\rangle^{G(n)}$ quickly yields that $[G(n):\langle R^{-1}SR,S\rangle]=2$. So each family member $\mathbf{R}(n)$ has a diagonal map $D(\mathbf{R}(n))$, giving rise to a polynomial family $\mathcal{F}^{(8n,8n)}_{16n-7}$ of type $(8n,8n)$. It has the standard map presentation

$$\mathrm{Aut}^+(D(\mathbf{R}(n)))=\left\langle R,S\,\middle|\,R^{8n},S^{8n},(RS)^2,(R^2S^2)^2,R^4S^4\right\rangle$$

of order $64n$ with parameters $v=8$, $e=32n=2(g+7)$, $f=8$. Some members from the Conder list are: $\mathbf{R}_{9.23}$, $\mathbf{R}_{25.37}$, $\mathbf{R}_{41.67}$.

2.7.4 — The polynomial family $\mathcal{F}^{(4,12n)}_{24n-7}$ of type $(4,\frac{1}{2}(g+7))$

Proposition 2.7.4. *The groups*

$$G(n)=\left\langle R,S\,\middle|\,R^4,S^{12n},(RS)^2,(RS^{-5})^2,R^2S^3R^2S^{6n-3}\right\rangle$$

have order $192n$*. They define a polynomial family of reflexive platonic maps of genus* $24n-7$ *and type* $(4,12n)$*. These maps have parameters* $v=16$*,* $e=96n=4(g+7)$*,* $f=48n=2(g+7)$*.*

Proof. Parallel to the reasoning in Proposition 2.7.1 we deduce

$$R^{-1} = SRS \overset{e2}{=} S(S^5R^{-1}S^5)S = S^6R^{-1}S^6 \quad \Longrightarrow \quad R^{-1}S^6R = S^{-6},$$

showing that $\langle S^6 \rangle \lhd G$. The quotient $Q(n) = G(n)/\langle S^6 \rangle \cong G(1)$ has presentation

$$Q(n) = \left\langle \overline{R}, \overline{S} \,\middle|\, \overline{R}^4, \overline{S}^6, (\overline{RS})^2, (\overline{RS}^{-5})^2, [\overline{R}^2, \overline{S}^3] \right\rangle$$

It can be computed by coset enumeration and some analysis to be isomorphic to a semi-direct product $\mathbb{Z}_2^4 \rtimes \mathrm{Sym}_3$ of order 96. To complete the proof that the extension $G(n)$ of $Q(n)$ by $\langle S^6 \rangle$ has order $192n$, we construct a matrix representation for it by viewing the group as the equivalent

$$\widetilde{G}(n) = \left\langle R, S, T \,\middle|\, R^4, TS^{-6}, T^{2n}, (RS)^2, (RS^{-5})^2, [R^2, S^3]T^n \right\rangle.$$

We have shown above that $\langle T \rangle \lhd \widetilde{G}(n)$ is of index 96. Now again set

$$t(k) := \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/2n\mathbb{Z})$$

and consider the linear representation defined by $T^k \mapsto t(1)^k = t(k)$. We induce it up to a faithful representation ρ of $\widetilde{G}(n)$ having degree 192. This representation will have $\mathrm{ord}(\rho(T)) = 2n$ and fullfill all group relations; all this can readily be checked by taking a specific instance of n and considering that the form of $\rho(R)$ and $\rho(S)$ will be analogous for all n. This finishes the proof that $|G(n)| = 192n$ and defines a platonic map of the claimed type. The other parameters follow readily. Reflexivity of the maps follows from $(R^{-1}S^5)^2 = (R^{-1}(RS^{-5})^{-1}R)^2 = 1$ and $R^{-2}S^{-3}R^{-2}S^{-(6n-3)} \overset{e2}{=} S^{6n-3}S^{-(6n-3)} = 1$. □

Some members from the Conder list are: $\mathbf{R}_{17.11}$, $\mathbf{R}_{41.10}$, $\mathbf{R}_{65.43}$.

Computation of a presentation for the quotient group $G(n)/\langle R^{-1}SR, S\rangle^{G(n)}$ quickly yields that $[G(n) : \langle R^{-1}SR, S\rangle] = 2$. So each family member $\mathbf{R}(n)$ has a diagonal map $D(\mathbf{R}(n))$, giving rise to a polynomial family $\mathcal{F}_{24n-7}^{(12n,12n)}$ of type $(12n, 12n)$. It has the standard map presentation

$$\mathrm{Aut}^+(D(\mathbf{R}(n))) = \left\langle R, S \,\middle|\, R^{12n}, S^{12n}, (RS)^2, R^{-1}S^3RS^{-(6n+3)}, SRS^{-1}R^2SR^{-1}S \right\rangle$$

of order $96n$ with parameters $v = 8$, $e = 48n = 2(g+7)$, $f = 8$. Some members from the Conder list are: $\mathbf{R}_{17.35}$, $\mathbf{R}_{41.63}$, $\mathbf{R}_{65.136}$.

2.7.5 — The polynomial family $\mathcal{F}_{32n-7}^{(4,16n)(1)}$ and $\mathcal{F}_{32n-7}^{(4,16n)(2)}$

Proposition 2.7.5. *The groups*

$$G_1(n) = \left\langle R, S \,\middle|\, R^4, S^{16n}, (RS)^2, R^{-1}S^4RS^{-(8n-4)}, (RS^{-1})^4 \right\rangle,$$

$$G_2(n) = \left\langle R, S \,\middle|\, R^4, S^{16n}, (RS)^2, R^{-1}S^4RS^{-(8n-4)}, (RS^{-1})^4S^{8n} \right\rangle$$

both have order $256n$*. They define two polynomial families of reflexive platonic maps of genus* $32n-7$ *and type* $(4,16n)$*. These maps have parameters* $v=16$*,* $e=128n=4(g+7)$*,* $f=64n=2(g+7)$*.*

Proof. In both cases we learn from the relator $(e1)$ that $\langle S^4\rangle \lhd G_i(n)$ for $i=1,2$. Moreover, for either group the quotient $G_i(n)/\langle S^4\rangle$ has presentation

$$Q(n)=\left\langle \overline{R},\overline{S}\,\middle|\,\overline{R}^4,\overline{S}^4,(\overline{RS})^2,(\overline{RS}^{-1})^4\right\rangle .$$

By coset enumeration and some more computations one finds that $Q(n)$ is isomorphic to a semi-direct product $\mathbb{Z}_4^2\rtimes\mathbb{Z}_4$ of order 64. By the same procedure as in the proof of Proposition 2.7.4, with $T=S^4$, we construct a faithful linear representation for both $G_1(n)$ and $G_2(n)$ over $\mathbb{Z}/4n\mathbb{Z}$ that shows there is no more collapse than claimed and hence $|G_1(n)|=|G_2(n)|=256n$. Reflexivity follows from $RS^{-4}R^{-1}S^{8n-4}\overset{e1}{=}S^{-(8n-4)}S^{8n-4}=1$ and $(R^{-1}S)^4=(R^{-1}(RS^{-1})^{-1}R)^4=1$ in the first case; in the second

$$\begin{aligned}(R^{-1}S)^4S^{-8n}&=(R^{-1}(RS^{-1})^{-1}R)^4S^{-8n}=R^{-1}(RS^{-1})^4RS^{-8n}\\&\overset{e2}{=}RS^{-8n}RS^{-8n}\overset{e1}{=}S^{8n}S^{-8n}=1.\end{aligned}$$ □

Some members from the Conder list are: $\mathbf{R}_{25.10}$, $\mathbf{R}_{57.8}$, $\mathbf{R}_{89.11}$ for the family $\mathcal{F}_{32n-7}^{(4,16n)(1)}$ and $\mathbf{R}_{25.11}$, $\mathbf{R}_{57.6}$, $\mathbf{R}_{89.9}$ for $\mathcal{F}_{32n-7}^{(4,16n)(2)}$.

Computation of a presentation for the quotient group $G(n)/\langle R^{-1}SR,S\rangle^{G(n)}$ for both families yields that $[G(n):\langle R^{-1}SR,S\rangle]=2$. So each family member $\mathbf{R}(n)$ has a diagonal map $D(\mathbf{R}(n))$, giving rise to two polynomial families $\mathcal{F}_{32n-7}^{(16n,16n)(1)}$ and $\mathcal{F}_{32n-7}^{(16n,16n)(2)}$ of type $(16n,16n)$. The first has the standard map presentation

$$\mathrm{Aut}^+(D(\mathbf{R}(n))=\langle R,S\mid R^{16n},S^{16n},(RS)^2,R^{8n-4}S^{-4},[R,S]^2\rangle$$

and the second

$$\mathrm{Aut}^+(D(\mathbf{R}(n)))=\langle R,S\mid R^{16n},S^{16n},(RS)^2,R^{8n-4}S^{-4},[R^2,S^2]\rangle$$

Both groups are of order $128n$ with parameters $v=8$, $e=64n=2(g+7)$, $f=8$. Some members from the Conder list are: $\mathbf{R}_{25.36}$, $\mathbf{R}_{57.63}$, $\mathbf{R}_{89.67}$ for $\mathcal{F}_{32n-7}^{(16n,16n)(1)}$ and $\mathbf{R}_{25.39}$, $\mathbf{R}_{57.61}$, $\mathbf{R}_{89.65}$ for $\mathcal{F}_{32n-7}^{(16n,16n)(2)}$.

2.7.6 — The polynomial family $\mathcal{F}_{32n-23}^{(4,16n-8)(1)}$ and $\mathcal{F}_{32n-23}^{(4,16n-8)(2)}$

Proposition 2.7.6. *The groups*

$$\begin{aligned}G_1(n)&=\langle R,S\mid R^4,S^{16n-8},(RS)^2,R^{-1}S^8RS^8,R^2SR^{-1}S^{-5}RS^{8n-10}\rangle,\\G_2(n)&=\langle R,S\mid R^4,S^{16n-8},(RS)^2,R^{-1}S^8RS^8,[R,S]^2S^{8n-8}\rangle\end{aligned}$$

both have order $128(2n-1)$*. They define two polynomial families of reflexive platonic maps of genus* $32n-23$ *and type* $(4,16n-8)$*. These maps have parameters* $v=16$*,* $e=128n-64=4(g+7)$*,* $f=64n-32=2(g+7)$*.*

Proof. In both cases the relator $(e1)$ tells us that $\langle S^8\rangle \lhd G_.(n)$. The quotients that we find are

$$Q_1(n) = \langle R, S \mid R^4, S^8, (RS)^2, R^2SR^{-1}S^{-5}RS^{-2}\rangle\,,$$
$$Q_2(n) = \langle R, S \mid R^4, S^8, (RS)^2, [R,S]^2\rangle\,.$$

They are non-isomorphic 2-groups (order 128). Each is isomorphic to a central extension $(\mathbb{Z}_2^4 \rtimes \mathbb{Z}_4)\cdot\mathbb{Z}_2$. By the same procedure as in the proof of Proposition 2.7.4, this time with $T = S^8$, we construct a faithful linear representation for both $G_1(n)$ and $G_2(n)$ over $\mathbb{Z}/(2n-1)\mathbb{Z}$ that shows there is no more collapse than claimed and hence $|G_1(n)| = |G_2(n)| = 128(2n-1)$. Reflexivity is established as follows. For both we compute $RS^{-8}R^{-1}S^{-8} \stackrel{e1}{=} S^8S^{-8} = 1$. The relator $(e2)$ of $G_1(n)$ is the most troublesome. We note that it equals $R(RS)R^{-1}\cdot S^{-5}(RS)S^5\cdot S^{8n-16}$ and compute its image under $(R,S)\mapsto(R^{-1},S^{-1})$ to be

$$((RS)^{-1})^R\cdot((RS)^{-1})^{S^5}\cdot S^{-(8n-16)} \stackrel{e2}{=} (RS)^R\cdot(RS)^{S^5}\cdot(RS)^{S^{-5}}\cdot(RS)^{R^{-1}} = (RS)^R\cdot(RS)^{R^{-1}} = 1.$$

For the group $G_2(n)$ we check that

$$[R^{-1},S^{-1}]^2S^{-(8n-8)} \stackrel{e2}{=} [R^{-1},S^{-1}]^2[R,S]^2 = (RSR^{-1}S^{-1})^2(R^{-1}S^{-1}RS)^2$$
$$= (RS^2R)^2(R^{-1}S^{-2}R^{-1})^2 = 1. \qquad \square$$

Some members from the Conder list are: $\mathbf{R}_{9.5}$, $\mathbf{R}_{41.12}$, $\mathbf{R}_{73.31}$ for $\mathcal{F}_{32n-23}^{(16n-8,16n-8)(1)}$ and $\mathbf{R}_{9.6}$, $\mathbf{R}_{41.13}$, $\mathbf{R}_{73.32}$ for $\mathcal{F}_{32n-23}^{(16n-8,16n-8)(2)}$.

Computation of a presentation for the quotient group $G_.(n)/\langle R^{-1}SR, S\rangle^{G(n)}$ for both families yields that $[G_.(n) : \langle R^{-1}SR, S\rangle] = 2$. So each family member $\mathbf{R}(n)$ has a diagonal map $D(\mathbf{R}(n))$, giving rise to two polynomial families $\mathcal{F}_{32n-23}^{(16n-8,16n-8)(1)}$ and $\mathcal{F}_{32n-23}^{(16n-8,16n-8)(2)}$ of type $(16n-8, 16n-8)$. Their standard map presentations are

$$\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^{16n-8}, S^{16n-8}, (RS)^2, R^{-1}S^4RS^{-4}, [R^2,S^2], S^{-1}R^2S^{-3}R^{8n-10}\rangle$$

for the first and

$$\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^{16n-8}, S^{16n-8}, (RS)^2, R^{-1}S^4RS^{-4}, [R^2,S^2], S^{-1}RS^{-1}R^{8n-7}\rangle$$

for the second. Both these families have order $128n-64$ and parameters $v = 8$, $e = 64n-32 = 2(g+7)$, $f = 8$. Some members from the Conder list are: $\mathbf{R}_{9.21}$, $\mathbf{R}_{41.68}$, $\mathbf{R}_{73.115}$ for $\mathcal{F}_{32n-23}^{(16n-8,16n-8)(1)}$ and $\mathbf{R}_{9.19}$, $\mathbf{R}_{41.64}$, $\mathbf{R}_{73.111}$ for $\mathcal{F}_{32n-23}^{(16n-8,16n-8)(2)}$.

2.8 Polynomial families with more vertices

2.8.1 — The polynomial family $\mathcal{F}_{100n-99}^{(4,8n-4)}$ and four subfamilies

Conjecture 2.8.1. The groups

$$G(n) = \langle R, S \mid R^4, S^{8n-4}, (RS)^2, (RS^{-3})^2, [R^2,S]^5\rangle$$

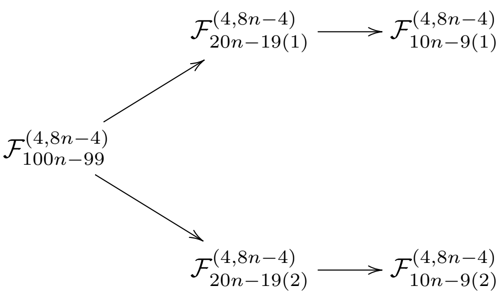
have order $400(2n-1)$. They define a polynomial family of reflexive platonic maps of genus $100n-99$ and type $(4, 8n-4)$ with data $v = 100$, $e = 200(2n-1)$, $f = 100(2n-1)$. Four suitable quotients yield four families of chiral platonic maps with $\mathrm{Aut} = \mathrm{Aut}^+$ of sizes $80(2n-1)$, $80(2n-1)$, $40(2n-1)$, and $40(2n-1)$ respectively:

$$
\begin{aligned}
\mathrm{Aut}^+(\mathcal{F}^{(4,8n-4)}_{100n-99}(n)) &= \left\langle R, S \,\middle|\, R^4, S^{8n-4}, (RS)^2, (RS^{-3})^2, [R^2, S]^5 \right\rangle \\
\mathrm{Aut}(\mathcal{F}^{(4,8n-4)}_{20n-19(1)}(n)) &= \left\langle R, S \,\middle|\, R^4, S^{8n-4}, (RS)^2, R^2S(R^{-1}S)^3SR^{-1}S \right\rangle \\
\mathrm{Aut}(\mathcal{F}^{(4,8n-4)}_{20n-19(2)}(n)) &= \left\langle R, S \,\middle|\, R^4, S^{8n-4}, (RS)^2, (SR^{-1})^2(S^{-1}R)^4 \right\rangle \\
\mathrm{Aut}(\mathcal{F}^{(4,8n-4)}_{10n-9(1)}(n)) &= \left\langle R, S \,\middle|\, R^4, S^{8n-4}, (RS)^2, R^2S^2R^{-1}S^{-2}RS^{4k-6} \right\rangle \\
\mathrm{Aut}(\mathcal{F}^{(4,8n-4)}_{10n-9(2)}(n)) &= \left\langle R, S \,\middle|\, R^4, S^{8n-4}, (RS)^2, RSR^{-1}S^{-2}RS^2R^{-1}S^{-(4k-7)} \right\rangle
\end{aligned}
$$

The combinatorial properties of the four quotient families are:

$$
\begin{aligned}
\mathcal{F}^{(4,8n-4)}_{20n-19(1)} &: \quad v = 20, \quad e = 40(2n-1), \quad f = 20(2n-1) \\
\mathcal{F}^{(4,8n-4)}_{20n-19(2)} &: \quad v = 20, \quad e = 40(2n-1), \quad f = 20(2n-1) \\
\mathcal{F}^{(4,8n-4)}_{10n-9(1)} &: \quad v = 10, \quad e = 20(2n-1), \quad f = 10(2n-1) \\
\mathcal{F}^{(4,8n-4)}_{10n-9(2)} &: \quad v = 10, \quad e = 20(2n-1), \quad f = 10(2n-1)
\end{aligned}
$$

We can summarize the data as a *family tree* (pun intended) of sorts.



2.9 Polynomial families with v non-constant

2.9.1 — The polynomial family $\mathcal{F}^{(3,2n)}_{\binom{n-1}{2}}$ (Fermat family)

This family of platonic maps was discovered by Coxeter and Moser [CM1980]. There is a surprise that we already spoiled in the heading: the family members have the Fermat curves as planar algebraic models. This is discussed in Chapter 5. Here we content ourselves with a group-theoretical analysis.

Proposition 2.9.1. *The groups*

$$G(n) := \langle R, S \mid R^3, S^{2n}, (RS)^2, (R^2S^2)^3 \rangle$$

are of order $6n^2$. *They define a polynomial family* **Fer** *of reflexive platonic maps of genus* $\binom{n-1}{2}$ *and type* $(3, 2n)$. *The maps have parameters* $v = 3n$, $e = 3n^2$, $f = 2n^2$.

Proof. Define the subgroup $H(n) := \langle S^2, R^{-1}S^2R\rangle$. The computations

$$R^{-1}(R^{-1}S^2R)R = RS^2R^{-1} = RS^2R^2 \overset{r2}{=} R(RS^{-2})^2 = (R^{-1}S^2R)^{-1}S^{-2}$$
$$S^{-1}(R^{-1}S^2R)S = RSS^2S^{-1}R^{-1} = RS^2R^{-1} = (R^{-1}S^2R)^{-1}S^{-2}$$

show that $H(n) \triangleleft G(n)$. To analyze the structure of $H(n)$, we prove that $[S^2, R^{-1}S^2R] = 1$:

$$S^{-2}(R^{-1}S^2R)S^2 = S^{-2}R(RS^2R)S^2 = S^{-2}RS^{-1}R^{-1}R^{-1}S^{-1}S^2 = S^{-2}R(S^{-1}RS) =$$
$$= S^{-2}RS^{-2}R^{-1} = (S^{-2}R)^2R \overset{e2}{=} R^{-1}S^2R.$$

So $H(n)$ is abelian. It is certainly a quotient of $\mathbb{Z}_n^2$, but we will see it is in fact isomorphic to it. It has the complement

$$H^\perp(n) := \langle RS, SR^2S\rangle \cong \mathrm{Sym}_3.$$

That $H^\perp(n) \cong \mathrm{Sym}_3$ is proved by noting

$$(SR^2S)^3 = S(R^2S^2)^3S^{-1} \overset{e1}{=} 1,$$
$$(RS)^{-1}(SR^2S)(RS) = S^{-1}R^{-1}SR(RS)^2 = S^{-1}R^{-2}S^{-1} = (SR^2S)^{-1}.$$

By writing down the presentation $G(n)/H(n) = \left\langle \overline{R}, \overline{S} \,\middle|\, \overline{R}^3, \overline{S}^2, (\overline{RS})^2 \right\rangle$ we see that the cosets of $H(n)$ all contain a representative lying in $H^\perp(n)$, showing that $H(n) \cap H^\perp(n) = 1$. We have thereby determined that $G(n) \cong H(n) \rtimes \mathrm{Sym}_3$. A little extra computation tells us that the conjugation action of Sym_3 on $H(n)$ is given by

$$\mathrm{con}_{RS}: \ (S^2, R^{-1}S^2R) \mapsto ((R^{-1}S^2R)S^{-2}, R^{-1}S^2R),$$
$$\mathrm{con}_{SR^{-1}S}: \ (S^2, R^{-1}S^2R) \mapsto (R^{-1}S^2R, (R^{-1}S^2R)S^2).$$

Now take $\mathbb{Z}_n^2 := \langle x, y\rangle$ and let $\mathrm{Sym}_3 = \left\langle z_1, z_2 \,\middle|\, z_1^2, z_2^3, (z_1z_2)^2 \right\rangle$ act on $\mathbb{Z}_n^2$ by $z_1^{-1}xz_1 = yx^{-1}$, $z_1^{-1}yz_1 = y$, $z_2^{-1}xz_2 = y$, $z_2^{-1}yz_2 = yx$. If we then set $R := z_1z_2^{-1}xz_1^{-1}$ and $S := z_1x^{-1}z_2$, we find back our original relators for R and S. This finishes the demonstration that $G(n) \cong \mathbb{Z}_n^2 \rtimes \mathrm{Sym}_3$ and proves all numerical claims. To see that the maps defined are reflexive, note that $R^{-2}S^{-2} = R^{-2}(R^2S^2)^{-1})R^2$ has order 3 by $(e1)$. Hence $G(n)$ satisfies the chirality criterion. □

The proof showed that $[G(n) : H(n)] = 6$. Since $H(n) = \cap_{g\in\langle R\rangle}\langle S, R^{-1}S^2R\rangle^g$, we find $[G(n) : \langle S, R^{-1}S^2R\rangle] = 2$. So each Fermat family member has a diagonal D_2-map (see Section 3.2) with standard map presentation

$$\mathrm{Aut}^+(D(\mathbf{Fer}(n))) = \langle R, S \mid R^n, S^{2n}, (RS)^2, [R, S^2]\rangle$$

of order $2n^2$ and parameters $v = n$, $e = n^2$, $f = 2n$. As $\mathbf{Fer}(n)$ has a diagonal map, we infer that its graph $\Gamma(\mathbf{Fer}(n))$ is tripartite. As a consequence of $v = 3n$ and $e = 3n^2$ we find that $\Gamma(\mathbf{Fer}(n)) = K_{n,n,n}$, the complete tripartite graph on n vertices.

Some members of $\mathbf{Fer}$ from the Conder list are: $\mathbf{Dih}(3)$, $\mathbf{Oct}$, $\mathbf{R}_{1.2:3}$, $\mathbf{R}_{3.2}$ (Dyck's map), $\mathbf{R}_{6.1}$, $\mathbf{R}_{10.2}$, $\mathbf{R}_{15.2}$. Some members of $D(\mathbf{Fer})$ are: $\mathbf{Hos}(1) = \mathbf{Dih}(1)$, $\mathbf{Hos}(4)$, $\mathbf{R}_{1.1:1}$, $\mathbf{R}_{3.5}$, $\mathbf{R}_{6.6}$, $\mathbf{R}_{10.16}$, $\mathbf{R}_{15.10}$.

Remark 2.9.2. The extra relator $(R^2S^2)^3$ forces Petrie-polygon length at most 6. In fact equality holds.

2.9.2 — The polynomial family $\mathcal{F}^{(4,2n)}_{(n-1)^2}$

This family was also discovered by Coxeter and Moser [CM1980]. Our statement is in terms of $\mathrm{Aut}(\mathbf{R})$ instead of $\mathrm{Aut}^+(\mathbf{R})$ because the group structure of the latter is less straightforward.

Proposition 2.9.3. *The groups*

$$G(n) := \left\langle a, b, c \,\middle|\, a^2, b^2, c^2, (ab)^4, (bc)^{2n}, (ca)^2, (abc)^4 \right\rangle$$

are of order $16n^2$*. They define a polynomial family of reflexive platonic maps of genus* $(n-1)^2$ *and type* $(4, 2n)$*. The maps have parameters* $v = 4n$*,* $e = 4n^2$*,* $f = 2n^2$*. Furthermore, the family members have standard map presentation*

$$\mathrm{Aut}^+(\mathbf{R}(n)) \cong \left\langle R, S \,\middle|\, R^4, S^{2n}, (RS)^2, (R^2S^2)^2 \right\rangle.$$

Proof. Define the subgroup $H(n) := \langle c, bcb, abcba \rangle$. Conjugations of its generators by a, b, c all lie in $H(n)$ again. The most difficult computation involved is

$$(abcba)^b = b(cba)^3bab = (bcb)acbac(babab) = (bcb)cabac(aba) = (bcb)c(abcba).$$

We conclude that $H(n) \lhd G(n)$, and we analyze $H(n)$ as follows. Use the abbreviations $R := ab$, $S := bc$ for simplicity. The extra relator translates to $(R^2S^2)^2 = 1$ and within the group $H(n) \cap \langle R, S \rangle = \langle (bcb)c, (abcba)c \rangle = \langle S^2, RS^{-1}RS \rangle$ we deduce the identity

$$\begin{aligned}(RS^{-1}RS)^{S^2} &= S^{-2}RS^{-1}(R)SS^2 = S^{-2}R(S^{-1}R^{-1})(R^2S^2)S = S^{-2}R(RS)(S^{-2}R^2)S \\ &= S^{-2}R^2SS^{-2}R^2S = R^2S^2SS^{-2}R^2S = R(RSR)RS = RS^{-1}RS.\end{aligned}$$

So $H(n) \cap \langle R, S \rangle$ is abelian. It is clear that $\langle c \rangle$ is a complement in $H(n)$ of this subgroup. The conjugation action is computed to be

$$\mathrm{con}_c : \; (S^2, RS^{-1}RS)) \mapsto (S^{-2}, (RS^{-1}RS)^{-1}).$$

Within $G(n)$, the subgroup $H(n)$ in turn has the complement $\langle ac, cbc \rangle$. This is readily seen to be isomorphic to Dih_8. We conclude that $G(n) \cong (H(n) \cap \langle R, S \rangle \rtimes \mathbb{Z}_2) \rtimes \mathrm{Dih}_8$.

A few extra computations yield the conjugation action

$$\begin{aligned} \mathrm{con}_{ac} &: \ (S^2, RS^{-1}RS, c) \mapsto (RS^{-1}RS, S^2, c), \\ \mathrm{con}_{cbc} &: \ (S^2, RS^{-1}RS, c) \mapsto (S^{-2}, RS^{-1}RS, cS^2). \end{aligned}$$

Now substitute the group $\mathbb{Z}_n^2 = \langle x, y\rangle$ for $H(n)$ and extend it first by $\mathbb{Z}_2 = \langle z\rangle$ and then by $\mathrm{Dih}_8 = \langle w_1, w_2\rangle$ with conjugation actions corresponding to the above. We can compute that the triple $(w_1 z, z w_2 z, z)$ satisfies the same relations as (a, b, c). So apparently $16n^2$ is a lower bound for the order and hence $H(n) \cong \mathbb{Z}_2$ and equality of orders holds. □

The quotient group $\mathrm{Aut}^+(\mathbf{R}(n))/\langle S, R^{-1}SR\rangle^{\mathrm{Aut}^+(\mathbf{R}(n))}$ is quickly found to be isomorphic to $\mathbb{Z}_2$ by computing a presentation for it. Thus $\langle S, R^{-1}SR\rangle \lhd \mathrm{Aut}^+(\mathbf{R}(n))$ of index 2. So each family member has a diagonal map $D(\mathbf{R}(n))$ with standard map presentation

$$\mathrm{Aut}^+(D(\mathbf{R}(n))) = \left\langle R, S \,\middle|\, R^{2n}, S^{2n}, (RS)^2, (RS^{-1})^2 \right\rangle$$

of order $4n^2$ and parameter $v = 2n$, $e = 2n^2$, $f = 2n$. Some members of $\mathcal{F}^{(4,2n)}_{(n-1)^2}$ from the Conder list are: $\mathbf{Dih}(4)$, $\mathbf{R}_{1.4:2}$, $\mathbf{R}_{4.3}$, $\mathbf{R}_{9.6}$, and some members of $D(\mathcal{F}^{(4,2n)}_{(n-1)^2})$ are: $\mathbf{Dih}(2)$, $\mathbf{R}_{1.3:2}$, $\mathbf{R}_{4.7}$, $\mathbf{R}_{9.19}$.

Remark 2.9.4. The extra relator $(R^2S^2)^2$ forces Petrie-polygon length at most 4. In fact equality holds.

Conjecture 2.9.5. Could this family have a simple corresponding family of algebraic planar models just like the Fermat family? One hunch would be that they are polynomial lemniscates. A *polynomial lemniscate* is an algebraic curve constructed as follows. Let $n \in \mathbb{Z}_{\geq 0}$ and pick a polynomial $p \in \mathbb{C}[z]$ with $\deg(p) = n$. For $c \in \mathbb{R}$, the level set $|p(x + yi)| = c$ is defined by a polynomial equation $P = 0$ with $P \in \mathbb{C}[x, y]$, as one finds by expanding. The curve $P = 0$ is generically of degree $2n$ and genus $(n-1)^2$. Polynomial lemniscates therefore seem a reasonable candidate space to search for planar models of this family of maps.

2.10 Two-parameter families

2.10.1 — The polynomial family $\mathcal{F}^{(m(2n-1),2m(2n-1))}_{m^2(n-1)+\frac{1}{2}(m-2)(m-1)}$

Proposition 2.10.1. *The groups*

$$G(m, n) := \left\langle R, S \,\middle|\, R^{m(2n-1)}, S^{2m(2n-1)}, (RS)^2, [R^m, S], [R, S^2] \right\rangle$$

satisfy $\mathrm{ord}(R) = m(2n-1)$*,* $\mathrm{ord}(S) = 2m(2n-1)$*, and* $|G(m, n)| = 2m^2(2n-1)$*. They define a polynomial family of reflexive platonic maps of genus* $m^2(n-1) + \binom{m-1}{2}$ *and* *type* $(m(2n-1), 2m(2n-1))$. *These maps have parameters* $v = m$, $e = m^2(2n-1)$, $f = 2m$.

Proof. The last relator implies that $\langle S^2 \rangle \lhd G(m,n)$. The quotient group $Q(m,n) := G(m,n)/\langle S^2 \rangle$ permits of the presentation

$$\left\langle \overline{R}, \overline{S} \,\middle|\, \overline{R}^{m(2n-1)}, \overline{S}^2, (\overline{RS})^2, [\overline{R}^m, \overline{S}] \right\rangle.$$

We infer that $\overline{S}^{-1}\overline{RS} = \overline{SRS} = \overline{R}^{-1}$, so that $\langle \overline{R} \rangle$ is a normal subgroup of $Q(m,n)$. One readily computes that

$$Q(m,n)/\langle \overline{R} \rangle \cong \mathbb{Z}_2.$$

Moreover, the relators imply

$$\overline{R}^m = \overline{S}^{-1}\overline{R}^m\overline{S} = (\overline{S}^{-1}\overline{RS})^m = \overline{R}^{-m}$$

so that $\overline{R}^{2m} = 1$. But since also $\overline{R}^{(2n-1)m} = 1$, we must have $\overline{R}^m = 1$. We can now refine the presentation of $Q(m,n)$ to

$$Q(m,n) = \left\langle \overline{R}, \overline{S} \,\middle|\, \overline{R}^m, \overline{S}^2, (\overline{RS})^2 \right\rangle$$

which shows that $Q(m,n) \cong \mathrm{Dih}_{2m}$. This implies the exactness of the sequence

$$1 \longrightarrow \langle S^2 \rangle \longrightarrow G(m,n) \xrightarrow{\pi} \mathrm{Dih}_{2m} \longrightarrow 1$$

and proves all the numerical claims of the proposition. The chirality criterion is satisfied, since $[x,y] = 1$ implies $[x^{-1}, y^{-1}] = 1$ and this can be applied to both extra relators. So the family is reflexive. □

Remark 2.10.2. When m is odd, we can elucidate the structure of $G(m,n)$ more. The subgroup $\langle RS, S^{m(2n-1)} \rangle$ then satisfies $\pi(RSS^{m(2n-1)}) = \overline{R}$ and $\pi(RS) = \overline{RS}$, whence it has π-image Dih_{2m}. Its order is therefore at least $2m$. We can bound it from above by exploiting the group relators. First off,

$$R^m = S^{-1}R^mS = (S^{-1}RS)^m = (S^{-2}R^{-1})^m = S^{-2m}R^{-m}.$$

This implies $R^{2m} = S^{-2m}$ and hence

$$R^m = R^{2mn} = S^{-2mn}.$$

We can use this to show (using that m is odd in the second and third step):

$$(RSS^{m(2n-1)})^m = (RS^{m(2n-1)+1})^m = R^mS^{m^2(2n-1)+m} = R^mS^{m(2n-1)+m} = R^mS^{2mn} = 1.$$

So the group $\langle RS, S^{m(2n-1)} \rangle$ has two generators of order 2 with product of order at most m. Combining that with our lower bound on its size, we see that Dih_{2m}. We know that $|G(m,n)| = m|\langle S \rangle|$, so $\langle R \rangle \cap \langle S \rangle = \langle R^m \rangle$. This means that $(RSS^{m(2n-1)})^j = R^jS^{m(2n-1)j+j} \notin \langle S \rangle$ for $j = 1, \ldots, m-1$. Neither can a product of such an element with $S^{m(2n-1)}$ lie in $\langle S\rangle$. We conclude that $\langle RS, S^{m(2n-1)}\rangle \cap \langle S^2\rangle = \{1\}$. Because $S^2 \in Z(G(m,n))$, the structure of the group $G(m,n)$ for odd m is

$$G(m,n) = \langle S^2\rangle \times \langle RS, S^{m(2n-1)}\rangle \cong \mathbb{Z}_{m(2n-1)} \times \mathrm{Dih}_{2m}.$$

Remark 2.10.3. The 1-parameter subfamily obtained by setting $n=1$ is $D(\mathbf{Fer}(n))$. The 1-parameter family obtained by setting $n=2$ also satisfies the conditions stated in Section 3.2 for it to be the diagonal map of another map (of type $(3,6m)$). This leads to another polynomial family $\mathcal{F}^{(3,6m)}_{\frac{3}{2}m^2-\frac{3}{2}m+1}$ with standard map presentation

$$\mathrm{Aut}^+(\mathcal{F}^{(3,6m)}_{\frac{3}{2}m^2-\frac{3}{2}m+1}(m)) = \left\langle R, S \,\middle|\, R^3, S^{6m}, (RS)^2, (RS^{-2})^3, [R, S^{2m}]\right\rangle.$$

The two mentioned 1-parameter subfamilies of $\mathcal{F}^{(m(2n-1),2m(2n-1))}_{m^2(n-1)+\frac{1}{2}(m-2)(m-1)}$ are its only maps that satisfy the D_2-map conditions. The proof of this is as follows. Define $x := R$, $y := S^2$, $z := SRS^{-1}$. Now consider $H = \langle R, S^2, SRS^{-1}\rangle = \langle x,y,z\rangle$ and rewrite the relators of $G(m,n)$ in (x,y,z). The relation $(RS)^2 = 1$ translates into $z = x^{-1}y^{-1}$, and $[R^m, S] = 1$ into $z^m x^{-m} = 1$. We therefore write down a presentation for H in terms of x and y only:

$$H = \left\langle x, y \,\middle|\, x^{m(2n-1)}, y^{m(2n-1)}, [x,y], y^{-m}x^{-2m}\right\rangle.$$

So in this abelian subgroup we have $y^m = x^{-2m}$. If the permutation (xyz) defines an automorphism of H, as is necessary if it is to be a D_2-map, then also $(x^{-1}y^{-1})^m = z^m = y^{-2m}$, so $x^{-m} = y^{-m} = x^{-2m}$. This implies $x^{3m} = 1$, and we infer that $2n-1 \mid 3$, which only happens if $n \in \{1,2\}$.

Remark 2.10.4. This family produces platonic maps with any number m of vertices, something our previous families had not yet achieved. We will see in Section 4.3 that if m is an odd prime, the maps produced by this family are all maps with $v = m$.

Remark 2.10.5. There are 161 maps belonging to this family with $2 \le g \le 101$, which is 4.77% of the total.

2.10.2 — The polynomial family $\mathcal{F}^{(2mn,2mn)}_{m^2n-2m+1}$

For this family, we work with $\mathrm{Aut}(\mathbf{R})$ instead of $\mathrm{Aut}^+(\mathbf{R})$ since the former has a less complicated group structure.

Proposition 2.10.6. *The groups*

$$G(m,n) := \left\langle a,b,c \,\middle|\, a^2, b^2, c^2, (ab)^{2mn}, (bc)^{2mn}, (ca)^2, [(ab)^m, bc], [ab, (bc)^2]\right\rangle$$

have order $8m^2n$. They define a polynomial family of reflexive platonic maps of genus $m^2n-2m+1$ and type $(2mn, 2mn)$. These maps have parameters $v = 2m$, $e = 2m^2n$, $f = 2m$. Furthermore

$$\mathrm{Aut}^+(\mathbf{R}(n)) = \left\langle R, S \,\middle|\, R^{2mn}, S^{2mn}, (RS)^2, [R^m, S], [R, S^2]\right\rangle.$$

Proof. Let $H(m,n) := \langle S^2, abc\rangle$. The computations

$$\begin{aligned} &aS^2a = abc(bca) = abcS^2(abc)^{-1} && a(abc)a = bca = S^2(abc)^{-1} \\ &bS^2b = cbcb = S^{-2} && b(abc)b = bac(cbcb) = bcaS^{-2} = S^2(abc)^{-1}S^{-2} \\ &cS^2c = cbcb = S^{-2} && c(abc)c = cab = abcS^{-2} \end{aligned}$$

show that $H(m,n) \lhd G(m,n)$. Note that $R^2 = (abc)^2S^{-2} \in H(m,n)$. Therefore the quotient group $Q(m,n) := G(m,n)/H(m,n)$ has presentation

$$\begin{aligned} Q(m,n) &= \langle a,b,c \mid a^2, b^2, c^2, (ab)^2, (bc)^2, (ca)^2, [(ab)^m, bc], [ab, (bc)^2], abc\rangle \\ &= \langle a,b,c \mid a^2, b^2, c^2, [a,b], [b,c], [c,a], abc\rangle \cong \mathbb{Z}_2^3/\mathbb{Z}_2 \cong \mathbb{Z}_2^2. \end{aligned}$$

Indeed, the subgroup $H^\perp(m,n) := \langle a, c\rangle$ is a complement, since its elements represent all four cosets of $H(m,n)$. To analyze the structure of $H(m,n)$, first note that the second extra relator implies that

$$(abc)^{-1}S^2abc = cba(bcbc)(ab)c = cba(ab)(bcbc)c = cbcb = S^{-2}.$$

We therefore have $\langle S^2\rangle \lhd H(m,n)$. A concern is whether $\langle S^2\rangle \cap \langle abc\rangle = 1$. But if we take $\mathbb{Z}_{mn} = \langle x\rangle$ and extend it first by $\mathbb{Z}_{2m} = \langle y\rangle$ acting by $x^y = x^{-1}$ and then by $\mathbb{Z}_2^2 = \langle z_1, z_2\rangle$ with (z_1, z_2) acting on x and y like (a, c) on S^2 and abc, then the triple (z_1, z_1yz_2, z_2) can be computed to satisfy the same relations as (a, b, c). We conclude that $H(m,n) = \langle S^2\rangle \rtimes \langle abc\rangle$ and hence

$$G(m,n) = (\langle S^2\rangle \rtimes \langle abc\rangle) \rtimes \langle a, c\rangle \cong (\mathbb{Z}_{mn} \rtimes \mathbb{Z}_{2m}) \rtimes \mathbb{Z}_2^2. \qquad \square$$

Remark 2.10.7. The 1-parameter subfamilies obtained by setting $m = 1$ or $m = 2$ are readily seen to consist of self-dual maps (by the self-duality criterion from Section 1.4). In fact, $G(1,n) = \mathcal{F}_{n-1}^{(2n,2n)} = D(\mathcal{F}_{n-1}^{(4,2n)})$ and $G(2,n) = \mathcal{F}_{4n-3}^{(4n,4n)} = D(\mathcal{F}_{4n-3}^{(4,4n)})$. These are in fact the only self-dual maps of this 2-parameter family. For suppose that $(R, S) \mapsto (R^{-1}, S^{-1})$ defines an automorphism of $G^+(m,n)$. Then $[R^2, S] = 1$ and so $|G(m,n)| = 8m^2n$ divides the order of $G(2,n) = 32n$. Apparently therefore $m \mid 2$, and the only self-dual family members are the ones discussed.

2.10.3 — The polynomial family $\mathcal{F}_{mn}^{(2m+2,2n+2)}$ (also Coxeter's $\{p, q \mid 2\}$)

Proposition 2.10.8. *The groups*

$$G(m,n) = \langle R, S \mid R^{2m+2}, S^{2n+2}, (RS)^2, (R^{-1}S)^2\rangle \qquad (m, n \geq 1)$$

have order $4(m+1)(n+1)$. They define a polynomial family of reflexive platonic maps of genus mn and type $(2m+2, 2n+2)$. These maps have parameters $v = 2m+2$, $e = 2(m+1)(n+1)$, $f = 2n+2$.

Proof. The computations

$$\begin{aligned} &S^{-1}R^2S \overset{e1}{=} R^{-1}SRS = R^{-1}R^{-1}S^{-1}S = R^{-2} \\ &R^{-1}S^2R \overset{e1}{=} S^{-1}RSR = S^{-1}S^{-1}R^{-1}R = S^{-2} \\ &R^{-2}S^2R^2 = R^{-1}(R^{-1}S^2R)R = R^{-1}S^{-2}R = (R^{-1}S^2R)^{-1} = S^2 \end{aligned}$$

prove that $\langle R^2, S^2\rangle = \langle R^2\rangle \times \langle S^2\rangle$ is normal in $G(m,n)$. Write down a presentation for the quotient group to conclude $G(m,n)/\langle R^2, S^2\rangle \cong \mathbb{Z}_2^2$. In general, the subgroup $\langle R^2, S^2\rangle$ has no $\mathbb{Z}_2^2$ complement. The best we can do is to write $G(m,n)$ as a group extension

$$1 \longrightarrow \langle R^2, S^2\rangle \longrightarrow G(m,n) \longrightarrow \mathbb{Z}_2^2 \longrightarrow 1$$

described by the conjugation action computed above and 2-cocycle (after some more work)

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & R^2 & \cdot & R^2 \\ \cdot & R^{-2}S^2 & S^2 & R^{-2} \\ \cdot & S^{-2} & S^{-2} & \cdot \end{bmatrix}$$

where rows and columns are indexed by the elements $(1, [R], [S], [RS])$ of the quotient group $G(m,n)/\langle R^2, S^2\rangle$. We can instantiate a group with this presentation as an extension of $\mathbb{Z}_2^2$ by $\mathbb{Z}_{m+1} \times \mathbb{Z}_{n+1}$, which shows that $|G(m,n)| = 2(m+1)(n+1)$ as claimed. The rest of the map parameters follow. The mapping on $\mathrm{Aut}^+(\mathbf{R}(m,n))$ induced by $(R,S) \mapsto (R^{-1}, S^{-1})$ sends $(R^{-1}S)^2$ to $(RS^{-1})^2 = SR^{-1}RS^{-1} = 1$, so the chirality criterion is satisfied and the maps are reflexive. □

The number of maps of genus g in $\mathcal{F}_{mn}^{(2m+2,2n+2)}$ is the number of ways to write g as a product of two numbers: $|\{d \in \mathbb{Z} :\ d \mid g \wedge d \leq \lfloor \sqrt{g} \rfloor\}|$.

Remark 2.10.9. This family was discovered by Coxeter and Moser [CM1980]. They called it $\{p, q \mid 2\}$. The reason is that the relator $(R^{-1}S)^2$ (or equivalently, $(RS^{-1})^2$) signifies geometrically that all the "2-turn-paths" have length 2. To construct such a path, walk along the graph of the map and at each vertex, take the second turn to your left. Because the map is reflexive, the same holds for the -2-turn-paths, where you take the second turn right instead of left.

Remark 2.10.10. This family contains the 1-parameter subfamilies $\mathcal{F}_n^{(4,4n)}$ and $\mathcal{F}_{2n-2}^{(6,2n)}$ by setting $m = 1$ and $m = 2$, respectively.

2.10.4 — The polynomial family $\mathcal{F}_{1+\gcd(m,n)(mn-m-n)}^{(2m,2n)}$

Proposition 2.10.11. *The groups*

$$G(m,n) := \left\langle R, S \,\middle|\, R^{2m}, S^{2n}, (RS)^2, [R^2, S^2] \right\rangle.$$

are of order $4mn\gcd(m,n)$*. They define a polynomial family of reflexive platonic maps of genus* $1 + \gcd(m,n)(mn - m - n)$ *and type* $(2m, 2n)$*. These maps have parameters* $v = 2m\gcd(m,n)$*,* $e = 2mn\gcd(m,n)$*,* $f = 2n\gcd(m,n)$*.*

Because this thesis deals with topology, we include the following proof of genus 1, i.e. a proof with a hole in it.

Proof. With a healthy appetite for computation, we deduce four identities within such a group:

$$\begin{aligned}
&R^{-1}S^2R = R^{-2}RSSR = R^{-2}S^{-1}R^{-1}SR = R^{-2}[R,S]^{-1},\\
&R^{-1}[R,S]R = R^{-2}S^{-1}(RSR) = R^{-2}S^{-2},\\
&S^{-1}R^2S = S^{-2}SRRS = S^{-2}R^{-1}S^{-1}RS = S^{-2}[R,S],\\
&S^{-1}[R,S]S = S^{-1}R^{-1}S^{-1}RS^2 = RSS^{-1}RS^2 = R^2S^2.
\end{aligned}$$

Together, these prove that $H(m,n) := \langle R^2, S^2, [R,S]\rangle \lhd G(m,n)$. The quotient is quickly seen to be $\mathbb{Z}_2^2$. The structure of $H(m,n)$ is analyzed by computing

$$\begin{aligned}
&R^{-2}[R,S]R^2 = R^{-2}R^{-1}S^{-1}S^{-1}R^{-1}R^2 = R^{-1}R^{-2}S^{-2}R^2R^{-1} = R^{-1}S^{-2}R^{-1} = [R,S],\\
&S^{-2}[R,S]S^2 = S^{-2}SRRSS^2 = SS^{-2}R^2S^2S = SR^2S = [R,S].
\end{aligned}$$

Thus, all three of the generators of $H(m,n)$ commute, i.e. $H(m,n)$ is abelian. This is, alas, where we fail to demonstrate that $\langle R^2\rangle \cap \langle S^2\rangle = 1$. If we could, we would continue by computing

$$\begin{aligned}
&[R,S]^k = (SR^2S)^k = SR^2S^2R^2\cdots R^2S = (S^{-1}R^{2k}S)S^{2k},\\
&[R,S]^k = (R^{-1}S^{-2}R^{-1})^k = R^{-1}S^{-2}R^{-2}S^{-2}\cdots S^{-2}R^{-1} = (RS^{-2k}R^{-1})R^{-2k}.
\end{aligned}$$

This leads to $([R,S]R^2)^k = R^{-1}S^{-2k}R = (R^{-1}S^{-2}R)^k$, showing that $\operatorname{ord}([R,S]R^2) = n$. Similarly, $\operatorname{ord}([R,S]S^{-2}) = m$. Also,

$$(S^{-1}R^{2k}S)S^{2k} = 1 \Leftrightarrow R^{2k}S^{2k} = 1 \Leftrightarrow R^{2k} = S^{2k} = 1 \Leftrightarrow k \equiv 0 \bmod \operatorname{lcm}(m,n),$$

which tells us that $\operatorname{ord}([R,S]) = \operatorname{lcm}(m,n)$. Suppose $a,b \in \mathbb{Z}$ satisfy $am + bn = \gcd(m,n)$. Then

$$[R,S]^{\gcd(m,n)} = [R,S]^{am+bn} = S^{2am}R^{-2bn} = S^{2\gcd(m,n)}R^{-2\gcd(m,n)}.$$

Therefore, $\operatorname{ord}([R,S]R^2S^{-2}) = \gcd(m,n)$. We have now analyzed the subgroup as $H(m,n) \cong \mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_{\gcd(m,n)}$. The extension

$$1 \longrightarrow H(m,n) \longrightarrow G(m,n) \longrightarrow \mathbb{Z}_2^2 \longrightarrow 1$$

is described by the following conjugation action (as proven above) and 2-cocycle (by some additional effort, rows/columns indexed by $(1,[R],[S],[RS])$):

$$\begin{aligned}
&\text{con}_R : (R^2, S^2, [R,S]) \mapsto (R^2, [R,S]^{-1}R^2, R^2S^{-2})\\
&\text{con}_S : (R^2, S^2, [R,S]) \mapsto ([R,S]^{-1}S^2, S^2, R^2S^2)
\end{aligned}
\qquad
\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & R^2 & \cdot & R^2 \\ \cdot & RS^{-2}R & S^2 & R^2 \\ \cdot & S^{-2} & RS^2R^{-1} & \cdot \end{bmatrix}$$

As a byproduct of our analysis of the group $G(m,n)$, we have proved $|G(m,n)| = 4mn\gcd(m,n)$, $\operatorname{ord}(R) = 2m$, $\operatorname{ord}(S) = 2n$. That the chirality criterion is satisfied is immediate from the presentation. □

Remark 2.10.12. Clearly,

$$\mathcal{F}^{(2m,2n)}_{1+\gcd(m,n)(mn-m-n)}(n,m) = \mathcal{F}^{(2m,2n)}_{1+\gcd(m,n)(mn-m-n)}(m,n)^{\vee}.$$

For $m = 1$ we find $\mathcal{F}^{(2m,2n)}_{1+\gcd(m,n)(mn-m-n)}(1,n) = \mathbf{Hos}(2n)$. For $m, n \geq 2$, we have

$$g(m,n) \geq 1 + mn - m - n = (m-1)(n-1) \geq \max\{m-1, n-1\},$$

and we can use this to list the number of family members occurring in each genus up to a given bound (within reason). Below is a table with this data for small genera (counting duals).

genus	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
# maps	1	2	0	2	2	4	0	4	2	3	0	6	2	4	0

Since $mn - m - n \equiv 0 \bmod 2$ if and only if $m, n \equiv 0 \bmod 2$, we see that $g(m,n) = 1 + \gcd(m,n)(mn - m - n) \not\equiv 3 \bmod 4$, also visible in the above table. The number of family members with $2 \leq g \leq 101$ is 349 out of a total of 6104, or about 5.72% of all reflexive maps in these genera (counting duals).

3

Triangle groups and diagonal maps

In this chapter we study three geometric procedures that create new (reflexive) platonic maps from existing ones. These procedures stem from inclusions of triangle groups. Each inclusion leads to a group-theoretic link between two maps of certain types. The geometry behind the constructions is surprisingly elegant. We will name them the D_1-map, D_2-map and D_4-map, or 'diagonal maps'. Like polynomial families, these constructs help in detecting structure within the set of platonic maps.

We describe each of the three diagonal maps in the three respective Sections 3.1, 3.2, and 3.3. One may compare the diagonal maps to the ideas of Singerman and Syddall from [Syd1997] and [BCM2001, p.64]. They already made efforts to relate triangle group inclusions to platonic maps. We believe however, that our procedures have a broader range of applicability. Furthermore, the discussions shed some new light on the triangle group inclusions under consideration.

An application of diagonal maps will appear in Section 3.4. The construction of a diagonal map out of the original can be executed with geodesic edges, as will follow naturally from the discussion. As a consequence, the diagonal map has the same platonic surface as the original. In fact, Proposition 3.4.1 resolves the matter of which platonic maps have the same platonic surface. This knowledge saves us work when constructing algebraic models for platonic surfaces in Chapter 6.

Triangle group inclusions

As mentioned in Section 1.2, triangle groups are examples of Fuchsian groups. All inclusions between Fuchsian groups have been determined by David Singerman in [Sin1972]. The ones relevant to platonic maps are those where both groups are triangle groups of the form $\Delta(p, q, 2)$. Going over all the triangle groups in Singerman's list case by case, it is easy to determine all inclusions $\Delta(p', q', 2) < \Delta(p, q, 2)$ and $\Delta^+(p', q', 2) < \Delta^+(p, q, 2)$. There are three inclusions between elliptic triangle groups, which give group inclusions for platonic maps of genus 0. We list them with subgroup index indicated, and consider the group theory and geometry behind the inclusion in detail:

$$\Delta(2,2,2) \overset{3}{<} \Delta(3,3,2), \qquad \Delta(2,n,2) \overset{2}{\lhd} \Delta(2,2n,2), \qquad \Delta(2,2,2) \overset{6}{<} \Delta(3,4,2).$$

For the first inclusion, we have $\mathrm{Aut}(\mathbf{Tet}) = \Delta(3,3,2) \cong \mathrm{Sym}_4$, which contains three conjugate self-normalizing subgroups $\mathbb{Z}_2^2 \rtimes \mathbb{Z}_2$ of order 8/index 3. One can partition the vertex set of $\mathbf{Tet}$ into two pairs of (adjacent) vertices. The three subgroups correspond to the three possible partitions. For such a partition, the two vertices in each set bound an edge. Take the two midpoints of these edges. One of the three subgroups leaves this set of two points invariant and contains the rotation over π fixing them both, resulting in the group structure of $\mathrm{Aut}(\mathbf{Hos}(2))$. We can realize that map by choosing one of the two vertices, dropping perpendiculars to opposite sides of the adjacent triangles, and continuing these to the other vertex.

The second inclusion is instantiated by $\mathrm{Aut}(\mathbf{Hos}(2n)) = \Delta(2,2n,2) \cong \mathrm{Dih}_{2n} \times \mathbb{Z}_2$, which contains $\mathrm{Dih}_n \times \mathbb{Z}_2 \cong \mathrm{Aut}(\mathbf{Hos}(n))$ as a normal subgroup. To construct the latter out of the former geometrically, partition the $2n$ digons of $\mathbf{Hos}(2n)$ into n adjacent pairs. Joining up each pair to one new digon yields $\mathbf{Hos}(n)$.

The third inclusion is best understood by seeing it as part of the chain of inclusions $\Delta(2,2,2) \lhd \Delta(2,4,2) < \Delta(3,4,2)$. The first of these finer inclusions is the one between hosohedra we just described. The second arises from $\mathrm{Aut}(\mathbf{Oct}) = \Delta(3,4,2) \cong \mathrm{Sym}_4 \times \mathbb{Z}_2$. It contains three conjugate self-normalizing subgroups $\mathrm{Aut}(\mathbf{Hos}(4))$ of order 16. Geometrically speaking, there are three pairs of opposite vertices. Each pair has a corresponding subgroup leaving it invariant. That subgroup contains four rotations fixing them both, resulting in the group structure of $\mathbf{Hos}(4)$. Let the vertex pair be $\{v_1, v_2\}$. It is tempting to view the faces of $\mathbf{Hos}(4)$ as the amalgam of two faces of $\mathbf{Oct}$ with a common edge, one face adjacent to v_1, the other to v_2. In fact, the way that we can generalize this construction is by drawing new edges: drop perpendiculars from v_1, v_2 onto the opposite sides of the triangles they are in to get four edges.

There are also two inclusions between the parabolic triangle groups:

$$\Delta(4,4,2) \overset{4}{<} \Delta(4,4,2) \quad \text{and} \quad \Delta(3,6,2) \overset{4}{<} \Delta(3,6,2).$$

These give rise to infinite chains of group inclusions of platonic maps of genus 1. We leave them till the end of Sections 3.1 and 3.2, for they are part of the general correspondences that follow. The other inclusions include hyperbolic triangle groups, and certainly the first two have a much broader range of application to platonic maps. These two are:

$$\Delta(n,2n,2) \overset{3}{<} \Delta(3,2n,2) \quad \text{and} \quad \Delta(n,n,2) \overset{2}{<} \Delta(4,n,2).$$

Finally, one inclusion applies only to an orientation preserving triangle group:

$$\Delta(7,7,2)^+ \overset{9}{<} \Delta(3,7,2)^+.$$

The relevant construction for platonic maps is slightly different from the two previous ones. It is the last three that we will discuss in the coming sections. The only chain of multiple inclusions that can be built up is

$$\Delta(8,8,2) \overset{2}{<} \Delta(4,8,2) \overset{3}{<} \Delta(3,8,2)$$

and it is easy to check in any particular case whether this happens.

3.1 $\Delta(4, n, 2)$ and the D_1-map

Consider the triangle group

$$\Delta(4,n,2) = \langle a,b,c \mid a^2, b^2, c^2, (ab)^4, (bc)^n, (ca)^2 \rangle.$$

The subgroup $\langle aba, c, b \rangle$ is readily seen to be normal. Computing the quotient group $\Delta(4,n,2)/\langle aba, c, b \rangle$, we find that it has index 2. The subgroup has complement $\langle a \rangle$, so we find

$$\Delta(4,n,2) = \langle aba, c, b \rangle \rtimes \langle a \rangle,$$

with a acting as the permutation (13) on the given sequence of generators of the normal subgroup. Moreover, our index 2 subgroup satisfies all relations of $\Delta(n,n,2) = \langle a', b', c' \rangle$, and is in fact isomorphic to that group by $(a', b', c') \mapsto (aba, c, b)$. Injectivity of this mapping follows because the subgroup generators are the reflections in a $(n,n,2)$-triangle created as an amalgam of two $(4,n,2)$-triangles adjacent along an 'a-side' (opposite the π/n angle). Abstractly therefore,

$$\Delta(4,n,2) \cong \Delta(n,n,2) \rtimes \mathbb{Z}_2.$$

Because $\Delta(n,n,2) = \langle a', b', c' \rangle$ admits the involution $(a', b', c') \mapsto (c', b', a')$, we could also have utilized the isomorphism $(a', b', c') \mapsto (b, c, aba)$ with our subgroup. For the orientation preserving subgroup

$$\Delta^+(4,n,2) = \langle R, S \mid R^4, S^n, (RS)^2 \rangle,$$

generated by $R := ab$ and $S := bc$, the situation is comparable but a little more complicated. The index 2 subgroup

$$\Delta^+(n,n,2) = \Delta^+(4,n,2) \cap \Delta(n,n,2) = \langle S, R^{-1}SR \rangle$$

gives the same quotient $\Delta^+(4,n,2)/\Delta^+(n,n,2) \cong \mathbb{Z}_2$, but this time there is no complementary $\mathbb{Z}_2$. Instead, we can recover $G = \Delta^+(4,n,2)$ as an extension

$$1 \longrightarrow \Delta^+(n,n,2) \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow 1$$

in the following way: Let $F = F(r)$ be the free group on the indicated symbol, $\mathbb{Z}_2 = \langle r \mid r^2 \rangle$ and $\pi_F : F \to \mathbb{Z}_2$ the natural projection. Define $\phi : F \to \mathrm{Aut}(\Delta^+(n,n,2))$ by

$$\phi(r) : (S, R^{-1}SR) \mapsto (R^{-1}SR, (R^{-1}SR)S(R^{-1}SR)^{-1})$$

Note that $\mathrm{ord}(\phi(r)) = 4$. This follows from the fact that it mimics conjugation in $\Delta^+(4,n,2)$ by R. Also, $\mathrm{ord}(\mathrm{con}_S) = n$ and $\mathrm{ord}(\phi(r)\mathrm{con}_S) = 2$. Let

$$\sigma : \mathrm{Aut}(\Delta^+(n,n,2)) \to \mathrm{Out}(\Delta^+(n,n,2))$$

be the natural projection. Because $\phi(r)^2 \neq 1$, we do not get a homomorphism $\mathbb{Z}_2 \to \mathrm{Aut}(\Delta^+(n,n,2))$. This was to be expected, otherwise we *would* get a split sequence and hence a semi-direct product. But since $\phi(r)^2 = \mathrm{con}_{S^{-1}(R^{-1}SR)^{-1}}$,

$$\sigma\phi : F \to \mathrm{Out}(\Delta^+(n,n,2))$$

satisfies $\mathrm{ord}((\sigma\phi)(r)) = 2$ and this mapping factors through $\mathbb{Z}_2$ as $\sigma\phi = \pi_F\overline{\phi}$. Put otherwise, ϕ is a lift of $\pi_F\overline{\phi}$ over σ. This allows us to define

$$G := (F \,{}_{\phi}\!\ltimes \Delta^+(n,n,2))/N$$

by dividing out the normal closure N of $\langle (r^2, R^{-2}) \rangle$. We get an isomorphism $G \to \Delta^+(4,n,2)$ determined by $(r,1) \bmod N \mapsto R$ and $(1,S) \bmod N \mapsto S$. That this construction works is a basic result on group extensions, see e.g. [Rob1982, Ch. 11].

To understand the geometry behind both group inclusions better, consider that an inclusion $\Delta(4,n,2) < \mathrm{Aut}(\mathbb{H}^2)$ induces a tiling of $\mathbb{H}^2$ by quadrilaterals. The graph of that tiling is bipartite. Choosing one of the vertex sets of the partition, one can draw diagonals between two vertices in this set that lie across a square to create a $\Delta(n,n,2)$-tiling. The two choices of vertex set correspond to the two isomorphisms $\Delta(n,n,2) \to \langle aba, c, b \rangle < \Delta(4,n,2)$ discussed above. The inclusion is illustrated in Figure 3.1.

Suppose now that we have a reflexive platonic map $\mathbf{R}$ defined by $\Gamma \lhd \Delta(4,n,2)$ of finite index. We always have $\Gamma \cap \Delta(n,n,2) \lhd \Delta(n,n,2)$ and get a reflexive platonic map $\mathbf{R}'$ of type (n,n) and genus

$$g' = 1 + 2\frac{|\Delta(n,n,2) : \Gamma \cap \Delta(n,n,2)|}{|\Delta(4,n,2) : \Gamma|}(g-1)$$

defined by $\mathrm{Aut}(\mathbf{R}') = \Delta(n,n,2)/(\Gamma \cap \Delta(n,n,2))$. Whenever $\Gamma < \Delta(n,n,2)$, we get $g' = g$. We delve deeper into the latter situation.

Proposition 3.1.1. *Let* $\mathbf{R}$ *be a reflexive platonic map of genus* g *and type* $(4,n)$*, and let* (a,b,c) *be standard generators of* $\mathrm{Aut}(\mathbf{R})$*. If*

$$[\mathrm{Aut}(\mathbf{R}) : \langle aba, c, b \rangle] = 2$$

then with $a' = aba$*,* $b' = c$*,* $c' = b$*, the group* $\langle a', b', c' \rangle$ *is the standard map presentation of a new reflexive platonic map* $\mathbf{R}'$ *of genus* g *and type* (n,n)*. The map* $\vee : (a',b',c') \mapsto (c',b',a')$ *defines an automorphism of* $\mathrm{Aut}(\mathbf{R}')$*.*

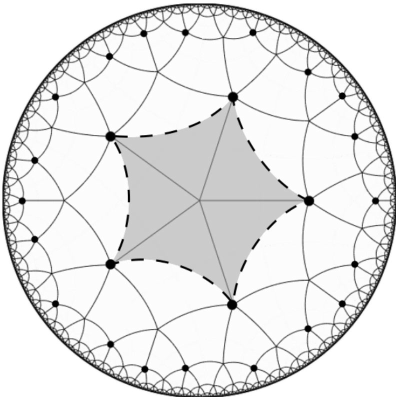


Figure 3.1: The inclusion $\Delta(n, n, 2) < \Delta(4, n, 2)$ for the case $n = 5$. A fundamental $(2, 5, 5)$-triangle inside the 5-gon consists of two adjacent $(4, 5, 2)$-triangles. Note the bipartition of the vertex set for the $(2, 4, 5)$-tiling, one set indicated by black dots. Each choice corresponds to an embedding $\Delta(n, n, 2) < \Delta(4, n, 2)$.

Conversely, let $\mathbf{R}'$ *be a reflexive platonic map of genus* g *and type* (n, n) *such that* $\mathrm{Aut}(\mathbf{R}') = \langle a', b', c' \rangle$ *admits the automorphism* $\vee : (a', b', c') \mapsto (c', b', a')$. *Then the semi-direct product* $\mathrm{Aut}(\mathbf{R}') \rtimes \mathbb{Z}_2$ *with* $\mathbb{Z}_2 = \langle a \rangle$ *and conjugation action*

$$a^{-1}a'a = c' \qquad a^{-1}b'a = b' \qquad a^{-1}c'a = a'$$

defines a reflexive platonic map of genus g *and type* $(4, n)$ *with standard generators* (a, a', b').

Proof. Let $\Delta(4, n, 2) = \langle \tilde{a}, \tilde{b}, \tilde{c} \rangle$ and $\mathrm{Aut}(\mathbf{R}) = \langle \tilde{a}, \tilde{b}, \tilde{c} \rangle / \Gamma$, where $\Gamma \lhd \Delta(4, n, 2)$. By the subgroup correspondence theorem $\Gamma < \langle \tilde{a}\tilde{b}\tilde{a}, \tilde{c}, \tilde{b} \rangle \lhd \Delta(4, n, 2)$. The second inclusion is of index 2. So the middle group is isomorphic to $\Delta(n, n, 2)$ and we identify it as such. Since $\Gamma \lhd \Delta(4, n, 2)$ it is certainly normal in $\Delta(n, n, 2)$, and we get a reflexive platonic map $\Delta(n, n, 2)/\Gamma$ with standard generators $(a', b', c') = (\tilde{a}\tilde{b}\tilde{a}\Gamma, \tilde{c}\Gamma, \tilde{b}\Gamma) = (aba, c, b)$. The type is (n, n) because $\mathrm{ord}(abac) = \mathrm{ord}(abca) = \mathrm{ord}(bc) = n$ and $\mathrm{ord}(cb) = n$.

For the proof of the second statement, note that the inner automorphism $\mathrm{con}_{\tilde{a}}$ of $\Delta(4, n, 2)$ normalizes $\Delta(n, n, 2) = \langle \tilde{a}', \tilde{b}', \tilde{c}' \rangle$ with $\tilde{a}' = \tilde{a}\tilde{b}\tilde{a}$, $\tilde{b}' = \tilde{c}$, $\tilde{c}' = \tilde{b}$, and acts on it as the permutation $(\tilde{a}' \; \tilde{c}')$. Since $\mathrm{con}_{\tilde{a}}$ normalizes Γ as well, it factors through $\pi : \Delta(n, n, 2) \to \Delta(n, n, 2)/\Gamma$, yielding the automorphism $\vee$.

The converse assumes that we have $\Gamma \lhd \Delta(n, n, 2)$. We embed $\Delta(n, n, 2) < \Delta(4, n, 2)$ as above. That $\vee$ is an automorphism implies that $\mathrm{con}_{\tilde{a}}$ restricted to $\Delta(n, n, 2)$ factors through $\pi : \Delta(n, n, 2) \to \Delta(n, n, 2)/\Gamma$, hence $\mathrm{con}_{\tilde{a}}(\Gamma) = \Gamma$. It follows that $\Gamma \lhd \Delta(n, n, 2) \cdot \langle a \rangle = \Delta(4, n, 2)$. We thus have a platonic map $\mathbf{R} = \Delta(4, n, 2)/\Gamma$. The existence of $\vee$ also implies that $a := \tilde{a}$ acts on $\mathrm{Aut}(\mathbf{R}) = \mathrm{Aut}(\mathbf{R}') \rtimes \langle a \rangle$ with conjugation action as claimed. Obviously a is an involution, and $(aa')^2 = \vee(a')a' = c'a'$ and $(ab')^2 = \vee(b')b' = (b')^2 = 1$, so that $\mathrm{ord}(aa') = 4$ and $\mathrm{ord}(ab') = 2$, proving that

$\mathbf{R}$ is of type $(4, n)$ with map presentation as claimed. □

The same remarks and reasoning as above apply mutatis mutandis for any platonic map $\mathbf{M}$ and its group $\mathrm{Aut}^+(\mathbf{M})$, with some extra cumbersome details. We leave its proof to the reader.

Proposition 3.1.2. *Let $\mathbf{M}$ be a platonic map of genus g and type $(4, n)$, and let (R, S) be standard generators of $\mathrm{Aut}^+(\mathbf{M})$. If*

$$[\mathrm{Aut}^+(\mathbf{M}) : \langle S, R^{-1}SR \rangle] = 2$$

then with $R' = S$ and $S' = R^{-1}SR$, the group $\langle R', S' \rangle$ is the standard map presentation of a new platonic map $\mathbf{M}'$ of genus g and type (n, n). The map $\vee : (R', S') \mapsto ((S')^{-1}, (R')^{-1})$ defines an automorphism of $\mathrm{Aut}^+(\mathbf{M}')$.

Conversely, let $\mathbf{M}'$ be a platonic map of genus g and type (n, n) such that $\mathrm{Aut}^+(\mathbf{M}') = \langle R', S' \rangle$ admits the automorphism $\vee : (R', S') \mapsto ((S')^{-1}, (R')^{-1})$. Then the group

$$G := F(R) \,{}_{\phi}\!\ltimes \mathrm{Aut}^+(\mathbf{M}) \;\; / \;\; \langle (R^2, S'R') \rangle^{F(R) \,{}_{\phi}\!\ltimes \mathrm{Aut}^+(\mathbf{M})}$$

with conjugation action

$$\phi(R) : (R', S') \mapsto (S', S'R'(S')^{-1})$$

is a degree 2 extension of $\mathrm{Aut}^+(\mathbf{M})$ by $\mathbb{Z}_2$ that defines a new platonic map of genus g and type $(4, n)$ with standard map generators (R, R'). □

The geometric interpretation of these propositions is more enlightening.

Proposition 3.1.3. *Let $\mathbf{M}$ be a platonic map of genus g and type $(4, n)$ with a bipartite graph $(\mathrm{cells}_0, \mathrm{cells}_1)$. There is a unique self-dual platonic map $\mathbf{M}'$ of genus g and type (n, n) with standard map presentation for $\mathrm{Aut}^+(\mathbf{M}')$ that of $\langle S, R^{-1}SR \rangle$. If $\mathbf{M}$ is reflexive, then so is $\mathbf{M}'$ and we have $\mathrm{Aut}(\mathbf{M}') = \langle aba, c, b \rangle$.*

Conversely, if $\mathbf{M}'$ is a self-dual platonic map of genus g and type (n, n), then there is a unique platonic map of type $(4, n)$ and genus g with a bipartite graph and standard map presentation as found in Proposition 3.1.2. If $\mathbf{M}'$ is reflexive so is $\mathbf{M}$, and then $\mathrm{Aut}(\mathbf{M})$ has standard map presentation as found in Proposition 3.1.1.

Again we only prove the reflexive version.

Proof. Suppose $\mathbf{R}$ is of type $(4, n)$ and choose a fundamental triangle T with reflections in its sides (a, b, c) generating $\mathrm{Aut}(\mathbf{R})$. The elements b and c fix the vertex v of this triangle that is also a vertex of the map. The element aba fixes the vertex of the map opposite to v on the 4-gon in which T lies. Consequently, if $V_1 \sqcup V_2$ (disjoint union) is a bipartition of the vertices and $v \in V_1$ (without loss of generality), then $\langle aba, c, b \rangle$ fixes V_1 setwise and is thus a strict subgroup of $\mathrm{Aut}(\mathbf{R})$. It must have index 2. Apply the previous proposition and construct $\mathbf{R}'$. Taking duals induces the homomorphism $\mathrm{Aut}(\mathbf{R}') \to \mathrm{Aut}((\mathbf{R}')^\vee)$ with $(a', b', c') \mapsto (c', b', a')$. According to the previous proposition, this is an automorphism. Thus, $\mathbf{R}' \cong (\mathbf{R}')^\vee$.

Conversely, for a self-dual platonic map $\mathbf{R}'$, the map $(a', b', c') \mapsto (c', b', a')$ induces a group automorphism. The previous propositions then allow the construction of the platonic map $\mathbf{R}$ of type $(4, n)$ whose existence is claimed now. That map will have a bipartite graph, since $\mathrm{Aut}(\mathbf{R})$ contains the index two subgroup $\mathrm{Aut}(\mathbf{R}')$. □

Definition 3.1.4. *Given a platonic map* $\mathbf{M}$ *of type* $(4, n)$ *with a bipartite graph, we define the* D_1-map $D_1(\mathbf{M})$ *(diagonal map across one face) to be the platonic map of type* (n, n) *constructed above.*

The name D_1-map is justified by the geometric interpretation of the construction. Suppose you have a platonic map $\mathbf{M}$ of type $(4, n)$ with a bipartite 1-skeleton and vertex set $V = V_1 \sqcup V_2$. Across every 4-gon, draw the diagonal between the two vertices of V_1 incident to it. Each vertex of V_2 will be the center of an n-gon defined by these diagonals. Around each vertex of V_1 lie n such 2-cells, so we get a new map of type (n, n). The proposition says this map is again platonic and self-dual. A fundamental triangle consists of two fundamental triangles of $\mathbf{M}$, glued together along their a-side. One can also choose V_2 and glue along c-sides. This yields an isomorphic map. Conversely, given a self-dual platonic map $\mathbf{M}'$ of type (n, n), draw the barycentric subdivision. The four fundamental triangles that are incident to a given edge of the map form a regular 4-gon with angles $2\pi/n$. The map of type $(4, n)$ so obtained will be platonic because self-duality allows for a reflection that fixes such a square but interchanges vertices and face centers of $\mathbf{M}'$. One could write down an explicit combinatorial definition of the cells and the incidence relation of $\mathbf{M}'$ in terms of that of $\mathbf{M}$ and vice versa, but we leave that to the reader.

The two constructions described are each others inverses and we have proved that

$$D_1 : \{\text{Type } (4, n) \text{ maps with a bipartite graph}\} \to \{\text{self-dual maps of type } (n, n)\}$$

is a bijection (graded by the genus).

Remark 3.1.5. The condition of bipartiteness can be reformulated as the existence of a coclique of size $v/2$ inside the graph $(\mathrm{cells}_0, \mathrm{cells}_1)$ of $\mathbf{R}$. Compare the D_2-map, described in the next section.

Remark 3.1.6. Since $\mathbf{M}_r = (D_1(\mathbf{M}))_r$, the D_1-map construction apparently yields geodesic edges again when applied to a platonic map on a Riemann surface. This 'miracle' occurs because two hyperbolic $(4, n, 2)$-triangles glued along their 'a-side' yield a $(n, n, 2)$-triangle.

Example 3.1.7. Not all platonic maps of type $(4, n)$ are bipartite. A counterexample is $\mathbf{R}_{6.2}$, of type $(4, 6)$. Indeed, there is no self-dual map of genus 6 and type $(6, 6)$. And not all platonic maps of type (n, n) are self-dual. The smallest example of that is $\mathbf{R}_{4.8}$ of type $(6, 6)$.

Example 3.1.8. In the case $n = 3$, the correspondence is between any map with a bipartite graph of type $(4, 3)$ and the set of self-dual maps of type $(3, 3)$. Indeed we find that $D_1(\mathbf{Cub}) = \mathbf{Tet}$. One can actually execute the procedure for the cube by cutting off four pyramids to get a tetrahedron, as is well known. In the case $n = 4$, there is the infinite normal chain $\mathrm{Aut}(\mathbf{R}_{1.3:1}) \lhd \mathrm{Aut}(\mathbf{R}_{1.4:1}) \lhd \mathrm{Aut}(\mathbf{R}_{1.3:2}) \lhd \cdots \lhd \mathrm{Aut}(\mathbf{R}_{1.3:n}) \lhd \mathrm{Aut}(\mathbf{R}_{1.4:n}) \lhd \mathrm{Aut}(\mathbf{R}_{1.3:n+1}) \lhd \cdots$

3.2 $\Delta(3, 2n, 2)$ and the D_2-map

Consider the triangle group

$$\Delta(3, 2n, 2) = \langle a, b, c \mid a^2, b^2, c^2, (ab)^3, (bc)^{2n}, (ca)^2 \rangle$$

and its action on $\mathbb{H}^2$ after fixing a $(3, 2n, 2)$-triangle. The subgroup $\langle abcba, c, b\rangle$ is generated by reflections in a $(n, 2n, 2)$-triangle consisting of three suitable chosen adjacent $(3, 2n, 2)$-triangles, and so $\langle abcba, c, b\rangle \cong \Delta(n, 2n, 2)$. Study Figure 3.2 to see this. The geometry also proves that the subgroup is of index 3, and that it is not normal. In fact,

$$\mathrm{Core}(\Delta(n, 2n, 2)) = \bigcap_{k=0}^{2} (ab)^{-k} \Delta(n, 2n, 2)(ab)^k = \langle bcb, c, abcba\rangle.$$

The normal core has a complement $\langle a, b\rangle \cong \langle a, b \mid a^2, b^2, (ab)^3\rangle \cong \mathrm{Sym}_3$ and we can have the isomorphism

$$\Delta(3, 2n, 2) = \mathrm{Core}(\Delta(n, 2n, 2)) \rtimes_\phi \mathrm{Sym}_3$$

with conjugation action $\phi : (a, b) \mapsto ((13), (23))$, the images permuting the given generators of the normal core. For the orientation preserving subgroup

$$\Delta^+(3, 2n, 2) = \langle R, S \mid R^3, S^{2n}, (RS)^2\rangle$$

the situation is comparable, but a little more complicated. The index 3 subgroup

$$\Delta^+(n, 2n, 2) = \Delta^+(3, 2n, 2) \cap \Delta(n, 2n, 2) = \langle S^2, R^{-1}SR\rangle$$

has normal core

$$\mathrm{Core}(\Delta^+(n, 2n, 2)) = \bigcap_{k=0}^{2} R^{-k}\Delta(n, 2n, 2)R^k = \langle S^2, R^{-1}S^2R, R^{-2}S^2R^2\rangle.$$

We still obtain the quotient $\Delta^+(3, 2n, 2)/\mathrm{Core}(\Delta^+(n, 2n, 2)) \cong \mathrm{Sym}_3$, but this time there is no complementary Sym_3. Instead, we can recover $\Delta^+(3, 2n, 2)$ as an extension

$$1 \longrightarrow \mathrm{Core}(\Delta^+(n, 2n, 2)) \longrightarrow G \longrightarrow \mathrm{Sym}_3 \longrightarrow 1$$

in the following way: Let $F = F(r, s)$ be the free group on the indicated symbols, $\mathrm{Sym}_3 = \langle r, s \mid r^3, s^2, (rs)^2\rangle$ and $\pi_F : F \to \mathrm{Sym}_3$ the natural projection. Define $\phi : F \to \mathrm{Aut}(\mathrm{Core}(\Delta^+(n, 2n, 2)))$ by

$$\phi(r) : (S^2, R^{-1}S^2R, R^{-2}S^2R^2) \mapsto (R^{-1}S^2R, R^{-2}S^2R^2, S^2),$$
$$\phi(s) : (S^2, R^{-1}S^2R, R^{-2}S^2R^2) \mapsto (S^2, R^{-2}S^2R^2, S^{-2}(R^{-1}S^2R)S^2).$$

Note that $\operatorname{ord}(\phi(r)) = 3$, $\operatorname{ord}(\phi(s)) = 2n$ and $\operatorname{ord}(\phi(rs)) = 2$. This follows from the fact that they mimic conjugation in $\Delta^+(3, 2n, 2)$ by R, S, and RS respectively. It is also visible directly from their definition. Let

$$\sigma : \operatorname{Aut}(\operatorname{Core}(\Delta^+(n, 2n, 2))) \to \operatorname{Out}(\operatorname{Core}(\Delta^+(n, 2n, 2)))$$

be the natural projection. Because $\operatorname{ord}(\phi(s)) \neq 2$, conjugation in $\Delta^+(3, 2n, 2)$ restricted to $\operatorname{Core}(\Delta^+(n, 2n, 2))$ does not factor through a homomorphism $\operatorname{Sym}_3 \to \operatorname{Aut}(\operatorname{Core}(\Delta^+(n, 2n, 2)))$. But since $\phi(s)^2 = \operatorname{con}_{S^2}$,

$$\sigma\phi : F \to \operatorname{Out}(\operatorname{Core}(\Delta^+(n, 2n, 2)))$$

satisfies $\operatorname{ord}((\sigma\phi)(s)) = 2$, and this map does factor through Sym_3 as $\sigma\phi = \pi_F\overline{\phi}$. Put otherwise, ϕ is a lift of $\pi_F\overline{\phi}$ over σ. This allows us to define

$$G := (F \,{}_{\phi}\!\ltimes \operatorname{Core}(\Delta^+(n, 2n, 2)))/N$$

by dividing out the normal subgroup

$$N := \langle (r^3, 1), (s^2, S^{-2}), ((rs)^2, 1)\rangle^{F\,{}_{\phi}\!\ltimes \operatorname{Core}(\Delta^+(n,2n,2))}.$$

We have an isomorphism $G \to \Delta^+(3, 2n, 2)$ by $(r, 1) \bmod N \mapsto R$ and $(s, 1) \bmod N \mapsto S$. That this construction works is a basic result on group extensions, see e.g. [Rob1982, Ch. 11].

To understand the geometry behind both group inclusions better, consider that an inclusion $\Delta(3, 2n, 2) < \operatorname{Aut}(\mathbb{H}^2)$ induces a tiling of $\mathbb{H}^2$ by triangles. The graph of this tiling is tripartite. Choosing one of the vertex sets of the partition, one can draw diagonals between vertices in this set lying across two triangles to create a $\Delta(n, 2n, 2)$-tiling. This is illustrated in Figure 3.2.

Suppose now that we have a reflexive platonic map $\mathbf{R}$ defined by $\Gamma \lhd \Delta(3, 2n, 2)$, $\Gamma \leq \Delta^+(3, 2n, 2)$ of finite index. Writing $\Delta_k = (ab)^{-k}\Delta(n, 2n, 2)(ab)^k$, we have $\Gamma \cap \Delta_k \lhd \Delta_k$ ($k = 0, 1, 2$). Thus, we get three new reflexive platonic maps $\mathbf{R}'_k$ of type $(n, 2n)$ and genus

$$g' = 1 + 3\frac{|\Delta_k : \Gamma \cap \Delta_k|}{|\Delta(3, 2n, 2) : \Gamma|}(g - 1),$$

where $\operatorname{Aut}(\mathbf{R}'_k) = \Delta_k/(\Gamma \cap \Delta_k)$. Since all three subgroups Δ_k are conjugate within $\Delta(3, 2n, 2)$, these three maps are identical. In the case where $\Gamma < \Delta_k$ for some k, it holds for all k, and we get a map with $g' = g$. We delve deeper into the latter situation.

Proposition 3.2.1. *Let $\mathbf{R}$ be a reflexive platonic map of genus g and type $(3, 2n)$, and let (a, b, c) be standard generators of $\operatorname{Aut}(\mathbf{R})$. If $\langle abcba, c, b\rangle$ has index 3 in $\operatorname{Aut}(\mathbf{R})$, then it generates a standard map presentation of a new reflexive platonic map $\mathbf{R}'$ of genus g and type $(n, 2n)$. Let $(a', b', c') = (abcba, c, b)$. The subgroup $\langle a', b', c'b'c'\rangle < \operatorname{Aut}(\mathbf{R}')$ has index 2 and the automorphism con_{ab} restricts to this subgroup and cyclically permutes its three generators.*

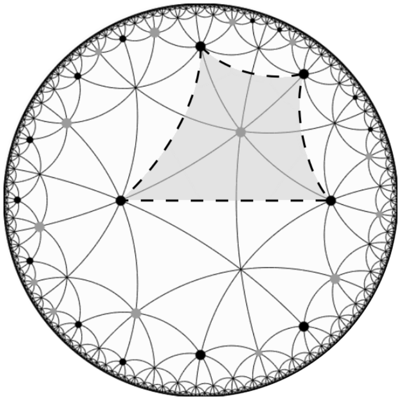


Figure 3.2: The inclusion $\Delta(n,2n,2) < \Delta(3,2n,2)$ for the case $n=4$. A fundamental $(4,8,2)$-triangle inside the 4-gon consists of three adjacent $(3,8,2)$-triangles. Note the three cocliques of vertices for the $(3,8,2)$-tiling, one set indicated by black, another by grey dots. Each choice corresponds to an embedding $\Delta(n,2n,2) < \Delta(3,2n,2)$.

Conversely, let $\mathbf{R}'$ *be a reflexive platonic map of genus* g *and type* $(n,2n)$ *such that*

$$[\mathrm{Aut}(\mathbf{R}') : \langle a', b', c'b'c'\rangle] = 2.$$

Suppose that the subgroup permits of the automorphisms of order 3 *that cyclically permute its generators. Then the semi-direct product* $\langle a', b', c'b'c'\rangle \rtimes \mathrm{Sym}_3$ *with* $\mathrm{Sym}_3 = \langle a, b \,|\, a^2, b^2, (ab)^3\rangle$ *and conjugation action*

$$a : (a', b', c'b'c') \mapsto (c'b'c', b', a') \qquad b : (a', b', c'b'c') \mapsto (a', c'b'c', b')$$

defines a reflexive platonic map $\mathbf{R}$ *of genus* g *and type* $(3,2n)$ *with standard generators* (a,b,b')*.*

Proof. Let $\Delta(3,2n,2) = \langle \tilde{a}, \tilde{b}, \tilde{c}\rangle$ and $\mathrm{Aut}(\mathbf{R}) = \langle \tilde{a}, \tilde{b}, \tilde{c}\rangle/\Gamma$. By the subgroup correspondence theorem from group theory, the assumption implies that $\Gamma < \langle \tilde{a}\tilde{b}\tilde{c}\tilde{b}\tilde{a}, \tilde{c}, \tilde{b}\rangle < \Delta(3,2n,2)$, the second inclusion being of index 3. This index 3 subgroup is isomorphic to $\Delta(n,2n,2)$ and we identify it as such. Since $\Gamma \lhd \Delta(3,2n,2)$, certainly $\Gamma \lhd \Delta(n,2n,2)$ and we get a platonic map $\Delta(n,2n,2)/\Gamma$ with standard generators $(a',b',c') = (\tilde{a}\tilde{b}\tilde{c}\tilde{b}\tilde{a}\Gamma, \tilde{c}\Gamma, \tilde{b}\Gamma) = (abcba, c, b)$. The automorphism $\mathrm{con}_{\tilde{a}\tilde{b}}$ of $\Delta(3,2n,2)$ normalizes the subgroup

$$\bigcap_{k\in\{0,1,2\}} (\tilde{a}\tilde{b})^{-k}\Delta(n,2n,2)(\tilde{a}\tilde{b})^{k} = \langle \tilde{a}\tilde{b}\tilde{c}\tilde{b}\tilde{a}, \tilde{c}, \tilde{b}\tilde{c}\tilde{b}\rangle = \langle \tilde{a}', \tilde{b}', \tilde{c}'\tilde{b}'\tilde{c}'\rangle$$

and so induces an automorphism of order 3 on it. Since $\mathrm{con}_{\tilde{a}\tilde{b}}$ also normalizes Γ, it factors through the projection $\pi : \Delta(n,2n,2) \to \Delta(n,2n,2)/\Gamma$, yielding con_{ab} of $\langle a', b', c'b'c'\rangle$. It is easy to check that $[\langle a', b', c'\rangle : \langle a', b', c'b'c'\rangle] = 2$ and that the action cyclically permutes the three generators of the smaller group.

The converse assumes that we have $\Gamma \lhd \Delta(n,2n,2)$. We embed $\Delta(n,2n,2)$ in $\Delta(3,2n,2)$ as Δ_1 above. The conjugation $\mathrm{con}_{\tilde{a}\tilde{b}}$ normalizes $\tilde{H} = \Delta_1 \cap \Delta_2 \cap \Delta_3$ and permutes its generators $(\tilde{a}\tilde{b}\tilde{c}\tilde{b}\tilde{a}, \tilde{c}, \tilde{b}\tilde{c}\tilde{b})$ as a cycle (123). The assumption on the existence of the order 3 automorphisms of $H = \tilde{H}/\Gamma$ implies that $\mathrm{con}_{\tilde{a}\tilde{b}}$ factors through $\pi : \tilde{H} \to \tilde{H}/\Gamma$, hence that $\mathrm{con}_{\tilde{a}\tilde{b}}(\Gamma) = \Gamma$. Also, $\Gamma^{\tilde{b}} = \Gamma$ because $\tilde{b} \in \Delta_1$. Taken together, we see that $\Gamma \lhd \Delta_1 \cdot \langle \tilde{a}\tilde{b}, \tilde{b}\rangle = \Delta(3,2n,2)$. We thus have a platonic map $\mathbf{R} = \Delta(3,2n,2)/\Gamma$. The structure of the second factor is Sym_3, since $\mathrm{ord}(\tilde{a}\tilde{b}) = 3$, $\mathrm{ord}(\tilde{b}) = 2$, and $\mathrm{ord}(\tilde{a}\tilde{b}\tilde{b}) = \mathrm{ord}(\tilde{a}) = 2$. Setting $a := (\tilde{a}'\tilde{b}')\tilde{b}'$ and $b = \tilde{b}'$ yields the presentation of Sym_3 as in the proposition. Obviously a and b are involutions and $(ab)^3 = 1$. Also, $(ab')^2 = (ab'a)b' = (b')^2 = 1$ and $(bb')^2 = (bb'b)b' = c'b'c'b'$, so that $\mathrm{ord}(ab') = 2$ and $\mathrm{ord}(bb') = 2n$, proving that $\mathbf{R}$ is of type $(3,2n)$ with the map presentation as claimed. □

The same construction and reasoning applies mutatis mutandis for a general platonic map $\mathbf{M}$ and its group $\mathrm{Aut}(\mathbf{M}) = \mathrm{Aut}^+(\mathbf{M})$. We therefore leave the proof of the following proposition to the reader.

Proposition 3.2.2. *Let $\mathbf{M}$ be a platonic map of genus g and type $(3,2n)$, and let (R,S) be standard generators of $\mathrm{Aut}(\mathbf{M})$. If $\langle S^2, R^{-1}SR\rangle$ has index 3 in $\mathrm{Aut}(\mathbf{M})$, then it is the automorphism group of a new platonic map $\mathbf{M}'$ of genus g and type $(n,2n)$. Writing $(R',S') = (S^2, R^{-1}SR)$, the subgroup $\langle R', (S')^2, S'R'(S')^{-1}\rangle = \langle S^2, R^{-1}S^2R, R^{-2}S^2R^2\rangle$ of $\mathrm{Aut}(\mathbf{M}')$ has index 2 and the automorphism con_R restricts to this subgroup and cyclically permutes its three generators.*

Conversely, let $\mathbf{M}'$ be a platonic map of genus g and type $(n,2n)$ with standard map generators (R',S'), such that the subgroup

$$H := \langle R', (S')^2, S'R'(S')^{-1}\rangle < \mathrm{Aut}(\mathbf{M}')$$

is of index 2. Suppose H permits of the automorphisms of order 3 that cyclically permute its generators. Then we can form a group extension G of H by Sym_3 in the following way: Let $F = F(R,S)$ be the free group on the indicated symbols, and $\mathrm{Sym}_3 = \langle R, S \mid R^3, S^2, (RS)^2\rangle$. Define $\phi : F \to \mathrm{Aut}(H)$ by

$$\phi(R) : (R', (S')^2, S'R'(S')^{-1}) \mapsto ((S')^2, S'R'(S')^{-1}, R')$$
$$\phi(S) : (R', (S')^2, S'R'(S')^{-1}) \mapsto (R', S'R'(S')^{-1}, (R')^{-1}(S')^2R')$$

For $\pi : F(R,S) \to \mathrm{Sym}_3$ and $\pi' : \mathrm{Aut}(H) \to \mathrm{Out}(H)$ we have $\phi\pi' = \pi\phi'$ Define $G := (F(R,S) \ {}_{\phi}\!\ltimes H)/N$, where

$$N = \langle (R^3, 1), (S^2, (R')^{-1}), ((RS)^2, 1)\rangle.$$

Then G is the map automorphism group of a new platonic map of type $(3,2n)$ with standard map generators $(R,1) \bmod N$ and $(S,1) \bmod N$. □

Remark 3.2.3. Working with platonic maps of low genus, one at first gets the idea that the subgroup $\langle S^2, R^{-1}S^2R, R^{-2}S^2R^2\rangle$ always has a complement isomorphic to Sym_3 in $\mathrm{Aut}^+(\mathbf{M})$. The first counterexamples are the maps $\mathbf{R}_{28.2}$ and $\mathbf{C}_{28.2}$.

Again, a geometric version of these propositions can help us attain a better grasp of what's going on. We introduce new terminology to facilitate formulation of the result.

Definition 3.2.4. *Let* $\mathbf{M}$ *be a platonic map of type* $(n, 2n)$ *whose dual has a bipartite graph. A* map trial $\mathbf{M}^{\vee 3}$ *of* $\mathbf{M}$ *is formed as follows. Consider the Riemann surface* $(\mathbf{M})_r = (\mathbf{M}^\vee)_r$ *and realize the two maps on it. Take the vertices of one set of the bipartition of* $\mathrm{cells}_0(\mathbf{M}^\vee)$*. Whenever two of these vertices lie in* 2*-cells of* $\mathbf{M}$ *incident to a common vertex of* $\mathbf{M}$*, connect them by a geodesic segment that lies within the fundamental triangles of* $\mathbf{M}$ *incident to this common vertex. If* $\mathbf{M}^{\vee 3} \cong \mathbf{M}$*, then* $\mathbf{M}$ *is called* self-trial*.*

The map trial $\mathbf{M}^{\vee 3}$ is seen to be platonic and reflexive if $\mathbf{M}$ is, because we can apply the elements of $\mathrm{Aut}(\mathbf{M})$ to it, and this action is transitive on the oriented $(0,1)$-flags of $\mathbf{M}^{\vee 3}$, and on the oriented $(0,1,2)$-flags if $\mathbf{M}$ is reflexive. Further consideration of Figure 3.2 makes this clear.

Proposition 3.2.5. *Let* $\mathbf{M}$ *be a platonic map of genus* g *and type* $(3, 2n)$ *with a tripartite graph* $(\mathrm{cells}_0(\mathbf{M}), \mathrm{cells}_1(\mathbf{M}))$*. There is a unique self-trial platonic map* $\mathbf{M}'$ *of genus* g *and type* $(n, 2n)$ *with standard map presentation* $\mathrm{Aut}^+(\mathbf{M}) = \langle S^2, R^{-1}SR\rangle$*. If* $\mathbf{M}$ *is reflexive, so is* $\mathbf{M}'$*, and then* $\mathrm{Aut}(\mathbf{M}') = \langle abcba, c, b\rangle$*.*

Conversely, if $\mathbf{M}'$ *is a self-trial platonic map of type* $(n, 2n)$ *and genus* g*, there is a unique platonic map* $\mathbf{M}$ *of type* $(3, 2n)$ *and genus* g *with a bipartite graph and standard map presentation given by the group extension of Proposition 3.2.2. If* $\mathbf{M}'$ *is reflexive, so is* $\mathbf{M}$*, and then* $\mathrm{Aut}(\mathbf{M})$ *is as found in Proposition 3.2.1.*

Proof. If $\mathbf{M}$ is of type $(3, 2n)$ and has a tripartite graph, then $\langle S^2, R^{-1}SR\rangle$ is the subgroup of all elements fixing one specific maximal vertex coclique. It therefore has index 3 in $\mathrm{Aut}^+(\mathbf{M})$ and Proposition 3.2.2 applies. Self-triality of the resulting map $\mathbf{M}'$ of type $(n, 2n)$ is evident by the existence of the rotation R on $(\mathbf{M})_r$: it permutes the three map trials of $\mathbf{M}'$. Conversely, if $\mathbf{M}'$ is self-trial, the Riemann surface $\mathbf{M}'_r$ apparently has an order 3 automorphism that permutes the three point sets formed by the vertices of $\mathbf{M}'_r$ and either set of the bipartition that the set of face centers allows. The index 2 subgroup H of Proposition 3.2.2 consists of the automorphism fixing each of those three sets setwise, and we can apply this proposition to extend $\mathrm{Aut}^+(\mathbf{M}')$. □

Definition 3.2.6. *Given a platonic map* $\mathbf{M}$ *of type* $(3, 2n)$ *with a tripartite graph, we define the* D_2-map $D_2(\mathbf{M})$ *(diagonal map across two faces) to be the platonic map of type* $(n, 2n)$ *constructed in the foregoing propositions.*

The name D_2-map is justified by the geometric interpretation of the construction. The edges of $D_2(\mathbf{M})$ are (geodesic) segments crossing two triangles of $\mathbf{M}$. They connect vertices of one of the three cocliques of $\mathrm{cells}_0(\mathbf{M})$. The construction forms a bijection

$$D_2 : \{\text{Type } (3, 2n) \text{ maps with a tripartite graph}\} \to \{\text{self-trial maps of type } (n, 2n)\}.$$

We can also rephrase the result in terms of platonic covers by forming the quotient of a reflexive platonic map $\mathbf{R}$ by the normal index 6 subgroup $\langle abcba, c, bcb\rangle$ appearing in the proof of Proposition 3.2.1.

Proposition 3.2.7. *If a reflexive platonic map $\mathbf{R}$ of genus g and type $(3, 2n)$ allows a platonic cover $\pi : \mathbf{R} \to \mathbf{Dih}(3)$, then it has a diagonal map $D_2(\mathbf{R})$. The map $D_2(\mathbf{R})$ is a platonic cover of $\mathbf{Dih}(1)$, and there is such a platonic cover π' for which the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Aut}(\mathbf{R}) & \xrightarrow{\ \pi\ } & \mathrm{Aut}(\mathbf{Dih}(3)) \\ \uparrow & & \uparrow \\ \mathrm{Aut}(D_2(\mathbf{R})) & \xrightarrow{\ \pi'\ } & \mathrm{Aut}(\mathbf{Dih}(1)) \end{array}$$

For chiral platonic maps, one replaces Aut *by* Aut^+ *in the formulation.* □

The simplest example of this proposition is

$$\begin{array}{ccc} \mathbf{Oct} & \longrightarrow & \mathbf{Dih}(3) \\ {\scriptstyle D_2}\downarrow & & \downarrow{\scriptstyle D_2} \\ \mathbf{Hos}(4) & \longrightarrow & \mathbf{Dih}(1) \end{array}$$

Example 3.2.8. Not all maps of type $(3, 2n)$ have a coclique necessary for the above construction. The smallest counterexamples are $\mathbf{R}_{3.3}$ of type $(3, 12)$ and $\mathbf{R}_{5.2}$ of type $(3, 10)$. And not all maps of type $(n, 2n)$ are self-trial. For example, none of the maps in the polynomial family $\mathbf{Wi1}(n)$ is. Two other small counterexamples are $\mathbf{R}_{5.5}$ and $\mathbf{R}_{8.8}$.

Example 3.2.9. In the case $n = 2$, we get the genus 0 example $\mathbf{Hos}(4) = D_2(\mathbf{Oct})$. To construct the diagonal map, start with $\mathbf{Oct}$ realized on the sphere, take the north and south pole to be two (opposite) vertices, and draw the four meridians through the midpoints of the equatorial edges. In the case $n = 3$, there are infinitely many inclusions $\mathrm{Aut}(\mathbf{R}_{1.1:n}) \lhd \mathrm{Aut}(\mathbf{R}_{1.2:3n})$ and $\mathrm{Aut}(\mathbf{R}_{1.2:n}) \lhd \mathrm{Aut}(\mathbf{R}_{1.1:n})$, all of index 3.

Remark 3.2.10. In Chapter 2 we saw many polynomial families that have diagonal D_1-maps and D_2-maps, so there are infinitely many instances of both these diagonal maps. We do not know if there are infinitely many instances of the composition of a D_2-map and a D_1-map, which can only happen by the inclusion $\Delta(8, 8, 2) < \Delta(4, 8, 2) < \Delta(3, 8, 2)$.

3.3 $\Delta^+(3, 7, 2)$ and the D_4-map

Consider the triangle group $\Delta^+(3, 7, 2) = \langle R, S \mid R^3, S^7, (RS)^2\rangle$ and its action on $\mathbb{H}^2$ given by choosing a $(3, 7, 2)$-triangle. Computation most easily executed by a computer shows that the normal core of the subgroup

$$H := \langle S, S^{R^{-1}S^3R} \rangle = \langle S, (R^{-1}S^3R)^{-1}S(R^{-1}S^3R) \rangle$$

with respect to either $\Delta^+(3,7,2)$ or $\Delta(3,7,2)$ is the intersection of the subgroups H^g, where $g \in \{1, R, R^2, R^2SR, R^2S^2R, R^2SR^2, R^2S^3R, R^2S^4R^2, R^2S^4R\}$. This normal core can in fact be written as the normal closure of a cyclic subgroup:

$$\mathrm{Core}_{\Delta(3,7,2)}(H) = \langle\, (R^2S^4RS^4R^2S)^2 \,\rangle^{\Delta^+(3,7,2)}.$$

Its index in $\Delta(3,7,2)$ is 1008, and $\Delta(3,7,2)/\mathrm{Core}_{\Delta(3,7,2)}(H) \cong \mathrm{PSL}(2,8) \times \mathbb{Z}_2$. In fact, $\mathbb{H}^2/\mathrm{Core}_{\Delta(3,7,2)}(H) \cong (\mathbf{R}_{7.1})_r$, and the quotient group acts by the standard map presentation of this platonic map, the Fricke-Macbeath map. It is quite remarkable that this platonic map is so closely connected to the present triangle group inclusion. Additional computation tells us that $[\Delta^+(3,7,2) : H] = 9$, and that the conjugates we found form the whole $\Delta^+(3,7,2)$-orbit of H under conjugation. Surprisingly, they even form the whole $\Delta(3,7,2)$-orbit, because conjgation by the hyperbolic isometry $(abc)^9$ also fixes H. If we embed $\Delta^+(3,7,2) \rightarrowtail \mathrm{Aut}^+(\mathbb{H}^2)$, the action of H contains rotations about the center and vertices of a regular hyperbolic 7-gon as depicted in Figure 3.3. Because the angles of the 7-gon are readily seen to be $2\pi/7$, the line segments to its center divide it into seven equilateral $(7,7,7)$-triangles, and we find that

$$\langle S, S^{R^{-1}S^3R} \rangle \cong \Delta^+(7,7,2).$$

There is no complementary $\mathrm{PSL}(2,8) \times \mathbb{Z}_2$ to $\Delta^+(7,7,2)$ in $\Delta(3,7,2)$, nor a complementary $\mathrm{PSL}(2,8)$ to it in $\Delta^+(3,7,2)$. If there were, all its non-trivial elements would be of finite order, so would have to be elliptic isometries of $\mathbb{H}^2$ around the same point. The point stabilizers of $\Delta(3,7,2)$ have order at most 14, though. Instead, we can recover $\Delta^+(3,7,2)$ as an extension of $\mathrm{PSL}(2,8)$ by $\mathrm{Core}(\Delta^+(7,7,2))$ in the following way. First, we elaborate on the description above of $\mathrm{Core}(\Delta^+(7,7,2))$ as a normal closure by giving explicit generators. If we set $t := (R^2S^4RS^4R^2S)^2$, then

$$\mathrm{Core}(\Delta^+(7,7,2)) = \langle t^h : h = R^iS^j,\ i \in \{0,1,2\},\ j \in \{0,\dots,6\} \rangle.$$

We can even generate it by 14 elements, but the above generating set is sufficient for our purpose of succinctly specifying a conjugation action on the subgroup. Let $F = F(r,s)$ be the free group on the indicated symbols, consider

$$\mathrm{PSL}(2,8) = \langle r, s \mid r^3, s^7, (rs)^2, (r^2s^4rs^4r^2s)^2 \rangle$$

and let $\pi_F : F \to \mathrm{PSL}(2,8)$ be the natural projection. In order to write down a homomorphism $\phi : F \to \mathrm{Aut}(\mathrm{Core}(\Delta^+(7,7,2)))$, we leave out mention of the element t and define ϕ on the two generators while writing e.g. $(t^R)^3(t^{RS})^{-1}$ as $3R - RS$. The automorphism $\phi(r)$ is given as by the following table:

w	1	R	R^2	S	RS	R^2S	S^2	RS^2	R^2S^2	S^3
$\phi(r)(w)$	R	R^2	1	R^2S^6	S^6	RS^6	$-R^2S^5+RS$	$-R^2S^6$	$-R^2+S^4$	$-R^2S^6+RS^2+R^2S^4-RS^6+1+RS$

w	RS^3	R^2S^3	S^4	RS^4
$\phi(r)(w)$	$-RS+R^2S^5$	$-R^2S-S^3+R^2S^6$	$-RS-R^2S^2-S^4+R^2+1+RS$	$-RS-1+RS^6-R^2S^4-RS^2+R^2S^6$

w	R^2S^4	S^5	RS^5	R^2S^5
$\phi(r)(w)$	$-R^2S^3+1+RS$	$-S^5+R^2S+1+RS$	$-RS-1-R^2+S^4+R^2S^2+RS$	$-R^2S^5+S^2+R^2+1+RS$

w	S^6	RS^6	R^2S^6
$\phi(r)(w)$	$-RS-1-R^2+1+RS$	$-RS-1-R^2S+S^5$	$-R^2S^6-S+1+RS$

The definition of $\phi(s)$ is easier in comparison: $R^iS^j \mapsto R^iS^{j+1 \bmod 7}$. Some orders of ϕ-images are clear: $\mathrm{ord}(\phi(r)) = 3$, $\mathrm{ord}(\phi(s)) = 7$ and $\mathrm{ord}(\phi(rs)) = 2$. This follows from the fact that they mimic conjugation in $\Delta^+(3,7,2)$ by R, S, and RS respectively. The order of $\phi(r^2s^4rs^4r^2s)$ is bigger than 2, so conjugation in $\Delta^+(3,7,2)$ restricted to $\mathrm{Core}(\Delta^+(7,7,2))$ does not factor through a homomorphism $\mathrm{PSL}(2,8) \to \mathrm{Aut}(\mathrm{Core}(\Delta^+(7,7,2)))$. Its square is an inner automorphism of $\mathrm{Core}(\Delta^+(7,7,2))$, though:

$$\phi((r^2s^4rs^4r^2s))^2 = \mathrm{con}_{t^{-2}(t^{RS})^{-3}}.$$

Therefore, letting

$$\sigma : \mathrm{Aut}(\mathrm{Core}(\Delta^+(7,7,2))) \to \mathrm{Out}(\mathrm{Core}(\Delta^+(7,7,2)))$$

be the natural projection, $\sigma\phi : F \to \mathrm{Out}(\mathrm{Core}(\Delta^+(7,7,2)))$ satisfies $\mathrm{ord}((\sigma\phi)(s)) = 2$, and this homomorphism does factor through $\mathrm{PSL}(2,8)$ as $\sigma\phi = \pi_F\overline{\phi}$. Put otherwise, ϕ is a lift of $\pi_F\overline{\phi}$ over σ. This allows us to define

$$G := (F_\phi \ltimes \mathrm{Core}(\Delta^+(7,7,2)))/N$$

by dividing out the normal closure N of the subgroup

$$\langle (r^3,1), (s^7,1), ((rs)^2,1), ((r^2s^4rs^4r^2s)^2, (t^{RS})^3t^2) \rangle.$$

We have an isomorphism $G \to \Delta^+(3,7,2)$ by $(r,1) \bmod N \mapsto R$ and $(s,1) \bmod N \mapsto S$. That this construction works is a basic result on group extensions, see e.g. [Rob1982, Ch. 11].

The geometry of the inclusion $\Delta^+(7,7,2) \subset \Delta(3,7,2)^+$ is illustrated in Figure 3.3. Note that there is no reflection in $\Delta(3,7,2)$ that fixes $\Delta^+(7,7,2)$. This is in contrast to the previous two triangle group inclusions. In spite of this, there is no mirror version of the group inclusion, because of the orientation reversing isometry $(abc)^9$ mentioned above.

Suppose now that we have a platonic map $\mathbf{M}$ defined by $\Gamma \lhd \Delta(3,7,2)^+$ of finite index. For any $\Delta \cong \Delta(7,7,2)^+$ embedded in $\Delta(3,7,2)^+$, we have $\Gamma \cap \Delta \lhd \Delta$. Thus, we get a new platonic map $\mathbf{M}' = \Delta/(\Gamma \cap \Delta)$ of type $(7,7)$ and genus

$$g' = 1 + 9\frac{|\Delta : \Gamma \cap \Delta|}{|\Delta(3,7,2)^+ : \Gamma|}(g-1).$$

When $\Gamma < \Delta$, then $g' = g$. In this case, $\Gamma < \Delta^h$ for any conjugate Δ^h of Δ by $h \in \Delta^+(3,7,2)$, since conjugation leaves Γ invariant. So $\Gamma < \mathrm{Core}_{\Delta^+(3,7,2)}(\Delta)$. On the level of platonic maps, the latter situation has a different description.

Proposition 3.3.1. *Let $\mathbf{M}$ be a platonic map of genus g and type $(3,7)$, and let (R,S) be standard map generators. If $\langle S, S^{R^{-1}S^3R}\rangle$ has index 9 in $\mathrm{Aut}^+(\mathbf{M})$, then its generators form a standard generator pair (r,s) for a chiral platonic map $\mathbf{M}'$ of genus g and type $(7,7)$. We have $(R^2S^4RS^4R^2S)^2 = s^4rs^{-1}r^2$ and the group $\mathrm{Aut}(\mathbf{M}')$ has the normal index 56 subgroup*

$$H := \langle (s^4rs^{-1}r^2)^h : h = R^iS^j,\ i \in \{0,1,2\},\ j \in \{0,\dots,6\}\rangle$$

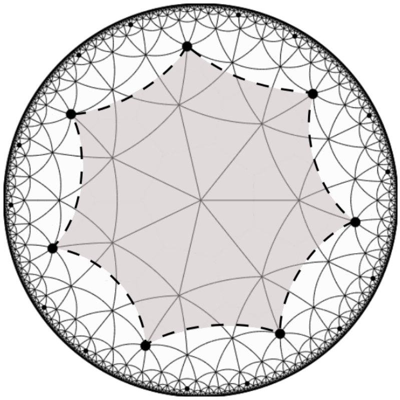


Figure 3.3: The inclusion $\Delta(7,7,2)^+ < \Delta(3,7,2)^+$. A fundamental $(7,7,2)$-triangle inside the 7-gon has the same area as nine $(3,7,2)$-triangles. The vertices for the $(3,7,2)$-tiling are partitioned into nine equivalence classes mod $\Delta^+(7,7,2)$, one of which (indicated by black dots) forms the vertices for the $(7,7,2)$-tiling. The vertices inside the 7-gon are representatives of the other eight.

on which the automorphisms con_R *and* con_S *act as* ϕ *described on the previous page. Conversely, let* $\mathbf{M}'$ *be a chiral platonic map of genus* g *and type* $(7,7)$ *such that the subgroup* H *has index* 56 *in* $\mathrm{Aut}(\mathbf{M}')$ *and permits of two automorphisms* ϕ_R *and* ϕ_S *acting as* con_R *and* con_S *respectively. Suppose also that* $\mathrm{ord}(\phi_R) = 3$*,* $\mathrm{ord}(\phi_S) = 7$ *and* $\mathrm{ord}(\phi_R\phi_S) = 2$*. Then there is a reflexive platonic map of genus* g *and type* $(3,7)$ *defined by the group extension of* H *described on the previous page.*

Proof. Let $\Delta^+(3,7,2) = \langle \tilde{R}, \tilde{S}\rangle$ and $\mathrm{Aut}^+(\mathbf{M}) = \langle \tilde{R}, \tilde{S}\rangle/\Gamma$. By the subgroup correspondence theorem, the assumptions imply that

$$\Gamma < \langle \tilde{S}, \tilde{S}^{\tilde{R}^{-1}\tilde{S}^3\tilde{R}}\rangle \overset{9}{<} \Delta^+(3,7,2).$$

This index 9 subgroup is isomorphic to $\Delta^+(7,7,2)$ and we label it as such. Since $\Gamma \lhd \Delta^+(3,7,2)$, certainly $\Gamma \lhd \Delta^+(7,7,2)$ and we get a chiral platonic map $\Delta^+(7,7,2)/\Gamma$ with standard generators

$$(R', S') = (\tilde{S}\Gamma, \tilde{S}^{\tilde{R}^{-1}\tilde{S}^3\tilde{R}}\Gamma) = (S, S^{R^{-1}S^3R}).$$

The conjugations $\mathrm{con}_{\tilde{R}}$ and $\mathrm{con}_{\tilde{S}}$ induce automorphisms on $\mathrm{Core}(\Delta^+(7,7,2))$ as described in the analysis of the inclusion $\Delta^+(7,7,2) < \Delta^+(3,7,2)$. Since $\Gamma < \Delta^+(7,7,2)$ is normal in $\Delta^+(3,7,2)$ we find $\Gamma < \mathrm{Core}(\Delta^+(7,7,2))$ and the conjugations factor through respective conjugations con_R and con_S of $H = \mathrm{Core}(\Delta^+(7,7,2))/\Gamma$, exhibiting the claimed behavior.

For the converse, the existence of the prescribed automorphisms allows us define the group extension of H discussed before the proposition. □

The condition on a platonic map of type $(3,7)$ from Proposition 3.3.1 has a geometric equivalent:

(*) its graph has a nine-coloring for which 'being of the same color' is $\mathrm{Aut}(\mathbf{M})$-invariant and such that it contains a subgraph isomorphic to that contained in the grey part of Figure 3.3, such that the black vertices in the figure correspond to graph vertices of the same color.

Whether it is enough for the graph to be nine-colorable, we do not know. Still, the proposition leads us to a diagonal map construction, just like in the previous two sections.

Definition 3.3.2. *Given a platonic map* $\mathbf{M}$ *of type* $(3,7)$ *with graph satisfying* **(*)***, we define the* D_4-map $D_4(\mathbf{M})$ *(diagonal map across four faces) to be the platonic map of type* $(7,7)$ *constructed in Proposition 3.3.1.*

One could work out how to construct eight 'map nonals' of type $(7,7)$ out of a given platonic map of type $(7,7)$ in a way similar to the construction of map trials in the previous section, and go on to define '*self-nonality*' for platonic maps of type $(7,7)$. We have not done so and settle instead for a proposition relating the D_4-map to platonic covers, analogous to Proposition 3.2.7.

Proposition 3.3.3. *If* $\mathbf{M}$ *is a platonic map of genus* g *and type* $(3,7)$ *with a platonic cover* $\pi : \mathbf{M} \to \mathbf{R}_{7.1}$*, then it has a diagonal map* $D_4(\mathbf{M})$*. The map* $D_4(\mathbf{M})$ *is a platonic cover of* $\mathbf{C}_{7.2}$*, and there is such a cover* π' *for which the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Aut}(\mathbf{M}) & \xrightarrow{\pi} & \mathrm{Aut}(\mathbf{R}_{7.1}) \\ \uparrow & & \uparrow \\ \mathrm{Aut}(D_4(\mathbf{M})) & \xrightarrow{\pi'} & \mathrm{Aut}(\mathbf{C}_{7.2}) \end{array}$$

□

The morphism π' is determined up to an element of $\mathrm{Aut}(\mathbf{C}_{7.2})$. This is the reason for the existential statement in the proposition.

The Fricke-Macbeath map $\mathbf{R}_{7.1}$ is the *only* example of the D_4-map that we know of. One can check there is no other type $(3,7)$ map satisfying the condition **(*)** for $2 \leq g \leq 301$. We expect there to be an infinite number of examples, since it seems reasonable that $\mathrm{Core}(\Delta^+(7,7,2))$ contains many subgroups normal in $\Delta^+(3,7,2)$.

Problem 3.3.4. To construct an infinite sequence of platonic maps of type $(3,7)$ that have a D_4-map.

We remark that no map with automorphism group $\mathrm{Aut}^+(\mathbf{R}) = \mathrm{PSL}(2,q)$ except $\mathbf{R}_{7.1}$ will have a D_4-map. The explanation is that an epimorphism $\phi : \mathrm{PSL}(2,q) \twoheadrightarrow \mathrm{PSL}(2,8)$ would give rise to the normal subgroup $\ker(\phi)$ of the simple group $\mathrm{PSL}(2,q)$. Since ϕ is supposed to be surjective, $\ker(\phi) = 1$ and $q = 8$. Since a platonic cover of $\mathbf{R}_{7.1}$ satisfies $3 \mid \mathrm{ord}(R)$ and $7 \mid \mathrm{ord}(S)$, we must in fact have $\mathrm{ord}(R) = 3$, $\mathrm{ord}(S) = 7$ and hence $g = 7$; we conclude that $\mathbf{R} = \mathbf{R}_{7.1}$.

3.4 Application: platonic maps vs. platonic surfaces

With the diagonal map constructions in hand, we can now analyze the relationship between the map automorphism group $\mathrm{Aut}^+(\mathbf{M})$ of an orientable platonic map and the group $\mathrm{Aut}(\mathbf{M}_r)$ of Riemann surface automorphisms of the corresponding platonic surface.

Proposition 3.4.1. *The group $\mathrm{Aut}^+(\mathbf{M})$ of a platonic map $\mathbf{M}$ on an orientable surface of genus $g \geq 2$ is also the full automorphism group of $\mathbf{M}_r$, except when $\mathbf{M}$ satisfies the converse conditions of one of the Propositions 3.1.2, 3.2.2, 3.3.1. Equivalently, the exceptional cases can be described as:*

1. *$\mathbf{M}$ is of type (n, n) and self-dual;*
2. *$\mathbf{M}$ is of type $(n, 2n)$ and self-trial (or equivalently, a platonic cover of $\mathbf{Dih}(1)$);*
3. *$\mathbf{M}$ is of type $(7, 7)$ and a platonic cover of $\mathbf{C}_{7.2}$.*

In the first case, $\mathrm{Aut}(\mathbf{M}_r)$ is an extension of $\mathbb{Z}_2$ by $\mathrm{Aut}^+(\mathbf{M})$ of degree 2. In the second case, $\mathrm{Aut}(\mathbf{M})$ contains the index 2 subgroup $\langle S^2, R^{-1}S^2R, R^{-2}S^2R^2\rangle < \mathrm{Aut}^+(\mathbf{M})$ and there is an extension of Sym_3 by this subgroup that contains $\mathrm{Aut}(\mathbf{M})$ and acts on $\mathbf{M}_r$. This extension is either the full group $\mathrm{Aut}(\mathbf{M}_r)$, or exceptional case 1 applies. In the third case, $\mathrm{Aut}(\mathbf{M}_r)$ is a degree 2 extension of $\mathrm{PSL}(2, 8)$ by the index 56 subgroup $H\mathrm{Aut}^+(\mathbf{M})$ from proposition 3.3.1.

Proof. Let $\mathbf{M}_r$ be uniformized as $\mathbb{H}^2/\Gamma$, where $\Gamma \lhd \Delta^+(p, q, 2)$ and $\Delta^+(p, q, 2)/\Gamma \cong \mathrm{Aut}^+(\mathbf{M})$. If $\mathrm{Aut}(\mathbf{M}_r) > \Delta^+(p, q, 2)/\Gamma$, then there is a Fuchsian group $F > \Delta^+(p, q, 2)$ such that $\mathrm{Aut}(\mathbf{M}_r) = F/\Gamma$ and $[F : \Delta^+(p, q, 2)] < \infty$. But by [Sin1972], $\Delta^+(p, q, 2) < F$ must be one of the triangle group inclusions discussed in Sections 3.1, 3.2, or 3.3. Hence, we find ourselves in one of the exceptional cases. Because $g \geq 2$, the type of the triangle group F is different from that of $\Delta^+(p, q, 2)$ and either F is a finitely maximal Fuchsian group, or it is $\Delta^+(4, 8, 2)$ and we can extend once more to $\Delta^+(3, 8, 2)$. □

4

Geometric classification theorems

In this chapter we undertake several classification attempts. After looking at the simple cases of platonic maps with one or two vertices in Sections 4.1 and 4.2, we classify all reflexive platonic maps whose vertex number is an odd prime in Section 4.3. In Section 4.4 we study the concept of *density* of a platonic map, which we import from graph theory. This leads to a combinatorial characterization of the Fermat maps. In the last few sections we investigate maps of various types that occur a lot. We then show that such maps must belong to one of a certain set of polynomial families described in Chapter 2. Tools we use are the genus formula (see Section 1.4), multiplicity quotients and the study of the local incidence structure.

The notation we use througout is V for the vertex set, sometimes F for the set of faces. These are indexed, but we tailor the indices to the task at hand.

4.1 One-vertex maps

Platonic maps with few vertices are easiest to study. Of course the same holds for their duals, maps with few faces. So we start with one-vertex maps.

Theorem 4.1.1. *A platonic map with $v = 1$ is a member of one of the two polynomial families $\mathbf{Wi1}(n)$ and $D_1(\mathbf{Wi2}(n))$.*

Proof. For any standard map presentation, R fixes the unique vertex, whence $R = S^k$ for some $k \in \mathbb{Z}$. This implies that $RS = S^{k+1}$ and this element has to have order 2. Since $\langle R, S\rangle$ is cyclic and generated by S, q must be even and $R = S^{\frac{1}{2}q-1}$. If $q = 4n+2$, then $\gcd(q, \frac{1}{2}q-1) = 2$ and $p = q/2 = 2n+1$. Since we now find the relator RS^{-2n}, the map is a member of $\mathbf{Wi1}(n)$. If $q = 4n$, then $\gcd(q, \frac{1}{2}q-1) = 1$

and $p = q = 4n$. Because we find the relator $RS^{-(2n-1)}$, we end up with a member of $D_1(\mathbf{Wi2}(n))$. □

Remark 4.1.2. One-vertex maps are the only platonic maps that are not loop-free. If a platonic map has a vertex v_1 with an incident loop e_1, then $S_1^k(e_1)$ is a loop for each k, so v_1 is not connected to other vertices. Connectedness of the map graph implies $V = \{v_1\}$. Dually, the only platonic maps that are not dual-loop-free are one-face maps, and such a map must be a member of $\mathbf{Wi1}(n)^\vee$ or $D_1(\mathbf{Wi2}(n))^\vee$.

4.2 Two-vertex maps

After one-vertex maps come two-vertex maps.

Theorem 4.2.1. *A platonic map with $v = 2$ has the map presentation*

$$\mathrm{Aut}^+(\mathbf{R}) = \left\langle R, S \,\middle|\, R^p, S^q, (RS)^2, R^{-1}SRS^{-k} \right\rangle$$

with $k^2 \equiv 1 \bmod q$ and $p = \frac{2q}{\gcd(k+1,q)}$. Every pair (q,k) satisfying these conditions yields a reflexive platonic map $v = 2$.

Proof. Let the vertex set be $\{v_1, v_2\}$. Any rotation around a vertex is one around both vertices, so $S_2 = S_1^k$. By platonicity, also $S_1 = S_2^k$ so that $S_1 = S_1^{k^2}$, entailing $k^2 \equiv 1 \bmod q$. Since $S_1^k = S_2 = R^{-1}S_1R$, the map has the relator $R^{-1}SRS^{-k}$. From this we derive

$$(RS)^2 = 1 \implies SR = R^{-1}S^{-1} \implies 1 = R^{-1}SRS^{-k} = R^{-2}S^{-k-1} \implies R^2 = S^{-k-1}.$$

It follows that $p = 2q/\gcd(k+1,q)$. Also, since $[\mathrm{Aut}^+(\mathbf{R}) : \langle S\rangle] = 2$ we know that $\langle S\rangle \lhd \mathrm{Aut}^+(\mathbf{R})$ with complement $\langle RS\rangle$, and hence

$$\mathrm{Aut}^+(\mathbf{R}) \cong \langle S\rangle \rtimes \langle RS\rangle$$

of order $2q$. Because the presentation in the theorem yields the same group structure (i.e. no additional relators are needed), we have shown that the map group is isomorphic to the group presented. Conversely, a group G with such a presentation defines a platonic map with two vertices, since $[G : \langle S\rangle] = 2$.

Using the chirality criterion (Section 1.4), we note that $(R,S) \mapsto (R^{-1}, S^{-1})$ maps the extra relator of the presentation to $RS^{-1}R^{-1}S^k = RRSS^k = S^{-k-1}SS^k = 1$. It is thus a group homomorphism and the map must be reflexive. □

The computation $(RS)^{-1}S(RS) = (RS)S(S^{-1}R^{-1}) = RSR^{-1} = S^k$ completes the description of $\mathrm{Aut}^+(\mathbf{R})$ as the semi-direct product displayed above. We also see

$$e = \frac{|\mathrm{Aut}^+(\mathbf{R})|}{2} = q, \qquad f = \frac{|\mathrm{Aut}^+(\mathbf{R})|}{p} = \gcd(k+1,q)$$

and from this derive

$$g = 1 - \frac{1}{2}(v - e + f) = \frac{1}{2}\left(q - \gcd(k+1, q)\right).$$

Remark 4.2.2. There are two solutions for k that work for any q, namely $k \equiv \pm 1 \bmod q$. The solution $k \equiv -1 \bmod q$ leads to $R^2 = 1$, and we see that the map is $\mathbf{Hos}(q)$. The solution $k \equiv 1 \bmod q$ yields the family $\mathcal{F}_{n-1}^{(2n,2n)}$. The number of possible k for a given q is well known. It is

$$m(q)2^{\#\text{ odd prime divisors of } q}$$

with $m(q) = 1$ if $\mathrm{val}_2(q) \leq 1$, $m(q) = 2$ if $\mathrm{val}_2(q) = 2$ and $m(q) = 4$ if $\mathrm{val}_2(q) \geq 3$, where val_2 is valuation at the prime 2. For example, for $q = 2^3 \cdot 5 \cdot 23$, there are $4 \cdot 2 \cdot 2 = 16$ solutions. To express the number of solutions for a given g in closed form is not so easy, but all solutions are quickly computed using the fact that $q \leq 4g + 2$.

4.3 Reflexive platonic maps with v an odd prime

After looking at the set of platonic maps for some time, one notices that for most maps v is even. A problem that might thus be amenable to solution is to classify all platonic maps for which v is odd. We take a first step by determining all reflexive platonic maps for which v is an odd prime.

Theorem 4.3.1. *Let $v \geq 3$ be an odd prime. The unique platonic map with v vertices and $g = 0$ is the platonic map $\mathbf{Dih}(v)$. The only platonic maps with v vertices and genus $g \geq 1$ are the members $\mathbf{R}_{(v,n)}$ of the family $\mathcal{F}^{(v(2n-1),2v(2n-1))}_{v^2n-v^2+\frac{1}{2}(v-2)(v-1)}$ of type $(\frac{1}{v}(2g+3v-2), \frac{2}{v}(2g+3v-2))$ defined by:*

$$\mathrm{Aut}(\mathcal{F}^{(v(2n-1),2v(2n-1))}_{v^2n-v^2+\frac{1}{2}(v-2)(v-1)})^+ = \left\langle R, S \,\middle|\, R^{v(2n-1)}, S^{2v(2n-1)}, (RS)^2, [R^v, S], [R, S^2] \right\rangle$$

See Subsection 2.10.1 for a proof that the group $G(v, n)$ defines a platonic map with the parameters displayed.

Proof. To start with, we form the μ_{00}-quotient map $\overline{\mathbf{R}}$ of $\mathbf{R}$, see Section 1.4. The map $\overline{\mathbf{R}}$ still has v vertices. But it also satisfies $\overline{\mu}_{00} = \overline{\mu}_{02} = 1$. Let us therefore classify the reflexive platonic maps with this extra constraint first. We will show that $\mathbf{R} = \mathbf{Dih}(v)$. The action of $\phi \in \mathrm{Aut}^+(\mathbf{R})$ on the vertex set V determines the local action of ϕ around a vertex unequivocally and therefore the automorphism itself, because $\mu_{00} = 1$. Thus, $\mathrm{Aut}^+(\mathbf{R})$ is a permutation group on V. Since $v \mid qv = |\mathrm{Aut}^+(\mathbf{R})|$ and v is prime, there is an orientation preserving automorphism T of order v. The map $\mathbf{R}$ satisfies $q < v$, so none of $T, T^2, \ldots, T^{v-1}$ can be a vertex rotation, all of them being of order v. Again using $\mu_{00} = 1$, it follows that T acts as a v-cycle on the vertex set. Without loss of generality, we now label the vertices using $\mathbb{Z}/v\mathbb{Z}$ and replace T by a suitable power such that $T : i \mapsto i + 1$ and there is an edge e_{01}.

Since $|\mathrm{Aut}^+(\mathbf{R})| \le v(v-1)$, the group $\langle T\rangle$ is a Sylow v-subgroup of it. Because its order is prime, it is disjoint from its conjugates. This number of conjugates is $1 \bmod v$ by the Sylow theorems, but it cannot be bigger than v, again by the size constraint on $\mathrm{Aut}^+(\mathbf{R})$. We find that $\langle T\rangle \lhd \mathrm{Aut}^+(\mathbf{R})$ with complement the rotation subgroup around one vertex. In other words,

$$\mathrm{Aut}^+(\mathbf{R}) \cong \langle T\rangle \rtimes \langle S\rangle.$$

Let S_i be the rotation around vertex i. For some $k \in \mathbb{Z}/v\mathbb{Z}$ then, $S_0^{-1}TS_0 = T^k$. Thus, for any vertex i we have $S_0(i+1) = S_0(i)+k$. Applying the knowledge that $S_0(0) = 0$ we find $S_0(i) = ki$ for all $i \in \mathbb{Z}/v\mathbb{Z}$. The action of a power is therefore defined by $S_0^j : i \mapsto k^j i$, and because $\mathrm{ord}(S_0) = q$ we get the equality $k^q \equiv 1 \bmod v$. Now q is even because $qv = 2e$ and v is odd. The fact that v is prime implies that $1 \bmod v$ only has two square roots, and considering the order of S_0 we see that $k^{q/2} \equiv -1 \bmod v$. But geometrically, this means that -1 lies straight across from 1 when considered as neighbors of vertex 0. The transformation T thus preserves the geodesic wall through e_{01}, and the vertices on this wall are exactly $0, 1, 2, \ldots, v-1$. This finishes the game: there is supposedly a reflection in this geodesic wall, but we also know it fixes all vertices. The two vertices $S_0(1)$ and $S_0^{-1}(1)$ are therefore identical and hence $q \le 2$. We conclude that $\mathbf{R} = \mathbf{Dih}(v)$.

We return to the situation of the platonic cover π of $\mathbf{R}$ to its μ_{00}-quotient $\overline{\mathbf{R}}$. It is now clear that $\overline{\mathbf{R}} = \mathbf{Dih}(v)$ and we work our way back up to $\mathbf{R}$ in two steps. We have the group homomorphisms

$$\mathrm{Aut}(\mathbf{R}) \xrightarrow{\pi_1} \mathrm{Aut}(\mathbf{R})/\langle S^{q/\mu_{02}}\rangle \xrightarrow{\pi_2} \mathrm{Aut}(\mathbf{R})/\langle S^{q/\mu_{00}}\rangle = \mathrm{Aut}(\mathbf{Dih}(v)).$$

The platonic cover π_2 is branched over cells_0. The platonic map $\mathbf{R}'$ with automorphism group $\mathrm{Aut}(\mathbf{R})/\langle S^{q/\mu_{02}}\rangle$ has parameter $\mu'_{00} = \deg \pi_2$. Also, no two vertices of $\mathbf{R}'$ are identified by π_2, hence $v' = v$, $e' = v\mu'_{00}$, $f' = 2\mu_{00}$, $p' = v$, $q' = 2\mu_{00}$, $\mu'_{02} = 1$. The graph $\Gamma(V', E')$ is a v-cycle, with μ'_{00} edges $e_{i,i+1}$ between to successive vertices. Therefore, S_i^2 fixes all vertices. Platonicity implies that $S_i^2 = S_{i\pm1}^{2k}$ for some $k \in \mathbb{Z}/\mu'_{00}\mathbb{Z}$, independently of i. Applying this twice going from i to $i+1$ and back yields $S_0^2 = S_1^{2k} = S_0^{2k^2}$. Going around the whole v-cycle yields $S_0^2 = S_1^{2k} = \cdots = S_{v-1}^{2k^{v-1}} = S_0^{2k^v}$. So $k^2 \equiv 1 \bmod \mu'_{00}$ and $k^v \equiv 1 \bmod \mu'_{00}$. Since v is odd, this means $k \equiv 1 \bmod \mu'_{00}$ and hence

$$S_i^2 = S_j^2 \qquad \text{for all } i, j.$$

We see that each face appears around all v vertices exactly once, since $p' = v'$ and $\mu'_{02} = 1$. Let us number the faces around v_0 as $f_1, \ldots, f_{2\mu_{00}}$ counterclockwise. The relationship just derived implies that the odd-numbered faces are grouped in exactly the same way around all v_i: spaced two apart, in ascending numerical order $f_1, f_3, \ldots, f_{2\mu_{00}-1}$ counterclockwise. The same holds for the even-numbered faces. Consider an edge $e_{i\ i+1}$. Suppose faces f_{2j-1}, and f_{2k} border it right and left respectively, around v_i. Then around v_{i+1}, f_{2k} borders it on the right, so the edge it borders on the left around v_{i+1} is f_{2j+1}. One sees that the even-numbered faces are 'rotated counterclockwise by two' relative to the odd-numbered ones, when passing from v_i

to v_{i+1}. Following f_{2k} along its boundary, the neighbor sequence must repeat after p' steps, whence $v = p' \equiv 0 \bmod \mu'_{00}$, or in other words $\mu'_{00} \mid v$. Since v is prime, the only non-trivial such covering has degree $\mu'_{00} = v$ and its incidence structure was just described. These maps are in fact those in the proposition with parameters $(m, n) = (v, 1)$.

The map π_1 is branched over $\mathrm{cells}_0 \cup \mathrm{cells}_2$ of the map $\mathbf{R}'$, and $\deg_{v_i}(\pi_1) = \deg_{f_j}(\pi_1)$. The covering is of degree μ_{02}, we may therefore conclude we still have v vertices, and $e = v^2\mu_{02}$, $f = 2v$, $p = v\mu_{02}$, $q = 2v\mu_{02}$. It follows that $2-2g(\mathbf{R}) = \chi(\mathbf{R}) = 3v - v^2\mu_{02}$. So μ_{02} has to be odd for $\mathbf{R}$ to be a well-defined (platonic) map.

Lastly, let (R, S) be a standard generator pair for $\mathbf{R}$. By steps 1 and 2 we know that $R \bmod \langle S^{q/\mu_{00}} \rangle$ has order v. That implies $R^v \in \langle S^{q/\mu_{00}} \rangle$, so $[R^v, S] = 1$. Also, since $S^2 = S^{q/\mu_{00}} \in Z(\mathrm{Aut}(\mathbf{R})^+)$, we have $[R, S^2] = 1$. This finishes the proof that $\mathbf{R}$ belongs to the family described in our theorem. □

That reflexivity is necessary is shown e.g. by chiral torus maps $\mathbf{M}_{1.1}(m,n)$ with $v = m^2 + mn + n^2$ or chiral chiral torus maps $\mathbf{M}_{1.2}(m,n)$ with $v = m^2 + n^2$. One can generate maps with p vertices for any prime $p \equiv 1 \bmod 4$ in this way. There are also plenty of higher genus chiral maps satisfying the condition, although more structure might be discovered in this set yet.

Remark 4.3.2. The reflexive platonic maps with v and odd prime and $2 \leq g \leq 101$ are $\mathbf{R}_{10.20}$, $\mathbf{R}_{19.31}$, $\mathbf{R}_{28.34}$, $\mathbf{R}_{37.50}$, $\mathbf{R}_{46.34}$, $\mathbf{R}_{55.52}$, $\mathbf{R}_{64.39}$, $\mathbf{R}_{73.116}$, $\mathbf{R}_{82.78}$, $\mathbf{R}_{91.67}$, $\mathbf{R}_{100.49}$ ($v = 3$); $\mathbf{R}_{6.6}$, $\mathbf{R}_{31.15}$, $\mathbf{R}_{56.18}$, $\mathbf{R}_{81.170}$ ($v = 5$); $\mathbf{R}_{15.10}$, $\mathbf{R}_{64.36}$ ($v = 7$); $\mathbf{R}_{45.25}$ ($v = 11$); and $\mathbf{R}_{66.12}$ ($v = 13$).

4.4 Dense platonic maps

We take up the study of the *density* of a platonic map, which is meant to measure the ratio of its number of edges to its number of vertices. We will classify maps with high density. In general, the graph of a platonic map is not a simple graph. For example, for a polynomial family of platonic maps with increasing genus the number of edges increases without bound, whereas we have seen families for which the number of vertices stays fixed. Thus, their ratio tends to infinity. A more informative measure is the number of *different* vertices to which a vertex is connected. This suggests the following definition.

Definition 4.4.1. *The* (graph) density *of a platonic map* $\mathbf{M}$ *is defined to be* $\delta(\mathbf{M}) := d_{00}/v$.

First, it is obvious that $\delta(\mathbf{M}) \in (0, 1]$, and that the only way to get density 1 is if $\Gamma(\mathbf{M})$ has loops. These are the one-vertex maps, see Section 4.1. We will exclude them from further consideration.

Second, if a platonic map $\mathbf{M}$ of type (p,q) has multiplicity $\mu_{00} > 1$, we can form the μ_{00}-quotient map $\mathbf{M}'$, as described in Section 1.6. The quotient map has the same density as the original:

$$d'_{00}/v' = q'/v' = (q/\mu_{00})/v = d_{00}/v$$

Therefore we assume in this section that $\mu_{00} = 1$, in which case $\delta(\mathbf{M}) = q/v$. We can generalize results for such maps to all platonic maps afterwards via platonic covers branched over cells_0. Because our graph is loop-free, the μ_{00}-quotient map has a simple graph.

Third, reflexivity turns out to be a necessary and sufficient condition to get a handle on matters. To summarize, we may and will now assume that we have a reflexive platonic map $\mathbf{R}$ for which $\Gamma(\mathbf{R})$ is simple, $v \geq 2$, $p \leq q$, and $\delta(\mathbf{R}) \in (0,1)$.

How close can we get to $\delta(\mathbf{R}) = 1$? The densest simple graph on any number v of vertices is of course the complete graph K_v, with density $\delta(K_v) = (v-1)/v$. But the number of reflexive platonic maps with a complete graph is very limited.

Proposition 4.4.2. *Let* $\mathbf{R}$ *be a reflexive platonic map that has graph* $\Gamma(\mathbf{R}) \cong K_v$. *Then* $\mathbf{R}$ *is one of* $\mathbf{Hos}(2)$ *(and* $v = 2$*),* $\mathbf{Dih}(3)$ *(and* $v = 3$*) or* $\mathbf{Tet}$ *(and* $v = 4$*).*

Proof. If $v \leq 3$ then $q \leq 2$ so we have a genus 0 map. This case is quickly dealt with and yields the first two maps from the theorem. Assume $v \geq 4$. Two vertices v_i, v_j have the same set of neighbors, excepting each other. We define the permutation $\pi_{ij} \in \mathrm{Sym}(\mathbb{Z}/(v-1)\mathbb{Z}^\times)$ by the equation

$$\text{end point of } S_i^k(e_{ij}) = \text{end point of } S_j^{\pi_{ij}(k)}(e_{ji}) \qquad (k = 1, \ldots, v-2).$$

Platonicity implies that $\pi_{ij} = \pi$ is independent of the pair (i,j). Also, it is obvious

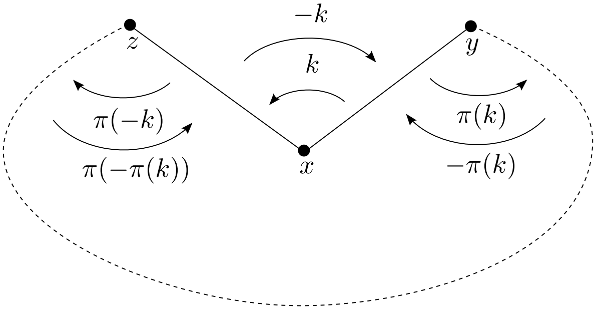


Figure 4.1: A constraint on the neighbor permutation π

that $\pi_{ji} = \pi_{ij}^{-1}$, so $\pi^2 = 1$. Now consider a vertex x and two of its neighbors y, z with $S_x^k(e_{xy}) = e_{xz}$; refer to Figure 4.1. The vertex z is also a neighbor of y, and using $\pi = \pi_{xy}$ we find $S_y^{\pi(k)}(e_{yx}) = e_{yz}$. So $e_{yx} = S_y^{-\pi(k)}(e_{yz})$. Since x is not only a neigbor of y but also of z, it follows that $S_z^{\pi(-\pi(k))}(e_{zy}) = e_{zx}$ by using π_{yz}. But considering π_{xz} we find that $S_z^{\pi(-k)}(e_{zx}) = e_{zy}$. Since y and z can be chosen arbitrarily, this implies the equality

$$\pi(-\pi(k)) = -\pi(-k)$$

for all k. Platonicity simplifies this equation for us, since reflection in an edge shows us that $\pi(-k) = -\pi(k)$. Combining this with the previous equality and $\pi^2 = 1$, we conclude that $\mathbf{R}$ satisfies

$$\pi(k) = -k.$$

Now label the vertices $p_1, \ldots, p_v$ such that the end points of edges around p_1 are $p_2, \ldots, p_v$ counterclockwise. The end points occurring around p_2 are then $p_v, \ldots, p_3, p_1$ counterclockwise. Around p_3 the end points must then be $p_2, p_4, \ldots, p_v, p_1$. Focussing on the edge e_{31} we see, using $\pi_{31}(-1) = 1$, that the edge e_{14} must equal the edge e_{3v}. Hence $v = 4$. The construction also gave us a triangle $p_1 p_2 p_3$, so $p = 3$. We quickly find $f = 4$ and recover the map $\mathbf{Tet}$. □

Remark 4.4.3. Numerous chiral platonic maps have graph K_v. For $2 \leq g \leq 101$ these are $\mathbf{C}_{7.2}$ ($v = 8$), $\mathbf{C}_{10.3}$ ($v = 9$), $\mathbf{C}_{12.1}/\mathbf{C}_{12.2}$ ($v = 11$), $\mathbf{C}_{27.7}$ ($v = 13$), $\mathbf{C}_{45.2}$ ($v = 16$), $\mathbf{C}_{52.2}/\mathbf{C}_{52.3}$ ($v = 17$), $\mathbf{C}_{58.6}/\mathbf{C}_{58.7}/\mathbf{C}_{58.8}$ ($v = 19$), and $\mathbf{C}_{93.2}/\mathbf{C}_{93.3}/\mathbf{C}_{93.4}/\mathbf{C}_{93.5}/\mathbf{C}_{93.6}$ ($v = 23$). Presumably there is no bound on the genera for which a complete graph occurs.

In the preceding proposition, the highest density occurring was $\frac{3}{4}$. Any platonic map with this density will have graph K_4 if its graph is simple, so this value only occurs for the tetrahedron. In fact, it seems to be a global maximum, as a glance at a plot of the density spectra (Figure 4.2) for $1 \leq g \leq 101$ indicates. We will proceed to prove

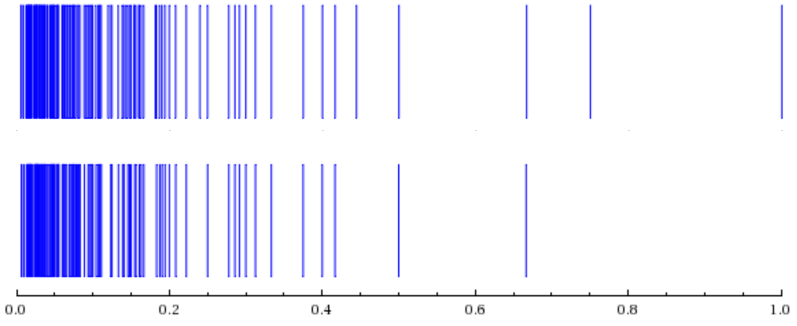


Figure 4.2: Density spectra of reflexive platonic maps of genus $1 \leq g \leq 101$ (top) and those with simple graph (bottom).

this and more in the next propositions, where we study reflexive platonic maps with density $\delta(\mathbf{R}) > \frac{1}{2}$. The plan is as follows:

1. prove that such a map consists of triangles (like in the previous proof);
2. show that the only such density occurring for $g \geq 1$ is $\frac{2}{3}$;
3. classify all maps of density $\frac{2}{3}$ with simple graph.

For convenience, denote the set of vertices of Γ at graph distance at most r from a given vertex v by $D(v, r)$, and the vertices at distance exactly r by $\partial D(v, r)$. We start with a graph-theoretic lemma.

Lemma 4.4.4. *Let* Γ *be a* d*-regular simple graph on* v*-vertices. If* $d \geq \lfloor \frac{1}{2}v \rfloor$*, then* $\operatorname{diam}(\Gamma) \leq 2$*.*

Proof. Take a vertex $v \in V(\Gamma)$. The assumption implies $|D(v,1)| \geq \lfloor \frac{1}{2}v \rfloor + 1$, so that $|V(\Gamma) - D(v,1)| \leq \lfloor \frac{1}{2}v \rfloor$. Thus, any vertex in $V(\Gamma) - D(v,1)$ must be connected to some vertex in $\partial D(v,1)$ and therefore has distance 2 to v. □

We proceed to prove that reflexive platonic maps of density greater than $\frac{1}{2}$ have triangular faces.

Proposition 4.4.5. *Let* $\mathbf{R}$ *be a platonic map with simple graph and* $\delta(\mathbf{R}) > 1/2$*. Then* $p = 3$*.*

Proof. Take a vertex v and an incident face f, and suppose that $v \not\sim R_f^2(v)$. By platonicity this non-adjacency holds for any choice of v and f such that $f * v$. We count all triples $(v, f, R_f^2(v))$. Around any vertex we have q different choices for f, since $\mu_{02} = 1$. If no pair of vertices occurs twice, then there are at least qv triples. This means that the complementary graph $\Gamma(\mathbf{R})^c$ has at least qv edges. However, since $\Gamma(\mathbf{R})$ is simple and $\delta(\mathbf{R}) > \frac{1}{2}$, we also know that it has at most $\frac{1}{2}v(v-1) - \frac{1}{2}qv$, which leads to the inequality

$$\frac{3}{2}qv \leq \frac{1}{2}v(v-1) \qquad \Longleftrightarrow \qquad q \leq \frac{1}{3}(v-1).$$

This is contrary to assumption. Apparently some pair of vertices $(v, R_f^2(v))$ occurs in

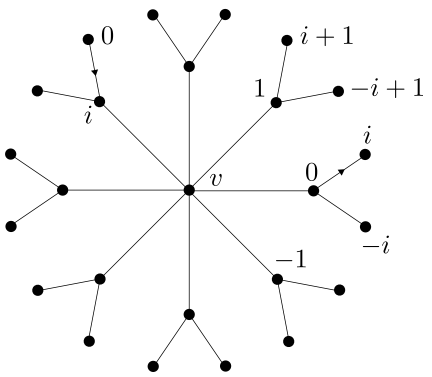


Figure 4.3: Vertex i and the edge $\vec{e}_{0i}$ occur twice.

more than one triple. Therefore, there is a $j \in \{1, \ldots, q-1\}$ such that S_v^j fixes $R_f^2(v)$. But by platonicity, $S_v^j(R_f^2(v)) = R_f^2(v)$ for all v, f such that $v * f$. So let $w \sim v$ be a neighbor. By induction, going around w in clockwise order, it follows that S_v^j fixes all neighbors of w. This implies that all these neighbors lie in $\partial D(v,2)$, since S_v^j fixes no vertex in $\partial D(v,1)$. However, because $\Gamma(\mathbf{R})$ is simple, we now conclude $|\partial D(v,2)| \geq q-1$. Since $|\partial D(v,2)| = v - q - 1$, we find that $v \geq 2q$, again a contradiction to our assumption on the density.

The inevitable conclusion is that $v \sim R_f^2(v)$ for all vertices v and incident faces f. Number the vertices around v counterclockwise with $\mathbb{Z}/q\mathbb{Z}$ and consider the face with counterclockwise vertex sequence starting $(v, 0, i, \ldots)$. Using the rotations and reflections fixing v, the numbers of all the vertices $R_f^{\pm 2}(v)$ for incident f then becomes clear, as illustrated in Figure 4.3. The (oriented) edge $\vec{e}_{0i}$ occurs again in this figure, at neighor i of v. Since the surface is orientable, the face locally to its left must be the same for both occurrences. Thus v is doubly incident to that face – which $\mu_{02} = 1$ tells us is false – unless the two occurrences of the edge are in fact the same. We conclude that $i = 1$ and $p = 3$. □

Example 4.4.6. A counterexample to this theorem for chiral platonic maps is given by $\mathbf{C}_{41.22}$. It has a simple graph, is of type $(12, 12)$ and has $v = 20$, wherefore $\delta(\mathbf{C}_{41.22}) = 3/5$.

In the situation $p = 3$ the following notion is a handy abbreviation.

Definition 4.4.7. *Two vertices of a map with $p = 3$ are* diagonal neighbors *if they are non-adjacent but are incident to two adjacent faces.*

We continue by showing that the graph of a high-density map is tripartite, unless the map is the tetrahedron. This also gives an upperbound on the density.

Proposition 4.4.8. *Let $\mathbf{R}$ be a reflexive platonic map with simple graph and $\delta(\mathbf{R}) > \frac{1}{2}$. Either $\mathbf{R}$ is $\mathbf{Tet}$, or its vertex set is the union of three $\Gamma(\mathbf{R})$-cocliques of size $\frac{v}{3}$ and $\delta(\mathbf{R}) \leq \frac{2}{3}$.*

Proof. By the previous proposition we know that $p = 3$. Take a vertex v and consider a diagonal neighbor w across triangles $\Delta_1 * v$ and $\Delta_2 * w$. There are three possibilities for the distance $d(v, w)$. If $w = v$, maybe we should have said that v had no diagonal neighbors at all. Still, consider a vertex u, common to Δ_1 and Δ_2. There are two instances of the edge e_{uv}, one incident to each Δ_i. They must coincide because $\Gamma(\mathbf{R})$ is simple, and we find $q = 2$. This implies $\mathbf{R} = \mathbf{Dih}(v)$. We already know $v \geq 3$, and because $\delta(\mathbf{R}) > \frac{1}{2}$ we conclude $v = 3$ and hence $\mathbf{R} = \mathbf{Dih}(3)$ of density $\frac{2}{3}$. The coclique property of the graph is clear.

If $w \in \partial D(v, 1)$, then the reflection in the angle bisector of Δ_1 through v fixes w, so the edge vw lies on the reflection axis (which is this bisector) on the opposite side from Δ_1, and so q must be odd. We number the vertices of $\partial D(v, 1)$ with $\mathbb{Z}/q\mathbb{Z}$ counterclockwise and apply rotations around v to arrive at Figure 4.4. The oriented edge $\vec{e}_{0\frac{q-1}{2}}$ appears twice, so the triangles appearing to its right are identical. Considering their vertices, we find the equality

$$\frac{q+1}{2} \equiv q - 1 \bmod q \qquad \Longleftrightarrow \qquad 3 \equiv 0 \bmod q \qquad \Longleftrightarrow \qquad q = 3.$$

We conclude that $\mathbf{R} = \mathbf{Tet}$.

The remaining possibility is that $w \in \partial D(v, 2)$. Now consider the set $W = \{S_v^i(w) : i = 0, \ldots, q-1\}$. If these are all different vertices, then $|V(\mathbf{R})| \geq |\partial D(v, 1)| + |W| = 2q$,

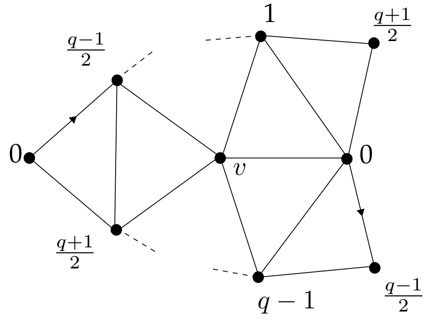


Figure 4.4: If $w \in \partial D(v, 1)$, we find the tetrahedron.

which contradicts our assumption. So there is a well-defined minimal $J \in \{1, \ldots, q-1\}$ such that $S_v^J(w) = w$. The diagonal neighbors of v then repeat with primitive period J. Also, $S_v^J = S_w^{kJ}$ for some $k \in \mathbb{Z}/\frac{q}{J}\mathbb{Z}$. By platonicity therefore, S_v^J fixes every vertex that can be reached by a path consisting of steps from a vertex to a diagonal neighbor. Specifically, we can walk around a neighbor $u \sim v$ with such steps. This forces q to be even, for otherwise S_v^J would fix all neighbors of u, hence also the common neighbors of v and u, of which there are at least two. But then $S_v^J = 1$, which is not the case. Still, S_v^J fixes at least the $\frac{q}{2}$ different vertices $S_u^{2m}(v)$, $m = 0, \ldots, \frac{1}{2}q - 1$, which shows that $|\partial D(v, 2)| \geq \frac{q}{2}$. This implies $q \leq \frac{2}{3}v$.

Moreover, the primitive period J allows us to define a relation $\approx$ on $V(\Gamma(\mathbf{R}))$ by $v \approx w$ whenever $S_v^J(w) = w$. It is not hard to see that this is an equivalence relation. Because adjacent vertices are not equivalent, each equivalence class is a coclique. Also, since $p = 3$, the number of equivalence classes it at least three. Pick a base triangle and consider a vertex of any other triangle. We can move from that vertex to a $\approx$-equivalent vertex of an adjacent triangle by either doing nothing or taking a diagonal neighbor. Since the surface is connected, we can repeat this to produce an equivalent vertex of the base triangle. The number of $\approx$-equivalence classes is thus exactly three. By platonicity, all three have the same cardinality and we have partitioned the vertex set of $\Gamma(\mathbf{R})$ into three cocliques of size $\frac{v}{3}$ as promised. □

The last case in the above proof is in need of further inspection. Let us restate the assumptions we can now make. We have $p = 3$, $2 \mid q$ and the vertex set $V(\mathbf{R})$ has a tripartition into cocliques of $\Gamma(\mathbf{R})$. These cocliques have size $v/3$. We will name them A, B, C and denote their respective vertices by (indexed) a's, b's and c's. We remark that the tripartition allows us to form the diagonal map $D(\mathbf{R})$, and that diagonal neighbors are indeed neighbors in this map if we choose their coclique to be the vertex set. The primitive period J is of prime significance. A convenient number for us will be $j := \operatorname{lcm}(J, 2)$, instead of J itself. It will avoid complications, since at various points we will use that a rotation S_v^j fixes A, B, C setwise. Note that $j \mid q$. Our primary goal now will be to establish $j = 2$. As a byproduct, it will become clear that there exist no reflexive platonic maps with density in the interval $(\frac{1}{2}, \frac{2}{3})$. The steps we take are:

1. demonstration that $S_a^j = S_{a'}^j$ for diagonal neighbors a, a';
2. Proof of the formula $S_a^j S_b^j S_c^j = 1$;
3. construction of a quotient $\mathbf{R}/H$, leading by descent to $q = \frac{2}{3}v$;
4. proof that $j = 2$;
5. classification of the maps with $j = 2$.

One more remark before we execute this program. If $a \in A$ is a vertex then S_a^j fixes all diagonal neighbors of a, and by induction the whole coclique A. Because it is not the identity, it cannot fix any vertex from either B or C ($\Gamma(\mathbf{R})$ is simple). If a and a' are two diagonal neighbors, then $S_a^j = S_{a'}^{kj}$ for some $k \in \mathbb{Z}/\frac{q}{j}\mathbb{Z}$. By platonicity, also $S_{a'}^j = S_a^{kj}$ and hence $S_a^j = S_a^{k^2 j}$, from which we deduce the equation $k^2 \equiv 1 \bmod \frac{q}{j}$. It will be our job to prove $k \equiv 1 \bmod \frac{q}{j}$. We do already know that if a'' is a diagonal neighbor of a' and a' of a, then $S_a^j = S_{a''}^j$. Also, $[S_a^j, S_b^j] = 1$ because this automorphism fixes all vertices from A and B. And for two vertices in different cocliques, say $a \in A$ and $b \in B$, we can compute $[S_a^j, S_b^4] = S_a^{-j} S_{S_b^4(a)}^j = S_a^{-j} S_a^j = 1$.

Demonstration that $S_a^j = S_{a'}^j$. Take $a \in A$, $b \in B$, $c \in C$ such that $a \sim b$, $a \sim c$. Suppose that $S_b^j(c) \not\sim a$. Certainly $S_b^j = S_{b'}^j$ for any $b' = S_a^{4m}(b)$, $m = 0, \ldots, \frac{q}{4} - 1$. Hence by platonicity, $S_{b'}^j(S_a^{4m}(c)) \not\sim a$. So none of the images of the vertices $S_a^{4m}(c)$ under S_b^j are connected to a, and they are all different. Because j is even all these vertices lie in C. Adding the vertices of $\partial D(a,1) \cap C$, we find that

$$|C| \geq \frac{q}{2} + \frac{q}{4} = \frac{3}{4}q > \frac{3}{8}v > \frac{v}{3},$$

an impossibility. If follows that $S_b^j(c) \sim a$. So take a vertex $a \in A$ and number its neighbors counterclockwise $b_0, c_0, b_1, c_1, \ldots, b_{q/2-1}, c_{q/2-1}$. The automorphism $S_{b_0}^j$ fixes all neighbors b_i and permutes the neighbors c_i. By platonicity, there are $m_o, m_e \in \mathbb{Z}/\frac{q}{2}\mathbb{Z}$ such that $S_{b_k}^j(c_l) = c_{l+m_o}$ if $|k-l|$ is odd and $S_{b_k}^j(c_l) = c_{l+m_e}$ if $|k-l|$ is even. Thus, $S_{b_0}^j$ also acts on the parity of the indices of the vertices c_i, wherefore $m_o \equiv m_e \bmod 2$. We prove that $2m_e \equiv 0 \bmod j$ by distinguishing the two parity cases.

<u>$m_o, m_e \equiv 0 \bmod 2$</u>**.** It now follows that $S_{b_0}^{nj}(c_0) = c_{nm_e}$. Consideration of the order q/j of $S_{b_0}^j$ implies directly that $2m_e \equiv 0 \bmod j$.

<u>$m_o, m_e \equiv 1 \bmod 2$</u>**.** The computations

$$S_{b_0}^{(k-1)j}(c_0) = S_{b_0}^{kj} S_{b_0}^{-j}(c_0) = S_{b_0}^{kj}(c_{-m_o}) = S_{b_1}^j(c_{-m_o}) = c_{m_e - m_o}$$
$$S_{b_0}^{(1-k)j}(c_0) = S_{b_0}^{-kj} S_{b_0}^j(c_0) = S_{b_0}^{-kj}(c_{m_e}) = S_{b_1}^{-j}(c_{m_e}) = c_{m_e - m_o}$$

force us to conclude $S_{b_0}^{(k-1)j} = S_{b_0}^{(1-k)j}$, since both automorphism fix B vertexwise and have the same effect on c_0. Hence, $2(k-1) \equiv 0 \bmod q/j$. If $q/j \equiv 1 \bmod 2$ this immediately implies $k \equiv 1 \bmod q/j$ already; we can skip that case. If $q/j \equiv 0 \bmod 2$, we note that $k \equiv 1 \bmod 2$, since $k^2 \equiv 1 \bmod q/j$. We then compute $S_{b_0}^{(k+1)j}(c_0) = S_{b_0}^{kj}(c_{m_e}) = S_{b_1}^j(c_{m_e}) = c_{2m_e}$, and again consideration of the order (of $S_{b_0}^{(k+1)j}$) implies $2m_e \equiv 0 \bmod (q/2)/(q/2j)$, which is the claim.

From $2m_e \equiv 0 \bmod j$ we deduce that $S_a^{-2m_e} S_{b_0}^j \in \langle S_{c_0} \rangle$ and that $[S_a^{-2m_e}, S_{b_0}^j] = 1$. The order of $S_a^{-2m_e} S_{b_0}^j$ divides q/j by the commutation relation, and order consideration tells us $S_a^{-2m_e} S_{b_0}^j = S_{c_0}^{nj}$. This implies that the automorphism also fixes c_1 and we see

$$c_1 = S_{c_0}^{nj}(c_1) = S_a^{-2m_e} S_{b_0}^j(c_1) = S_a^{-2m_e}(c_{1+m_o}) = c_{1+m_o-m_e},$$

making it clear that $m_o = m_e$. This knowledge in turn leads to $S_{b_0}^j(c_0) = c_{m_e} = c_{m_o} = S_{b_1}^j(c_0)$. But then $S_{b_1}^{-j} S_{b_0}^j$ fixes c_0 and all vertices of B, so it must be the identity. The upshot is that $S_b^j = S_{b'}^j$ for any two diagonal neighbors, or in other words $k \equiv 1 \bmod q/j$.

Proof that $S_a^j S_b^j S_c^j = 1$. We show that $S_a^j S_b^j S_c^j = 1$ for any choice of vertices in the respective cocliques. Label the vertices, this time the neighbors of $a_0 \in A$ as $b_0, c_0, b_1, c_1, \ldots, b_{q/2-1}, c_{q/2-1}$ and the vertices of $\partial D(b_0, 1) \cap A$ as $a_0, a_1, \ldots, a_{q/2-1}$, both sets counterclockwise. One the one hand $S_{c_0}^j(b_i) = b_{i+m}$ for some $m \in \mathbb{Z}/\frac{q}{2}\mathbb{Z}$ (look around a_0) and $S_{c_0}^j(a_i) = a_{i+m}$ (look around b_0). Applying that automorphism i times yields the first part of Figure 4.5. Consideration of $\operatorname{ord}(S_{c_0}^j)$ implies that we may choose i to obtain $im = \frac{j}{2} \bmod \frac{q}{2}$. On the other hand, applying $S_{b_0}^j$ to results in the transformation shown in the second part of Figure 4.5. We see that the triangles $\Delta a_{j/2} b_{j/2} c_0$ and $\Delta a_{j/2} b_{j/2} c_{j/2+m}$ have to be the same, so that $m \equiv -\frac{j}{2} \bmod \frac{q}{2}$. We now compute

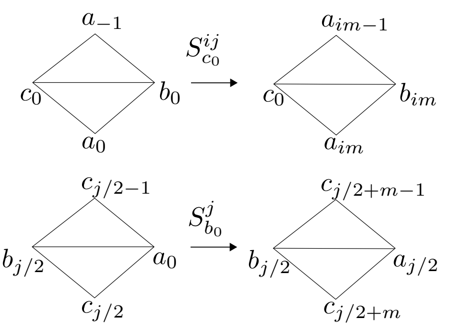


Figure 4.5: Diagrams helping to prove that $S_a^j S_b^j S_c^j = 1$.

$$S_a^j S_b^j S_c^j(a_i) = S_a^j S_b^j(a_{i+m}) = S_a^j(a_{i+m+j/2}) = a_{i+m+j/2} = a_i.$$

The second step is by definition of the numbering. Similar computations yield that $S_a^j S_b^j S_c^j$ fixes b_i and c_i, so $S_a^j S_b^j S_c^j = 1$.

Reduction to the quotient $\mathbf{R}/H$. Our next step is to define the subgroup

$$H = \langle S_v^j : v \in V(\mathbf{R}) \rangle.$$

Its non-trivial elements are those automorphisms that fix all three cocliques setwise and fix all the vertices in one of them. Clearly then, $H \lhd \operatorname{Aut}(\mathbf{R})$ and H contains no edge rotations. By the previous step $H = \langle S_a^j, S_b^j \rangle$ for any fixed $a \in A$, $b \in B$. Because $[S_a^j, S_b^j] = 1$ this group is abelian. We form the quotient map $\overline{\mathbf{R}} := \mathbf{R}/H$. First of all $\overline{q} = j$: take the vertex labelling from the previous paragraph and write $H = \langle S_{a_0}^j, S_{b_0}^j \rangle$. Since $\Gamma(\mathbf{R})$ is simple the only edges $e_{a_0 \cdot}$ identified with $e_{a_0 b_0}$ under

H are the edges $e_{a_0 b_i}$ with i a multiple of j. Second, $\overline{\mathbf{R}}$ has a simple graph: b_0 gets identified only with those same neighbors b_i of a_0 with i a multiple of j for which the edges are also identified. Third, this same argument shows $\overline{v} = vj/q$. As a consequence, we find the density of the quotient map:

$$\delta(\overline{\mathbf{R}}) = \frac{\overline{q}}{\overline{v}} = \frac{j}{vj/q} = \frac{q}{v} = \delta(\mathbf{R}).$$

We can repeat this procedure until at some point $q/j = 1$. In this case we must have $J \in \{q, q/2\}$, but the first possibility was excluded in the proof of Proposition 4.4.8. If $J = q/2$, form the diagonal map $D(\mathbf{R})$, with primitive vertex rotation S'. Take its μ_{00}-quotient $\overline{D(\mathbf{R})} = D(\mathbf{R})/\langle S'^J \rangle$. This has a simple graph because S'^J fixes all vertices of $D(\mathbf{R})$ and the edges identified are precisely those between the same vertices. Its parameters satisfy $\overline{v} = v/3$ and $\overline{q} = q/2 > \frac{1}{4}v = \frac{3}{4}\overline{v}$. We proved that this density is not possible unless $\overline{v} = 1$. So apparently we had $v = 3$ and thus $q = 2$, which together lead to $\mathbf{R} = \mathbf{Dih}(3)$. In this terminal situation a vertex coincides with its diagonal neighbors, but that is irrelevant. The important conclusion is that by this descent $\delta(\mathbf{R}) = \frac{2}{3}$ holds for the original map. As a consequence, any two vertices of different cocliques are connected by an edge. On a side note, this implies that there are no reflexive platonic maps with density in the interval $(\frac{1}{2}, \frac{2}{3})$.

Demonstration that $j = 2$. Label the vertices as before. Since S_{a_0} fixes A setwise and all vertices of A have been labelled (because $q = \frac{2}{3}v$), for any $i \neq 0$ we have $S_{a_0}(a_i) = a_k$ for some $k \neq 0$. Let $c = S_{a_0}(b_0)$. We deduce

$$S_{a_0}(a_{i+lj/2}) = S_{a_0} S_{b_0}^{lj}(a_i) = S_{S_{a_0}(b_0)}^{lj} S_{a_0}(a_i) = S_c^{lj}(a_k) = a_{k-lj/2}.$$

The last step uses $S_{a_0}^j S_{b_0}^j S_c^j = 1$. So S_{a_0} induces an action on the sets of vertices in A with the same residue mod $j/2$. By definition $S_{a_0}(a_1) = a_{-1}$: look at the local picture around b_0. Therefore $S_{a_0}(a_{1+lj/2}) = a_{-1-lj/2} = a_{-(1+lj/2)}$, or in other words $S_{a_0}(a_i) = a_{-i}$ if $i \equiv 1 \bmod j/2$. The residue classes repeat with some primitive period $i > 0$, i.e. i is the smallest such number for which $S_{a_0}^i(a_1) = a_k$ with $k \equiv 1 \bmod j/2$. Obviously, $i \leq j/2$. But because $k \equiv 1 \bmod j/2$ we find $S_{a_0}^{i+1}(a_1) = S_{a_0}(a_k) = a_{-k} = S_{a_0}^{-i+1}(a_1)$. Therefore $S_{a_0}^{2i}(a_1) = a_1$ and hence $J \mid 2i$, so $j = \operatorname{lcm}(J, 2) \mid 2i$. We find that $i \geq j/2$, so equality holds: $i = j/2$. All residue classes therefore appear as diagonal neighbors of a_0. However, residue class $0 \bmod j/2$ is fixed by S_{a_0}, so this must be the only class present, which implies $j/2 = 1$.

The conclusion is now inescapable: $j = 2$. This means that $J \in \{1, 2\}$. The maps with $J = 1$ are easy to determine, because all diagonal neighbors of a vertex a are now identical. If such a diagonal neighbor is a itself (as in the first case treated in Proposition 4.4.8) we find, like we did there, $\mathbf{R} = \mathbf{Dih}(3) = \mathbf{Fer}(1)$. Otherwise, we see that $\operatorname{ord}(S_b) = 4$ for a neighboring vertex $b \sim a$, which leads to $q = 4$ and hence $\mathbf{R} = \mathbf{Oct} = \mathbf{Fer}(2)$. What about the remaining maps, for which $J = 2$? We just hinted at the outcome...

Classification of the maps with $J = 2$. Draw the familiar local picture around a_0 with the knowledge $J = 2$, see Figure 4.6. We alternately apply $S_{b_0}^{-1}$ and S_{a_0} and see what happens to the vertices in the figure:

$$a_0 \mapsto c_0 \mapsto b_1 = S_{b_0}^{-1}(b_{-1}) \mapsto S_{b_0}^{-2}(b_{-1}) = b_{-1} \mapsto c_{-1} \mapsto a_0 \mapsto a_0,$$
$$b_0 \mapsto b_0 \mapsto c_0 \mapsto a_{-1} \mapsto a_k = a_1 \mapsto c_{-1} \mapsto b_0.$$

This shows that $(S_{b_0}^{-1} S_{a_0})^3$ fixes both a_0 and b_0, and hence that $(S_{b_0}^{-1} S_{a_0})^3 = 1$. If we choose the standard generator pair $(R_{\triangle a_0 b_0 c_0}, S_{a_0})$, then $(S_{b_0}^{-1} S_{a_0})^3 = 1$ is equivalent to $[R, S]^3 = 1$. Taken together with $p = 3$ and $q = 2n$, we conclude that $\mathrm{Aut}^+(\mathbf{R})$ must be a quotient of

$$\left\langle R, S \,\middle|\, R^3, S^{2n}, (RS)^2, [R, S]^3 \right\rangle.$$

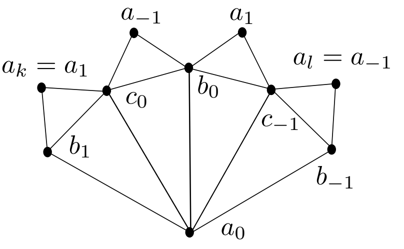


Figure 4.6: We can derive a relator when $J = 2$.

Referring back to polynomial family 2.9.1, we see the presented group is $\mathrm{Aut}^+(\mathbf{Fer}(n))$ of order $6n^2$. But since $q = \frac{2}{3}v$, we must have $|\mathrm{Aut}^+(\mathbf{R})| = qv = 2n \cdot 3n = 6n^2$. So the relators found form a complete set, and $\mathbf{R} = \mathbf{Fer}(n)$.

Combining all the work, we have proved the following characterization of the Fermat maps as satisfying a combinatorial extremal property.

Platonic density theorem. *The only reflexive platonic maps with simple graph and density $\delta > \frac{1}{2}$ are the tetrahedron ($\delta = \frac{3}{4}$) and the Fermat maps ($\delta = \frac{2}{3}$).* □

Remark 4.4.9. The assumption $\delta(\mathbf{R}) > \frac{1}{2}$ was used once more in the induction step to conclude $\delta(\mathbf{R}) = \frac{2}{3}$. We mention this because there are other reflexive platonic maps with graph diameter 2, $p = 3$ and three cocliques of vertices, as exemplified by $\mathbf{R}_{13.2}$ with density exactly $\frac{1}{2}$. The reflexivity was essential again in the proof that $j = 2$. For example the chiral map $\mathbf{C}_{28.2}$ of type $(3, 18)$ has $\delta = 2/3$ and simple graph with its vertex set divided into three cocliques, but satisfies $j = 6$.

4.5 Platonic maps of type $(4, 2g + 2)$

Proposition 4.5.1. *A platonic map of genus g and type $(4, 2g + 2)$ is a member of $\mathbf{AM}(n)$ or $\mathbf{Kul}(n)$.*

Proof. With the genus formula, we compute that a platonic map of type $(4, 2g + 2)$ has automorphism group $\mathrm{Aut}^+(\mathbf{M})$ of order $8(g + 1)$. It follows that $v = 4$, $e = 4(g + 1)$, and $f = 2(g + 1)$. Suppose that a vertex is adjacent to all three others. Then the edges $e_0, \ldots, e_{2g+1}$ around v_1 will alternately have v_2, v_3, v_4 as their other end point. Platonicity implies that each quadrangle has all four vertices on its boundary. Three cycles of vertices will now occur when traversing the boundary of each face counterclockwise: $(v_1 v_2 v_4 v_3)$, $(v_1 v_3 v_2 v_4)$, and $(v_1 v_4 v_3 v_2)$. A rotation R around a face of the first type acts sends a face of the third type to a face with vertices $(v_2v_3v_1v_4)$ on its boundary. This cycle will occur counterclockwise, since R preserves orientation. But such a face does not exist on the map.

We conclude that a vertex cannot be connected to all others by edges. Since it must be connected to at least two in order for $\Gamma(\mathbf{M})$ to be connected, the graph is bipartite. Proposition 3.1.2 implies that we can construct a (self-dual) platonic D_1-map of type $(2g+2, 2g+2)$ out of $\mathbf{M}$. This map will have two vertices. We have already classified all such platonic maps. The only solutions are from the families $D_1(\mathbf{AM}(n))$ and $D_1(\mathbf{Kul}(n))$. This in turn implies that any map of type $(4, 2g+2)$ belongs to one of the two families $\mathbf{AM}(n)$ or $\mathbf{Kul}(n)$. □

4.6 Platonic maps of type $(g+3, g+3)$

We would like to classify all platonic maps of type $(g+3, g+3)$. With the genus formula, we find that such a map has automorphis group $\mathrm{Aut}^+(\mathbf{M})$ of order $4(g+3)$, and so $v=4$, $e=2g+6$, $f=4$. The most difficult thing is then to find the right conjecture to prove. We first introduce some new families by their map automorphism groups:

$$\mathrm{Aut}^+\left(\mathcal{F}_{3n-3}^{(3n,3n)}\right) = \left\langle R, S \mid R^{3n}, S^{3n}, (RS)^2, R^3S^3 \right\rangle,$$
$$\mathrm{Aut}^+\left(\mathcal{F}_{4n-3}^{(4n,4n)}\right) = \left\langle R, S \mid R^{4n}, S^{4n}, (RS)^2, R^4S^4, [R^2, S], [R, S^2] \right\rangle,$$
$$\mathrm{Aut}^+\left(\mathcal{F}_{8n-3}^{(8n,8n)}\right) = \left\langle R, S \mid R^{8n}, S^{8n}, (RS)^2, R^4S^4, R^{4n-3}S^{-1}RS^{-1}, S^{4n-3}R^{-1}SR^{-1} \right\rangle,$$
$$\mathrm{Aut}^+\left(\mathcal{F}_{16n-3}^{(16n,16n)}\right) = \left\langle R, S \mid R^{16n}, S^{16n}, (RS)^2, [R^2, S], R^{8n-4}S^{-4} \right\rangle,$$
$$\mathrm{Aut}^+\left(\mathcal{F}_{12n-3}^{(12n,12n)}\right) = \left\langle R, S \mid R^{12n}, S^{12n}, (RS)^2, [R^3, S], [R, S^3], R^{6n-3}S^{-3} \right\rangle,$$
$$\mathrm{Aut}^+\left(\mathcal{F}_{24n-3}^{(24n,24n)(1)}\right) = \left\langle R, S \mid R^{24n}, S^{24n}, (RS)^2, [R^3, S], [R, S^3], R^{18n-3}S^{-3} \right\rangle,$$
$$\mathrm{Aut}^+\left(\mathcal{F}_{24n-3}^{(24n,24n)(2)}\right) = \left\langle R, S \mid R^{24n}, S^{24n}, (RS)^2, [R^3, S], [R, S^3], R^{6n-3}S^{-3} \right\rangle.$$

The notation $\mathrm{Aut}^+(\mathcal{F}_n^{(p(n),q(n))})$ is a slight abuse, but of course we mean the automorphism group of the n-th family member.

Remark 4.6.1. Some members of the Conder list are:

$\mathcal{F}_{3n-3}^{(3n,3n)}$	$\mathbf{R}_{3.8}$, $\mathbf{R}_{6.9}$, $\mathbf{R}_{9.27}$, $\mathbf{R}_{12.8}$	$\mathcal{F}_{12n-3}^{(12n,12n)}$	$\mathbf{R}_{9.26}$, $\mathbf{R}_{21.34}$, $\mathbf{R}_{33.78}$
$\mathcal{F}_{24n-3}^{(24n,24n)(1)}$	$\mathbf{R}_{21.33}$, $\mathbf{R}_{45.35}$, $\mathbf{R}_{69.43}$	$\mathcal{F}_{24n-3}^{(24n,24n)(2)}$	$\mathbf{R}_{21.32}$, $\mathbf{R}^{\vee}_{45.35}$, $\mathbf{R}_{69.44}$
$\mathcal{F}_{4n-3}^{(4n,4n)}$	$\mathbf{R}_{5.13}$, $\mathbf{R}_{9.28}$, $\mathbf{R}_{13.19}$	$\mathcal{F}_{8n-3}^{(8n,8n)}$	$\mathbf{R}_{5.12}$, $\mathbf{R}_{13.18}$, $\mathbf{R}_{21.37}$
$\mathcal{F}_{16n-3}^{(16n,16n)}$	$\mathbf{R}_{13.17}$, $\mathbf{R}_{29.28}$, $\mathbf{R}_{45.39}$		

Not all relators displayed are necessary to define the families. Some are chosen to make the presentation symmetric in (R, S), so self-duality can be read off directly. This makes the burden of proof heavier in computing the group orders for these families. Alas, we did not finish proofs of the orders of all of them. Listing many cases by computer did assure us that they are correct, though. Assuming this, the members of the two families $\mathcal{F}_{24n-3}^{(24n,24n)(1)}$ and $\mathcal{F}_{24n-3}^{(24n,24n)(2)}$ coincide when n is odd. Apart from that, the families are disjoint. Now we can state the classification.

Proposition 4.6.2. *A platonic map of type* $(g+3, g+3)$ *belongs to one of the self-dual families* $\mathbf{M}_{1.2}(m,n)$ *(torus maps),* $\mathcal{F}_{3n-3}^{(3n,3n)}$*,* $\mathcal{F}_{4n-3}^{(4n,4n)}$*,* $\mathcal{F}_{8n-3}^{(8n,8n)}$*, or* $\mathcal{F}_{12n-3}^{(12n,12n)}$*, or it's a member of one of the families* $\mathcal{F}_{24n-3}^{(24n,24n)(1)}$*,* $\mathcal{F}_{24n-3}^{(24n,24n)(2)}$*, or* $\mathcal{F}_{16n-3}^{(16n,16n)}$ *or their respective dual families.*

Proof. Topologically, we want to glue together four $(g+3)$-gons around four vertices. Each vertex is either connected to two or three others by edges, i.e. $d_{00} \in \{2,3\}$. First we consider the case $d_{00} = 2$, which will yield $\mathcal{F}_{4n-3}^{(4n,4n)}$, $\mathcal{F}_{8n-3}^{(8n,8n)}$, $\mathcal{F}_{16n-3}^{(16n,16n)}$, and $\mathcal{F}_{16n-3}^{(16n,16n)\vee}$. We see that a face must be incident to all four vertices by following its boundary. That implies that around each vertex, the faces alternate with period 4, and hence $g+3 \equiv 0 \bmod 4$. Let us number them like in Figure 4.7. We see that around

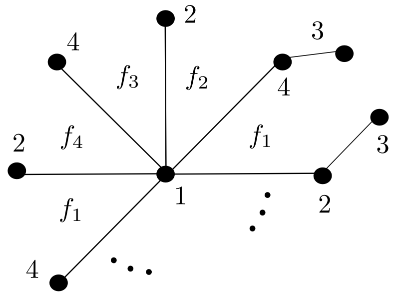


Figure 4.7: A local figure if $d_{00} = 2$.

v_2 and v_4, the faces will alternate as f_1, f_4, f_3, f_2 counterclockwise, and around v_1 and v_3 as f_1, f_2, f_3, f_4 counterclockwise. Likewise, on ∂f_1 and ∂f_3 the vertices alternate as v_1, v_2, v_3, v_4 counterclockwise, and on ∂f_2 and ∂f_4 as v_1, v_4, v_3, v_2 counterclockwise. From the local figure, we quickly see that

$$R_2^2 = S_{j+1}^{-1} S_j^{-1} = R_4^2 \qquad R_1^2 = S_j^{-1} S_{j+1}^{-1} = R_3^2$$
$$S_2^2 = R_{j+1}^{-1} R_j^{-1} = S_4^2 \qquad S_1^2 = R_j^{-1} R_{j+1}^{-1} = S_3^2$$

where the indices are taken from $\mathbb{Z}/4\mathbb{Z}$. Since R_i switches f_{i+1} and f_{i-1} (deduce this from the local figures around $v_{1,2}$), it fixes both f_i and f_{i+2}. Analogously, S_i fixes both v_i and v_{i+2}. We find

$$R_i = R_{i+2}^{u_1}$$
$$S_i = S_{i+2}^{u_2}$$

and squaring these equations implies $u_{1,2} \in \{1, \frac{g+3}{2}+1\} \bmod (g+3)$. (One may also note that the squared equations imply that $[R_i^2, S_j^2] = 1$ for all (i,j).) Taking the dual of the map under consideration doesn't change any of the assumptions, so we are left with the following trichotomy:

$u_1 = u_2 = 1$ Because $S_{i+2} = R_j^{-2} S_i R_j^2$ (for any j) we find that $[R_j^2, S_i] = 1$. Similarly, $[R_j, S_i^2] = 1$. We conclude from this that we have a member of $\mathcal{F}_{4n-3}^{(4n,4n)}$. The relators defining the map are symmetric in (R, S), so the maps are self-dual.

$u_1 = \frac{g+3}{2} + 1, u_2 = 1$ Just like in the first case we deduce $[R_i^2, S_j] = 1$. The equation $R_i = R_{i+2}^{\frac{g+3}{2}+1}$ implies that $g+3 \equiv 0 \bmod 8$, since the action on the vertices (permutation (v_1, v_2, v_3, v_4)) implies that $1 \equiv \frac{g+3}{2} + 1 \bmod 4$. Notice that R_i^4 fixes all vertices, and hence $R_i^4 = S_j^{4k}$, k being independent of (i,j) by platonicity. And k must be odd because $\operatorname{ord}(R_i^4) = \operatorname{ord}(S_j^4)$. Since $4 \mid \frac{g+3}{2}$, we can write $R_i^{\frac{1}{2}(g+3)} = S_j^{\frac{1}{2}(g+3)k} = S_j^{\frac{1}{2}(g+3)}$. We now deduce

$$S_1^2 = R_1^{-1} R_2^{-1} = R_1^{-1} R_4^{-1} R_4^{\frac{1}{2}(g+3)} = S_2^2 S_2^{\frac{1}{2}(g+3)} = S_2^{\frac{1}{2}(g+7)}.$$

This generalizes to $S_i^2 = S_{i\pm1}^{\frac{1}{2}(g+7)}$. Easier is that

$$R_1^2 = S_1^{-1} S_2^{-1} = S_1^{-1} S_4^{-1} = R_2^2,$$

which generalizes to $R_i^2 = R_j^2$. This allows us to prove

$$S_1^4 = S_1^{-2} S_1^{-2} = R_1^{-1} R_2^{-1} R_2^{-1} R_3^{-1} = R_1^{-1} R_1^{-2} R_1^{-\frac{1}{2}(g+3)-1} = R_1^{\frac{1}{2}(g-5)},$$

which generalizes by platonicity to $S_i^4 = R_j^{\frac{1}{2}(g-5)}$. Finally, from this last equation we glean that $\frac{1}{2}(g-5) = (2m+1) \cdot 4$, so $g+3 = (2m+2) \cdot 8 \equiv 0 \bmod 16$, since the orders of both elements must be equal. We must therefore have a member of the family $\mathcal{F}_{16n-3}^{(16n,16n)}$ or its dual.

$u_1 = u_2 = \frac{g+3}{2} + 1$ Like in the second case we know that $g+3 \equiv 0 \bmod 8$, $R_i^4 = S_j^{4k}$, $R_i^{\frac{1}{2}(g+3)} = S_j^{\frac{1}{2}(g+3)}$, and $S_i^2 = S_{i\pm1}^{\frac{1}{2}(g+7)}$. This time, because the conditions are symmetric in R_i and S_i, also $R_i^2 = R_{i\pm1}^{\frac{1}{2}(g+7)}$. This allows us to prove

$$S_1^{-4} = S_1^{-2} S_1^{-2} = R_4 R_3 R_3 R_2 = R_4 R_4^{\frac{1}{2}(g+7)} R_4^{\frac{1}{2}(g+3)+1} = R_4^4,$$

which generalizes by platonicity to $R_i^4 = S_j^{-4}$. Furthermore,

$$S_1 R_1^{-1} S_1 = S_1^2 (S_1^{-1} R_1^{-1} S_1) = S_1^2 R_2^{-1} = R_1^{-1} R_2^{-2} = R_1^{-1-\frac{1}{2}(g+7)} = R_1^{\frac{1}{2}(g-3)},$$

generalizing to $S_i R_j^{-1} S_i = R_j^{\frac{1}{2}(g-3)}$. This demonstrates that all relations of $\mathcal{F}_{8n-3}^{(8n,8n)}$ hold.

This finishes treatment of the case $d_{00} = 2$. We now proceed with the case $d_{00} = 3$, implying that any two vertices are connected by edges. Let the end points of edges incident to v_1 be v_2, v_3, v_4 periodically, counterclockwise. Note that the condition thus implies $3 \mid \operatorname{ord}(S_1) = g+3$. We proceed to prove that exactly three faces are incident to v_1. If there were only 2, then they would repeat with period 2 around, and a face would be incident to all vertices. By platonicity, all four faces would then be incident to v_1, contrary to assumption. Assume that indeed all four faces are incident to v_1. Then a face f would again be incident to all four vertices, and these would thus repeat with period 4 around ∂f. But there would be a face with both (v_3, v_1, v_2) and (v_4, v_1, v_3) as part of this pattern (counterclockwise), which is impossible. It follows that exactly three faces are incident with v_1, and hence three vertices occur on a face boundary ∂f. We number the faces so that f_i is *not* incident to v_i. We can now fill in the local figures (stars) around each vertex and face. Next, since S_i^3 fixes all

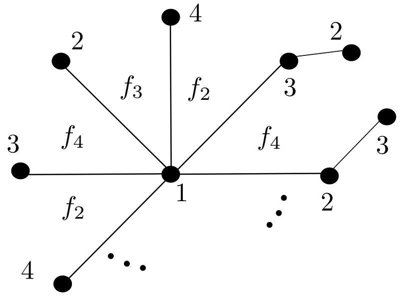


Figure 4.8: A local figure if $d_{00} = 3$.

vertices, $S_i^3 = S_j^{3k}$ ($k \in \mathbb{Z}/\left(\frac{1}{3}(g+3)\right)\mathbb{Z}$ independent of (i,j) by platonicity). We have $S_1^3 = S_2^{3k} = S_1^{3k^2}$ and $S_1^3 = S_2^{3k} = S_3^{3k^2} = S_1^{3k^3}$, whence $k \equiv 1 \bmod \frac{1}{3}(g+3)$ and

$$S_i^3 = S_j^3 \text{ for all } i, j.$$

Similarly, we find that

$$R_i^3 = R_j^3 \text{ for all } i, j.$$

Now we compute

$$\begin{aligned} R_1^4 S_1^4 &= R_1 R_4^3 S_1^4 = R_1 R_4 (S_3^{-1} S_1^{-1}) S_1^4 = R_1 R_4 S_3^{-1} S_1^3 \\ &= R_1 R_4 S_3^2 = R_1 R_4 R_4^{-1} R_1^{-1} = 1 \end{aligned}$$

where the equalities $R_4^2 = S_3^{-1} S_1^{-1}$ and $S_3^2 = R_4^{-1} R_1^{-1}$ follow by looking at the local figures. Of course this generalizes to $R_i^4 S_i^4 = 1$. The crucial step for determining the structure of these maps completely is to notice that R_i fixes v_i, and thus $R_i = S_i^k$ (with $k \in \mathbb{Z}/(g+3)\mathbb{Z}$ independent of i by platonicity). First of all, this implies $R_i^3 = S_i^{3k} = S_j^{3k}$, which shows that $[R_i^3, S_j] = 1$ and $[R_i, S_j^3] = 1$ for all i, j. Second, we can deduce that

$$S_1^{4(k+1)} = R_1^4 S_1^4 = 1,$$

so that $4(k+1) \equiv 0 \bmod (g+3)$. There are four possibilities:

<u>$k \equiv \frac{1}{4}(g+3) - 1 \bmod (g+3)$</u> We must have $g+3 \equiv 0 \bmod 4$ for this value of k to be defined. But then $\mathrm{ord}(R_i) = g+3$ is even, and because $\mathrm{ord}(S_i^k) = \mathrm{ord}(R_i)$, k must be odd. This means that $g+3 \equiv 0 \bmod 8$ and factoring in the congruence modulo 3, we find that $g+3 \equiv 0 \bmod 24$. The relators found now imply that we must have a member of $\mathcal{F}_{24n-3}^{(24n,24n)(1)}$ or $\mathcal{F}_{24n-3}^{(24n,24n)(2)}$.

$k \equiv \frac{2}{4}(g+3) - 1 \equiv \frac{1}{2}(g-3) \bmod (g+3)$ We must have $g+3 \equiv 0 \bmod 2$, and again k must therefore be odd, so that we get the condition $g+3 \equiv 0 \bmod 12$. This yields a member of the family $\mathcal{F}^{(12n,12n)}_{12n-3}$.

$k \equiv \frac{3}{4}(g+3) - 1 \bmod (g+3)$ The same reasoning as in the first case implies we have a member of $\mathcal{F}^{(24n,24n)(1/2)}_{24n-3}$.

$k \equiv -1 \bmod (g+3)$ The presence of the relator $R_i^3 S_j^3$ with $i \neq j$ immediately implies we have a member of $\mathcal{F}^{(3n,3n)}_{3n-3}$.

□

Remark 4.6.3. The families $\mathcal{F}^{(3n,3n)}_{3n-3}$, $\mathcal{F}^{(4n,4n)}_{4n-3}$, $\mathcal{F}^{(8n,8n)}_{8n-3}$, and $\mathcal{F}^{(12n,12n)}_{12n-3}$ are self-dual. This implies that they are families of D_1-maps of families $\mathcal{F}^{(4,3n)}_{3n-3}$, $\mathcal{F}^{(4,4n)}_{4n-3}$, $\mathcal{F}^{(4,8n)}_{8n-3}$, and $\mathcal{F}^{(4,12n)}_{12n-3}$, respectively. The two families $\mathcal{F}^{(24n,24n)(1)}_{24n-3}$ and $\mathcal{F}^{(24n,24n)(2)}_{24n-3}$ show special behavior: their members coincide and are self-dual for n odd, but for n even they are each other's duals and different. By the D_1-map construction, we know that there is a related family $\mathcal{F}^{(4,48n-24)}_{48n-27}$.

Remark 4.6.4. The families $\mathcal{F}^{(3n,3n)}_{3n-3}$, $\mathcal{F}^{(12n,12n)}_{12n-3}$, $\mathcal{F}^{(24n,24n)(1)}_{24n-3}$, and $\mathcal{F}^{(24n,24n)(2)}_{24n-3}$ are covers of **Tet**, branched over $\mathrm{cells}_0 \cup \mathrm{cells}_2$.

4.7 Platonic maps of type $(4, g+3)$

We want to classify all platonic maps of type $(4, g+3)$. For this purpose, we again introduce a few new polynomial families, by standard map presentations:

$$\mathrm{Aut}^+\left(\mathcal{F}^{(4,3n)}_{3n-3}\right) = \left\langle R, S \mid R^4, S^{3n}, (RS)^2, R^{-1}S^3RS^3 \right\rangle,$$
$$\mathrm{Aut}^+\left(\mathcal{F}^{(4,4n)}_{4n-3}\right) = \left\langle R, S \mid R^4, S^{4n}, (RS)^2, R^{-1}S^4RS^4, [S^2, R^{-1}SR], [S, R^{-1}S^2R] \right\rangle$$
$$\mathrm{Aut}^+\left(\mathcal{F}^{(4,8n)}_{8n-3}\right) = \left\langle R, S \mid R^4, S^{8n}, (RS)^2, R^{-1}S^4RS^4, S^{4n-2}R^2S^2R^2 \right\rangle,$$
$$\mathrm{Aut}^+\left(\mathcal{F}^{(4,12n)}_{12n-3}\right) = \left\langle R, S \mid R^4, S^{12n}, (RS)^2, [S^3, R^{-1}SR], [S, R^{-1}S^3R], S^{6n-3}R^{-1}S^{-3}R \right\rangle,$$
$$\mathrm{Aut}^+\left(\mathcal{F}^{(4,48n-24)(1)}_{48n-27}\right) = \left\langle R, S \,\middle|\, R^4, S^{48n-24}, (RS)^2, [S^3, R^{-1}SR], [S, R^{-1}S^3R], S^{3(12n-7)}R^{-1}S^{-3}R \right\rangle,$$
$$\mathrm{Aut}^+\left(\mathcal{F}^{(4,48n-24)(2)}_{48n-27}\right) = \left\langle R, S \,\middle|\, R^4, S^{48n-24}, (RS)^2, [S^3, R^{-1}SR], [S, R^{-1}S^3R], S^{3(4n-3)}R^{-1}S^{-3}R \right\rangle.$$

Remark 4.7.1. Some members of the Conder list are:

$\mathcal{F}^{(4,3n)}_{3n-3}$	$\mathbf{R}_{3.4}$, $\mathbf{R}_{6.3}$, $\mathbf{R}_{9.11}$, $\mathbf{R}_{12.1}$	$\mathcal{F}^{(4,4n)}_{4n-3}$	$\mathbf{R}_{5.6}$, $\mathbf{R}_{9.10}$, $\mathbf{R}_{13.4}$
$\mathcal{F}^{(4,8n)}_{8n-3}$	$\mathbf{R}_{5.5}$, $\mathbf{R}_{13.5}$, $\mathbf{R}_{21.11}$	$\mathcal{F}^{(4,12n)}_{12n-3}$	$\mathbf{R}_{9.9}$, $\mathbf{R}_{21.6}$, $\mathbf{R}_{33.27}$

Now we are in a position to state:

Proposition 4.7.2. *A platonic map of type* $(4, g+3)$ *belongs to one of the families* $\mathbf{M}_{1.2}(m,n)$, $\mathcal{F}^{(4,3n)}_{3n-3}$, $\mathcal{F}^{(4,4n)}_{4n-3}$, $\mathcal{F}^{(4,8n)}_{8n-3}$, $\mathcal{F}^{(4,12n)}_{12n-3}$, $\mathcal{F}^{(4,48n-24)(1)}_{48n-27}$, *or* $\mathcal{F}^{(4,48n-24)(2)}_{48n-27}$.

Proof. With the genus formula, we find that platonic maps of type $(4, g+3)$ satisfy $|\mathrm{Aut}^+(\mathbf{M})| = 8(g+3)$ and hence $v = 8$. Its μ_{00}-quotient map $\mathbf{M}/\langle S^{d_{00}}\rangle$ will have $\overline{p} \mid 4$, so $\overline{p} \neq 3$. It will thus have density at most $\frac{1}{2}$ by our density theorems of Section 4.4. We conclude that the reduced graph $\overline{\Gamma}(\mathbf{M})$ is an arc-transitive graph on eight vertices with valency $\overline{q} \in \{2,3,4\}$. The only such graphs are the cyclic graph C_8 ($\overline{q} = 2$), the cubic graph Q_3 ($\overline{q} = 3$), and the complete bipartite graph $K_{4,4}$ ($\overline{q} = 4$). All three are bipartite, and we can lift the bipartition to $\mathbf{M}$. So we know that we can form the D_1-map of $\mathbf{M}$, which will be of type $(g+3, g+3)$. We have classified those maps in Section 4.6 and from the presentations for the families listed there we derive presentations for those claimed and completeness of the list. We note that the cyclic graph C_8 does not in fact occur. □

Remark 4.7.3. The families $\mathcal{F}^{(4,3n)}_{3n-3}$ $\mathcal{F}^{(4,12n)}_{12n-3}$, $\mathcal{F}^{(4,48n-24)(1)}_{48n-27}$, and $\mathcal{F}^{(4,48n-24)(2)}_{48n-27}$ consists of platonic covers of **Cub** branched over cells_0. The last two demonstrate that a platonic cover is not uniquely determined by its degree and the branching orders for vertices and faces.

4.8 Platonic maps of type $(6, g+2)$

We want to classify all platonic maps of type $(6, g+2)$. With the genus formula, we find that such a map has map automorphism group $\mathrm{Aut}(\mathbf{M})$ of order $12(g+2)$. It follows that $v = 6$, $e = 3g+6$, and $f = g+2$. The most difficult step is again formulating the right proposition. We introduce a few more polynomial families:

$$\mathrm{Aut}^+\left(\mathcal{F}^{(6,2n)}_{2n-2}\right) = \left\langle R, S \mid R^6, S^{2n}, (RS)^2, (RS^{-1})^2 \right\rangle,$$
$$\mathrm{Aut}^+\left(\mathcal{F}^{(6,3n)}_{3n-2}\right) = \left\langle R, S \mid R^6, S^{3n}, (RS)^2, [R^2, S] \right\rangle,$$
$$\mathrm{Aut}^+\left(\mathcal{F}^{(6,18n-12)}_{18n-14}\right) = \left\langle R, S \mid R^6, S^{18n-12}, (RS)^2, [R^2, S^2], RS^{-3}RS^{6n-7} \right\rangle,$$
$$\mathrm{Aut}^+\left(\mathcal{F}^{(6,18n-6)}_{18n-8}\right) = \left\langle R, S \mid R^6, S^{18n-6}, (RS)^2, RS^{-1}RS^{6n-3} \right\rangle,$$
$$\mathrm{Aut}\left(\mathcal{F}^{(6,18n-9)}_{18n-11}\right) = \left\langle R, S \;\middle|\; R^6, S^{18n-9}, (RS)^2, R^3SR^{-1}S^{-(6n-4)} \right\rangle,$$
$$\mathrm{Aut}\left(\mathcal{F}^{(6,18n)}_{18n-2}\right) = \left\langle R, S \mid R^6, S^{18n}, (RS)^2, R^3S^{-1}RS^{6n-1} \right\rangle.$$

Note that the last two families are chiral. Again, we have not verified the group orders for all these families, but computer experiments give us confidence of the correctness. This puts us in a position to state a conjecture, but we did not manage to finish a proof.

Conjecture 4.8.1. A platonic map of type $(6, g+2)$ is either $\mathbf{R}_{6.8}$ or $\mathbf{R}_{22.11}$, which are defined by

$$\begin{aligned}\mathrm{Aut}^+(\mathbf{R}_{6.8}) &= \left\langle R, S \,\middle|\, R^6, S^8, (RS)^2, (R^2S^{-1})^2, R^3S^4 \right\rangle, \\ \mathrm{Aut}^+(\mathbf{R}_{22.11}) &= \left\langle R, S \,\middle|\, R^6, S^{24}, (RS)^2, (R^2S^{-1})^2, RS^{-3}RS^5 \right\rangle.\end{aligned}$$

or is a member of one of the eight families $\mathcal{F}^{\vee}_{1.1:n}$, $\mathcal{F}^{\vee}_{1.2:n}$, $\mathcal{F}^{(6,2n+2)}_{2n}$, $\mathcal{F}^{(6,3n+3)}_{3n+1}$, $\mathcal{F}^{(6,18n+6)}_{18n+4}$, $\mathcal{F}^{(6,18n+12)}_{18n+10}$, $\mathcal{F}^{(6,18n-9)}_{18n-11}$ (chiral), or $\mathcal{F}^{(6,18n)}_{18n-2}$ (chiral).

Remark 4.8.2. The map $\mathbf{R}_{22.11}$ is a platonic cover of $\mathbf{R}_{6.8}$ branched over cells_0, corresponding to the exact sequence

$$1 \longrightarrow \langle S^8 \rangle \longrightarrow \mathrm{Aut}(\mathbf{R}_{22.11}) \longrightarrow \mathrm{Aut}(\mathbf{R}_{6.8}) \longrightarrow 1$$

The map $\mathbf{R}_{22.11}$ is thereby also seen to be a platonic cover of $\mathbf{Oct}$, branched over $\mathrm{cells}_0 \cup \mathrm{cells}_2$, by

$$1 \longrightarrow \langle R^3, S^8 \rangle \longrightarrow \mathrm{Aut}(\mathbf{R}_{22.11}) \longrightarrow \mathrm{Aut}(\mathbf{Oct}) \longrightarrow 1$$

5

From algebraic curves to platonic maps

SEVERAL algebraic curves well known to the mathematical community are platonic surfaces. This chapter lists the ones known to the author at the time of writing. In Sections 5.1–5.7 we treat the Wiman curves of type I/II, the Accola-Maclachlan curves, Kulkarni curves, Fermat curves and modular curves $X(N)$.

The strategy to prove that a curve C is indeed platonic is to exhibit an $\mathrm{Aut}(C)$-invariant Belyĭ function on it. In each case we construct such a function, with three branch points, such that their pre-images form one $\mathrm{Aut}(C)$-orbit. The ramification is then the same in the whole orbit. In general, this gives a (p, q, r)-hypermap structure, but we will find $r = 2$ in all cases. Moreover, our branch points will be found to lie on a straight line of $\mathbb{P}^1 \cong \widehat{\mathbb{C}}$. The pre-image of this line is then the graph of our platonic map. So we are in each case assigned the task of constructing a smart function and computing its ramification. For a more hands-on understanding of each curve as a platonic surface, we also try to compute the explicit cell structure by using the extended automorphism group $\mathrm{Aut}^*(C)$. This enables us to present a valid standard map presentation for the curve.

To give a Belyĭ function on the Fermat curves is not that hard, see e.g. [LZ2004, Sec.2.5.3], but we have not seen an $\mathrm{Aut}(\mathbf{Fer}(n))$-invariant one. I therefore believe Section 5.5 to be slightly more comprehensive than previous treatments. The modular curves have been added for completeness.

5.1 The Wiman type I maps $\mathbf{Wi1}(n)$

Proposition 5.1.1 (Wiman type I maps)**.** *The algebraic curves $y^2 = x^{2n+1} - 1$ are platonic surfaces. They support a polynomial family of reflexive platonic maps $\mathbf{Wi1}(n)$*

of genus n and type $(2n+1, 4n+2)$, the Wiman type I maps.

Proof. We projectivize to $y^2z^{2n-1} = x^{2n+1} - z^{2n+1}$. This hyperelliptic curve has a singular point at $(0:1:0)$ and is of genus n. The automorphism group is cyclic of order $4n+2$:

$$\mathrm{Aut}(\mathbf{Wi1}(n)) = \langle (x:y:z) \mapsto (\zeta_{2n+1}x : -y : z) \rangle.$$

Define the function $\beta : \mathbf{Wi1}(n) \to \mathbb{P}^1$ by $(x:y:z) \mapsto (y^2 : z^2)$. It is well-defined everywhere on the curve and its invariance under $\mathrm{Aut}(\mathbf{Wi1}(n))$ is immediate. The function's degree is $4n+2$: for a generic point $(y^2:1) \in \mathbb{P}^1$ there are two choices for y and then $2n+1$ choices for x according to the equation. It remains to calculate the ramification. Locally x is a function of y on the curve, unless $x=0$ (with the implicit function theorem). So on $\{z=1\}$ we find the critical points with $x \neq 0$ where

$$0 = \frac{d}{dy}\beta(x(y):y:1) = 2y.$$

The solutions to this equation are the $2n+1$ points $(\zeta_{2n+1}^k : 0 : 1)$, which all map to $(0:1) \in \mathbb{P}^1$. The other possibilities $x=0$ and $z=0$ also turn out to be critical points. They result in the branch data

Vertex	$\beta^{-1}\big((1:0)\big) = \{(0:1:0)\}$
Edge centers	$\beta^{-1}\big((0:1)\big) = \big\{(\zeta_{2n+1}^k : 0 : 1) \mid k = 1, \ldots, 2n+1\big\}$
Face centers	$\beta^{-1}\big((-1:1)\big) = \big\{(0 : \pm i : 1)\big\}$

We have thus exhibited β as a Belyĭ function defining a platonic map of type $(2n+1, 4n+2)$. □

5.2 The Wiman type II maps Wi2(n)

Proposition 5.2.1 (Wiman type II maps). *The algebraic curves $y^2 = x(x^{2n}-1)$ are platonic surfaces. They support a polynomial family of reflexive platonic maps $\mathbf{Wi2}(n)$ of genus n and type $(4, 4n)$, the Wiman type II maps.*

Proof. The cases $n=0$ and $n=1$ are trivial, so let $n \geq 2$. We projectivize to $y^2z^{2n-1} = x(x^{2n} - z^{2n})$. This curve has a singular point at $(0:1:0)$ and is of genus n. Its automorphism group contains the order $8n$ subgroup (and is in fact equal to it) generated by

$$R : (x:y:z) \mapsto (-x^nz : yz^n : x^{n+1}) \quad \text{and} \quad S : (x:y:z) \mapsto (\zeta_{2n}x : \zeta_{4n}y : z).$$

One computes that $R^2 = S^{2n}$ and so $\mathrm{ord}(R) = 4$. Let us define the function $\beta_0(x : y:z) = (y^2 : xz)$, which is invariant under S. Because apart from a finite number of points, we can consider β_0 to be a function into $\mathbb{C}$, we can average it and set

$$\beta := \sum_{k=0}^{3} \beta_0 \circ R^k : \ (x:y:z) \mapsto (y^4z^{2n-2} : x^{2n+2}).$$

This expression for β is well-defined everywhere on the curve except at $(0:0:1)$ and $(0:1:0)$. The function is R-invariant and also still S-invariant, because $\langle S\rangle \lhd \langle R, S\rangle$; or one verifies it directly. To compute the ramification, we first establish that the degree of β is $8n$: without loss of generality suppose $z=1$ and $\beta((x:y:1)) = (\lambda:1)$. Then $y^4 = \lambda x^{2n-2}$ together with $y^2 = x(x^{2n}-1)$ lead to $x^2(x^{4n}-2x^{2n}+1) = \lambda x^{2n-2}$. Since $x=0$ (and $z=1$) implies $(x:y:z) = (0:0:1)$ and this is not generically in the fiber $\beta^{-1}((\lambda:1))$, we are left with $x^{4n}-2x^{2n}+1 = \lambda x^{2n-4}$. There are $4n$ solutions for x and to each belong two solutions for y.

Now we compute the critical points. We stay in the chart $\{z=1\}$. Locally y is a function of x, unless $y=0$ (with the implicit function theorem). The critical points of β with $x, y \neq 0$ are those for which

$$0 = \frac{d}{dx}\beta(x:y(x):1) = \frac{d}{dx}\left(\frac{y^4}{x^{2n+2}}\right) = \frac{d}{dx}\left(\frac{x^2(x^{2n}-1)}{x^{2n+2}}\right).$$

This is equivalent to

$$x^{2n}(4nx^{4n-1} - 4nx^{2n-1}) - 2nx^{2n-1}(x^{4n} - 2x^{2n} + 1) = 0$$

which reduces to $x = 0$ or $x = \zeta_{4n}^k$ (with $k = 0, \ldots, 4n-1$). We had excluded the first possibility. The others yield $y^2 = \zeta_{4n}^k((-1)^k - 1)$ and separate into two sets. When k is even, then $y = 0$ and β maps these points to $(0:1)$. At each the ramification is 4-fold. When k is odd, then $y = \pm\sqrt{2}\zeta_{8n}^{k+2n}$ and β maps these points to $(-4:1)$. At each the ramification is 2-fold. The remaining points to be investigated are $(0:0:1)$ and $(0:1:0)$. For these one can use the alternative expressions $\beta(x:y:z) = ((x^{2n}-z^{2n})^2 : x^{2n}z^{2n})$ and $\beta(x:y:z) = (y^4 : xz(y^2+zx))$ respectively. We find $\beta((0:0:1)) = \beta((0:1:0)) = (1:0)$ and the ramification at both is $4n$. Summing up, there are exactly three branch points with the following inverse images:

Vertices	$\beta^{-1}\big((1:0)\big) = \{(0:1:0), (0:0:1)\}$,
Edge centers	$\beta^{-1}\big((-4:1)\big) = \big\{(\zeta_{4n}^{2k+1} : \pm\sqrt{2}\zeta_{8n}^{2n+2k+1} : 1) \mid k = 1, \ldots, 2n\big\}$,
Face centers	$\beta^{-1}\big((0:1)\big) = \big\{(\zeta_{2n}^{k} : 0 : 1) \mid k = 1, \ldots, 2n\big\}$.

We have thus exhibited this algebraic curve as a platonic map of type $(4, 4n)$. □

Remark 5.2.2. As we know from Chapters 2 and 3, the family $\mathbf{Wi2}(n)$ has a polynomial family of diagonal maps $D_1(\mathbf{Wi2}(n))$.

5.3 The Accola-Maclachlan maps $\mathrm{AM}(n)$

Proposition 5.3.1 (Accola-Maclachlan maps)**.** *The algebraic curves $y^2 = x^{2n+2} - 1$ are platonic surfaces. They support a polynomial family of reflexive platonic maps $\mathbf{AM}(n)$ of genus n and type $(4, 2n+2)$, the Accola-Maclachlan maps.*

Proof. We projectivize to $y^2 z^{2n} = x^{2n+2} - z^{2n+2}$. This hyperelliptic curve has a unique singular point at $(0 : 1 : 0)$ and is of genus n. Its automorphism group contains the order $8n + 8$ subgroup (and is in fact equal to it) generated by

$$R : (x : y : z) \mapsto (-x^n z : iyz^n : x^{n+1}) \quad \text{and} \quad S : (x : y : z) \mapsto (\zeta_{2n+2} x : y : z)$$

A function invariant under these two operations is quickly found to be $\beta : (x : y : z) \mapsto (x^{2n+2} : y^4 z^{2n-2})$. The expression is well-defined everywhere except at the unique singular point $(0 : 1 : 0) \in \mathbf{AM}(n)$. The degree of β is $8n + 8$: without loss of generality assume that $z = 1$ and $\beta((x : y : 1)) = (\lambda : 1)$. Then $x^{2n+2} = \lambda y^4$. Combine this with $y^2 = x^{2n+2} - 1$ to find $y^2 = \lambda y^4 - 1$. So generically there are 4 solutions for y, and to each belong $2n + 2$ values of x.

We compute the critical points on $\{z = 1\}$. Locally, we can write $x = f(y)$ (with the implicit function theorem) unless $x = 0$. Thus, the critical points for which $x \neq 0$ are those for which

$$0 = \frac{d}{dy}\beta(x(y) : y : 1) = \frac{d}{dy}\left(\frac{y^4}{x^{2n+2}}\right) = \frac{d}{dy}\left(\frac{y^4}{y^2 + 1}\right).$$

This is equivalent to

$$4y^3(y^2 + 1) = 2y^5$$

which reduces to $y = 0$ or $y = \pm i\sqrt{2}$. The first case yields the $2n + 2$ points $(\zeta_{2n+2}^k : 0 : 1)$ that β all maps to $(1 : 0)$. At each the ramification is 4-fold. The second case leads to the $4n + 4$ points $(\zeta_{4n+4}^{2k+1} : \pm i\sqrt{2} : 1)$. They are all mapped to $(-1 : 4)$ by β, and at each the ramification is 2-fold. Scheduled for separate consideration were the points with $x = 0$. These are $(0 : \pm i : 1)$ and $(0 : 1 : 0)$, which together form $\beta^{-1}((0 : 1))$. To see that indeed $\beta((0 : 1 : 0)) = (0 : 1)$, use the alternative expression $\beta((x : y : z)) = (z^2(y^2 + z^2) : y^4)$. At the first two the ramification is $2n + 2$-fold, at the third $4n + 4$-fold; two of the four vertices of the platonic map lie on top of each other. To summarize:

Vertices	$\beta^{-1}\big((0 : 1)\big) = \{(0 : 1 : 0), (0 : \pm i : 1)\}$
Edge centers	$\beta^{-1}\big((-1 : 4)\big) = \big\{(\zeta_{4n+4}^{2k+1} : \pm i\sqrt{2} : 1) \mid k = 1, \ldots, 2n + 2\big\}$
Face centers	$\beta^{-1}\big((1 : 0)\big) = \big\{(\zeta_{2n+2}^{k} : 0 : 1) \mid k = 1, \ldots, 2n + 2\big\}$

We have thus exhibited this algebraic curve as a platonic map of type $(4, 4n+4)$. □

Remark 5.3.2. As we know from Chapters 2 and 3, the family $\mathbf{AM}(n)$ has a polynomial family of diagonal maps $D_1(\mathbf{AM}(n))$.

Remark 5.3.3. For infinitely many n, the Accola-Maclachlan curve $\mathbf{AM}(n)$ has the largest automorphism group (order $|\mathrm{Aut}(\mathbf{AM}n)| = 8n + 8$) of all curves of genus n, cf. [Acc1968]. The genera $g \leq 101$ for which this occurs form the set $\{20, 23, 32, 35, 38, 44, 47, 59, 62, 68, 74, 80, 83, 88, 95, 98\}$.

5.4 The Kulkarni maps Kul(n)

In 1991, Ravi Kulkarni [Kul1991] discovered the only Riemann surfaces X besides the Accola-Maclachlan surface $(\mathbf{AM}(n))_r$ for which $|\mathrm{Aut}(X)| = 2g + 2$. In [Tur1997], Peter Turbek found the following planar algebraic models $y^{2g+2} - x(x-1)^{g-1}(x+1)^{g+2} = 0$ for these Kulkarni curves and computed explicit expressions for their automorphisms. They form the polynomial family of reflexive platonic maps $\mathbf{Kul}(n)$ that we have met in Chapter 2.

Proposition 5.4.1. *The algebraic curves $y^{8n} = x(x-z)^{4n-2}(x+z)^{4n+1}$ are platonic surfaces. They support the polynomial family of reflexive platonic maps $\mathbf{Kul}(n)$ of genus $4n-1$ and type $(4, 8n)$. Standard complex conjugation is a reflection in the real wall of each map.* □

We have not computed a standard map presentation. The article [Tur1997] explicit expressions for the automorphisms.

5.5 The Fermat maps Fer(n)

In Section 2.9.1 we described the family of platonic maps $\mathbf{Fer}(n)$ of type $(3, 2n)$ and its family of diagonal maps $D_2(\mathbf{Fer}(n))$ of type $(n, 2n)$. This paragraph justifies the names of these families by demonstrating the following proposition.

Proposition 5.5.1. *The algebraic curves $x^n + y^n + z^n = 0$ are platonic surfaces. They support the polynomial family of reflexive platonic maps $\mathbf{Fer}(n)$ of genus $\binom{n-1}{2}$ and type $(3, 2n)$. A standard map presentation of $\mathbf{Fer}(n)$ on this model is defined by*

$$R = ab : (x : y : z) \mapsto (\zeta_n z : x : y),$$
$$S = bc : (x : y : z) \mapsto (y : \zeta_n^{-1} x : z).$$

The complex conjugation $c : (x : y : z) \mapsto (\zeta_n \overline{x} : \overline{y} : \overline{z})$ defines a reflection of the map.

Proof. Let us temporarily use $F(n)$ to denote the n-th Fermat curve. The group $\mathrm{Aut}(F(n))$ consists entirely of the following projectivities (see [Tze1995]). There is the normal subgroup of diagonal elements of the form

$$\mathrm{DiaMat}(\zeta_n^i, \zeta_n^j, \zeta_n^k).$$

This subgroup is naturally isomorphic to $\mathbb{Z}_n^3/\mathbb{Z}_n = \mathbb{Z}_n^2$. It has a complementary subgroup consisting of the coordinate permutations, which we will denote by permutations. As a consequence, $\mathrm{Aut}(F(n)) \cong \mathbb{Z}_n^2 \rtimes \mathrm{Sym}_3 \cong \mathrm{Aut}^+(\mathbf{Fer}(n))$. We remind the reader that every member of $\mathbf{Fer}(n)$ has a D_2-map (see Chapter 2). Each of the three transpositions $\sigma \in \mathrm{Sym}_3$ gives rise to one of the three conjugate subgroup $\mathbb{Z}_n^2 \rtimes \langle\sigma\rangle$ that will be the groups $\mathrm{Aut}^+(D_2(\mathbf{Fer}(n)))$ for the three conjugate realizations of the map $D_2(\mathbf{Fer}(n))$ on the planar algebraic curve $F(n)$. We pick one and set

$\mathrm{Aut}^+(D_2(\mathbf{Fer}(n))) = \mathbb{Z}_n^2 \rtimes \langle(23)\rangle$. To construct an $\mathrm{Aut}(\mathbf{Fer}(n))$-invariant Belyǐ function β on $F(n)$ we work our way up. First we exhibit an $\mathrm{Aut}^+(D_2(\mathbf{Fer}(n)))$-invariant Belyǐ function β_D. We then average β_D to construct β. The points for which (at least) one coordinate is zero will be important for us. The only such points on the curve are easily seen to be the $3n$ points

$$(0 : \zeta_{2n}^{2k-1} : 1) \qquad (\zeta_{2n}^{2k-1} : 0 : 1) \qquad (\zeta_{2n}^{2k-1} : 1 : 0) \qquad (\text{for } k = 1, \dots, n).$$

Without further ado, we present:

$$\beta_D : (x : y : z) \mapsto (x^{2n} : 4y^n z^n).$$

What about well-definedness? The expressions are certainly homogeneous of the same degree. Suppose that $x^{2n} = 0$. Then $x = 0$, so $(y : z) = (\zeta_{2n}^{2k-1} : 1)$ and $4y^n z^n = -4 \neq 0$. Therefore, β_D is indeed a well-defined function $\mathbf{Fer}(n)_a \to \mathbb{P}^1$. It is also clearly $\mathrm{Aut}^+(D(\mathbf{Fer}(n)))$-invariant. Now restrict attention to the affine subspace $\{x = 1\}$. Then we can compute

$$\beta_D(1 : y : z) = (1 : 4y^n z^n) = (1 : 4y^n(-1 - y^n)) = (1 : -4y^{2n} - 4y^n).$$

This shows that generically, a point of $\mathbb{P}^1$ has $2n^2$ inverse images under β_D: there are $2n$ possible values for y, and for each an additional n values for z. Hence, $\deg(\beta_D) = 2n^2$. The critical points on the curve that lie in $\{x = 1\}$ are those for which

$$-4ny^{n-1}(2y^n + 1) = \frac{d}{dy}\beta_D(1 : y : z(y)) = 0.$$

The equation implies that either $y = 0$ or $y^n = -\frac{1}{2}$. The first option yields $(1 : 0)$ as critical value, the second $(1 : 4 \cdot -\frac{1}{2}(\frac{1}{2} - 1)) = (1 : 1)$. So the critical values on the whole of $\mathbb{P}^1$ are at most these two and $(0 : 1)$. In fact, we have

$$\begin{aligned}
(\beta_D)^{-1}\big((0:1)\big) &= \big\{(0 : \zeta_{2n}^{2k-1} : 1) \mid k = 1, \dots, n\big\} \\
(\beta_D)^{-1}\big((1:0)\big) &= \big\{(\zeta_{2n}^{2k-1} : 0 : 1) \mid k = 1, \dots, n\big\} \cup \big\{(\zeta_{2n}^{2k-1} : 1 : 0) \mid k = 1, \dots, n\big\} \\
(\beta_D)^{-1}\big((1:1)\big) &= \big\{(2^{1/n} : \zeta_{2n}^{2k-1} : \zeta_{2n}^{2l-1}) \mid k, l = 1, \dots, n\big\}
\end{aligned}$$

of size n, $2n$ and n^2 respectively. So they are indeed critical values. This proves that β_D is an $\mathrm{Aut}^+(D_2(\mathbf{Fer}(n)))$-invariant Belyǐ function. We can use $(\beta_D)^{-1}((0:1))$ as vertex set, $(\beta_D)^{-1}((1:0))$ as the set of face centers and $(\beta_D)^{-1}((1:1))$ as the edge centers. The graph embedded on $F(n)$ is the inverse image $(\beta_D)^{-1}\big(\{(t : 1) : t \in [0, 1]\}\big)$.

To obtain an $\mathrm{Aut}^+(\mathbf{Fer}(n))$-invariant Belyǐ function we average β_D, just like we did earlier for the Wiman type II maps in Section 5.2. Addition on $\mathbb{P}^1$ is not defined, but it is on $\{y = 1\} = \mathbb{C}$. The resulting formula can be checked for correctness afterwards. The first attempt

$$\sum_{\sigma \in \langle(123)\rangle} \beta \circ \sigma$$

is a constant function. Using $(\beta_D)^2$ is the next best thing, and we set

$$\beta := \frac{1}{3} \sum_{\sigma \in \langle (123) \rangle} (4\beta_D)^2 \circ \sigma \quad : \quad (x : y : z) \mapsto (x^{6n} + y^{6n} + z^{6n} : 3x^{2n}y^{2n}z^{2n}).$$

If $3x^{2n}y^{2n}z^{2n} = 0$, then at least one coordinate is zero and hence $x^{6n}+y^{6n}+z^{6n} = 0$ as well. So β is well-defined. It is also clear that β is $\mathrm{Aut}^+(\mathbf{Fer}(n))$-invariant, and that $\deg(\beta) = \max\{6n, 2n\} \cdot n = 6n^2$. For the latter, given a generic point $(\lambda : \mu) \in \mathbb{P}^1$, set $z = 1$, solve $\beta((x : y : z)) = (\lambda : \mu)$ for y and then compute x using $x^n + y^n + z^n = 0$. The computation of the critical points and critical values of β is somewhat more involved. To make the formulas less cumbersome, we introduce $a := x^n$, $b := y^n$, $c := z^n$. Our algebraic curve is given by $a + b + c = 0$ and our image point is $(a^6 + b^6 + c^6 : 3a^2b^2c^2)$. For what $(\lambda : \mu) \in \mathbb{P}^1$ are there fewer than $6n^2$ pre-images? This certainly happens for $(1 : 0)$, which yields the $3n$ points for which one of the coordinates is zero. To find the other critical values, assume $\mu = 1$. The question is then when

$$a^6 + b^6 + c^6 - 3\lambda a^2b^2c^2$$

has fewer than 6 solutions. We know that $c \neq 0$, since points with $c = z^n = 0$ are in the pre-image of $(1 : 0)$. So we assume $z = 1$ and hence $c = 1$. We can now rewrite the above polynomial modulo $\langle a + b + c, c - 1\rangle$ as

$$f_\lambda(a) = 2a^6 + 6a^5 + (15 - 3\lambda)a^4 + (20 - 6\lambda)a^3 + (15 - 3\lambda)a^2 + 6a + 2$$

and the question is for which λ it has multiple roots. We are thus lead to compute

$$f'_\lambda(a) = 12a^5 + 30a^4 + 4(15 - 3\lambda)a^3 + 3(20 - 6\lambda)a^2 + 2(15 - 3\lambda)a + 6.$$

The ideal $\langle f_\lambda(a), f'_\lambda(a)\rangle$ of $\mathbb{C}[a, \lambda]$ contains

$$\lambda^2 - \frac{13}{2}\lambda + \frac{11}{2} = (\lambda - 1)\left(\lambda - \frac{11}{2}\right),$$

so there are only two more critical values, $(1 : 1)$ and $(11 : 2)$. This proves we have a Belyĭ function. The first solution for λ leads us to $f(a) = (a^2 + a + 1)^3$. The only solutions for $(a : b : c)$ are then $(\zeta_3 : \zeta_3^2 : 1)$ and $(\zeta_3^2 : \zeta_3 : 1)$. This results in $2n^2$ pre-images. At each of these the ramification index is 3, as we can read off from the multiplicity of the factor in $f(a)$. The second solution for λ brings us to $f(a) = (a - 1)^2(a + 2)^2(a + \frac{1}{2})^2$. We find the solutions $(1 : -2 : 1)$, $(-2 : 1 : 1)$ and $(1 : 1 : -2)$ for $(a : b : c)$, which give rise to $3n^2$ pre-images. At each of these, the ramification index is 2. Summarizing, we have constructed the platonic map $\mathbf{Fer}(n)$ on $F(n)$ with vertices, edge centers and face centers the following sets:

Vertices	$\beta^{-1}\big((1 : 0)\big) = \big\{(0 : \zeta_{2n}^{2k-1} : 1) \mid k = 1, \ldots, n\big\}^{\mathrm{Sym}_3}$
Edge centers	$\beta^{-1}\big((11 : 2)\big) = \big\{(2^{1/n} : \zeta_{2n}^{2k-1} : \zeta_{2n}^{2l-1}) \mid k, l = 1, \ldots, n\big\}^{\mathrm{Sym}_3}$
Face centers	$\beta^{-1}\big((1 : 1)\big) = \big\{(\zeta_{3n}^{3k\pm1} : \zeta_{3n}^{3l\mp1} : 1) \mid k, l = 1, \ldots, n\big\}$

Because both $F(n)$ and $\mathbb{P}^1$ are invariant under standard complex conjugation, our two Belyĭ functions satisfy

$$\beta^{(D)}(\overline{z}) = \overline{\beta^{(D)}(z)}.$$

Hence, they are $\mathrm{Aut}(\mathbf{Fer}(n))$-invariant and both platonic maps must be reflexive. The transformation $c : (x : y : z) \mapsto (\zeta_n \overline{x} : \overline{y} : \overline{z})$ is an involution in $\mathrm{Aut}^-(\mathbf{Fer}(n))$ and fixes points, so it is a reflection. We determine the exact cell structure of $\mathbf{Fer}(n)$ on $F(n)$. The invariant set of the reflection c from the proposition is

$$\big\{(t_1\zeta_{2n} : t_2 : t_3) \mid (t_1 : t_2 : t_3) \in \mathbb{P}^2(\mathbb{R})\big\} \cap \mathbf{Fer}(n),$$

and this is the solution set of the real algebraic equation

$$t_2^n + t_3^n = t_1^n.$$

It turns out that the solution set is connected, so it is in fact only one geodesic wall of $\mathbf{Fer}(n)$. Since it clearly contains none of the face centers, it is a wall that consists entirely of edges. For even n, the wall has length 4 and the vertices on it are $(\pm\zeta_{2n} : 0 : 1)$ and $(\pm\zeta_{2n} : 1 : 0)$. For odd n, the wall has length 3, and the vertices on it are $(0 : -1 : 1)$, $(\zeta_{2n} : 1 : 0)$ and $(\zeta_{2n} : 0 : 1)$. In both cases, there is an edge between $(\zeta_{2n} : 0 : 1)$ and $(\zeta_{2n} : 1 : 0)$. It is definable as

$$\big\{(\zeta_{2n} : y : z) \mid y^n + z^n = 1,\ y, z \in [0,1]\big\}.$$

All this is illustrated in Figure 5.1. There is a unique involution that fixes the wall

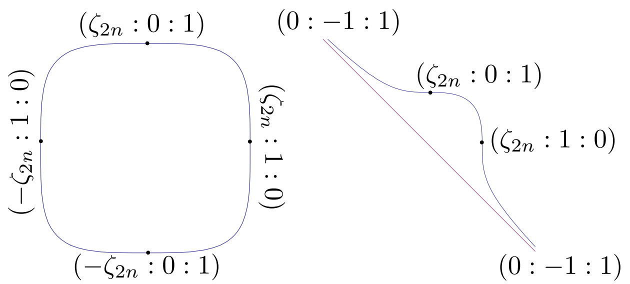


Figure 5.1: The geodesic walls of the map $\mathbf{Fer}(n)$, for n even (left) and n odd, with a point at infinity in the chart $\{x = \zeta_{2n}\}$ indicated by the asymptote (right).

$\mathrm{Fix}(c)$ and switches $(\zeta_{2n} : 1 : 0)$ and $(\zeta_{2n} : 0 : 1)$. We can freely choose this to be the reflection a for a standard map presentation in terms of reflections (a, b, c):

$$a : (x : y : z) \mapsto (\zeta_n \overline{x} : \overline{z} : \overline{y}).$$

Next, we derive that a point $(\zeta_{2n} : y : z) \in \mathrm{Fix}(a)$ satisfies $y = \overline{z}$ and since it has to lie on $\mathbf{Fer}(n)$ we have $\overline{z}^n + z^n = 1$. This is equivalent to $\mathrm{Re}(z^n) = \frac{1}{2}$ and we find that the invariant set consists of n geodesic walls (numbered with variable k) parametrized as

$$\left\{\left(\zeta_{2n} : \sqrt[2n]{\tfrac{1}{4}+t^2}\, e^{-\arctan(2t)i/n}\zeta_n^{-k} : \sqrt[2n]{\tfrac{1}{4}+t^2}\, e^{\arctan(2t)i/n}\zeta_n^{k}\right) \;\middle|\; t \in \mathbb{R}\right\}.$$

Our wall has the edge center $\mathrm{Fix}(a) \cap \mathrm{Fix}(c) = \{(\zeta_{2n} : 2^{-1/n} : 2^{-1/n})\}$ at value $t = 0$, which forces the choice $k = 0$. The face centers that this wall contains are $(\zeta_{2n} : \zeta_{6n}^{-1} : \zeta_{6n}) = (1 : \zeta_{3n}^{-2} : \zeta_{3n}^{-1})$, reached at $t = \frac{1}{2}\sqrt{3}$, and $(\zeta_{2n} : \zeta_{6n} : \zeta_{6n}^{-1}) = (1 : \zeta_{3n}^{-1} : \zeta_{3n}^{-2})$, reached at $t = -\frac{1}{2}\sqrt{3}$. Let's pick the first one for our fundamental triangle; this fixes an orientation on $x^n + y^n + z^n = 0$. The final result is that

$$a : (x : y : z) \mapsto (\zeta_n \overline{x} : \overline{z} : \overline{y})$$
$$b : (x : y : z) \mapsto (\zeta_n \overline{y} : \zeta_n \overline{x} : \overline{z})$$
$$c : (x : y : z) \mapsto (\zeta_n \overline{x} : \overline{y} : \overline{z})$$

represent three standard reflections in a fundamental triangle of $\mathbf{Fer}(n)$. Simple checks yield that $a^2 = b^2 = c^2 = 1$, $\mathrm{ord}(ab) = 3$, $\mathrm{ord}(bc) = 2n$ and $\mathrm{ord}(ac) = 2$. The formules for the rotations $R = ab$ and $S = bc$ are easy to compute. □

When n is odd, standard complex conjugation $\mathrm{con} = S^{-2} \circ c = cbcbc$ is also a reflection, this time in the real geodesic wall. This gives an even simpler standard map presentation of $\mathrm{Aut}(\mathbf{Fer}(2n+1))$ than the one given in the proof. The vertices on the real wall are then $(0 : -1 : 1)$, $(-1 : 0 : 1)$, and $(-1 : 1 : 0)$. But this is a less uniform description, for $\mathbf{Fer}(2n)$ map has no real points. Complex conjugation is then a fixed point free involution. Incidentally, we can compute the number of such fixed point free involutions. This is done by calculating the centralizer $C_{\mathrm{Aut}(\mathbf{Fer}(n))}(\mathrm{con})$. Obviously permutations of the coordinates commute with con. The diagonal automorphisms $(x : y : z) \mapsto (\zeta_n^{e_1} x : \zeta_n^{e_2} y : \zeta_n^{e_3} z)$ that commute with con are those for which $\zeta_n^{e_i} : \zeta_n^{e_j} = \zeta_n^{-e_i} : \zeta_n^{-e_j}$ for all choices of (i,j). This is equivalent to $e_i + e_j \in \{0, n/2\}$ for all these pairs, and we find (up to a scalar) the diagonal automorphisms with factors $(1 : \pm 1 : \pm 1)$. Adding con itself as a last generator of this centralizer, we conclude

$$|C_{\mathrm{Aut}(\mathbf{R})}(\mathrm{con})| = 6 \cdot 4 \cdot 2 = 48$$

from which we conclude that the size of the conjugacy class of con is $n^2/4$.

Remark 5.5.2. In the foregoing description of the Belyĭ function β we encounter the hyperplane $\{a+b+c=0\} \subset \mathbb{P}^2$ with 3 vertices, 3 edge centers and 2 face centers on it. This hyperplane is itself a $\mathbb{P}^1$, and the Belyĭ function $(a : b : c) \mapsto (a^6 + b^6 + c^6 : 3a^2b^2c^2)$ yields the platonic map $\mathbf{Dih}(3)$ on it. This shows directly that there is the platonic cover $(x : y : z) \mapsto (x^n : y^n : z^n)$ of $\mathbf{Fer}(n) \to \mathbf{Dih}(3)$, for each $n \in \mathbb{N}$. This is an instance of Proposition 3.2.7.

Remark 5.5.3. In Chapter 7 we will see the notion of a Weierstraß point. The points $\beta^{-1}((1 : 0))$ and $\beta^{-1}((11 : 2))$ are Weierstraß points of $\mathbf{Fer}(n)$. Members of the first set are sometimes called *trivial Weierstraß points*, the others the *Leopoldt points*. For uniformity, we simply refer to them as the vertices and edge centers of the (platonic map described by β on the) Fermat curve.

Problem 5.5.4. A fascinating problem which we consider to be in the intersection of mathematics and art is: can we comprehensibly visualize the Fermat maps? The maps $\mathbf{Fer}(1) = \mathbf{Dih}(3)$, $\mathbf{Fer}(2) = \mathbf{Oct}$ and $\mathbf{Fer}(3) = \mathbf{R}_{1.2:3}$ are not so difficult. The first non-trivial case is $\mathbf{Fer}(4)$, which is Dyck's platonic map $\mathbf{R}_{3.2}$. A linear realization of this map was made in 1987 by Ulrich Brehm [Bre1987] and is shown in Figure 5.2. Can we give an elegant (linear) realization for higher n?

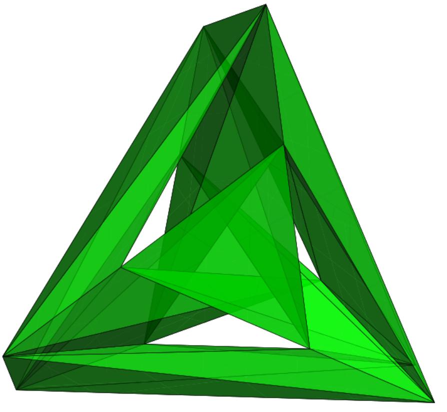


Figure 5.2: Brehm's linear realization in $\mathbb{R}^3$ of the map $\mathbf{Fer}(4)$.

5.6 The Humbert maps $\mathbf{Hum}_k(n)$

In [CGAHR2008], a family of algebraic curves is defined by a property of their automorphism group.

Definition 5.6.1. *Let $n \in \mathbb{Z}_{\geq 3}$ and let $\mathcal{M}_{g(n)}$ be the moduli space of curves of genus $g(n) = 1 + 2^{n-2}(n-3)$. The set of* generalized Humbert curves *of size n is the locus in $\mathcal{M}_{g(n)}$ consisting of those curves X for which $\mathbb{Z}_2^n$ is a subgroup of $\mathrm{Aut}(X)$.*

The authors show that the generalized Humbert curves of size n are parametrized by $\lambda_1, \ldots, \lambda_{n-2} \in \mathbb{C}$, and thus we can denote such a curve as $\mathbf{Hum}(\lambda_1, \ldots, \lambda_{n-2})$. Furthermore, for each n the locus contains at least three members uniformized by a normal subgroup of a triangle group. These special curves form three polynomial families of reflexive platonic maps $\mathbf{Hum}_1(n)$, $\mathbf{Hum}_2(n)$, and $\mathbf{Hum}_3(n)$, that we term *Humbert maps*. They are defined by their map automorphism groups as follows:

$$\mathrm{Aut}^+(\mathbf{Hum}_1(n)) = \left\langle R, S \,\middle|\, R^4, S^{n+1}, (RS)^2, (S^j R)^4 \;:\; 3 \leq j \leq 2^{n+2}(n+1) \right\rangle,$$
$$\mathrm{Aut}^+(\mathbf{Hum}_2(n)) = \left\langle R, S \,\middle|\, R^k, S^{2n}, (RS)^2, (S^{n+1}R)^2, (S^j R^j)^2 \;:\; 1 \leq j \leq 2^n(n+1) \right\rangle,$$
$$\mathrm{Aut}^+(\mathbf{Hum}_3(n)) = \left\langle R, S \,\middle|\, R^4, S^{2(n-1)}, (RS)^2, (S^{n-1}R^2)^2, (S^j R)^4 \;:\; 3 \leq j \leq 2^{n+1}(n-1)+1 \right\rangle.$$

Reflexivity of the maps follows from the fact that for any relator of the form $(S^a R^b)^c$ we have $(S^{-a}R^{-b})^c = (R^b(S^a R^b)^{-1}R^{-b})^c = R^b(S^a R^b)^{-c}R^{-b} = 1$, using the chirality criterion. The isomorphism types of these map automorphism groups are:

$$\mathrm{Aut}^+(\mathbf{Hum}_1(n)) \cong \mathbb{Z}_2^n \rtimes \mathrm{Dih}_{2(n+1)},$$
$$\mathrm{Aut}^+(\mathbf{Hum}_2(n)) \cong \mathbb{Z}_2^n \rtimes \mathbb{Z}_n,$$
$$\mathrm{Aut}^+(\mathbf{Hum}_3(n)) \cong \mathbb{Z}_2^n \rtimes \mathrm{Dih}_{2(n-1)}.$$

Every generalized Humbert curve $\mathbf{Hum}(\lambda_1, \ldots, \lambda_{n-2})$ has an elegant model in $\mathbb{P}^n$, defined by the following ideal in $\mathbb{C}[x_1, \ldots, x_{n+1}]$:

$$I(\lambda_1, \ldots, \lambda_{n-2}) = \langle \lambda_i x_1^2 + x_2^2 + x_{i+3}^2 \ : \ 0 \le i \le n-2, \ \lambda_0 := 1 \rangle.$$

5.7 The modular maps $\mathbf{Mod}(n)$

Another intriguing family of reflexive platonic maps is the family $\mathbf{Mod}(n)$ of *modular maps*. We will shortly see that the (reflexive) platonic surfaces of these maps are the modular curves $X(n)$. The modular map family has the property that

$$\mathrm{Aut}^+(\mathbf{Mod}(n)) \cong \mathrm{SL}(2, \mathbb{Z}/n\mathbb{Z})/\langle -1 \rangle \qquad (n \ge 1)$$

We refrain from calling this group $\mathrm{PSL}(2, \mathbb{Z}/n\mathbb{Z})$, since we do not necessarily factor out the whole center of $\mathrm{SL}(2, \mathbb{Z}/n\mathbb{Z})$.

Proposition 5.7.1. *Denote the equivalence class mod $\langle -1 \rangle$ of a matrix in $\mathrm{SL}(2, \mathbb{Z}/n\mathbb{Z})$ by writing the matrix with square brackets. Let*

$$R = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

These two elements generate a map presentation of type $(3, n)$ of $\mathrm{SL}(2, \mathbb{Z}/n\mathbb{Z})/\langle -1 \rangle$.

Proof. The computations proving that $\mathrm{ord}(R) = 3$, $\mathrm{ord}(S) = n$ and $\mathrm{ord}(RS) = 2$ are left to the reader. It is a well-known fact that the displayed matrices generate $\mathrm{SL}(2, \mathbb{Z})$, and hence the quotient under discussion. □

Knowing that this group with the given generators defines a platonic map of type $(3, n)$, we would like to see a map presentation. Presentations of special linear groups are not trivial to come by (and prove correct). We will use the work of Mennicke. By exploiting the Steinberg relations, he constructed presentations $\mathrm{SL}(2, \mathbb{Z}/p^n\mathbb{Z})$ (with p prime) in [Men1967], building on work of Todd and Coxeter [CM1980]. In the subsequent paper [BM1968] Mennicke together with Behr improved upon this work and gave a short presentation for all odd n. The even case requires (at least for now) a more complicated presentation, apparently stemming from the fact that $(\mathbb{Z}/2^k\mathbb{Z})^\times = \langle -1, 5 \rangle$ is not cyclic for $n \ge 3$. For odd n, the Mennicke presentation is

$$\mathrm{SL}(2, \mathbb{Z}/n\mathbb{Z})/\langle -1 \rangle = \left\langle R, S \,\middle|\, R^3, S^n, (RS)^2, (S^2 R S^{\frac{1}{2}(n+3)} RS)^3 \right\rangle.$$

For $n = 2^k$ we proceed as follows. For $k = 1$ we have $-1 = 1$ and find

$$\mathrm{SL}(2, \mathbb{Z}/2\mathbb{Z})/\langle -1 \rangle = \mathrm{SL}(2, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Sym}_3.$$

Its standard map presentation is of type $(3, 2)$, so $\mathbf{Mod}(1) = \mathbf{Dih}(3)$. For $k = 2$ we get $\mathrm{SL}(2, \mathbb{Z}/4\mathbb{Z})/\langle -1 \rangle \cong \mathrm{Sym}_4$ with standard map presentation of type $(3, 4)$, whence

$\mathbf{Mod}(2) = \mathbf{Oct}$. Now suppose that $k \geq 3$. Write $E_{12} = S$ and $E_{21} = R^2S$ for the two unit transvection matrices. We introduce the following element of order 2^{k-2} generating the diagonal subgroup:

$$T = \begin{bmatrix} 5 & 0 \\ 0 & 5^{-1} \end{bmatrix} \in \mathrm{SL}(2, \mathbb{Z}/n\mathbb{Z})/\langle -1 \rangle.$$

Direct computation shows that for any $k \in (\mathbb{Z}/n\mathbb{Z})^\times$:

$$\begin{bmatrix} k & 0 \\ 0 & n^{-1} \end{bmatrix} = E_{12}^{k^2-k} E_{21}^{k^{-1}} E_{12}^{-(k-1)} E_{21}^{-1} = S^{k^2-k}(R^2S)^{k^{-1}} S^{-k} R^{-2}.$$

Specializing to the case $k = 5$, we find $T = S^{20}(R^2S)^{5^{-1}} S^{-5} R^{-2}$. Using T as an abbreviation, the Mennicke presentation for $\mathrm{SL}(2, \mathbb{Z}/2^k\mathbb{Z})/\langle -1 \rangle$ is then given by

$$\mathrm{SL}(2, \mathbb{Z}/2^k\mathbb{Z})/\langle -1 \rangle = \left\langle R, S \,\middle|\, R^3, S^{2^k}, (RS)^2, (TRS)^2, T^{-1}R^2ST(R^2S)^{-25}, (T(R^2S)^5RS)^3 \right\rangle.$$

The previous two presentations suffice to assemble one for $n = 2^k m$ ($k \geq 1$, m odd). The Chinese Remainder Theorem transfers to the groups under consideration:

$$\mathrm{SL}(2, \mathbb{Z}/n\mathbb{Z})/\langle -1 \rangle \cong \mathrm{SL}(2, \mathbb{Z}/2^k\mathbb{Z})/\langle -1 \rangle \times \mathrm{SL}(2, \mathbb{Z}/m\mathbb{Z})/\langle -1 \rangle.$$

To make this isomorphism concrete, define $e_1 \in \mathbb{Z}/n\mathbb{Z}$ by $e_1 \equiv 1 \bmod 2^k$, $e_1 \equiv 0 \bmod m$, and likewise $e_2 \in \mathbb{Z}/n\mathbb{Z}$ by $e_2 \equiv 1 \bmod m$, $e_2 \equiv 0 \bmod 2^k$. Then the two pairs $(R_i, S_i) := (S^{e_i}RS^{e_i}R^{-1}, S^{e_i})$, where $i = 1, 2$, generate the two direct factors. We use these as abbreviations, as well as $T = S_1^{20}(R_1^2S_1)^{-5^{-1}} S_1^{-5} R_1^{-2}$. The map presentation that results is:

$$\begin{aligned}\mathrm{SL}(2, \mathbb{Z}/2^k m\mathbb{Z}) = \langle R, S \,| R^3, S^n, (RS)^2, S_2^2 R_2 S_2^{\frac{1}{2}(k+3)} R_2 S_2, \\ (TR_1S_1)^2, T^{-1}R_1^2S_1T(R_1^2S_1)^{-25}, (T(R_1^2S_1)^5R_1S_1)^3 \rangle.\end{aligned}$$

For computational purposes one might prefer a permutation representation. The group order we find is

$$|\mathrm{Aut}^+(\mathbf{Mod}(n))| = \frac{1}{2}|\mathrm{SL}(2, \mathbb{Z}/n\mathbb{Z})| = \frac{n^3}{2}\prod_{p|n}(1 - p^{-2}) \qquad (n \geq 3),$$

and hence

$$g(\mathbf{Mod}(n)) = 1 + \frac{n^4 - 6n^3}{24n}\prod_{p|n}(1 - p^{-2}) \qquad (n \geq 3).$$

If $n = p$ is a prime, this expression reduces to

$$g(\mathbf{Mod}(p)) = \frac{1}{24}(p + 2)(p - 3)(p - 5).$$

The platonic surface of the map $\mathbf{Mod}(n)$ is in fact the modular curve $X(n)$. Reduction mod n induces a natural map $\pi_N : \mathrm{SL}(2, \mathbb{Z})/\langle -1 \rangle \to \mathrm{SL}(2, \mathbb{Z}/n\mathbb{Z})/\langle -1 \rangle$. Define the congruence subgroup $\Gamma(N) := \ker \pi_n$. The natural action of $\Gamma(1) =$

$SL(2,\mathbb{Z})/\langle -1\rangle$ on the extended upper half-plane $\mathbb{H}^2 \cup \{\infty\}$ by Möbius transformations has fundamental domain as illustrated in Figure 5.3. The quotient $\mathbb{H}^2/\Gamma(1)$ is isomorphic to $\mathbb{P}^1$. Interpreted as Möbius transformations, we find $R : z \mapsto (1-z)^{-1}$ to be a rotation around ζ_6 over $2\pi/3$, $S : z \mapsto z+1$ to be translation by 1, and $RS : z \mapsto -z^{-1}$ to be a rotation around i over π. Thus, the transformations R and RS exhibit the same local behaviour on $\mathbb{H}^2/\Gamma(n)$ and its compactification $X(n)$. The transformation S becomes a rotation around $[\infty]$ of order n. The natural quotient map $\mathbb{H}^2/\Gamma(n) \to \mathbb{H}^2/\Gamma(1) \cong \mathbb{P}^1$ has i, ζ_6 and ∞ as its only branch points. It is therefore a Belyĭ function, and $X(n)$ has a platonic map structure induced by (R,S) with the given presentation. This knowledge also enables us to see that these maps are

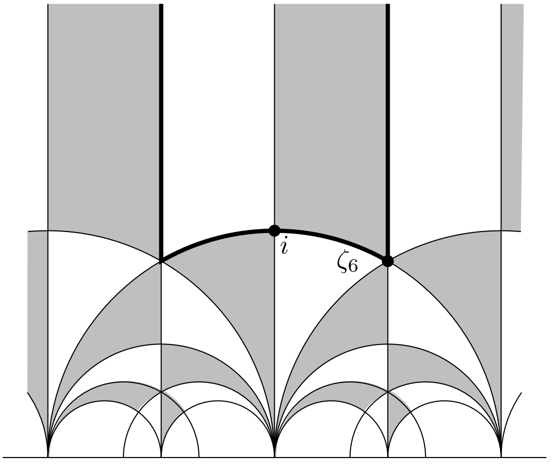


Figure 5.3: The (extended) upper half-plane with a fundamental domain of $\mathrm{SL}(2,\mathbb{Z})/\langle -1\rangle$, (demarcated by a thicker line). The domain folds up into $\mathbb{P}^1$ covered by one grey and one white triangle under the group action.

reflexive, a fact not so easy to glean from the Mennicke presentations. The antiholomorphic transformation of the upper half-plane $c : z \mapsto -\overline{z}$ satisfies

$$c(R(z)) = -(1-\overline{z})^{-1} = (1-(-\overline{z}+2))^{-1} = R(S^2(c(z)))$$
$$c(S(z)) = -\overline{z} - 1 = S^{-1}(c(z))$$

So c indeed induces an antiholomorphic automorphism on $Y(n) = \mathbb{H}^2/\langle R,S\rangle$ (and thus on $X(n)$) that functions precisely like the c from a standard reflexive map presentation. In general, we have no better description of $\mathrm{Aut}(\mathbf{Mod}(n))$ than as a semidirect product $\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z}) \rtimes \mathbb{Z}_2$. When $n = p$ is a prime, then $\mathrm{SL}(2,p)/\langle -1\rangle = \mathrm{PSL}(2,p)$, and because $c \in \mathrm{SL}^-(2,\mathbb{Z})$ we find $\mathrm{Aut}(\mathbf{Mod}(n)) \cong \mathrm{PGL}(2,p)$.

The modular maps $\mathbf{Mod}(n)$ with genus at most 101 are listed in Table 5.1.

Remark 5.7.2. When n is even, the subgroup $\langle S^2, R^{-1}SR\rangle < \mathrm{Aut}(X(n))$ seems to be of index 3, so then $\mathbf{Mod}(n)$ would have a D_2-map (see Chapter 3). But this remains to be proven.

n	$g(n)$	$\lvert\mathrm{Aut}(\mathbf{Mod}(n))\rvert$	Reflexive platonic map in Conder's list
2	0	12	$\mathbf{Dih}(3)$
3	0	24	**Tet**
4	0	48	**Oct**
5	0	120	**Ico**
6	1	144	$\mathbf{R}_{1.1:2}$
7	3	336	$\mathbf{R}_{3.1}$ (Klein quartic)
8	5	384	$\mathbf{R}_{5.1}$ (Wiman-del Centina curve)
9	10	648	$\mathbf{R}_{10.1}$
10	13	720	$\mathbf{R}_{13.1}$
11	26	1320	$\mathbf{R}_{26.2}$ (Klein)
12	25	1152	$\mathbf{R}_{25.3}$
13	50	2184	$\mathbf{R}_{50.1}$
14	49	2016	$\mathbf{R}_{49.3}$
15	73	2880	$\mathbf{R}_{73.5}$
16	81	3072	$\mathbf{R}_{81.1}$

Table 5.1: Modular maps $\mathbf{Mod}(n)$ of genus at most 101.

Example 5.7.3. Historically, the modular maps were the first examples of platonic maps. Indeed, by studying the congruence subgroups $\Gamma(5)$, $\Gamma(7)$, and $\Gamma(11)$, Felix Klein constructed first the algebraic curve $X(7)$ in [Kle1878], now known as the Klein quartic, and then $X(11)$ in [Kle1879]. For completeness, we briefly discuss this latter curve. It has automorphism group $\mathrm{Aut}(\mathbf{Mod}(11)) = \mathrm{PGL}(2, 11)$. Klein constructed a beautiful non-singular model in $\mathbb{P}^4$. First consider the algebraic variety defined by the polynomial

$$f = x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_4 + x_4^2 x_5 + x_5^2 x_1.$$

The polynomial has Hessian matrix

$$\mathrm{Hess}(f) = 2\begin{pmatrix} x_2 & x_1 & 0 & 0 & x_5 \\ x_1 & x_3 & x_2 & 0 & 0 \\ 0 & x_2 & x_4 & x_3 & 0 \\ 0 & 0 & x_3 & x_5 & x_4 \\ x_5 & 0 & 0 & x_4 & x_1 \end{pmatrix}$$

The set of all (4×4)-minors of the Hessian defines the algebraic curve corresponding to $\mathbf{Mod}(11)$:

$$X(11) \cong \left\{(x_1 : x_2 : x_3 : x_4 : x_5) \in \mathbb{P}^4 \mid \mathrm{rank}(\mathrm{Hess}(f)) \leq 3\right\}.$$

The automorphism group $\mathrm{Aut}^+(\mathbf{Mod}(11))$ is generated by the following three elegant transformations:

$$T_1 = \mathrm{PerMat}([2,3,4,5,1])$$
$$S = \mathrm{DiaMat}([\zeta_{11}, \zeta_{11}^9, \zeta_{11}^4, \zeta_{11}^3, \zeta_{11}^5])$$
$$T_2 = \frac{1}{\sqrt{-11}} \begin{pmatrix} \zeta_{11}^9 - \zeta_{11}^2 & \zeta_{11}^5 - \zeta_{11}^6 & \zeta_{11}^4 - \zeta_{11}^7 & \zeta_{11} - \zeta_{11}^{10} & \zeta_{11}^3 - \zeta_{11}^8 \\ \zeta_{11}^5 - \zeta_{11}^6 & \zeta_{11}^4 - \zeta_{11}^7 & \zeta_{11} - \zeta_{11}^{10} & \zeta_{11}^3 - \zeta_{11}^8 & \zeta_{11}^9 - \zeta_{11}^2 \\ \zeta_{11}^4 - \zeta_{11}^7 & \zeta_{11} - \zeta_{11}^{10} & \zeta_{11}^3 - \zeta_{11}^8 & \zeta_{11}^9 - \zeta_{11}^2 & \zeta_{11}^5 - \zeta_{11}^6 \\ \zeta_{11} - \zeta_{11}^{10} & \zeta_{11}^3 - \zeta_{11}^8 & \zeta_{11}^9 - \zeta_{11}^2 & \zeta_{11}^5 - \zeta_{11}^6 & \zeta_{11}^4 - \zeta_{11}^7 \\ \zeta_{11}^3 - \zeta_{11}^8 & \zeta_{11}^9 - \zeta_{11}^2 & \zeta_{11}^5 - \zeta_{11}^6 & \zeta_{11}^4 - \zeta_{11}^7 & \zeta_{11} - \zeta_{11}^{10} \end{pmatrix}$$

Though not immediately apparent, $\mathrm{ord}(T_2) = 2$.

Since the model is defined over $\mathbb{Q}$, the curve has the standard complex conjugation as antiholomorphic automorphism, completing the map automorphism group.

6

From platonic maps to algebraic curves

THIS chapter traces the construction of algebraic models for all reflexive platonic maps of genus at most 8 and some assorted platonic maps of genus between 9 and 15. Most of these models are canonical models, and otherwise they are planar curves. This extends the realm of platonic surfaces with known algebraic descriptions. Historically, this study started with Felix Klein's 1878 momentous discovery of the quartic $x^3y + y^3z + z^3x = 0$, documented in [Kle1878]. Somewhat later Anders Wiman constructed several other algebraic curves with many automorphisms, some of which re-appear in this book. Robert Fricke [Fri1899] studied the map $\mathbf{R}_{7.1}$, but this work was overlooked. In 1963, Alexander Murray Macbeath [Mac1965] brought it back to attention with his construction of a canonical model for what is termed here $\mathbf{R}_{7.1}$ or the Fricke-Macbeath curve. Since then, Riemann surfaces with non-trivial automorphisms have never been out of fashion. From the late 1970s on, a 'Japanese school' has constructed models for all Riemann surfaces with a non-trivial automorphism group of genus 2 and 3, and canonical representations for all group actions up to genus 5. See [KK1977], [KK1978], [KK1987], [KK1990], [Kim1991], [Kim2003].

We start with an introduction of some notions from algebraic geometry that we need in Section 6.1. Then we set down our strategy for construction of the models in Section 6.2. After returning from the detour of dealing with hyperelliptic maps and their models in Section 6.3, we deal with genus 0 and 1 in the last two sections of the chapter. The discussion of all higher genus models, to which our strategies apply, has been relegated to Appendix A. Among the new results are a model of $\mathbf{R}_{7.1}$ over $\mathbb{Q}$, models for $X(9)$ and $X(10)$, and models for the first Hurwitz triplet ($\mathbf{R}_{14.1}$, $\mathbf{R}_{14.2}$, $\mathbf{R}_{14.3}$).

6.1 Curves and canonical models

Algebraic geometry, the study of algebraic varieties, is a subtle subject, and we stick to the basics. Let $\mathbb{A}^n$ be the complex affine space $\mathbb{C}^n$, and $\mathbb{P}^n$ be the complex projective space of dimension n, i.e. $(\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*$ where the action is by scalar multiplication. Let $\mathrm{Sym}(V)$ denote the symmetric algebra of a vector space V. The symmetric algebra is naturally graded and we denote the degree d part by $\mathrm{Sym}^d(V)$. The space $\mathrm{Sym}((\mathbb{C}^n)^\vee)$ is isomorphic as a graded $\mathbb{C}$-algebra to the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. We also denote the degree d part of the latter by $\mathbb{C}[x_1, \ldots, x_n]_d$.

Definition 6.1.1. *An ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$ is* homogeneous *if for any $p \in I$ and $d \geq 0$, the homogeneous degree d part of p is also in I.*

A homogeneous ideal is the direct sum of its degree d parts: $I = \bigoplus_{d \geq 0} I_d$, where $I_d := I \cap \mathbb{C}[x_1, \ldots, x_n]_d$. This also means a homogeneous ideal is generated as an ideal by homogeneous polynomials.

Definition 6.1.2. *A* (complex) affine algebraic variety *$V \subset \mathbb{C}^n$ is the zero set of an ideal $I(V) \subset \mathbb{C}[x_1, \ldots, x_n]$. A* (complex) projective algebraic variety *$V \subset \mathbb{P}^n$ is the zero set of a homogeneous ideal $I(V) \subset \mathbb{C}[x_1, \ldots, x_{n+1}]$. An affine or projective variety V is* irreducible *if $I(V)$ is a prime ideal.*

We want to restrict to *irreducible* varieties, that is, varieties that are not the union of two closed proper subsets (in the Zariski topology). This property has an algebraic formulation: a variety V is irreducible if and only if $I(V)$ is prime. A notion that is intuitively clear but, as many notions in algebraic geometry, surprisingly hard to define, is that of dimension. A possible rigorous definition is as the degree of the Hilbert polynomial, described in Definition 6.1.16, but we leave out further details. We call an irreducible complex projective algebraic variety a *(complex projective algebraic) curve* if its dimension is 1, and a *(complex projective algebraic) surface* if its dimension is 2.

A function $\phi : X \to Y$ from one complex projective variety $X \subset \mathbb{P}^m$ to $Y \subset \mathbb{P}^n$ defined on a Zariski open subset of X by polynomials that do not vanish simultaneously for any point of X are called *regular mappings*. For proper treatment of this notion and the concomitant one of *birational mapping*, see e.g. [Har1992] or [Har1977].

Varieties important for our story

A few sets of varieties inevitably come into play in our story of algebraic models for platonic maps. Our next examples are projective spaces that we embed into higher-dimensional projective spaces.

Definition 6.1.3. *A* Veronese mapping *is an embedding $v_{n,d} : \mathbb{P}^n \rightarrowtail \mathbb{P}^N$ with $N = \binom{n+d}{d} - 1$ defined by*

$$(x_1 : \cdots : x_{n+1}) \mapsto (\cdots : x^\alpha : \cdots)$$

with $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ *ranging over the multi-indices* α *of degree* d *with* n *components. Its image is the* Veronese variety $V_{n,d}$. *A Veronese variety with* $n = 1$ *is called a* rational normal curve.

Every component of the mapping is a monomial in the $n+1$ variables $x_0, \ldots, x_n$. The Veronese embedding is obviously dependent on the ordering of the monomials x^α, but the images are all isomorphic by an isomorphism of $\mathbb{P}^N$ permuting its coordinates.

Example 6.1.4. The Veronese mapping $v_{2,2} : \mathbb{P}^2 \to \mathbb{P}^5$ is given by

$$v_{2,2} : (x_1 : x_2 : x_3) \mapsto (x_1^2 : x_1x_2 : x_1x_3 : x_2^2 : x_2x_3 : x_3^2).$$

Let us denote the standard homogeneous coordinates in $\mathbb{P}^5$ with $(y_1 : \cdots : y_6)$. The image $V_{2,2}$ of $v_{2,2}$, called the *Veronese surface*, then turns out to be defined by the following set of quadrics:

$$I(V_{2,2}) = (y_1y_4 - y_2^2, y_1y_6 - y_3^2, y_4y_6 - y_5^2, y_2y_3 - y_1y_5, y_2y_5 - y_3y_4, y_3y_5 - y_2y_6).$$

The second set of varieties is built up by connecting up available ones, cf. [Har1992, p. 92f.]. Given two algebraic varieties $X, Y \subset \mathbb{P}^n$ and a regular mapping $\phi : X \to Y$ such that $\forall x \in X, \phi(x) \neq x$, one can create a new algebraic variety $S(X, Y, \phi)$ as the union $\cup_{x \in X} \mathrm{line}(x, \phi(x))$. We are especially interested in algebraic surfaces formed in this way between two rational normal curves.

Definition 6.1.5. *Let* Λ_1, Λ_2 *be linear subspaces of* $\mathbb{P}^n$ *of dimension* $k, l \geq 0$ *that are disjoint and of complementary dimension* ($k + l = n - 1$)*. Suppose that* $X \subset \Lambda_1$ *and* $Y \subset \Lambda_2$ *are rational normal curves, or a point if the linear space has dimension* 0*. Assume that* $\phi : X \to Y$ *is an isomorphism. A* rational normal scroll *is the variety* $S_{k,l} := S(X, Y, \phi)$.

The rational normal scroll $S_{k,l}$ is uniquely defined up to projective isomorphism by the pair (k, l): as a first step, the linear subspaces Λ_1, Λ_2 can be mapped to any other pair Λ_1', Λ_2' of the same dimensions simultaneously with an automorphism of $\mathbb{P}^n$. As a second step, any two rational normal curves $C, C' \subset \Lambda_i$ are also isomorphic ($i = 1, 2$), and we can find an isomorphism of $\mathbb{P}^n$ accomplishing the transformation on both subspaces. Rational normal scrolls are *determinantal varieties*. That is, there is a matrix with homogeneous polynomials of the same degree such that the variety is defined by the minors of a certain rank of this matrix.

Proposition 6.1.6. *A rational normal scroll* $S_{k,l} \subset \mathbb{P}^n$ *can be described as the set*

$$\{x \in \mathbb{P}^n : \mathrm{rank}(M(x)) \leq 1\}, \qquad M(x) = \begin{pmatrix} L_{11}(x) & \cdots & L_{1(n-1)}(x) \\ L_{21}(x) & \cdots & L_{2(n-1)}(x) \end{pmatrix}$$

for appropriate linear forms $L_{ij} : \mathbb{C}^n \to \mathbb{C}$.

The third set we discuss is that of *hyperelliptic* curves. These are planar algebraic curves. As affine curves, the appear as defined by a single equation $y^2 = h(x)$, where

h is a polynomial with distinct roots. Their automorphism group always contains the involution $(x : y) \mapsto (x : -y)$. Hyperelliptic curves have a special Riemann surface structure, as made evident by the following theorem.

Theorem 6.1.7. *Let X be a Riemann surface of genus $g \geq 2$. The following are equivalent:*

1. *There is a meromorphic function on X with precisely two poles;*
2. *X admits a degree 2 map to $\widehat{\mathbb{C}}$. This map will necessarily branch at precisely $2g + 2$ points;*
3. *There is an involution $J : X \to X$ with precisely $2g + 2$ fixed points. This involution is necessarily central in $\mathrm{Aut}(X)$;*
4. *X is isomorphic to an algebraic curve in $\mathbb{P}^2$ defined by an equation $y^2 = h(x)$, where h is a polynomial of degree $2g + 1$ or $2g + 2$ with distinct roots.*

If X satisfies (one of) these criteria, X is said to be hyperelliptic*. The central involution is called the* hyperelliptic involution *of the curve.*

Proof. See e.g. [FK1980] or [Mir1995]. □

The canonical model

Algebraic varieties can have points at which the tangent space behaves unexpectedly, for example points of self-intersection. They deserve special attention, as witnessed already e.g. for the models we gave of the Accola-Maclachlan maps in Section 5.3.

Definition 6.1.8. *Let $C \subset \mathbb{A}^n$ be an affine algebraic curve defined by the ideal $I(C) = (f_1, \ldots, f_m)$. A point $x \in C$ is* smooth *if the rank of the $m \times n$-matrix $\mathrm{d}f(x) = (\partial f_i / \partial x_j)$ is $n - 1$. A point that is not smooth is called* singular*. The curve is smooth if each of its points is smooth.*

For projective curves, one can check smoothness in every standard affine subset $\{x_i = 1\}$. Smoothness can be described more intrinsically, but we will not need this abstraction. The smooth points of a complex algebraic curve form a Zariski-open set, and hence are dense in the usual topology on $\mathbb{P}^n$.

A fundamental result is that we can embed any compact Riemann surface into a projective space as a smooth algebraic curve. This is a consequence of the GAGA theorem [Ser1955]. The conditions for stating a precise result about a specific embedding employ the concepts of a divisor and of holomorphic differentials. Divisors are introduced in the next chapter. For the definitions of a *linear system*, a *very ample* divisor, and *holomorphic differentials*, see [Mir1995].

Theorem 6.1.9. *Let X be a compact Riemann surface. To each divisor D on X with linear system $|D|$ of dimension n that is base-point-free, one can associate a holomorphic mapping $\phi_D : X \to \mathbb{P}^n$. If D is very ample, then ϕ_D is an embedding onto a smooth complex projective algebraic curve.*

If $\deg(D) \geq 2g+1$, then ϕ_D is very ample. Hence, we can always embed a Riemann surface of genus g into $\mathbb{P}^{2g+1}$. But in fact, we can do better most of the time.

Theorem 6.1.10 (Canonical model)**.** *Let X be a compact Riemann surface and K a canonical divisor. The associated map ϕ_K is called the* canonical map*. If X is not hyperelliptic, then $\phi_K : X \to \mathbb{P}(\Omega^1(X)) \cong \mathbb{P}^{g-1}$ is an embedding as a smooth projective algebraic curve into the projective space over the holomorphic differentials. The name of this curve is the* canonical model *or* canonical curve *of X.*

A basic fact is that the elements of $\mathrm{Aut}(X)$ induce a linear action on $\Omega^1(X)$ and hence on the canonical model (cf. [Bre2000]). This is of vital importance to our work in Section 6.2. To deal with $\mathrm{Aut}^*(X)$, we need to extend this result by introducing complex conjugations.

Complex conjugations and the canonical representation

Let V be a complex vector space. We can extend the notion of complex conjugation from $\mathbb{C}$ to V.

Definition 6.1.11. *An $\mathbb{R}$-linear transformation $T : V \to V$ is called* a complex conjugation *if it is an involution ($\mathrm{ord}(T) = 2$) and it is* antilinear*: $T(zv) = \overline{z}\,T(v)$ for all $z \in \mathbb{C}$, $v \in V$.*

Example 6.1.12. The map $\mathrm{con}_n : \mathbb{C}^n \to \mathbb{C}^n$ defined by $(z_1, \ldots, z_n) \mapsto (\overline{z_1}, \ldots, \overline{z_n})$ is a complex conjugation. We call it the *standard complex conjugation* on $\mathbb{C}^n$. It leaves the $\mathbb{R}$-linear subspace $\mathbb{R}^n$ invariant. On the space $\mathrm{Mat}_n(\mathbb{C})$ of $n \times n$ complex matrices, the conjugate transpose is a complex conjugation. The map of $\mathbb{C}^n \to \mathbb{C}^n$ defined by $(z_1, \ldots, z_n) \mapsto (\overline{z_1}, z_2, \ldots, z_n)$ is not a complex conjugation for $n > 1$ since it is not antilinear.

Because of antilinearity, a complex conjugation $c : V \to V$ also induces an involution $\mathbb{P}(V) \to \mathbb{P}(V)$, which we denote by the same symbol and also call a complex conjugation. The image of a complex submanifold X of V or $\mathbb{P}(V)$ under a complex conjugation c is again a complex submanifold. This holds in particular for a projective algebraic curve. The pullback $c^*(f)$ of a holomorphic (respectively antiholomorphic) function $f \in \mathcal{O}^*(X)$ is antiholomorphic (respectively holomorphic) on $c(X)$. In the special case where $c(X) = X$, the mapping c^* switches the holomorphic and antiholomorphic functions on X. Important is that we can likewise pull back holomorphic differentials along a complex reflection. Combining this fact with the preceding remarks on the action of $\mathrm{Aut}(X)$ on $\Omega^1(X)$, we get a representation of $\mathrm{Aut}^*(X)$ acting on the canonical model. We first add one more remark for its proper formulation.

The linear mapping on V defined by $v \mapsto iv$ induces a bijection between the $\mathbb{R}$-linear eigenspaces $E(c, 1)$ and $E(c, -1)$. Together these eigenspaces span V, since $v = \frac{1}{2}(v + c(v)) + \frac{1}{2}(v - c(v))$ and the two terms lie in the respective aforementioned eigenspaces. We conclude that $E(c, \pm 1)$ has real dimension $\frac{1}{2}\dim(V)$ and that for any two complex conjugations c_1, c_2 there is an $A \in \mathrm{GL}(n, \mathbb{C})$ such that $c_2 = A^{-1}c_1A$.

Definition 6.1.13. *The* extended complex general linear group $\mathrm{GL}^*(n, \mathbb{C})$ *and* extended projective complex general linear group $\mathrm{PGL}^*(n, \mathbb{C})$ *are defined as*

$$\mathrm{GL}^*(n, \mathbb{C}) := \mathrm{GL}(n, \mathbb{C}) \cup \{g \circ \mathrm{con}_n : g \in \mathrm{GL}(n, \mathbb{C})\},$$
$$\mathrm{PGL}^*(n, \mathbb{C}) := \mathrm{PGL}(n, \mathbb{C}) \cup \{g \circ \mathrm{con}_n : g \in \mathrm{PGL}(n, \mathbb{C})\}.$$

Now we can state:

Theorem 6.1.14. *Let X be a Riemann surface with canonical model C. The action of $\mathrm{Aut}^*(X)$ on $\Omega^1(X)$ induces an (anti)linear representation*

$$\rho_c^* : \mathrm{Aut}^*(X) \to \mathrm{PGL}^*(g-1, \mathbb{C}).$$

This representation leaves C invariant and restricts to a linear representation

$$\rho_c : \mathrm{Aut}(X) \to \mathrm{PGL}(g-1, \mathbb{C}).$$

We call either one the canonical representation. *We name the character χ_c of ρ_c the* canonical character.

The canonical ideal

To our great fortune, there is a small and absolute bound on the degree(s) of polynomials necessary to generate the radical ideal of a canonical model.

Theorem 6.1.15 (Canonical ideal generation)**.** *Let X be a non-hyperelliptic complex algebraic curve of genus $g \geq 3$ with canonical model C. If $g = 3$, then $I(C)$ is generated by one element in degree 4. If $g \geq 4$, it is generated by I_2, except in the following two cases:*

- *$g = 6$ and X is birationally equivalent to a non-singular planar quintic;*
- *X is trigonal.*

In both cases $I(C) = (I_2, I_3)$. The ideal (I_2) then defines a surface: the Veronese surface $V_{2,2}$ in the first case and a rational normal scroll $S_{k,l}$ in the second.

This theorem arose from work of Dennis Babbage, Federigo Enriques, Max Noether, and Karl Petri. Confer Bernard Saint-Donat [SD1973] for a proof that even holds over any algebraically closed field of arbitrary characteristic.

The ideal of a canonical model of a non-hyperelliptic genus 4 curve is generated by three quadrics and one cubic (see [Mir1995]). Thus, all curves of genus 4 are either hyperelliptic or trigonal. It is no longer true for $g \geq 5$ that the canonical curve can be cut out from $\mathbb{P}^{g-1}$ by $g-2$ polynomials; they are not *complete intersection* curves.

When we try to construct a canonical model it would be advantageous to know how many polynomials we need. We can actually compute the dimensions of I_2 and I_3 as $\mathbb{C}$-vector spaces in advance.

Definition 6.1.16. *The* Hilbert-Poincaré series $\mathrm{HP}(I,t)$ *of a homogeneous ideal* $I \subset \mathbb{C}[x_1,\ldots,x_{n+1}]$ *is defined as*

$$\mathrm{HP}(I,t) = \sum_{k\geq 0} \dim(\mathbb{C}[x_1,\ldots,x_{n+1}]_k/I_k)t^k.$$

In general, for small k the coefficients of the Hilbert-Poincaré series may behave erratically, but for large enough k, their values coincide with those of a polynomial function $\mathrm{H}(I,k)$. The polynomial is called the *Hilbert polynomial* of I. An important fact is that $\dim(V(I)) = \deg(\mathrm{H}(I,k))$. When speaking of a canonical curve, large enough k are still pretty small.

Proposition 6.1.17. *For a non-hyperelliptic canonical curve, the Hilbert-Poincaré series of its radical ideal* I *is*

$$\mathrm{HP}(I,t) = 1 + gt + \sum_{k\geq 2}(g-1)(2k-1)t^k.$$

This entails that

$$\dim I_2 = \frac{1}{2}g(g+1) - 3(g-1) = \frac{1}{2}(g-2)(g-3),$$
$$\dim I_3 = \frac{1}{6}(g-1)g(g+2) - 5(g-1) = \frac{1}{6}(g-1)(g^2+2g-30).$$

Also, for a canonical ideal $\mathrm{H}(I,k) = (g-1)(2k-1)$. We see that indeed $\deg(\mathrm{H}(I,k)) = 1$ as it should be, and also that we can read off the genus from the Hilbert polynomial. In the next section we will lay out a strategy to construct for a non-hyperelliptic platonic map various possible ideals (I_2). Such an ideal will be generated by the number of independent quadrics the Hilbert-Poincaré series predicted, and will be our first attempt at finding the canonical ideal. We then check whether $\deg(\mathrm{H}((I_2),k)) = 1$, and if so, whether the curve has the correct genus and (I_2) is prime. If one of these last two checks fail, we recognize (I_2) as a blind alley.

In the exceptional cases of Theorem 6.1.15, (I_2) will not define a curve but a surface, and we will find $\deg(\mathrm{H}((I_2),k)) = 2$. For curves equivalent to a planar quintic, this surface will be the Veronese surface $V_{2,2} \subset \mathbb{P}^5$. The Hilbert polynomial for the Veronese surface is $\mathrm{H}(I(V_{2,2}),k) = 2k^2+3k+1$, so its Hilbert-Poincaré series starts as

$$\mathrm{HP}(I(V_{2,2}),t) = \sum_{k\geq 0}(2k^2+3k+1)t^k = 1 + 6t + 15t^2 + 28t^3 + \cdots.$$

Since in this case $g = 6$, the Hilbert-Poincaré series for the canonical curve is

$$\mathrm{HP}(I,t) = 1 + 6t + 15t^2 + 25t^3 + \cdots$$

and one needs three extra polynomials of degree 3 to generate the canonical ideal. For trigonal curves, (I_2) will define a rational normal scroll $S_{k,l}$. The Hilbert-Poincaré series for the rational normal scroll $S_{k,l}$ is the convolution of those for its defining rational normal curves (or a curve and a point if k or l is zero), and we can compute that

$$\mathrm{HP}(S_{k,l}, t) = 1 + (k + l + 2)t + (kl + 3k + 3l + 3)t^2 + (4kl + 6k + 6l + 4)t^3 + \cdots$$

One thus needs $4kl + 6k + 6l + 4 - 5(g - 1)$ extra polynomials of degree 3 for the canonical ideal.

6.2 A construction strategy for a canonical model

Our strategy to compute a canonical model for a non-hyperelliptic platonic map $\mathbf{M}$ consists of several parts.

1. Take a standard presentation of $\mathrm{Aut}(\mathbf{M})$ and apply the fixed point counting lemma (Lemma 1.4.3) to compute the fixed points and rotation indices for a representative of each conjugacy class.
2. Calculate the canonical character χ_c of $\mathbf{M}$ with the Eichler trace formula.
3. Construct a canonical linear representation ρ_c on $V = \mathbb{C}^g$.
4. Calculate the linear representation $(\rho_c^\vee)^{2+}$, which acts on the space $(V^\vee)^{2+} \cong \mathbb{C}[x_1, \ldots, x_g]_2$.
5. Decompose $(\rho_c^\vee)^{2+}$ into its isotypic components with the Reynolds operator.
6. Apply various methods (e.g. truncated Gröbner bases, the centralizer trick, radicals, the primary decomposition, fixed points as listed below) to determine necessary and sufficient invariant pieces of $(V^\vee)^{2+}$ that generate $I_2(\mathbf{M}_a)$.
7. If the ideal is not of dimension 2, then repeat steps 4–6 with $(\rho_c^\vee)^{3+}$ to compute necessary and sufficient invariant pieces of $(V^\vee)^{3+}$ that generate $I_3(\mathbf{M}_a)$.

We stress that steps 6 and 7 are not automatic and require creativity for each separate case. They consist of excluding some isotypic pieces and determining invariant subspaces of the rest that together define a prime ideal with the correct Hilbert-Poincaré series. We doubt that the tricks below will always lead to a solution. For this reason we refrain from calling the above an algorithm. Some remarks detailing the different steps are in order.

The Eichler trace formula. Let X be a Riemann surface of genus $g \geq 2$ and $\sigma \in \mathrm{Aut}(X)$ of order $n > 1$. The canonical character χ_c of the action of $\mathrm{Aut}(X)$ on $H^1(X)$ satisfies

$$\chi(\sigma) = 1 + \sum_{p \in \mathrm{Fix}(\sigma)} \frac{\zeta_n^{-\mathrm{rot}(\sigma,p)}}{1 - \zeta_n^{-\mathrm{rot}(\sigma,p)}} = 1 + \sum_{\substack{1 \leq m < n \\ \gcd(m,n)=1}} |\mathrm{Fix}(\sigma, m)| \, \frac{\zeta_n^{-m}}{1 - \zeta_n^{-m}}.$$

where $\mathrm{Fix}(\sigma, m) = \{x \in \mathrm{Fix}(\sigma) \mid \mathrm{rot}(\sigma, x) = m\}$ is the set of fixed points of σ with rotation index m. See [Bre2000, Theorem 12.1 and Lemma 11.5].

The Construction of ρ_c. Once the canonical character χ_c has been computed, we can construct a canonical representation ρ_c by hook or crook. The representations we used were obtained in the following ways:

1. From the articles of the Japanese school for genus $3, 4, 5$;
2. From the ATLAS of finite group representations [W+1996];
3. With the help of the GAP package REPSN by John D. Dixon and Vahid Dabbaghian;
4. By hand, for example with induction from a subgroup or by a lucky guess.

The Reynolds operator. Let G be a finite group, $\rho : G \to V$ any finite-dimensional representation and χ and irreducible character. The generalized Reynolds operator $\mathrm{Rey} = \mathrm{Rey}(\rho, \chi)$ is defined as

$$\mathrm{Rey}(\rho, \chi) = \frac{\dim(\chi)}{|G|} \sum_{g \in G} \chi(g^{-1})\rho(g).$$

This operator is an element of $\mathrm{End}(V)$ that satisfies $\mathrm{Rey}^2 = \mathrm{Rey}$ and $g \circ \mathrm{Rey} = \mathrm{Rey} \circ g$ for all $g \in G$. In other words, it is a G-invariant projection. Its image is the G-invariant subspace called the *isotypic χ-component* V_χ of V, which is the unique maximal subspace of V with character $k\chi$ ($k \in \mathbb{Z}$). Of course $\dim V_\chi = \langle \rho, \chi \rangle$, where $\langle \cdot, \cdot \rangle$ is the standard inner product on the character ring.

Because it is a projection, we can use Rey on $(\rho_c^\vee)^{2+} : \mathrm{Aut}^+(\mathbf{M}) \to \mathrm{GL}(\binom{g+1}{2}, \mathbb{C})$ to compute the isotypic components for all irreducible representations χ. Either we apply it to a basis of V to get a spanning subset of V_χ or, because that is time consuming, we apply it to a small number of vectors and spin those with $\mathrm{Aut}^+(\mathbf{M})$ afterwards. The same applies to $(\rho_c^\vee)^{3+}$.

Truncated Gröbner bases. It can be hard to decide whether a whole isotypic component of the decomposition $(V^\vee)^{2+}$ should be used or not. To prove that we cannot use the whole component it suffices to exhibit a polynomial in it that factors linearly over $\mathbb{C}$, since in the final prime ideal no linear factors will be present. To find such a certificate polynomial quickly, we can compute truncated Gröbner bases, successively increasing the degree if necessary. Since our ideal is homogeneous, any truncated Gröbner basis will be the truncation of the actual Gröbner basis so a product of d linear polynomials will show up in the d-truncated basis.

The centralizer trick. The centralizer $C_{\mathrm{GL}(g,\mathbb{C})}(\mathrm{Aut}^+(\mathbf{M}))$ is of use if the representation ρ_c is reducible. Note that an element of $C_{\mathrm{GL}(g,\mathbb{C})}(\mathrm{Aut}^+(\mathbf{M}))$ induces an $\mathrm{Aut}^+(\mathbf{M})$-invariant automorphism on each of the subrepresentations of ρ_c, so by Schur's lemma it is scalar on each of them. An element of the centralizer maps a G-invariant curve to another one by a projectivity. The dimension of this group of projectivities is # components of $\rho_c - 1$. With help of the centralizer we can make extra assumptions about the ideal we are searching for. Specifically, given that we have to pick the subspace $m\chi < V_\chi$ we can (partially) normalize it.

Example 6.2.1. For $\mathbf{R}_{5.11}$ the space $(V^\vee)^{2+}$ has a 1-parameter family of invariant subspaces $2_2(t) = \langle x_1x_2 + tx_3x_5, x_1x_3 - t\zeta_3x_2x_4\rangle$. We have to choose one, a hard task since there is an infinite number of options. The centralizer $C_{\mathrm{GL}(5,\mathbb{C})}(\mathrm{Aut}^+(\mathbf{R}_{5.11}))$ rescues us. The space $V = \mathbb{C}^5$ decomposes into invariant subspaces $\langle e_1\rangle$, $\langle e_2, e_3\rangle$ and $\langle e_4, e_5\rangle$. This means we can perform normalization by any transformation of the form $\mathrm{DiaMat}(\lambda_1, \lambda_2, \lambda_2, \lambda_3, \lambda_3)$. So we may set $\lambda_1 = t$ or $\lambda_3 = t$.

Radicals and the primary decomposition trick. Suppose that we already have a subspace $J \subset I_2(\mathbf{M}_a)$ of positive dimension. Certainly the radical $\mathrm{Rad}(J) \subseteq I(\mathbf{M}_a)$. So we may try to compute it to enlarge the subspace we know. We have a bigger chance if we succeed in decomposing J into primary ideals. An ideal Q of a ring R is a *primary ideal* when $xy \in Q \implies x \in Q \vee \exists n \in \mathbb{N}, y^n \in Q$. The following theorem stresses the importance of primary ideals.

Theorem 6.2.2 (Lasker-Noether (Emmy))**.** *Let $J \subseteq R$ be an ideal of a noetherian ring R. Then there is a finite set of primary ideals $Q_1, \dots, Q_n$ such that $J = \cap_{k=1}^n Q_k$. The set is unique up to ordering, if we take the decomposition to be irredundant.*

Irredundancy means that $Q_1 \cap \cdots \cap \widehat{Q_k} \cap \cdots \cap Q_n \not\subseteq Q_k$ (for $k = 1, \dots, n$). A further theorem says that if $J \subseteq I$ and I is prime and minimal with this property, then I is contained in the set $\{\mathrm{Rad}(Q_k) : k = 1, \dots, n\}$, all of which are prime, and called the *associated primes* of J.

So if we can compute the associated primes of J, we know that at least one of them will be contained in $I(\mathbf{M}_a)$. This gives us a finite number of cases to investigate further.

Using fixed points. As described in the corollary to the fixed point counting lemma, we know that conjugates of powers of R, S and RS (and only those) have fixed points on $\mathbf{M}_a$, and we know how many. Since $\mathrm{Aut}^+(\mathbf{M}_a)$ acts linearly on the canonical model for $\mathbf{M}_a$, we can search for a set of vectors of the right size in the eigenspaces $E(R, \zeta_p^k)$ of R, and similarly for S and RS. Typically, we apply this strategy in tandem with the other strategies: a non-empty subset $J \subseteq I_2(\mathbf{M}_a)$ will give constraints on the possible fixed points. We might also use the centralizer to pick a vector we want. As soon as we have obtained a vector we know has to lie on the projective curve, we spin it with $\mathrm{Aut}^+(\mathbf{M}_a)$. This then yields a set of points that have to lie on $\mathbf{M}_a$, giving constraints on the pieces present in $I_2(\mathbf{M}_a)$. The use of fixed points is also described in [FGT2012].

Remark 6.2.3. In addition to showing that a group acts on the canonical model, to get a standard map presentation we need to know which automorphisms are primitive rotations and then choose a standard generator pair from those. For example, consider the canonical model of $\mathbf{R}_{14.1}$. If one chooses a standard generator pair (R, S) suitable for $\mathbf{R}_{14.2}$ or $\mathbf{R}_{14.3}$, then on the model for $\mathbf{R}_{14.1}$ S is not a primitive rotation. A pair that satisfies all relations of the standard map presentation of $\mathbf{R}_{14.1}$ must be a standard generator pair, though. For some maps even this is false. The group $\mathrm{Aut}(\mathbf{R}_{7.1})$ has outer automorphism group of order three. For a standard generator pair (R, S), e.g. the pair $(R, S[R, S]^2)$ also satisfies all group relations, but is *not* a standard generator pair on the same canonical model, but on an algebraic conjugate.

The field of definition

Belyĭ's theorem [Bel1979] ensures us that for a platonic map $\mathbf{M}$ there exists an algebraic model $\mathbf{M}_a$ defined over a number field $K \subset \overline{\mathbb{Q}}$. But what will K be? To answer this question, let V be a projective algebraic variety defined over $\overline{\mathbb{Q}}$ and consider

$$\Sigma(V) = \{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \mid V^\sigma \cong V\}.$$

The action of σ is on the coefficients of the polynomials in the ideal $I(V)$. If V is defined over K, then certainly all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K) \subseteq \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ are elements of $\Sigma(V)$. This suggests the following definition.

Definition 6.2.4. *The field*

$$M_{\overline{\mathbb{Q}}}(V) := \{\alpha \in \overline{\mathbb{Q}} \mid \forall \sigma \in \Sigma(V), \sigma(\alpha) = \alpha\}$$

of all algebraic numbers fixed by $\Sigma(V)$ *is called the* field of moduli *of* V.

The field of moduli is contained in all possible fields of definition. Equality need not always hold, but in a special case that applies for platonic surfaces, it does. Confer [CH1985] and [DE1999].

Theorem 6.2.5 (Coombes and Harbater)**.** *Let* K *be a number field. A Galois cover* $\beta : X \to \mathbb{P}^1$ *over* $\overline{K}$ *with* K*-base* $\mathbb{P}^1$ *is defined over its field of moduli* $M_{\overline{\mathbb{Q}}}(X)$ *relative to the extension* $K < \overline{\mathbb{Q}}$.

The field of moduli of a variety V defined over $K < \overline{\mathbb{Q}}$ is computable if one can establish for all elements $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ whether $V^\sigma \cong V$. For our platonic surfaces, we have knowledge helping us to answer that question of isomorphy. If we know that a platonic surface has no siblings (cf. Section 1.4), then all conjugates C^σ of an algebraic model C must be isomorphic to C. There are platonic surface with siblings that occur as algebraic conjugates. Examples of this are the first tuplet ($\mathbf{R}_{8.1}$ and $\mathbf{R}_{8.2}$) and the first Hurwitz triplet (see Chapter 7). Still, the field of moduli is always the unique minimal field of definition.

The minimal field of definition of a canonical curve is determined by the isotypic components of $(\rho^\vee)^{2+}$ and $(\rho^\vee)^{3+}$ that contain polynomials of the canonical ideal. Each such component has a corresponding field, namely that over which its irreducible representation is defined. This field must be a subfield of the minimal field of definition, since the canonical ideal is invariant under the group action. When applying the fixed point strategy to construct a canonical ideal, we may introduce superfluous field extensions necessary to write down the fixed points, but not for the definition of the curve. If the constructed canonical ideal is invariant under conjugation of these extraneous algebraic numbers, then we can average all ideal generators with the appropriate algebraic conjugates to get polynomials over a smaller field.

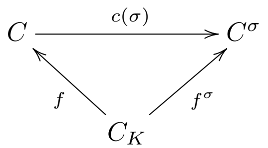
But if the ideal is not invariant under these conjugations yet the resulting curves are all isomorphic, how can we transform the curve to be definable over the field of moduli? A procedure accomplishing this is *Galois descent*, which we now outline. We re-use the concepts from group cohomology introduced in Section 2.1. For our purpose, we must alter them to accomodate a non-abelian target group for cochains.

Definition 6.2.6. *Let N and G be groups, with a right action $N \times G \to N$. We write the image of (n, g) as n^g. A mapping $c : G \to N$ is called a 1-cochain. The set $Z^1(G, N)$ of* 1-cocycles *consists of the 1-cochains c that satisfy $c(g_1 g_2) = c(g_2)^{g_1} c(g_1)$ for all $g_1, g_2 \in G$. Two 1-cocycles c_1, c_2 are said to be equivalent if $c_1(g) = n^{-1} c_2(g) n^g$ for all $g \in G$. The set of equivalence classes is the called the* first cohomology group *$H^1(G, N)$. The set $B^1(G, N)$ of* 1-coboundaries *consists of the equivalence class of the trivial 1-cocycle defined by $c(g) = 1$ for all $g \in G$.*

Now let $K \subset L$ be a field extension that is Galois, with Galois group $G := \mathrm{Gal}(L/K)$. Suppose that the curve $C_L \subset \mathbb{P}^n$ is defined over L and that there is a curve $C_K \subset \mathbb{P}^n$ defined over K that is isomorphic to C_L by the projective isomorphism $f : C_K \to C_L$. Then f can be conjugated to yield

$$f^\sigma := \sigma \circ f \circ \sigma^{-1} : C_K \to C_L^\sigma$$

for any $\sigma \in G$. Concretely, if f is given by the matrix (f_{ij}), then f^σ is defined by the matrix $(\sigma(f_{ij}))$. We find the following commutative diagram:



where $c(\sigma) := f^\sigma \circ f^{-1}$. It follows that $c : G \to \mathrm{GL}(n, L)$ is an element of the 1-coboundary set $B^1(G, \mathrm{GL}(n, L))$. We now utilize the fact that $H^1(G, \mathrm{GL}(n, L)) = 0$, see [Ser1997, Lemma III.1.1]. It implies that every 1-cocycle $c \in Z^1(G, \mathrm{GL}(n, L))$ is a 1-coboundary. So given isomorphic algebraic varieties C_L^σ for all $\sigma \in G$, we are in search of a mapping $c : G \to \mathrm{GL}(n, L)$ satisfying the 1-cocycle condition, for which each $c(\sigma)$ permutes the set $\{C^\tau : \tau \in G\}$. We have no general recipe for how to find such a 1-cocycle, but have managed this in various cases with an outer automorphism of $\mathrm{Aut}^+(\mathbf{R})$. Compare e.g. the model constructions of $\mathbf{R}_{7.1}$ in Appendix A and the first Hurwitz triplet in Section 7.2. If we manage to find such a 1-cocycle, we need to split it, i.e. write it as a 1-coboundary, to find a projective isomorphism f to a model over K. One approach to splitting is to look at the equation $c(\sigma) = f^\sigma \circ f^{-1}$ for one value of σ. Write $L = K(\alpha)$ and substitute $f = \sum_{k=1}^{d} f_k \alpha^k$ in the equation, where $d = [L : K]$ and each f_k is defined over K. Considering the different α^k-components yields a system of linear equations over K, which can readily be solved. A small difficulty resides in finding an *invertible* solution, but a solution will generically have full rank.

Question 6.2.7. A problem related to Question 1.4 arises from the above. Non-isomorphic platonic curves resulting from different embeddings $K \rightarrowtail \mathbb{C}$ of a field of definition for a canonical ideal give rise to a tuplet. What about the other direction? Assuming that all members of a tuplet have the same canonical character, is there necessarily one canonical ideal defined over an algebraic number field K, such that the different embeddings $K \rightarrowtail \mathbb{C}$ yield all tuplet members?

Simplification of coefficients

Mathematicians being the hairsplitters they are, they do not content themselves with just any algebraic model, even if it is over the field of moduli. Preferably, the coefficients are small too. It is intuitively clear what is meant by this, we need not bother about a rigorous definition. The strategy sketched at the beginning of the section often results in models that are suboptimal with respect to coefficient size. We have no proven method of amelioration. If the model is over an algebraic number field $\mathbb{Q}(\alpha)$, the coefficients depend on the extension element α. We know of no algorithm to choose an α yielding small $\mathbb{Q}$-coefficients, itself having a minimal polynomial over $\mathbb{Q}$ with small coefficients. In practice though, we often found that we could improve the size of the rational numbers appearing in a set of polynomial generators, until they were to our satisfaction.

Suppose that you have a homogeneous ideal $I \subset \mathbb{Q}(\alpha)[x_1, \ldots, x_n]$, with $[\mathbb{Q}(\alpha) : \mathbb{Q}] = k$. Let $(\alpha_1, \ldots, \alpha_k)$ be a $\mathbb{Q}$-basis of $\mathbb{Q}(\alpha)$, for example the power basis $(1, \alpha, \alpha^2, \ldots, \alpha^{k-1})$. Treat each homogeneous piece I_d of degree d separately. In our cases this would be I_2 or I_3. Write out a $\mathbb{Q}(\alpha)$-spanning set of I_d as N-dimensional $\mathbb{Q}$-vectors, where $N = k \dim(\mathbb{C}[x_1, \ldots, x_n]_d)$. Add copies of the spanning set so that you get a $\mathbb{Q}$-spanning multiples. In the case of a power basis, this can be done by multiplying with powers of α. Then multiply to get rid of denominators. The result is a $k \dim(I_d)$-dimensional lattice $\Lambda \subset \mathbb{Z}^N$. The goal is now to find a short basis for Λ. One iteratively applies two operations to Λ:

1. compute a short $\mathbb{Z}$-basis using the LLL-algorithm;
2. compute the kernel of the reduction of Λ modulo p for a prime p. If this kernel is non-empty, say $v_1, \ldots, v_t$, then construct the set of vectors $\{\frac{1}{p}v_1, \ldots, \frac{1}{p}v_t\}$. If desired, expand it by re-interpreting the set as polynomials, multiplying by a set of elements of $\mathbb{Q}(\alpha)$, and convert back to $\mathbb{Z}$-vectors. Add the resulting vectors to Λ.

The second step is warranted because polynomials may be scaled. It helps because it introduces $\mathbb{Z}$-vectors outside of Λ, which shrinks its covolume. Primes p for which the kernel is non-trivial are factors of the lattice determinant $\det(\Lambda)$. This leads to the problem of factoring this determinant, a hard nut to crack. Also, not every prime gives a non-trivial kernel.

A priori, we saw no reason to expect good results from the method, since there seems to be no guarantee, even if Λ has a $\mathbb{Q}$-basis with small coefficients, that it would have a small $\mathbb{Z}$-basis: take $\langle \frac{1}{2}x_1 + \frac{1}{3}x_2 + \frac{1}{5}x_3 + \cdots + \frac{1}{p_n}x_n \rangle$ with the first n primes in the denominators as an example. To our surprise, although the LLL-algorithm did not accomplish improvement on its own, supplemented by the second technique it became very effective. Suffice it to say that for the first Hurwitz triplet, the reduction was from 30-digits numbers to 2-digit numbers.

6.3 Hyperelliptic platonic maps

Hyperelliptic curves were explicitly excluded from Theorem 6.1.15. A hyperelliptic curve has no canonical model. Natural questions are, for which platonic surfaces this situation arises and what to do if it does.

How to detect a hyperelliptic map. From Theorem 6.1.7 we learned that a curve is hyperelliptic if and only if it has a conformal automorphism with $2g + 2$ fixed points, the *hyperelliptic involution*. Also, we have seen in Theorem 3.4.1 that the group $\mathrm{Aut}(\mathbf{M}_a)$ will be realized by a platonic map $\mathbf{M}'$ on $\mathbf{M}_a$, which is either $\mathbf{M}$ itself or such that $\mathbf{M} = D(\mathbf{M}')$. We can therefore compute $\mathrm{Aut}(\mathbf{M}')$ explicitly. Then we can use the fixed point counting lemma (Lemma 1.4.3) to determine whether $\mathbf{M}'$ has a hyperelliptic involution. The detection problem can thus be solved in a case-by-case fashion. But in fact, we can determine all hyperelliptic platonic maps explicitly, and will do so below. Let us assume that our platonic map contains all conformal automorphisms of the platonic surface: $\mathrm{Aut}(\mathbf{M}_a) = \mathrm{Aut}^+(\mathbf{M})$.

First, we remark that even though hyperelliptic platonic maps have no canonical model, the canonical map is still useful. Let us see what actually happens.

Theorem 6.3.1 (Miranda, Theorem VII.2.2)**.** *For a hyperelliptic curve X of genus g, given as a planar curve by the equation $y^2 = p(x)$ with $p(x)$ of degree $2g+1$ or $2g+2$ with distinct roots, the canonical map ϕ_K is the composition of the double covering map $(x, y) \mapsto x$ and a Veronese map. In particular, the image is a rational normal curve Y of degree $g - 1$ in $\mathbb{P}^{g-1}$ and the map $\phi_K : X \to Y$ has degree 2.*

We compute the canonical representation ρ_c of a hyperelliptic map all the same. The hyperelliptic involution h is a central element that is the identity on $\phi_K(\mathbf{M}_a)$, and ρ_c will exhibit this fact by $\rho_c(h) = -I$. One way forward is to find an invariant $\mathbb{P}^1 \subset \mathbb{P}^{g-1}$ with the fixed points of $\rho_c(h)$ on it. The ramification points of the natural projection $\pi : \mathbf{M}_a \to \mathbf{M}_a/\langle h\rangle \cong \mathbb{P}^1$ are precisely these $2g+2$ fixed points of h. But for a projectivized planar model $y^2 z^{d-2} = p(x, z)$, where p is homogeneous of degree $d \in \{2g + 1, 2g + 2\}$, this natural projection is the map $(x : y : z) \mapsto (x : z)$, and its branch points are the zeroes of p. Therefore, the branch data we get from the fixed points of h on $\mathbb{P}^1$ allows us to write down p. We carried out this procedure for various hyperelliptic platonic maps such as $\mathbf{R}_{3.4}$, $\mathbf{R}_{5.2}$, and $\mathbf{R}_{6.8}$. It is still described for the first of these three in Appendix A. However, there is no need to go through this trouble, which will become clear after making an inventory of all hyperelliptic maps.

Determination of all hyperelliptic platonic maps

Theorem 6.3.2. *The hyperelliptic platonic maps are precisely the following polynomial families and exceptional maps (or their duals):*

1. *the family* $\mathbf{AM}(n)$ *: platonic* 2*-covers of* $\mathbf{Dih}(n)$ *branched over faces;*
2. *the family* $D(\mathbf{AM}(n))$ *(self-dual);*
3. *the family* $\mathbf{Wi2}(n)$ *: platonic* 2*-covers of* $\mathbf{Dih}(2n)$ *branched over vertices and faces;*
4. *the family* $D(\mathbf{Wi2}(n))$ *(self-dual);*
5. *the family* $\mathbf{Wi1}(n)$*;*
6. $\mathbf{R}_{2.1}$ *: a platonic* 2*-cover of* $\mathbf{Oct}$ *branched over vertices;*
7. $\mathbf{R}_{3.4}$ *: a platonic* 2*-cover of* $\mathbf{Cub}$ *branched over vertices;*
8. $\mathbf{R}_{3.8}$ *: a platonic* 2*-cover of* $\mathbf{Tet}$ *branched over vertices and faces;*
9. $\mathbf{R}_{5.2}$ *: a platonic* 2*-cover of* $\mathbf{Ico}$ *branched over vertices;*
10. $\mathbf{R}_{6.8}$ *: a platonic* 2*-cover of* $\mathbf{Oct}$ *branched over vertices and faces;*
11. $\mathbf{R}_{9.15}$ *: a platonic* 2*-cover of* $\mathbf{Dod}$ *branched over vertices;*
12. $\mathbf{R}_{15.9}$ *: a platonic* 2*-cover of* $\mathbf{Ico}$ *branched over vertices and faces.*

Proof. We may assume that $\mathrm{Aut}(\mathbf{M}_a) = \mathrm{Aut}^+(\mathbf{M})$, diagonal maps can be included afterwards. The fixed points of the hyperelliptic involution h must be cell centers of $\mathbf{M}_a$, as we discussed above. Take y to be such a fixed point. Because h is central in the map automorphism group, for any $g \in \mathrm{Aut}^+(\mathbf{M})$ we find $h(g(y)) = g(h(y)) = g(y)$, in other words $g(y)$ is again a fixed point of h. If follows that the ramification locus $\mathrm{Fix}(h)$ of the natural projection $\mathbf{M} \to \mathbf{M}/\langle h \rangle$ is a union of all the cell centers for a certain combination of cell types. The problem at hand splits naturally into five cases, according to whether $\mathrm{Fix}(h)$ is cells_0, cells_1, $\mathrm{cells}_0 \cup \mathrm{cells}_1$, $\mathrm{cells}_0 \cup \mathrm{cells}_2$, or $\mathrm{cells}_0 \cup \mathrm{cells}_1 \cup \mathrm{cells}_2$. One can eliminate the cases cells_2 and $\mathrm{cells}_1 \cup \mathrm{cells}_2$ by dualizing the map. We will treat each case separately, starting from the equation

$$|\mathrm{Fix}(h)| = 2g + 2 = 4 - \chi(\mathbf{M}) = 4 - v + e - f.$$

For the different fixed point loci, the left hand side is equal to $v, e, v + e, v + f$, and $v + e + f$ respectively.

<u>$\mathrm{Fix}(h) = \mathrm{cells}_0$</u> We have $v = 2g + 2$ and or $f - e = 4 - 2v = -4g$. Since $p = 2e/f = 2e/(e - 4g)$ must be an integer, and $p \leq 2$ can be excluded since it would imply $g = 0$, we have $e - 4g \leq \frac{2}{3}e$, or $e \leq 12g$. On the other hand, $e = \frac{1}{2}qv = q(g + 1)$, so $q \in \{3, \ldots, 11\}$ and

$$p = 2e/f = 2q(g + 1)/(q(g + 1) - 4g).$$

All integer values of the right hand side can be determined with a reasonable calculation effort, for all the available values of q. The only solutions (g, p, q) with $q \geq 5$ (and $g \geq 2$) are $(3, 5, 5)$, $(5, 6, 5)$, $(15, 8, 5)$, $(35, 9, 5)$, $(3, 4, 6)$, $(9, 5, 6)$, $(7, 4, 7)$, $(2, 3, 8)$, $(3, 3, 9)$, $(5, 3, 10)$, and $(11, 3, 11)$. Not all of these types are realized by platonic maps, only $\mathbf{R}_{3.4}$ (type $(4, 6)$), $\mathbf{R}_{9.15}$ (type $(5, 6)$), $\mathbf{R}_{2.1}$ (type $(3, 8)$), and $\mathbf{R}_{5.2}$ (type $(3, 10)$). These four maps are all hyperelliptic. The value

$q = 3$ is clearly impossible because $p > 0$, and for $q = 4$ we get $p = 2g + 2$. The duals of such maps were classified in Section 4.5; they are the families $\mathbf{AM}(n)^\vee$ and $\mathbf{Kul}(n)^\vee$. Only the first one consists of hyperelliptic maps. We conclude that the only hyperelliptic maps for the case $q = 4$ are those of $\mathbf{AM}(n)$.

$\underline{\mathrm{Fix}(h) = \mathrm{cells}_1}$ We have $e = 2g + 2$ and $v + f = 4$. Hence $(v, f) \in \{(1, 3), (2, 2)\}$ by using duality. The first option yields $q = 2e/v = 4g + 4 > 4g + 2$, which is impossible by the well-known bound $4g+2$ on the order of an automorphism of a Riemann surface. The second option implies that the map belongs to either $D(\mathbf{AM}(n))$ or $D(\mathbf{Kul}(n))$, as derived in Section 4.2. Only the former family consists of hyperelliptic maps.

$\underline{\mathrm{Fix}(h) = \mathrm{cells}_0 \cup \mathrm{cells}_1}$ We have $v+e = 2g+2$ and deduce $2v+f = 4$. It follows that $v = 1$ and $f = 2$, so we must be dealing with maps of $\mathbf{Wi1}(n)$ (consult Section 4.1). These maps are indeed hyperelliptic.

$\underline{\mathrm{Fix}(h) = \mathrm{cells}_0 \cup \mathrm{cells}_2}$ We have $v + f = 2g + 2$ and deduce $2v - e + 2f = 4$. This implies $4 + e = 2(v + f) = 4 + 4g$, so $e = 4g$. Now we see that $p = 8g/f$ and $q = 8g/v$, so that

$$\frac{1}{p} + \frac{1}{q} = \frac{v + f}{8g} = \frac{2g + 2}{8g} = \frac{1}{4} + \frac{2}{8g} > \frac{1}{4}.$$

Assuming $p \leq q$, this means that $p \in \{3, \ldots, 7\}$. The case $p = 3$ violates $v > 0$ for $g > 1$ so is ruled out. The case $p = 4$ gives $(p, q) = (4, 4g)$. The members of $\mathbf{Wi2}(n)$ were determined in Section 4.2 to be the only such maps, and they are indeed hyperelliptic. The other cases each yield a linear fractional expression for $q(g)$ of which the integral solutions can quickly be enumerated: (g, p, q) must be one of $(5, 5, 10)$, $(15, 5, 15)$, $(20, 5, 16)$, $(45, 5, 18)$, $(95, 5, 19)$, $(3, 6, 6)$, $(6, 6, 8)$, $(9, 6, 9)$, $(15, 6, 10)$, $(33, 6, 11)$, $(7, 7, 7)$, $(14, 7, 8)$, $(63, 7, 9)$, $(2, 8, 4)$, $(6, 8, 6)$, or $(14, 8, 7)$. The only triples occurring for actual platonic maps are $(3, 6, 6)$, $(6, 6, 8)$, $(7, 7, 7)$, $(15, 6, 10)$ and $(63, 7, 9)$. By looking at Conder's list, we find that the only corresponding hyperelliptic maps are $\mathbf{R}_{3.8}$, $\mathbf{R}_{6.8}$, and $\mathbf{R}_{15.9}$.

$\underline{\mathrm{Fix}(h) = \mathrm{cells}_0 \cup \mathrm{cells}_1 \cup \mathrm{cells}_2}$ We have $v + e + f = 2g + 2$ and deduce $2v + 2f = 4$. It follows that $v = f = 1$ and that we are dealing with a map of $D(\mathbf{Wi2}(n))$, all of which are hyperelliptic.

Not all of our solutions turn out to satisfy $\mathrm{Aut}(\mathbf{M}_a) = \mathrm{Aut}^+(\mathbf{M})$, but that is not important. The diagonal maps of the maps in our solution set have also arisen already. This finishes the classification. □

Remark 6.3.3. As expected, we get all platonic maps of genus 2, since any algebraic curve of genus 2 is hyperelliptic. We also note that all solutions are reflexive.

To find an algebraic model is not too hard for any of the hyperelliptic maps. The exceptional maps are platonic covers of the platonic solid maps **Tet**, **Cub**, **Oct**, **Ico**, and **Dod**. In Section 6.4 we will compute the configuration that the vertices and face centers of these genus 0 maps form in $\mathbb{P}^1$, after which we can use that as branch data to write down planar models for the exceptional hyperelliptic platonic surfaces. In the rest of this section we consider the hyperelliptic polynomial families.

Models for the hyperelliptic polynomial families

We now construct planar algebraic models for the hyperelliptic polynomial families appearing in Theorem 6.3.2. These are the families $\mathbf{Wi1}(n)$ (and their duals), $\mathbf{Wi2}(n)$ (and their duals and diagonal maps), and $\mathbf{AM}(n)$ (and their duals and diagonal maps). These algebraic models are of course already known, and we treated them extensively in Sections 5.1–5.3. The constructions presented below furnish a complement to that treatment, pointing out an alternative road to their discovery if one did not already know these models.

To express the values of the canonical character in a way that tells us what the canonical representation is, we will avail to a small lemma on cyclotomic fields.

Lemma 6.3.4.
$$\frac{1}{1-\zeta_n} = \frac{1}{n} \cdot \sum_{k=1}^{n-1} k\zeta_n^{n-1-k}.$$

Proof.
$$\begin{aligned}(1-\zeta_n)\cdot\sum_{k=1}^{n-1} k\zeta_n^{n-1-k} &= \sum_{k=1}^{n-1} k\zeta_n^{n-1-k} - \sum_{k=1}^{n-1} k\zeta_n^{n-k}\\ &= (n-1) + \sum_{k=1}^{n-2} k\zeta_n^{n-1-k} - \zeta_n^{n-1} - \sum_{k=1}^{n-2}(k+1)\zeta_n^{n-1-k}\\ &= (n-1) - \sum_{k=1}^{n-1}\zeta_n^k = n.\end{aligned}$$
□

Now we are ready to proceed.

Proposition 6.3.5 (Model construction for the Wiman type I maps)**.** *A member of the polynomial family* $\mathbf{Wi1}(n)$ *has planar algebraic model*

$$y^2 = x^{2n+1} - 1.$$

Proof. Let us assume that we have the map $\mathbf{Wi1}(n)$ with $n \geq 2$; the case $n = 0$ is trivial and $n = 1$ is dealt with in Section 6.5. We remember that $\mathbf{Wi1}(n)$ has standard map presentation $\langle R, S \mid R^{2n+1}, S^{4n+1}, (RS)^2, RS^{-2n}\rangle$ (cf. Section 2.2) and that hence $\mathrm{Aut}^+(\mathbf{Wi1}(n)) = \langle S\rangle$ with $R = S^{2n}$. Obviously all automorphisms fix the one vertex of the map, and only the elements of $\langle S^2\rangle$ fix the two faces. By using the fixed point counting lemma (Lemma 1.4.3), we see that S^i fixes $\langle RS\rangle h = \langle S^{2n+1}\rangle h$ (with $h \in \{S^k : k \in \{0,\ldots,2n\}\}$) precisely if $i \in \{0, 2n+1\}$. Since the element S^{2n+1} fixes all edges and the vertex, it has $2n+2 = 2g+2$ fixed points, which confirms again that the map is hyperelliptic. By using the lemma with a little more care, we can compute the rotation indices and canonical character value for the non-trivial elements:

$$\chi_c(S^k) = \begin{cases} 1 + (2n+2)\cdot \frac{\zeta_{4n+2}^{-(2n+1)}}{1-\zeta_{4n+2}^{-(2n+1)}} = -n & k = 2n+1 \\ 1 + 1\cdot \frac{\zeta_{4n+2}^{-k}}{1-\zeta_{4n+2}^{-k}} = \frac{1}{\zeta_{4n+2}^{k}-1} & k \neq 2n+1 \text{ odd} \\ 1 + 3\cdot \frac{\zeta_{4n+2}^{-k}}{1-\zeta_{4n+2}^{-k}} & k \text{ even} \end{cases}$$

By Lemma 6.3.4 this is equal to $\sum_{i=1}^{2n-1} \zeta_{4n+2}^{ik}$ in all cases. A canonical linear representation ρ_c afforded by χ_c is defined by extending the assignment

$$S \mapsto \mathrm{DiaMat}(\zeta_{4n+2}^{2n-1}, \zeta_{4n+2}^{2n-3}, \ldots, \zeta_{4n+2}),$$

which implies $RS = S^{2n+1} = -I$ and

$$R \mapsto \mathrm{DiaMat}\big(-\zeta_{4n+2}^{-(2n-1)}, -\zeta_{4n+2}^{-(2n-3)}, \ldots, -\zeta_{4n+2}^{-1}\big).$$

To define a planar model, we restrict the action to the rational normal curve defined by $\langle e_1, e_2\rangle \subset \mathbb{P}^{g-1}$. Identify it with $\widehat{\mathbb{C}}$ by $z = (z:1)$ and $\infty = (1:0)$. The branch data for the hyperelliptic involution is given by the fixed points of conjugates of S and RS. The unique fixed point of type S on the rational normal curve can be chosen to be either e_1 or e_2; we choose $e_1 = (1:0) = \infty$. The fixed points of $RS = -I$ are undetermined, we may choose any $\mathrm{Aut}^+(\mathbf{Wi1}(n))$-orbit of the correct size. Picking

$$(1:1)^{\mathrm{Aut}^+(\mathbf{Wi1}(n))} = \{(\zeta_{4n+2}^{k} : \zeta_{4n+2}^{3k}) : k \in \{0,\ldots,2n\}\} = \{(\zeta_{2n+1}^{k} : 1) : k \in \{0,\ldots,2n\}\}$$

we find the planar model

$$y^2 = \prod_{k=0}^{2n}(x - \zeta_{2n+1}^{k}) = x^{2n+1} - 1. \qquad \square$$

Proposition 6.3.6 (Model construction for the Wiman type II maps). *A member of the polynomial family* $\mathbf{Wi2}(n)$ *has planar algebraic model*

$$y^2 = x(x^{2n} - 1).$$

Proof. Remember that $\mathbf{Wi2}(n)$ has standard map presentation

$$\left\langle R, S \,\middle|\, R^4, S^{4n}, (RS)^2, R^{-1}SRS^{-(2n-1)} \right\rangle.$$

We find that $R^2 = S^{2n}$. Since $v = 2$, any power of S fixes both vertices. By the fixed point counting lemma (Lemma 1.4.3), the only non-trivial power that fixes other cells is S^{2n}: it fixes all $2n$ faces. More computation shows that $R^{\pm 1}$ fixes two faces. The element $R^2 = S^{2n}$ therefore has $2n + 2 = 2g + 2$ fixed points; this re-affirms that the map is hyperelliptic. The edge rotation RS fixes only edges. To count how many, we note that any element of $\mathrm{Aut}^+(\mathbf{Wi2}(n))$ can be written either as S^i or RS^i, and solve

$$\begin{aligned} S^i RSS^{-i} = RS &\iff S^{(2n-1)i-i} = R^{-1}S^iRS^{-i} = 1 \\ &\iff (n-1)i \equiv 0 \bmod 2n, \\ RS^jRS(RS^j)^{-1} = RS &\iff S^{(2n-2)(1-j)} = R^{-1}S^{1-j}RS^{j-1} = 1 \\ &\iff (n-1)(1-j) \equiv 0 \bmod 2n. \end{aligned}$$

There are always the solutions $i = 0, 2n$ and $j = 1, 2n+1$, but if n is odd also four additional ones, the given ones multiplied by $\frac{1}{2}(n-1)$. This shows that RS fixes 2 edges if n is odd, 4 when n is even. Extracting the rotation indices from the computations, we find the values of the canonical character χ_c on powers of rotations:

$$\chi_c(R^{\pm 1}) = \begin{cases} \mp i & \text{if } n \equiv 1 \bmod 2 \\ 0 & \text{if } n \equiv 0 \bmod 2 \end{cases}, \quad \chi_c(S^k) = 1 + 1 \cdot \frac{\zeta_{4n}^{-k}}{1-\zeta_{4n}^{-k}} + 1 \cdot \frac{\zeta_{4n}^{-(2n-1)k}}{1-\zeta_{4n}^{-(2n-1)k}},$$
$$\chi_c(RS) = \begin{cases} -1 & \text{if } n \equiv 1 \bmod 2 \\ 0 & \text{if } n \equiv 0 \bmod 2 \end{cases}, \quad \chi_c(R^2) = \chi_c(S^{2n}) = -n.$$

By using Lemma 6.3.4 we find that $\chi_c(S^k) = -\sum_{i=1}^n \zeta_{4n}^{(2i-1)k}$ for all k. An easy canonical representation ρ_c that this character affords the representation generated by

$$R \mapsto \begin{pmatrix} & & & -\zeta_{4n} \\ & & -\zeta_{4n}^3 & \\ & \iddots & & \\ -\zeta_{4n}^{2n-1} & & & \end{pmatrix}, \qquad S \mapsto \begin{pmatrix} -\zeta_{4n} & & & \\ & -\zeta_{4n}^3 & & \\ & & \ddots & \\ & & & -\zeta_{4n}^{2n-1} \end{pmatrix},$$

within which an invariant rational normal curve is defined by the subspace $\langle e_1, e_n \rangle$. We identify it with $\widehat{\mathbb{C}}$ in the usual way, using $\infty = (1:0)$. The branch data for the hyperelliptic involution S^{2n} is given by the vertices and face centers on this rational normal curve, i.e. the fixed points of conjugates of S and R. Eigenspace computations tell us that the two vertices are $(1:0)$ and $(0:1)$, and the face centers form the $\mathrm{Aut}^+(\mathbf{Wi2}(n))$-orbit of $(\pm\zeta_{4n}^{n-1} : 1)$, which consists of all points $(\pm\zeta_{4n}^{(n-1)(1-2i)} : 1)$, where $i = 0, \ldots, 2n-1$. Thus, the map has a planar model

$$y^2 = x \prod_{i=0}^{n-1} \left(x - \zeta_{4n}^{(n-1)(1-2i)}\right)\left(x + \zeta_{4n}^{(n-1)(1-2i)}\right) = x \prod_{i=0}^{n-1} \left(x^2 + \zeta_{2n}^{2i-1}\right).$$

The right hand side is equal to $x(x^{2n}-1)$ if n is odd and $x(x^{2n}+1)$ when n is even, and scaling x by ζ_{4n} in the latter case, the claim is proved. □

Proposition 6.3.7 (Model construction for the Accola-Maclachlan maps)**.** *A member of the polynomial family* $\mathbf{AM}(n)$ *has planar algebraic model*

$$y^2 = x^{2n+2} - 1.$$

Proof. We make it easy for ourselves and work with the family $D(\mathbf{AM}(n))$. It has standard map presentation

$$\langle R, S \mid R^{2n+2}, S^{2n+2}, (RS)^2, [R, S] \rangle$$

and satisfies $R^2 = S^{-2}$, $v = f = 2$, and $e = 2n + 2$. With the fixed point counting lemma (Lemma 1.4.3) we quickly find that R^{2k+1} fixes the two faces and no other cells, S^{2k+1} the two vertices and no other cells, $R^{2k} = S^{2k}$ the vertices and faces, but no edges, and RS all the edges and nothing else. This confirms that the map is hyperelliptic, with RS the hyperelliptic involution. Computing the rotation indices, we find that the canonical character is defined by

$$\begin{aligned}
\chi_c(R^{2k+1}) &= \chi_c(S^{2k+1}) = 1 + 2 \cdot \frac{\zeta_{2n+2}^{-(2k+1)}}{1 - \zeta_{2n+2}^{-(2k+1)}}, \overset{\text{Lemma 6.3.4}}{=} \sum_{i=1}^{n} \zeta_{2n+2}^{-i(2k+1)}, \\
\chi_c(R^{2k}) &= 1 + 4 \cdot \frac{\zeta_{2n+2}^{-2k}}{1 - \zeta_{2n+2}^{-2k}}, \\
\chi_c(RS) &= 1 + (2n+2) \cdot \frac{\zeta_{2n+2}^{-(n+1)}}{1 - \zeta_{2n+2}^{-(n+1)}} = -n, \\
\chi_c(R^{2k+1}S) &= 1 \qquad (k \in \{0, \ldots, n\}).
\end{aligned}$$

A linear representation ρ_c afforded by χ_c can be generated by

$$\begin{aligned}
R &\mapsto \operatorname{DiaMat}(\zeta_{2n+2}^{-1}, \zeta_{2n+2}^{-2}, \ldots, \zeta_{2n+2}^{-n}), \\
S &\mapsto \operatorname{DiaMat}(-\zeta_{2n+2}, -\zeta_{2n+2}^{3}, \ldots, -\zeta_{2n+2}^{n}).
\end{aligned}$$

To define the hyperelliptic curve, restrict to the rational normal curve defined by the subspace $\langle e_1, e_2 \rangle$, and identify it with $\widehat{\mathbb{C}}$ in the usual way. The branch data for the covering of $\mathbb{P}^1$ consists of the edge centers that lie in this space. But since RS fixes all points, we can freely choose any orbit of the right size, so we pick

$$(1:1)^{\operatorname{Aut}^+(D(\mathbf{AM}(n)))} = \{(\zeta_{2n+2}^{k} : 1) : k \in \{0, \ldots, 2n+1\}\}.$$

This gives us a planar model

$$y^2 = \prod_{k=0}^{2n+1} (x - \zeta_{2n+2}^{k}) = x^{2n+2} - 1. \qquad \square$$

6.4 Genus 0

The platonic maps of genus zero have algebraic realizations as the only algebraic curve of genus zero, the projective line $\mathbb{P}^1$. As a Riemann surface, $\mathbb{P}^1 \cong \widehat{\mathbb{C}}$, the Riemann sphere. Remember that $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We identify $z \in \mathbb{C}$ with $(z:1) \in \mathbb{P}^1$ and ∞ with $(1:0) \in \mathbb{P}^1$. The Riemann sphere has the chordal metric ρ as natural metric, under which it has diameter 2. The Möbius transformations, which constitute the automorphism group $\mathrm{Aut}(\mathbb{P}^1) \cong \mathrm{PGL}(2,\mathbb{C})$, are isometries with respect to ρ. Using this information, we can compute the cell structure of the algebraic model for the spherical platonic map $\mathbf{R}$ without herculean effort, as well as a concrete representation $\mathrm{Aut}(\mathbf{R}) \rightarrowtail \mathrm{PGL}(2,\mathbb{C})$. We start with the object in a polyhedral shape in $\mathbb{R}^3$ so that the vertices lie on the unit sphere S^2. We then transfer the vertices, edge centers and face centers to $\widehat{\mathbb{C}}$ by stereographic projection st from the point $(0,0,1)$:

$$\mathrm{st} : S^2 \longrightarrow \widehat{\mathbb{C}}$$
$$(x,y,z) \mapsto \frac{1}{1-z}(x,y)$$

All this material is standard. The only small innovation we might claim is the presentation of cell data with a polynomial. In this form, the information can be used to construct planar models for other platonic maps. See Appendix A.

Hos(n) **type** $(2,n)$ **#cells** $(2,n,n)$ **map group size** $4n$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^2, S^n, (RS)^2 \rangle$

For the algebraic hosohedron $\mathbf{Hos}(n)$ we choose $v_0 = 0$, $v_1 = \infty$ and edges

$$e_k = \mathbb{R} \cdot \zeta_{2n}^{2k+1} = \{z \in \widehat{\mathbb{C}} : \mathrm{Arg}(z) = (2k+1)\pi/2n\}, \qquad (k \in \{0,\dots,n-1\}),$$

each with edge center ζ_{2n}^{2k+1}. The face centers are then ζ_n^k. The map $\mathbf{Hos}(6)$ is illustrated in Figure 6.1. A standard map presentation of $\mathrm{Aut}^+(\mathbf{Hos}(n))$ acting on this

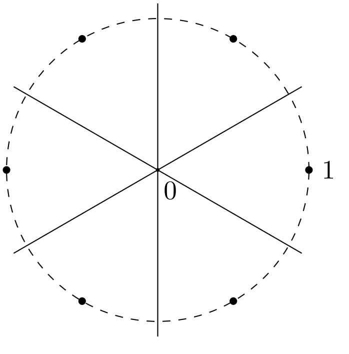


Figure 6.1: The hosohedron $\mathbf{Hos}(6)$ and its dual dihedron (dashed) realized on $\widehat{\mathbb{C}}$.

model can be realized by using the following Möbius transformations:

$$R : z \mapsto \zeta_n/z, \qquad S : z \mapsto \zeta_n z.$$

For the dihedron $\mathbf{Dih}(n)$, vertices and face centers swap places. The edge centers stay the same, but the edges are then the circle segments

$$\{z \in \widehat{\mathbb{C}} : |z| = 1 \ \wedge\ k\pi/n < \mathrm{Arg}(z) < (k+1)\pi/n\}, \qquad (k \in \{0,\dots,n-1\}).$$

All information about the vertex sets of a hosohedron and its dual dihedron is summarized by the polynomials

$$\prod_{c\in\mathrm{cells}_0(\mathbf{Hos}(n))} (c_2x - c_1z) = -xz,$$
$$\prod_{c\in\mathrm{cells}_0(\mathbf{Dih}(n))} (c_2x - c_1z) = x^n - z^n.$$

Tet type $(3,3)$ **#cells** $(4,6,4)$ **map group size** 24
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^3, S^3, (RS)^2 \rangle$

The vertices of the tetrahedron can be taken to be $v_k = \sqrt{2}\zeta_3^k$ for $k = 0,1,2$ and $v_3 = 0$. The edge length of this tetrahedron with respect to the chordal metric ρ is $\frac{2}{3}\sqrt{6}$. Label the faces with the same number as the opposite vertex. Then these are $c(f_3) = \infty$ and $c(f_k) = -\frac{1}{2}\sqrt{2}\zeta_3^k$ for $k = 0,1,2$. They form the vertices of a dual tetrahedron, having the same edge length. The map is illustrated in Figure 6.2. The automorphism group $\mathrm{Aut}^+(\mathbf{Tet})$ can be realized by using the following Möbius

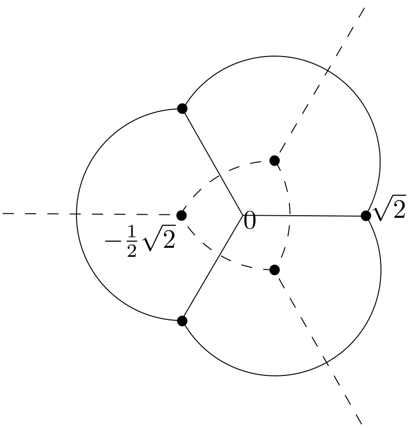


Figure 6.2: The platonic map **Tet** and its dual tetrahedron (dashed) realized on $\widehat{\mathbb{C}}$.

transformations for a standard map presentation:

$$R_0 : z \mapsto \frac{-\zeta_3 z + \sqrt{2}}{\sqrt{2}z + \zeta_3^2}, \qquad S_3 : z \mapsto \zeta_3 z.$$

All information about the vertex sets of the tetrahedron and its dual tetrahedron is summarized by the polynomials

$$\prod_{c\in\mathrm{cells}_0(\mathbf{Tet})} (c_2x - c_1z) = x(x^3 - 2\sqrt{2}z^3),$$
$$\prod_{c\in\mathrm{cells}_0(\mathbf{Tet}^\vee)} (c_2x - c_1z) = -z(x^3 + \frac{1}{4}\sqrt{2}z^3).$$

Remark 6.4.1. A peculiar detail of Figure 6.2 is that each face center lies on a euclidean straight line between two vertices.

Oct type $(3,4)$ **#cells** $(6,12,8)$ **map group size** 48
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^3, S^4, (RS)^2 \rangle$

We can take the vertices to be $v_k = i^k$ (with $k = 1,2,3,4$) plus $v_0 = 0$ and $v_5 = \infty$. The edges are the line segments from v_0 and v_5 to each of v_1, v_2, v_3, v_4, and the four circle segments of $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ that connect v_1, v_2, v_3, v_4. In the chordal metric, the edge length is $\sqrt{2}$. The face centers of the octahedron can be computed by considering the octahedron realized on the unit sphere $S^2 \subset \mathbb{R}^3$ with vertices $\pm e_k$ ($k = 1,2,3$). In $\mathbb{R}^3$ the face centers must then obviously be $\frac{1}{3}\sqrt{3}(\pm 1, \pm 1, \pm 1)$. Using stereographic projection, we obtain the complex numbers

$$\left(s_1 \frac{1}{2} + \frac{1}{2}\sqrt{3}\right)(s_2 + s_3 i), \qquad s_1, s_2, s_3 \in \{\pm 1\}.$$

We number the faces according to the triple (s_1, s_2, s_3) defining their center. The face centers form the vertices of a dual cube with edge length $\frac{2}{3}\sqrt{3}$. The map is illustrated in Figure 6.3. The automorphism group $\mathrm{Aut}^+(\mathbf{Oct})$ can be realized by

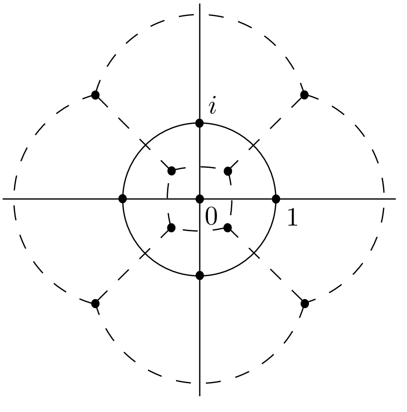


Figure 6.3: The platonic maps **Oct** and **Cub** (dashed) realized on $\widehat{\mathbb{C}}$.

using the following Möbius transformations for a standard map presentation:

$$R_{(-1,1,1)} : z \mapsto \frac{iz+1}{-iz+1}, \qquad S_0 : z \mapsto iz.$$

All information about the vertex sets of the octahedron and its dual cube is summarized by the polynomials

$$\prod_{c \in \mathrm{cells}_0(\mathbf{Oct})} (c_2 x - c_1 z) = -xz(x^4 - z^4),$$

$$\prod_{c \in \mathrm{cells}_0(\mathbf{Cub})} (c_2 x - c_1 z) = x^8 + 14x^4 z^4 + z^8.$$

Remark 6.4.2. A peculiar detail of the figure is that the euclidean circle center of each big circular arc representing an edge of the cube is a vertex of the octahedron, the one closest to the edge center. And the euclidean circle center of each small circular arc is also a vertex of the octahedron, the one on the unit circle farthest away from the edge center.

Ico type $(3, 5)$ **#cells** $(12, 30, 20)$ **map group size** 120
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^3, S^5, (RS)^2 \rangle$

To find algebraic expressions for the vertices is harder for the icosahedron and dual dodecahedron than it was for the previous maps. Two antipodal vertices of the icosahedron can be chosen as $v_0 = 0$, $v_{12} = \infty$. The other vertices then lie in two concentric circles around 0. The radius of the inner circle can be calculated by solving

$$\rho(0, v_1) = \rho(v_1, \zeta_5 v_1)$$

By rotation, we may choose one vertex on this inner circle to be in $\mathbb{R}_{>0}$. This results in the inner and outer circle

$$\left(-\frac{11}{2} + \frac{5}{2}\sqrt{5}\right)\zeta_5^k \text{ and } \left(-\frac{11}{2} - \frac{5}{2}\sqrt{5}\right)\zeta_5^k \text{ respectively} \quad (0 \leq k \leq 4)$$

For the vertices of the dual dodecahedron, start with the well-known polyhedral realization in $\mathbb{R}^3$ that has vertices

$$(\pm 1, \pm 1, \pm 1), \quad (\pm 1/\phi, \pm\phi, 0), \quad (0, \pm 1/\phi, \pm\phi), \quad (\pm\phi, 0, \pm 1/\phi),$$

where $\phi = \frac{1}{2} + \frac{1}{2}\sqrt{5}$ is the golden ratio (cf. Heath's comment in [Eὐκ300BCE]). To get the correct dual after stereographic projection, we first rotate this dodecahedron around the y-axis over $\arctan(1/\phi)$. This is the linear transformation

$$v \mapsto v \begin{pmatrix} \frac{\phi}{\sqrt{1+\phi^2}} & 0 & \frac{1}{\sqrt{1+\phi^2}} \\ 0 & 1 & 0 \\ \frac{-1}{\sqrt{1+\phi^2}} & 0 & \frac{\phi}{\sqrt{1+\phi^2}} \end{pmatrix}$$

The top and bottom face are now horizontal and the dodecahedron is aligned correctly for our purpose. Stereographic projection results in four sets of five points in $\widehat{\mathbb{C}}$, each set on a circle with center 0. We list them in order of increasing circle radius, using $\psi = \sqrt{3}\sqrt{1+\phi^2}$ for brevity:

$$\frac{1}{2}(1 + \phi - \psi)\zeta_5^k, \quad \frac{1}{2}(2 - \phi + (1-\phi)\psi)\zeta_5^k, \quad \frac{1}{2}(2 - \phi - (1-\phi)\psi)\zeta_5^k, \quad \frac{1}{2}(1 + \phi + \psi)\zeta_5^k$$

These points form the face centers of our icosahedron; see Figure 6.4. The automorphism group $\mathrm{Aut}^+(\mathbf{Ico})$ can be realized by using the following Möbius transformations for a standard map presentation:

$$R : z \mapsto \frac{-z + \zeta_5(5\phi - 8)}{(2\phi + 1)z + \zeta_5}, \qquad S : z \mapsto \zeta_5 z.$$

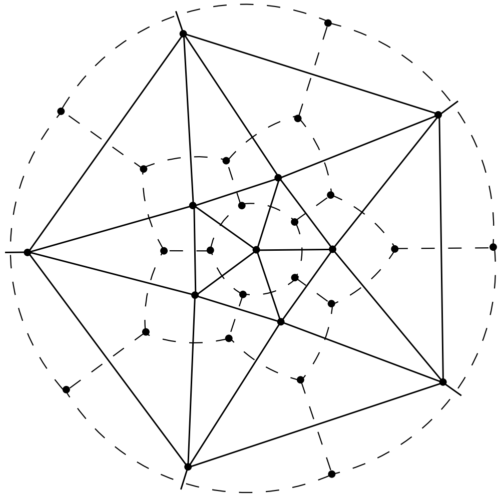


Figure 6.4: The platonic maps **Ico** and **Dod** (dashed) realized on $\widehat{\mathbb{C}}$. The picture was distorted so as to fit the page.

All information about the vertex sets of the icosahedron and its dual dodecahedron is summarized by the polynomials

$$\prod_{c\in\mathrm{cells}_0(\mathbf{Ico})} (c_2x - c_1z) = -xz(x^{10} + 11x^5z^5 - z^{10}),$$
$$\prod_{c\in\mathrm{cells}_0(\mathbf{Dod})} (c_2x - c_1z) = x^{20} - 228x^{15}z^5 + 494x^{10}z^{10} + 228x^5z^{15} + z^{20}.$$

6.5 Genus 1

A Riemann surface of genus 1 is a complex torus $\mathbb{C}/\Lambda$, where $\Lambda \subset \mathbb{C}$ is a lattice. A torus with lattice $\langle z_1, z_2\rangle$ is isomorphic to the torus with lattice $\langle 1, z_2/z_1\rangle$: the isomorphism is induced by the entire holomorphic function $z \mapsto z/z_1$. We can choose the basis $\langle z_1, z_2\rangle$ so that $\mathrm{Arg}(z_2 - z_1) < \pi$ and thereby ensure that we have a standard basis $\Lambda = \langle 1, \tau\rangle$ with τ in the upper-halfplane $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. The number τ is called the *modulus* of the torus. It is well known that two complex tori are isomorphic if and only if $\tau_2 = T(\tau_1)$ with $T \in \mathrm{SL}(2,\mathbb{R})$. The *moduli space* $\mathcal{M}_1$ parametrizing complex tori is thus

$$\mathcal{M}_1 = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\} \;/\; \mathrm{SL}(2,\mathbb{Z}).$$

An algebraic curve that serves as a model for a complex torus is an *elliptic curve*. Historically, this name derives from the study of integrals relevant to ellipses. An elliptic curve can be brought into several standard forms, one of which is the planar model called the *Weierstraß normal form*:

$$y^2 = x^3 + ax + b \qquad (a, b \in \mathbb{C}).$$

Given a complex torus with lattice Λ, the *modular invariants* $a = a(\Lambda)$ and $b = b(\Lambda)$ can be computed from the lattice as follows:

$$a(\Lambda) = -240 \sum_{w \in \Lambda - \{0\}} w^{-4}, \qquad b(\Lambda) = -2240 \sum_{w \in \Lambda - \{0\}} w^{-6}.$$

Platonic maps make use of only two lattices, as witnessed in Section 1.7. Interpreted as complex lattices, these are the lattice $\langle 1, \zeta_6 \rangle$ (the *Eisenstein integers*) for maps of type $(3, 6)$ and their duals, and the lattice $\langle 1, i \rangle$ (the *Gaussian integers*) for maps of type $(4, 4)$. Computing the above sums is difficult in general, but for these two lattices it is still tractable. We find that all platonic maps of type $(3, 6)$ or $(6, 3)$ have the algebraic model

$$y^2 = x^3 + 1$$

whereas all platonic maps of type $(4, 4)$ have the model

$$y^2 = x^3 + 3 \cdot 1727x + 2 \cdot 1727^2.$$

One can check these calculations afterwards by making use of the *j-function*

$$j(a, b) = 1728 \frac{4a^3}{4a^3 + 27b^2}.$$

It can be proved that the j-invariant is in fact a holomorphic function $j(\tau)$ on the upper-halfplane that depends only on the isomorphism class of the torus with lattice $\langle 1, \tau \rangle$. The j-invariant maps the moduli space $\mathcal{M}_1$ biholomorphically onto $\mathbb{C}$, so its value uniquely determines the torus. The relevant values are $j(\zeta_6) = 0$ and $j(i) = 1$, and these are indeed what the elliptic curves above yield.

7

Weierstraß points & The first Hurwitz triplet

HAVING a canonical model for a platonic map enables one to solve new problems concerning it. In this chapter, we take up the study of Weierstraß points on Riemann surfaces. Section 7.1 presents an introduction to the subject. In the rest of the chapter we document our study of the *first Hurwitz triplet*. In Section 7.2 we detail the construction of a nice canonical model for it, which was particularly hard. Then the extra problem arose, as we shall see, of distinguishing between the triplet members. The topic is treated in Section 7.3. We finish with an investigation of the Weierstraß points on the triplet in Section 7.4.

7.1 Weierstraß points

A Riemann surface X of genus $g \geq 2$ has certain special points, called Weierstraß points. They are key to the understanding of the function field $\mathcal{M}(X)$ of meromorphic functions on X. To explain the relevant concepts, we take a short detour through this whole field of study. For all the details, cf. [FK1980] or [Mir1995].

Definition 7.1.1. *A* divisor *is a formal sum* $D = \sum_{p \in X} n_p \cdot p$ *of points on* X*, where* $n_p \in \mathbb{Z}$ *is non-zero for only a finite number of* p*. The* degree *of a divisor is* $\deg(D) = \sum_{p \in X} n_p$.

A function $f \in \mathcal{M}(X)$ defines a divisor $\operatorname{div}(f) := \sum_{p \in X} \operatorname{ord}_p(f)$, where $\operatorname{ord}_p(f)$ is the usual order of a function at a point p and we set $\operatorname{ord}_p(0) = \infty$. For any non-constant f, we have $\deg(\operatorname{div}(f)) = 0$. A divisor coming from a meromorphic function in this way is called a *principal divisor*. We take a look at the relationship on all divisors induced by the notion of principal divisor.

Definition 7.1.2. *Two divisors are defined to be* linearly equivalent *if their difference is a principal divisor.*

This is indeed an equivalence relation. Linearly equivalent divisors have the same degree, but the converse is false. A special equivalence class of divisors, all linearly equivalent, is that of *canonical divisors*. A canonical divisor is one that comes from a meromorphic differential form $\omega \in \Omega^1(X)$ in a way analogous to a principal divisor: use a local expression $\omega = f(z)\mathrm{d}z$ on a chart and count zero and pole orders. One has to check that this is well-defined and that all canonical divisors are linearly equivalent. A canonical divisor K has degree $\deg(K) = 2g - 2$.

We will interpret divisors as restrictions on sets of meromorphic functions. This is their raison d'être. First, a partial order on divisors is obtained by setting $D_1 \geq D_2$ if this holds pointwise.

Definition 7.1.3. *Let D be a divisor on X. We set $L(D) := \{f \in \mathcal{M}(X) \mid \mathrm{div}(f) \geq -D\}$.*

The spaces $L(D)$ are complex vector spaces. They always contain the zero function. But what about more interesting functions on X with the prescribed order behavior? We want to know the dimensions of these spaces. The theorem that governs them is the famed Riemann-Roch theorem.

Theorem 7.1.4 (Riemann-Roch)**.** *Let X be a Riemann surface of genus g, D a divisor and K a canonical divisor. Then*

$$\dim L(D) - \dim L(K - D) = \deg(D) - (g - 1).$$

If E is a divisor and $\deg(E) < 0$, then $f \in L(E)$ must satisfy $\deg(\mathrm{div}(f)) \geq \deg(-E) > 0$, but if f is non-constant its degree is zero. Hence, $L(E) = \{0\}$. As a corollary, whenever D is a divisor of degree $\deg(D) \geq 2g - 1$, we know $\deg(K - D) < 0$, whence $\dim L(K - D) = 0$ and $\dim L(D) = \deg(D) - (g - 1)$. A particular instance of this is when $D = nP$ for a point $P \in X$. The space $L(np)$ (with $n \in \mathbb{Z}_{\geq 0}$) is that of meromorphic functions f holomorphic on $X - \{p\}$ and with $\mathrm{ord}_p(f) \geq -n$. We will focus on such divisors. The dimensions $d_n = \dim L(np)$ apparently form a sequence

$$1 = d_0, d_1, d_2, \ldots, d_{2g-2}, d_{2g-1} = g, g + 1, g + 2, \ldots$$

So the only information missing is what $d_2, \ldots, d_{2g-2}$ are. A Laurent expansion around p shows that $d_{i+1} \leq d_i + 1$, so that $d_{i+1} \in \{d_i, d_i + 1\}$. If follows that there are exactly g integers $m_1, \ldots, m_g \in \{1, \ldots, 2g - 1\}$ for which $d_{m_i} = d_{m_i - 1}$. These are the *gap numbers* at p, forming the *gap sequence*. If m is a gap number, there is no function holomorphic outside of p and of pole order exactly m at p. For almost all points of X, the gap sequence is $1, 2, \ldots, g$: there are usually no functions with small pole order at p and no other poles. This motivates the following definitions.

Definition 7.1.5. *Let X be a Riemann surface. The* Weierstraß weight *of a point $p \in X$ is $\mathrm{wt}(p) := \sum_{i=1}^{2g-1}(m_i - i)$. The point is a* Weierstraß point *if it has non-zero Weierstraß weight, i.e. if the gap sequence $(m_1, \ldots, m_g)$ at p is not equal to $(1, 2, \ldots, g)$.*

Theorem 7.1.6. *Let X be a Riemann surface. Its total Weierstraß weight is*

$$\sum_{p \in X} \mathrm{wt}(p) = g(g-1)(g+1).$$

Our goal is to find (all) Weierstraß points of a Riemann surface $\mathbf{M}_r$ and compute their weights. We need a more algebraic handle on things to compute them. Let $F = (f_1, \ldots, f_s)$ be a sequence of meromorphic functions $f_i(z)$ defined on an open set $U \subset \mathbb{C}$, and let $m = (m_1, \ldots, m_s)$ be an increasing sequence of non-negative integers. The *generalized Wronskian determinant* of the two sequences is defined to be

$$W(F, m) = \det \begin{pmatrix} \frac{d^{m_1} f_1}{dz^{m_1}} & \cdots & \frac{d^{m_1} f_s}{dz^{m_1}} \\ \frac{d^{m_2} f_1}{dz^{m_2}} & \cdots & \frac{d^{m_2} f_s}{dz^{m_2}} \\ \vdots & & \vdots \\ \frac{d^{m_s} f_1}{dz^{m_s}} & \cdots & \frac{d^{m_s} f_s}{dz^{m_s}} \end{pmatrix}.$$

The generalized Wronskian for the sequence $m_0 = (0, 1, \ldots, s-1)$ is the determinant classically known as the Wronskian. The classical Wronskian exhibits a particular form of behavior.

Definition 7.1.7. *Let X be a Riemann surface. Suppose we have a set of expressions $f_i(z)(dz_i)^n$, where $i = 1, \ldots, s$, given on the domains $V_i \subset \mathbb{C}$ of charts $z_i : U_i \to V_i$ covering X. If $f(z) = \tilde{f}(T(z))(dT/dz)^n$ holds for every transition map $T = \tilde{z} \circ z^{-1}$, then we call the set an n-*fold differential *on X.*

The classical Wronskian $W(F, m_0)$ defined on a chart z of X turns out to extend to an $s(s-1)/2$-fold differential on X ([Mir1995, Lemma VII.4.9]). This implies that it has a well-defined zero locus. More generally one can speak of the order of vanishing of the classical Wronskian differential. The Wronskian determinant is relevant to us because of the following classical theorem.

Theorem 7.1.8 (Hurwitz)**.** *Let X be a Riemann surface of genus g, $p \in X$ with a neighborhood U and chart $z : U \to \mathbb{C}$. Choose a sequence of meromorphic functions $F = (f_1, \ldots, f_g)$ on $z(U)$ such that $f_i \mathrm{d}z$ defines a meromorphic 1-form on X for $i = 1, \ldots, g$, and such that $(f_i \mathrm{d}z)_{i=1}^{g}$ is a basis of $\Omega^1(X)$. Then the Weierstraß weight $\mathrm{wt}(p)$ is equal to the vanishing order of the Wronskian $W(F, m_0)$ in p.*

An extra tool for computation is provided by Christopher Towse in [Tow2000]. We define the weight of the sequence m just like the Weierstraß weight; as $\mathrm{wt}(m) := \sum_{i=1}^{s}(m_i - i)$.

Theorem 7.1.9 (Towse)**.** *Let F be as in the previous theorem. The vanishing order of the classical Wronskian $W(F, m_0)$ in a point z is equal to the minimal number in $\mathbb{Z}_{\geq 0}$ for which there is a sequence m of that weight such that the generalized Wronskian $W(F, m)(z)$ is non-zero.*

As a corollary, a point on X is Weierstraß precisely when the generalized Wronskian determinant with respect to the sequence $m = (0, \dots, g-1)$ is zero.

The easiest candidates for Weierstraß points on a platonic surface $\mathbf{M}_r$ are algebraic points on the curve that we can compute, such as fixed points of automorphisms. Following Singerman and Watson [SW1997], we define:

Definition 7.1.10 (Singerman, Watson)**.** *A Weierstraß point of the Riemann surface X is* geometric *if it is fixed by an automorphism of X.*

That these are in fact viable candidates is made believable by the following theorem, originally proved in [Sch1951] and [Lew1963]. Read [FK1980, Theorem V.I.7].

Theorem 7.1.11 (Lewittes, Schoeneberg)**.** *Let X be a Riemann surface. If an automorphism fixes more than 4 points, then all of its fixed points are Weierstraß.* □

So we can even read off from the combinatorics of the map, using the Fixed Point Counting Lemma 1.4.3, that certain cell centers are Weierstraß.

Example 7.1.12. A theorem of Hasse [Has1950] says that all vertices of the map $\mathbf{Fer}(n)$ (all points on the curve $x^n + y^n + z^n = 0$ with $xyz = 0$) have Weierstraß weight

$$\frac{1}{24}(n-1)(n-2)(n-3)(n+4).$$

Towse [Tow2000] also computes lower bounds for the Weierstraß weight of the edge centers of $\mathbf{Fer}(n)$.

For a hyperelliptic map (see Section 6.3), the Weierstraß weights are known in advance.

Fact 7.1.13 ([FK1980, III.7])**.** *Let X be a hyperelliptic Riemann surface. Its Weierstraß points are the $2g+2$ fixed points of the hyperelliptic involution, each of Weierstraß weight $\frac{1}{2}g(g-1)$.*

The weight $\frac{1}{2}g(g-1)$ is the maximal possible weight of a point.

Example 7.1.14. The Weierstraß points of the hyperelliptic platonic map families (see Sections 6.3 and 5.1–5.3) are the centers of cells with the following dimensions:

$\mathcal{F}_n^{(2n+1,4n+2)}$	$\mathcal{F}_n^{(4,4n)}$	$\mathcal{F}_n^{(4n,4n)}$	$\mathcal{F}_n^{(4,2n+2)}$	$\mathcal{F}_n^{(2n+2,2n+2)}$
0,1	0,2	0,1,2	2	1

We calculated the Weierstraß weights of cell centers for all reflexive platonic maps of genus at most 8, listed in Appendix B. In general, one cannot expect all Weierstraß points to be geometric. The reflexive platonic surfaces with non-geometric Weierstraß points of genus $g \leq 8$ are those of the maps $\mathbf{R}_{5.11}$, $\mathbf{R}_{6.2}$, $\mathbf{R}_{6.3}$, $\mathbf{R}_{6.7}$, $\mathbf{R}_{6.10}$, $\mathbf{R}_{7.3}$, $\mathbf{R}_{7.6}$, $\mathbf{R}_{7.8}$, $\mathbf{R}_{8.1}$, $\mathbf{R}_{8.2}$, $\mathbf{R}_{8.5}$, $\mathbf{R}_{8.6}$, $\mathbf{R}_{8.7}$, and $\mathbf{R}_{8.8}$. Calculation of non-geometric

Weierstraß points is a major challenge. We take it up for the first Hurwitz triplet in Section 7.4.

7.2 A canonical model for the first Hurwitz triplet

The *first Hurwitz triplet* is the triple of reflexive platonic maps $\mathbf{R}_{14.1}$, $\mathbf{R}_{14.2}$ and $\mathbf{R}_{14.3}$. Why do they bear this name? They are maps of type $(3,7)$, and hence give rise to Hurwitz surfaces, introduced in Section 1.4. We let $\mathbf{R}$ stand for a yet undetermined triplet member. The group $\mathrm{Aut}^+(\mathbf{R})$ is isomorphic to $\mathrm{PSL}(2,13)$. The triplet is the first occurrence of an infinite collection of Hurwitz surfaces.

Theorem 7.2.1 (Macbeath)**.** *The group* $\mathrm{PSL}(2,q)$ *is a Hurwitz group precisely when one of the following holds:*

- $q = 7$*;*
- $q \equiv \pm 1 \bmod 7$ *and* q *is prime;*
- $q = p^3$*,* $p \in \{\pm 2, \pm 3\} \bmod 7$ *and* p *is prime.*

In the first and third case, there is a unique normal torsion free subgroup $\Gamma \lhd \Delta(3,7,2)$ *with quotient* $\Delta(3,7,2)/\Gamma \cong \mathrm{PSL}(2,q)$*. In the second case, there are exactly three such subgroups.*

We know that the case $q = 7$ yields the Klein quartic $\mathbf{R}_{3.1}$. The Fricke-Macbeath curve $\mathbf{R}_{7.1}$ is the first example of the third case, with $p = 2$. The next in line is the prime $q = 13$, which belongs to the second case. That is why the first Hurwitz triplet deserves its name. The connection between the algebraic curves of each Hurwitz triplet is one of algebraic conjugacy. We shall see this exemplified below. For more details, see [Mac1969], [Str2000], and [Dža2007].

Applying the construction strategy from Section 6.2 to the first Hurwitz triplet is quite a challenge. We describe the quest for a clean and simple model in more detail. The canonical character χ_c for all three triplet members $\mathbf{R}_{14.1}$, $\mathbf{R}_{14.2}$, $\mathbf{R}_{14.3}$ is the irreducible character of $\mathrm{PSL}(2,13)$ defined by:

Element order	1	2	3	6	7	13
χ_c	14	-2	-1	1	0	1

The group $\mathrm{PSL}(2,13)$ has three different standard map presentations for a genus 14 map, distinguishable by $\mathrm{ord}([R,S])$. Geometrically, this order equals half the Petrie length (cf. Section 1.4). It is 6 for $\mathbf{R}_{14.1}$, 13 for $\mathbf{R}_{14.2}$, and 7 for $\mathbf{R}_{14.3}$. Several canonical representations are promising candidates to work with. The canonical character is in fact monomial, induced from a 1-dimensional character of a Borel subgroup of $\mathrm{PSL}(2,13)$. This leads to the following canonical representation ρ_c:

$$R \mapsto \mathrm{MonMat}([1,\zeta_6,-1,-\zeta_6,\zeta_6^2,-\zeta_6^2,1,1,1,-\zeta_6^2,\zeta_6^2,-\zeta_6,-1,\zeta_6],[7,5,2,4,3,14,8,1,6,13,11,10,12,9])$$
$$S \mapsto \mathrm{MonMat}([\zeta_6,-1,1,\zeta_6,-\zeta_6^2,\zeta_6,1,-1,-\zeta_6^2,-\zeta_6,\zeta_6^2,\zeta_6^2,-\zeta_6,-\zeta_6^2],[4,14,6,8,1,13,2,7,3,11,12,9,10,5])$$

Its virtue is of course its sparsity, but a field extension is used. The canonical representation is in fact rational, and another canonical representation ρ_c, which we owe to Stephen Glasby [private communication], sends the generator pair (R, S) to:

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&-1&-1&-1\\ -1&0&-1&0&0&0&0&0&1&0&0&-1&0&-1\\ -1&1&0&0&1&1&0&0&0&0&-1&1&1&0\\ 0&-1&1&0&0&0&1&-1&0&0&0&0&0&0\\ 0&1&-1&1&1&-1&0&1&-1&1&-1&0&0&0\\ 1&-1&0&0&-1&-1&0&0&0&0&0&-1&0&0\\ 0&-1&0&-1&-1&0&0&0&1&-1&1&-1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0\\ 0&1&0&1&1&0&-1&0&-1&1&0&0&-1&-1\\ 0&1&0&0&0&0&-1&0&0&0&0&0&0&0\\ 0&0&-1&0&0&-1&0&0&0&0&0&-1&0&0\\ 0&0&1&0&0&1&0&-1&0&0&1&0&-1&0\\ 0&0&0&0&1&0&1&0&0&0&-1&0&0&0\\ 0&0&0&1&0&-1&0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&0&0&0&-1&0&0&0&0&0&0&0&0\\ 0&0&-1&0&0&-1&-1&0&0&0&0&-1&0&-1\\ 0&0&1&0&0&1&0&-1&0&-1&1&0&-1&0\\ -1&0&0&0&0&0&1&0&1&0&0&0&1&0\\ 0&0&1&-1&0&1&0&0&0&-1&0&1&0&1\\ 0&0&1&0&0&0&1&0&0&0&0&1&1&1\\ 1&-1&0&0&0&-1&0&0&0&0&0&0&0&0\\ 0&-1&2&0&0&1&1&-1&0&-1&1&1&0&1\\ 0&0&0&0&-1&0&-1&0&0&0&1&-1&-1&-1\\ 1&0&0&0&-1&0&-1&0&0&0&1&-1&-1&0\\ 0&-1&0&-1&-1&0&0&0&1&-1&0&0&1&1\\ 0&0&-2&0&-1&-1&-1&1&0&1&0&-1&0&-1\\ -1&0&1&-1&0&1&1&-1&1&-1&0&1&1&1\\ 0&0&0&0&0&0&0&0&0&0&0&-1&0&0 \end{pmatrix}$$

The matrix pairs displayed both form standard generator pairs for $\mathbf{R}_{14.1}$, but that is irrelevant for the representation as a whole. Continuing with this rational representation, we compute that $(\rho_c^\vee)^{2+}$ decomposes into isotypic components as

$$1 \oplus 12_1 \oplus 12_2 \oplus 12_3 \oplus 2 \times 13 \oplus 3 \times 14_2.$$

Here 14_2 is the irreducible degree 14 representation with $\chi_{14_2}(RS) = 2$. With the Hilbert-Poincaré series known (see Section 6.1) we see that $I_2(\mathbf{R})$ is a 66-dimensional subspace of $\mathbb{C}[x_1, \ldots, x_{14}]_2$. There are three possible shapes that $I_2(\mathbf{R})$ can take:

- $1 + 12_x + 12_y + 13 + 2 \times 14_2$
- $12_x + 12_y + 3 \times 14_2$
- $12_x + 2 \times 13 + 2 \times 14_2$

In each case, we certainly have some isotypic piece $12_x < I_2(\mathbf{R})$. The three representations 12_1, 12_2, 12_3 are algebraically conjugate, all defined over $\mathbb{Q}(\alpha)$ with $\alpha = \zeta_7 + \zeta_7^{-1}$. They correspond to the three different embeddings of this field into $\mathbb{C}$, namely $\alpha = 2\cos(2\pi/7)$, $\alpha = 2\cos(4\pi/7)$, and $\alpha = 2\cos(6\pi/7)$. Could it be that the choice between these components determines the three members of the triplet? This is indeed the case, as we will make explicit. In any case, let us without loss of generality choose 12_1. We now use our fixed point method. The Fixed Point Counting Lemma 1.4.3 shows that S fixes two vertices on a triplet member. The condition $12_1 \subset I_2(\mathbf{R})$ puts constraints on these fixed points by intersecting their zero sets with the eigenspaces of S. The eigenspaces $E(S, \zeta_7^k)$ are all of dimension 2, for $k = 0, \ldots, 6$. This gives a quadratic equation in one variable for each of them. Precisely two of those equations have a solution, and they have a unique one. The corresponding eigenvalues are each other's inverses. So we can write down the fixed points of S, but it requires a quadratic field extension $\mathbb{Q}(\alpha) < \mathbb{Q}(\alpha, \beta)$ with $\beta^2 = -3\alpha - 2$. Spinning with $\mathrm{Aut}^+(\mathbf{R})$ creates a set V of 156 points that have to form the vertex set of the canonical model. We now compute a basis of the linear subspace

$$\{f \in \mathbb{C}[x_1, \ldots, x_{14}]_2 \mid \forall p \in V, f(p) = 0\}$$

of $\mathbb{C}[x_1, \ldots, x_{14}]_2$, merely a matter of linear algebra. The subspace has dimension 66, and so we have found $I_2(\mathbf{R})$. A further check assures us that $(I_2(\mathbf{R}))$ is a prime ideal with the correct Hilbert polynomial $\mathrm{H}(I(\mathbf{R}), k) = 13(2k-1)$. The ideal is in fact of the form $1 + 12_x + 12_y + 13 + 2 \times 14_2$. One interesting detail that follows is that the curve is contained in a unique $\mathrm{Aut}(\mathbf{R})$-invariant quadric, which we will show later on.

Several things are still wrong with the model: the field of definition contains an unwanted β, and we cannot display any of the defining polynomials we computed due to size limitations. These two issues must and will be addressed.

Galois descent. First we try to apply Galois descent, discussed in Section 6.2, to get rid of β. This solves two problems at once, because the suspicious reader will have noticed that $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}] = 6$. The embeddings into $\mathbb{C}$ give rise to more curves than we bargained for. The solution lies therein, that the field automorphism defined by $\sigma : (\alpha, \beta) \mapsto (\alpha, -\beta)$ can be realized by a projectivity $[A] \in \mathrm{PGL}(14, \mathbb{C})$ carrying each curve C of our set of six into its algebraic conjugate C^σ. We can in fact choose $A \in \mathrm{GL}(14, \mathbb{Q})$:

$$A = \begin{pmatrix}
0 & 0 & 3 & 1 & 2 & 2 & 2 & -2 & 0 & 0 & 0 & 1 & -1 & 1 \\
0 & 1 & 2 & -1 & 0 & 2 & 0 & -1 & -1 & 0 & 1 & 2 & 0 & 1 \\
-1 & 0 & 0 & -1 & -2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & -1 & 0 & -1 & 0 & 1 & -1 & -2 & -1 \\
1 & -1 & -1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\
1 & 0 & 1 & 2 & 2 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\
0 & -1 & 4 & -1 & 1 & 2 & 2 & -2 & 0 & -2 & 1 & 3 & 0 & 2 \\
-1 & -1 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & -2 & 0 & -1 & -1 & -1 & 0 & 0 & 2 & 0 & -2 & 0 & -2 \\
0 & 0 & -2 & -1 & -2 & -1 & -1 & 0 & 2 & 0 & 0 & -2 & 1 & -1 \\
1 & -1 & 1 & 1 & -1 & -1 & 0 & -1 & 0 & -1 & 2 & -1 & -2 & 0 \\
-1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & -2 \\
1 & -1 & 3 & 1 & 0 & 1 & 1 & -1 & 0 & -1 & 1 & 0 & -2 & 1 \\
1 & 0 & -3 & -1 & -2 & -2 & -2 & 1 & 0 & 0 & 0 & -2 & 1 & 0
\end{pmatrix}$$

We have $A/\sqrt{3} \in \mathrm{SL}(14, \mathbb{C})$. For the projectivity, the scalar is of course irrelevant. But we want to use $[A]$ for Galois descent. Remark that $\mathrm{Gal}(\mathbb{Q}(\alpha, \beta)/\mathbb{Q}(\alpha)) = \langle \sigma \rangle \cong \mathbb{Z}_2$, and the homomorphism $\sigma \mapsto [A]$ induces a 1-cocycle $c \in Z^1(\langle \sigma \rangle, \mathrm{PGL}(14, \mathbb{Q}))$. In order to lift this to a 1-cocycle $\widetilde{c} \in Z^1(\langle \sigma \rangle, \mathrm{GL}(14, \mathbb{Q}))$, we need a scalar $\lambda \in \mathbb{Q}(\alpha, \beta)$ for which $\widetilde{c}(\sigma) = \lambda A$ satisfies

$$I = \widetilde{c}(\sigma^2) = \widetilde{c}(\sigma)^\sigma \widetilde{c}(\sigma) = 3\lambda\lambda^\sigma A^\sigma A = 3\lambda\lambda^\sigma I,$$

so we need to solve $\lambda\lambda^\sigma = 1/3$. Write $\lambda = \lambda_0 + \lambda_1\beta$ with $\lambda_0, \lambda_1 \in \mathbb{Q}(\alpha)$. The equation transforms to

$$\lambda_0^2 + (3\alpha + 2)\lambda_1^2 = \frac{1}{3}.$$

This is the equation for a conic. Because we don't want to introduce new algebraic (or transcendental) numbers into the model, our solution better be a $\mathbb{Q}(\alpha)$-rational point on this conic. Deciding whether a conic contains such a point and finding one are possible, and in this case a solution is $(\lambda_0, \lambda_1) = (\frac{1}{6}(\alpha-3), \frac{1}{6}\alpha)$, giving rise to the scalar $\lambda = \frac{1}{6}(\alpha - 3 + \alpha\beta)$.

After we have lifted our 1-cocycle, we now want to split it. Splitting must be possible because $H^1(\langle\sigma\rangle, \mathrm{GL}(14, \mathbb{Q})) = 0$. Writing $\widetilde{c}$ as coboundary is equivalent to solving

$$\lambda A = B^{-1} B^{\sigma}.$$

This problem can be reduced to a linear system of equations over $\mathbb{Q}(\alpha)$ by writing $B = B_0 + \beta B_1$, where $B_0, B_1 \in GL(14, \mathbb{Q}(\alpha))$. The solution space is 196-dimensional, and a new concern arises: to find a solution with small coefficients for B^{-1}, in order to keep coefficients small when substituting in the ideal generators. The best solution we found is

$$B^{-1} = (3 - \alpha + \alpha\beta)\beta I + 2\beta A.$$

Transforming the curve by the transformation B results in a new curve that is σ-invariant. We construct a spanning set over $\mathbb{Q}(\alpha)$ for the new ideal $(I_2(\mathbf{R}))$ by averaging polynomials with σ. The three embeddings of $\mathbb{Q}(\alpha) \rightarrowtail \mathbb{C}$ must define non-isomorphic curves, since we know that there have to be at least three. So the three choices yield the three triplet members. In Section 7.3 we find out which one is which…

New canonical representation. There is also a trade-off to the Galois descent. The transformation is over $\mathbb{Q}(\alpha, \beta)$, and the canonical representation for the new model is not defined over $\mathbb{Q}$ anymore. It is generated by $B^{-1}RB$ and $B^{-1}SB$, matrices best left to the computer.

Coefficient size reduction. The coefficients of the model over $\mathbb{Q}(\alpha)$ that we obtained after Galois descent were rather large, some 30 digits. So we applied the coefficient reduction strategy described in Section 6.2. To this end, we wrote each polynomial $\sum_{1\le i\le j\le 14} c_{ij} x_i x_j$ as a $\mathbb{Q}(\alpha)$-vector by choosing a basis (monomial ordering) of $S^{2+}(\mathbb{C}^{14}[x_1, \ldots, x_{14}])$. Then we added the multiples αv and $\alpha^2 v$ for each vector v of the set and expanded all $\mathbb{Q}(\alpha)$-coefficients $c_{ij} = c_{ij2}\alpha^2 + c_{ij1}\alpha + c_{ij0}$ in terms of the power basis $(1, \alpha, \alpha^2)$ for $\mathbb{Q}(\alpha)$. As a last step, get rid of denominators. The end result was a basis for a $3 \cdot 66$-dimensional integral lattice $\Lambda \subset \mathbb{Z}^{3\cdot 105}$.

The lattice determinant $\det(\Lambda)$ had an order of magnitude 10^{300}. Computing some small prime factors p of $\det(\Lambda)$ (up to 10^8) resulted in multiple non-trivial kernels modulo $(p\mathbb{Z})^{3\cdot 105}$. This led to a reduction that gave the polynomial listed in Appendix A. Although it seemed obvious that we could not do much better, since its $\mathbb{Z}$-coefficients have at most two digits, in order to ensure an optimal set of generators we wanted to know all the prime factors of $\det(\Lambda)$. The new (finer) integral lattice after the reductions had determinant

$$
\begin{gathered}
660140937923054491378810970382293896213219016350956261682271255565499373888 \\
2031788879569117350720871571688748633436086555831108448077393479408580178 7 \\
77951466189845275968141149021205279444622111072514117037566829163810 1.
\end{gathered}
$$

Factoring numbers is a specialist's job. We had the good fortune that our colleague Dan Bernstein wanted to help us out. He computed that the above number is prime. Using it for reduction $\bmod p$ yielded a trivial kernel.

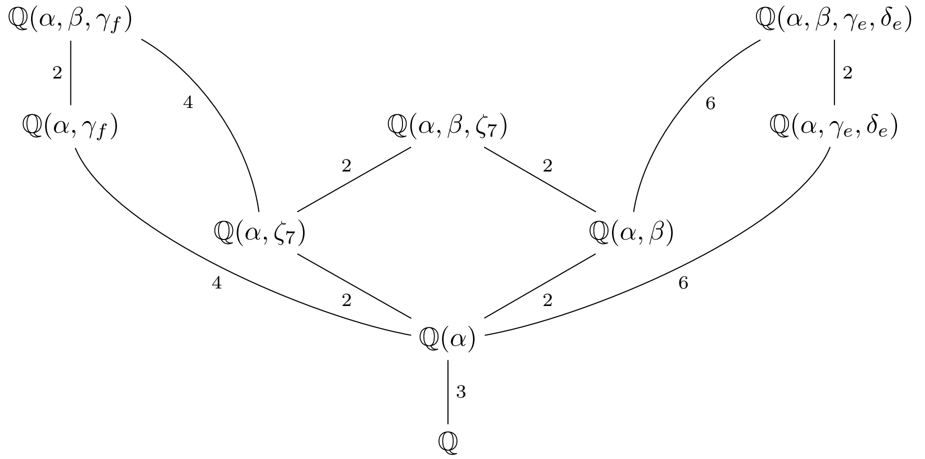
The resulting presentation. A generic polynomial from $I(\mathbf{R})_2$ to generate the canonical model of a Hurwitz triplet member is listed in Appendix A; we will not repeat it here. Instead, we document the unique invariant quadric containing all three triplet members.

$$
\begin{gathered}
(15\alpha^2 - 9\alpha + 9)x_1^2 + (8\alpha^2 - 6\alpha + 4)x_1x_2 + (15\alpha^2 - 9\alpha + 9)x_2^2 + (-8\alpha^2 + 6\alpha - 4)x_1x_3 + (15\alpha^2 - 9\alpha + 9)x_3^2 + \\
(8\alpha^2 - 6\alpha + 4)x_1x_4 + (-6\alpha^2 - 6)x_2x_4 + (-2\alpha^2 + 6\alpha + 2)x_3x_4 + (12\alpha^2 - 9\alpha + 6)x_4^2 + (6\alpha^2 + 6)x_1x_5 + \\
(-8\alpha^2 + 6\alpha - 4)x_2x_5 + (-12\alpha^2 - 12)x_3x_5 + (4\alpha^2 + 6\alpha + 8)x_4x_5 + (6\alpha^2 - 9\alpha)x_5^2 + (14\alpha^2 - 6\alpha + 10)x_1x_6 + \\
(-8\alpha^2 + 6\alpha - 4)x_2x_6 + (-2\alpha^2 + 6\alpha + 2)x_3x_6 + (18\alpha^2 - 9\alpha + 12)x_4x_6 + (6\alpha^2 + 6)x_5x_6 + (12\alpha^2 - 9\alpha + 6)x_6^2 + \\
(8\alpha^2 - 6\alpha + 4)x_1x_7 + (12\alpha^2 - 9\alpha + 6)x_2x_7 + (-8\alpha^2 + 6\alpha - 4)x_3x_7 + (-6\alpha^2 - 6)x_4x_7 + (-2\alpha^2 + 6\alpha + 2)x_5x_7 + \\
(-6\alpha^2 - 6)x_6x_7 + (12\alpha^2 - 9\alpha + 6)x_7^2 + (-14\alpha^2 + 6\alpha - 10)x_1x_8 + (-10\alpha^2 + 3\alpha - 8)x_2x_8 + (18\alpha^2 - 9\alpha + 12)x_3x_8 + \\
(6\alpha^2 + 6)x_4x_8 + (-6\alpha^2 - 6)x_5x_8 + (4\alpha^2 - 3\alpha + 2)x_6x_8 + (-6\alpha^2 - 6)x_7x_8 + (12\alpha^2 - 9\alpha + 6)x_8^2 + (-6\alpha^2 - 6)x_2x_9 + \\
(4\alpha^2 - 3\alpha + 2)x_3x_9 + (-6\alpha^2 - 6)x_4x_9 + (-8\alpha^2 + 6\alpha - 4)x_6x_9 + (-8\alpha^2 + 6\alpha - 4)x_7x_9 + (-6\alpha^2 - 6)x_8x_9 + \\
(12\alpha^2 - 9\alpha + 6)x_9^2 + (-2\alpha^2 + 6\alpha + 2)x_1x_{10} + (-2\alpha^2 + 6\alpha + 2)x_2x_{10} + (8\alpha^2 - 6\alpha + 4)x_3x_{10} + \\
(-12\alpha^2 + 9\alpha - 6)x_4x_{10} + (-2\alpha^2 - 3\alpha - 4)x_5x_{10} + (-10\alpha^2 + 3\alpha - 8)x_6x_{10} + (-8\alpha^2 + 6\alpha - 4)x_7x_{10} + \\
(-6\alpha^2 - 6)x_8x_{10} + (20\alpha^2 - 6\alpha + 16)x_9x_{10} + (12\alpha^2 - 9\alpha + 6)x_{10}^2 + (6\alpha^2 + 6)x_2x_{11} + (-14\alpha^2 + 6\alpha - 10)x_3x_{11} + \\
(6\alpha^2 + 6)x_4x_{11} + (6\alpha^2 - 9\alpha)x_5x_{11} + (-6\alpha^2 - 6)x_6x_{11} + (10\alpha^2 - 3\alpha + 8)x_7x_{11} + (-6\alpha^2 - 6)x_8x_{11} + \\
(-4\alpha^2 + 3\alpha - 2)x_9x_{11} + (-6\alpha^2 - 6)x_{10}x_{11} + (15\alpha^2 - 9\alpha + 9)x_{11}^2 + (-6\alpha^2 - 6)x_1x_{12} + (-8\alpha^2 + 6\alpha - 4)x_3x_{12} + \\
(-8\alpha^2 - 3\alpha - 10)x_4x_{12} + (-2\alpha^2 + 6\alpha + 2)x_5x_{12} + (-14\alpha^2 + 6\alpha - 10)x_6x_{12} + (6\alpha^2 + 6)x_7x_{12} + \\
(-8\alpha^2 + 6\alpha - 4)x_8x_{12} + (12\alpha^2 - 9\alpha + 6)x_9x_{12} + (6\alpha^2 + 6)x_{10}x_{12} + (6\alpha^2 + 6)x_{11}x_{12} + (12\alpha^2 - 9\alpha + 6)x_{12}^2 + \\
(12\alpha^2 - 9\alpha + 6)x_1x_{13} + (6\alpha^2 + 6)x_3x_{13} + (14\alpha^2 - 6\alpha + 10)x_4x_{13} + (2\alpha^2 - 6\alpha - 2)x_5x_{13} + \\
(14\alpha^2 - 6\alpha + 10)x_6x_{13} + (6\alpha^2 + 6)x_8x_{13} + (-8\alpha^2 + 6\alpha - 4)x_9x_{13} + (-8\alpha^2 + 6\alpha - 4)x_{10}x_{13} + \\
(2\alpha^2 - 6\alpha - 2)x_{11}x_{13} + (-14\alpha^2 + 6\alpha - 10)x_{12}x_{13} + (9\alpha^2 - 9\alpha + 3)x_{13}^2 + (-2\alpha^2 + 6\alpha + 2)x_1x_{14} + \\
(6\alpha^2 + 6)x_2x_{14} + (-6\alpha^2 - 6)x_3x_{14} + (-4\alpha^2 + 3\alpha - 2)x_4x_{14} + (4\alpha^2 - 3\alpha + 2)x_5x_{14} + (-8\alpha^2 + 6\alpha - 4)x_7x_{14} + \\
(-6\alpha^2 - 6)x_8x_{14} + (-6\alpha^2 - 6)x_9x_{14} + (2\alpha^2 - 6\alpha - 2)x_{10}x_{14} + (-14\alpha^2 + 6\alpha - 10)x_{12}x_{14} + \\
(-2\alpha^2 + 6\alpha + 2)x_{13}x_{14} + (15\alpha^2 - 9\alpha + 9)x_{14}^2.
\end{gathered}
$$

Vertices, edges and face centers. Let us construct a vertex, edge center and face center of each of the triplet members. A fixed point of an automorphism corresponds to a 1-dimensional eigenspace of the action of this automorphism on the canonical model. We intersect the eigenspaces of the respective automorphisms S, RS and R with the curve model C over $\mathbb{Q}(\alpha)$, where we choose the isotypic components 12_1 and 12_2 for the model, and define ζ_7 by

$$f_{\min;\mathbb{Q}(\alpha)}(\zeta_7)(X) = X^2 - \alpha X + 1.$$

The eigenspaces of S, RS and R that result in points on the curve are $E(S, \zeta_7^{\pm 1})$ of dimension 2, $E(RS, -1)$ of dimension 8, and $E(R, \zeta_3^{\pm 1})$ of dimension 5. The fixed points on the curve of each of these automorphisms can now be found, but they require field extensions. The diagram below gives an overview of them.



We add β and ζ_7 to define a vertex v, γ_e and δ_e to define an edge center and γ_f to define a face center. The reason for choosing two extending elements instead of one to define an edge center $e \in E(RS, -1) \cap C$ is succinctness of the descriptions we could find.

Remark 7.2.2. The degree of the extension over the field of definition of the curve necessary to define a fixed point of an automorphism is the number of fixed points of that automorphism on the curve. Indeed, R fixes 4 points on the triplet members, RS fixes 6, and S fixes 2.

To find simple forms of the points we obtained on the curve, we first sought field elements with reasonable minimal polynomials by brute force, trying to factor polynomials over the field. Having chosen a suitable extension element, we simplified coefficients with the same lattice reduction technique we applied to the ideal generators. As a final result, we can define a map vertex v by:

$$\begin{pmatrix}
((\alpha^2 - 2\alpha + 2)\beta + (-3\alpha^2 + 9\alpha + 20))\zeta_7 + (9\alpha^2 - 5\alpha - 15)\beta + -12\alpha^2 - 3\alpha + 4 \\
((2\alpha^2 + 4\alpha)\beta + (-2\alpha - 8))\zeta_7 + (-\alpha^2 - \alpha + 1)\beta + 2\alpha^2 + 11\alpha + 4 \\
((-6\alpha^2 + 3\alpha)\beta + (14\alpha^2 - \alpha - 3))\zeta_7 + (-12\alpha^2 + 5\alpha + 10)\beta + 22\alpha^2 - 19\alpha - 23 \\
((-11\alpha^2 + 10\alpha + 7)\beta + (12\alpha^2 - 6\alpha - 11))\zeta_7 + (-11\alpha^2 + 8\alpha + 8)\beta + 9\alpha^2 - 9\alpha \\
((-7\alpha^2 + 12\alpha + 6)\beta + (11\alpha^2 - 9\alpha - 18))\zeta_7 + (-12\alpha^2 + 6\alpha + 9)\beta + 19\alpha^2 + 5\alpha - 5 \\
((11\alpha^2 - \alpha - 10)\beta + (-13\alpha^2 - 1))\zeta_7 + (-\alpha^2 - 2\alpha + 5)\beta + 10\alpha^2 + 18\alpha - 7 \\
((-4\alpha^2 + 3\alpha - 1)\beta + (5\alpha^2 - 2\alpha - 12))\zeta_7 + (-9\alpha^2 + 5\alpha + 11)\beta + 10\alpha^2 - 3\alpha - 6 \\
((4\alpha^2 + 4\alpha + 2)\beta + (-8\alpha^2 - 4\alpha + 4))\zeta_7 + (11\alpha^2 - 4\alpha - 14)\beta + -9\alpha^2 + 23\alpha + 16 \\
((6\alpha^2 - 13\alpha - 7)\beta + (-11\alpha^2 + 22\alpha + 6))\zeta_7 + (5\alpha^2 - 7\alpha + 2)\beta + -19\alpha^2 - 2\alpha - 1 \\
((-5\alpha^2 + 12\alpha + 4)\beta + (7\alpha^2 - 17\alpha - 12))\zeta_7 + (-5\alpha^2 + 7\alpha + 1)\beta + 10\alpha^2 + 7\alpha + 6 \\
((5\alpha^2 - 6\alpha - 6)\beta + (-5\alpha^2 + 7\alpha + 12))\zeta_7 + (4\alpha^2 - 2\alpha - 4)\beta + -2\alpha^2 - 6\alpha - 8 \\
((\alpha^2 + 12\alpha + 6)\beta + (3\alpha^2 - 21\alpha - 10))\zeta_7 + (-3\alpha^2 + \alpha - 1)\beta + 28\alpha^2 + 19\alpha \\
(4\beta + (4\alpha^2 - 6\alpha - 12))\zeta_7 + (\alpha^2 - 2\alpha + 1)\beta + 2\alpha^2 + 6\alpha + 7 \\
6\alpha^2 - 4
\end{pmatrix}$$

To define the edge center e, we need an extension of degree 6 over $\mathbb{Q}(\alpha)$ obtained by adding a square root and a root of a polynomial of degree 3. The simplest description we found is:

$$f_{\min;\mathbb{Q}(\alpha)}(\gamma_e)(X) := X^2 - (3\alpha + 2)$$
$$f_{\min;\mathbb{Q}(\alpha)}(\delta_e)(X) = X^3 + (1 + 3\alpha - \alpha^2)X + (-2 - 2\alpha^2)$$

So in fact $\gamma_e = -i\beta$. We can write down an edge center e as:

$$\begin{pmatrix}
((-2\alpha^2+\alpha+2)\gamma_e+(\alpha+1))\delta_e^2+((8\alpha^2+\alpha-15)\gamma_e-7\alpha^2-9\alpha+9)\delta_e+(7\alpha^2-5\alpha-4)\gamma_e+7\alpha^2+10\alpha-6 \\
((\alpha^2-\alpha-1)\gamma_e-\alpha^2-\alpha)\delta_e^2+((-2\alpha+1)\gamma_e+(5\alpha^2-\alpha-7))\delta_e+(-\alpha^2+3\alpha+2)\gamma_e+2\alpha^2-7\alpha+3 \\
((-\alpha^2+2\alpha+3)\gamma_e+(6\alpha^2+3\alpha-6))\delta_e^2+((-4\alpha^2+\alpha+3)\gamma_e-9\alpha^2+5\alpha+15)\delta_e+(-\alpha^2-2)\gamma_e-12\alpha^2+13\alpha-5 \\
((2\alpha^2-\alpha+1)\gamma_e+(3\alpha-6))\delta_e^2+((-2\alpha^2+\alpha+4)\gamma_e+(3\alpha^2-5\alpha-4))\delta_e+(-8\alpha^2+6\alpha+2)\gamma_e+11\alpha^2-3\alpha \\
((2\alpha^2-\alpha)\gamma_e+(2\alpha^2+2\alpha-6))\delta_e^2+((-2\alpha^2-\alpha+4)\gamma_e-2\alpha-2)\delta_e+(-6\alpha^2+7\alpha+4)\gamma_e+\alpha^2+\alpha+2 \\
((2\alpha-3)\gamma_e+(2\alpha^2-\alpha+2))\delta_e^2+((-2\alpha^2-2\alpha+2)\gamma_e-5\alpha^2+9\alpha+6)\delta_e+(2\alpha^2-3\alpha+6)\gamma_e-16\alpha^2+2\alpha-2 \\
((2\alpha^2-\alpha-2)\gamma_e+(2\alpha^2+3\alpha-3))\delta_e^2+((-4\alpha^2+\alpha+9)\gamma_e-\alpha^2+\alpha-3)\delta_e+(-7\alpha^2+3\alpha+4)\gamma_e-3\alpha^2+2\alpha+4 \\
((\alpha^2-3)\gamma_e-4\alpha^2+3)\delta_e^2+((8\alpha^2-\alpha-12)\gamma_e-10\alpha-4)\delta_e+(2\alpha^2-2\alpha+6)\gamma_e+14\alpha^2-4\alpha-3 \\
((-\alpha^2-\alpha+1)\gamma_e-\alpha^2-2\alpha+3)\delta_e^2+((-4\alpha^2+2\alpha+8)\gamma_e+(4\alpha^2+6\alpha-4))\delta_e+(-\alpha-6)\gamma_e-4\alpha^2-2\alpha+5 \\
((3\alpha^2-\alpha-5)\gamma_e-2\alpha^2+3\alpha)\delta_e^2+((4\alpha^2-3)\gamma_e-\alpha^2-9\alpha-7)\delta_e+(-5\alpha^2+\alpha+10)\gamma_e+11\alpha^2-2\alpha+1 \\
((-2\alpha^2+2\alpha+2)\gamma_e-2\alpha+2)\delta_e^2+((2\alpha^2+2\alpha-6)\gamma_e-4\alpha^2+8)\delta_e+(4\alpha^2-6\alpha-4)\gamma_e-2\alpha^2+6\alpha-4 \\
((3\alpha^2+2\alpha-6)\gamma_e+(5\alpha^2+4\alpha-4))\delta_e^2+((-2\alpha^2-3\alpha+3)\gamma_e-10\alpha^2+4\alpha+7)\delta_e+(-5\alpha^2+\alpha+16)\gamma_e-15\alpha^2+6\alpha-3 \\
((3\alpha^2-\alpha-7)\gamma_e-\alpha^2+2\alpha+3)\delta_e^2+((-6\alpha^2-2\alpha+14)\gamma_e+(6\alpha^2+8\alpha-12))\delta_e+(-6\alpha^2+\alpha+10)\gamma_e-6\alpha^2-14\alpha+5 \\
((\alpha^2+\alpha-3)\gamma_e+(4\alpha^2+5\alpha-2))\delta_e^2+((-2\alpha^2-2\alpha+3)\gamma_e-5\alpha^2+3\alpha+5)\delta_e+(-\alpha^2-\alpha+6)\gamma_e-9\alpha^2+2\alpha-3
\end{pmatrix}$$

The extension element γ_f is defined by

$$f_{\min;\mathbb{Q}(\alpha)}(\gamma_f)(X) = X^4+(-2\alpha^2+6)X^3+(-\alpha^2+4\alpha+4)X^2+(9\alpha^2+7\alpha-15)X+946\alpha^2+518\alpha-2116.$$

Then a face center f is given by:

$$\begin{pmatrix}
-\gamma_f^3 + (2\alpha^2 - \alpha - 6)\gamma_f^2 + (71\alpha^2 + 33\alpha - 167)\gamma_f + 198\alpha^2 + 110\alpha - 457 \\
71\alpha^2 + 41\alpha - 153 \\
(-9\alpha^2 - 8\alpha + 18)\gamma_f^2 + (-26\alpha^2 - 17\alpha + 53)\gamma_f - 456\alpha^2 - 256\alpha + 1015 \\
-\gamma_f^3 + (4\alpha^2 + 3\alpha - 7)\gamma_f^2 + (5\alpha^2 - 2\alpha - 19)\gamma_f + 465\alpha^2 + 259\alpha - 1045 \\
(-3\alpha^2 - 3\alpha + 7)\gamma_f^2 + (-10\alpha^2 - 6\alpha + 21)\gamma_f + 63\alpha^2 + 41\alpha - 139 \\
\gamma_f^3 + (-11\alpha^2 - 7\alpha + 24)\gamma_f^2 + (-26\alpha^2 - 9\alpha + 67)\gamma_f - 460\alpha^2 - 254\alpha + 1043 \\
(-a^2 + \alpha + 6)\gamma_f^2 + (-5\alpha^2 + 16)\gamma_f + 65\alpha^2 + 37\alpha - 145 \\
(a^2 + 2\alpha)\gamma_f^3 + (6\alpha^2 + 7\alpha - 7)\gamma_f^2 + (109\alpha^2 + 66\alpha - 232)\gamma_f + 472\alpha^2 + 268\alpha - 1051 \\
(-3\alpha^2 - 2\alpha + 4)\gamma_f^2 + (-6\alpha^2 - 5\alpha + 11)\gamma_f - 541\alpha^2 - 308\alpha + 1211 \\
(5\alpha^2 + 6\alpha - 5)\gamma_f^2 + (11\alpha^2 + 11\alpha - 16)\gamma_f + 737\alpha^2 + 416\alpha - 1646 \\
(-6\alpha^2 - 5\alpha + 11)\gamma_f^2 + (-16\alpha^2 - 11\alpha + 32)\gamma_f - 325\alpha^2 - 185\alpha + 725 \\
(a^2 + 2\alpha + 1)\gamma_f^3 + (5\alpha^2 + 6\alpha - 4)\gamma_f^2 + (110\alpha^2 + 73\alpha - 224)\gamma_f + 201\alpha^2 + 123\alpha - 441 \\
\gamma_f^3 + (11\alpha^2 + 10\alpha - 22)\gamma_f^2 + (37\alpha^2 + 30\alpha - 66)\gamma_f + 122\alpha^2 + 70\alpha - 266 \\
(71\alpha^2 + 41\alpha - 153)\gamma_f
\end{pmatrix}$$

7.3 Distinguishing the triplet members

Our current predicament is this: we have constructed the first Hurwitz triplet ($\mathbf{R}_{14.1}$, $\mathbf{R}_{14.2}$, $\mathbf{R}_{14.3}$) as a set of three algebraic curves, but we do not know which embedding of $\mathbb{Q}(\alpha)$ into $\mathbb{C}$, where $\alpha = \zeta_7 + \zeta_7^{-1}$, corresponds to which curve. Let C_k be the curve resulting from the embedding

$$\alpha \mapsto \alpha_k := 2\cos(2k\pi/7).$$

We have $k \in \{1, 2, 3\}$. Important about the numbers α_k is that $-3\alpha_k - 2$ is negative for $k = 1, 2$ and positive for $k = 3$.

The extended automorphism group. One curve is easiest to distinguish, and that is $\mathbf{R}_{14.2}$. The reason is that although $\mathrm{Aut}^+(\mathbf{R}) \cong \mathrm{PSL}(2, 13)$ for all three maps, we have

$$\begin{aligned} \mathrm{Aut}(\mathbf{R}_{14.1}) &\cong \mathrm{Aut}(\mathbf{R}_{14.3}) \cong \mathrm{PGL}(2, 13), \\ \mathrm{Aut}(\mathbf{R}_{14.2}) &\cong \mathrm{PSL}(2, 13) \times \mathbb{Z}_2. \end{aligned}$$

In particular, $\mathrm{Aut}^-(\mathbf{R}_{14.2})$ contains a central involution. This element is not a reflection, as we can deduce directly from its centrality, or by applying Theorem 1.4.3. Both show us that it fixes no cells at all, and hence no points on the algebraic curve it belongs to.

We can exploit this. Since our algebraic curves are defined over $\mathbb{Q}(\alpha) < \mathbb{R}$, standard complex conjugation $\mathrm{con} = \mathrm{con}_{14}$ leaves them all invariant. Remembering Theorem 6.1.14, we see this map is an antiholomorphism of each curve. It acts as an outer automorphism on $\mathrm{Aut}(C) = \mathrm{Aut}^+(\mathbf{R}) = \langle R, S\rangle$ for each triplet member. The new canonical representation for our models over $\mathbb{Q}(\alpha)$ is defined over $\mathbb{Q}(\alpha, \beta)$, where $\beta = \sqrt{-3\alpha - 2}$. For $k \in \{1, 2\}$, this is a purely imaginary number, for $k = 3$ it is real. Hence, we know that $\overline{\alpha} = \alpha$ in any case, but $\overline{\beta} = \beta$ for $k = 3$ and $\overline{\beta} = -\beta$ otherwise. With this knowledge we can compute g^{con} for all $g \in \mathrm{Aut}(C)$, so we know the structure of $\mathrm{Aut}(\mathbf{R}_k) = \mathrm{Aut}^*(C_k)$. Inevitably, we find $\mathrm{Aut}^*(C_1) \cong \mathrm{Aut}^*(C_2) \cong \mathrm{PGL}(2, 13)$ and $\mathrm{Aut}^*(C_3) \cong \mathrm{PSL}(2, 13) \times \mathbb{Z}_2$. We conclude that $C_3 = (\mathbf{R}_{14.2})_a$.

Remark 7.3.1. The central involution behaves like the classical antipodal mapping of the sphere. We say that the map $\mathbf{R}_{14.2}$ is *antipodal*: every point has a unique point associated to it by the antipodal mapping, its *antipode*. Expressed in a standard map presentation extended to $\mathrm{Aut}^-(\mathbf{R}_{14.2})$ (cf. Section 1.1), our specific central element is $\mathrm{con} = S(R^{-1}S^2)^6 a \in \mathrm{Aut}^-(\mathbf{R}_{14.2})$. We also see that our curve C_3 has no real points. It twists around the subspace $\mathbb{P}^{13}(\mathbb{R}) \subset \mathbb{P}^{13}(\mathbb{C})$ without intersecting it.

Petrie paths. To distinguish $\mathbf{R}_{14.1}$ and $\mathbf{R}_{14.3}$ from each other, we must resort to other means. Our approach is to explicity compute (enough of) the cell structure on each curve C_1 and C_2 to determine the Petrie path length of each curve, as introduced in Section 1.4. As mentioned before, this length is 12 for $\mathbf{R}_{14.1}$ and 14 for $\mathbf{R}_{14.3}$.

What we need are three involutions a, b, c in $\mathrm{Aut}^*(C_k)$ of which we know they are reflections in the three sides of some fundamental triangle. To this end, we want to be able to decide adjacency for vertices of the barycentric graph. One way to do this is to plot geodesic walls. For our two remaining curves, we can consider the real wall. As alluded to in Section 6.1, the invariant set of con_{14} is $\mathbb{P}^{13}(\mathbb{R}) \subset \mathbb{P}^{13}(\mathbb{C})$. This subspace is well-defined as the set of points x with ratios $x_i/x_j \in \mathbb{R}$ whenever they are defined. More generally, the invariant set V^σ of a complex conjugation $\sigma : V \to V$ on a complex vector (projective) space V is a real n-dimensional vector (projective) subspace. Intersecting this $\mathbb{R}$-linear subspace with the curve gives the real wall. To algebraicize matters, let us say that our original equations were over $\mathbb{C}[z_1, \dots, z_{14}]$. We are now led to converting them to a real form by substituting $z_k = x_k + y_k i$ and splitting each equation into its real and imaginary part. We can then express con (or any other complex conjugation on $\mathbb{P}^{13}(\mathbb{C})$) as an $\mathbb{R}$-linear transformation in $(x_1, y_1, \dots, x_{14}, y_{14})$. Adding invariance under this transformation to the ideal yields a real algebraic description of the real wall.

The next step is to compute which barycentric vertices lie on the real wall. For this, the embedding $\mathbb{Q}(\alpha) \rightarrowtail \mathbb{C}$ needs to be extended to the field $\mathbb{Q}(\alpha, \gamma)$ over which such a point is defined, and complex conjugation has to be defined on this field. If we enlarge the field so that it is invariant under complex conjugation, the possibilities for complex conjugation are field involutions that send γ to another root of $f_{\mathrm{min};\mathbb{Q}(\alpha)}(\gamma)$. There may be several choices, in effect differing by a permutation of the vertex set. But the end result is the same.

Incidentally, using the formula from Section 1.4 we can compute in advance that the number of geodesic walls on both $\mathbf{R}_{14.1}$ and $\mathbf{R}_{14.3}$ is 78. Each wall has the repeating barycentric vertex pattern "vevfef" and contains 14 vertices of each type. This contrasts with $\mathbf{R}_{14.2}$, which has 91 geodesic walls, and only has 12 vertices of each type on a wall. Thus, computing the vertices on a geodesic wall also singles out $\mathbf{R}_{14.2}$. Also, on the two remaining curves a reflection fixes precisely one wall, as can be computed following the method outlined in Section 1.4. Projections of the real walls of these two curves and their barycentric vertices are plotted in Figures 7.2 and 7.4 in the next section. This enables one to solve the problem of finding two adjacent barycentric vertices visually.

One strategy to compute a fundamental triangle is now to pick an edge center and plot the unique other geodesic wall going through it, and the barycentric vertices on that. One can then choose a neighboring barycentric vertex on each wall to define a fundamental triangle and compute the Petrie path length.

We also document a different strategy. We applied this before having succeeded in plotting the geodesic walls. There is always a unique (primitive) rotation around an edge center. We are in the lucky situation that the faces of the platonic maps are triangles ($p = 3$), so there are only two non-trivial rotations fixing a face center and they are also both primitive rotations. Therefore, suppose you know one edge center e and one face center f that are adjacent. Then the rotation R_e, any of the two possible rotations R_f and the reflection a in the geodesic wall common to e and f determine a fundamental triangle with reflections $b := aR_f$ and $c := aR_e$. We can now compute the Petrie path length as

$$2\mathrm{ord}(abc) = 2\mathrm{ord}(a(aR_f)(aR_e)) = 2\mathrm{ord}(R_f a R_e).$$

To compute an adjacent edge and face center, restrict to an affine chart and pick an edge center e. Compute the euclidean distances of each face center to e. It is then tempting to assume that the closest such point will be adjacent to e. This need not be the case, as illustrated in Figure 7.1. But the figure also shows the way to a topological proof if it holds. Compute a small sphere intersecting the geodesic wall in only two points, containing in the ball bounded by it exactly one edge center and one face center. These two then have to be neighbors. To execute this proof computationally, one adds a real equation for the sphere to the ideal for the geodesic wall. The result is that $C_1 = (\mathbf{R}_{14.1})_a$ and $C_2 = (\mathbf{R}_{14.3})_a$.

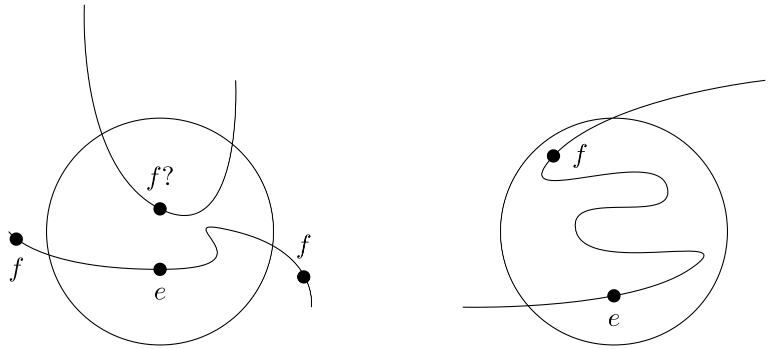


Figure 7.1: Two vertices on a geodesic wall that are close need not be adjacent (left), but they are if one can find a ball containing the two vertices whose boundary sphere intersects the wall only twice (right).

7.4 Weierstraß points on the first Hurwitz triplet

One of our primary reasons to construct canonical models for the first Hurwitz triplet is a problem by Kay Magaard and Helmut Völklein that was still open. The question is whether the automorphism group of a Hurwitz surface acts transitively on its set of Weierstraß points. This is indeed the case for the Klein quartic $\mathbf{R}_{3.1}$ and the Macbeath curve $\mathbf{R}_{7.1}$. In almost all other cases it proved to be false, see [MV2006] and [LS2012].

Theorem 7.4.1 (Magaard, Völklein)**.** *Let X be a Hurwitz surface of genus $g > 14$. The group $\mathrm{Aut}(X)$ does not act transitively on the set of Weierstraß points of X.*

The only omission from the theorem are the members of the first Hurwitz triplet. So far, group theoretic attempts to settle the question for the triplet (e.g. [LS2012]) have not reached a conclusion. With a canonical model in hand, a different attack becomes possible. Using the Fixed Point Counting Lemma 1.4.3 and applying the Lewittes-Schoeneberg theorem 7.1.11, we know in advance that the edge centers of each of the triplet members are Weierstraß points. We computed these edge centers explicitly back in Section 7.2. Thus by Theorem 7.1.9 a computation of Wronskians

(MAGMA code available from the author) tells us what the Weierstraß weight of an edge center is.

Proposition 7.4.2. *The Weierstraß weight of each edge center of one of the platonic maps* $\mathbf{R}_{14.1}$*,* $\mathbf{R}_{14.2}$ *and* $\mathbf{R}_{14.3}$ *is* 1*.* □

As a corollary, we can extend the Magaard-Völklein theorem.

Theorem 7.4.3. *The* X *be a Hurwitz surface. The group* $\mathrm{Aut}(X)$ *does not act transitively on the set of Weierstraß points of* X*, except if* X *is* $\mathbf{R}_{3.1}$ *or* $\mathbf{R}_{7.1}$*.* □

Non-geometric Weierstraß points

Although the ideal for the canonical model together with the action of its automorphism group and explicit expressions of the edge centers present a computable certificate for the claim we just staked, the evidence leaves something to be desired. It would be preferable if we could actually point to another orbit (or all orbits) of Weierstraß points on the triplet members. Repeating the computation of Wronskians for the vertices and face centers shows that they are not Weierstraß points. So we will have to search for non-geometric Weierstraß points, but how and where?

Let us first note that the total Weierstraß weight on each curve is $14 \cdot (14^2 - 1) = 2730$ by Theorem 7.1.6. Since the Weierstraß weight of the edge centers turns out to be 1, we are left with a contribution of $2184 = |\mathrm{Aut}(\mathbf{R})|$ by unknown points, for any triplet member $\mathbf{R}$. There are three possible weight distributions:

- two $\mathrm{Aut}(\mathbf{R})$-orbits of 1092 points on geodesic walls, each point of weight 1;
- one $\mathrm{Aut}(\mathbf{R})$-orbit of 1092 points on geodesic walls, each point of weight 2;
- one $\mathrm{Aut}(\mathbf{R})$-orbit of 2184 points not on geodesic walls, each point of weight 1.

One possible attempt at computing the non-geometric Weierstraß points is to compute the abstract Wronskian and try to find all its solutions. However, even explicitly writing down the Wronskian turned out to be too taxing for our computer resources. We then took up the following approach.

Approximating Weierstraß points on a geodesic wall. Let us search for Weierstraß points on a geodesic wall. This makes the problem more tractable, since a geodesic wall is real 1-dimensional. As the type of the triplet members is $(3, 7)$, there is only one type of geodesic wall. Moreover, the maps $\mathbf{R}_{14.1}$ and $\mathbf{R}_{14.3}$ have a real wall, and the field $\mathbb{Q}(\alpha)$ is totally real. So the Wronskian is a continuous real-valued function on the real walls for these two siblings. By computing enough if its values, we can find sign changes and ascertain the existence of a zero. If this indeed occurs, it proves the existence of non-geometric Weierstraß points on the geodesic walls. With the most basic root approximation algorithm possible (bisection) we can even approximate the real Weierstraß points.

There is a snag that we already encountered when trying to distinguish the triplet members, which is that one has to be able to decide betweenness on the real wall. This means completely understanding its topology as a space curve, which is quite hard in $\mathbb{P}^{13}$. To make it easier, we look at a projection to $\mathbb{P}^2$, by computing an elimination ideal $\langle P(x_i, x_j, x_k)\rangle$ for a suitable choice of (i, j, k). Then we consider an affine part of this planar curve.

Remark 7.4.4. This is computationally non-trivial, the computation only ending within reasonable time for certain choices of variables. In point of fact we computed $I(\mathbf{R}) \cap \mathbb{Q}(\alpha)[x_1, x_{13}, x_{14}]$ and took the affine part $\{x_{14} = 1\}$. We will not display the resulting planar model of the curve, since its defining polynomial P of degree $2g - 2 = 26$ had very large coefficients that we were unable to simplify enough.

Now consider the real wall of the planar model. It forms the real solution set of $P(x_1, x_{13}, x_{14}) = 0$. From the previous section, we know that we have the embedding $\alpha \mapsto 2\cos(2\pi/7)$ of $\mathbb{Q}(\alpha)$ into $\mathbb{C}$ for $\mathbf{R}_{14.1}$ and $\alpha \mapsto 2\cos(4\pi/7)$ for $\mathbf{R}_{14.3}$. This allows one to actually graph (an affine part of) the real wall. The result can be seen in Figures 7.2 and 7.4. The next step is to compute all 42 vertices, edge centers and

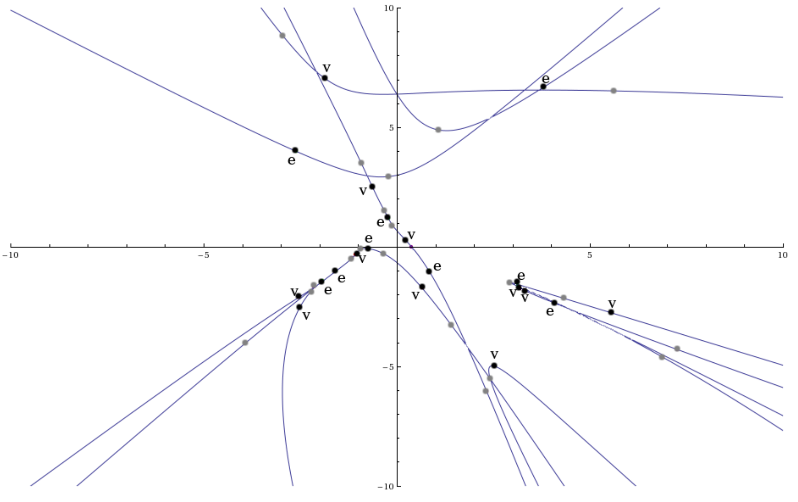


Figure 7.2: The affine part of the real wall on the curve with $\alpha = 2\cos(2\pi/7)$, projected to (x_1, x_{13}). Shown are the vertices (v) and edge centers (e). In between one can see the two orbits of non-geometric Weierstrass points (grey points). The segment of the wall with $(x_1, x_{13}) \in [-1, 1] \times [0, 2.9]$ is easy to distinguish computationally.

face centers that lie on the real wall of $\mathbf{R}_{14.1}$ and $\mathbf{R}_{14.3}$. To maximize efficacy, we choose a small segment that satisfies two conditions. First, it has to contain an edge and a triangle altitude of the platonic map, so that we cannot miss any Weierstraß points on the wall. Second, we must be able to distinguish (preferably automatically using a simple criterion) points lying on the segment from points on nearby branches. In Figure 7.2, such a segment is defined by points on the real wall with $(x_1, x_{13}) \in [-1, 1] \times [0, 29/10]$. After selecting a suitable segment, one can start to compute Wronskian values. The graph of the Wronskian for $\mathbf{R}_{14.1}$ is shown in Figure 7.3. Apart from the zeroes at the edge centers (visible in the graph at $x_{13} \approx 1.2$) the

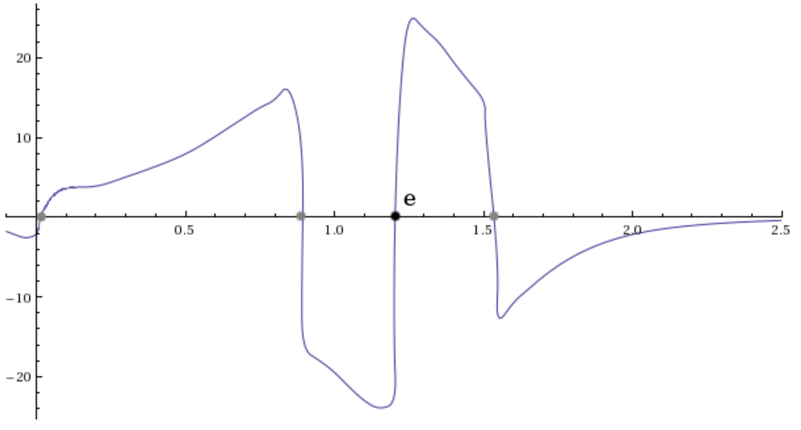


Figure 7.3: The Wronskian determinant $W(x_{13})$ on the real wall segment of $\mathbf{R}_{14.1}$ with $(x_1, x_{13}) \in [-1, 1] \times [0, 29/10]$. As the Wronskian varies so wildly, the function $x \mapsto \operatorname{sign}(x)|x|^{1/30}$ was applied to its values to get a clear picture.

graph proves a sign change on the wall between a vertex and a face center ($x_{13} \approx 0.89$ and $x_{13} \approx 1.53$, related by the rotation around the real edge they lie on), and also one between a vertex and an edge center ($x_{13} \approx 0.009$). This forces the first option above: we have two non-geometric $\operatorname{Aut}(\mathbf{R}_{14.1})$-orbits of Weierstraß points, each point of weight 1. Because the Weierstraß points are defined algebraically by equations over $\mathbb{Q}(\alpha)$, their weight is independent of the embedding $\mathbb{Q}(\alpha) \rightarrowtail \mathbb{C}$. The computation for $\mathbf{R}_{14.1}$ therefore entails a conclusion for all three triplet members that is more far-reaching than Proposition 7.4.2.

Theorem 7.4.5. *The Weierstraß points of each member $\mathbf{R}$ of the first Hurwitz triplet lie in three $\operatorname{Aut}^+(\mathbf{R})$-orbits, two of size 1092 and of size 546. All Weierstraß points have weight 1.* □

This re-establishes the non-transitivity of $\operatorname{Aut}(\mathbf{R})$ on the Weierstraß points with greater vigor and rigor. What we have not established is that the non-geometric Weierstraß points lie on geodesic walls for all triplet members. This is equivalent to these points being invariant under a complex conjugation in $\operatorname{Aut}^*(C_k)$. For example, since C_3 has no real points, their field of definition cannot be totally real. It is therefore no certainty that the real wall of C_2 contains non-geometric Weierstraß points. But let us record the specific result just obtained.

Proposition 7.4.6. *On the platonic map $\mathbf{R}_{14.1}$, there are two $\operatorname{Aut}(\mathbf{R}_{14.1})$-orbits of non-geometric Weierstraß points, both lying on the geodesic walls of $(\mathbf{R}_{14.1})_a$.* □

After getting a rough idea of the position of the zeroes on the real wall of $\mathbf{R}_{14.1}$, we use bisection to compute accurate approximations of the non-geometric Weierstraß points on the segment under consideration. We record the number up to 30-digit precision:

$$\begin{aligned} x_{13} &= 0.891162135033780174753084747614\ldots && \text{on a barycentric “vf” segment} \\ x_{13} &= 0.009206167466405642102383818784\ldots && \text{on a barycentric “ve” segment} \end{aligned}$$

These values yield the following respective Weierstraß points p_{vf} and p_{ve} on our original canonical model for $\mathbf{R}_{14.1}$:

$$p_{\mathrm{vf}} = \begin{pmatrix} -0.110498918008641056910528586910500847759977975956603336 \\ -0.628778052752740137663775191353613981012077484710756425 \\ 0.993813745433539958705693642249538854601432192395889419 \\ -1.73968071591864699547491020463330155520325643640945311 \\ -1.132424326592612084136581798579737394941188638083406 94 \\ 1.327213213956011143446246934717857057063072999994760 01 \\ -0.078908031235586559331144013500317713645371676486400 0380 \\ -0.001635185591178687213131009629661891619511399532626985 95 \\ 1.231446873014084309133458020620320340709937038691221 80 \\ -1.032464497865691843605626478400810036654844338909464 37 \\ 1.112755160671027672806611638206881320275682525348185 08 \\ 1.094973296789867371137471634185464979457638681916375 98 \\ 0.891162135033780174753084747614890896426222169431854 7271 \\ 1 \end{pmatrix}$$

$$p_{\mathrm{ve}} = \begin{pmatrix} 0.373819682424353593275641296829713052968329955 \\ -0.406506596046647128369352813696360444949207807 \\ 1.226332613947928877756647269041857148376934 13 \\ -1.765664763966190734441526989574716507284736 36 \\ -1.683321041981857520108586324632623430111934 96 \\ 1.666083692213249177893789283012952027993846 08 \\ -0.453486633086269285935192939070656083827189149 \\ -3.101502506108493300688953210896158315331106 73 \cdot 10^{-6} \\ 1.151291581439961545454640939453830044468698 24 \\ -0.911907180963972701472255498968424201127448053 \\ 1.556627360891611083004402469013576508102154 96 \\ 0.959514553604818239746933865683497387812538720 \\ 0.009206167466405642102383818784843417375542915 70 \\ 1 \end{pmatrix}$$

The platonic map $\mathbf{R}_{14.3}$, corresponding to the choice $\alpha = 2\cos(4\pi/7)$, gives a different result. In Figure 7.4 the (affine part $\{x_{14} = 1\}$ of) its real wall is drawn. In Figure 7.5 the graph of the Wronskian is drawn for a segment of the wall lying in $[0, 2] \times [-7/2, -1/4]$. The segment contains two vertices ($x_{13} \approx -3.48$ and $x_{13} \approx -0.28$) and an edge center ($x_{13} \approx -1.28$) in between. The figure has only one discernable zero, namely at the edge center. Heuristically, this implies that the non-geometric Weierstraß points do not lie on geodesic walls for $\mathbf{R}_{14.3}$. It is no proof however, especially considering the wild behavior of the Wronskian.

The algebraic model for $\mathbf{R}_{14.2}$, corresponding to the choice $\alpha = 2\cos(6\pi/7)$, has no real points. We have not plotted a geodesic wall on it.

The non-geometric Weierstraß points over $\overline{\mathbb{Q}}$. The best result we could hope for is to find the non-geometric Weierstraß points exactly, as algebraic numbers. One method would be to guess a minimal polynomial for the given approximation of a coordinate. There is a method for this, which, given a degree n, creates a best fit for a minimal polynomial over $\mathbb{Q}$ with a zero that lies close to a rational number one feeds it. The algorithm uses lattice base reduction. It is implemented in PARI/GP, for example. But although we used an approximation of x_{13} on $\mathbf{R}_{14.1}$ accurate to

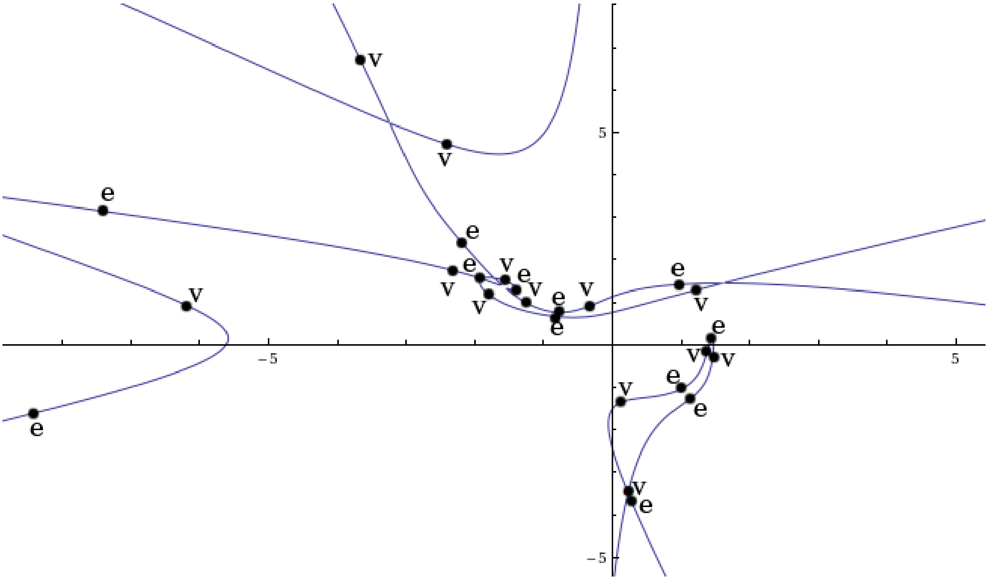


Figure 7.4: The affine part of the real wall on the curve with $\alpha = 2\cos(4\pi/7)$, projected to (x_1, x_{13}). Shown are the vertices (v) and edge centers (e) on it. The segment of the wall with $x_{13} \in [-3.4, -0.25]$ and the largest possible $x_1 \in [0, 2]$ is easy to distinguish computationally.

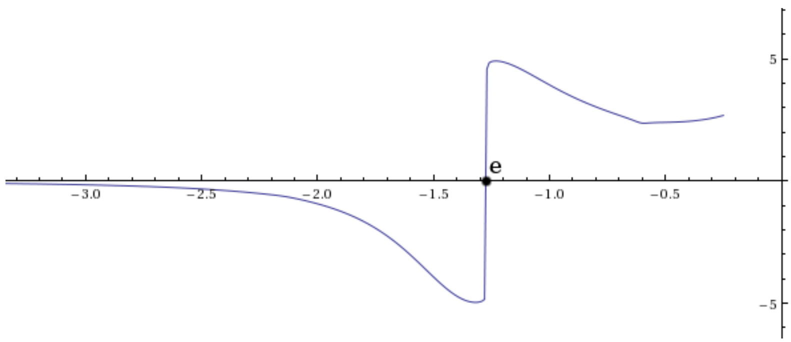


Figure 7.5: The Wronskian determinant $W(x_{13})$ on the real wall segment of $\mathbf{R}_{14.3}$ with $(x_1, x_{13}) \in [0, 2] \times [-3.4, -0.25]$ with maximal x_1. As the Wronskian varies so wildly, the function $x \mapsto \text{sign}(x)|x|^{1/30}$ was applied to its values to get a clear picture.

54 digits, the best fit minimal polynomials of degree at most 70 sadly did not yield algebraic Weierstraß points. On a more positive note, this is good material for further research.

A

Algebraic models for $2 \leq g \leq 15$

THIS appendix contains descriptions and construction details of algebraic models for reflexive platonic maps of low genus. All reflexive platonic surfaces of genus at most 8 are present, and in addition selected examples of genera up to and including 15. Most of the models are canonical models, constructed using the strategy described in Section 6.2. For some maps we also describe a planar model; for hyperelliptic maps this is the only model we present. We list some data for each map, for example a standard map representation under the heading "SMP". Throughout, a monomial matrix will be denoted as $\mathrm{MonMat}([\lambda_1, \ldots, \lambda_n], [c_1, \ldots, c_n])$, meaning that in row i, the only non-zero entry is λ_i in column c_i. Similarly, a diagonal matrix will be denoted $\mathrm{DiaMat}(\lambda_1, \ldots, \lambda_n)$ with the obvious interpretation. The coordinates of $\mathbb{P}^2$ will be written as x, y, z to avoid clutter, but as $x_1, \ldots, x_{n+1}$ for $\mathbb{P}^n$ with $n \geq 3$. The canonical ideal will be denoted by I in each case, without further mention.

A.2 Genus 2

Prior remarks. The Hilbert-Poincaré series predicts (cf. Section 6.1) that all curves of genus 2 are hyperelliptic. The canonical representation is an action on $\mathbb{P}^1$ and obviously $\mathbb{P}^1$ does not contain an embedded model of the algebraic curve of a genus 2 platonic map. But as described in Section 6.3 we can still use the canonical representation to determine branch data for a planar model. For genus 2 such a model will be of the form $y^2 = h(x)$ with $\deg(h) \in \{5, 6\}$. All these hyperelliptic curves have a polynomial h with distinct zeroes, and therefore each has the unique singular point $(0 : 1 : 0)$. To create a suitable set of branch points in $\mathbb{C}$, we identify the complex numbers with the chart $U := \{(x:1) \mid x \in \mathbb{C}\} = \mathbb{P}^1 - \{(1:0)\}$.

$\mathbf{R}_{2.1}$ type $(3, 8)$ **#cells** $(6, 24, 16)$ **map group size** 96 **Bolza's map**
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^3, S^8, (RS)^2, (RS^{-3})^2 \rangle$

The canonical representation ρ_c can be generated by

$$R \mapsto \frac{1}{1+i}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \qquad S \mapsto \begin{pmatrix} -\zeta_8 & 0 \\ 0 & \zeta_8^{-1} \end{pmatrix}.$$

The only $\mathrm{Aut}^+(\mathbf{R})$-orbit of size at most 6 is that of vertices of the map, so these have to be the fixed points of the hyperelliptic involution, and therefore yield the branch data of the mapping $\mathbf{R}_{2.1} \to \mathbb{P}^1$. Computing the eigenvectors of S and spinning by $\mathrm{Aut}^+(\mathbf{R})$, we see that this orbit consists of $(1:0)$, $(0:1)$ and $(i^k:1)$, where $k \in \{0, \ldots, 3\}$. We multiply to find the algebraic curve

$$y^2 z^4 = xz \cdot \prod_{k \in \{0,\ldots,3\}} (x - i^k z).$$

This is not quite right, since it is reducible, containing the extra component $z = 0$. We cut that out to arrive at the planar model

$$y^2 z^3 = x \cdot \prod_{k \in \{0,\ldots,3\}} (x - i^k z) = x^5 - xz^4.$$

This is the well-known Bolza curve, and so we give $\mathbf{R}_{2.1}$ the name *Bolza's map*. Note that it is the same curve as that of the Wiman type II map **Wi2**(2), stemming from the fact that $\mathbf{Wi2}(2) = D_2(\mathbf{R}_{2.1})$. Also, comparing with the polynomial data from Section 6.4 confirms that this is a platonic 2-cover of **Oct**. The projectivities of the canonical representation translate into the following automorphisms generating $\mathrm{Aut}(\mathbf{R}_a) = \mathrm{Aut}^+(\mathbf{R})$:

$$R : (x:y:z) \mapsto (x - iz : yz^2 : x + iz),$$
$$S : (x:y:z) \mapsto (ix : \zeta_8^{-3} y : z).$$

Complex conjugation induces a reflection.

$\mathbf{R}_{2.2}$ type $(4, 6)$ **#cells** $(4, 12, 6)$ **map group size** 48 **AM**(2)
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^4, S^6, (RS)^2, (RS^{-1})^2 \rangle$

Our efforts in Chapters 2, 5, and 6 already establish this is the Accola-Maclachlan map **AM**(2). To repeat the work from Chapter 6: the canonical representation ρ_c can be generated by

$$R \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad S \mapsto \begin{pmatrix} -\zeta_6 & 0 \\ 0 & \zeta_6^{-1} \end{pmatrix}.$$

The only $\mathrm{Aut}^+(\mathbf{R})$-orbit of size 5 or 6 is that of cell centers of the map, so these have to be the fixed points of the hyperelliptic involution, and therefore yield the branch data of the mapping $\mathbf{R}_{2.2} \to \mathbb{P}^1$. Computing the eigenvectors of R and spinning by $\mathrm{Aut}^+(\mathbf{R})$, we see that this orbit consists of $(\zeta_6^k : 1)$, where $k \in \{0, \ldots, 5\}$. We multiply to find the planar model

$$y^2 z^4 = \prod_{k \in \{0,\ldots,5\}} (x - \zeta_6^k z) = x^6 - z^6.$$

$\mathbf{R}_{2.3}$ type $(4, 8)$ **#cells** $(2, 8, 4)$ **map group size** 32 D(**Bolza's map**) **/ Wi2**(2)
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^4, S^8, (RS)^2, R^{-1}SRS^{-3} \rangle$

This map is $D_2(\mathbf{R}_{2.1}) = D_2(\text{Bolza's map})$ and therefore has the same planar model $y^2 z^3 = x^5 - xz^4$, the Bolza curve. At the same time, it is the Wiman type II map $\mathbf{Wi2}(2)$. For more information, see Section 5.2.

$\mathbf{R}_{2.4}$ type $(5, 10)$ **#cells** $(1, 5, 2)$ **map group size** 20 $\mathbf{Wi1}(2)$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^5, S^{10}, (RS)^2, RS^{-4} \rangle$

This map is the Wiman type I map $\mathbf{Wi1}(2)$ with planar model $y^2 z^3 = x^5 - z^5$. For more information, see Section 5.1.

$\mathbf{R}_{2.5}$ type $(6, 6)$ **#cells** $(2, 6, 2)$ **map group size** 24 $D_1(\mathbf{AM}(2))$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^6, S^6, (RS)^2, [R, S] \rangle$

This map is $D_1(\mathbf{R}_{2.2}) = D_1(\mathbf{AM}(2))$ and therefore also has planar model $y^2 z^4 = x^6 - z^6$. For more information, see Section 5.3.

$\mathbf{R}_{2.6}$ type $(8, 8)$ **#cells** $(1, 4, 1)$ **map group size** 16 $D_1(\mathbf{Wi2}(2))$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^8, S^8, (RS)^2, RS^{-3} \rangle$

This map is $D_1(\mathbf{R}_{2.3}) = D_1(\mathbf{Wi2}(2)) = D_1(D_2(\text{Bolza's map}))$ and therefore also has planar model $y^2 z^3 = x^5 - xz^4$. This is again the Bolza curve, since $\mathbf{R}_{2.3}$ is itself $D_2(\mathbf{R}_{2.1}) = D_2(\text{Bolza's map})$. For more information, see Section 5.2.

A.3 Genus 3

Prior remarks. The Hilbert-Poincaré series predicts (cf. Section 6.1) that the canonical model of a non-hyperelliptic platonic map of genus 3 is defined by a single quartic.

$\mathbf{R}_{3.1}$ type $(3,7)$ **#cells** $(24, 84, 56)$ **map group size** 336 **Klein quartic / Mod**(7)
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^3, S^7, (RS)^2, (RS^{-2})^4 \rangle$

The canonical model of this map is the well-known Klein quartic

$$x^3y + y^3z + z^3x = 0.$$

The canonical representation acting as automorphism group $\mathrm{Aut}^+(\mathbf{R})$ on this curve is determined by

$$R \mapsto \mathrm{MonMat}([1,1,1],[2,3,1]),$$

$$S \mapsto \frac{1}{7}\begin{pmatrix} -2\zeta_7^5 - 2\zeta_7^4 - 3\zeta_7^2 - 4\zeta_7 - 3 & -2\zeta_7^5 - \zeta_7^3 + 2\zeta_7^2 + 2\zeta_7 - 1 & -2\zeta_7^5 - 3\zeta_7^4 - 3\zeta_7^3 - 2\zeta_7^2 - 4 \\ -\zeta_7^5 + 2\zeta_7^4 + 2\zeta_7^3 - \zeta_7^2 - 2 & 2\zeta_7^5 - 2\zeta_7^4 + 2\zeta_7^3 - \zeta_7 - 1 & \zeta_7^5 + 4\zeta_7^4 + 2\zeta_7^3 + 2\zeta_7^2 + 4\zeta_7 + 1 \\ \zeta_7^5 + 3\zeta_7^4 - \zeta_7^3 + 3\zeta_7^2 + \zeta_7 & 3\zeta_7^5 + \zeta_7^4 + \zeta_7^3 + 3\zeta_7^2 - 1 & -3\zeta_7^4 - 2\zeta_7^3 - 4\zeta_7^2 - 2\zeta_7 - 3 \end{pmatrix}.$$

Standard complex conjugation defines a reflection. For more information, see the entire book [Lev2001] that has been written about this curve.

$\mathbf{R}_{3.2}$ type $(3,8)$ **#cells** $(12, 48, 32)$ **map group size** 192 **Dyck's map / Fer**(4)
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^3, S^8, (RS)^2, (S^2R^{-1})^3 \rangle$

This is the Fermat map **Fer**(4), introduced in Chapter 2. We treated the Fermat maps extensively in Section 5.5. We can also (re)derive the planar model by computing that the canonical representation ρ_c can be generated by

$$R \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ i & 0 & 0 \end{pmatrix}, \qquad S \mapsto \begin{pmatrix} 0 & -i & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The representation $(\rho_c^\vee)^{4+}$ has exactly one invariant 1-dimensional subspace, namely $\langle x^4 + y^4 + z^4 \rangle$.

$\mathbf{R}_{3.3}$ type $(3,12)$ **#cells** $(4, 24, 16)$ **map group size** 96
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^3, S^{12}, (RS)^2, [R, S^3] \rangle$

From [KK1990] we obtain a canonical representation ρ_c generated by

$$R \mapsto \frac{1}{2}\begin{pmatrix} i-1 & -i+1 & 0 \\ -i-1 & -i-1 & 0 \\ 0 & 0 & 2\zeta_3 \end{pmatrix}, \qquad S \mapsto \frac{1}{2}\begin{pmatrix} i+1 & i-1 & 0 \\ -i-1 & i-1 & 0 \\ 0 & 0 & 2(\zeta_3+1) \end{pmatrix}.$$

The representation $(\rho_c^\vee)^{4+}$ has two isotypic components for 1-dimensional irreducible representations of $\mathrm{Aut}^+(\mathbf{R}_{3.3})$ in its decomposition. One is $\langle x^4 + (4\zeta_6 - 2)x^2y^2 + y^4\rangle$, but that consists of reducible polynomials. The other is

$$\langle x^4 - (4\zeta_6 - 2)x^2y^2 + y^4, z^4\rangle.$$

All subspaces of the latter are $\mathrm{Aut}^+(\mathbf{R})$-invariant. The two basis vectors presented yield reducible curves, so we must use a non-degenerate linear combination. By exploiting the centralizer $C_{\mathrm{GL}(3,\mathbb{C})}(\mathrm{Aut}^+(\mathbf{R}))$ we may choose

$$x^4 - (4\zeta_6 - 2)x^2y^2 + y^4 + z^4,$$

which is irreducible and gives the canonical model we sought.

Remark A.3.1. Another planar model for $\mathbf{R}_{3.3}$ is $y^3 = x^4 - 1$, already computed in [KK1978] but without recognition of its platonicity. An obvious advantage is that the field of definition is $\mathbb{Q}$. This model also exhibits the curve explicitly as a trigonal curve. However, it is not a canonical model: $\mathrm{Aut}^+(\mathbf{R})$ does not act linearly on it. It admits the standard map presentation

$$\begin{aligned} R:&\quad (x:y:z) \mapsto ((-\zeta_6+2)xz(y-\zeta_6 z)(y-\zeta_6^{-1}z) : x^4 - 3y^2z^2 + 3yz^3 - 3z^4 : \zeta_6 x^4), \\ S:&\quad (x:y:z) \mapsto (-ix : -\zeta_6 y : z). \end{aligned}$$

Yet another way to obtain an algebraic model is to look back at Section 6.4 and consider the polynomial data for the tetrahedron. The map $\mathbf{R}_{3.3}$ is a platonic 4-cover of $\mathbf{Tet}$, leading to the model $y^4 = x(x^3 - 2\sqrt{2}z^3)$, which in turn is isomorphic to $y^4 = x(x^3 - z^3)$.

$\mathbf{R}_{3.4}$ **type** $(4,6)$ **#cells** $(8,24,12)$ **map group size** 96
SMP $\mathrm{Aut}^+(\mathbf{R}) = \left\langle R, S \,\middle|\, R^4, S^6, (RS)^2, (RS^{-2})^2 \right\rangle$

This map is hyperelliptic, as revealed by Theorem 6.3.2 or the table in Appendix C. The canonical representation ρ_c is generated by

$$R \mapsto \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad S \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Hyperellipticity is confirmed by the fact that $S^3 = -I$. The branch points are therefore the vertices of the map. We now search for an $\mathrm{Aut}^+(\mathbf{R})$-invariant $\mathbb{P}^1$. Since the representation is irreducible, we try quadrics. Either by computing $(\rho_c^\vee)^{2+}$ or looking at the above matrices, one finds the easy invariant quadric

$$Q : x^2 + y^2 + z^2 = 0.$$

The eigenspaces of S are $\langle(1,-1,-1)\rangle$, $\langle(1,\zeta_6,\zeta_6^{-1})\rangle$, and $\langle(1,\zeta_6^{-1},\zeta_6)\rangle$. We need $2g + 2 = 8$ branch points and the $\mathrm{Aut}^+(\mathbf{R})$-orbit of the first is too small. The other two lie in one orbit, which is thus the set of branch points. The parametrization of this conic by

$$\begin{aligned} \phi, \ \mathbb{P}^1 &\longrightarrow Q \\ (s:t) &\mapsto (s^2 - t^2 : 2st : i(s^2+t^2)) \end{aligned}$$

allows us to pull back these points to $\mathbb{P}^1$ with its inverse $(x : y : z) \mapsto (x - iz : y)$. The resulting set of complex numbers brings us to the canonical model

$$y^2z^6 = (x - (\pm\zeta_{12}^5 \pm \zeta_{12}^4)z)(x - (\pm\zeta_{12}^2 \pm \zeta_{12})z) = x^8 + 14x^4z^4 + z^8.$$

You may recognize the right hand side as the polynomial data for $\text{cells}_0(\mathbf{Cub})$ from Section 6.4, and indeed we saw in Theorem 6.3.2 that $\mathbf{R}_{3.4}$ is a platonic 2-cover of **Cub** branched over its vertices.

$\mathbf{R}_{3.5}$ **type** $(4,8)$ **#cells** $(4,16,8)$ **map group size** 64 — $\mathbf{Kul}(1)$ / $D_2(\mathbf{Fer}(4))$
SMP $\text{Aut}^+(\mathbf{R}) = \langle R, S \mid R^4, S^8, R^{-1}S^2RS^{-2}\rangle$

This map is $D_2(\mathbf{R}_{3.2}) = D_2(\mathbf{Fer}(4))$ and therefore also has the planar model $x^4 + y^4 + z^4 = 0$. For more information, see Section 5.5.

$\mathbf{R}_{3.6}$ **type** $(4,8)$ **#cells** $(4,16,8)$ **map group size** 64 — $\mathbf{AM}(3)$
SMP $\text{Aut}^+(\mathbf{R}) = \langle R, S \mid R^4, S^8, (RS)^2, (RS^{-1})^2\rangle$

This is the Accola-Maclachlan map $\mathbf{AM}(3)$ with planar model $y^2z^6 = x^8 - z^8$. For more information, see Section 5.3.

$\mathbf{R}_{3.7}$ **type** $(4,12)$ **#cells** $(2,12,6)$ **map group size** 48 — $\mathbf{Wi2}(3)$
SMP $\text{Aut}^+(\mathbf{R}) = \langle R, S \mid R^4, S^{12}, (RS)^2, R^{-1}SRS^{-5}\rangle$

This is the Wiman type II map $\mathbf{Wi2}(3)$ with planar model $y^2z^5 = x(x^6 - z^6)$. For more information, see Section 5.2.

$\mathbf{R}_{3.8}$ **type** $(6,6)$ **#cells** $(4,12,4)$ **map group size** 48
SMP $\text{Aut}^+(\mathbf{R}) = \langle R, S \mid R^6, S^6, (RS)^2, R^3S^3\rangle$

This map is $D_1(\mathbf{R}_{3.4})$ and therefore also has planar model $y^2z^6 = x^8 + 14x^4z^4 + z^8$. As a member of $\mathcal{F}_{3n-3}^{(3n,3n)}$ discussed in Chapter 2, it is a platonic 2-cover of **Tet**.

$\mathbf{R}_{3.9}$ **type** $(7,14)$ **#cells** $(1,7,2)$ **map group size** 28 — $\mathbf{Wi1}(3)$
SMP $\text{Aut}^+(\mathbf{R}) = \langle R, S \mid R^7, S^{14}, RS^{-6}\rangle$

This is the Wiman type I map $\mathbf{Wi1}(3)$ with planar model $y^2z^5 = x^7 - z^7$. For more information, see Section 5.1.

$\mathbf{R}_{3.10}$ **type** $(8,8)$ **#cells** $(2,8,2)$ **map group size** 32 $D(\mathbf{Kul}(1))$ / $D_1(D_2(\mathbf{Fer}(4)))$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^8, S^8, (RS)^2, R^{-1}SRS^{-5} \rangle$

This map is $D_1(\mathbf{R}_{3.5}) = D_1(D_2(\mathbf{R}_{3.2})) = D_1(D_2(\mathbf{Fer}(4)))$. It therefore also has planar model $x^4 + y^4 + z^4 = 0$. For more information, see Section 5.5.

$\mathbf{R}_{3.11}$ **type** $(8,8)$ **#cells** $(2,8,2)$ **map group size** 32 $D(\mathbf{AM}(3))$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^8, S^8, (RS)^2, [R,S] \rangle$

This map is $D_1(\mathbf{R}_{3.6}) = D_1(\mathbf{AM}(3))$ and therefore also has planar model $y^2z^6 = x^8 - z^8$. For more information, see Section 5.3.

$\mathbf{R}_{3.12}$ **type** $(12,12)$ **#cells** $(1,6,1)$ **map group size** 24 $D(\mathbf{Wi2}(3))$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^{12}, S^{12}, (RS)^2, RS^{-5} \rangle$

This map is $D_1(\mathbf{R}_{3.7}) = D_1(\mathbf{Wi2}(3))$ and therefore also has planar model $y^2z^5 = x(x^6 - z^6)$. For more information, see Section 5.2.

A.4 Genus 4

Prior remarks. The Hilbert-Poincaré series predicts (cf. Section 6.1) that the canonical model of a non-hyperelliptic platonic map of genus 4 is defined by one quadric and one cubic. These maps are therefore all trigonal.

$\mathbf{R}_{4.1}$ type $(3, 12)$ **#cells** $(6, 36, 24)$ **map group size** 144
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^3, S^{12}, (RS)^2, (S^2R^{-1})^3, [R, S^4]\rangle$

The canonical representation ρ_c can be generated by

$$R \mapsto \begin{pmatrix} 0 & 0 & \zeta_3+1 & 0 \\ \zeta_3+1 & 0 & 0 & 0 \\ 0 & \zeta_3^2 & 0 & 0 \\ 0 & 0 & 0 & \zeta_3 \end{pmatrix}, \qquad S \mapsto \begin{pmatrix} 0 & \zeta_3 & 0 & 0 \\ -\zeta_3 & 0 & 0 & 0 \\ 0 & 0 & \zeta_3 & 0 \\ 0 & 0 & 0 & \zeta_3+1 \end{pmatrix}.$$

Computing the actions $(\rho_c^\vee)^{2+}$ and $(\rho_c^\vee)^{3+}$, we find that the first has only $x_1^2 + x_2^2 + x_3^2$ and x_4^2 as 1-dimensional invariant subspaces. Since the latter polynomial is reducible, we have no choice but to use

$$I_2 = \langle\, x_1^2 + x_2^2 + x_3^2 \,\rangle.$$

The representation $(\rho_c^\vee)^{3+}$ has two isotypic components of 1-dimensional irreducible components:

$$\langle\, x_4(x_1^2 + x_2^2 + x_3^2) \,\rangle \text{ and } \langle\, x_1x_2x_3, x_4^3 \,\rangle.$$

Because the first space visibly contains a reducible polynomial, we must use the second. For the same reason, the linear combination we pick inside that space has to be non-degenerate. Exploiting the centralizer $C_{\mathrm{GL}(4,\mathbb{C})}(\mathrm{Aut}^+(\mathbf{R}))$, we may without loss of generality pick $x_1x_2x_3 + x_4^3$, and so we find

$$I = \langle x_1^2 + x_2^2 + x_3^2, x_1x_2x_3 + x_4^3\rangle.$$

as the solution. This ideal defines the non-singular irreducible curve that is our platonic surface.

Remark A.4.1. Another take on this map is to note that $\mathbf{R}_{4.1}$ is a platonic 3-cover of **Oct** branched over cells_0. Referring back to Section 6.4, we take $y^2z^3 = -xz(x^4 - z^4)$ and remove the component $z = 0$ to arrive at the planar model

$$y^3z^2 = -x(x^4 - z^4).$$

A standard map presentation is generated by

$$\begin{aligned} R:\ (x : y : z) &\mapsto (ix^2 - iz^2 : 2(\zeta_{12} - i)yz : x^2 + 2xz + z^2), \\ S:\ (x : y : z) &\mapsto (ix : \zeta_{12}y : z). \end{aligned}$$

Standard complex conjugation is a reflection on the map.

$\mathbf{R}_{4.2}$ type $(4,5)$ **#cells** $(24,60,30)$ **map group size** 240 **Bring's map**
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^4, S^5, (RS)^2, (RS^{-1}RS^{-2})^2 \rangle$

The canonical representation ρ_c (taken from [KK1990]) can be generated by

$$R \mapsto \frac{1}{5}\begin{pmatrix} 2\zeta_5^3+3\zeta_5^2+3\zeta_5+2 & 2\zeta_5^2+\zeta_5+2 & \zeta_5^3+2\zeta_5+2 & -\zeta_5^3-\zeta_5^2-3 \\ 2\zeta_5^2+\zeta_5+2 & -3\zeta_5^3-\zeta_5-1 & \zeta_5^3+\zeta_5^2-2 & -2\zeta_5^3-\zeta_5^2-2\zeta_5 \\ \zeta_5^3+2\zeta_5+2 & \zeta_5^3+\zeta_5^2-2 & \zeta_5^3-2\zeta_5^2+\zeta_5 & \zeta_5^3-\zeta_5^2-\zeta_5+1 \\ -\zeta_5^3-\zeta_5^2-3 & -2\zeta_5^3-\zeta_5^2-2\zeta_5 & \zeta_5^3-\zeta_5^2-\zeta_5+1 & -\zeta_5^2-3\zeta_5-1 \end{pmatrix},$$
$$S \mapsto \mathrm{DiaMat}(\zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4).$$

Computing the actions $(\rho_c^\vee)^{2+}$ and $(\rho_c^\vee)^{3+}$, we find that each has a unique invariant 1-dimensional subspace. This results in the ideal

$$I = \langle x_1x_4 + x_2x_3, x_1^2x_3 - x_1x_2^2 + x_2x_4^2 - x_3^2x_4 \rangle,$$

which indeed defines a non-singular irreducible curve, giving us the platonic surface we sought.

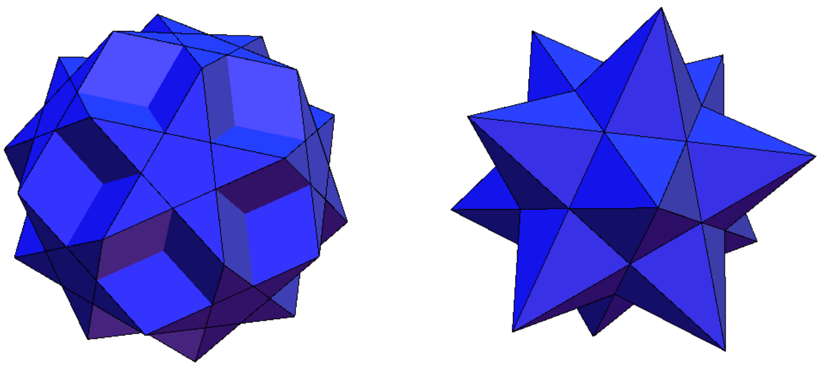


Figure A.1: The dodecadodecahedron is an embedding into $\mathbb{R}^3$ of $\mathbf{R}_{4.2}^\vee$ (left) and the small stellated dodecahedron is one of $\mathbf{R}_{4.6}^\vee = D_1(\mathbf{R}_{4.2})^\vee$ (right).

Remark A.4.2. This platonic surface is known as Bring's curve, and we therefore baptize it *Bring's map*. Bring's curve appears naturally in the solution of the reduced quintic equation $x^5 + ax + b = 0$. The roots $x_1, \dots, x_5$ of this equation satisfy

$$\sum_{i=1}^{5} x_i = 0, \qquad \sum_{i=1}^{5} x_i^2 = 0, \qquad \sum_{i=1}^{5} x_i^3 = 0,$$

and these three equations define a non-singular genus 4 curve B in $\mathbb{P}^4$. The natural permutation action of Sym_5 on $\mathbb{P}^4$ acts on B. Because $120 > 72 = 24(g-1)$, the relevant remark in Section 1.4 already shows B is a platonic surface. Explicitly, the Belyĭ function

$$\beta : (x_1 : x_2 : x_3 : x_4 : x_5) \mapsto (x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 : x_1x_2x_3x_4x_5)$$

is well-defined, Sym_5-invariant and of degree 120, since almost all points on B have five different coordinates. Because of its Sym_5-invariance, it follows that the ramification points of β are precisely those points of B that are invariant under an element of this group. A long winded case elimination reveal that only these Sym_5-orbits of points on B are invariant under an element of Sym_5:

$$\begin{aligned} &(1 : \zeta_5 : \zeta_5^2 : \zeta_5^3 : \zeta_5^4), && \text{which is invariant under } (12345);\\ &(1 : 1 : \alpha_1 : \alpha_2 : \alpha_3), && \text{which is invariant under } (12);\\ &(1 : i : -1 : -i : 0), && \text{which is invariant under } (1234). \end{aligned}$$

Here $\alpha_1, \alpha_2, \alpha_3$ are the three complex roots of the polynomial $X^3 + 2X^2 + 3X + 4$. The ramification orders of the points in these three orbits are $5, 2, 4$ and they map to $(5 : 1)$, $(-11 : 2)$, and $(5 : 1)$, respectively. This shows we have an $\mathrm{Aut}^+(\mathbf{R})$-invariant Belyĭ function and hence a platonic surface, notwithstanding β having only two branch points: pull back the extended real line of $\widehat{\mathbb{C}}$ to get the barycentric graph of $\mathbf{R}_{4.2}$ on B. The curve B contains no real points (consider its second defining equality) and hence standard complex reflection – which leaves B invariant – is not a reflection, but the unique *antipodal mapping* of $\mathbf{R}_{4.2}$.

A beautiful surprise is that the dual of Bring's map admits a topological immersion into $\mathbb{R}^3$ as the dodecadodecahedron. The diagonal map $D_1(\mathbf{R}_{4.2})^\vee$ can be immersed in $\mathbb{R}^3$ as the small stellated dodecahedron, one of the Kepler-Poinsot polyhedra. They are shown in Figure A.1. More details on this curve and its Riemann surface properties can be found in the lovely [Web2005] by Matthias Weber, giving further references to the history of Bring's curve. Weber also gives an elegant planar model:

$$y^5 = (x+1)x^2(x-1)^{-1}.$$

$\mathbf{R}_{4.3}$ **type** $(4, 6)$ **#cells** $(12, 36, 18)$ **map group size** 144 $\mathcal{F}^{(4,2n)}_{(n-1)^2}(2)$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^4, S^6, (RS)^2, [RS, SR] \rangle$

The canonical representation ρ_c can be generated by

$$R \mapsto \begin{pmatrix} 0 & 0 & 0 & \zeta_3^2 \\ 0 & 0 & \zeta_3 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad S \mapsto \begin{pmatrix} 0 & 0 & 0 & -\zeta_3 \\ 0 & 0 & -\zeta_3^2 & 0 \\ 0 & -\zeta_3^2 & 0 & 0 \\ -\zeta_3 & 0 & 0 & 0 \end{pmatrix}.$$

Computing the actions $(\rho_c^\vee)^{2+}$ and $(\rho_c^\vee)^{3+}$, we find that the first contains only $\langle x_1x_2 + x_3x_4 \rangle$ and $\langle x_1x_2 - x_3x_4 \rangle$ as 1-dimensional invariant subspaces, the second only $x_1^3 +$

$x_2^3+x_3^3+x_4^3$ and $x_1^3+x_2^3-x_3^3-x_4^3$. Of the four possible combinations, the ones using $\langle x_1x_2-x_3x_4\rangle$ are not prime. Furthermore, the projectivity

$$(x_1:x_2:x_3:x_4)\mapsto(x_1:x_2:-x_3:-x_4)$$

stabilizes the invariant subspaces of $\mathbb{C}[x_1,\dots,x_4]_2$ but interchanges the two subspaces of $\mathbb{C}[x_1,\dots,x_4]_3$, so up to isomorphism we have one remaining candidate:

$$I=\langle x_1x_2+x_3x_4, x_1^3+x_2^3+x_3^3+x_4^3\rangle.$$

This does indeed define the platonic surface $(\mathbf{R}_{4.3})_a$.

$\mathbf{R}_{4.4}$ **type** $(4,10)$ **#cells** $(4,20,10)$ **map group size** 80 **AM**(4)
SMP $\mathrm{Aut}^+(\mathbf{R})=\langle R,S\mid R^4,S^{10},(RS)^2,(RS^{-1})^2\rangle$

This is the Accola-Maclachlan map **AM**(4) with planar model $y^2z^8=x^{10}-z^{10}$. For more information, see Section 5.3.

$\mathbf{R}_{4.5}$ **type** $(4,16)$ **#cells** $(2,16,8)$ **map group size** 64 **Wi2**(4)
SMP $\mathrm{Aut}^+(\mathbf{R})=\langle R,S\mid R^4,S^{16},(RS)^2,R^{-1}SRS^{-7}\rangle$

This is the Wiman type II map **Wi2**(4) with planar model $y^2z^7=x(x^8-z^8)$. For more information, see Section 5.2.

$\mathbf{R}_{4.6}$ **type** $(5,5)$ **#cells** $(12,30,12)$ **map group size** 120 D(**Bring's map**)
SMP $\mathrm{Aut}^+(\mathbf{R})=\langle R,S\mid R^5,S^5,(RS)^2,(RS^{-1})^3\rangle$

This map is $D_1(\mathbf{R}_{4.2})$. It therefore has the same canonical model as Bring's map, and can also be realized on Bring's curve.

$\mathbf{R}_{4.7}$ **type** $(6,6)$ **#cells** $(6,18,6)$ **map group size** 72
SMP $\mathrm{Aut}^+(\mathbf{R})=\langle R,S\mid R^6,S^6,(RS)^2,(RS^{-1})^2\rangle$

This map is $D_1(\mathbf{R}_{4.3})$ and therefore has the same canonical model as $\mathbf{R}_{4.3}$.

$\mathbf{R}_{4.8}$ **type** $(6,6)$ **#cells** $(6,18,6)$ **map group size** 72
SMP $\mathrm{Aut}^+(\mathbf{R})=\langle R,S\mid R^6,S^6,(RS)^2,[R^2,S],(RS^{-2})^2\rangle$

The canonical representation ρ_c (obtained from [KK1990]) can be generated by

$$R\mapsto\begin{pmatrix}0&0&0&-1\\0&\zeta_6-1&0&0\\0&0&\zeta_6&0\\-\zeta_6+1&0&0&0\end{pmatrix},\qquad S\mapsto\begin{pmatrix}\zeta_6&0&0&0\\0&\zeta_6&0&0\\0&0&\zeta_6-1&0\\0&0&0&-1\end{pmatrix}.$$

This has invariant subspaces $\langle e_1, e_4\rangle$, $\langle e_2\rangle$ and $\langle e_3\rangle$. We compute the actions $(\rho_c^\vee)^{2+}$ and $(\rho_c^\vee)^{3+}$. The only invariant 1-dimensional subspaces of the first are $\langle x_2x_3\rangle$, $\langle x_2^2\rangle$ and the 1-dimensional subspaces of $\langle x_1x_4, x_3^2\rangle$. Primality of the canonical ideal forces us to choose a non-degenerate linear combination in the third space. Using the centralizer $C_{\mathrm{GL}(4,\mathbb{C})}(\mathrm{Aut}^+(\mathbf{R}))$ to scale the x_3^2-component, we may pick

$$I_2 = \langle x_1x_4 + x_3^2\rangle.$$

The representation $(\rho_c^\vee)^{3+}$ leaves $\langle x_1^3 + x_4^3\rangle$, $\langle x_2^2x_3\rangle$ and the 1-dimensional subspaces of $\langle x_1^3 - x_4^3, x_2^3\rangle$, $\langle x_1x_3x_4, x_3^3\rangle$, $\langle x_1x_2x_4 + x_2x_3^2\rangle$ invariant. Only the third space contains polynomials irreducible over $\mathbb{C}$, and again we exploit $C_{\mathrm{GL}(4,\mathbb{C})}(\mathrm{Aut}^+(\mathbf{R}))$, this time scaling the x_2^3-component. Up to isomorphism, we thus find the unique candidate

$$I = \langle x_1x_4 + x_3^2, x_1^3 + x_2^3 - x_4^3\rangle.$$

This indeed yields the non-singular irreducible curve of genus 4 we sought.

$\mathbf{R}_{4.9}$ type $(6, 12)$ **#cells** $(2, 12, 4)$ **map group size** 48
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^6, S^{12}, (RS)^2, R^{-1}SRS^{-7}\rangle$

This map is $D_2(\mathbf{R}_{4.1})$ and therefore has the same canonical model as $\mathbf{R}_{4.1}$.

$\mathbf{R}_{4.10}$ type $(9, 18)$ **#cells** $(1, 9, 2)$ **map group size** 36 **Wi1**(4)
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^9, S^{18}, (RS)^2, RS^{-8}\rangle$

This is the Wiman type I map **Wi1**(4) with planar model $y^2z^7 = x^9 - z^9$. For more information, see Section 5.1.

$\mathbf{R}_{4.11}$ type $(10, 10)$ **#cells** $(2, 10, 2)$ **map group size** 40 $D(\mathbf{AM}(4))$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^{10}, S^{10}, (RS)^2, [R, S]\rangle$

This map is $D_1(\mathbf{R}_{4.4}) = D_1(\mathbf{AM}(4))$ and therefore also has planar model $y^2z^8 = x^{10} - z^{10}$. For more information, see Section 5.3.

$\mathbf{R}_{4.12}$ type $(16, 16)$ **#cells** $(1, 8, 1)$ **map group size** 32 $D(\mathbf{Wi2}(4))$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^{16}, S^{16}, (RS)^2, RS^{-7}\rangle$

This map is $D_1(\mathbf{R}_{4.5}) = D_1(\mathbf{Wi2}(4))$ and therefore also has planar model $y^2z^7 = x(x^8 - z^8)$. For more information, see Section 5.2.

A.5 Genus 5

Prior remarks. The Hilbert-Poincaré series predicts (cf. Section 6.1) that the canonical model of a non-hyperelliptic, non-trigonal platonic map of genus 5 is a complete intersection curve, defined by three quadrics. If it is trigonal, some additional cubics will be needed.

$\mathbf{R}_{5.1}$ **type** $(3,8)$ **#cells** $(24,96,64)$ **map group size** 384 **Mod**(8)
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^3, S^8, (RS)^2, (RS^3R^{-1}S^{-2})^2 \rangle$

We can generate the canonical representation ρ_c by

$$R \mapsto \frac{1}{2}\begin{pmatrix} -1-i & -1+i \\ 1+i & -1+i \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad S \mapsto \frac{\zeta_8}{2}\begin{pmatrix} -1-i & -1+i \\ -1+i & -1-i \end{pmatrix} \oplus \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & -1 \\ 0 & i & 0 \end{pmatrix}.$$

The representation $(\rho_c^\vee)^{2+}$ decomposes into irreducibles as

$$2 \times 3_1 \oplus 3_6 \oplus \text{higher-dimensional subspaces.}$$

The second isotypic component contains reducibles and therefore cannot be included in the ideal as a whole. Hence, we must use an appropriate subspace of $2 \times 3_1$. Its 3-dimensional invariant subspaces are of the form

$$\langle x_1^2 + \lambda(x_3^2 - ix_4^2), x_1x_2 + \lambda x_5^2, x_2^2 - \lambda(x_3^2 + ix_4^2) \rangle.$$

The centralizer $C_{\mathrm{GL}(5,\mathbb{C})}(\mathrm{Aut}(\mathbf{R}_{5.1})^+)$ enables us to choose $\lambda = 1$ without loss of generality. Up to isomorphism, therefore, we can generate I_2 by

$$\langle x_1^2 + x_3^2 - ix_4^2,\ x_1x_2 + x_5^2,\ x_2^2 - x_3^2 - ix_4^2 \rangle.$$

This ideal indeed defines the platonic surface for $\mathbf{R}_{5.1}$.

Remark A.5.1. This curve has been studied by Wiman and Del Centina (see [Wim1895], [KM2010]), and has been referred to by some as Wiman's curve. The number of curves Wiman studied being rather large, this might not be a fortunate denomination. Another candidate name seems more suitable to us: $\mathbf{R}_{5.1}$ is the modular map **Mod**(8). For more information, see Section 5.7.

Remark A.5.2. The map $\mathbf{R}_{5.1}^\vee$ has a beautiful topological realization in $\mathbb{R}^3$ with all faces isometric to each other. This was discovered by Jarke van Wijk [Wij2009] and is shown in Figure A.2.

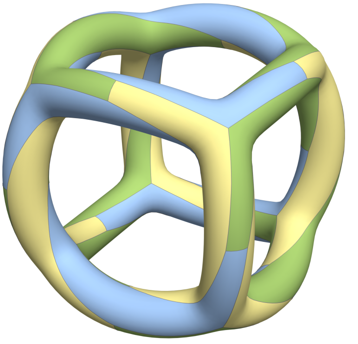


Image courtesy
Jarke J. van Wijk
Eindhoven University of Technology

Figure A.2: The map $\mathbf{R}_{5.1}^\vee$ embedded in $\mathbb{R}^3$ with isometric faces.

Remark A.5.3. The group $\mathrm{Aut}(\mathbf{R}_{5.1})$ contains an orientation preserving *antipodal map* σ. This is a central element of $\mathrm{Aut}(\mathbf{R}_{5.1})$ and the quotient mapping is an unbranched platonic 2-cover $\mathbf{R}_{5.1} \to \mathbf{R}_{5.1}/\langle\sigma\rangle = \mathbf{R}_{3.2}$ of the map $\mathbf{R}_{3.2} = \mathbf{Fer}(4)$.

$\mathbf{R}_{5.2}$ **type** $(3,10)$ **#cells** $(12,60,40)$ **map group size** 240
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^3, S^{10}, (RS)^2, (RS^{-4})^2 \rangle$

Some consideration of $\mathrm{Aut}(\mathbf{R}_{5.1})$ leads us to the discovery that there is a platonic 2-cover $\pi : \mathbf{R}_{5.1} \to \mathbf{R}_{5.1}/\langle S^5\rangle = \mathbf{Ico}$, branched over cells_0. The map is therefore hyperelliptic, and indeed we saw it in Theorem 6.3.2. Referring back to Section 6.4, we construct the planar curve $y^2z^{10} = -xz(x^{10} + 11x^5z^5 - z^{10})$. It is reducible, but dividing out the component $z = 0$ and transforming by $y \mapsto iy$, we find the planar model

$$y^2z^9 = x^{11} + 11x^6z^5 - xz^{10}.$$

This model has the standard map presentation

$$R : (x : y : z) \mapsto \left(-\zeta_5(\zeta_5^2+\zeta_5+1)(x - (\zeta_5+1)z)^5(x + (\zeta_5^3+\zeta_5+1)z) : 5(4\zeta_5^3+4\zeta_5^2-3)yz^5 : (x - (\zeta_5+1)z)^6\right),$$
$$S : (x : y : z) \mapsto (\zeta_5 x : -\zeta_5^3 y : z).$$

Standard complex conjugation con_5 is a reflection of the map.

$\mathbf{R}_{5.3}$ **type** $(4,5)$ **#cells** $(32,80,40)$ **map group size** 320 $\mathbf{Hum}_1(4)$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^4, S^5, (RS)^2, (RS^{-1})^4 \rangle$

The canonical representation ρ_c can be generated by

$$R \mapsto \mathrm{MonMat}([1,-1,-1,1,-1],[2,1,4,3,5]),$$
$$S \mapsto \mathrm{MonMat}([1,1,1,1,1],[2,4,1,5,3]).$$

The representation $(\rho_c^\vee)^{2+}$ decomposes as $1_1 \oplus 2_1 \oplus 2_2 \oplus$ other subspaces, so we need

$$1_1 = \langle x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 \rangle$$

and one of the 2-dimensional isotypic components. The latter are

$$2_1 = \langle x_1^2 + (\zeta_5^3+\zeta_5^2)x_3^2 - x_4^2 - (\zeta_5^3+\zeta_5^2)x_5^2,\ x_2^2 - x_3^2 + (\zeta_5^3+\zeta_5^2)x_4^2 - (\zeta_5^3+\zeta_5^2)x_5^2 \rangle,$$
$$2_2 = \langle x_1^2 - (\zeta_5^3+\zeta_5^2+1)x_3^2 - x_4^2 + (\zeta_5^3+\zeta_5^2+1)x_5^2,\ x_2^2 - x_3^2 - (\zeta_5^3+\zeta_5^2+1)x_4^2 + (\zeta_5^3+\zeta_5^2+1)x_5^2 \rangle.$$

They are both defined over $\mathbb{Q}(\zeta_5 + \zeta_5^{-1})$, and are related by the Galois automorphism of this field that sends $\zeta_5 + \zeta_5^{-1}$ to $\zeta_5^2 + \zeta_5^{-2}$. Both choices of component yield a viable candidate for the platonic surface $(\mathbf{R}_{5.3})_a$, and standard complex conjugation is a reflection of the map on either curve.

The presence of two solutions is explained by investigating the outer automorphism group $\mathrm{Out}(\mathrm{Aut}(\mathbf{R}_{5.3})) \cong \mathrm{Sym}_3$. The action of an outer element on $\mathrm{Aut}^+(\mathbf{R}_{5.3})$ allows us to set up linear equations to construct a projectivity accomplishing the same on

$\rho_c(\mathrm{Aut}^+(\mathbf{R}_{5.3}))$. It turns out we cannot extend ρ_c projectively to represent the order three outer elements, but we can represent an outer involution by

$$\mathrm{MonMat}([1,1,1,1,1],[5,1,2,4,3]).$$

This projectivity swaps the two curves, so the conclusion is that up to isomorphism, our platonic surface is defined by $1_1 \oplus 2_1$ as defined above.

Remark A.5.4. This map is the Humbert map $\mathbf{Hum}_1(4)$, and as such we can also use the algebraic model in $\mathbb{P}^4$ described in Section 5.6.

$\mathbf{R}_{5.4}$ **type** $(4,6)$ **#cells** $(16,48,24)$ **map group size** 192 $\mathbf{Hum}_3(4)$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^4, S^6, (RS)^2, (R^2S^3)^2 \rangle$

The canonical representation ρ_c can be generated by

$$R \mapsto \mathrm{MonMat}([-1,1,-1,-1,1],[2,1,5,4,3]),$$
$$S \mapsto \mathrm{MonMat}([-\zeta_3,\zeta_3^2,-1,1,1],[1,2,4,5,3]).$$

It has invariant subspaces $\langle e_1, e_2\rangle$ and $\langle e_3, e_4, e_5\rangle$. The representation $(\rho_c^\vee)^{2+}$ decomposes as $1_1 \oplus 1_4 \oplus 2 \times 2_1 \oplus 3_2 \oplus 6_1$. The isotypic components $1_4 = \langle x_1x_2\rangle$ and $3_2 = \langle x_3x_4, x_4x_5, x_5x_3\rangle$ both contain a multiple of a linear form and hence are excluded. We must therefore use

$$1_1 = \langle x_3^2 + x_4^2 + x_5^2\rangle$$

and an invariant 2-dimensional subspace of $2 \times 2_1$. All these subspaces are of the form

$$\langle x_1^2 + \lambda(\zeta_3 x_3^2 + x_4^2 + \zeta_3^2 x_5^2),\ x_2^2 + \lambda(\zeta_3^2 x_3^2 + x_4^2 + \zeta_3 x_5^2)\rangle.$$

Exploiting the centralizer $C_{\mathrm{GL}(5,\mathbb{C})}(\mathrm{Aut}^+(\mathbf{R}_{5.4}))$ we may set $\lambda = 1$. By subtracting the above generator of 1_1 we arrive at the unique candidate ideal

$$I = \langle x_3^2 + x_4^2 + x_5^2,\ x_1^2 + (\zeta_3 - 1)x_3^2 + (\zeta_3^2 - 1)x_5^2,\ x_2^2 + (\zeta_3^2 - 1)x_3^2 + (\zeta_3 - 1)x_5^2\rangle.$$

This ideal does define the correct platonic surface.

Remark A.5.5. This map is the Humbert map $\mathbf{Hum}_3(4)$, and as such we can also use the algebraic model in $\mathbb{P}^4$. For more information, see Section 5.6.

$\mathbf{R}_{5.5}$ **type** $(4,8)$ **#cells** $(8,32,16)$ **map group size** 128 $\mathbf{Hum}_2(4)$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^4, S^8, (RS)^2, [RS, SR], (RS^{-3})^2 \rangle$

The canonical representation ρ_c can be generated by

$$R \mapsto \mathrm{MonMat}([-i,1,1,1,1],[1,5,4,2,3]),$$
$$S \mapsto \mathrm{MonMat}([-i,-1,-1,-1,1],[1,4,5,3,2]).$$

We note that ρ_c has invariant subspaces $\langle e_1 \rangle$ and $\langle e_2, e_3, e_4, e_5 \rangle$. The representation $(\rho_c^\vee)^{2+}$ decomposes into irreducibles as

$$1_1 \oplus 2 \times 1_2 \oplus 1_5 \oplus 1_6 \oplus 2_1 \oplus 4_2 \oplus 4_3.$$

The 4-dimensional components are too big, and $2_1 = \langle x_2x_3, x_4x_5 \rangle$ contains a multiple of a linear form, so we must use three 1-dimensional pieces. Also, the pieces

$$1_5 = \langle x_2^2 - x_3^2 - ix_4^2 + ix_5^2 \rangle,$$
$$1_6 = \langle x_2^2 - x_3^2 + ix_4^2 - ix_5^2 \rangle$$

cannot be combined, since $(x_4 - x_5)(x_4 + x_5) \in 1_5 \oplus 1_6$. Furthermore,

$$2 \times 1_2 = \langle x_1^2, x_2^2 + x_3^2 - x_4^2 - x_5^2 \rangle,$$

so we must use $1_1 = \langle x_2^2 + x_3^2 + x_4^2 + x_5^2 \rangle$, an invariant subspace $1_2 < 2 \times 1_2$, and either 1_5 or 1_6. The subspace $\langle x_1^2 \rangle < 2 \times 1_2$ is obviously incorrect, as is $\langle x_2^2 + x_3^2 - x_4^2 - x_5^2 \rangle$, which results in $(x_4 - x_5)(x_4 + x_5) \in 1_1 \oplus 1_2$. With the help of the centralizer $C_{\mathrm{GL}(5,\mathbb{C})}(\mathrm{Aut}^+(\mathbf{R}_{5.5}))$ we may normalize any non-degenerate linear combination in $2 \times 1_2$ to

$$1_2 = \langle x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2 \rangle.$$

The only choice left is now between 1_5 and 1_6. Like for $\mathbf{R}_{5.3}$, both solutions define valid curves to serve as our canonical model. The components 1_5 and 1_6 are related by the field automorphism of $\mathbb{Q}(i)$ induced by $i \mapsto -i$. This field automorphism can be effected by standard complex conjugation con_4 on $\mathbb{P}^4$, switching the two curves. But they are also clearly switched by the projectivity

$$T : (x_1 : x_2 : x_3 : x_4 : x_5) \mapsto (x_1 : x_2 : x_3 : x_5 : x_4).$$

This not only shows us that up to isomorphism the canonical model is defined by $1_1 \oplus 1_2 \oplus 1_5$. It also means that $T \circ \mathrm{con}_4$ is an antiholomorphism of the model. In fact this mapping defines a reflection of the platonic map $\mathbf{R}_{5.5}$.

Remark A.5.6. This map is the Humbert map $\mathbf{Hum}_2(4)$, and as such we can also use the algebraic model in $\mathbb{P}^4$. For more information, see Section 5.6.

$\mathbf{R}_{5.6}$ type $(4, 8)$ **#cells** $(8, 32, 16)$ **map group size** 128 $D(\mathbf{Mod}(8))$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^4, S^8, (RS)^2, [R^2, S^2] \rangle$

This map is $D_2(\mathbf{R}_{5.1}) = D_2(\mathbf{Mod}(8))$ and thus has the same canonical model as $\mathbf{R}_{5.1}$.

$\mathbf{R}_{5.7}$ type $(4, 12)$ **#cells** $(4, 24, 12)$ **map group size** 96 $\mathbf{AM}(5)$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^4, S^{12}, (RS)^2, (RS^{-1})^2 \rangle$

This is the Accola-Maclachlan map **AM**(5) with planar model $y^2z^{10} = x^{12} - z^{12}$. For

more information, see Section 5.3.

$\mathbf{R}_{5.8}$ type $(4, 20)$ **#cells** $(2, 20, 10)$ **map group size** 80 $\mathbf{Wi2}(5)$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^4, S^{20}, (RS)^2, R^{-1}SRS^{-9} \rangle$

This map is the Wiman type II map $\mathbf{Wi2}(5)$ with planar model $y^2z^9 = x(x^{10} - z^{10})$. For more information, see Section 5.2.

$\mathbf{R}_{5.9}$ type $(5, 5)$ **#cells** $(16, 40, 16)$ **map group size** 160 $D(\mathbf{Hum}_1(4))$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^5, S^5, (RS)^2, [RS, SR] \rangle$

This map is $D(\mathbf{R}_{5.3}) = D(\mathbf{Hum}_1(4))$ and thus has the same canonical model as $\mathbf{R}_{5.3}$.

$\mathbf{R}_{5.10}$ type $(6, 6)$ **#cells** $(8, 24, 8)$ **map group size** 96 $D(\mathbf{Hum}_3(4))$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^6, S^6, (RS)^2, (RS^{-2})^2, (R^2S^{-1})^2 \rangle$

This map is $D(\mathbf{R}_{5.4}) = D(\mathbf{Hum}_3(4))$ and thus has the same canonical model as $\mathbf{R}_{5.4}$.

$\mathbf{R}_{5.11}$ type $(6, 15)$ **#cells** $(2, 15, 5)$ **map group size** 60
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^6, S^{15}, (RS)^2, R^{-1}SRS^{-4} \rangle$

The canonical linear representation ρ_c can be generated by

$$R \mapsto \mathrm{MonMat}([-\zeta_3, 1, \zeta_3, 1, \zeta_3^2], [1, 3, 2, 5, 4]),$$
$$S \mapsto \mathrm{MonMat}([\zeta_{15}^{10}, \zeta_{15}^{11}, \zeta_{15}^{14}, \zeta_{15}^{13}, \zeta_{15}^{7}], [1, 2, 3, 4, 5]).$$

It has invariant subspaces $\langle e_1 \rangle$, $\langle e_2, e_3 \rangle$, and $\langle e_4, e_5 \rangle$. The representation $(\rho_c^\vee)^{2+}$ decomposes as

$$1_5 \oplus 2 \times 1_6 \oplus 2_1 \oplus 2 \times 2_2 \oplus 2_3 \oplus 2_4 \oplus 2_6.$$

All these pieces contain irreducibles, leaving us to select invariant subspaces $1_6 < 2 \times 1_6$ and $2_2 < 2 \times 2_2$. These invariant subspaces all have the following shape:

$$1_6 = \langle x_1^2 + \lambda_1 x_4 x_5 \rangle,$$
$$2_2 = \langle x_1x_2 + \lambda_2 x_3 x_5,\ x_1x_3 - \lambda_2 \zeta_3 x_2 x_4 \rangle.$$

The centralizer $C_{\mathrm{GL}(5,\mathbb{C})}(\mathrm{Aut}^+(\mathbf{R}_{5.11}))$ allows us to scale the coordinates of either subspace $\langle e_1 \rangle$ or $\langle e_4, e_5 \rangle$. Scaling within $\langle e_2, e_3 \rangle$ however does not change the invariant subspaces. We may therefore set $\lambda_2 = 1$, but the centralizer is as yet of no further avail. We are left with the task of finding a suitable 1_6-subspace. After pondering this for a while, we compute that

$$(\lambda_1 - \zeta_3)x_2^2x_4x_5 = x_2^2(x_1^2 + \lambda_1 x_4x_5) - x_1x_2(x_1x_2 + x_3x_5) + x_2x_5(x_1x_3 - \zeta_3 x_2x_4) \in I,$$

so that if $\lambda_1 \neq \zeta_3$, we have a multiple of a linear form in the ideal. The only option is thus $\lambda_1 = \zeta_3$ and we conclude that

$$I_2 = \langle x_1^2 + \zeta_3 x_4x_5,\ x_1x_2 + x_3x_5,\ x_1x_3 - \zeta_3 x_2x_4 \rangle.$$

The plot thickens, because I_2 defines a surface, not a curve. This must be a rational normal scroll and the map $\mathbf{R}_{5.11}$ must be trigonal, as described in Section 6.1.

Remark A.5.7. This can be confirmed group theoretically, since the map admits the branched 3-cover

$$\pi : \mathbf{R}_{5.11} \longrightarrow \mathbf{R}_{5.11}/\langle R^3, S^3 \rangle \cong \mathbb{P}^1.$$

The cover π is not platonic, according to our definition in Section 1.6, since the subgroup contains RS, but that is of little import.

The way forward is to search in $(\rho_c^\vee)^{3+}$ for the additional ideal generators. Comparing the Hilbert Poincaré series of the ideal (I_2) to the one for our canonical ideal I reveals that I_3 is still missing a 2-dimensional subspace of irreducible degree 3 polynomials. The representation $(\rho_c^\vee)^{3+}$ decomposes as

$$2 \times 1_2 \oplus 1_3 \oplus 2 \times 1_4 \oplus 1_5 \oplus 1_6 \oplus 3 \times 2_1 \oplus 3 \times 2_2 \oplus 2_3 \oplus 2 \times 2_4 \oplus 3 \times 2_5 \oplus 2 \times 2_6.$$

All isotypic components contain reducibles, so we cannot use any of them completely. Computing the intersection of each component with $(I_2) \cap \mathbb{C}[x_1, \ldots, x_5]_3$, only the possibilities 1_3, 1_5 and subspaces of 2_1 or 2_2 remain. The first two are ruled out by application of the radical decomposition trick. All invariant subspaces of the latter two have the form

$$\begin{aligned} 2_1 &= \langle x_1^2 x_4 + \lambda_1 x_2^3 + \lambda_2 \zeta_3 x_4^2 x_5,\ x_1^2 x_5 + \lambda_1 \zeta_3^2 x_3^3 + \lambda_2 \zeta_3 x_4 x_5^2 \rangle, \\ 2_2 &= \langle x_1 x_4^2 + \lambda_1 x_2^2 x_3 + \lambda_2 x_5^3,\ x_1 x_5^2 - \lambda_1 x_2 x_3^2 - \lambda_2 \zeta_3 x_4^3 \rangle. \end{aligned}$$

Now we can use the remaining freedom of the centralizer $C_{\mathrm{GL}(5,\mathbb{C})}(\mathrm{Aut}^+(\mathbf{R}_{5.11}))$, which does not affect I_2, to set $\lambda_2 = 1$. This leads to the elimination of the first option, since choosing it would force

$$\lambda_1 x_2^3 = x_1^2 x_4 + \lambda_1 x_2^3 + \zeta_3 x_4^2 x_5 - x_4(x_1^2 + \zeta_3 x_4 x_5) \in I.$$

So we must use a 2_2 subspace and we have the parameter λ_1 left. It is time to apply the fixed point strategy. We compute the eigenspaces of R, which are $\langle e_2 + \zeta_3 e_3 \rangle$, $\langle e_2 - \zeta_3 e_3 \rangle$, $\langle e_4 - \zeta_6 e_5 \rangle$, and $\langle e_1, e_4 + \zeta_6 e_5 \rangle$. None of the 1-dimensional eigenspaces lies on the zero set of any choice of 2_2, unless $\lambda_1 = 0$. But that choice of parameter results in a non-prime ideal again. Hence, the fixed point p_R of R must correspond to an eigenvector $e_1 + \mu(e_4 + \zeta_6 e_5)$. Substituting this point into $x_1^2 + \zeta_3 x_4 x_5 \in I_2$, we find that $\mu = \pm 1$. This enables us to calculate λ_1 and check the resulting ideal. We find the unique solution $p_R = e_1 + e_4 + \zeta_6 e_5$ and the homogeneous ideal part

$$I_3 = \langle x_1 x_4^2 + x_2^2 x_3 + x_5^3,\ x_1 x_5^2 - x_2 x_3^2 - \zeta_3 x_4^3 \rangle.$$

The ideal (I_2, I_3) indeed defines a non-singular, irreducible curve of genus 5 that is our platonic surface.

$\mathbf{R}_{5.12}$ type $(8, 8)$ **#cells** $(4, 16, 4)$ **map group size** 64 $D(\mathbf{Hum}_2(4))$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^8, S^8, (RS)^2, (RS^{-1})^2, RS^3R^{-3}S^{-1} \rangle$

This map is $D(\mathbf{R}_{5.5}) = D(\mathbf{Hum}_2(4))$ and thus has the same canonical model as $\mathbf{R}_{5.5}$.

$\mathbf{R}_{5.13}$ type $(8, 8)$ **#cells** $(4, 16, 4)$ **map group size** 64 $D_1(D_2(\mathbf{Mod}(8)))$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^8, S^8, (RS)^2, [R^2, S], [R, S^2], R^4S^{-4} \rangle$

This map is $D_1(\mathbf{R}_{5.6}) = D_1(D_2(\mathbf{R}_{5.1}))$ and therefore has the same canonical model as $\mathbf{R}_{5.1}$, which is also $\mathbf{Mod}(8)$.

$\mathbf{R}_{5.14}$ type $(11, 22)$ **#cells** $(1, 11, 2)$ **map group size** 44 $\mathbf{Wi1}(5)$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^{11}, S^{22}, (RS)^2, RS^{-10} \rangle$

This map is the Wiman type I map $\mathbf{Wi1}(5)$ with planar model $y^2z^9 = x^{11} - z^{11}$. For more information, see Section 5.1.

$\mathbf{R}_{5.15}$ type $(12, 12)$ **#cells** $(2, 12, 2)$ **map group size** 48 $D(\mathbf{AM}(5))$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^{12}, S^{12}, (RS)^2, [R, S] \rangle$

This map is $D(\mathbf{R}_{5.7}) = D(\mathbf{AM}(5))$ and therefore also has planar model $y^2z^{10} = x^{12} - z^{12}$. For more information, see Section 5.3.

$\mathbf{R}_{5.16}$ type $(20, 20)$ **#cells** $(1, 10, 1)$ **map group size** 40 $D(\mathbf{Wi2}(5))$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^{20}, S^{20}, (RS)^2, RS^{-9} \rangle$

This map is $D(\mathbf{R}_{5.8}) = D(\mathbf{Wi2}(5))$ and therefore also has planar model $y^2z^9 = x(x^{10} - z^{10})$. For more information, see Section 5.2.

A.6 Genus 6

Prior remarks. The Hilbert-Poincaré series predicts (cf. Section 6.1) that the canonical model of a non-hyperelliptic platonic map of genus 6 satisfies $\dim(I_2) = 6$. If the map is trigonal, some additional cubics will be needed to define the model. Also, as we noted in Section 6.1, for $g = 6$ we have another possible exceptional situation besides the occurrence of trigonal curves: algebraic curves birationally equivalent to a non-singular planar quintic. That this does indeed occur is exemplified by $\mathbf{Fer}(5)$. For other platonic maps we encounter, we distinguish between these two cases by exhibiting explicitly either a 3-cover of $\mathbb{P}^1$ or a birational mapping to a non-singular planar quintic.

R$_{6.1}$ **type** $(3, 10)$ **#cells** $(15, 75, 50)$ **map group size** 300 $\mathbf{Fer}(5)$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^3, S^{10}, (RS)^2, (S^2R^{-1})^3 \rangle$

From the standard map presentation we see this is the platonic map $\mathbf{Fer}(5)$, introduced in Section 2.9. As we elaborated in Section 5.5, the corresponding platonic surface is defined by the Fermat curve $x^5 + y^5 + z^5 = 0$. Still, for the record and to demonstrate our construction strategy once more, we proceed to discover a (relatively nice) canonical model for it.

The canonical representation ρ_c can be generated by

$$R \mapsto \mathrm{MonMat}([1, 1, 1, 1, 1, 1], [2, 3, 1, 5, 6, 4]),$$
$$S \mapsto \mathrm{MonMat}([-\zeta_5^4, -1, -\zeta_5, -\zeta_5^2, -\zeta_5^3, -1], [2, 1, 3, 4, 6, 5]).$$

and has invariant subspaces $\langle e_1, e_2, e_3 \rangle$ and $\langle e_4, e_5, e_6 \rangle$. The representation $(\rho_c^\vee)^{2+}$ decomposes as

$$2 \times 3_1 \oplus 3_2 \oplus 2 \times 3_5 \oplus 6_1.$$

All isotypic pieces contain reducibles, so we must use an invariant $3_1 < 2 \times 3_1$ and an invariant $3_5 < 2 \times 3_5$. The two isotypic components are

$$2 \times 3_1 = \langle x_1x_5, x_2x_6, x_3x_4, x_4x_5, x_4x_6, x_5x_6 \rangle,$$
$$2 \times 3_5 = \langle x_1x_2, x_1x_3, x_2x_3, x_4^2, x_5^2, x_6^2 \rangle.$$

All their 1-dimensional invariant subspaces are of the form

$$3_1 = \langle x_1x_5 + \lambda_1x_4x_6, x_2x_6 + \lambda_1x_4x_5, x_3x_4 + \lambda_1x_5x_6 \rangle,$$
$$3_5 = \langle x_1x_2 + \lambda_2x_4^2, x_1x_3 + \lambda_2x_6^2, x_2x_3 + \lambda_2x_5^2 \rangle.$$

With the centralizer trick we set $\lambda_1 = -1$. Now we compute the eigenspaces of R to apply the fixed point strategy:

$$E(R, \zeta_3) = \langle e_1 - \zeta_6e_2 + \zeta_6^2e_3, e_4 - \zeta_6e_5 + \zeta_6^2e_6 \rangle,$$
$$E(R, \zeta_3^2) = \langle e_1 + \zeta_6^2e_2 - \zeta_6e_3, e_4 + \zeta_6^2e_5 - \zeta_6e_6 \rangle,$$
$$E(R, 1) = \langle e_1 + e_2 + e_3, e_4 + e_5 + e_6 \rangle.$$

Any point in the third space yields an orbit size smaller than 50, so we must use one of the other two. These two are each others complex conjugates. The constraints that our choice of 3_1 lays on them leaves one suitable eigenvector in each of them: $-\zeta_6 b_1 + b_2$ and $-\zeta_6^2 b_1 - b_2$ in terms of their respective basis vectors. The two vectors generate the same $\mathrm{Aut}^+(\mathbf{R}_{6.1})$-orbit, which has to be the set of cell centers of the canonical model, 50 in total. These points also force $\lambda_2 = -1$, so that we have determined I_2. As expected, this does not define a curve, but the Veronese surface $V_{2,2}$.

The Hilbert-Poincaré series of (I_2) tells us we need three more irreducible polynomials to generate I_3. The representation $(\rho_c^\vee)^{3+}$ decomposes as

$$3 \times 1_2 \oplus 2_1 \oplus 3_1 \oplus 2 \times 3_4 \oplus 2 \times 3_6 \oplus 2 \times 3_7.$$

All the isotypic components contain reducibles, and (I_2) already contains invariant subspaces $2 \times 1_2$, 2_1, 3_4, and 3_6. The only remaining option is to add a $3_7 < 2 \times 3_7$. The cell centers yield enough constraints on the component to single out an invariant 3_7, and we find

$$\begin{aligned} I = \langle\ & x_1x_5 - x_4x_6,\ x_2x_6 - x_4x_5,\ x_3x_4 - x_5x_6,\ x_1x_2 - x_4^2,\ x_1x_3 - x_6^2,\ x_2x_3 - x_5^2, \\ & x_1^3 + x_2^2x_4 + x_3^2x_6,\ x_1^2x_4 + x_2^3 + x_3^2x_5,\ x_1^2x_6 + x_2^2x_5 + x_3^3\ \rangle. \end{aligned}$$

The elimination ideal to the variables (x_1, x_4, x_6) gives us back the standard planar curve in the shape of $x_1^5 + x_4^5 + x_6^5 = 0$.

$\mathbf{R}_{6.2}$ type $(4,6)$ **#cells** $(20,60,30)$ **map group size** 240 **Wiman's 1st sextic map**
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^4, S^6, (RS)^2, (RS^{-1})^3 \rangle$

The canonical presentation ρ_c can be induced from an index 6 subgroup and generated by

$$\begin{aligned} R &\mapsto \mathrm{MonMat}([i, 1, -i, 1, i, -i], [1, 3, 5, 2, 4, 6]), \\ S &\mapsto \mathrm{MonMat}([-i, -1, 1, i, 1, -1], [2, 4, 1, 5, 6, 3]). \end{aligned}$$

It is the unique irreducible 6-dimensional representation of $\mathrm{Aut}^+(\mathbf{R}_{6.2})$. The representation $(\rho_c^\vee)^{2+}$ decomposes as

$$1_1 \oplus 1_2 \oplus 4_1 \oplus 2 \times 5_1 \oplus 5_2.$$

We must have either $I_2 = 1_1 \oplus 1_2 \oplus 4_1$ or $I_2 = 1_k \oplus 5_l$. The first possibility does not generate a prime ideal, and neither do the two ideals involving 5_2. The conclusion is that we need to find a suitable invariant subspace $5_1 < 2 \times 5_1$ and add 1_1 or 1_2 to it.

We now apply the fixed point strategy. The automorphism S has six 1-dimensional eigenspaces. The ones at ± 1 yield an orbit that is smaller than 20 and therefore unsuitable. There are two other $\mathrm{Aut}^+(\mathbf{R}_{6.2})$-orbits: an orbit containing one point each from $E(S, \zeta_6^{\pm 1})$ and an orbit with one point each from $E(S, \zeta_6^{\pm 2})$. To be explicit:

$$\begin{aligned} E(S, \zeta_6) &= \langle (1, -\zeta_{12}, \zeta_{12}^2, \zeta_{12}^{-1}, 1, \zeta_{12}^{-2}) \rangle, \\ E(S, \zeta_6^2) &= \langle (1, \zeta_{12}^5, \zeta_{12}^4, -\zeta_{12}, -1, \zeta_{12}^2) \rangle \end{aligned}$$

Either one of the two orbits yields enough conditions to determine a correct $5_1 < 2 \times 5_1$. And in each case, only $1_1 \oplus 5_1$ gives a prime ideal. Those two prime ideals indeed define irreducible, non-singular curves of genus 6 modelling our platonic surface. The two solutions are isomorphic (as they should be), being interchanged by the transformation $\mathrm{DiaMat}(-1,-1,1,1,1,-1) \circ \mathrm{con}_6$. Picking the first one, we can write down the canonical ideal as

$$
\begin{aligned}
I = \langle\, & x_1x_2 - ix_1x_3 + ix_1x_4 - ix_1x_5 + x_1x_6 - ix_2x_3 - ix_2x_4 + ix_2x_5 + x_2x_6 + x_3x_4 + x_3x_5 + \\
& ix_3x_6 + x_4x_5 - ix_4x_6 - ix_5x_6, \qquad 25x_1^2 - 5x_2^2 - (6i+12)x_2x_3 + (6i+12)x_2x_4 + \\
& (-6i-12)x_2x_5 + (-12i+6)x_2x_6 + 5x_3^2 - (12i-6)x_3x_4 + (12i-6)x_3x_5 - \\
& (6i+12)x_3x_6 + 5x_4^2 - (12i-6)x_4x_5 + (6i+12)x_4x_6 + 5x_5^2 + (-6i-12)x_5x_6 - 5x_6^2, \\
& 5x_1x_2 - 5x_1x_6 - (2i+1)x_2^2 + 3ix_2x_3 + 3ix_2x_4 + ix_2x_5 - (4i+2)x_3^2 + 4x_3x_4 - ix_3x_6 - \\
& 4x_4x_5 - 3ix_4x_6 + (4i+2)x_5^2 - 3ix_5x_6 + (2i+1)x_6^2, \quad 5x_1x_3 - 5ix_1x_6 - 3x_2x_3 - 4x_2x_4 \\
& - 4x_2x_5 + 3ix_2x_6 - (i-2)x_3^2 + ix_3x_4 - 3ix_3x_5 - (2i-4)x_4^2 + 3x_4x_6 + (2i-4)x_5^2 - \\
& x_5x_6 - (i-2)x_6^2, \quad 5x_1x_4 + 5ix_1x_6 + (2i-4)x_2^2 + 3x_2x_4 + 4x_2x_5 + ix_2x_6 + (2i-4)x_3^2 \\
& - ix_3x_4 + 4ix_3x_5 + 3x_3x_6 + (i-2)x_4^2 + 3ix_4x_5 - 3x_5x_6 + (i-2)x_6^2, \\
& 5x_1x_5 - 5ix_1x_6 - (2i-4)x_2^2 - 4x_2x_3 - x_2x_5 + 3ix_2x_6 + 4ix_3x_4 - 3ix_3x_5 - 3x_3x_6 - \\
& (2i-4)x_4^2 - 3ix_4x_5 - x_4x_6 + (i-2)x_5^2 + (i-2)x_6^2 \,\rangle.
\end{aligned}
$$

By Galois descent it should be possible to define the canonical model over $\mathbb{Q}$, but that is a story yet to unfold.

Remark A.6.1. We call $\mathbf{R}_{6.2}$ *Wiman's first sextic map*, because its platonic surface has got a rather famous planar model, discovered by Anders Wiman ([Wim1897]):

$$x^6 + y^6 + z^6 + (x^2 + y^2 + z^2)(x^4 + y^4 + z^4) - 12x^2y^2z^2 = 0.$$

The adjective 'first' is used to avoid confusion with $\mathbf{R}_{10.6}$. The planar model above is singular, with four nodes $(\pm 1 : \pm 1 : 1)$. A natural road to its construction, starting from the permutation action of Sym_5 on $\mathbb{P}^4$ is presented in [IK2005]. This sextic curve is sure to be a platonic surface, since its automorphism group has size greater than $24(g-1)$ (see Section 1.4). It has the following standard map presentation:

$$
\begin{aligned}
R : & \ (x : y : z) \mapsto \\
& (x^2 - xy - y^2 - xz + yz + z^2 : x^2 - xy + y^2 - xz + yz - z^2 : x^2 + xy - y^2 + xz - yz - z^2) \\
S : & \ (x : y : z) \mapsto \\
& (-x^2 + xy - y^2 + xz - yz + z^2 : -x^2 - xy + y^2 - xz + yz + z^2 : x^2 - xy - y^2 - xz + yz + z^2)
\end{aligned}
$$

This group contains the transformations permuting the variables and diagonal transformations $\mathrm{DiaMat}(\pm 1 \pm 1, \pm 1)$. Since the curve is defined over $\mathbb{Q}$, standard complex conjugation con_2 on $\mathbb{P}^2$ is an antiholomorphic automorphism. The curve has no real wall, but it does have four real points: its four nodes. As points they are fixed by con_2, but locally the two branches meeting at a node get switched; con_2 is in fact the antipodal mapping $R^{-1}c[R,S]^2 \in \mathrm{Aut}^-(\mathbf{R}_{6.2})$, generating the center of $\mathrm{Aut}(\mathbf{R}_{6.2})$. A reflection is defined by any composition of con_2 with a non-trivial element of $\mathrm{Aut}^+(\mathbf{R}_{6.2})$. Computation of Weierstraß weights (cf. Section 7.1) proves that there lie non-geometric Weierstraß points on the geodesic walls, but they have yet to be constructed.

$\mathbf{R}_{6.3}$ type $(4, 9)$ **#cells** $(8, 36, 18)$ **map group size** 144
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^4, S^9, (RS)^2, (RS^{-2})^2 \rangle$

The canonical representation ρ_c can be generated by:

$$R \mapsto \mathrm{MonMat}([1, 1, 1, -1, -1, 1], [4, 5, 6, 1, 2, 3]),$$
$$S \mapsto \mathrm{MonMat}([1, \zeta_3^2, 1, 1, -1, -\zeta_3], [2, 3, 1, 6, 4, 5]).$$

This is an irreducible 6_1. The representation $(\rho_c^\vee)^{2+}$ decomposes as

$$1_1 \oplus 2_1 \oplus 2_2 \oplus 2_3 \oplus 2_4 \oplus 2 \times 3_1 \oplus 6_1.$$

Since both 6_1 and $2 \times 3_1$ contain reducibles, I_2 is either some $2_x \oplus 2_y \oplus 2_z$ or some $1_1 \oplus 2_x \oplus 3_1$ for a suitable invariant $3_1 < 2 \times 3_1$. The first of these two options results in four possible ideals, which all yield ideals with multiples of linear forms. The second option must hold true. We apply the fixed point strategy to find the cell centers of $\mathbf{R}_{6.3}$, and compute the eigenspaces of R:

$$E(R, i) = \langle e_1 - ie_4, e_2 - ie_5 \rangle, \qquad E(R, -1) = \langle e_3 - e_6 \rangle,$$
$$E(R, -i) = \langle e_1 + ie_4, e_2 + ie_5 \rangle, \qquad E(R, 1) = \langle e_3 + e_6 \rangle.$$

The 1_1 isotypic component gives no constraints on these fixed points. But the component $2_1 = \langle x_1x_5 + \zeta_3x_3x_6, x_2x_4 - \zeta_3x_3x_6 \rangle$ does. It forces us to choose one of the two basis vectors of $E(R, \pm i)$. Those generate two different $\mathrm{Aut}^+(\mathbf{R}_{6.3})$-orbits, but both yield constraints on $2 \times 3_1$ that result in a 3_1-piece containing reducibles. The upshot is that we cannot use 2_1.

The three representations 2_2, 2_3, and 2_4 are algebraically conjugated, and hence the isotypic components we found are as well. Since we know that there is only one type $(4, 9)$ platonic map of genus 6, we know in advance that the three options will lead to isomorphic curves. Let us choose the first,

$$2_2 = \langle x_1^2 + \zeta_9x_2^2 + \zeta_9^{-1}x_3^2, x_4^2 + \zeta_9x_5^2 + \zeta_9^{-1}x_6^2 \rangle.$$

Now the only possible cell centers are the orbits of $\pm\zeta_{36}^{11}b_1 + b_2$ in terms of the basis of $E(R, i)$. This in turn yields constraints on $2 \times 3_1$, and we find 3_1. The ideal (I_2) defines a surface instead of a curve, and $\dim((I_2) \cap \mathbb{C}[x_1, \ldots, x_6]_3) = 28$ tells us that we need 3 extra irreducible polynomials of degree 3 to define our canonical curve. The representation $(\rho_c^\vee)^{3+}$ decomposes as

$$2_1 \oplus 2_2 \oplus 2_3 \oplus 2_4 \oplus 3 \times 3_1 \oplus 3 \times 3_2 \oplus 5 \times 6_1,$$

so we need either an invariant $3_1 < 3 \times 3_1$ or a $3_1 < 3 \times 3_1$. We apply the fixed point strategy once more, this time computing the map vertices. The automorphism S has six 1-dimensional eigenspaces. Any one gives enough constraints to determine a unique $\mathrm{Aut}^+(\mathbf{R}_{6.3})$-invariant $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$. Only for the eigenvalues $\zeta_9^{\pm 1}$ does this result in a prime ideal, and the choice between $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$ gives two curves easily seen to be isomorphic. If we choose $\mathfrak{Z}_1$, we can write down the following canonical ideal:

$$\begin{aligned} I = \langle\, & x_1x_5 - x_2x_4 + \zeta_6^{-1}x_3x_6,\ x_1^2 + \zeta_9x_2^2 + \zeta_9^{-1}x_3^2,\ \ x_4^2 + \zeta_9x_5^2 + \zeta_9^{-1}x_6^2,\ x_1x_4 + \zeta_9x_2x_5, \\ & x_1x_6 + \zeta_9^5x_3x_5, \quad x_2x_6 - \zeta_9^4x_3x_4, \quad x_1^3 - \zeta_9^2x_1x_3^2 - \zeta_9^2x_4^2x_5 + x_5^3, \\ & x_1^2x_2 - \zeta_9^{-2}x_2^3 + \zeta_9^{-2}x_4^3 - x_4x_6^2, \quad x_2^2x_3 - \zeta_9^4x_3^3 - x_5^2x_6 + \zeta_9^4x_6^3\ \rangle. \end{aligned}$$

Is the curve trigonal, or the canonical model of a non-singular planar quintic? The MAGMA system computed a parametrization of the scroll S defined by (I_2) for us. There is a birational transformation $\phi : \mathbb{P}^2 \to S$ defined by

$$\begin{aligned} \phi :& (x:y:z) \mapsto \\ & (2\zeta_{36}^{31}x^2y : x^3 + \zeta_{36}^4xy^2 : \zeta_{36}^{31}x^3 + \zeta_{36}^{17}xy^2 : x^2z + \zeta_{36}^4y^2z : 2\zeta_{36}^9xyz : \zeta_{36}^{11}x^2z + \zeta_{36}^{33}y^2z), \end{aligned}$$

with inverse $\phi^{-1} : (x_1 : x_2 : x_3 : x_4 : x_5 : x_6) \mapsto (x_2 + \zeta_{36}^5x_3 : \zeta_{36}^5x_1 : x_4 - \zeta_{36}^7x_6)$. If one pulls back the ideal generators along ϕ this results in the following planar curve:

$$x^7 + 2(\zeta_{36}^{10} + \zeta_{36}^{16})x^5y^2 + \zeta_{36}^8x^3y^4 - \zeta_{36}^6x^4z^3 - 2(\zeta_{36}^{10} + \zeta_{36}^{16})x^2y^2z^3 + \zeta_{36}^{32}y^4z^3 = 0.$$

This curve has the obvious degree 3 mapping $(x : y : z) \mapsto (x : y)$ to $\mathbb{P}^1$. Thus the map $\mathbf{R}_{6.3}$ is trigonal, and we have found a rational mapping, defined on a Zariski open set of the curve by

$$(x_1 : x_2 : x_3 : x_4 : x_5 : x_6) \mapsto (x_2 + \zeta_{36}^5x_3 : \zeta_{36}^5x_1)$$

certifying trigonality.

$\mathbf{R}_{6.4}$ type $(4, 14)$ **#cells** $(4, 28, 14)$ **map group size** 112 **$\mathbf{AM}(6)$**
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^4, S^{14}, (RS)^2, (RS^{-1})^2 \rangle$

This is the Accola-Maclachlan map $\mathbf{AM}(6)$ with planar model $y^2z^{12} = x^{14} - z^{14}$. For more information, see Section 5.3.

$\mathbf{R}_{6.5}$ type $(4, 24)$ **#cells** $()$ **map group size** 96 **$\mathbf{Wi2}(6)$**
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^4, S^{24}, (RS)^2, R^{-1}SRS^{-11} \rangle$

This is the Wiman type II map $\mathbf{Wi2}(6)$ with planar model $y^2z^{11} = x(x^{12} - z^{12})$. For

more information, see Section 5.2.

$\mathbf{R}_{6.6}$ type $(5,10)$ **#cells** $(5,25,10)$ **map group size** 100 $D(\mathbf{Fer}(5))$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^5, S^{10}, (RS)^2, [R, S^2] \rangle$

This map is $D(\mathbf{R}_{6.1}) = D(\mathbf{Fer}(5))$ and thus has the same algebraic models as $\mathbf{Fer}(5)$.

$\mathbf{R}_{6.7}$ type $(6,8)$ **#cells** $(6,24,8)$ **map group size** 96
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^1, S^1, (RS)^2, (RS^{-1})^2 \rangle$

The canonical representation ρ_c can be generated by:

$$R \mapsto \mathrm{MonMat}([-\zeta_{12}^2 + 1, -\zeta_{12}^2, 1, \zeta_{12}^2 - 1, -\zeta_{12}^3 + \zeta_{12}, -\zeta_{12}^3], [1,2,4,3,6,5]),$$
$$S \mapsto \mathrm{MonMat}([-1, 1, 1, 1, \zeta_{12}^3, -\zeta_{12}^3], [2,1,5,6,3,4]).$$

It has invariant subspaces $\langle e_1, e_2 \rangle$ and $\langle e_3, e_4, e_5, e_6 \rangle$. The representation $(\rho_c^\vee)^{2+}$ decomposes as

$$1_1 \oplus 2 \times 1_4 \oplus 2 \times 2_1 \oplus 2_3 \oplus 2_4 \oplus 2_5 \oplus 2_6 \oplus 2_7 \oplus 4_1.$$

The pieces $2 \times 1_4$, $2 \times 2_1$, 2_3, 2_4, 2_5 and 4_1 contain multiples of linear forms, as does the space $2_6 \oplus 2_7$. We deduce that I_2 is of the form

$$1_1 \oplus 1_4 \oplus 2_1 \oplus 2_6 \quad \text{where } k \in \{6, 7\}.$$

The spaces 2_6 and 2_7 are

$$2_6 = \langle x_1x_3 + (\zeta_{24}^5 - \zeta_{24})x_2x_5, x_1x_4 - \zeta_{24}^3 x_2x_6 \rangle,$$
$$2_7 = \langle x_1x_3 - (\zeta_{24}^5 - \zeta_{24})x_2x_5, x_1x_4 + \zeta_{24}^3 x_2x_6 \rangle,$$

and are related by the projectivity mapping $x_2 \mapsto -x_2$ and fixing the other coordinates. This projectivity leaves 1_1, $2 \times 1_4$ and $2 \times 2_1$ invariant, so we may choose 2_6 up to an isomorphism of our final curve. To proceed we need to determine suitable invariant spaces $1_4 < 2 \times 1_4$ and $2_1 < 2 \times 2_1$. We have

$$1_1 = \langle x_3x_6 + ix_4x_5 \rangle,$$
$$2 \times 1_4 = \langle x_1x_2, x_3x_6 - ix_4x_5 \rangle,$$
$$2 \times 2_1 = \langle x_1^2, x_2^2, x_3x_4, x_5x_6 \rangle,$$

and with the centralizer $C_{\mathrm{GL}(6,\mathbb{C})}(\mathrm{Aut}^+(\mathbf{R}_{6.7}))$ we can choose $1_4 = \langle x_1x_2 + x_3x_6 - ix_4x_5 \rangle$. Now we compute the primary decomposition of the ideal $(1_1 \oplus 1_4 \oplus 2_6)$, which leaves one possibility for our 2_1 subspace. However, the resulting prime ideal defines a surface, and again we are either in the trigonal or non-singular planar quintic case. The Hilbert series tells us we need three extra irreducible polynomials in degree 3 to generate the canonical ideal. The representation $(\rho_c^\vee)^{3+}$ decomposes as

$$1_1 \oplus 1_2 \oplus 1_3 \oplus 1_4 \oplus 2 \times 2_1 \oplus 2 \times 2_2 \oplus 2 \times 2_3 \oplus 3 \times 2_4 \oplus 2_5 \oplus 3 \times 2_6 \oplus 3 \times 2_7 \oplus 5 \times 4_1.$$

Considering $\dim((I_2) \cap \mathbb{C}[x_1, \ldots, x_6]_3)$ and considering multiples of linear forms in the isotypic pieces, we come to the conclusion that we can only use 1_k, 2_6 and 2_7, so we certainly need a 1-dimensional piece. The four possibilities are

$$\pm\zeta_6 x_1 x_5^2 + x_1 x_6^2 \pm \zeta_6 x_2 x_3^2 + x_2 x_4^2$$

All four choices of 1_k yield a canonical model of $\mathbf{R}_{6.7}$ when computing the radical of $(I_2 \oplus 1_k)$. Choosing 1_1 we can write down this model as

$$\begin{aligned} I = \langle\ & x_1^2 - \zeta_8 x_5 x_6,\ \ x_2^2 - \zeta_8 x_3 x_4,\ \ x_1 x_2 - x_4 x_5,\ \ -\zeta_8^2 x_3 x_6 + x_4 x_5,\ \ \zeta_8^5 x_1 x_3 + x_2 x_5, \\ & \zeta_8^{-1} x_1 x_4 - x_2 x_6, -\zeta_{24}^4 x_1 x_5^2 + x_1 x_6^2 - \zeta_{24}^4 x_2 x_3^2 + x_2 x_4^2, \\ & -\zeta_{24}^3 x_3^3 + \zeta_{24}^{-1} x_3 x_4^2 - x_5^3 + \zeta_{24}^{-4} x_5 x_6^2, -\zeta_{24}^3 x_3^2 x_4 + \zeta_{24}^{-1} x_4^3 - \zeta_{24}^6 x_5^2 x_6 + \zeta_{24}^2 x_6^3\ \rangle. \end{aligned}$$

Is the curve trigonal, or the canonical model of a non-singular planar quintic? The MAGMA system computed a parametrization of the scroll S defined by (I_2) for us. There is a birational transformation $\phi : \mathbb{P}^2 \to S$ defined by

$$(x:y:z) \mapsto (4x(x^2-y^2) : 4z(x^2-y^2) : 8iz(x+y)^2 : \zeta_8^3 z(x-y)^2 : 8\zeta_8^3 x(x+y)^2 : ix(x-y)^2)$$

with inverse $(x_1 : x_2 : x_3 : x_4 : x_5 : x_6) \mapsto (4x_1 - \zeta_8 x_5 - 8ix_6 : -\zeta_8 x_5 + 8ix_6 : 4x_2 - ix_3 - 8\zeta_8 x_4)$. If one pulls back the ideal generators along ϕ this results in the following planar curve, where $\alpha = 4/16773121(-128\zeta_{12}^3 + 8192\zeta_{12}^2 - 524160\zeta_{12} + 16773119)$:

$$x^7 + \alpha x^6 y + 6x^5 y^2 + \alpha x^4 y^3 + x^3 y^4 - \frac{1}{4} i\alpha x^4 z^3 - 4ix^3 y z^3 - \frac{3}{2} i\alpha x^2 y^2 z^3 - 4ixy^3 z^3 - \frac{1}{4} i\alpha y^4 z^3 = 0.$$

This curve has the obvious degree 3 mapping $(x:y:z) \mapsto (x:y)$ to $\mathbb{P}^1$. Thus, the map $\mathbf{R}_{6.7}$ is trigonal, and we have found a rational mapping, defined on a Zariski open set of the curve by

$$(x_1 : x_2 : x_3 : x_4 : x_5 : x_6) \mapsto (4x_1 - \zeta_8 x_5 - 8ix_6 : -\zeta_8 x_5 + 8ix_6)$$

certifying trigonality.

$\mathbf{R}_{6.8}$ type $(6,8)$ **#cells** $(6,24,8)$ **map group size** 96
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^6, S^8, (RS)^2, (R^2 S^{-1})^2, R^3 S^4 \rangle$

Consideration of the structure of $\mathrm{Aut}^+(\mathbf{R}_{6.8})$ leads us to the conclusion that there is a platonic 2-cover $\pi : \mathbf{R}_{6.8} \to \mathbf{R}_{6.8}/\langle R^3, S^4\rangle = \mathbf{Oct}$, branched over $\mathrm{cells}_0 \cup \mathrm{cells}_2$. The map is therefore hyperelliptic, and indeed we saw it in Theorem 6.3.2. Referring back to Section 6.4, we construct a planar curve by multiplying the polynomial (branch) data for the vertices of **Oct** and its dual **Cub** to arrive at $y^2 z^{12} = -xz(x^{12} - 13x^8 z^4 + 13x^4 z^8 + z^{12})$. It is reducible, but dividing out the component $z = 0$ and transforming by $y \mapsto iy$, we find the planar model

$$y^2 z^{11} = x^{13} + 13x^9 z^4 - 13x^5 z^8 - xz^{12}.$$

This model has the standard map presentation

$$\begin{aligned} R :&\quad (x:y:z) \mapsto (-i(x-z)^6(x+z) : 8(i-1)yz^6 : (x-z)^7), \\ S :&\quad (x:y:z) \mapsto (ix : \zeta_8 y : z). \end{aligned}$$

Standard complex conjugation con_6 is a reflection on the curve.

$\mathbf{R}_{6.9}$ **type** $(9,9)$ **#cells** $(4,18,4)$ **map group size** 72
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^9, S^9, (RS)^2, R^3S^3 \rangle$

This map is $D(\mathbf{R}_{6.3})$ and therefore has the same canonical model as $\mathbf{R}_{6.3}$.

$\mathbf{R}_{6.10}$ **type** $(10,15)$ **#cells** $(2,15,3)$ **map group size** 60
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^{10}, S^{15}, (RS)^2, R^{-1}SRS^{-11} \rangle$

The canonical representation ρ_c can be generated by

$$R \mapsto \mathrm{MonMat}([-\zeta_{15}^3, -\zeta_{15}^6, 1, \zeta_{15}^9, 1, \zeta_{15}^{12}], [1,2,4,3,6,5]),$$
$$S \mapsto \mathrm{MonMat}([\zeta_{15}^{12}, \zeta_{15}^9, \zeta_{15}^8, \zeta_{15}^{13}, \zeta_{15}^{14}, \zeta_{15}^4], [1,2,3,4,5,6]).$$

It has invariant subspaces $\langle e_1 \rangle$, $\langle e_2 \rangle$, $\langle e_3, e_4 \rangle$, and $\langle e_5, e_6 \rangle$. The representation $(\rho_c^\vee)^{2+}$ decomposes as

$$1_5 \oplus 2 \times 1_6 \oplus 2 \times 1_7 \oplus 1_9 \oplus 1_{10} \oplus 2 \times 2_3 \oplus 2 \times 2_4 \oplus 2 \times 2_5.$$

All pieces but 1_9 and 1_{10} contain multiples of linear forms, restricting our options. We have

$$2 \times 1_6 = \langle x_2^2, x_5x_6 \rangle, \qquad 1_9 = \langle x_3x_6 - \zeta_5^2 x_4x_5 \rangle,$$
$$2 \times 1_7 = \langle x_1x_2, x_3x_4 \rangle, \qquad 1_{10} = \langle x_3x_6 + \zeta_5^2 x_4x_5 \rangle,$$

and the invariant subspaces of the 2_k are all of the form

$$2_3 = \langle x_1x_5 + \lambda_1 x_4^2, x_1x_6 - \lambda_1 x_3^2 \rangle,$$
$$2_4 = \langle x_2x_5 + \lambda_2\zeta_5 x_6^2, x_2x_6 - \lambda_2 x_5^2 \rangle,$$
$$2_5 = \langle x_2x_3 + \lambda_3 x_4x_6, x_2x_4 - \lambda_3 x_3x_5 \rangle.$$

Since $x_3x_6 \in 1_9 \oplus 1_{10}$ we cannot choose both these pieces, and the shape of I_2 is either $1. \oplus 1. \oplus 2. \oplus 2.$ or $2. \oplus 2. \oplus 2.$. Assuming the first shape, we need at least one of 1_7, 1_9, or 1_{10}. Suppose we choose 1_7 and 2_4. They can be normalized simultaneously (with the centralizer trick), so we can assume $\lambda_2 = 1$ and $1_7 = \langle x_1x_2 + x_3x_4 \rangle$. Computing its primary decomposition gives associated primes with too many quadrics. The same is true when adding 1_9 or 1_{10} to 2_4. Therefore we cannot use 2_4 if we choose the first shape, and apparently must use $2_3 \oplus 2_5$. These two components also appear for the second ideal shape, so we are obliged to utilize them, and may normalize them simultaneously. We choose $\lambda_1 = \lambda_3 = 1$ and compute the primary decomposition of $2_3 \oplus 2_5$, which yields

$$I_2 = \langle x_2^2 + x_5x_6,\ \ x_1x_2 + x_3x_4,\ \ x_1x_5 + x_4^2,\ \ x_1x_6 - x_3^2,\ \ x_2x_3 + x_4x_6,\ \ x_2x_4 - x_3x_5 \rangle,$$

or in other words $I_2 = 1_6 \oplus 1_7 \oplus 2_3 \oplus 2_5$, and we incorporate the sum of the two basis vectors shown for both 1_6 and 1_7. The ideal (I_2) defines a surface, however, and we find ourselves again in either the trigonal or non-singular planar quintic case. The Hilbert series tells us we need three extra irreducible polynomials in degree 3 to generate the canonical ideal. The representation $(\rho_c^\vee)^{3+}$ decomposes into a great number isotypic components. A number of them are excluded because extending $(I_2) \cap \mathbb{C}[x_1, \ldots, x_6]_3$ already forces us to use the whole component and that contains a multiple of a linear form. Also, all irreducible representations of $\mathrm{Aut}^+(\mathbf{R}_{6.10})$ are 1- or 2-dimensional, so we will certainly use a 1-dimensional piece. The 1-dimensional subspaces we may select are

$$\begin{aligned}
1_3 &< 3 \times 1_3 = \langle x_1^3, x_2x_3x_6 + \zeta_5^2 x_2x_4x_5, x_3x_5^2 + \zeta_{10}x_4x_6^2 \rangle, \\
1_9 &< 3 \times 1_9 = \langle x_2^3, x_2x_5x_6, x_5^3 - \zeta_5 x_6^3 \rangle, \\
1_{10} &= \langle x_5^3 + \zeta_5 x_6^3 \rangle.
\end{aligned}$$

The associated primes resulting from adding 1_{10} all contain linear forms. We now apply the fixed point strategy. The polynomials of I_2 give constraints on the eigenspaces of R, S and RS. In fact, they yield immediately that the vertices are $\{e_5, e_6\}$ and the face centers form the $\mathrm{Aut}^+(\mathbf{R}_{6.10})$-orbit of $\zeta_5^{-1}e_2 + e_5 + \zeta_{10}e_6$. These points in turn give constraints on the polynomials in I_3, and leave only one polynomial in $3 \times 1_9$ available, which is already in (I_2). Therefore, we are searching for a $1_3 < 3 \times 1_3$. Next, we try to find the 15 edge centers of the map. The eigenspaces of RS are $E(RS, -1) = \langle e_1, e_2, e_3 + \zeta_{30}^{11}e_4, e_5 + \zeta_{30}^{-7}e_6 \rangle$ and $E(RS, 1) = \langle e_3 - \zeta_{30}^{11}e_4, e_5 - \zeta_{30}^{-7}e_6 \rangle$. The ideal (I_2) allows us to eliminate $E(RS, 1)$ at once. Within $E(RS, -1)$, the constraints say a vector $\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3(e_3 + \zeta_{30}^{11}e_4) + \lambda_4(e_5 + \zeta_{30}^{-7}e_6)$ must satisfy $\lambda_2^2 = \zeta_{30}^8 \lambda_4^2$. If $\lambda_2 = \zeta_{30}^4 \lambda_4$, then $\lambda_3 = 0$ and $\lambda_1\lambda_4 = 0$. Each of these two avenues leads to an orbit size that is too small. We conclude that $\lambda_2 = -\zeta_{30}^2\lambda_4$ and may set $\lambda_4 = 1$ by scaling. Any fixed point of RS can thus be written in the form

$$\lambda^2\zeta_{30}^7 e_1 - \zeta_{30}^4 e_2 + \lambda(e_3 + \zeta_{30}^{11}e_4) + e_5 + \zeta_{30}^{-7}e_6.$$

There are going to be five solutions, for RS fixes five points on the curve, as one can compute with the fixed point counting lemma. For any λ we get an $\mathrm{Aut}^+(\mathbf{R}_{6.10})$-orbit, and the points give constraints on $3 \times 1_3$, resulting in an additional invariant subspace

$$1_3 = \langle x_1^3 + \tfrac{\lambda^5}{2}(\zeta_5 x_3x_5^2 - \zeta_5^{-1}x_4x_6^2) \rangle.$$

We now use the remaining freedom from the centralizer $C_{\mathrm{GL}(6,\mathbb{C})}(\mathrm{Aut}^+(\mathbf{R}_{6.10}))$ and consider the effect of the transformation $\mathrm{DiaMat}(\lambda^{-2}, 1, \lambda^{-1}, \lambda^{-1}, 1, 1)$ on $\mathbb{P}^5$. It leaves I_2 invariant and normalizes the above component to that for which $\lambda^5 = 1$. So without loss of generality we assume this. It means that indeed there are five solutions $\lambda \in \{1, \zeta_5, \ldots, \zeta_5^4\}$ that give fixed points of RS on the curve. The primary decomposition of the new ideal $(I_2 \cup 1_3)$ contains a unique ideal defining our curve, with

$$I_3 = \left\langle\, x_1^3 + \tfrac{1}{2}(\zeta_5 x_3x_5^2 - \zeta_5^{-1}x_4x_6^2),\ x_1^2x_3 + \tfrac{1}{2}\zeta_5^{-1}x_2x_6^2 + \tfrac{1}{2}\zeta_5 x_5^2x_6,\ x_1^2x_4 - \tfrac{1}{2}\zeta_5 x_2x_5^2 + \tfrac{1}{2}\zeta_5^{-1}x_5x_6^2 \,\right\rangle.$$

Is this curve trigonal, or birationally isomorphic to a non-singular plane quintic? The answer becomes clear by computing an elimination ideal to the variables (x_2, x_4, x_5), which tells us that $x_2^4x_5 - \zeta_5^2x_2x_5^4 + 2\zeta_5x_4^5 \in I$. The mapping

$$(x_1 : x_2 : x_3 : x_4 : x_5 : x_6) \mapsto (\zeta_{150}x_2 : \sqrt[5]{2}\zeta_{25}x_4 : \zeta_{150}^{-4}x_5)$$

is a birational isomorphism onto the non-singular planar quintic $x^4z + xz^4 + y^5 = 0$. A standard map presentation for this planar model is readily found to be

$$R: \quad (x:y:z) \mapsto (\zeta_3^2 z : \zeta_{15}^{-1} y : z),$$
$$S: \quad (x:y:z) \mapsto (\zeta_3 x : \zeta_{15} y : z).$$

Standard complex conjugation con_6 is a reflection of it.

$\mathbf{R}_{6.11}$ **type** $(13, 26)$ **#cells** $(1, 13, 2)$ **map group size** 52 $\mathbf{Wi1}(6)$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^{13}, S^{26}, (RS)^2, RS^{-12} \rangle$

This map is the Wiman type I curve $\mathbf{Wi1}(6)$, with planar model $y^2z^{11} = x^{13} - z^{13}$. For more information, see Section 5.1.

$\mathbf{R}_{6.12}$ **type** $(14, 14)$ **#cells** $(2, 14, 2)$ **map group size** 56 $D(\mathbf{AM}(6))$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^{14}, S^{14}, (RS)^2, [R, S] \rangle$

This map is $D(\mathbf{R}_{6.4}) = D(\mathbf{AM}(6))$ and therefore also has planar model $y^2z^{12} = x^{14} - z^{14}$. For more information, see Section 5.3.

$\mathbf{R}_{6.13}$ **type** $(24, 24)$ **#cells** $(1, 12, 1)$ **map group size** 48 $D(\mathbf{Wi2}(6))$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^{24}, S^{24}, (RS)^2, RS^{-11} \rangle$

This map is $D(\mathbf{R}_{6.5}) = D(\mathbf{Wi2}(6))$ and therefore also has planar model $y^2z^{11} = x(x^{12} - z^{12})$. For more information, see Section 5.2.

A.7 Genus 7

Prior remarks. The Hilbert-Poincaré series predicts (cf. Section 6.1) that the canonical model of a non-hyperelliptic platonic map of genus 7 satisfies $\dim(I_2) = 10$. If the map is trigonal, some additional cubics will be needed to define the model. In any case $\dim(I_3) = 26$.

$\mathbf{R}_{7.1}$ **type** $(3,7)$ **#cells** $(72,252,168)$ **map group size** 1008	**Fricke-Macbeath map**
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^3, S^7, (RS)^2, ((S^{-3}R)^2 S^{-1}R)^2 \rangle$	

In [Mac1965], Alexander Murray Macbeath constructed a canonical model of $\mathbf{R}_{7.1}$ with the following polynomials in $\mathbb{C}[x_0, x_1, \ldots, x_6]$ generating its canonical ideal:

$$\sum_{i=0}^{6} x_i^2, \qquad \sum_{i=0}^{6} \zeta_7^i x_i^2, \qquad \sum_{i=0}^{6} \zeta_7^{-i} x_i^2,$$

$$(\zeta_7 - \zeta_7^{-1}) x_k x_{k+2} + (\zeta_7^2 - \zeta_7^{-2}) x_{k+1} x_{k+5} - (\zeta_7^3 - \zeta_7^{-3}) x_{k+3} x_{k+4}, \qquad \text{for } k = 0, \ldots, 6.$$

The indices of the variables are taken mod 7, which is why numbering them from zero to six is practical. The canonical representation that belongs to this model is generated by

$$R \mapsto \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & -1 & 1 & 0 & 1 \\ 0 & -1 & -1 & -1 & 0 & -1 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & -1 & 1 \\ 1 & 0 & 0 & 1 & 1 & -1 & 0 \end{pmatrix}, \qquad S \mapsto \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

Standard complex conjugation leaves the curve invariant and is a reflection. It seems it was unknown to Macbeath that this platonic surface had also been studied by Robert Fricke in [Fri1899], and henceforth the platonic surface has been dubbed the *Fricke-Macbeath curve*. Staying close to this nomenclature, we call $\mathbf{R}_{7.1}$ the *Fricke-Macbeath map*.

Because the map has no siblings (cf. Section 1.4), we remind the reader of Section 6.2, which states the result that there must be a canonical model over the field of moduli of the curve, $\mathbb{Q}$. We attempt Galois descent from the Macbeath model to find such a rational model. This was also attempted in [Hid2012]. The result obtained here is somewhat more concise.

To apply Galois descent, we first remark the curve as presented above seems defined over $\mathbb{Q}(\zeta_7)$, but is in fact invariant under the Galois automorphism $\zeta_7 \mapsto \zeta_7^{-1}$. We could therefore also write down a set of generators over $\mathbb{Q}(\zeta_7 + \zeta_7^{-1})$, even though we let this be. Our goal is to find and then split a 1-cocycle in $Z_1(\mathrm{Gal}(\mathbb{Q}(\zeta_7 + \zeta_7^{-1})/\mathbb{Q}), \mathrm{GL}(7, \mathbb{C}))$. Let us first remark that

$$\mathrm{Gal}(\mathbb{Q}(\zeta_7 + \zeta_7^{-1})/\mathbb{Q}) = \langle \sigma \rangle \cong \mathbb{Z}_3, \text{ where } \sigma : \zeta_7 + \zeta_7^{-1} \mapsto \zeta_7^2 + \zeta_7^{-2}.$$

Calling the original curve C, we thus have three Galois conjugates: C, C^σ, and C^{σ^2}.

Next, we note that the group $\mathrm{Aut}^+(\mathbf{R}_{7.1}) \cong \mathrm{PSL}(2,8)$ has outer automorphisms of order 3. It has 168 of them, to be precise. We can write down the action for such an outer automorphism on a standard generator pair (R, S). This can for example be done by writing down $\mathrm{PSL}(2,8)$ in its original form, as matrices over $\mathbb{F}_8$. An outer automorphism of order 3 is then the Frobenius $x \mapsto x^2$ applied to the matrix entries, and we can explicitly compute what it does to a standard generator pair. We picked an automorphism, let us call it t, with action

$$\begin{aligned} R &\mapsto S^{-1}(RS^{-2})^2RS^2, \\ S &\mapsto S^{-1}[R^{-1}, S^{-2}]RS^{-1}. \end{aligned}$$

We can set up a linear system of equations to solve

$$\begin{cases} T^{-1}\rho_c(R)T = \rho_c(S^{-1}(RS^{-2})^2RS^2) \\ T^{-1}\rho_c(S)T = \rho_c([R^{-1}, S^{-2}]RS^{-1}) \end{cases}.$$

The unique solution with determinant 1 is

$$T = \mathrm{MonMat}([1,1,1,1,1,1,1],[3,7,4,1,5,2,6]).$$

The elements R, S and t generate $\mathrm{Aut}(\mathrm{Aut}(\mathbf{R}_{7.1})) \cong \mathrm{P\Gamma L}(2,8)$ and we can extend the representation ρ_c to $\overline{\rho_c} : \mathrm{Aut}(\mathrm{Aut}(\mathbf{R}_{7.1})) \to \mathrm{GL}(7,\mathbb{Q})$ by setting $\overline{\rho_c}(t) = T$. The projectivity T permutes the three conjugates of C cyclically in the same way that σ does. Therefore, the homomorphism $c : \langle\sigma\rangle \to \mathrm{GL}(7,\mathbb{Q})$ defined by $\sigma \mapsto T$ can be construed as a 1-cocycle. This cocycle can be split by solving the equation $U = T^{-1}T^\sigma$. We want an invertible solution for which U^{-1} has small coefficients, to get that same property for the transformed ideal generators. The best we came up with is

$$U = \begin{pmatrix} 0 & \zeta_7+\zeta_7^{-1} & 0 & 0 & 0 & \zeta_7^2+\zeta_7^{-2} & \zeta_7^3+\zeta_7^{-3} \\ \zeta_7^3+\zeta_7^{-3} & 0 & \zeta_7^2+\zeta_7^{-2} & \zeta_7+\zeta_7^{-1} & 0 & 0 & 0 \\ 0 & \zeta_7^3+\zeta_7^{-3} & 0 & 0 & 0 & \zeta_7+\zeta_7^{-1} & \zeta_7^2+\zeta_7^{-2} \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ \zeta_7+\zeta_7^{-1} & 0 & \zeta_7^3+\zeta_7^{-3} & \zeta_7^2+\zeta_7^{-2} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 2 & 2 \\ \zeta_7^2+\zeta_7^{-2} & 0 & \zeta_7+\zeta_7^{-1} & \zeta_7^3+\zeta_7^{-3} & 0 & 0 & 0 \end{pmatrix}.$$

The transformed curve $U(C)$ is indeed definable over $\mathbb{Q}$. Applying the coefficient reduction strategy from Section 6.2, we arrive at an alternative canonical model for

$\mathbf{R}_{7.1}$:

$$\begin{aligned}
I = \langle & -x_1x_2 + x_1x_7 + x_2x_6 + x_3x_4 - x_3x_5 - x_3x_7 - x_4x_6 - x_5x_6, \\
& x_1x_3 + x_1x_6 - x_2^2 + 2x_2x_5 + x_2x_7 - x_3^2 + x_4x_5 - x_4x_7 - x_5^2, \\
& x_1^2 - x_1x_3 + x_2^2 - x_2x_4 - x_2x_5 - x_2x_7 - x_3^2 + x_3x_6 + 2x_5x_7 - x_7^2, \\
& x_1x_4 - 2x_1x_5 + 2x_1x_7 - x_2x_6 - x_3x_4 - x_3x_5 + x_5x_6 + x_6x_7, \\
& x_1^2 - 2x_1x_3 - x_2^2 - x_2x_4 - x_2x_5 + 2x_2x_7 + x_3^2 + x_3x_6 + x_4x_5 + x_5^2 - x_5x_7 - x_6^2, \\
& x_1x_2 - x_1x_5 - 2x_1x_7 + 2x_2x_3 - x_3x_7 - x_5x_6 + 2x_6x_7 \\
& -2x_1x_2 - x_1x_4 - x_1x_5 + 2x_1x_7 + 2x_2x_3 - 2x_3x_7 + 2x_5x_6 - x_6x_7, \\
& 2x_1^2 + x_1x_3 - x_1x_6 + 3x_2x_7 + x_4x_5 - x_4x_7 - x_5^2 + x_6^2 - x_7^2, \\
& 2x_1^2 - x_1x_3 + x_1x_6 + x_2^2 + x_2x_7 + x_3^2 - 2x_3x_6 + x_4x_5 - x_4x_7 + x_5^2 - 2x_5x_7 + x_6^2 + x_7^2, \\
& x_1^2 + x_1x_3 - x_1x_6 + 2x_2x_5 - 3x_2x_7 + 2x_3x_6 + x_4^2 + x_4x_5 - x_4x_7 + x_6^2 + 3x_7^2 \ \rangle.
\end{aligned}$$

There is an obvious trade-off to this model as compared to Macbeath's: the standard map presentation of the new model is not over $\mathbb{Q}$ anymore, but over $\mathbb{Q}(\zeta_7 + \zeta_7^{-1})$. One can easily compute it as (URU^{-1}, USU^{-1}). The involution $URSU^{-1}$ is even defined over $\mathbb{Q}$.

Remark A.7.1. The map $\mathbf{R}_{7.1}$ is antipodal, with the orientation reversing antipodal mapping defined by $(abc)^9$ in terms of a standard generator triple (a, b, c).

$\mathbf{R}_{7.2}$ **type** $(3, 12)$ **#cells** $(12, 72, 48)$ **map group size** 288
SMP $\mathrm{Aut}^+(\mathbf{R}) = \left\langle R, S \,\middle|\, R^3, S^{12}, (RS)^2, RS^{-2}RS^2R^{-1}S^2R^{-1}S^{-2} \right\rangle$

The canonical representation ρ_c can be generated by

$$R \mapsto (\zeta_3^2) \oplus \frac{1}{2}\begin{pmatrix} -1+i & -1-i \\ 1-i & -1-i \end{pmatrix} \oplus \frac{1}{2}\begin{pmatrix} -\zeta_{12}^4 + \zeta_{12}^7 & \zeta_{12}^4 - \zeta_{12}^7 & 0 & 0 \\ -\zeta_{12}^4 - \zeta_{12}^7 & -\zeta_{12}^4 - \zeta_{12}^7 & 0 & 0 \\ 0 & 0 & -1+i & -1+i \\ 0 & 0 & 1+i & -1-i \end{pmatrix},$$

$$S \mapsto (-\zeta_3) \oplus \frac{1}{2}\begin{pmatrix} -1-i & 1+i \\ 1-i & 1-i \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2}\zeta_{12}^8 + \frac{1}{2}\zeta_{12}^{11} & \frac{1}{2}\zeta_{12}^8 + \frac{1}{2}\zeta_{12}^{11} & 0 & 0 \\ -\frac{1}{2}\zeta_{12}^8 + \frac{1}{2}\zeta_{12}^{11} & \frac{1}{2}\zeta_{12}^8 - \frac{1}{2}\zeta_{12}^{11} & 0 & 0 \end{pmatrix}.$$

The representation $(\rho_c^\vee)^{2+}$ decomposes as

$$2 \times 1_3 \oplus 2_3 \oplus 2_9 \oplus 2 \times 3_2 \oplus 4_2 \oplus 2 \times 6_1.$$

All pieces except 2_3 contain reducibles, so we cannot use 2_9 or 4_2. The space I_2 must therefore be of the form $1_3 \oplus 3_2 \oplus 6_1$. All invariant $6_1 < 2 \times 6_1$ subspaces are of the form

$$\begin{aligned}1_6 = \langle\ & \lambda x_2x_4 - \zeta_{12}^2x_6^2 + 2x_6x_7 - \zeta_{12}^2x_7^2,\\ & \lambda x_2x_5 + \zeta_{12}^3x_3x_4 + 2\zeta_{12}x_6^2 - 2\zeta_{12}x_7^2,\\ & \lambda x_2x_6 - x_3x_7 + (\zeta_{12}^3+1)x_4^2 + (-2\zeta_{12}^2 - 2\zeta_{12}+2)x_4x_5 + (\zeta_{12}^3+1)x_5^2,\\ & \lambda x_2x_7 + x_3x_7 + (-\zeta_{12}^3 - \zeta_{12}^2 + \zeta_{12})x_4^2 + (2\zeta_{12}^2 + 2\zeta_{12} - 2)x_4x_5 + (\zeta_{12}^3 + \zeta_{12}^2 - \zeta_{12})x_5^2,\\ & \lambda x_3x_5 + (-\zeta_{12}^3 + \zeta_{12})x_6^2 - 2\zeta_{12}^3x_6x_7 + (-\zeta_{12}^3 + \zeta_{12})x_7^2,\\ & \lambda x_3x_6 - x_3x_7 + (2\zeta_{12}^3 + \zeta_{12}^2 - \zeta_{12} + 1)x_4^2 + (-\zeta_{12}^2 + \zeta_{12} + 1)x_5^2\ \rangle,\end{aligned}$$

and with the centralizer $C_{\mathrm{GL}(7,\mathbb{C})}(\mathrm{Aut}^+(\mathbf{R}_{7.2}))$ we can set $\lambda = 1$. These polynomials then constrain the possibilities for the vertices and face centers enough to determine them. We compute the eigenspaces of R and S, intersect with the zero locus of the 6_1 component and find that the vertices must form the $\mathrm{Aut}^+(\mathbf{R}_{7.2})$-orbit of

$$(0 : -2(\zeta_{12}^3 - \zeta_{12}^2 - \zeta_{12} + 2) : 2(\zeta_{12}^3 - \zeta_{12}^2 - \zeta_{12}) : 1 : -\zeta_{12}^2 + \zeta_{12} : \zeta_{12} : -\zeta_{12}^3 + \zeta_{12}^2)$$

and the face centers that of

$$(1 : -2(2\zeta_{12}^3 - 2\zeta_{12}^2 + 1)\alpha^2 : 2(\zeta_{12}^3 - \zeta_{12}^2 - \zeta_{12} + 1)\alpha^2 : \alpha : (\zeta_{12}^3 - \zeta_{12}^2 - \zeta_{12} + 1)\alpha : 1 : -\zeta_{12}^2 - \zeta_{12}),$$

where $\alpha^3 = 3\zeta_{12}^3 - 6\zeta_{12} - 5 = -3\sqrt{3} - 5$. In advance one can compute that R should fix 6 face centers on the curve, and this is now confirmed to be true on the model. The choice inherent in α is thus no cause for worries. These orbits of points in turn constrain the possible $1_3 < 2 \times 1_3$ and $3_2 < 2 \times 3_2$ pieces, and we find:

$$\begin{aligned}1_3 &= \langle\ 2(10\zeta_{12}^3 + 9\zeta_{12}^2 - 5\zeta_{12} - 9)x_1^2 + \zeta_{12}^3\alpha^2x_4x_6 - \zeta_{12}^3\alpha^2x_4x_7 + \alpha^2x_5x_6 + \alpha^2x_5x_7\ \rangle\\ 3_2 &= \langle\ (-\zeta_{12}^2 - \zeta_{12} + 1)x_2^2 + 4x_4x_6 + 4x_4x_7 + 4(\zeta_{12}^3 - 2\zeta_{12})x_5x_6 - 4(\zeta_{12}^3 - 2\zeta_{12})x_5x_7,\\ &\quad (-\zeta_{12}^2 - \zeta_{12} + 1)x_2x_3 + 4x_4x_6 - 4x_4x_7 + 4\zeta_{12}^3x_5x_6 + 4\zeta_{12}^3x_5x_7,\\ &\quad (-\zeta_{12}^2 - \zeta_{12} + 1)x_3^2 + 4(2\zeta_{12}^2 - 1)x_4x_6 + 4(2\zeta_{12}^2 - 1)x_4x_7 + 4\zeta_{12}^3x_5x_6 - 4\zeta_{12}^3x_5x_7\ \rangle.\end{aligned}$$

Together, the three indicated pieces define a canonical model for $\mathbf{R}_{7.2}$.

$\mathbf{R}_{7.3}$ **type** $(4, 16)$ **#cells** $(4, 31, 16)$ **map group size** 128 **Kul**(2)
SMP $\mathrm{Aut}^+(\mathbf{R}) = \left\langle R, S \,\middle|\, R^4, S^{16}, (RS)^2, R^{-1}S^2RS^{-6} \right\rangle$

This map is **Kul**(2), as we deduced in Chapter 2. A planar model for it is discussed in Section 5.4. We still applied our construction strategy to find a nice canonical model, possibly offering an alternative to study the curve further. The canonical representation ρ_c is generated by

$$\begin{aligned}R &\mapsto \mathrm{MonMat}([-i, 1, -1, 1, 1, \zeta_8^3, \zeta_8], [1, 3, 2, 5, 4, 7, 6]),\\ S &\mapsto \mathrm{MonMat}([-i, -\zeta_8, -\zeta_8^3, -\zeta_8, -\zeta_8^3, 1, 1], [1, 2, 3, 6, 7, 4, 5]),\end{aligned}$$

which has invariant subspaces $\langle e_1\rangle$, $\langle e_2, e_3\rangle$ and $\langle e_4, e_5, e_6, e_7\rangle$. The representation $(\rho_c^\vee)^{2+}$ decomposes as

$$2 \times 1_3 \oplus 1_6 \oplus 1_8 \oplus 2 \times 2_1 \oplus 2_4 \oplus 2_5 \oplus 2 \times 2_6 \oplus 2 \times 4_1 \oplus 4_2.$$

All isotypic pieces except 1_6 and 1_8 contain multiples of linear forms. Also, since

$$1_6 = \langle x_4x_7 - \zeta_8 x_5x_6 \rangle,$$
$$1_8 = \langle x_4x_7 + \zeta_8 x_5x_6 \rangle,$$

we find that $x_4x_7 \in 1_6 \oplus 1_8$, so we cannot use both pieces. These restrictions dictate that the shape of I_2 must be $1_3 \oplus 1_k \oplus 2_1 \oplus 2_6 \oplus 4_1$, where the 1_k is either 1_6 or 1_8. We now use the centralizer $C_{\mathrm{GL}(7,\mathbb{C})}(\mathrm{Aut}^+(\mathbf{R}_{7.3}))$ to normalize the last two of these invariant subspaces, which are of the form

$$2_6 = \langle \lambda_1 x_1x_2 + x_5^2 - \zeta_8 x_7^2, x_1x_3 - ix_4^2 + \zeta_8 x_6^2 \rangle,$$
$$4_1 = \langle \lambda_2 x_1x_4 + x_2x_5, x_1x_5 - ix_3x_4, x_1x_6 - \zeta_8^3 x_2x_7, x_1x_7 - \zeta_8^3 x_3x_6 \rangle,$$

by setting $\lambda_1 = \lambda_2 = 1$. The primary decomposition of the ideal $(2_6 \oplus 4_1)$ contains a unique prime ideal defining our canonical model. The remaining isotypic pieces of I_2 are:

$$1_3 = \langle x_1^2 + \zeta_8^2 x_2x_3 \rangle,$$
$$1_6 = \langle x_4x_7 - \zeta_8 x_5x_6 \rangle,$$
$$2_1 = \langle x_2^2 - x_4x_5 + \zeta_8^2 x_6x_7, x_3^2 - x_4x_5 - \zeta_8^2 x_6x_7 \rangle.$$

$\mathbf{R}_{7.4}$ type $(4,16)$ **#cells** $(4,32,16)$ **map group size** 128 **$\mathbf{AM}(7)$**
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^4, S^{16}, (RS)^2, (RS^{-1})^2 \rangle$

This is the Accola-Maclachlan map $\mathbf{AM}(7)$ with planar model $y^2z^{14} = x^{16} - z^{16}$. For more information, see Section 5.3.

$\mathbf{R}_{7.5}$ type $(4,28)$ **#cells** $(2,28,14)$ **map group size** 112 **$\mathbf{Wi2}(7)$**
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^4, S^{28}, (RS)^2, R^{-1}SRS^{-13} \rangle$

This is the Wiman type II map $\mathbf{Wi2}(7)$ with planar model $y^2z^{13} = x(x^{14} - z^{14})$. For more information, see Section 5.2.

$\mathbf{R}_{7.6}$ type $(6,9)$ **#cells** $(6,27,9)$ **map group size** 108
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^6, S^9, (RS)^2, [R^2, S], (RS^{-2})^2 \rangle$

The canonical representation ρ_c can be generated by

$$R \mapsto \mathrm{MonMat}([-\zeta_3, 1, \zeta_3^2, 1, \zeta_3^2, 1, \zeta_3], [1,3,2,5,4,7,6]),$$
$$S \mapsto \mathrm{MonMat}([\zeta_9^6, \zeta_9^5, \zeta_9^7, \zeta_9^4, \zeta_9^8, \zeta_9^8, \zeta_9^7], [1,2,3,4,5,6,7]),$$

with invariant subspaces $\langle e_1 \rangle$, $\langle e_2, e_3 \rangle$, $\langle e_4, e_5 \rangle$, and $\langle e_6, e_7 \rangle$. The representation $(\rho_c^\vee)^{2+}$ decomposes as

$$1_3 \oplus 3 \times 1_6 \oplus 2 \times 2_1 \oplus 2_3 \oplus 2_4 \oplus 2 \times 2_6 \oplus 2 \times 2_7 \oplus 2_9 \oplus 2 \times 2_{10} \oplus 2_{12}.$$

All isotypic components contain multiples of linear forms, which forces I_2 to take on the shape

$$2 \times 1_6 \oplus 2_1 \oplus 2_6 \oplus 2_7 \oplus 2_{10}.$$

The general form of these invariant subspaces is

$$\begin{aligned}
2 \times 1_6 &= \langle \lambda_1 x_1^2 + \zeta_9^{-1} x_2 x_3, \lambda_2 x_2 x_3 - \zeta_{18}^5 x_4 x_5 \rangle, \\
2_1 &= \langle \lambda_3 x_2 x_7 + \zeta_{36}^{-1} x_4 x_6, \lambda_3 x_3 x_6 + \zeta_{36}^{11} x_5 x_7 \rangle, \\
2_6 &= \langle \lambda_4 x_1 x_6 - \zeta_{36}^5 x_3 x_7, \lambda_4 x_1 x_7 + \zeta_{36}^{17} x_2 x_6 \rangle, \\
2_7 &= \langle \lambda_5 x_1 x_2 + \zeta_9 x_3 x_4, \lambda_5 x_1 x_3 + \zeta_{18}^5 x_2 x_5 \rangle, \\
2_{10} &= \langle \lambda_6 x_1 x_4 + x_2^2, \lambda_6 x_1 x_5 - \zeta_3 x_3^2 \rangle.
\end{aligned}$$

The centralizer trick allows us to normalize (only) two of these. We choose to set $\lambda_5 = \lambda_6 = 1$. The primary decomposition of $(2_7 \oplus 2_{10})$ contains only one primary ideal without non-trivial linear forms, and we find that we are obliged to use $\lambda_1 = \lambda_2 = 1$. Now we apply the fixed point strategy and compute the 27 edge centers of the curve by studying the eigenspaces of RS. The constraints imposed by $2 \times 1_6 \oplus 2_7 \oplus 2_{10}$ force a fixed point of RS on the curve to lie in $E(RS, -1)$ and be of the form

$$\zeta_9^{-1} e_1 \pm \zeta_{36}^7 (e_2 + \zeta_{18}^5 e_3) + e_4 + \zeta_{18}^7 e_5 + \mu(e_6 + \zeta_{18}^5 e_7).$$

Again we can utilize $C_{\mathrm{GL}(7,\mathbb{C})}(\mathrm{Aut}^+(\mathbf{R}_{7.6}))$, since we still have the freedom to scale $\langle e_6, e_7 \rangle$. If $\mu = 0$, then the orbit size is smaller than 27. Thus, we set $\mu = 1$. For either choice of sign in the expression above, we find a suitable orbit of edge centers. Let us pick a $+$ sign. Then the points determine that we must use $\lambda_3 = \lambda_4 = 1$.

In this way we have completed the construction of I_2, but the ideal (I_2) defines a surface, not a curve. Hence, $\mathbf{R}_{7.6}$ is trigonal. The Hilbert-Poincaré series of (I_2) tells us we need four more irreducible polynomials of degree 3 to define the canonical ideal.

The representation $(\rho_c^\vee)^{3+}$ decomposes into a lot of isotypic components of high multiplicity. All irreducible representations of $\mathrm{Aut}^+(\mathbf{R}_{7.6})$ are 1- or 2-dimensional. The codimension of $(I_2) \cap \mathbb{C}[x_1, \ldots, x_7]_3$ in any of the multiples of 1-dimensional components is at most 1, and the option of adding any full isotypic piece can be discarded by computing the primary decompositions: all arising ideals contain too many quadrics. Similarly, the codimension of $(I_2) \cap \mathbb{C}[x_1, \ldots, x_7]_3$ in most of the multiples of 2-dimensional invariant subspaces is 2 and the same technique excludes those pieces as well. We are left with isotypic components $5 \times 2_1$ and $4 \times 2_6$. The constraints imposed by the edge centers brings each of the two dimensions down by two, resulting in:

$$\begin{aligned}
4 \times 2_1 = \langle & x_1 x_2 x_4 + \zeta_{36}^{15} x_5^3, x_1 x_3 x_5 + \zeta_{36}^{9} x_4^3, x_2^3 - \zeta_{36}^{15} x_5^3, x_2 x_5^2 + \zeta_{36}^{17} x_4^3, x_3^3 - \zeta_{36}^{15} x_4^3, \\
& x_3 x_4^2 - \zeta_{36}^{11} x_5^3, x_6^3, x_7^3 \rangle, \\
3 \times 2_6 = \langle & x_1 x_4^2 + \zeta_{36}^5 x_3 x_5^2, x_1 x_5^2 - \zeta_{36}^{12} x_3^2 x_5, x_2^2 x_4 - \zeta_{36}^5 x_3 x_5^2, x_2 x_4^2 + \zeta_{36}^{13} x_3^2 x_5, x_6^2 x_7, x_6 x_7^2 \rangle.
\end{aligned}$$

Since (I_2) already contains a $2 \times 2_1 < 4 \times 2_1$ and a $2_6 < 3 \times 2_6$, we must choose an extra 2-dimensional subspace in each of these components. To progress, we compute the eigenspaces of R, which are

$$E(R,\zeta_6) = \langle e_6 + \zeta_6^{-1} e_7 \rangle, \qquad E(R,\zeta_6^2) = \langle e_2 - \zeta_6 e_3, e_4 - \zeta_6 e_5 \rangle,$$
$$E(R,\zeta_6^{-2}) = \langle e_6 - \zeta_6^{-1} e_7 \rangle, \qquad E(R,\zeta_6^{-1}) = \langle e_1, e_2 + \zeta_6 e_3, e_4 + \zeta_6 e_5 \rangle.$$

Again I_2 restricts the options, ruling out $E(R,\zeta_6^2)$ and leaving only the two vectors $-\zeta_{36}^{10} e_1 \pm \zeta_{36}^5 (e_2 + \zeta_6 e_3) + e_4 + \zeta_6 e_5$ within $E(R,\zeta_6^{-1})$. We try the now finite number of possible ideals. It turns out we only get an irreducible curve of genus 7 if we use a fixed point from $E(R,\zeta_6^{-1})$ and choose the same sign chosen for the edge center, so a $+$ for us. All in all we must add the following $\mathrm{Aut}^+(\mathbf{R}_{7.6})$-invariant pieces to I_2:

$$2_1 = \langle x_2^3 + \zeta_{12}^5 x_5^3 + 2\zeta_{12} x_6^3, x_3^3 + \zeta_{12}^5 x_4^3 + 2\zeta_{12} x_7^3 \rangle,$$
$$2_6 = \langle x_1 x_4^2 - \zeta_{36}^5 x_3 x_5^2 + 2\zeta_{36}^4 x_6^2 x_7, x_2^2 x_4 + \zeta_{36}^5 x_3 x_5^2 - 2\zeta_{36}^4 x_6^2 x_7 \rangle.$$

We have now constructed the the canonical ideal I. Let us certify trigonality of the map. Computation of various elimination ideals quickly reveals that $x_1^9 + \zeta_{36}^9 x_2^9 + 2\zeta_3 x_2^6 x_6^3 \in I$. One can check that the map $(x_1 : \cdots : x_7) \mapsto (x_1 : x_2 : x_6)$ is a birational isomorphism onto the planar curve defined by this polynomial. A degree 3 map from the canonical curve to $\mathbb{P}^1$ is therefore defined by $(x_1 : \cdots : x_7) \mapsto (x_1 : x_2)$. Furthermore, scaling x_2 and x_6, we see that our curve is in fact birationally equivalent to the planar curve

$$x^9 + y^9 + x^6 z^3 = 0.$$

A standard map presentation of this planar curve is generated by

$$R : (x : y : z) \mapsto (\zeta_9^{-1} x^2 z : \zeta_9^{-3} y z^2 : x^3),$$
$$S : (x : y : z) \mapsto (\zeta_9 x : \zeta_3 y : z),$$

and standard complex conjugation is a reflection of the map $\mathbf{R}_{7.6}$ on it.

$\mathbf{R}_{7.7}$ type $(6, 12)$ **#cells** $(4, 24, 8)$ **map group size** 96
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^6, S^{12}, (RS)^2, (R^2S^{-1})^2, S^{-2}RS^{-2}R^{-2} \rangle$

This map is $D(\mathbf{R}_{7.2})$ and therefore has the same canonical model as $\mathbf{R}_{7.2}$.

$\mathbf{R}_{7.8}$ type $(6, 21)$ **#cells** $(2, 21, 7)$ **map group size** 84
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^6, S^{21}, (RS)^2, R^{-1}SRS^{-13} \rangle$

The canonical representation ρ_c can be generated by

$$R \mapsto \mathrm{MonMat}([-\zeta_3, 1, \zeta_3, 1, \zeta_3^2, 1, \zeta_3^2], [1, 3, 2, 5, 4, 7, 6]),$$
$$S \mapsto \mathrm{MonMat}([\zeta_{21}^{14}, \zeta_{21}^{16}, \zeta_{21}^{19}, \zeta_{21}^{17}, \zeta_{21}^{11}, \zeta_{21}^{20}, \zeta_{21}^{8}], [1, 2, 3, 4, 5, 6, 7]),$$

with invariant subspaces $\langle e_1 \rangle$, $\langle e_2, e_3 \rangle$, $\langle e_4, e_5 \rangle$, and $\langle e_6, e_7 \rangle$ respectively. The representation $(\rho_c^\vee)^{2+}$ decomposes as

$$1_3 \oplus 3 \times 1_6 \oplus 2 \times 2_1 \oplus 2 \times 2_2 \oplus 2 \times 2_3 \oplus 2 \times 2_4 \oplus 2 \times 2_5 \oplus 2_6 \oplus 2_7.$$

All contain reducibles, so we cannot use 1_3, 2_6 or 2_7, and must pick at least four 2-dimensional irreducibles. Hence, we can leave out at most one of the five present. The invariant 2-dimensional irreducible subspaces are of the form

$$\begin{aligned}
2_1 &= \langle \lambda_1 x_1 x_2 + x_3 x_5, \lambda_1 x_1 x_3 + \zeta_6^{-1} x_2 x_4 \rangle, \\
2_2 &= \langle \lambda_2 x_1 x_4 + \zeta_6 x_5 x_6, \lambda_2 x_1 x_5 + \zeta_6^2 x_4 x_7 \rangle, \\
2_3 &= \langle \lambda_3 x_1 x_6 - \zeta_6 x_4^2, \lambda_3 x_1 x_7 - x_5^2 \rangle, \\
2_4 &= \langle \lambda_4 x_2 x_5 + x_3 x_7, \lambda_4 x_2 x_6 + \zeta_6^2 x_3 x_4 \rangle, \\
2_5 &= \langle \lambda_5 x_4 x_6 + x_7^2, \lambda_5 x_5 x_7 - \zeta_6 x_6^2 \rangle.
\end{aligned}$$

The centralizer trick allows us to normalize (only) two of these, since none varies along with the coordinates of $\langle e_2, e_3 \rangle$. Our first attempt is to use $2_5 \oplus 2_4$, so we set $\lambda_4 = \lambda_5 = 1$. But computation of the primary decomposition of $(2_4 \oplus 2_5)$ yields only associated primes containing linear forms. The same holds if we try $2_5 \oplus 2_3$. The conclusion that 2_5 must be excluded is inevitable. Therefore, I_2 is of the form

$$2 \times 1_6 \oplus 2_1 \oplus 2_2 \oplus 2_3 \oplus 2_4.$$

We set $\lambda_1 = \lambda_4 = 1$. The ideal $(2_1 \oplus 2_4)$ has exactly one associated prime with the correct number of quadrics. That ideal satisfies $\lambda_2 = \lambda_3 = 1$. Furthermore, I_2 contains the subspace

$$2 \times 1_6 = \langle x_1^2 + x_6 x_7, x_4 x_5 + \zeta_6 x_6 x_7 \rangle.$$

As we have seen happen before, (I_2) defines a surface, not a curve. We conclude that $\mathbf{R}_{7.8}$ is trigonal. The Hilbert-Poincaré series implies that we need four extra generators in degree 3 for the canonical ideal. The representation $(\rho_c^\vee)^{3+}$ decomposes into several large isotypic components. All irreducible representations of $\mathrm{Aut}^+(\mathbf{R}_{7.8})$ are 1- or 2-dimensional. The isotypic components for 1-dimensional irreducibles are excluded from participation because $(I_2) \cap \mathbb{C}[x_1, \ldots, x_7]_3$ has codimension at most 1 in each, and all yield only associated primes containing linear forms when added to I_2.

Similar reasoning applies to some isotypic components of 2-dimensional irreducibles, and we are left with $5 \times 2_1$, $6 \times 2_2$, $6 \times 2_3$ and $5 \times 2_7$, of which (I_2) already contains subspaces of dimension 6, 6, 6, and 4 respectively. We compute the 21 edge centers of the map to progress. The polynomials in I_2 impose constraints on the eigenspaces of RS. There is only one orbit of points on (I_2) with $\mathrm{Aut}^+(\mathbf{R}_{7.8})$-orbit size 21, and this is the orbit of

$$(\zeta_{42}^{29} : 1 : \zeta_{42}^{17} : \zeta_{42}^{32} : \zeta_{42}^{33} : 1 : \zeta_{42}^{37}).$$

The edge centers shave two off the dimension of each of the four available isotypic components. Either of the two spaces

$$\begin{aligned}
4 \times 2_1 = \langle\, & x_1^2 x_4 + x_7^3,\ x_1^2 x_5 - \zeta_6 x_6^3,\ x_1 x_4 x_7 + \zeta_6^{-1} x_6^3,\ x_1 x_5 x_6 + \zeta_6^2 x_7^3,\ x_4^2 x_5 + \zeta_6 x_7^3, \\
& x_4 x_5^2 + \zeta_6^{-1} x_6^3,\ x_4 x_6 x_7 - x_7^3,\ x_5 x_6 x_7 + \zeta_6 x_6^3 \,\rangle, \\
4 \times 2_7 = \langle\, & x_1 x_2 x_7 - \zeta_6 x_3 x_6^2,\ x_1 x_3 x_6 - \zeta_6 x_3 x_4^2,\ x_2 x_4 x_6 + \zeta_6^2 x_3 x_4^2,\ x_2 x_5^2 - \zeta_6 x_3 x_6^2, \\
& x_2 x_7^2 + \zeta_6^2 x_3 x_4^2,\ x_3 x_5 x_7 + \zeta_6 x_3 x_6^2 \,\rangle,
\end{aligned}$$

is therefore excluded or included whole. Computation of the Hilbert-Poincaré series of $(I_2 + 4 \times 2_1)$ and $(I_2 + 4 \times 2_7)$ precludes these options, however. Similarly, the remaining $5 \times 2_2$ and $5 \times 2_3$ cannot be used completely, so the canonical ideal must be generated by $I_2 + 4 \times 2_2 + 4 \times 2_3$. We now compute the possible face centers of the map by considering the eigenspaces

$$E(R,\zeta_7) = \langle e_2 + \zeta_6^{-1} e_3 \rangle, \qquad E(R,\zeta_7^2) = \langle e_4 - \zeta_6 e_5, e_6 - \zeta_6 e_7 \rangle$$
$$E(R,\zeta_7^4) = \langle e_2 - \zeta_6^{-1} e_3 \rangle, \qquad E(R,\zeta_7^5) = \langle e_1, e_4 + \zeta_6 e_5, e_6 + \zeta_6 e_7 \rangle.$$

The polynomials of I_2 again impose constraints, leaving only four possible orbits, namely those of:

$$e_2 + \zeta_6^{-1} e_3, \qquad \zeta_6^{-1} e_1 - \zeta_6^{-1} e_4 - e_5 + e_6 + \zeta_6 e_7,$$
$$e_2 - \zeta_6^{-1} e_3, \qquad \zeta_6^{-1} e_1 + \zeta_6^{-1} e_4 + e_5 + e_6 + \zeta_6 e_7.$$

Any of these orbits determines a unique $\mathrm{Aut}^+(\mathbf{R}_{7.8})$-invariant $4 \times 2_2$, and for all four we compute the associated primes. It turns out we only get an irreducible curve of genus 7 if we pick the orbit of $\zeta_6^{-1} e_1 + \zeta_6^{-1} e_4 + e_5 + e_6 + \zeta_6 e_7$. The polynomials we must add to I_2 to generate the canonical ideal are:

$$2_2 = \langle\, x_2^2 x_3 - \tfrac{1}{2}\zeta_{42}^{-2} x_5^2 x_7 + \tfrac{1}{2}\zeta_{42}^{5} x_5 x_6^2,\ x_2 x_3^2 + \tfrac{1}{2}\zeta_{42}^{-9} x_4^2 x_6 + \tfrac{1}{2}\zeta_{42}^{-2} x_5^3 \,\rangle,$$
$$2_3 = \langle\, x_2^3 + \tfrac{1}{2}\zeta_{42}^{-2} x_5 x_7^2 - \tfrac{1}{2}\zeta_{42}^{5} x_6^2 x_7,\ x_3^3 + \tfrac{1}{2}\zeta_{42}^{-2} x_4 x_6^2 + \tfrac{1}{2}\zeta_{42}^{-2} x_6 x_7^2 \,\rangle.$$

We have now constructed the the canonical ideal I. Let us certify trigonality of the map. Computation of various elimination ideals quickly reveals that $x_3^3 x_6^5 - \frac{1}{2}\zeta_{42}^5 x_4^8 + \frac{1}{2}\zeta_{42}^{-2} x_4 x_6^7 \in I$. One can check that the map $(x_1 : \cdots : x_7) \mapsto (x_3 : x_4 : x_6)$ is a birational isomorphism onto the planar curve defined by this polynomial. A degree 3 map from the canonical curve to $\mathbb{P}^1$ is therefore defined by $(x_1 : \cdots : x_7) \mapsto (x_4 : x_6)$. Furthermore, scaling x_4 and x_6, we see that our curve is in fact birationally equivalent to the planar curve

$$y^3 z^5 + x^8 + x z^7 = 0.$$

A standard map presentation of this planar curve is generated by

$$R : (x : y : z) \mapsto (\zeta_{21}^{-3} x^2 z : \zeta_{21}^{-1} y z^2 : x^3),$$
$$S : (x : y : z) \mapsto (\zeta_{21}^{3} x : \zeta_{21} y : z),$$

and standard complex conjugation is a reflection of the map $\mathbf{R}_{7.8}$ on it.

$\mathbf{R}_{7.9}$ type $(15,30)$ **#cells** $(1,15,2)$ **map group size** 60 **Wi1**(7)
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^{15}, S^{30}, (RS)^2, RS^{-14} \rangle$

This is the Wiman type I map $\mathbf{Wi1}(7)$ with planar model $y^2 z^{13} = x^{15} - z^{15}$. For more information, see Section 5.1.

$\mathbf{R}_{7.10}$ **type** $(16, 16)$ **#cells** $(2, 16, 2)$ **map group size** 64 $D(\mathbf{AM}(7))$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^{16}, S^{16}, (RS)^2, [R, S] \rangle$

This map is $D(\mathbf{R}_{7.4}) = D(\mathbf{AM}(7))$ and therefore also has planar model $y^2 z^{14} = x^{16} - z^{16}$. For more information, see Section 5.3.

$\mathbf{R}_{7.11}$ **type** $(16, 16)$ **#cells** $(2, 16, 2)$ **map group size** 64 $D(\mathbf{Kul}(2))$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^{16}, S^{16}, (RS)^2, R^{-1}SRS^{-9} \rangle$

This map $D(\mathbf{R}_{7.3}) = D(\mathbf{Kul}(2))$. It therefore has the same canonical and planar models. For more information on the planar model, see Section 5.4.

$\mathbf{R}_{7.12}$ **type** $(28, 28)$ **#cells** $(1, 14, 1)$ **map group size** 56 $D(\mathbf{Wi2}(7))$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^{28}, S^{28}, (RS)^2, RS^{-13} \rangle$

This map is $D(\mathbf{R}_{7.5}) = D(\mathbf{Wi2}(7))$ and therefore also has planar model $y^2 z^{13} = x(x^{14} - z^{14})$. For more information, see Section 5.2.

A.8 Genus 8

Prior remarks. The Hilbert-Poincaré series predicts (cf. Section 6.1) that the canonical model of a non-hyperelliptic platonic map of genus 8 satisfies $\dim(I_2) = 15$. If the map is trigonal, some additional cubics will be needed to define the model. In any case $\dim(I_3) = 85$.

$\mathbf{R}_{8.1}$ and $\mathbf{R}_{8.2}$ type $(3, 8)$ **#cells** $(42, 168, 112)$ **map group size** 672 **First tuplet**
SMP $\mathbf{R}_{8.1}$: $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^3, S^8, (RS)^2, (RS^{-2})^4 \rangle$
SMP $\mathbf{R}_{8.2}$: $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^3, S^8, (RS)^2, [RS, S^3RS^{-2}] \rangle$

The canonical characters of $\mathbf{R}_{8.1}$ and $\mathbf{R}_{8.2}$ turn out to be identical. This is the first occurrence of a tuplet, a phenomenon we described in Section 1.4. We therefore baptize these two maps *the first tuplet*, and denote an as yet undetermined member by $\mathbf{R}$. The group $\mathrm{Aut}^+(\mathbf{R})$ is isomorphic to $\mathrm{PGL}(2, 7)$, and $\mathrm{Aut}(\mathbf{R})$ to $\mathrm{PGL}(2, 7) \times \mathbb{Z}_2$; both maps are antipodal. The canonical representation ρ_c is irreducible, and can be found by induction from a 1-dimensional representation over $\mathbb{Q}(\zeta_3)$ of an index 8 subgroup. It can be generated by

$$R \mapsto \mathrm{MonMat}([-\zeta_3, 1, \zeta_3, -\zeta_3^2, -\zeta_3, \zeta_3^2, -\zeta_3, \zeta_3], [2, 4, 3, 1, 8, 6, 5, 7]),$$
$$S \mapsto \mathrm{MonMat}([\zeta_3^2, \zeta_3^2, \zeta_3, \zeta_3, \zeta_3^2, -\zeta_3, -\zeta_3^2, \zeta_3], [5, 1, 2, 3, 7, 8, 6, 4]).$$

The representation $(\rho_c^\vee)^{2+}$ decomposes as

$$1_1 \oplus 6_2 \oplus 6_3 \oplus 7_2 \oplus 2 \times 8_1.$$

This implies that I_2 is generated by either $7_2 \oplus 8_1$ or $1_1 \oplus 6_k \oplus 8_1$, where $k \in \{2, 3\}$. We apply the fixed point strategy and compute the vertices of the map. The eigenspaces of S are all 1-dimensional, so there are at most 8 possible $\mathrm{Aut}^+(\mathbf{R})$-orbits. The orbits for $E(S, \zeta_8^k)$ and $E(S, \zeta_8^{-k})$ are identical, and the orbits for the eigenvalues ± 1 are smaller than 42, whence they can be excluded. The variety defined by the isotypic component 7_2 however, contains only those two small orbits. We conclude that $I_2 = 1_1 \oplus 6_k \oplus 8_1$. The two components 6_2 and 6_3 each contain precisely one possible $\mathrm{Aut}^+(\mathbf{R})$-orbit of vertices, namely

$$(1, -\zeta_{24}^{-1}, \zeta_{24}^6, \zeta_{24}, -\zeta_{24}, \zeta_{24}^3, \zeta_{24}^2, \zeta_{24}^{-4}) \in E(S, \zeta_8) \text{ and}$$
$$(1, \zeta_{24}^{17}, \zeta_{24}^{18}, \zeta_{24}^{19}, \zeta_{24}^7, \zeta_{24}^9, \zeta_{24}^{14}, \zeta_{24}^{20}) \in E(S, \zeta_8^3)$$

respectively. In both cases, the resulting 42 points yield enough conditions to single out a unique $8_1 < 2 \times 8_1$, and we can check that we have found a canonical ideal for one of our two tuplet members.

$$\begin{aligned}1_1 = \langle &x_1x_2 - \zeta_3^2 x_1x_3 + x_1x_4 + \zeta_3^2 x_1x_5 - \zeta_3^2 x_1x_6 - \zeta_3 x_1x_7 - \zeta_3^2 x_1x_8 + x_2x_3 - \\ &\zeta_3^2 x_2x_4 - \zeta_3^2 x_2x_5 + \zeta_3^2 x_2x_6 + x_2x_7 + x_2x_8 + \zeta_3^2 x_3x_4 + x_3x_5 - \zeta_3^2 x_3x_6 - \\ &\zeta_3^2 x_3x_7 - \zeta_3 x_3x_8 - \zeta_3^2 x_4x_5 + x_4x_6 + \zeta_3^2 x_4x_7 + \zeta_3 x_4x_8 + x_5x_6 + \zeta_3 x_5x_7 + \\ &\zeta_3^2 x_5x_8 - x_6x_7 - x_6x_8 - x_7x_8\rangle\end{aligned}$$

The invariant 6_2 and 6_3 are obtained by applying $\mathrm{Aut}^+(\mathbf{R})$ to spin one of the following two polynomials (with the $+$-sign and $-$-sign in the middle, respectively):

$$\Big(7x_1x_2 - 7\zeta_6x_1x_8 - x_2x_3 - \zeta_6x_2x_4 + \zeta_6x_2x_5 + 3\zeta_6x_2x_6 - 3x_2x_7 - 2\zeta_6x_3x_4 - 2\zeta_6x_3x_6 +$$
$$2\zeta_6x_3x_7 + \zeta_3x_3x_8 - 2\zeta_6x_4x_5 + 4x_4x_6 + 4\zeta_6x_4x_7 + \zeta_3x_4x_8 - 2x_5x_6 + 2\zeta_3x_5x_7 - \zeta_6x_5x_8 -$$
$$3x_6x_8 - 3x_7x_8\Big) \pm \Big(2(-\zeta_{24}^5+\zeta_{24}^3+\zeta_{24})x_2x_3 + 2(\zeta_{24}^7+\zeta_{24})x_2x_4 - 2(\zeta_{24}^7+\zeta_{24})x_2x_5 + (\zeta_{24}^7+\zeta_{24})x_2x_6 +$$
$$(\zeta_{24}^5-\zeta_{24}^3-\zeta_{24})x_2x_7 - 3(\zeta_{24}^7+\zeta_{24})x_3x_4 - 3(\zeta_{24}^7+\zeta_{24})x_3x_6 - 4(\zeta_{24}^7+\zeta_{24})x_3x_7 - 2(\zeta_{24}^7+\zeta_{24}^5-\zeta_{24}^3)x_3x_8 -$$
$$3(\zeta_{24}^7+\zeta_{24})x_4x_5 + (\zeta_{24}^5-\zeta_{24}^3-\zeta_{24})x_4x_6 + (-\zeta_{24}^7-\zeta_{24})x_4x_7 - 2(\zeta_{24}^7+\zeta_{24}^5-\zeta_{24}^3)x_4x_8 - 4(\zeta_{24}^5-\zeta_{24}^3-\zeta_{24})x_5x_6$$
$$+ 3(\zeta_{24}^7+\zeta_{24}^5-\zeta_{24}^3)x_5x_7 + 2(\zeta_{24}^7+\zeta_{24})x_5x_8 + (\zeta_{24}^5-\zeta_{24}^3-\zeta_{24})x_6x_8 + (\zeta_{24}^5-\zeta_{24}^3-\zeta_{24})x_7x_8\Big).$$

The invariant 8_1 is obtained by spinning the following polynomial with $\mathrm{Aut}^+(\mathbf{R})$:

$$49x_1^2 + (-5\zeta_{24}^7-4\zeta_{24}^5-\zeta_{24}^4+4\zeta_{24}^3-\zeta_{24}-9)x_2x_3 + (5\zeta_{24}^7+4\zeta_{24}^5+\zeta_{24}^4-4\zeta_{24}^3+\zeta_{24}+9)x_2x_4 +$$
$$(-4\zeta_{24}^7+\zeta_{24}^5+9\zeta_{24}^4-\zeta_{24}^3-5\zeta_{24}-10)x_2x_5 + (4\zeta_{24}^7-\zeta_{24}^5-9\zeta_{24}^4+\zeta_{24}^3+5\zeta_{24}+10)x_2x_6 +$$
$$(4\zeta_{24}^7-\zeta_{24}^5-9\zeta_{24}^4+\zeta_{24}^3+5\zeta_{24}+10)x_2x_7 + (-5\zeta_{24}^7-4\zeta_{24}^5-\zeta_{24}^4+4\zeta_{24}^3-\zeta_{24}-9)x_2x_8 +$$
$$(4\zeta_{24}^7-\zeta_{24}^5-9\zeta_{24}^4+\zeta_{24}^3+5\zeta_{24}+10)x_3x_4 + (4\zeta_{24}^7-\zeta_{24}^5-9\zeta_{24}^4+\zeta_{24}^3+5\zeta_{24}+10)x_3x_5 +$$
$$(-\zeta_{24}^7-5\zeta_{24}^5-10\zeta_{24}^4+5\zeta_{24}^3+4\zeta_{24}+1)x_3x_6 + (5\zeta_{24}^7+4\zeta_{24}^5+\zeta_{24}^4-4\zeta_{24}^3+\zeta_{24}+9)x_3x_7 +$$
$$(5\zeta_{24}^7+4\zeta_{24}^5+\zeta_{24}^4-4\zeta_{24}^3+\zeta_{24}+9)x_3x_8 + (-4\zeta_{24}^7+\zeta_{24}^5+9\zeta_{24}^4-\zeta_{24}^3-5\zeta_{24}-10)x_4x_5 +$$
$$(-5\zeta_{24}^7-4\zeta_{24}^5-\zeta_{24}^4+4\zeta_{24}^3-\zeta_{24}-9)x_4x_6 + (\zeta_{24}^7+5\zeta_{24}^5+10\zeta_{24}^4-5\zeta_{24}^3-4\zeta_{24}-1)x_4x_7 +$$
$$(\zeta_{24}^7+5\zeta_{24}^5+10\zeta_{24}^4-5\zeta_{24}^3-4\zeta_{24}-1)x_4x_8 + (4\zeta_{24}^7-\zeta_{24}^5-9\zeta_{24}^4+\zeta_{24}^3+5\zeta_{24}+10)x_5x_6 +$$
$$(4\zeta_{24}^7-\zeta_{24}^5-9\zeta_{24}^4+\zeta_{24}^3+5\zeta_{24}+10)x_5x_7 + (\zeta_{24}^7+5\zeta_{24}^5+10\zeta_{24}^4-5\zeta_{24}^3-4\zeta_{24}-1)x_5x_8 +$$
$$(-\zeta_{24}^7-5\zeta_{24}^5-10\zeta_{24}^4+5\zeta_{24}^3+4\zeta_{24}+1)x_6x_7 + (-4\zeta_{24}^7+\zeta_{24}^5+9\zeta_{24}^4-\zeta_{24}^3-5\zeta_{24}-10)x_6x_8 +$$
$$(-\zeta_{24}^7-5\zeta_{24}^5-10\zeta_{24}^4+5\zeta_{24}^3+4\zeta_{24}+1)x_7x_8.$$

Remark A.8.1. The two canonical curves are related by the Galois automorphism $\zeta_{24} \mapsto -\zeta_{24}$ of $\mathrm{Gal}(\mathbb{Q}(\zeta_{24})/\mathbb{Q}(\zeta_3))$, which switches the 6_2 and 6_3 parts. Which of the choices defines $(\mathbf{R}_{8.1})_a$ and which $(\mathbf{R}_{8.2})_a$ is still an open problem. To identify which is which one could compute the length of a Petrie path, i.e. two times the order of $[R, S]$, which is 4 for $\mathbf{R}_{8.1}$ and 7 for $\mathbf{R}_{8.2}$. This entails picking a generator pair (R, S) of $\mathrm{Aut}^+(\mathbf{R})$ that provably acts on one of the two algebraic curves under consideration as a pair of rotations around a vertex and the center of an adjacent face. One should therefore compute a fundamental triangle on one of the two curves.

$\mathbf{R}_{8.3}$ type $(4, 18)$ **#cells** $(4, 36, 18)$ **map group size** 144 **AM**(8)
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^4, S^{18}, (RS)^2, (RS^{-1})^2 \rangle$

This is the Accola-Maclachlan map $\mathbf{AM}(8)$ with planar model $y^2z^{16} = x^{18} - z^{18}$. For more information, see Section 5.3.

$\mathbf{R}_{8.4}$ type $(4, 32)$ **#cells** $(2, 32, 16)$ **map group size** 128 **Wi2**(8)
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^4, S^{32}, (RS)^2, R^{-1}SRS^{-15} \rangle$

This is the Wiman type II map $\mathbf{Wi2}(8)$ with planar model $y^2z^{15} = x(x^{16} - z^{16})$. For

more information, see Section 5.2.

$\mathbf{R}_{8.5}$ type $(6, 10)$ **#cells** $(6, 30, 10)$ **map group size** 120
SMP $\mathrm{Aut}^+(\mathbf{R}) = \left\langle R, S \,\middle|\, R^6, S^{10}, (RS)^2, (RS^{-1})^2 \right\rangle$

The canonical representation ρ_c can be generated by

$$R \mapsto \mathrm{MonMat}([1, \zeta_{15}^{10}, \zeta_{15}^{14}, \zeta_{15}^{6}, 1, \zeta_{15}^{10}, \zeta_{15}^{2}, \zeta_{15}^{3}], [2, 1, 4, 3, 6, 5, 8, 7]),$$
$$S \mapsto \mathrm{MonMat}([1, 1, \zeta_5^3, \zeta_5^2, 1, 1, \zeta_5^4, \zeta_5], [3, 4, 1, 2, 7, 8, 5, 6]),$$

with invariant subspaces $\langle e_1, e_2, e_3, e_4\rangle$, $\langle e_5, e_6, e_7, e_8\rangle$. The representation $(\rho_c^\vee)^{2+}$ decomposes as

$$2 \times 1_1 \oplus 2 \times 1_2 \oplus 2 \times 2_2 \oplus 2_3 \oplus 2_4 \oplus 2 \times 2_5 \oplus 2 \times 2_6 \oplus 2 \times 4_1 \oplus 2 \times 4_2.$$

All isotypic components that are not irreducible representations contain multiples of linear forms, and as a consequence we must use some 4-dimensional irreducible isotypic components. The invariant subspaces of these components all have the form

$$4_1 = \langle \lambda_1 x_1 x_5 + x_6^2, \lambda_1 x_2 x_6 - \zeta_6 x_5^2, \lambda_1 x_3 x_7 + x_8^2, \lambda_1 x_4 x_8 - \zeta_6 x_7^2\rangle,$$
$$4_2 = \langle \lambda_2 x_1^2 + x_2 x_5, \lambda_2 x_1 x_6 - \zeta_6 x_2^2, \lambda_2 x_3^2 + x_4 x_7, \lambda_2 x_3 x_8 - \zeta_6 x_4^2\rangle.$$

Each eigenspace of S is one-dimensional and yields a unique orbit as candidate vertex set. Every orbit puts constraints on $2 \times 4_1$ and $2 \times 4_2$, and for any orbit one of the two is excluded. Hence, we must choose exactly one of these components. Using the centralizer $C_{\mathrm{GL}(8,\mathbb{C})}(\mathrm{Aut}^+(\mathbf{R}_{8.5}))$, we may make this choice under the assumption $\lambda_1 = \lambda_2 = 1$. Next, calculation of the primary decomposition of $(2_3 \oplus 2_4 \oplus 4_k)$ yields exactly one ideal with $\dim(I_2) = 15$ for either choice. And in both these prime ideals $2 \times 1_2$ is present. This implies that we cannot include invariant subspaces 2_k for all $k \in \{2, 3, 4, 5, 6\}$, since that would imply $\dim(I_2) > 15$. As a consequence, at least one of

$$2 \times 1_1 = \langle x_1 x_4 - \zeta_{10} x_2 x_3, x_5 x_8 - \zeta_{10}^3 x_6 x_7\rangle,$$
$$2 \times 1_2 = \langle x_1 x_4 + \zeta_{10} x_2 x_3, x_5 x_8 + \zeta_{10}^3 x_6 x_7\rangle$$

must be included. Each of the four ideals $(2 \times 1_i \oplus 4_j)$ has a unique associated prime $I(i, j)$ that satisfies $\dim I(i, j)_2 = 15$, and the four primes each omit a different $2 \times 2_k$. For example, $I(1, 1)$ contains the additional pieces

$$2_2 = \langle x_1 x_2 + \zeta_6^{-1} x_5 x_6, x_3 x_4 + \zeta_6^{-1} x_7 x_8\rangle,$$
$$2_3 = \langle x_1 x_7 + \zeta_{10}^{-1} x_3 x_5, x_2 x_8 + \zeta_{10} x_4 x_6\rangle,$$
$$2_4 = \langle x_1 x_8 - \zeta_5 x_3 x_6, x_2 x_7 + \zeta_{10}^3 x_4 x_5\rangle.$$

All four ideals $I(i, j)$ define a corresponding surface $S(i, j)$. We conclude that our platonic surface must be trigonal, and that we need five more irreducible degree 3 irreducible polynomials to define it. The transformation $\mathrm{DiaMat}(1, 1, -1, 1, 1, 1, -1, 1)$ swaps $S(1, 1)$ and $S(2, 1)$, and similarly $\mathrm{DiaMat}(1, -1, 1, 1, -1, 1, 1, 1)$ swaps $S(1, 2)$

and $S(2,2)$. We must admit we have not found a projectivity relating $S(1,1)$ to $S(1,2)$, but it must exist, since all four choices lead to the unique surface $(\mathbf{R}_{8.5})_a$. Let us take $I = I(1,1)$ to continue the search.

We still need five irreducible polynomials of degree 3 to define the canonical curve. The representation $(\rho_c^\vee)^{3+}$ splits into many isotypic components. We compute the edge centers of the canonical model by imposing the constraints of I_2 on $E(RS, \pm 1)$. There are only two viable $\mathrm{Aut}^+(\mathbf{R}_{8.5})$-orbits, namely those of:

$$(\zeta_{60}^{31} : \zeta_{60}^{8} : \zeta_{60}^{18} : \zeta_{60} : \zeta_{60}^{-1} : 1 : \zeta_{60}^{10} : -\zeta_{60}^{-1}) \in E(RS, -1),$$
$$(\zeta_{60}^{31} : \zeta_{60}^{8} : -\zeta_{60}^{18} : -\zeta_{60} : \zeta_{60}^{-1} : 1 : -\zeta_{60}^{10} : \zeta_{60}^{-1}) \in E(RS, 1).$$

These orbits are interchanged by the projectivity $\mathrm{DiaMat}(-1,-1,1,1,-1,-1,1,1)$, which leaves the surface $S(1,1)$ invariant. Thus we choose the first without loss of generality. This point set puts enough restrictions on all isotypic components to yield a finite number of choices. All these choices yield a unique solution. This multitude of options seems to plague this specific map at every step, for some reason. One way to supplement I_2 with degree 3 polynomials to generate the canonical ideal is to add:

$$1_1 = \langle x_1x_6^2 - \zeta_6 x_2 x_5^2 + x_3 x_8^2 - \zeta_6 x_4 x_7^2 \rangle,$$
$$2_3 = \langle x_1^2 x_6 - \zeta_5 x_3^2 x_8 + \zeta_6^2 x_5^3 + \zeta_{30} x_7^3,\ x_2^2 x_5 + \zeta_{10}^3 x_4^2 x_7 + x_6^3 + \zeta_{10}^3 x_8^3 \rangle,$$
$$2_5 = \langle x_1^3 + \zeta_5^2 x_3^3 + \zeta_6^{-1} x_5^2 x_6 + \zeta_{30}^7 x_7^2 x_8,\ x_2^3 - \zeta_{10} x_4^3 + \zeta_6 x_5 x_6^2 + \zeta_{30}^{-7} x_7 x_8^2 \rangle.$$

$\mathbf{R}_{8.6}$ type $(6,24)$ **#cells** $(2,24,8)$ **map group size** 96
SMP $\mathrm{Aut}^+(\mathbf{R}) = \left\langle R, S \,\middle|\, R^6, S^{24}, (RS)^2, R^{-1}SRS^{-7} \right\rangle$

The canonical representation ρ_c can be generated by

$$R \mapsto \mathrm{MonMat}([\zeta_6^4, \zeta_6^{-1}, \zeta_6^{-1}, \zeta_6^2, 1, \zeta_6^2, 1, \zeta_6^4], [1,2,3,4,6,5,8,7]),$$
$$S \mapsto \mathrm{MonMat}([\zeta_6^{-1}, \zeta_6^4, \zeta_6^{-1}, 1, \zeta_{24}^3\zeta_6^{-1}, -\zeta_{24}^5, -\zeta_{24}^7, -\zeta_{24}], [1,2,4,3,5,6,7,8]),$$

with invariant subspaces $\langle e_1 \rangle$, $\langle e_2 \rangle$, $\langle e_3, e_4 \rangle$, $\langle e_5, e_6 \rangle$, $\langle e_7, e_8 \rangle$. The representation $(\rho_c^\vee)^{2+}$ decomposes as

$$2 \times 1_3 \oplus 1_4 \oplus 1_8 \oplus 3 \times 1_9 \oplus 2 \times 1_{10} \oplus 1_{11} \oplus 2 \times 2_1 \oplus 2_2 \oplus 2 \times 2_3 \oplus 3 \times 2_4 \oplus 2_5 \oplus 2 \times 2_6 \oplus 2_7 \oplus 2_9.$$

The isotypic components of all 2-dimensional irreducible representations and $2 \times 1_3$, $3 \times 1_9$, $2 \times 1_{10}$, 1_{11} contain multiples of linear forms. Hence, I_2 must certainly contain $2_1 \oplus 2_3 \oplus 2_6$. The general form of these invariant subspaces is

$$2_1 = \langle \lambda_1 x_1 x_3 + (i+1) x_5 x_7 - (i+1) x_6 x_8, \lambda_1 x_1 x_4 + (\zeta_6 + \zeta_{12}^{-1}) x_5 x_7 + (\zeta_6 + \zeta_{12}^{-1}) x_6 x_8 \rangle,$$
$$2_3 = \langle \lambda_2 x_2 x_3 + (i+1) x_7^2 + (-\zeta_6 + \zeta_{12}^{-1}) x_8^2, \lambda_2 x_2 x_4 + (\zeta_6 + \zeta_{12}^{-1}) x_7^2 + (\zeta_3 + \zeta_{12}) x_8^2 \rangle,$$
$$2_6 = \langle \lambda_3 x_2 x_7 - \zeta_{24}^5 x_3 x_8 - \zeta_{24}^7 x_4 x_8, \lambda_3 x_2 x_8 + \zeta_{24} x_3 x_7 - \zeta_{24}^3 x_4 x_7 \rangle.$$

Using the centralizer trick in a somewhat precarious balancing act of scaling coefficients, we first normalize 2_3 by scaling $\langle e_7, e_8 \rangle$. Second, we can normalize 2_6 by varying the scaling $\langle e_2 \rangle$ and $\langle e_3, e_4 \rangle$ inversely. This will not affect 2_3. Third, we normalize 2_1 by scaling $\langle e_1 \rangle$. The final outcome is that we may set $\lambda_1 = \lambda_2 = \lambda_3 = 1$. The primary decomposition of $(2_1 \oplus 2_3 \oplus 2_6)$ contains a unique prime ideal I that satisfies $\dim(I_2) = 15$. Its additional pieces are

$$\begin{aligned}
1_3 &= \langle x_1x_2 + (\zeta_{24}^7 + \zeta_{24}^5 - \zeta_{24}^3)x_5x_8 + (\zeta_{24}^7 + \zeta_{24})x_6x_7 \rangle, \\
1_4 &= \langle x_5x_8 + \zeta_3 x_6x_7 \rangle, \\
2 \times 1_9 &= \langle x_2^2 + 2(\zeta_{24}^7 + \zeta_{24}^5 - \zeta_{24}^3)x_7x_8,\ x_3^2 - \zeta_{24}^4 x_4^2 + 2(\zeta_{24}^7 - \zeta_{24}^5 - \zeta_{24}^3)x_7x_8 \rangle, \\
1_{10} &= \langle x_1^2 + 2(\zeta_{24}^7 + \zeta_{24})x_5x_6 \rangle, \\
2 \times 2_4 &= \langle x_1x_7 - \zeta_{24}x_3x_6 - \zeta_{24}^3 x_4x_6,\ x_1x_8 + \zeta_{24}x_3x_5 - \zeta_{24}^3 x_4x_5, \\
&\qquad x_2x_5 - \zeta_{24}x_3x_6 - \zeta_{24}^3 x_4x_6,\ x_2x_6 + \zeta_{24}^5 x_3x_5 - \zeta_{24}^7 x_4x_5 \rangle.
\end{aligned}$$

This ideal defines a surface, however, so the platonic surface is trigonal. The Hilbert-Poincaré series of I tells us we still need five extra irreducible polynomials of degree 3 to define the canonical ideal. We compute the decomposition of $(\rho_c^\vee)^{3+}$, which has a lot of big isotypic pieces. We supplement I_2 with each isotypic component in turn and consider the resulting ideals. Most of them only have associated primes with too many quadrics or cubics. The representation only has 1- and 2-dimensional irreducible representations, and the only remaining possibility for an additional 1-dimensional piece is restricted to one from

$$4 \times 1_4 = \langle x_1^3, x_1x_5x_6, x_2x_3x_4, x_3x_7^2 + \zeta_6 x_3x_8^2 + \zeta_{12}x_4x_7^2 - ix_4x_8^2 \rangle.$$

Parity forces the use of some $3 \times 1_4 < 4 \times 1_4$. To discover it, we compute the face centers and edge centers of the map, using the constraints imposed by I_2. The edge centers must lie in $E(RS, -1)$, which is spanned by $e_1, e_2, e_3 + \zeta_6 e_4, e_5 + \zeta_{24}^5 e_6$, and $e_7 + \zeta_{24}e_8$. With the last remaining freedom from the centralizer $C_{\mathrm{GL}(8,\mathbb{C})}(\mathrm{Aut}^+(\mathbf{R}_{8.6}))$, we may scale its $\langle e_5, e_6 \rangle$ component. The constraints of I_2 now leave only a finite number of possible $\mathrm{Aut}^+(\mathbf{R}_{8.6})$-orbits, and the only one with proper size is

$$((i-1)\sqrt[4]{2}\zeta_{16} : (i-1)\sqrt[4]{2}\zeta_{16} : \sqrt[4]{2}\zeta_{16} : \zeta_6\sqrt[4]{2}\zeta_{16} : 1 : \zeta_{24}^5 : 1 : \zeta_{24}).$$

Similarly, the orbit of face centers must be that of $\alpha(\zeta_8^3 e_2 + e_3) + e_7 + \zeta_6 e_8 \in E(R, \zeta_6^{-1})$, where $\alpha^2 = 2(\zeta_{24}^5 + \zeta_{24}^3 - \zeta_{24})$. These orbits of points give us enough constraints to determine $3 \times 1_4$. The ideal $(I_2 + 3 \times 1_4)$ has a unique associated prime that defines the canonical ideal. Polynomials of degree 3 we can add are:

$$\begin{aligned}
1_4 &= \langle 2x_1x_5x_6 + \zeta_{24}^{-1}x_3x_7^2 + \zeta_{24}^3 x_3x_8^2 + \zeta_{24}x_4x_7^2 - \zeta_{24}^5 x_4x_8^2 \rangle, \\
2_1 &= \langle 2x_1x_5^2 + (\zeta_{24}^{10} - \zeta_{24}^4)x_3x_4^2 - (\zeta_{24}^6 + 1)x_4^3 - 4\zeta_{24}x_4x_7x_8, \\
&\qquad 2x_1x_6^2 + (\zeta_{24}^{20} - \zeta_{24}^2)x_3x_4^2 + (\zeta_{24}^{10} + \zeta_{24}^4)x_4^3 + 4\zeta_{24}^5 x_4x_7x_8 \rangle, \\
2_4 &= \langle x_3x_4x_7 + (\zeta_{24}^{11} - \zeta_{24}^5)x_5^2x_6,\ x_3x_4x_8 + (\zeta_{24}^7 - \zeta_{24})x_5x_6^2 \rangle.
\end{aligned}$$

We arrive at a planar model for $\mathbf{R}_{8.6}$ by computing a suitable elimination ideal of the canonical ideal, for example to (x_1, x_2, x_5). This yields a birational mapping $(x_1 : \cdots : x_8) \mapsto (x_1 : x_2 : x_5)$. The image, when renaming the variables, is the planar model

$$y^3(x^8 - 64z^8) = -16ix^7z^4.$$

$\mathbf{R}_{8.7}$ type $(8,12)$ **#cells** $(4,24,6)$ **map group size** 96
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^8, S^{12}, (RS)^2, (RS^{-1})^2, RS^5R^{-3}S^{-1} \rangle$

The canonical representation ρ_c can be generated by

$$R \mapsto \mathrm{MonMat}([-1, 1, \zeta_8^{-1}, -\zeta_8, 1, -i, \zeta_{12}, \zeta_6], [2, 1, 3, 4, 6, 5, 8, 7]),$$
$$S \mapsto \mathrm{MonMat}([\zeta_6^{-1}, -\zeta_6, -1, 1, 1, 1, \zeta_6^{-1}, \zeta_6], [1, 2, 4, 3, 7, 8, 5, 6]).$$

It has invariant subspaces $\langle e_1, e_2 \rangle$, $\langle e_3, e_4 \rangle$, and $\langle e_5, e_6, e_7, e_8 \rangle$. The representation $(\rho_c^\vee)^{2+}$ decomposes as

$$1_1 \oplus 3 \times 1_3 \oplus 2 \times 2_1 \oplus \cdots \oplus 2 \times 2_5 \oplus 2_6 \oplus 2_7 \oplus 2 \times 4_1.$$

All isotypic pieces that are not irreducible representations contain multiples of linear forms. Moreover, of the three pieces

$$1_1 = \langle x_5x_8 + \zeta_6^{-1}x_6x_7 \rangle,$$
$$2_6 = \langle x_1x_5 - \zeta_8x_2x_6,\ x_1x_7 - \zeta_{24}^7x_2x_8 \rangle,$$
$$2_7 = \langle x_1x_5 + \zeta_8x_2x_6,\ x_1x_7 + \zeta_{24}^7x_2x_8 \rangle,$$

we can use only one, because any two yield no viable associated primes. Since $\dim I_2 = 15$ for the canonical ideal, an invariant $4_1 < 2 \times 4_1$ is necessary in the assembly of I_2. Any such subspace has the form

$$4_1 = \langle x_1x_3 + \lambda x_2x_5, x_1x_4 - \lambda\zeta_6x_2x_7,\ \lambda x_1x_6 - \zeta_8x_2x_3, \lambda x_1x_8 + \zeta_{24}^7x_2x_4 \rangle.$$

We employ the centralizer $C_{\mathrm{GL}(8,\mathbb{C})}(\mathrm{Aut}^+(\mathbf{R}_{8.7}))$ to set $\lambda = 1$. The ideal (4_1) has one associated prime without linear forms, and this prime ideal J contains 1_1 and invariant subspaces 1_3, 2_1, and 2_2. Hence, $2_6, 2_7 \not\subseteq I_2$, and I_2 must be of the form

$$1_1 \oplus 2 \times 1_3 \oplus 2_1 \oplus 2_2 \oplus 2_k \oplus 2_l \oplus 4_1,$$

where $k, l \in \{3, 4, 5\}$. To progress, we compute the vertices and face centers of the map. The only orbits generated by an eigenvector of R that satisfies the constraints imposed by J and of the right orbit size are those of $e_1 + ie_2$, $e_1 - ie_2$ and $\zeta_{24}^{10}e_4 + e_7 + \zeta_{24}^{11}e_8$. But the first two impose constraints on

$$3 \times 1_3 = \langle x_1x_2,\ x_3x_4,\ x_5x_8 + \zeta_3x_6x_7 \rangle$$

that only lead to prime ideals with linear forms. There are two possible vertex orbits, that of $e_6 + \zeta_{12}^{-1}e_8$ and that of $e_5 - \zeta_{12}e_7$. Each turns out a posteriori to be a correct path. We choose the first. This choice of vertices and face centers rules out a 2_5 and forces

$$2_4 = \langle x_3x_5 + \zeta_6x_4x_7 - x_6^2 + \zeta_6x_8^2,\ x_3x_6 + \zeta_3x_4x_8 + \zeta_8^{-1}x_5^2 - \zeta_{24}^5x_7^2 \rangle.$$

Now we look at the general form of an invariant $2_3 < 2 \times 2_3$, which is

$$2_3 = \langle \mu x_1^2 + x_3x_7 + \zeta_3x_4x_5,\ \mu x_2^2 + \zeta_{24}x_3x_8 - \zeta_{24}^5x_4x_6 \rangle.$$

The centralizer trick can be used again to scale the $\langle e_1, e_2 \rangle$ subspace. Miraculously, this does not affect the ideal part we already have: look back at $1_1 \oplus 4_1$. Therefore, we can set $\mu = 1$. The ideal $(1_1 \oplus 2_3 \oplus 2_4 \oplus 4_1)$ has one associated prime, and this defines the canonical model. Its additional invariant subspaces are

$$\begin{aligned}
2 \times 1_3 &= \langle x_1x_2 - \zeta_{24}x_5x_8 + \zeta_8^{-1}x_6x_7,\ 2x_3x_4 + \zeta_{24}^5x_5x_8 - \zeta_{24}x_6x_7 \rangle, \\
2_1 &= \langle x_3x_7 - \zeta_3x_4x_5,\ x_3x_8 + \zeta_6x_4x_6 \rangle, \\
2_2 &= \langle x_3^2 + \zeta_8^{-1}x_5x_6,\ x_4^2 + \zeta_8^{-1}x_7x_8 \rangle.
\end{aligned}$$

$\mathbf{R}_{8.8}$ type $(10, 20)$ **#cells** $(2, 20, 4)$ **map group size** 80
SMP $\mathrm{Aut}^+(\mathbf{R}) = \left\langle R, S \,\middle|\, R^{10}, S^{20}, (RS)^2, R^{-1}SRS^{-11} \right\rangle$

The canonical representation ρ_c can be generated by

$$\begin{aligned}
R &\mapsto \mathrm{MonMat}(8, 8, [\zeta_5^3, \zeta_5^4, -\zeta_5, -\zeta_5^2, -\zeta_5^4, \zeta_5^4, -\zeta_5^2, \zeta_5^2], [1, 2, 3, 4, 5, 6, 7, 8]), \\
S &\mapsto \mathrm{MonMat}(8, 8, [-\zeta_5^2, -\zeta_5, \zeta_5^4, \zeta_5^3, 1, -\zeta_5^2, 1, -\zeta_5], [1, 2, 3, 4, 6, 5, 8, 7]).
\end{aligned}$$

It has invariant subspaces $\langle e_1 \rangle$, $\langle e_2 \rangle$, $\langle e_3 \rangle$, $\langle e_4 \rangle$, $\langle e_5, e_6 \rangle$, and $\langle e_7, e_8 \rangle$. The representation $(\rho_c^\vee)^{2+}$ decomposes into a great number of isotypic components. We note that $\mathrm{Aut}^+(\mathbf{R}_{8.8})$ has twenty 1-dimensional irreducible representations and five 2-dimensional; no others. All isotypic components of $(\rho_c^\vee)^{2+}$ that are not irreducible contain multiples of linear forms. The same holds for a few irreducible isotypic components, so those can be discarded immediately. The only irreducible isotypics left to us are

$$1_8 = \langle x_5x_8 + \zeta_{10}x_6x_7 \rangle \quad \text{and} \quad 1_{16} = \langle x_5x_7 + \zeta_{10}^3x_6x_8 \rangle.$$

All associated primes of $(1_8 \oplus 1_{16})$ contain linear forms, so we cannot use both 1_8 and 1_{16}. This already forces us to use a maximal invariant proper subspace of all other available isotypic components, and I_2 must have the shape

$$1_4 \oplus 1_5 \oplus 1_6 \oplus 1_7 \oplus 1_9 \oplus 1_i \oplus 1_{17} \oplus 2 \times 1_{18} \oplus 2_1 \oplus 2_2 \oplus 2_3,$$

where $i \in \{8, 16\}$. The relevant 2_k-pieces each come from an isotypic component $2 \times 2_k$, and the invariant 2_k subspaces are of the respective forms

$$\begin{aligned}
2_1 &= \langle \lambda_1x_1x_7 + \zeta_5x_3x_6,\ \lambda_1x_1x_8 + \zeta_5^2x_3x_5 \rangle, \\
2_2 &= \langle \lambda_2x_2x_5 - \zeta_{10}x_3x_8,\ \lambda_2x_2x_6 - \zeta_{10}^3x_3x_7 \rangle, \\
2_3 &= \langle \lambda_3x_2x_7 + \zeta_5x_4x_6,\ \lambda_3x_2x_8 + \zeta_5^2x_4x_5 \rangle.
\end{aligned}$$

We now employ the centralizer $C_{\mathrm{GL}(8,\mathbb{C})}(\mathrm{Aut}^+(\mathbf{R}_{8.8}))$ to scale $\langle e_1 \rangle$, $\langle e_3 \rangle$, and $\langle e_4 \rangle$ respectively, to set $\lambda_1 = \lambda_2 = \lambda_3 = 1$. The ideal $(2_1 \oplus 2_2 \oplus 2_3)$ has one associated prime without linear forms in it. This prime ideal tells us we need to use

$$1_4 = \langle x_1x_4 - x_2x_3 \rangle, \quad 1_6 = \langle x_1x_2 - x_3^2 \rangle, \quad 1_{16} = \langle x_5x_7 + \zeta_{10}^3x_6x_8 \rangle, \quad 1_{18} = \langle x_2^2 - x_3x_4 \rangle.$$

Hence, 1_8 is out of the picture. Mind that we have only found a $1 \times 1_{18}$ instead of the necessary $2 \times 1_{18}$. To continue, we compute the vertices of the model. The polynomials we have so far, along with orbit size computation, leave only the orbit consisting of $e_5 \pm \zeta_{20} e_6$. Next, we notice that the prime ideal we computed is invariant under the transformation $\mathrm{DiaMat}(1,1,1,1,\lambda,\lambda,\lambda,\lambda)$. This allows us to scale the invariant $1_7 < 2 \times 1_7 = \langle x_4^2, x_7^2 + \zeta_5 x_8^2 \rangle$ to be

$$1_7 = \langle x_4^2 - x_7^2 - \zeta_5 x_8^2 \rangle.$$

The ideal generated by all pieces determined so far has exactly one associated prime, giving us another two subspaces,

$$1_{17} = \langle x_2 x_4 + \zeta_5^3 x_5 x_8 + \zeta_5 x_6 x_7 \rangle \quad \text{and} \quad 2 \times 1_{18} = \langle x_2^2 - x_3 x_4, x_3 x_4 - x_5^2 - \zeta_5^2 x_6^2 \rangle.$$

All the polynomial data amassed now force the edge centers to be the orbit of a point of the form

$$-\mu^3 \sqrt{2} e_1 - \mu \sqrt{2} e_2 + \mu^2 \sqrt{2} e_3 + \sqrt{2} e_4 + \mu(e_5 + \zeta_5^{-1} e_6) + e_7 + \zeta_5^2 e_8.$$

Amazingly, the transformation $\mathrm{DiaMat}(t^3, t, t^2, 1, t, t, 1, 1)$ leaves the ideal that we computed invariant, so everything works out; we may set $\mu = 1$ in the formula above. The resulting orbit imposes constraints that uniquely define the canonical ideal, generated by I_2. The invariant subspaces that must be added to the ones already listed are:

$$1_5 = \langle x_1 x_3 + 2\zeta_5^3 x_7 x_8 \rangle \quad \text{and} \quad 1_9 = \langle x_1^2 - x_5 x_7 - \zeta_5^{-1} x_6 x_8 \rangle.$$

As an aside, the face centers form the orbit $e_4 + e_7$.

$\mathbf{R}_{8.9}$ type $(17, 34)$ **#cells** $(1, 17, 2)$ **map group size** 68 **$\mathbf{Wi1}(8)$**
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^{17}, S^{34}, (RS)^2, RS^{-16} \rangle$

This is the Wiman type I map $\mathbf{Wi1}(8)$ with planar model $y^2 z^{15} = x^{17} - z^{17}$. For more information, see Section 5.1.

$\mathbf{R}_{8.10}$ type $(18, 18)$ **#cells** $(2, 18, 2)$ **map group size** 72 $D(\mathbf{AM}(8))$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^{18}, S^{18}, (RS)^2, [R, S] \rangle$

This map is $D(\mathbf{R}_{8.3}) = D(\mathbf{AM}(8))$ and thefore also has planar model $y^2 z^{16} = x^{18} - z^{18}$. For more information, see Section 5.3.

$\mathbf{R}_{8.11}$ type $(32, 32)$ **#cells** $(1, 16, 1)$ **map group size** 64 $D(\mathbf{Wi2}(8))$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^{32}, S^{32}, (RS)^2, RS^{-15} \rangle$

This map is $D(\mathbf{R}_{8.4}) = D(\mathbf{Wi2}(8))$ and thefore also has planar model $y^2 z^{15} = x(x^{16} - z^{16})$. For more information, see Section 5.2.

A.9 Genus 9 (examples)

Prior remarks. The Hilbert-Poincaré series predicts (cf. Section 6.1) that the canonical model of a non-hyperelliptic platonic map of genus 9 satisfies $\dim(I_2) = 21$. If the map is trigonal, some additional cubics will be needed to define the model. In any case $\dim(I_3) = 125$.

$\mathbf{R}_{9.9}$ **type** $(4, 12)$ **#cells** $(8, 48, 24)$ **map group size** 192
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^4, S^{12}, (RS)^2, [R, S^3], (RS^{-1})^4 \rangle$

Some consideration of $\mathrm{Aut}(\mathbf{R}_{9.9})$ leads us to the discovery that there is a platonic 4-cover $\pi : \mathbf{R}_{9.9} \to \mathbf{R}_{9.9}/\langle S^3 \rangle = \mathbf{Cub}$, branched over cells_0. Referring back to the vertex data of **Cub** in Section 6.4, we write down the planar model

$$y^4 z^4 = x^8 + 14x^4 z^4 + z^8.$$

It has standard map presentation

$$R : (x : y : z) \mapsto (-ix : y : z),$$
$$S : (x : y : z) \mapsto ((x + z)^2 : 2yz : (x + iz)^2).$$

Standard complex conjugation con_9 is a reflection of the map on this curve.

Remark A.9.1. In fact, $\mathbf{R}_{9.11}$ is a platonic 4-cover $\pi : \mathbf{R}_{9.11} \to \mathbf{R}_{9.11}/\langle S^3 \rangle = \mathbf{Cub}$ of the cube as well. This demonstrates the point that a cover, even a platonic one, is not globally determined by its local behaviour. Computation of the automorphism group of the above planar model revealed *a posteriori* that that curve supports the standard map presentation of $\mathbf{R}_{9.9}$, not of $\mathbf{R}_{9.11}$. A model for the latter remains to be constructed, and this case deserves attention.

$\mathbf{R}_{9.15}$ **type** $(5, 6)$ **#cells** $(20, 60, 24)$ **map group size** 240
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^5, S^6, (RS)^2, (RS^{-2})^2 \rangle$

Some consideration of $\mathrm{Aut}(\mathbf{R}_{9.15})$ leads us to the discovery that there is a platonic 2-cover $\pi : \mathbf{R}_{9.15} \to \mathbf{R}_{9.15}/\langle S^3 \rangle = \mathbf{Dod}$, branched over cells_0. Referring back to the vertex data of **Dod** in Section 6.4, we write down the planar model

$$y^2 z^{18} = x^{20} - 228x^{15} z^5 + 494x^{10} z^{10} + 228x^5 z^{15} + z^{20}.$$

We have not computed a standard map presentation for, though.

$\mathbf{R}_{9.26}$ **type** $(12, 12)$ **#cells** $(4, 24, 4)$ **map group size** 96
SMP $\mathbf{R}_{9.26}$**:** $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^{12}, S^{12}, (RS)^2, R^3 S^{-3}, (RS^{-1})^3 \rangle$

This map is $D_1(\mathbf{R}_{9.9})$ and therefore has the same planar model. Also, there is a platonic 4-cover $\pi : \mathbf{R}_{9.26} \to \mathbf{R}_{9.26}/\langle R^3, S^3 \rangle = \mathbf{Tet}$. We have the following commutative diagram relating the four map automorphism groups:

$$\begin{array}{ccc} \mathrm{Aut}(\mathbf{R}_{9.9}) & \longrightarrow & \mathrm{Aut}(\mathbf{Cub}) \\ \uparrow & & \uparrow \\ \mathrm{Aut}(\mathbf{R}_{9.26}) & \longrightarrow & \mathrm{Aut}(\mathbf{Tet}) \end{array}$$

Remark A.9.2. Parallel to the above, $\mathbf{R}_{9.27} = D_1(\mathbf{R}_{9.11})$, and there is a platonic 4-cover $\pi : \mathbf{R}_{9.27} \to \mathbf{R}_{9.27}/\langle R^3, S^3\rangle = \mathbf{Tet}$. In fact, $\mathrm{Aut}(\mathbf{R}_{9.26}) \cong \mathrm{Aut}(\mathbf{R}_{9.27})$, both are isomorphic to a semi-direct product $(\mathbb{Z}_4 \times \mathbb{Z}_2^2) \rtimes \mathrm{Sym}_3$. The group $\mathrm{Aut}^+(\mathbf{R}_{9.26})$ has two $\mathrm{Aut}(\mathbf{R}_{9.26})$-orbits of standard generator pairs, resulting in the existence of these two platonic maps.

A.10 Genus 10 (examples)

Prior remarks. The Hilbert-Poincaré series predicts (cf. Section 6.1) that the canonical model of a non-hyperelliptic platonic map of genus 10 satisfies $\dim(I_2) = 28$. If the map is trigonal, some additional cubics will be needed to define the model. In any case $\dim(I_3) = 175$.

$\mathbf{R}_{10.1}$ **type** $(3, 9)$ **#cells** $(32, 162, 108)$ **map group size** 648 **Mod**(9)
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^3, S^9, (RS)^2, [RS^{-2}R, S^3] \rangle$

This is the modular map $\mathbf{Mod}(9)$. More information on the modular map family can be found in Section 5.7. The canonical representation ρ_c can be generated by

$$R \mapsto \mathrm{MonMat}([1, 1, 1, \zeta_3^2, 1, 1, 1, 1, 1, 1], [2, 3, 1, 4, 7, 8, 9, 10, 5, 6]),$$
$$S \mapsto \mathrm{MonMat}([1, \zeta_3^2, \zeta_3, \zeta_3, -1, -1, 1, 1, \zeta_6, \zeta_6^{-1}], [4, 2, 1, 3, 9, 10, 6, 5, 8, 7]).$$

It has invariant subspaces $\langle e_1, \dots, e_4 \rangle$ and $\langle e_5, \dots, e_{10} \rangle$. The representation $(\rho_c^\vee)^{2+}$ decomposes as

$$1_1 \oplus 1_2 \oplus 1_3 \oplus 4_1 \oplus 2 \times 4_4 \oplus 4_6 \oplus 2 \times 6_1 \oplus 2 \times 12_1.$$

Here,

$$\begin{aligned} 2 \times 4_4 &= \langle x_1^2,\ x_2^2,\ x_3^2,\ x_4^2,\ x_1x_5 + \zeta_3 x_2x_7 + \zeta_3^2 x_3x_9,\ x_1x_8 - x_2x_6 + \zeta_3 x_4x_9, \\ &\qquad x_1x_{10} - x_3x_6 - x_4x_7,\ x_2x_{10} - x_3x_8 + \zeta_3^2 x_4x_5 \rangle, \\ 2 \times 6_1 &= \langle x_1x_2,\ x_1x_3,\ x_1x_4,\ x_2x_3,\ x_2x_4,\ x_3x_4,\ x_5^2,\ x_6^2,\ x_7^2,\ x_8^2,\ x_9^2,\ x_{10}^2 \rangle, \\ 2 \times 12_1 &= \langle x_1x_6,\ x_1x_7,\ x_1x_9,\ x_2x_5,\ x_2x_8,\ x_2x_9,\ x_3x_5,\ x_3x_7,\ x_3x_{10},\ x_4x_6,\ x_4x_8, \\ &\qquad x_4x_{10},\ x_5x_7,\ x_5x_8,\ x_5x_9,\ x_5x_{10},\ x_6x_7,\ x_6x_8,\ x_6x_9,\ x_6x_{10},\ x_7x_9,\ x_7x_{10}, \\ &\qquad x_8x_9,\ x_8x_{10} \rangle. \end{aligned}$$

Because each of these three contains multiples of linear forms, the shape of I_2 is forced to be $12_1 \oplus 6_1 \oplus 4_i \oplus 4_j \oplus 1_k \oplus 1_l$. Furthermore,

$$\begin{aligned} 4_1 &= \langle x_1x_5 + x_2x_7 + x_3x_9,\ x_1x_8 - \zeta_3 x_2x_6 + x_4x_9,\ x_1x_{10} - \zeta_3^2 x_3x_6 - \zeta_3 x_4x_7, \\ &\qquad x_2x_{10} - \zeta_3 x_3x_8 + \zeta_3 x_4x_5 \rangle, \\ 4_6 &= \langle x_1x_5 + \zeta_3^2 x_2x_7 + \zeta_3 x_3x_9,\ x_1x_8 - \zeta_3^2 x_2x_6 + \zeta_3^2 x_4x_9,\ x_1x_{10} - \zeta_3 x_3x_6 - \zeta_3^2 x_4x_7, \\ &\qquad x_2x_{10} - \zeta_3^2 x_3x_8 + x_4x_5 \rangle, \end{aligned}$$

and $x_3x_9x_{10} \in (4_1 \oplus 4_6)$, so that we may assume $4_i = 4_4 < 2 \times 4_4$. All invariant spaces $12_1 < 2 \times 12_1$ are of the form

$$\begin{aligned} &\langle \lambda x_1x_6 - \zeta_3 x_8x_{10},\ \lambda x_1x_7 - x_5x_{10},\ \lambda x_1x_9 - \zeta_3^2 x_5x_8,\ \lambda x_2x_5 - \zeta_3^2 x_7x_{10}, \\ &\ \lambda x_2x_8 - \zeta_3 x_6x_{10},\ \lambda x_2x_9 - x_6x_7,\ \lambda x_3x_5 - x_8x_9,\ \lambda x_3x_7 - \zeta_3^2 x_6x_9, \\ &\ \lambda x_3x_{10} - \zeta_3 x_6x_8,\ \lambda x_4x_6 - \zeta_3^2 x_7x_9,\ \lambda x_4x_8 - \zeta_3 x_5x_9,\ \lambda x_4x_{10} - x_5x_7 \rangle \end{aligned}$$

for some $\lambda \in \mathbb{C}$. We now use the centralizer $C_{\mathrm{GL}(10,\mathbb{C})}(\mathrm{Aut}^+(\mathbf{R}_{10.1}))$ and set $\lambda = 1$. To this we add either 4_1 or 4_6 and compute the primary decomposition. Both choices yield exactly one ideal that defines a prime ideal with the correct Hilbert-Poincaré series. In each case, the invariant pieces

$$\begin{aligned}
1_1 &= \langle x_5x_6 + x_7x_8x_9x_{10}\rangle,\\
1_3 &= \langle x_5x_6 + \zeta_3^2x_7x_8 + \zeta_3x_9x_{10}\rangle,\\
1_6 &= \langle x_1x_2 - \zeta_3^2x_{10}^2,\ x_1x_3 - \zeta_3^2x_8^2,\ x_1x_4 - x_5^2,\\
&\qquad x_2x_3 - \zeta_3^2x_6^2,\ x_2x_4 - \zeta_3^2x_7^2,\ x_3x_4 - \zeta_3x_9^2\rangle
\end{aligned}$$

are the other polynomials sufficient to generate the canonical ideal. The choice between 4_1 and 4_6 does not matter: both yield a correct canonical model. We know they must be isomorphic, because the platonic surface is unique. However, we have not constructed a projectivity to transform one to the other, making this explicit.

$\mathbf{R}_{10.2}$ type $(3, 12)$ **#cells** $(18, 108, 72)$ **map group size** 432 **Fer**(6)
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^3, S^{12}, (RS)^2, (R^2S^2)^3\rangle$

This is the Fermat map **Fer**(6) with planar model $x^6 + y^6 + z^6 = 0$. For more information, see Section 5.5.

$\mathbf{R}_{10.3}$ type $(3, 15)$ **#cells** $(12, 90, 60)$ **map group size** 360
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^3, S^{15}, (RS)^2, [R, S^5], (RS^{-3})^3\rangle$

Some consideration of $\mathrm{Aut}(\mathbf{R}_{10.3})$ leads us to the discovery that there is a platonic 3-cover $\pi : \mathbf{R}_{10.3} \to \mathbf{R}_{10.3}/\langle S^5\rangle = \mathbf{Ico}$, branched over cells_0. Referring back to the vertex data of **Ico** in Section 6.4, we write down the planar curve $y^3z^9 = -xz(x^{10} + 11x^5z^5 - z^{10})$. It is reducible, but dividing out the component $z = 0$ and transforming by $y \mapsto iy$, we find the planar model

$$y^3z^8 = x^{11} + 11x^6z^5 - xz^{10}.$$

It has standard map presentation

$$\begin{aligned}
&R : (x : y : z) \mapsto\\
&\quad \left(-(\zeta_5^2+\zeta_5+1)(x - (\zeta_5+\zeta_5^3)z)(x - (1+\zeta_5^4)z)^3 : 5(\zeta_{15}^8-\zeta_{15}^{-1}-\zeta_{15}^2)yz^3 : (x - (1+\zeta_5^4)z)^4\right),\\
&S : (x : y : z) \mapsto (\zeta_5x : \zeta_{15}y : z).
\end{aligned}$$

Standard complex conjugation con_{10} is a reflection of the map on this curve.

$\mathbf{R}_{10.5}$ type $(3, 24)$ **#cells** $(6, 72, 48)$ **map group size** 288
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^3, S^{24}, (RS)^2, \rangle$

This map is a platonic 6-cover of **Oct** branched over cells_0. It therefore has the nonsingular planar model

$$y^6 = x(x \pm 1)(x \pm i) = x^5z - xz^5$$

This model has standard map presentation

$$\begin{aligned} R &: (x : y : z) \mapsto ((i-1)(x+z) : 2\zeta_{12} y : (i+1)(z-x)), \\ S &: (x : y : z) \mapsto (\zeta_{24}^5 x : y : \zeta_{24}^{-1} z). \end{aligned}$$

Standard complex conjugation is a reflection of the map on this model.

$\mathbf{R}_{10.6}$ **type** $(4, 5)$ **#cells** $(72, 180, 90)$ **map group size** 720 **Wiman's 2nd sextic map**
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^4, S^5, (RS)^2, (RS^{-1})^5 \rangle$

The group $\mathrm{Aut}^+(\mathbf{R}_{10.6})$ is isomorphic to Alt_6 (or equivalently, $\mathrm{PSL}(2, 9)$), and its canonical character is the unique 10-dimensional irreducible. The canonical representation ρ_c can be generated by sending a standard generator pair (R, S) to

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

Although it is not monomial, this representation (taken from [W$^+$1996]) does contain a monomial Alt_5 subgroup. The representation $(\rho_c^\vee)^{2+}$ decomposes as

$$1_1 \oplus 2 \times 5_1 \oplus 2 \times 5_2 \oplus 8_1 \oplus 8_2 \oplus 2 \times 9_1.$$

The pieces 8_1 and 8_2 are defined over $\mathbb{Q}(\alpha)$, where $\alpha = \zeta_5 + \zeta_5^{-1}$, and generated by spinning the following polynomials (using the $+$-sign and $-$-sign respectively) with $\mathrm{Aut}^+(\mathbf{R}_{10.6})$:

$$\Big(6x_2x_4 - 6x_2x_7 - x_2x_8 - 3x_2x_{10} - 6x_3x_4 + 6x_3x_7 + 3x_3x_8 + x_3x_{10} - 2x_4x_5 + 6x_4x_6 + 3x_4x_8 - 6x_4x_9 + 3x_4x_{10} - 2x_5x_7 + 6x_5x_8 - 6x_5x_{10} - 6x_6x_7 + 3x_6x_8 + x_6x_{10} + 3x_7x_8 + 6x_7x_9 + 3x_7x_{10} - x_8x_9 - 3x_9x_{10}\Big) \pm (2\alpha^2 - 3)\Big(- 2x_2x_5 + 4x_2x_6 + 2x_2x_7 + 3x_2x_8 + x_2x_{10} + 2x_3x_4 + 2x_3x_5 - x_3x_8 - 4x_3x_9 - 3x_3x_{10} - x_4x_8 - 2x_4x_9 + x_4x_{10} - 2x_5x_6 + 2x_5x_9 - 2x_6x_7 + x_6x_8 + 3x_6x_{10} + x_7x_8 - x_7x_{10} - 3x_8x_9 - x_9x_{10}\Big).$$

The ideal $(8_1 \oplus 8_2)$ is computed to have only associated primes with linear forms in them, so either 8_1 or 8_2 has to be excluded. The other isotypic components are defined over $\mathbb{Q}$. We will not display them, but one can compute

$$x_5^3 \in (2 \times 5_1 \oplus 2 \times 5_2)$$

and

$$(x_9 \pm x_{10})(x_9 + 3x_{10})(x_9 \pm ix_{10})(x_8 + \tfrac{1}{3}x_{10}) \in (2 \times 9_1).$$

This forces $I_2 = 1_1 \oplus 5_1 \oplus 5_2 \oplus 8_k \oplus 9_1$, with $k \in \{1, 2\}$. The field automorphism $\alpha \mapsto \alpha^2 - 2$ (which extends to $\zeta_5 \mapsto \zeta_5^2$ of $\mathbb{Q}(\zeta_5)$) interchanges the two pieces 8_1 and 8_2, so we can choose either one for the canonical ideal. The choices might result in two non-isomorphic curves, but we know a priori that $\mathbf{R}_{10.6}$ is not a member of a tuplet (cf. Section 1.4), so that will not happen. We choose 8_1 to form our canonical ideal and continue by using the fixed point strategy. The polynomials of $1_1 \oplus 8_1$ impose restrictions on the vertices on the canonical model. In fact, a single $\mathrm{Aut}^+(\mathbf{R}_{10.6})$-orbit remains, defined over $\mathbb{Q}(\zeta_5, \sqrt{-3})$, namely that of

$$\begin{gathered}\big(3\zeta_5^4\sqrt{-3} + \zeta_5^4 - 2\zeta_5^2 - 2\zeta_5 \;:\; -3\zeta_5^2\sqrt{-3} - 2\zeta_5^3 - 3\zeta_5^2 - 2\zeta_5 \;:\; 8\zeta_5^3 + 10\zeta_5^2 + 10\zeta_5 + 8 :\\ 3(\zeta_5^4 + 1)\sqrt{-3} - 3\zeta_5^3 - 5\zeta_5^2 - 3\zeta_5 \;:\; (3\sqrt{-3} - 7)(\zeta_5^2 + \zeta_5 + 1) \;:\; (-3\sqrt{-3} + 7)(\zeta_5 + 1) :\\ 3(\zeta_5 + 1)\sqrt{-3} - 2\zeta_5^3 + 3\zeta_5 + 3 \;:\; 2\zeta_5^3 + 2\zeta_5^2 - 8 \;:\; -3(\zeta_5^2 + \zeta_5 + 1)\sqrt{-3} - 3\zeta_5^2 - 5\zeta_5 - 3 :\\ 3\sqrt{-3} - 2\zeta_5^3 - 2\zeta_5^2 + 1\big).\end{gathered}$$

The 72 points of this set yield enough constraints to determine a unique $5_1 < 2 \times 5_1$, $5_2 < 2 \times 5_2$, and $9_1 < 2 \times 9_1$. Because of the approach using fixed points, the resulting canonical model is not defined over $\mathbb{Q}(\alpha)$, but over $\mathbb{Q}(\alpha, \sqrt{-3})$. The unique $\mathrm{Aut}^+(\mathbf{R}_{10.6})$-invariant quadric 1_1 is given by:

$$\begin{aligned}&2x_1^2 + x_1x_2 + x_1x_3 - x_1x_4 - x_1x_6 + x_1x_7 - x_1x_9 + 2x_2^2 + x_2x_5 + x_2x_6 - x_2x_7 - x_2x_9 +\\ &x_2x_{10} + 2x_3^2 + x_3x_4 + x_3x_5 - x_3x_6 + x_3x_8 + x_3x_9 + 2x_4^2 - x_4x_7 + x_4x_8 - x_4x_9 - x_4x_{10} +\\ &2x_5^2 + x_5x_6 - x_5x_8 + x_5x_9 - x_5x_{10} + 2x_6^2 + x_6x_7 + x_6x_8 + 2x_7^2 + x_7x_8 - x_7x_{10} + 2x_8^2 +\\ &x_8x_{10} + 2x_9^2 + x_9x_{10} + 2x_{10}^2.\end{aligned}$$

One gets the correct invariant 5_1, 5_2 and 9_1 by spinning the following polynomials with $\mathrm{Aut}^+(\mathbf{R}_{10.6})$, respectively:

$$\begin{aligned}&12x_2^2 + 8x_2x_5 + 8x_2x_6 - 8x_2x_7 + 8x_2x_{10} - 12x_3^2 - 8x_3x_4 - 8x_3x_5 - 8x_3x_8 - 8x_3x_9 -\\ &12x_4^2 - 8x_4x_8 + 8x_4x_9 + 8x_4x_{10} + 8x_5x_6 - 8x_5x_9 + 12x_6^2 + 8x_6x_7 + 8x_6x_8 + 12x_7^2 +\\ &8x_7x_8 - 8x_7x_{10} - 12x_9^2 - 8x_9x_{10} + (3\sqrt{-3} - \alpha^2 + 3)\big(x_2x_8 - x_3x_{10} - x_4x_5 - x_5x_7 -\\ &x_6x_{10} + x_8x_9\big),\\ &4x_1x_3 - 4x_1x_6 - 4x_3x_5 - 4x_3x_6 - 4x_3x_9 - 4x_5x_8 - 4x_5x_9 - 4x_5x_{10} - 4x_6x_7 -\\ &4x_6x_8 - 4x_7x_8 + 4x_8x_{10} + (3\sqrt{-3} - 2\alpha^2 + 3)\big(- x_1x_5 + x_1x_8 + 2x_1x_{10} + x_3x_7 +\\ &x_3x_{10} + x_5x_7 + x_6x_9 - x_6x_{10} + 2x_7x_9 + x_8x_9\big),\\ &2x_2x_4 - 2x_2x_8 + 2x_3x_7 + 2x_3x_{10} - 4x_4x_5 + 2x_4x_6 - 4x_5x_7 - 8x_5x_8 + 8x_5x_{10} +\\ &2x_6x_{10} + 2x_7x_9 - 2x_8x_9 + (3(\alpha^2 - 3)\sqrt{-3} - 7)\big(x_2x_7 + x_3x_4 + x_4x_9 + x_6x_7\big) +\\ &(3(\alpha^2 - 3)\sqrt{-3} + 1)\big(- x_2x_{10} + x_3x_8 + x_4x_8 + x_4x_{10} + x_6x_8 + x_7x_8 + x_7x_{10} -\\ &x_9x_{10}\big) + (3(\alpha^2 - 3)\sqrt{-3} + 5)\big(x_8^2 - x_{10}^2\big).\end{aligned}$$

It should be possible to obtain a canonical model over $\mathbb{Q}(\alpha)$ or even $\mathbb{Q}$ by applying Galois descent, but we leave this for future research.

Remark A.10.1. We designate this map as *Wiman's second sextic map*. The reason is that planar models of the corresponding platonic surface were studied first in [Wim1896]. The adjective 'second' distinguishes it from Wiman's first sextic map, $\mathbf{R}_{6.2}$. A planar model for Wiman's second sextic map is

$$x^6+y^6+z^6-\tfrac{3}{4}(5+i\sqrt{15})(x^2y^4+x^4y^2+y^2z^4+y^4z^2+z^2x^4+z^4x^2)+3(5-i\sqrt{15})x^2y^2z^2=0,$$

with standard map presentation

$$R \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad S \mapsto \begin{pmatrix} -\varsigma_3 & \varsigma_{15}^7-\varsigma_{15}^3+\varsigma_{15}^2-2 & -\varsigma_{15}^4-\varsigma_{15} \\ \varsigma_{15}^7-\varsigma_{15}^3+\varsigma_{15}^2-1 & \varsigma_3+1 & \varsigma_{15}^{-1}+\varsigma_{15}^{-4}-\varsigma_{15}{}^5 \\ \varsigma_{15}^5+\varsigma_{15}^4+\varsigma_{15}+1 & \varsigma_{15}^5-\varsigma_{15}^3+\varsigma_{15}^2-\varsigma_{15}^{-2} & 1 \end{pmatrix}.$$

We note that this standard map presentation only defines a 3-dimensional *projective* representation of $\mathrm{Aut}^+(\mathbf{R}_{10.6}) \cong \mathrm{Alt}(6)$, but does not lift to a linear representation of $\mathrm{Alt}(6)$. It does lift to a linear representation of a central extension $3 \cdot \mathrm{Aut}^+(\mathbf{R}_{10.6})$, which is traditionally called the 'Valentiner group'. In the same paper mentioned above, Wiman computes two more planar models for $\mathbf{R}_{10.6}$, so the reader can choose their own favorite:

$$z^6+30z^4xy-150x^2y^2z^2+100x^3y^3+15\sqrt{15}(z^2+2xy)(x^4-y^4)=0,$$
$$27z^6-135z^4xy-45x^2y^2z^2+9z(x^5+y^5)+10x^3y^3=0.$$

$\mathbf{R}_{10.9}$ **type** $(4,7)$ **#cells** $(24,84,42)$ **map group size** 336
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^4, S^7, (RS)^2, (RS^{-1})^3 \rangle$

The canonical representation ρ_c is generated by

$$R \mapsto \begin{pmatrix} 1 & -1 & \frac{1}{2}(\sqrt{-7}+1) \\ 0 & -1 & \frac{1}{2}(\sqrt{-7}-1) \\ 0 & \frac{1}{2}(\sqrt{-7}+1) & 1 \end{pmatrix} \oplus \begin{pmatrix} -1 & 1 & 0 & 0 & -1 & 1 & 1 \\ -1 & 0 & 0 & 1 & 0 & 1 & 0 \\ -1 & 2 & 1 & 1 & -1 & 2 & 1 \\ -1 & 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$S \mapsto \begin{pmatrix} 0 & -\frac{1}{2}(\sqrt{-7}+1) & -1 \\ \frac{1}{2}(\sqrt{-7}+1) & \frac{1}{2}(-\sqrt{-7}-3) & \frac{1}{2}(\sqrt{-7}-1) \\ 1 & \frac{1}{2}(\sqrt{-7}-1) & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ -2 & 2 & 1 & 1 & -1 & 3 & 1 \\ -1 & 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

It has invariant subspaces $\langle e_1, e_2, e_3 \rangle$ and $\langle e_4, \ldots, e_{10} \rangle$. The representation $(\rho_c^\vee)^{2+}$ decomposes as $1_1 \oplus 4 \times 6_1 \oplus 2 \times 7_1 \oplus 2 \times 8_1$. Except for the first, all these isotypic components contain reducibles: $x_1^2 \in (4 \times 6_1)$, $x_3x_{10}(x_8+x_{10})(x_7+x_8) \in (2 \times 7_1)$, and $x_3x_7^3 \in (2 \times 8_1)$. Since the canonical ideal I satisfies $\dim I_2 = 28$, I_2 must assume the shape

$$1_1 \oplus 2 \times 6_1 \oplus 7_1 \oplus 8_1.$$

The invariant quadric (unique up to a scalar) is determined by:

$$\begin{aligned}1_1 = \langle & 3x_4^2 - 2x_4x_5 + 2x_4x_6 + 2x_4x_7 - 2x_4x_8 + 2x_4x_9 + 2x_4x_{10} + 3x_5^2 - 2x_5x_6 - 2x_5x_7 + \\ & 2x_5x_8 - 2x_5x_9 - 2x_5x_{10} + 3x_6^2 + 2x_6x_7 - 2x_6x_8 - 2x_6x_9 + 2x_6x_{10} + 3x_7^2 - 2x_7x_8 - \\ & 2x_7x_9 + 2x_7x_{10} + 3x_8^2 + 2x_8x_9 - 2x_8x_{10} + 3x_9^2 - 2x_9x_{10} + 3x_{10}^2\rangle.\end{aligned}$$

All invariant subspaces 7_1 are generated by a set of vectors that one gets by spinning a linear combination of the two polynomials

$$\begin{gathered}2x_1x_4 - \tfrac{1}{2}(\sqrt{-7} - 1)x_2x_5 - \tfrac{1}{2}(\sqrt{-7} + 7)x_2x_6 + \tfrac{1}{2}(3\sqrt{-7} + 1)x_2x_7 + \tfrac{1}{2}(\sqrt{-7} - 1)x_2x_8 - \\ \tfrac{1}{2}(\sqrt{-7} - 1)x_2x_9 + (\sqrt{-7} + 1)x_2x_{10} - 2x_3x_5 + \tfrac{1}{2}(\sqrt{-7} - 1)x_3x_6 + 4x_3x_7 + \\ \tfrac{1}{2}(\sqrt{-7} - 1)x_3x_8 - 2x_3x_9 - (\sqrt{-7} - 3)x_3x_{10}, \qquad \text{and} \\ -(\sqrt{-7} - 5)x_4^2 + 8x_4x_6 + (2\sqrt{-7} + 6)x_4x_7 - 8x_4x_8 - (2\sqrt{-7} + 6)x_4x_9 + \\ (\sqrt{-7} + 11)x_4x_{10} + (\sqrt{-7} - 5)x_5^2 - (2\sqrt{-7} - 2)x_5x_6 + (2\sqrt{-7} - 2)x_5x_8 - \\ (\sqrt{-7} - 5)x_5x_{10} + (\sqrt{-7} + 3)x_6^2 + 8x_6x_7 - (2\sqrt{-7} + 6)x_6x_8 - 8x_6x_9 - \\ (\sqrt{-7} - 13)x_6x_{10} - (\sqrt{-7} - 5)x_7^2 - 8x_7x_8 + (2\sqrt{-7} - 10)x_7x_9 + (\sqrt{-7} + 11)x_7x_{10} + \\ (\sqrt{-7} + 3)x_8^2 + 8x_8x_9 - (\sqrt{-7} + 3)x_8x_{10} - (\sqrt{-7} - 5)x_9^2 + (\sqrt{-7} - 5)x_9x_{10}\end{gathered}$$

with $\mathrm{Aut}^+(\mathbf{R}_{10.9})$. Neither one of them alone yields a suitable prime ideal. We therefore apply the centralizer trick to scale the linear combination to be their sum, without loss of generality. Next, we turn to the fixed point strategy. We compute the eigenspaces of R and find that the polynomials of $1_1 \oplus 7_1$ yield enough constraints to determine a finite number of possible $\mathrm{Aut}^+(\mathbf{R}_{10.9})$-orbits of face centers. The only possible orbit in $E(R, 1)$ has an orbit that is too small. Each of $E(R, \pm i)$ contains three possible orbits. Trying each of these in turn, we compute what constraints they give on $4 \times 6_1$ and $2 \times 8_1$. Only the orbit of

$$\begin{gathered}(0 : -4(2\zeta_{28}^{11} - 2\zeta_{28}^{9} + \zeta_{28}^{7} - 2\zeta_{28} + 1) : 4(\zeta_{28}^{8} - 2\zeta_{28}^{7} + \zeta_{28}^{4} - \zeta_{28}^{2} + 2) : -\zeta_{28}^{11} + \zeta_{28}^{9} + \zeta_{28}^{8} - \zeta_{28}^{7} + \zeta_{28}^{4} - \zeta_{28}^{2} + \zeta_{28} - 3 : \\ 6 : 2(\zeta_{28}^{11} - \zeta_{28}^{9} + 2\zeta_{28}^{7} - \zeta_{28} + 2) : \zeta_{28}^{11} - \zeta_{28}^{9} - \zeta_{28}^{8} + \zeta_{28}^{7} - \zeta_{28}^{4} + \zeta_{28}^{2} - \zeta_{28} + 3 : -2(\zeta_{28}^{8} - 2\zeta_{28}^{7} + \zeta_{28}^{4} - \zeta_{28}^{2} + 2) : \\ 3(\zeta_{28}^{11} - \zeta_{28}^{9} - \zeta_{28}) + \zeta_{28}^{8} + \zeta_{28}^{7} + \zeta_{28}^{4} - \zeta_{28}^{2} + 11 : -\zeta_{28}^{11} + \zeta_{28}^{9} + \zeta_{28}^{8} - \zeta_{28}^{7} + \zeta_{28}^{4} - \zeta_{28}^{2} + \zeta_{28} + 3)\end{gathered}$$

is contained in the variety defined by $2 \times 8_1$, and it yields unique $\mathrm{Aut}^+(\mathbf{R}_{10.9})$-invariant subspaces $2 \times 6_1$ and 8_1. The first is spanned by spinning the following polynomials with $\mathrm{Aut}^+(\mathbf{R}_{10.9})$:

$$\begin{gathered}14x_1^2 + 14x_2x_4 + \tfrac{1}{2}(7\sqrt{-7} + 21)x_2x_6 - 7x_2x_8 + \tfrac{1}{2}(7\sqrt{-7} + 7)x_2x_{10} - 7\sqrt{-7}x_3x_4 - \\ (7\sqrt{-7} - 7)x_3x_6 + \tfrac{1}{2}(7\sqrt{-7} + 7)x_3x_8 + 7x_3x_{10} - (3\sqrt{-7} - 51)x_4^2 - \\ (8\sqrt{-7} + 24)x_4x_5 + (8\sqrt{-7} + 24)x_4x_6 - (24\sqrt{-7} + 8)x_4x_7 + (4\sqrt{-7} + 4)x_4x_8 + \\ (24\sqrt{-7} + 72)x_4x_9 - (6\sqrt{-7} - 38)x_4x_{10} + (14\sqrt{-7} - 46)x_5^2 - (16\sqrt{-7} - 32)x_5x_6 + \\ (12\sqrt{-7} + 148)x_5x_7 - 80x_5x_8 - (20\sqrt{-7} + 108)x_5x_9 + (-8\sqrt{-7} + 120)x_5x_{10} + \\ (7\sqrt{-7} + 5)x_6^2 - 80x_6x_7 + (14\sqrt{-7} + 170)x_6x_8 + (16\sqrt{-7} + 128)x_6x_9 - 80x_6x_{10} - \\ (2\sqrt{-7} + 62)x_7^2 + 80x_7x_8 - (20\sqrt{-7} - 52)x_7x_9 + (8\sqrt{-7} - 120)x_7x_{10} - (5\sqrt{-7} + 23)x_8^2 - \\ 80x_8x_9 - (16\sqrt{-7} - 32)x_8x_{10} + (18\sqrt{-7} - 2)x_9^2 + 80x_9x_{10} - (9\sqrt{-7} + 67)x_{10}^2.\end{gathered}$$

For the second, one has to spin

$$-\tfrac{1}{2}(7\sqrt{-7}+7)x_2x_4 - \tfrac{1}{2}(5\sqrt{-7}-7)x_2x_6 + \tfrac{1}{2}(\sqrt{-7}+21)x_2x_8 - \tfrac{1}{2}(5\sqrt{-7}-7)x_2x_{10} +$$
$$\tfrac{1}{2}(3\sqrt{-7}-21)x_3x_4 - \tfrac{1}{2}(3\sqrt{-7}+7)x_3x_6 - \tfrac{1}{2}(5\sqrt{-7}-7)x_3x_8 - \tfrac{1}{2}(\sqrt{-7}+21)x_3x_{10} +$$
$$4x_4^2 + 4x_4x_6 - 4x_4x_8 + 4x_4x_{10} + 8x_5^2 - 8x_5x_6 + 8x_5x_8 - 8x_5x_{10} - 8x_6x_7 + 8x_6x_9 +$$
$$8x_6x_{10} - 8x_7^2 + 8x_7x_8 + 16x_7x_9 - 8x_7x_{10} - 8x_8x_9 - 8x_9^2 + 8x_9x_{10}.$$

The constructed invariant subspaces indeed define the algebraic model $(\mathbf{R}_{10.9})_a$.

Remark A.10.2. The group $\mathrm{Aut}(\mathbf{R}_{10.9})$ is isomorphic to $\mathrm{PGL}(2,7)$, and $\mathrm{Aut}^+(\mathbf{R}_{10.9})$ therefore to its unique index 2 subgroup $\mathrm{PSL}(2,7)$.

$\mathbf{R}_{10.16}$ **type** $(6,12)$ **#cells** $(6,36,12)$ **map group size** 144 $D_2(\mathbf{Fer}(6))$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^6, S^{12}, (RS)^2, [R, S^2] \rangle$

This map is $D_1(\mathbf{R}_{10.2}) = D(\mathbf{Fer}(6))$ and thus has the same algebraic models as $\mathbf{Fer}(6)$.

A.11 Genus 11 (examples)

Prior remarks. The Hilbert-Poincaré series predicts (cf. Section 6.1) that the canonical model of a non-hyperelliptic platonic map of genus 11 satisfies $\dim(I_2) = 36$. If the map is trigonal, some additional cubics will be needed to define the model. In any case $\dim(I_3) = 236$.

$\mathbf{R}_{11.1}$ **type** $(4, 6)$ **#cells** $(40, 120, 60)$ **map group size** 480
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^4, S^6, (RS)^2, (R^{-1}S)^3(SR^{-1})^3 \rangle$

The canonical representation ρ_c is generated by

$$R \mapsto \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\alpha-2 & -1 & -\alpha-2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$S \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & -1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \alpha+2 & \alpha+1 & 2\alpha+2 \\ 0 & 0 & 0 & -1 & -\alpha-1 & -\alpha-1 \\ 2\alpha+2 & \alpha+1 & \alpha+2 & 0 & 0 & 0 \\ -\alpha-1 & -\alpha-1 & -1 & 0 & 0 & 0 \\ -\alpha-2 & -1 & -\alpha-2 & 0 & 0 & 0 \end{pmatrix}$$

where $\alpha = \zeta_5 + \zeta_5^{-1}$. The representation ρ_c^{2+} decomposes as

$$2 \times 1_1 \oplus 1_2 \oplus 2 \times 4_2 \oplus 4_3 \oplus 4_4 \oplus 3 \times 5_1 \oplus 5_2 \oplus 2 \times 5_3 \oplus 5_4 \oplus 2 \times 6_1.$$

Doing several truncated Gröbner basis computations, we find that $x_{11}^5 \in (3 \times 5_1)$, $x_2x_3x_5 \in (2 \times 5_3)$, $x_4x_5x_6(x_4 - x_5) \in (2 \times 6_1)$, $x_5^2x_7x_8x_9x_{11} \in (2 \times 1_1, 1_2, 4_3, 4_4)$,

$$\begin{aligned} &(x_4 - \zeta_6x_5)(x_4 + (\zeta_6 - 1)x_5)(x_3 + x_5)(x_3 + x_4 - x_5) \in (2 \times 4_2), \\ &x_5x_6x_7x_{10} + (-\alpha - 2)x_5x_7^2x_{10} + x_5x_7x_8x_{10} \in (5_2, 5_4), \\ &x_4^3x_5x_{10} - 3x_4x_5^3x_{10} - 2x_5^4x_{10} \in (4_3, 4_4, 5_2), \text{ and} \\ &x_4^3x_5x_{10} - 3x_4x_5^3x_{10} - 2x_5^4x_{10} \in (4_3, 4_4, 5_4). \end{aligned}$$

This means we cannot use the full isotypic components $2 \times 4_2$, $3 \times 5_1$, $2 \times 5_3$ or $2 \times 6_1$. We may use at most one of $\{5_2, 5_4\}$, and if we do, then at most one of $\{4_3, 4_4\}$. But if we leave out both 5_2 and 5_4, then we are obliged to utilize $2 \times 1_1 \oplus 1_2 \oplus 4_3 \oplus 4_4$, which is not an option. So we are forced to leave out one of the two, and the same for the set $\{4_3, 4_4\}$. We are thus certain that $5_3 < I_2$.

We normalize this subspace using the centralizer $C_{\mathrm{GL}(11,\mathbb{C})}(\mathrm{Aut}^+(\mathbf{R}_{11.1}))$. This reduces the number of possible invariant 5_3 subspaces to three. We combine each possibility with the four combinations choosing one each from $\{4_3, 4_4\}$ and $\{5_2, 5_4\}$.

For each case, we compute the possible vertices of the map. Only for one of the three invariant 5_3 subspaces do we get a suitable candidate, and then only when including 4_3 (depending on the numbering of the representations). The vertices are defined over $\mathbb{Q}(\zeta_{30})$, and form the orbit of

$$(15 : 15\zeta_6 : -15 : -5\zeta_6 + 10 : 0 : 6\zeta_{30}^7 - 12\zeta_{30}^6 - 7\zeta_{30}^5 - \zeta_{30}^4 + 4\zeta_{30}^3 + 8\zeta_{30}^2 + 13\zeta_{30} + 1 : \zeta_{30}^7 - 4\zeta_{30}^6 - \zeta_{30}^5 - 3\zeta_{30}^4 + 3\zeta_{30}^3 + \zeta_{30}^2 - \zeta_{30} - 1 : 9\zeta_{30}^7 + 12\zeta_{30}^6 + 2\zeta_{30}^5 - 4\zeta_{30}^4 + \zeta_{30}^3 - 13\zeta_{30}^2 - 8\zeta_{30} + 4 : -3\zeta_{30}^7 - 6\zeta_{30}^6 - 3\zeta_{30}^5 - 9\zeta_{30}^4 - 3\zeta_{30}^3 + 9\zeta_{30}^2 - 3\zeta_{30} : 3\zeta_{30}^7 - 6\zeta_{30}^6 - 6\zeta_{30}^5 - 3\zeta_{30}^4 - 3\zeta_{30}^3 + 9\zeta_{30}^2 + 9\zeta_{30} + 3 : 16\zeta_{30}^7 + 8\zeta_{30}^6 - 7\zeta_{30}^5 - 6\zeta_{30}^4 - 6\zeta_{30}^3 - 2\zeta_{30}^2 + 8\zeta_{30} + 11).$$

We still have the choice of an invariant subspace from $\{5_2, 5_4\}$, and either one yields a canonical model. We choose the first one. The vertices yield enough points to determine I_2. It turns out to have the shape

$$2 \times 1_1 \oplus 4_2 \oplus 4_3 \oplus 2 \times 5_1 \oplus 5_2 \oplus 5_3 \oplus 6_1.$$

The field of definition of the canonical ideal is still $\mathbb{Q}(\alpha)$, in spite of the use of $\mathbb{Q}(\zeta_{30})$. We find the following two polynomials spanning the $2 \times 1_1$:

$$3x_1^2 - 2x_1x_2 + 2x_1x_3 - 2x_1x_4 + 2x_1x_5 + 3x_2^2 + 2x_2x_3 + 2x_2x_4 + 2x_2x_5 + 3x_3^2 + 2x_3x_4 + 2x_3x_5 + 3x_4^2 - 2x_4x_5 + 3x_5^2, \text{ and}$$
$$3x_6^2 - 2x_6x_7 + (-4\alpha - 2)x_6x_8 + 3x_7^2 - 2x_7x_8 + 3x_8^2 + 3x_9^2 - 2x_9x_{10} + (-4\alpha - 2)x_9x_{11} + 3x_{10}^2 - 2x_{10}x_{11} + 3x_{11}^2$$

A complementary 34-dimensional subspace of I_2 can be obtained by spinning the following polynomial with $\mathrm{Aut}^+(\mathbf{R})$:

$$13x_1^2 + 2x_1x_2 + 2x_1x_5 + 30x_1x_6 - 5\alpha x_1x_8 + (20\alpha + 10)x_1x_9 + (-50\alpha - 60)x_1x_{10} + (25\alpha + 15)x_1x_{11} - 20x_2^2 + 9x_2x_3 - 6x_2x_4 - 14x_2x_5 + (7\alpha + 16)x_2x_7 + (-21\alpha - 23)x_2x_8 + (18\alpha + 29)x_2x_9 + (-34\alpha - 42)x_2x_{10} + (8\alpha + 9)x_2x_{11} + 12x_3^2 + 7x_3x_5 + (-4\alpha + 13)x_3x_6 + (25\alpha + 65)x_3x_7 + (-20\alpha - 40)x_3x_8 + (30\alpha + 55)x_3x_9 + (-55\alpha - 80)x_3x_{10} + (-11\alpha - 3)x_3x_{11} + 11x_4^2 - 18x_4x_5 + (14\alpha - 3)x_4x_6 + (-11\alpha + 37)x_4x_7 + (-4\alpha - 42)x_4x_8 + (16\alpha + 38)x_4x_9 + (-6\alpha - 8)x_4x_{10} + (-41\alpha - 23)x_4x_{11} - 12x_5^2 + (11\alpha + 43)x_5x_6 + (9\alpha - 3)x_5x_7 + (-29\alpha - 27)x_5x_8 + (31\alpha + 13)x_5x_9 + (-36\alpha - 28)x_5x_{10} + (22\alpha + 1)x_5x_{11} + (-2\alpha - 1)x_6^2 + (-4\alpha - 6)x_6x_7 + (6\alpha + 10)x_6x_8 + (\alpha + 2)x_6x_9 + (2\alpha + 3)x_6x_{10} + (-16\alpha - 49)x_6x_{11} + (118\alpha + 271)x_7^2 + (-296\alpha - 578)x_7x_8 + (-124\alpha - 184)x_7x_9 + (-334\alpha - 409)x_7x_{10} + (336\alpha + 559)x_7x_{11} + (158\alpha + 223)x_8^2 + (94\alpha + 101)x_8x_9 + (96\alpha + 141)x_8x_{10} + (-110\alpha - 223)x_8x_{11} + (106\alpha + 253)x_9^2 + (96\alpha + 86)x_9x_{10} + (-404\alpha - 510)x_9x_{11} + (162\alpha + 229)x_{10}^2 + (-330\alpha - 474)x_{10}x_{11} + (250\alpha + 443)x_{11}^2.$$

Remark A.11.1. One wonders whether there is a natural Sym_5-action from which this algebraic curve can be defined, like in the case of Wiman's first sextic map $\mathbf{R}_{6.2}$.

$\mathbf{R}_{11.5}$ **type** $(6,6)$ **#cells** $(20,60,20)$ **map group size** 240
SMP $\mathrm{Aut}^+(\mathbf{R}) = \left\langle R, S \,\middle|\, R^6, S^6, (RS)^2, RS^3RS^{-1}R^{-3}S^{-1} \right\rangle$

This map is $D_1(\mathbf{R}_{11.1})$ and therefore has the same canonical model as $\mathbf{R}_{11.1}$.

A.12 Genus 13 (examples)

Prior remarks. The Hilbert-Poincaré series predicts (cf. Section 6.1) that the canonical model of a non-hyperelliptic platonic map of genus 13 satisfies $\dim(I_2) = 55$. If the map is trigonal, some additional cubics will be needed to define the model. In any case $\dim(I_3) = 395$.

$\mathbf{R}_{13.1}$ type $(3, 10)$ **#cells** $(36, 180, 120)$ **map group size** 720 **Mod**(10)
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^3, S^{10}, (RS)^2, (RS^{-2}RS^{-3})^2 \rangle$

This is the modular map **Mod**(10). More information on the modular map family can be found in Section 5.7. The canonical representation ρ_c can be generated by

$$R \mapsto \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & -1 & -1 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} \zeta_3^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\zeta_3^2 & -\zeta_3^2 & -\zeta_3 & -\zeta_3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_3^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\zeta_3 & -\zeta_3 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & \zeta_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \zeta_3 & 0 \\ 0 & 0 & 0 & 0 & \zeta_3^2 & 0 & 0 & 0 \end{pmatrix}$$

$$S \mapsto \begin{pmatrix} 1 & 1 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \zeta_3^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta_3^2 & 0 & 0 & 0 & 0 & 0 \\ -\zeta_3 & -\zeta_3 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & \zeta_3 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The representation $(\rho_c^\vee)^{2+}$ decomposes as

$$2 \times 1_1 \oplus 2_1 \oplus 3_1 \oplus 3_2 \oplus 2 \times 4_1 \oplus 3 \times 5_1 \oplus 6_1 \oplus 6_2 \oplus 2 \times 8_1 \oplus 3 \times 10_1.$$

The certificates $x_2x_9^3 \in (2 \times 8_1)$, $x_4^3 \in (3 \times 5_1)$, $x_8(x_6 + \zeta_6^{-1}x_9) \in (3 \times 10_1)$,

$$x_5(x_2x_3 + x_2x_4 - x_2x_5 + x_3^2 - x_3x_5 + x_4^2) \in (2 \times 4_1), \text{ and}$$
$$x_4^2(x_7x_{12} - x_7x_{13} + \zeta_3x_9x_{13}) \in (6_1 \oplus 6_2)$$

show that I_2 must, for some $i, j \in \{1, 2\}$, have either one of the shapes

$$1_1 \oplus 3_1 \oplus 3_2 \oplus 4_1 \oplus 2 \times 5_1 \oplus 6_j \oplus 8_1 \oplus 2 \times 10_1, \text{ or}$$
$$2_1 \oplus 3_i \oplus 4_1 \oplus 2 \times 5_1 \oplus 6_j \oplus 8_1 \oplus 2 \times 10_1.$$

Since an invariant $8_1 < 2 \times 8_1$ must certainly be contained in the ideal, we compute the general form of such a space. It can be obtained by spinning the following polynomial with $\mathrm{Aut}^+(\mathbf{R}_{13.1})$:

$$\lambda_1\Big(5x_1x_{13} - 4\zeta_6x_2x_{10} + 4\zeta_6x_2x_{11} + 3x_2x_{13} + 2\zeta_6x_3x_{10} + 6x_3x_{12} - x_3x_{13} - 2\zeta_6x_4x_{11} - 6x_4x_{12} + 3x_4x_{13} - 4\zeta_6x_5x_{10} + 2\zeta_6x_5x_{11} + 2x_5x_{12} + x_5x_{13}\Big) + \lambda_2\Big(- 6\zeta_6x_6^2 + 6\zeta_6x_6x_7 - 4x_6x_9 - 3\zeta_6x_7^2 + 2x_7x_8 - 2x_7x_9 + 6\zeta_6^{-1}x_8^2 + 4\zeta_6^2x_8x_9\Big).$$

We use the centralizer $C_{\mathrm{GL}(13,\mathbb{C})}(\mathrm{Aut}^+(\mathbf{R}_{13.1}))$ to set either $\lambda_1 = 0$, $\lambda_2 = 0$. or $\lambda_1 = \lambda_2$ without loss of generality. Next, we apply the fixed point strategy and compute the vertices of the map. The polynomials of 8_1 impose constraints, leaving only two possible orbits of points. These orbits then impose enough constraints on $2 \times 4_1$, $3 \times 5_1$ and $3 \times 10_1$ to determine these pieces uniquely. For one orbit, one can obtain them by spinning the following respective polynomials:

$$2x_1x_5 + 2x_2x_3 - 2x_2x_4 + 2x_3x_5 + \zeta_6^{-1}x_6x_{13} + x_7x_{10} + x_7x_{11} + \zeta_6^2x_7x_{12} + \zeta_6x_8x_{13} + \zeta_6x_9x_{13},$$

$$8x_3^2 - 16x_3x_4 + 8x_4^2 - 8x_5^2 + (-6\zeta_{30}^7+6\zeta_{30}^3+6\zeta_{30}^2-3)x_6x_{10}+(-4\zeta_{30}^5-3\zeta_{30}^4+3\zeta_{30}+4)x_6x_{13}+ (3\zeta_{30}^7-3\zeta_{30}^3-3\zeta_{30}^2-1)x_7x_{10} + (3\zeta_{30}^7-3\zeta_{30}^3-3\zeta_{30}^2 + 4)x_7x_{11} + (3\zeta_{30}^5 + 6\zeta_{30}^4-6\zeta_{30}-3)x_7x_{13}+ (3\zeta_{30}^5 + 6\zeta_{30}^4 - 6\zeta_{30} - 3)x_8x_{11} + (-3\zeta_{30}^7 - \zeta_{30}^5 + 3\zeta_{30}^4 + 3\zeta_{30}^3 + 3\zeta_{30}^2 - 3\zeta_{30} - 3)x_8x_{13},$$

$$x_3x_{11} + 2\zeta_6^{-1}x_3x_{12} + x_4x_{10} + 2\zeta_6^{-1}x_4x_{12} + \zeta_6^2x_4x_{13} - x_5x_{11} + 2\zeta_6^2x_5x_{12} + \zeta_6^{-1}x_5x_{13}+ \tfrac{1}{4}(-6\zeta_{30}^7+6\zeta_{30}^3+6\zeta_{30}^2-3)x_6x_7+\tfrac{1}{4}(-\zeta_{30}^5+3\zeta_{30}^4-3\zeta_{30}+1)x_6x_8+\tfrac{1}{4}(-2\zeta_{30}^5-9\zeta_{30}^4+9\zeta_{30}+2)x_7x_8.$$

Now we compute that $(3_1 \oplus 3_2 \oplus 4_1 \oplus 2 \times 5_1 \oplus 8_1 \oplus 2 \times 10_1)$ is constant, forcing us to use $2 \times 1_1$ and 2_1, i.e. the ideal is of the second shape displayed above. These components are given by:

$$\begin{aligned} 2 \times 1_1 = \langle & 5x_1^2 - 2x_1x_2 + 2x_1x_3 + 2x_1x_4 + 2x_1x_5 + 5x_2^2 + 2x_2x_3 + 2x_2x_4 + 2x_2x_5+ \\ & 5x_3^2 - 2x_3x_4 - 2x_3x_5 + 5x_4^2 - 2x_4x_5 + 5x_5^2, \\ & x_6x_{10} - 4x_6x_{11} + \zeta_6^2x_6x_{12} + \zeta_6^2x_6x_{13} + x_7x_{10} + x_7x_{11} + \zeta_6^2x_7x_{12}+ \\ & \zeta_6^2x_7x_{13} + 4\zeta_6^{-1}x_8x_{10} + \zeta_6^2x_8x_{11} - \zeta_6x_8x_{12} - \zeta_6x_8x_{13} + \zeta_6^2x_9x_{10}+ \\ & \zeta_6^2x_9x_{11} + 4\zeta_6x_9x_{12} - \zeta_6x_9x_{13}\rangle \\ 2_1 = \langle & 2x_6^2 - x_6x_7 + \zeta_6^{-1}x_6x_8 + \zeta_6^{-1}x_6x_9 + 2x_7^2 + \zeta_6^{-1}x_7x_8 + \zeta_6^{-1}x_7x_9 - 2\zeta_6x_8^2+ \\ & \zeta_6x_8x_9 - 2\zeta_6x_9^2, \\ & 2x_{10}^2 - x_{10}x_{11} + \zeta_6^{-1}x_{10}x_{12} + \zeta_6^{-1}x_{10}x_{13} + 2x_{11}^2 + \zeta_6^{-1}x_{11}x_{12} + \zeta_6^{-1}x_{11}x_{13}- \\ & 2\zeta_6x_{12}^2 + \zeta_6x_{12}x_{13} - 2\zeta_6x_{13}^2\rangle. \end{aligned}$$

Depending on the fixed point orbit, there is now one suitable ideal choice. We used 3_1 and 6_2, which are spanned by the polynomials gotten when spinning the following two respective ones with $\mathrm{Aut}^+(\mathbf{R}_{13.1})$:

$$2x_6x_{10} - x_7x_{10} + 2x_7x_{11} + (\zeta_{30}^4 - \zeta_{30})x_7x_{12} + (-\zeta_{30}^5 - \zeta_{30}^4 + \zeta_{30} + 1)x_7x_{13} + 2\zeta_6^{-1}x_8x_{11}+$$
$$(\zeta_{30}^7 - \zeta_{30}^5 - \zeta_{30}^4 - \zeta_{30}^3 - \zeta_{30}^2 + \zeta_{30} + 1)x_8x_{12} + (-\zeta_{30}^7 + \zeta_{30}^4 + \zeta_{30}^3 + \zeta_{30}^2 - \zeta_{30} - 1)x_8x_{13}+$$
$$(-\zeta_{30}^5 - \zeta_{30}^4 + \zeta_{30} + 1)x_9x_{10} + (2\zeta_{30}^7 - \zeta_{30}^5 - 2\zeta_{30}^4 - 2\zeta_{30}^3 - 2\zeta_{30}^2 + 2\zeta_{30} + 2)x_9x_{13},$$

$$x_2x_{10} - x_2x_{11} + (-\zeta_{30}^5 - \zeta_{30}^4 + \zeta_{30} + 1)x_2x_{13} + (\zeta_{30}^7 - \zeta_{30}^3 - \zeta_{30}^2)x_3x_{10} + \zeta_6^{-1}x_3x_{12}+$$
$$\zeta_6^2x_3x_{13} + (-\zeta_{30}^7 + \zeta_{30}^3 + \zeta_{30}^2)x_4x_{11} + \zeta_6^2x_4x_{12} + (-\zeta_{30}^5 - \zeta_{30}^4 + \zeta_{30} + 1)x_4x_{13} + x_5x_{10}+$$
$$(\zeta_{30}^7 - \zeta_{30}^3 - \zeta_{30}^2)x_5x_{11} + (\zeta_{30}^5 + \zeta_{30}^4 - \zeta_{30} - 1)x_5x_{12} + \zeta_6^{-1}x_5x_{13}.$$

This finishes the construction of the canonical ideal. The choice between isotypic components 3_i and 6_j is explained by the fact that the two pieces of the same dimension are related by the field automorphism $\zeta_5 \mapsto \zeta_5^2$ of $\mathbb{Q}(\zeta_5)$. We know therefore, that they must both yield solutions, since ρ_c is defined over $\mathbb{Q}(\zeta_3)$.

A.13 Genus 14 (examples)

Prior remarks. The Hilbert-Poincaré series predicts (cf. Section 6.1) that the canonical model of a non-hyperelliptic platonic map of genus 14 satisfies $\dim(I_2) = 66$. If the map is trigonal, some additional cubics will be needed to define the model. In any case $\dim(I_3) = 495$.

$\mathbf{R}_{14.1}$, $\mathbf{R}_{14.2}$, $\mathbf{R}_{14.3}$ type $(3,7)$ **#cells** $(156, 546, 364)$ **map group size** 2184 **First Hurwitz triplet**
SMP $\mathbf{R}_{14.1}$: $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^3, S^7, (RS)^2, [R,S]^6 \rangle$
SMP $\mathbf{R}_{14.2}$: $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^3, S^7, (RS)^2, [R,S]^{13}, (S^3R^{-1}S^2R^{-1})^3 \rangle$
SMP $\mathbf{R}_{14.3}$: $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^3, S^7, (RS)^2, [R,S]^7 \rangle$

The first Hurwitz triplet is discussed at length in Chapter 7. For ease of reference we repeat the following. All three groups have $\mathrm{Aut}^+(\mathbf{R}) \cong \mathrm{PSL}(2,13)$. The full map automorphism groups are $\mathrm{PGL}(2,13)$ for $\mathbf{R}_{14.1}$ and $\mathbf{R}_{14.3}$, but $\mathrm{PSL}(2,13) \times \mathbb{Z}_2$ for $\mathbf{R}_{14.2}$. The canonical character is the same irreducible one for all three triplet members. Hence, the canonical representations for all three members can be chosen to have the same matrix group as the image, though the representations will differ in the assignment $(R,S) \mapsto (\rho_c(R), \rho_c(S))$. A canonical representation ρ_c for $\mathbf{R}_{6.1}$ can be generated by sending the pair (R,S) to:

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 \\
-1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 1 & 1 & -1 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & -1 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\quad
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 1 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & 0 & 0 & 1 & 1 & -1 & 0 & -1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & -1 & -1 \\
1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\
0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 1 \\
0 & 0 & -2 & 0 & -1 & -1 & -1 & 1 & 0 & 1 & 0 & -1 & 0 & -1 \\
-1 & 0 & 1 & -1 & 0 & 1 & 1 & -1 & 1 & -1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0
\end{pmatrix}
$$

After applying our construction strategy, Galois descent and coefficient simplification as described in the aforementioned chapter, we find that a generating set for the canonical ideal of any triplet member is produced by spinning the following polynomial with the matrix group $\langle \rho_c(R), \rho_c(S) \rangle$:

$$
\begin{aligned}
&(-3\alpha - 3)x_1^2 + (-6\alpha^2 - 5\alpha + 5)x_1x_2 + (-2\alpha^2 + 4\alpha - 1)x_2^2 + (3\alpha^2 + 9\alpha - 11)x_1x_3 + (7\alpha^2 - \alpha - 10)x_2x_3 + \\
&(-2\alpha + 1)x_3^2 + (5\alpha^2 - 2)x_1x_4 + (5\alpha^2 - 3\alpha - 10)x_2x_4 + (-2\alpha^2 - 2\alpha + 1)x_3x_4 + (3\alpha + 3)x_4^2 + (\alpha^2 - 2)x_1x_5 + \\
&(-\alpha^2 + \alpha + 1)x_2x_5 + (-2\alpha^2 + 3\alpha + 5)x_3x_5 + (-\alpha^2 + \alpha + 3)x_4x_5 + (-\alpha - 2)x_5^2 + (4\alpha^2 - \alpha - 2)x_1x_6 + \\
&(\alpha^2 - 9\alpha - 1)x_2x_6 + (-5\alpha^2 - 3\alpha)x_3x_6 + (2\alpha^2 - 1)x_4x_6 + (2\alpha^2 + 4\alpha - 1)x_5x_6 + (\alpha^2 + \alpha - 3)x_6^2 + \\
&(-8\alpha^2 - 15\alpha + 12)x_1x_7 + (-8\alpha^2 - 6\alpha + 11)x_2x_7 + (\alpha^2 + 11\alpha + 2)x_3x_7 + (5\alpha^2 - 2\alpha - 3)x_4x_7 + (-2\alpha^2 - \alpha)x_5x_7 + \\
&(-3\alpha + 8)x_6x_7 + (-5\alpha^2 - 7\alpha + 4)x_7^2 + (-\alpha^2 + \alpha + 5)x_1x_8 + (10\alpha^2 - 5\alpha - 15)x_2x_8 + (-5\alpha^2 - 3\alpha + 7)x_3x_8 + \\
&(-6\alpha^2 + 2\alpha + 9)x_4x_8 + (-4\alpha^2 + \alpha + 9)x_5x_8 + (-10\alpha^2 - 4\alpha + 12)x_6x_8 + (4\alpha^2 + 9\alpha - 3)x_7x_8 + (-5\alpha^2 + 6)x_8^2 + \\
&(8\alpha^2 + 10\alpha - 23)x_1x_9 + (4\alpha^2 - 2\alpha - 1)x_2x_9 + (4\alpha^2 + 4\alpha - 9)x_3x_9 + (\alpha^2 + 5\alpha - 3)x_4x_9 + (\alpha^2 - \alpha - 4)x_5x_9 + \\
&(4\alpha^2 + 4\alpha - 7)x_6x_9 + (2\alpha^2 + 2\alpha - 3)x_7x_9 + (2\alpha^2 + 5\alpha - 1)x_8x_9 + (\alpha^2 + 3\alpha - 7)x_9^2 + (7\alpha^2 + 7\alpha - 20)x_1x_{10} + \\
&(-\alpha^2 - 2\alpha + 5)x_2x_{10} + (5\alpha^2 + 3\alpha - 8)x_3x_{10} + (2\alpha^2 - 7\alpha - 4)x_4x_{10} + (2\alpha^2 - 5)x_5x_{10} + (4\alpha^2 + 3\alpha - 9)x_6x_{10} + \\
&(-\alpha^2 + 2\alpha + 10)x_7x_{10} + (3\alpha^2 + \alpha - 1)x_8x_{10} + (5\alpha^2 + 3\alpha - 21)x_9x_{10} + (3\alpha^2 + 7\alpha - 11)x_{10}^2 + (\alpha^2 - 10\alpha)x_1x_{11} + \\
&(3\alpha^2 + 6\alpha - 12)x_2x_{11} + (-\alpha^2 + 7\alpha + 4)x_3x_{11} + (-\alpha^2 - 9\alpha - 1)x_4x_{11} + (-2\alpha^2 - 6\alpha + 3)x_5x_{11} + \\
&(2\alpha^2 - 9\alpha + 1)x_6x_{11} + (-2\alpha^2 - 7\alpha + 2)x_7x_{11} + (-\alpha^2 + 4\alpha)x_8x_{11} + (2\alpha^2 + 2\alpha - 6)x_9x_{11} + (-\alpha^2 + 2\alpha + 11)x_{10}x_{11} + \\
&(-3\alpha^2 - 5\alpha + 5)x_{11}^2 + (-6\alpha^2 + 3\alpha + 11)x_1x_{12} + (-4\alpha^2 + 2\alpha + 12)x_2x_{12} + (-3\alpha^2 - \alpha + 5)x_3x_{12} + (3\alpha - 6)x_4x_{12} + \\
&(\alpha^2 + 2\alpha - 4)x_5x_{12} + (2\alpha^2 + 3\alpha)x_6x_{12} + (-2\alpha^2 - 2\alpha + 5)x_7x_{12} + (5\alpha^2 + 3\alpha - 11)x_8x_{12} + (-6\alpha^2 - 4\alpha + 8)x_9x_{12} + \\
&(-2\alpha^2 - 8\alpha + 7)x_{10}x_{12} + (6\alpha^2 + 2\alpha - 12)x_{11}x_{12} + (2\alpha^2 - 2\alpha - 3)x_{12}^2 + (3\alpha^2 + \alpha - 9)x_1x_{13} + (5\alpha^2 + 2\alpha - 13)x_2x_{13} + \\
&(\alpha^2 - 3)x_3x_{13} + (\alpha^2 + 2)x_4x_{13} + (-2\alpha^2 + 6)x_5x_{13} + (-2\alpha^2 - 3\alpha)x_6x_{13} + (-\alpha^2 + 3\alpha + 3)x_7x_{13} + \\
&(-7\alpha^2 - 3\alpha + 12)x_8x_{13} + (8\alpha^2 + 5\alpha - 15)x_9x_{13} + (6\alpha^2 + \alpha - 8)x_{10}x_{13} + (-2\alpha^2 - 2\alpha + 9)x_{11}x_{13} + \\
&(-\alpha^2 + 2\alpha)x_{12}x_{13} + (-\alpha^2 - 7\alpha + 2)x_1x_{14} + (2\alpha^2 + 10\alpha - 13)x_2x_{14} + (3\alpha^2 + 3\alpha - 3)x_3x_{14} + \\
&(-6\alpha^2 - 8\alpha + 15)x_4x_{14} + (2\alpha^2 - 8\alpha - 1)x_5x_{14} + (-6\alpha^2 - 8\alpha + 11)x_6x_{14} + (-2\alpha - 2)x_7x_{14} + (-6\alpha + 7)x_8x_{14} + \\
&(5\alpha^2 + \alpha - 6)x_9x_{14} + (11\alpha - 14)x_{10}x_{14} + (-6\alpha^2 - 2\alpha + 3)x_{11}x_{14} + (\alpha^2 - 4)x_{12}x_{14} + (-6\alpha^2 - 6\alpha + 12)x_{13}x_{14} + \\
&(5\alpha^2 + 10\alpha - 10)x_{14}^2.
\end{aligned}
$$

Here α is a root of the polynomial $X^3 + X^2 - 2X - 1$. That the one set of polynomials can define all three members stems from the fact that there are three choices for $\alpha \in \mathbb{C}$, namely $\zeta_7 + \zeta_7^{-1} = 2\cos(2\pi/7)$, $\zeta_7^2 + \zeta_7^{-2} = 2\cos(4\pi/7)$, and $\zeta_7^3 + \zeta_7^{-3} = 2\cos(6\pi/7)$. Each results in a complex projective algebraic curve that is the platonic surface of one of the three triplet members. As elaborated on in Chapter 7, this correspondence is as follows:

Choice for α	$\zeta_7 + \zeta_7^{-1}$	$\zeta_7^2 + \zeta_7^{-2}$	$\zeta_7^3 + \zeta_7^{-3}$
Platonic map	$\mathbf{R}_{14.1}$	$\mathbf{R}_{14.3}$	$\mathbf{R}_{14.2}$

A.14 Genus 15 (examples)

Prior remarks. The Hilbert-Poincaré series predicts (cf. Section 6.1) that the canonical model of a non-hyperelliptic platonic map of genus 15 satisfies $\dim(I_2) = 78$. If the map is trigonal, some additional cubics will be needed to define the model. In any case $\dim(I_3) = 610$.

$\mathbf{R}_{15.2}$ **type** $(3, 14)$ **#cells** $(21, 147, 98)$ **map group size** 588 **Fer**(7)
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^3, S^{14}, (RS)^2, (R^2S^2)^3 \rangle$

This is the map **Fer**(7) with planar model $x^7 + y^7 + z^7 = 0$. For more information, see Section 5.5.

$\mathbf{R}_{15.3}$ **type** $(3, 20)$ **#cells** $(12, 120, 80)$ **map group size** 480
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^3, S^{20}, (RS)^2, [R, S^5], (RS^{-2}RS^{-3})^2 \rangle$

This is a platonic 4-cover of **Ico** branched over cells_0. We suspect it has the planar model

$$y^4z^7 = x(x^{10} + 11x^5 - 1).$$

However, we have not yet succeeded in writing down a standard map presentation for it.

$\mathbf{R}_{15.9}$ **type** $(6, 10)$ **#cells** $(12, 60, 20)$ **map group size** 240
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^6, S^{10}, (RS)^2, (R^2S^{-1})^2, R^2S^4R^{-1}S^{-1} \rangle$

This map occurred in Section 6.3, being one of the exceptional hyperelliptic platonic maps. To be specific, the map has a platonic 2-cover $\pi : \mathbf{R}_{15.9} \to \mathbf{R}_{15.9}/\langle R^3, S^5 \rangle \cong$ **Ico** branched over $\mathrm{cells}_0 \cup \mathrm{cells}_2$. Referring back to Section 6.4, we write down the planar model:

$$y^2z^{29} = x^{31} - 217x^{26}z^5 - 2015x^{21}z^{10} + 5890x^{16}z^{15} + 2015x^{11}z^{20} - 217x^6z^{25} - xz^{30}.$$

However, we have not yet succeeded in writing down a standard map presentation.

$\mathbf{R}_{15.10}$ **type** $(7, 14)$ **#cells** $(7, 49, 14)$ **map group size** 196 $D_2(\mathbf{Fer}(7))$
SMP $\mathrm{Aut}^+(\mathbf{R}) = \langle R, S \mid R^7, S^{14}, (RS)^2, [R, S^2] \rangle$

This map is $D_1(\mathbf{R}_{15.2}) = D(\mathbf{Fer}(7))$ and thus has the same algebraic models as **Fer**(7).

B

Reflexive platonic maps of genus $g \leq 15$

LEGEND. Basic information for each reflexive platonic map of genus $g \leq 15$. Duals are not listed separately, but if the map is self-dual we note this with a superscript 'SD'. In the first column we also indicate, by ↓ followed by a map number, that the map has the indicated map as diagonal map (cf. Chapter 3). Similarly, ↑ signifies that the map is the diagonal map of the indicated map. The triple (v, e, f) indicating the number of vertices, edges and faces of the map is easily computed from $|\mathrm{Aut}(\mathbf{R})|$ and (p, q), but listed here for ease of reference.

If the platonic surface of a map has an existing name known to the author, or belongs to a polynomial family, this is listed in the 'name or family' column. The family $\mathcal{F}_{g(n)}^{(p(n),q(n))}$ is abbreviated in general as $g(n).(p(n), q(n))$, with an extra index in parentheses if we have defined multiple families of the same type. The exceptions are the families $\mathbf{Hos}(n)$ and those of genus 1.

In the last column, the relators R^p, S^q and $(RS)^2$ of the standard map presentation are not listed, only the extra relators. The map numbering is according to the list compiled by Conder & Dobscányi in [CD2001], but the extra relators are not per se the same as those listed there. The introduction of polynomial families (see Chapter 2) has led to many simplifications of the presentations, and some additional effort was made in the other instances to find as short a presentation as possible.

Map	Type	(v,e,f)	$\lvert\mathrm{Aut}(\mathbf{R})\rvert$	Name or family	Extra relators
$\mathbf{Hos}(n)/\mathbf{Dih}(n)$	(2,n)	(2,n,n)	$4n$	Hosohedron(n)/Dihedron(n)	
$\mathbf{Tet}^{\mathrm{SD}}$	(3,3)	(4,6,4)	24	Tetrahedron	
$\mathbf{Oct}$	(3,4)	(6,12,8)	48	Octahedron/Cube	

continued on next page…

Map	Type	(v, e, f)	$\lvert\mathrm{Aut}(\mathbf{R})\rvert$	Name or family	Extra relators
Ico	(3,5)	(12,30,20)	120	Icosahedron/Dodecahedron	
1.1:n	(3,6)	$(3n^2,9n^2,6n^2)$	$36n^2$		$[R,S^2]^n$
1.2:n	(3,6)	$(n^2,3n^2,2n^2)$	$12n^2$		$[R,S]^n$
1.3:n^{SD}	(4,4)	$(n^2,2n^2,n^2)$	$8n^2$		$(RS^{-1})^n$
1.4:n^{SD}	(4,4)	$(2n^2,4n^2,2n^2)$	$16n^2$		$[R,S]^n$
$2.1^{\downarrow 2.3}$	(3,8)	(6,24,16)	96	Bolza's map	$(RS^{-3})^2$
$2.2^{\downarrow 2.5}$	(4,6)	(4,12,6)	48	$\mathbf{AM}(2)$	$(RS^{-1})^2$
$2.3^{\uparrow 2.1,\downarrow 2.6}$	(4,8)	(2,8,4)	32	$\mathbf{Wi2}(2)$, $n.(n+2,2n+4)$	$R^{-1}SRS^{-3}$
2.4	(5,10)	(1,5,2)	20	$\mathbf{Wi1}(2)$	RS^{-4}
$2.5^{\mathrm{SD}\uparrow 2.2}$	(6,6)	(2,6,2)	24	$D_1(\mathbf{AM}(2))$	$[R,S]$
$2.6^{\mathrm{SD}\uparrow 2.3}$	(8,8)	(1,4,1)	16	$D_1(\mathbf{Wi2}(2))$	RS^{-3}
3.1	(3,7)	(24,84,56)	336	$\mathbf{Mod}(7)$, Klein quartic	$(RS^{-2})^4$
$3.2^{\downarrow 3.5}$	(3,8)	(12,48,32)	192	$\mathbf{Fer}(4)$, Dyck's map	$(S^2R^{-1})^3$
3.3	(3,12)	(4,24,16)	96	4-cover of **Tet**	$[R,S^3]$
$3.4^{\downarrow 3.8}$	(4,6)	(8,24,12)	96	$3n.(4,3n+3)$	$(RS^{-2})^2$
$3.5^{\uparrow 3.2,\downarrow 3.10}$	(4,8)	(4,16,8)	64	$\mathbf{Kul}(1), D_2(\mathbf{Fer}(4))$	$R^{-1}S^2RS^{-2}$
$3.6^{\downarrow 3.11}$	(4,8)	(4,16,8)	64	$\mathbf{AM}(3)$	$(RS^{-1})^2$
$3.7^{\downarrow 3.12}$	(4,12)	(2,12,6)	48	$\mathbf{Wi2}(3)$	$R^{-1}SRS^{-5}$
$3.8^{\mathrm{SD}\uparrow 3.4}$	(6,6)	(4,12,4)	48	$3n.(3n+3,3n+3)$	R^3S^3
3.9	(7,14)	(1,7,2)	28	$\mathbf{Wi1}(3)$	RS^{-6}
$3.10^{\mathrm{SD}\uparrow 3.5}$	(8,8)	(2,8,2)	32	$D_1(\mathbf{Kul}(1))$	$R^{-1}SRS^{-5}$
$3.11^{\mathrm{SD}\uparrow 3.6}$	(8,8)	(2,8,2)	32	$D_1(\mathbf{AM}(3))$	$[R,S]$
$3.12^{\mathrm{SD}\uparrow 3.7}$	(12,12)	(1,6,1)	24	$D_1(\mathbf{Wi2}(3))$	RS^{-5}
$4.1^{\downarrow 4.9}$	(3,12)	(6,36,24)	144		$(S^2R^{-1})^3, [R,S^4]$
$4.2^{\downarrow 4.6}$	(4,5)	(24,60,30)	240	Bring's map	$(RS^{-1}RS^{-2})^2$
$4.3^{\downarrow 4.7}$	(4,6)	(12,36,18)	144	$(n-1)^2.(4,2n)$	$[RS,SR]$
$4.4^{\downarrow 4.11}$	(4,10)	(4,20,10)	80	$\mathbf{AM}(4)$	$(RS^{-1})^2$
$4.5^{\downarrow 4.12}$	(4,16)	(2,16,8)	64	$\mathbf{Wi2}(4)$	$R^{-1}SRS^{-7}$
$4.6^{\mathrm{SD}\uparrow 4.2}$	(5,5)	(12,30,12)	120	D_1(Bring's map)	$(RS^{-1})^3$
$4.7^{\mathrm{SD}\uparrow 4.3}$	(6,6)	(6,18,6)	72	$2n.(6,2n+2),(n-1)^2.(2n,2n)$	$(RS^{-1})^2$
4.8	(6,6)	(6,18,6)	72	$3n+1.(6,3n+3)$	$[R^2,S],(RS^{-2})^2$
$4.9^{\uparrow 4.1}$	(6,12)	(2,12,4)	48	$3n+1.(6,9n+3),4n.(4n+2,8n+4)$	$R^{-1}SRS^{-7}$
4.10	(9,18)	(1,9,2)	36	$\mathbf{Wi1}(4)$	RS^{-8}
$4.11^{\mathrm{SD}\uparrow 4.4}$	(10,10)	(2,10,2)	40	$D_1(\mathbf{AM}(4))$	$[R,S]$
$4.12^{\mathrm{SD}\uparrow 4.5}$	(16,16)	(1,8,1)	32	$D_1(\mathbf{Wi2}(4))$	RS^{-7}
$5.1^{\downarrow 5.6}$	(3,8)	(24,96,64)	384	$\mathbf{Mod}(8)$	$(RS^3R^{-1}S^{-2})^2$
5.2	(3,10)	(12,60,40)	240	2-cover of **Ico**	$(RS^{-4})^2$
$5.3^{\downarrow 5.9}$	(4,5)	(32,80,40)	320	$\mathbf{Hum}_1(4)$	$(RS^{-1})^4$
$5.4^{\downarrow 5.10}$	(4,6)	(16,48,24)	192	$\mathbf{Hum}_3(4)$, $12n-7.(4,6n)$	$(R^2S^3)^2$
$5.5^{\downarrow 5.12}$	(4,8)	(8,32,16)	128	$\mathbf{Hum}_2(4)$	$[RS,SR],(RS^{-3})^2$
$5.6^{\uparrow 5.1,\downarrow 5.13}$	(4,8)	(8,32,16)	128	$D_2(\mathbf{Mod}(8)), 4n-3.(4,4n)$	$[R^2,S^2]$
$5.7^{\downarrow 5.15}$	(4,12)	(4,24,12)	96	$\mathbf{AM}(5)$	$(RS^{-1})^2$
$5.8^{\downarrow 5.16}$	(4,20)	(2,20,10)	80	$\mathbf{Wi2}(5)$	$R^{-1}SRS^{-9}$
$5.9^{\mathrm{SD}\uparrow 5.3}$	(5,5)	(16,40,16)	160	$D_1(\mathbf{Hum}_1(4))$	$[RS,SR]$
$5.10^{\mathrm{SD}\uparrow 5.4}$	(6,6)	(8,24,8)	96	$D_1(\mathbf{Hum}_3(4))$	$(RS^{-2})^2,(R^2S^{-1})^2$
5.11	(6,15)	(2,15,5)	60	$3n+2.(6,9n+6)$	$R^{-1}SRS^{-4}$
$5.12^{\mathrm{SD}\uparrow 5.5}$	(8,8)	(4,16,4)	64	$D_1(\mathbf{Hum}_2(4))$	$(RS^{-1})^2, RS^3R^{-3}S^{-1}$
$5.13^{\mathrm{SD}\uparrow 5.6}$	(8,8)	(4,16,4)	64	$4n-3.(4n,4n)$	$[R^2,S],[R,S^2],R^4S^{-4}$

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Map	Type	(v,e,f)	$\lvert\mathrm{Aut}(\mathbf{R})\rvert$	Name or family	Extra relators
5.14	(11,22)	(1,11,2)	44	$\mathbf{Wi1}(5)$	RS^{-10}
$5.15^{\mathrm{SD}\uparrow 5.7}$	(12,12)	(2,12,2)	48	$D_1(\mathbf{AM}(5))$	$[R,S]$
$5.16^{\mathrm{SD}\uparrow 5.8}$	(20,20)	(1,10,1)	40	$D_1(\mathbf{Wi2}(5))$	RS^{-9}
$6.1^{\downarrow 6.6}$	(3,10)	(15,75,50)	300	$\mathbf{Fer}(5)$	$(S^2R^{-1})^3$
6.2	(4,6)	(20,60,30)	240	Wiman's 1st sextic map	$(RS^{-1})^3$
$6.3^{\downarrow 6.9}$	(4,9)	(8,36,18)	144	$3n.(4,3n+3)$	$(RS^{-2})^2$
$6.4^{\downarrow 6.12}$	(4,14)	(4,28,14)	112	$\mathbf{AM}(6)$	$(RS^{-1})^2$
$6.5^{\downarrow 6.13}$	(4,24)	(2,24,12)	96	$\mathbf{Wi2}(6)$	$R^{-1}SRS^{-11}$
$6.6^{\uparrow 6.1}$	(5,10)	(5,25,10)	100	$D_2(\mathbf{Fer}(5))$	$[R,S^2]$
6.7	(6,8)	(6,24,8)	96	$2n.(6,2n+2)$	$(RS^{-1})^2$
6.8	(6,8)	(6,24,8)	96	2-cover of **Oct**	$(R^2S^{-1})^2, R^3S^4$
$6.9^{\mathrm{SD}\uparrow 6.3}$	(9,9)	(4,18,4)	72	$3n.(3n+3,3n+3)$	R^3S^3
6.10	(10,15)	(2,15,3)	60	$9n+6.(12n+10,18n+15)$	$R^{-1}SRS^{-11}$
6.11	(13,26)	(1,13,2)	52	$\mathbf{Wi1}(6)$	RS^{-12}
$6.12^{\mathrm{SD}\uparrow 6.4}$	(14,14)	(2,14,2)	56	$D_1(\mathbf{AM}(6))$	$[R,S]$
$6.13^{\mathrm{SD}\uparrow 6.5}$	(24,24)	(1,12,1)	48	$D_1(\mathbf{Wi2}(6))$	RS^{-11}
7.1	(3,7)	(72,252,168)	1008	Fricke-Macbeath	$(R^2S^4RS^4R^2S)^2$
$7.2^{\downarrow 7.7}$	(3,12)	(12,72,48)	288		$RS^{-2}RS^2R^{-1}S^2R^{-1}S^{-2}$
$7.3^{\downarrow 7.11}$	(4,16)	(4,32,16)	128	$\mathbf{Kul}(2)$	$R^{-1}S^2RS^{-6}$
$7.4^{\downarrow 7.10}$	(4,16)	(4,32,16)	128	$\mathbf{AM}(7)$	$(RS^{-1})^2$
$7.5^{\downarrow 7.12}$	(4,28)	(2,28,14)	112	$\mathbf{Wi2}(7)$	$R^{-1}SRS^{-13}$
7.6	(6,9)	(6,27,9)	108	$3n+1.(6,3n+3)$	$[R^2,S],(RS^{-2})^2$
$7.7^{\uparrow 7.2}$	(6,12)	(4,24,8)	96		$(R^2S^{-1})^2, S^{-2}RS^{-2}R^{-2}$
7.8	(6,21)	(2,21,7)	84	$3n+1.(6,9n+3)$	$R^{-1}SRS^{-13}$
7.9	(15,30)	(1,15,2)	60	$\mathbf{Wi1}(7)$	RS^{-14}
$7.10^{\mathrm{SD}\uparrow 7.4}$	(16,16)	(2,16,2)	64	$D_1(\mathbf{AM}(7))$	$[R,S]$
$7.11^{\mathrm{SD}\uparrow 7.3}$	(16,16)	(2,16,2)	64	$D_1(\mathbf{Kul}(2))$	$R^{-1}SRS^{-9}$
$7.12^{\mathrm{SD}\uparrow 7.5}$	(28,28)	(1,14,1)	56	$D_1(\mathbf{Wi2}(7))$	RS^{-13}
8.1	(3,8)	(42,168,112)	672	First tuplet	$(RS^{-2})^4$
8.2	(3,8)	(42,168,112)	672	First tuplet	$[RS,S^3RS^{-2}]$
$8.3^{\downarrow 8.10}$	(4,18)	(4,36,18)	144	$\mathbf{AM}(8)$	$(RS^{-1})^2$
$8.4^{\downarrow 8.11}$	(4,32)	(2,32,16)	128	$\mathbf{Wi2}(8)$	$R^{-1}SRS^{-15}$
8.5	(6,10)	(6,30,10)	120	$2n.(6,2n+2)$	$(RS^{-1})^2$
8.6	(6,24)	(2,24,8)	96	$3n+2.(6,9n+6)$	$R^{-1}SRS^{-7}$
8.7	(8,12)	(4,24,6)	96		$(RS^{-1})^2, RS^5R^{-3}S^{-1}$
8.8	(10,20)	(2,20,4)	80	$4n.(4n+2,8n+4)$	$R^{-1}SRS^{-11}$
8.9	(17,34)	(1,17,2)	68	$\mathbf{Wi1}(8)$	RS^{-16}
$8.10^{\mathrm{SD}\uparrow 8.3}$	(18,18)	(2,18,2)	72	$D_1(\mathbf{AM}(8))$	$[R,S]$
$8.11^{\mathrm{SD}\uparrow 8.4}$	(32,32)	(1,16,1)	64	$D_1(\mathbf{Wi2}(8))$	RS^{-15}
9.1	(3,12)	(16,96,64)	384		$(RS^{-5})^2,(RS^{-2})^4$
$9.2^{\downarrow 9.14}$	(4,5)	(64,160,80)	640		$R^{-2}S^2R^{-1}S^2R^2S^{-2}RS^{-2}$
$9.3^{\downarrow 9.17}$	(4,6)	(32,96,48)	384		$(RS^{-2})^2(R^{-1}S^2)^2$
$9.4^{\downarrow 9.18}$	(4,6)	(32,96,48)	384		$(RS^{-1})^4,[RS,SR^2S^2]$
$9.5^{\downarrow 9.21}$	(4,8)	(16,64,32)	256		$[RS^{-3},SR]$
$9.6^{\downarrow 9.19}$	(4,8)	(16,64,32)	256	$(n-1)^2.(4,2n)$	$(R^2S^2)^2$
$9.7^{\downarrow 9.22}$	(4,8)	(16,64,32)	256	$8n-7.(4,4n)$	$(RS^{-1})^4,(RS^{-3})^2$
$9.8^{\downarrow 9.23}$	(4,8)	(16,64,32)	256	$16n-7.(4,8n)$	$(RS^{-3})^2,(RS^{-1})^4S^4$
$9.9^{\downarrow 9.26}$	(4,12)	(8,48,24)	192	4-cover of **Oct**	$[R,S^3],(RS^{-1})^4$

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Map	Type	(v,e,f)	$\lvert\mathrm{Aut}(\mathbf{R})\rvert$	Name or family	Extra relators
$9.10^{\downarrow 9.28}$	(4,12)	(8,48,24)	192	$4n-3.(4,4n)$	$[R^2,S^2]$
$9.11^{\downarrow 9.27}$	(4,12)	(8,48,24)	192	$3n.(4,3n{+}3)$	$(RS^{-2})^2$
$9.12^{\downarrow 9.31}$	(4,20)	(4,40,20)	160	$\mathbf{AM}(9)$	$(RS^{-1})^2$
$9.13^{\downarrow 9.32}$	(4,36)	(2,36,18)	144	$\mathbf{Wi2}(9)$	$R^{-1}SRS^{-17}$
$9.14^{\mathrm{SD}\uparrow 9.2}$	(5,5)	(32,80,32)	320		$S^{-1}R^2S^2R^{-1}S^2R^2$
9.15	(5,6)	(20,60,24)	240	2-cover of $\mathbf{Ico}$	$(RS^{-2})^2$
9.16	(5,6)	(20,60,24)	240		$[RS,SR]$
$9.17^{\mathrm{SD}\uparrow 9.3}$	(6,6)	(16,48,16)	192		$[R^2,S^3],[R^3,S^2]$
$9.18^{\mathrm{SD}\uparrow 9.4}$	(6,6)	(16,48,16)	192		$(RS^{-1})^3,[RS,SR]$
$9.19^{\mathrm{SD}\uparrow 9.6}$	(8,8)	(8,32,8)	128	$(n{-}1)^2.(2n,2n)$	$(RS^{-1})^2$
9.20	(8,8)	(8,32,8)	128		$[R^2,S],(RS^{-3})^2$
$9.21^{\mathrm{SD}\uparrow 9.5}$	(8,8)	(8,32,8)	128	$32n{+}9.(16n{+}8,16n{+}8)(1)$	$RS^3R^{-3}S^{-1}$
$9.22^{\mathrm{SD}\uparrow 9.7}$	(8,8)	(8,32,8)	128	$8n-7.(4n,4n)$	$[RS,SR],R^4S^4$
$9.23^{\mathrm{SD}\uparrow 9.8}$	(8,8)	(8,32,8)	128	$16n-7.(8n,8n)$	$(R^2S^2)^2,R^4S^4$
9.24	(8,24)	(2,24,6)	96		$R^{-1}SRS^{-5}$
9.25	(8,24)	(2,24,6)	96	$9n.(6n+2,18n+6)$	$R^{-1}SRS^{-17}$
$9.26^{\mathrm{SD}\uparrow 9.9}$	(12,12)	(4,24,4)	96	4-cover of $\mathbf{Tet}$	$R^3S^{-3},(RS^{-1})^3$
$9.27^{\mathrm{SD}\uparrow 9.11}$	(12,12)	(4,24,4)	96	$3n.(3n{+}3,3n{+}3)$	R^3S^3
$9.28^{\mathrm{SD}\uparrow 9.10}$	(12,12)	(4,24,4)	96	$4n-3.(4n,4n)$	$[R^2,S],[R,S^2],R^4S^4$
9.29	(14,21)	(2,21,3)	84	$9n.(12n+2,18n+3)$	$R^{-1}SRS^{-8}$
9.30	(19,38)	(1,19,2)	76	$\mathbf{Wi1}(9)$	RS^{-18}
$9.31^{\mathrm{SD}\uparrow 9.12}$	(20,20)	(2,20,2)	80	$D_1(\mathbf{AM}(9))$	$[R,S]$
$9.32^{\mathrm{SD}\uparrow 9.13}$	(36,36)	(1,18,1)	72	$D_1(\mathbf{Wi2}(9))$	RS^{-17}
10.1	(3,9)	(36,162,108)	648	$\mathbf{Mod}(9)$	$[RS^{-2}R,S^3]$
$10.2^{\downarrow 10.16}$	(3,12)	(18,108,72)	432	$\mathbf{Fer}(6)$	$(R^2S^2)^3$
10.3	(3,15)	(12,90,60)	360	3-cover of $\mathbf{Ico}$	$[R,S^5],(RS^{-3})^3$
$10.4^{\downarrow 10.20}$	(3,18)	(9,81,54)	324		$(S^2R^{-1})^3,[R,S^6]$
$10.5^{\downarrow 10.21}$	(3,24)	(6,72,48)	288	6-cover of $\mathbf{Oct}$	$[R,S^4]$
10.6	(4,5)	(72,180,90)	720	Wiman's 2nd sextic map	$(RS^{-1})^5$
$10.7^{\downarrow 10.14}$	(4,6)	(36,108,54)	432		$(RS^{-1}RS^{-2})^2$
$10.8^{\downarrow 10.13}$	(4,6)	(36,108,54)	432		$[RS^{-1}R,S^2]$
10.9	(4,7)	(24,84,42)	336		$(RS^{-1})^3$
10.10	(4,12)	(9,54,27)	216		$(RS^{-1})^3,[R,S^4]$
$10.11^{\downarrow 10.23}$	(4,22)	(4,44,22)	176	$\mathbf{AM}(10)$	$(RS^{-1})^2$
$10.12^{\downarrow 10.24}$	(4,40)	(2,40,20)	160	$\mathbf{Wi2}(10)$	$R^{-1}SRS^{-19}$
$10.13^{\mathrm{SD}\uparrow 10.8}$	(6,6)	(18,54,18)	216		$[RS^{-1}R,S]$
$10.14^{\mathrm{SD}\uparrow 10.7}$	(6,6)	(18,54,18)	216		$RS^{-1}R^2S^{-1}RS^{-2}$
10.15	(6,6)	(18,54,18)	216		$(RS^{-2})^2,[R,S^3],(R^2S^2)^3$
$10.16^{\uparrow 10.2}$	(6,12)	(6,36,12)	144	$D_2(\mathbf{Fer}(6))$	$[R,S^2]$
10.17	(6,12)	(6,36,12)	144	$2n.(6,2n+2)$	$(RS^{-1})^2$
10.18	(6,12)	(6,36,12)	144	$3n+1.(6,3n+3)$	$[R^2,S],(RS^{-2})^2$
10.19	(6,30)	(2,30,10)	120	$3n+1.(6,9n+3)$	$R^{-1}SRS^{-19}$
$10.20^{\uparrow 10.4}$	(9,18)	(3,27,6)	108	$9n{+}1.(6n{+}3,12n{+}6)$	$[R,S^2],R^3S^{-6}$
$10.21^{\uparrow 10.5}$	(12,24)	(2,24,4)	96	$16n{+}10.(16n{+}12,32n{+}24)$	$R^{-1}SRS^{-19}$
10.22	(21,42)	(1,21,2)	84	$\mathbf{Wi1}(10)$	RS^{-20}
$10.23^{\mathrm{SD}\uparrow 10.11}$	(22,22)	(2,22,2)	88	$D_1(\mathbf{AM}(10))$	$[R,S]$
$10.24^{\mathrm{SD}\uparrow 10.12}$	(40,40)	(1,20,1)	80	$D_1(\mathbf{Wi2}(10))$	RS^{-19}
$11.1^{\downarrow 11.5}$	(4,6)	(40,120,60)	480		$(R^{-1}S)^3(SR^{-1})^3$

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Map	Type	(v,e,f)	$\lvert\mathrm{Aut}(\mathbf{R})\rvert$	Name or family	Extra relators
$11.2^{\downarrow 11.13}$	(4,24)	(4,48,24)	192	$\mathbf{Kul}(3)$	$R^{-1}S^2RS^{-10}$
$11.3^{\downarrow 11.12}$	(4,24)	(4,48,24)	192	$\mathbf{AM}(11)$	$(RS^{-1})^2$
$11.4^{\downarrow 11.14}$	(4,44)	(2,44,22)	176	$\mathbf{Wi2}(11)$	$R^{-1}SRS^{-21}$
$11.5^{\mathrm{SD}\uparrow 11.1}$	(6,6)	(20,60,20)	240		$RS^3RS^{-1}R^{-3}S^{-1}$
11.6	(6,8)	(12,48,16)	192		$(R^2S^{-1})^2, (RS^{-3})^2$
11.7	(6,8)	(12,48,16)	192		$R^2S^{-1}R^2S^3$
11.8	(6,33)	(2,33,11)	132	$3n+2.(6,9n+6)$	$R^{-1}SRS^{-10}$
11.9	(8,16)	(4,32,8)	128		$(RS^{-1})^2, RS^7R^{-3}S^{-1}$
11.10	(8,16)	(4,32,8)	128		$R^3SR^{-1}S, RS^6R^{-1}S^{-2}$
11.11	(23,46)	(1,23,2)	92	$\mathbf{Wi1}(11)$	RS^{-22}
$11.12^{\mathrm{SD}\uparrow 11.3}$	(24,24)	(2,24,2)	96	$D_1(\mathbf{AM}(11))$	$[R,S]$
$11.13^{\mathrm{SD}\uparrow 11.2}$	(24,24)	(2,24,2)	96	$D_1(\mathbf{Kul}(3))$	$R^{-1}SRS^{-13}$
$11.14^{\mathrm{SD}\uparrow 11.4}$	(44,44)	(1,22,1)	88	$D_1(\mathbf{Wi2}(11))$	RS^{-21}
$12.1^{\downarrow 12.8}$	(4,15)	(8,60,30)	240	$3n.(4,3n+3)$	$(RS^{-2})^2$
$12.2^{\downarrow 12.10}$	(4,26)	(4,52,26)	208	$\mathbf{AM}(12)$	$(RS^{-1})^2$
$12.3^{\downarrow 12.11}$	(4,48)	(2,48,24)	192	$\mathbf{Wi2}(12)$	$R^{-1}SRS^{-23}$
12.4	(6,14)	(6,42,14)	168	$2n.(6,2n+2)$	$(RS^{-1})^2$
12.5	(8,10)	(8,40,10)	160	$4n.(10,2n+2)^{\vee}$	$(RS^{-1})^2$
12.6	(10,30)	(2,30,6)	120	$9n+3.(6n+4,18n+12)$	$R^{-1}SRS^{-11}$
12.7	(14,28)	(2,28,4)	112	$4n.(4n+2,8n+4)$	$R^{-1}SRS^{-15}$
$12.8^{\mathrm{SD}\uparrow 12.1}$	(15,15)	(4,30,4)	120	$3n.(3n+3,3n+3)$	R^3S^3
12.9	(25,50)	(1,25,2)	100	$\mathbf{Wi1}(12)$	RS^{-24}
$12.10^{\mathrm{SD}\uparrow 12.2}$	(26,26)	(2,26,2)	104	$D_1(\mathbf{AM}(12))$	$[R,S]$
$12.11^{\mathrm{SD}\uparrow 12.3}$	(48,48)	(1,24,1)	96	$D_1(\mathbf{Wi2}(12))$	RS^{-23}
$13.1^{\downarrow 13.8}$	(3,10)	(36,180,120)	720	$\mathbf{Mod}(10)$	$(RS^{-2}RS^{-3})^2$
$13.2^{\downarrow 13.10}$	(3,12)	(24,144,96)	576		$[R,S^2]^2, (RS^2)^3$
$13.3^{\downarrow 13.15}$	(4,12)	(12,72,36)	288		$[RS,SR], (RS^{-5})^2$
$13.4^{\downarrow 13.19}$	(4,16)	(8,64,32)	256	$4n-3.(4,4n)$	$[R^2,S^2]$
$13.5^{\downarrow 13.18}$	(4,16)	(8,64,32)	256		$S^6R^2S^2R^2$
$13.6^{\downarrow 13.21}$	(4,28)	(4,56,28)	224	$\mathbf{AM}(13)$	$(RS^{-1})^2$
$13.7^{\downarrow 13.22}$	(4,52)	(2,52,26)	208	$\mathbf{Wi2}(13)$	$R^{-1}SRS^{-25}$
$13.8^{\uparrow 13.1}$	(5,10)	(12,60,24)	240	$D_2(\mathbf{Mod}(10))$	$(R^2S^{-2})^2$
13.9	(6,6)	(24,72,24)	288		$(RS^{-2})^2, [R^2SR^{-1},SR]$
$13.10^{\uparrow 13.2}$	(6,12)	(8,48,16)	192		$(RS^{-2})^2, [RS,SR]$
13.11	(6,12)	(8,48,16)	192		$(R^2S^{-1})^2, [R,S^3]$
13.12	(6,15)	(6,45,15)	180	$3n+1.(6,3n+3)$	$[R^2,S], (RS^{-2})^2$
13.13	(6,39)	(2,39,13)	156	$3n+1.(6,9n+3)$	$R^{-1}SRS^{-25}$
13.14	(9,18)	(4,36,8)	144		$R^3S^{-6}, [RS,SR]$
$13.15^{\mathrm{SD}\uparrow 13.3}$	(12,12)	(6,36,6)	144		$(RS^{-1})^2, RS^5R^{-5}S^{-1}$
13.16	(12,12)	(6,36,6)	144		$[R^2,S], [R,S^3], R^6S^{-6}$
13.17	(16,16)	(4,32,4)	128		$[R^2,S], R^4S^{-4}$
$13.18^{\mathrm{SD}\uparrow 13.5}$	(16,16)	(4,32,4)	128		$R^5S^{-1}RS^{-1}, RS^{-1}RS^{-5}$
$13.19^{\mathrm{SD}\uparrow 13.4}$	(16,16)	(4,32,4)	128	$4n-3.(4n,4n)$	$[R^2,S], [R,S^2], R^4S^4$
13.20	(27,54)	(1,27,2)	108	$\mathbf{Wi1}(13)$	RS^{-26}
$13.21^{\mathrm{SD}\uparrow 13.6}$	(28,28)	(2,28,2)	112	$D_1(\mathbf{AM}(13))$	$[R,S]$
$13.22^{\mathrm{SD}\uparrow 13.7}$	(52,52)	(1,26,1)	104	$D_1(\mathbf{Wi2}(13))$	RS^{-25}
14.1	(3,7)	(156,546,364)	2184	First Hurwitz triplet	$[R,S]^6$
14.2	(3,7)	(156,546,364)	2184	First Hurwitz triplet	$[R,S]^{13}, (S^3R^{-1}S^2R^{-1})^3$

continued on next page…

Map	Type	(v,e,f)	$\lvert\mathrm{Aut}(\mathbf{R})\rvert$	Name or family	Extra relators
14.3	(3,7)	(156,546,364)	2184	First Hurwitz triplet	$[R,S]^7$
$14.4^{\downarrow 13.11}$	(4,30)	(4,60,30)	240	$\mathbf{AM}(14)$	$(RS^{-1})^2$
$14.5^{\downarrow 13.12}$	(4,56)	(2,56,28)	224	$\mathbf{Wi2}(14)$	$R^{-1}SRS^{-27}$
14.6	(6,16)	(6,48,16)	192	$2n.(6,2n+2)$	$(RS^{-1})^2$
14.7	(6,42)	(2,42,14)	168	$3n+2.(6,9n+6)$	$R^{-1}SRS^{-13}$
14.8	(8,20)	(4,40,10)	160		$(RS^{-1})^2, RS^9R^{-3}S^{-1}$
14.9	(10,35)	(2,35,7)	140	$10n+4.(10,25n+10)$	$R^{-1}SRS^{-6}$
14.10	(29,58)	(1,29,2)	116	$\mathbf{Wi1}(14)$	RS^{-28}
$14.11^{\mathrm{SD}\uparrow 14.4}$	(30,30)	(2,30,2)	120	$D_1(\mathbf{AM}(14))$	$[R,S]$
$14.12^{\mathrm{SD}\uparrow 14.5}$	(56,56)	(1,28,1)	112	$D_1(\mathbf{Wi2}(14))$	RS^{-27}
15.1	(3,9)	(56,252,168)	1008		$(RS^{-2}RS^{-4})^2$
$15.2^{\downarrow 15.10}$	(3,14)	(21,147,98)	588	$\mathbf{Fer}(7)$	$(R^2S^2)^3$
15.3	(3,20)	(12,120,80)	480	4-cover of $\mathbf{Ico}$	$[R,S^5], (RS^{-2}RS^{-3})^2$
15.4	(4,6)	(56,168,84)	672		$(S^2R^{-1})^3$
$15.5^{\downarrow 15.18}$	(4,18)	(8,72,36)	288	$3n.(4,3n+3)$	$(RS^{-2})^2$
$15.6^{\downarrow 15.22}$	(4,32)	(4,64,32)	256	$\mathbf{Kul}(4)$	$R^{-1}S^2RS^{-14}$
$15.7^{\downarrow 15.21}$	(4,32)	(4,64,32)	256	$\mathbf{AM}(15)$	$(RS^{-1})^2$
$15.8^{\downarrow 15.23}$	(4,60)	(2,60,30)	240	$\mathbf{Wi2}(15)$	$R^{-1}SRS^{-29}$
15.9	(6,10)	(12,60,20)	240	2-cover of $\mathbf{Ico}$	$(R^2S^{-1})^2, R^2S^4R^{-1}S^{-1}$
$15.10^{\uparrow 15.2}$	(7,14)	(7,49,14)	196	$D_2(\mathbf{Fer}(7))$	$[R,S^2]$
15.11	(8,12)	(8,48,12)	192		$[R,S^3], R^3S^2R^{-1}S^2$
15.12	(8,12)	(8,48,12)	192		$(RS^{-2})^2, (R^2S^3)^2$
15.13	(8,12)	(8,48,12)	192		$(RS^{-1})^2$
15.14	(8,12)	(8,48,12)	192		$[R^2,S], (RS^{-3})^2$
15.15	(8,40)	(2,40,10)	160		$R^{-1}SRS^{-29}$
15.16	(8,40)	(2,40,10)	160		$R^{-1}SRS^{-9}$
15.17	(14,35)	(2,35,5)	140	$25n+15.(20n+14,50n+35)$	$R^{-1}SRS^{-29}$
$15.18^{\mathrm{SD}\uparrow 15.5}$	(18,18)	(4,36,4)	144	$3n.(3n+3,3n+3)$	R^3S^3
15.19	(22,33)	(2,33,3)	132	$9n+6.(12n+10,18n+15)$	$R^{-1}SRS^{-23}$
15.20	(31,62)	(1,31,2)	124	$\mathbf{Wi1}(15)$	RS^{-30}
$15.21^{\mathrm{SD}\uparrow 15.7}$	(32,32)	(2,32,2)	128	$D_1(\mathbf{AM}(15))$	$[R,S]$
$15.22^{\mathrm{SD}\uparrow 15.6}$	(32,32)	(2,32,2)	128	$D_1(\mathbf{Kul}(4))$	$R^{-1}SRS^{-17}$
$15.23^{\mathrm{SD}\uparrow 15.8}$	(60,60)	(1,30,1)	120	$D_1(\mathbf{Wi2}(15))$	RS^{-29}

C

Gonality bounds

LEGEND. Gonality bounds (cf. Section 1.6) of all reflexive platonic maps of genus $2 \leq g \leq 8$. If the gonality is known, we list only one number.

M	gon	M	gon	M	gon	M	gon	M	gon	M	gon	M	gon	M	gon
2.1	2	5.1	4	7.2	4	9.9	3-4	10.7	3-6	11.13	3-4	13.18	3-8	15.14	3-4
2.2	2	5.2	2	7.3	4	9.10	3-4	10.8	3-6	11.14	2	13.19	3-4	15.15	3-4
2.3	2	5.3	4	7.4	2	9.11	3-4	10.9	4-6	12.1	3-4	13.20	2	15.16	3-4
2.4	2	5.4	4	7.5	2	9.12	2	10.10	3-6	12.2	2	13.21	2	15.17	3-7
2.5	2	5.5	4	7.6	3	9.13	2	10.11	2	12.3	2	13.22	2	15.18	3-4
2.6	2	5.6	4	7.7	4	9.14	3-6	10.12	2	12.4	3	14.1	4-8	15.19	3-9
3.1	3	5.7	2	7.8	3	9.15	2	10.13	3-6	12.5	3-4	14.2	4-8	15.20	2
3.2	3	5.8	2	7.9	2	9.16	4-6	10.14	3-6	12.6	3-5	14.3	4-8	15.21	2
3.3	3	5.9	4	7.10	2	9.17	3-6	10.15	3-6	12.7	3-7	14.4	2	15.22	3-4
3.4	2	5.10	4	7.11	4	9.18	3-6	10.16	3-6	12.8	3-4	14.5	2	15.23	2
3.5	3	5.11	3	7.12	2	9.19	3-4	10.17	3	12.9	2	14.6	3		
3.6	2	5.12	4	8.1	4-5	9.20	3-4	10.18	3	12.10	2	14.7	3		
3.7	2	5.13	4	8.2	4-5	9.21	3-6	10.19	3	12.11	2	14.8	3-4		
3.8	2	5.14	2	8.3	2	9.22	3-6	10.20	3-6	13.1	4-6	14.9	3-5		
3.9	2	5.15	2	8.4	2	9.23	3-6	10.21	3-6	13.2	3-8	14.10	2		
3.10	3	5.16	2	8.5	3	9.24	3-4	10.22	2	13.3	3-6	14.11	2		
3.11	2	6.1	4	8.6	3	9.25	3-4	10.23	2	13.4	3-4	14.12	2		
3.12	2	6.2	4	8.7	4	9.26	3-4	10.24	2	13.5	3-8	15.1	3-9		
4.1	3	6.3	3	8.8	4-5	9.27	3-4	11.1	4-7	13.6	2	15.2	3-7		
4.2	3	6.4	2	8.9	2	9.28	3-4	11.2	3-4	13.7	2	15.3	3-4		
4.3	3	6.5	2	8.10	2	9.29	3-6	11.3	2	13.8	3-6	15.4	3-9		
4.4	2	6.6	4	8.11	2	9.30	2	11.4	2	13.9	3-8	15.5	3-4		
4.5	2	6.7	3	9.1	3-6	9.31	2	11.5	4-7	13.10	3-8	15.6	3-4		
4.6	3	6.8	2	9.2	3-6	9.32	2	11.6	3-4	13.11	3-8	15.7	2		
4.7	3	6.9	3	9.3	3-6	10.1	4-6	11.7	3-4	13.12	3	15.8	2		
4.8	3	6.10	4	9.4	3-6	10.2	3-6	11.8	3	13.13	3	15.9	2		
4.9	3	6.11	2	9.5	3-6	10.3	3	11.9	3-4	13.14	3-6	15.10	3-7		
4.10	2	6.12	2	9.6	3-4	10.4	3-6	11.10	3-4	13.15	3-6	15.11	3-4		
4.11	2	6.13	2	9.7	3-6	10.5	3-6	11.11	2	13.16	3-6	15.12	3-4		
4.12	2	7.1	4-5	9.8	3-6	10.6	4-6	11.12	2	13.17	3-8	15.13	3-4		

D

Geometric Weierstraß weights

LEGEND. Weierstraß weights w_i of the vertices (w_0), edge centers (w_1) and face centers (w_2) of the reflexive platonic maps of genus $2 \leq g \leq 8$. For convenience, we repeat some combinatorial data from Appendix B. A star indicates there are additional Weierstraß points. After a star we note the total Weierstraß weight that is still unaccounted for. If the map is a diagonal map (cf. Chapter 3) this weight might lie hidden in cell centers of the map with larger automorphism group. But if it is not a diagonal map, the star implies the existence of non-geometric Weierstraß points.

Map	Type	(v, e, f)	(w_0, w_1, w_2)
$2.1^{\downarrow 2.3}$	(3,8)	(6,24,16)	(1,0,0)
$2.2^{\downarrow 2.5}$	(4,6)	(4,12,6)	(0,0,1)
$2.3^{\uparrow 2.1, \downarrow 2.6}$	(4,8)	(2,8,4)	(1,0,1)
2.4	(5,10)	(1,5,2)	(1,1,0)
$2.5^{\mathrm{SD}\uparrow 2.2}$	(6,6)	(2,6,2)	(0,1,0)
$2.6^{\mathrm{SD}\uparrow 2.3}$	(8,8)	(1,4,1)	(1,1,1)
3.1	(3,7)	(24,84,56)	(1,0,0)
$3.2^{\downarrow 3.5}$	(3,8)	(12,48,32)	(2,0,0)
3.3	(3,12)	(4,24,16)	(2,0,1)
$3.4^{\downarrow 3.8}$	(4,6)	(8,24,12)	(3,0,0)
$3.5^{\uparrow 3.2, \downarrow 3.10}$	(4,8)	(4,16,8)	(2,0,2)
$3.6^{\downarrow 3.11}$	(4,8)	(4,16,8)	(0,0,3)
$3.7^{\downarrow 3.12}$	(4,12)	(2,12,6)	(3,0,3)
$3.8^{\mathrm{SD}\uparrow 3.4}$	(6,6)	(4,12,4)	(3,0,3)
3.9	(7,14)	(1,7,2)	(3,3,0)
$3.10^{\mathrm{SD}\uparrow 3.5}$	(8,8)	(2,8,2)	(2,2,2)
$3.11^{\mathrm{SD}\uparrow 3.6}$	(8,8)	(2,8,2)	(0,3,0)
$3.12^{\mathrm{SD}\uparrow 3.7}$	(12,12)	(1,6,1)	(3,3,3)

Map	Type	(v, e, f)	(w_0, w_1, w_2)
$4.1^{\downarrow 4.9}$	(3,12)	(6,36,24)	(4,1,0)
$4.2^{\downarrow 4.6}$	(4,5)	(24,60,30)	(0,1,0)
$4.3^{\downarrow 4.7}$	(4,6)	(12,36,18)	(2,1,0)
$4.4^{\downarrow 4.11}$	(4,10)	(4,20,10)	(0,0,6)
$4.5^{\downarrow 4.12}$	(4,16)	(2,16,8)	(6,0,6)
$4.6^{\mathrm{SD}\uparrow 4.2}$	(5,5)	(12,30,12)	(0,0,0)*60
$4.7^{\mathrm{SD}\uparrow 4.3}$	(6,6)	(6,18,6)	(2,0,2)*36
4.8	(6,6)	(6,18,6)	(3,1,4)
$4.9^{\uparrow 4.1}$	(6,12)	(2,12,4)	(4,1,4)*24
4.10	(9,18)	(1,9,2)	(6,6,0)
$4.11^{\mathrm{SD}\uparrow 4.4}$	(10,10)	(2,10,2)	(0,6,0)
$4.12^{\mathrm{SD}\uparrow 4.5}$	(16,16)	(1,8,1)	(6,6,6)
$5.1^{\downarrow 5.6}$	(3,8)	(24,96,64)	(5,0,0)
5.2	(3,10)	(12,60,40)	(10,0,0)
$5.3^{\downarrow 5.9}$	(4,5)	(32,80,40)	(0,0,3)
$5.4^{\downarrow 5.10}$	(4,6)	(16,48,24)	(3,0,3)
$5.5^{\downarrow 5.12}$	(4,8)	(8,32,16)	(3,3,0)
$5.6^{\uparrow 5.1, \downarrow 5.13}$	(4,8)	(8,32,16)	(5,0,5)

Map	Type	(v,e,f)	(w_0,w_1,w_2)
$5.7^{\downarrow 5.15}$	(4,12)	(4,24,12)	(0,0,10)
$5.8^{\downarrow 5.16}$	(4,20)	(2,20,10)	(10,0,10)
$5.9^{\mathrm{SD}\uparrow 5.3}$	(5,5)	(16,40,16)	(0,3,0)
$5.10^{\mathrm{SD}\uparrow 5.4}$	(6,6)	(8,24,8)	(3,3,3)
5.11	(6,15)	(2,15,5)	(5,0,4)*90
$5.12^{\mathrm{SD}\uparrow 5.5}$	(8,8)	(4,16,4)	(3,0,3)*96
$5.13^{\mathrm{SD}\uparrow 5.6}$	(8,8)	(4,16,4)	(5,5,5)
5.14	(11,22)	(1,11,2)	(10,10,0)
$5.15^{\mathrm{SD}\uparrow 5.7}$	(12,12)	(2,12,2)	(0,10,0)
$5.16^{\mathrm{SD}\uparrow 5.8}$	(20,20)	(1,10,1)	(10,10,10)
$6.1^{\downarrow 6.6}$	(3,10)	(15,75,50)	(9,1,0)
6.2	(4,6)	(20,60,30)	(0,1,1)*120
$6.3^{\downarrow 6.9}$	(4,9)	(8,36,18)	(6,0,1)*144
$6.4^{\downarrow 6.12}$	(4,14)	(4,28,14)	(0,0,15)
$6.5^{\downarrow 6.13}$	(4,24)	(2,24,12)	(15,0,15)
$6.6^{\uparrow 6.1}$	(5,10)	(5,25,10)	(9,1,9)*50
6.7	(6,8)	(6,24,8)	(3,0,6)*144
6.8	(6,8)	(6,24,8)	(15,0,15)
$6.9^{\mathrm{SD}\uparrow 6.3}$	(9,9)	(4,18,4)	(6,1,6)*144
6.10	(10,15)	(2,15,3)	(9,1,9)*150
6.11	(13,26)	(1,13,2)	(15,15,0)
$6.12^{\mathrm{SD}\uparrow 6.4}$	(14,14)	(2,14,2)	(0,15,0)
$6.13^{\mathrm{SD}\uparrow 6.5}$	(24,24)	(1,12,1)	(15,15,15)

Map	Type	(v,e,f)	(w_0,w_1,w_2)
7.1	(3,7)	(72,252,168)	(0,0,2)
$7.2^{\downarrow 7.7}$	(3,12)	(12,72,48)	(12,0,4)
$7.3^{\downarrow 7.11}$	(4,16)	(4,32,16)	(8,0,3)*256
$7.4^{\downarrow 7.10}$	(4,16)	(4,32,16)	(0,0,21)
$7.5^{\downarrow 7.12}$	(4,28)	(2,28,14)	(21,0,21)
7.6	(6,9)	(6,27,9)	(8,0,14)*162
$7.7^{\uparrow 7.2}$	(6,12)	(4,24,8)	(12,0,12)*192
7.8	(6,21)	(2,21,7)	(14,0,14)*210
7.9	(15,30)	(1,15,2)	(21,21,0)
$7.10^{\mathrm{SD}\uparrow 7.4}$	(16,16)	(2,16,2)	(0,21,0)
$7.11^{\mathrm{SD}\uparrow 7.3}$	(16,16)	(2,16,2)	(8,3,8)*256
$7.12^{\mathrm{SD}\uparrow 7.5}$	(28,28)	(1,14,1)	(21,21,21)
8.1	(3,8)	(42,168,112)	(0,1,0)*336
8.2	(3,8)	(42,168,112)	(0,1,0)*336
$8.3^{\downarrow 8.10}$	(4,18)	(4,36,18)	(0,0,28)
$8.4^{\downarrow 8.11}$	(4,32)	(2,32,16)	(28,0,28)
8.5	(6,10)	(6,30,10)	(4,0,12)*360
8.6	(6,24)	(2,24,8)	(16,1,14)*336
8.7	(8,12)	(4,24,6)	(6,0,8)*432
8.8	(10,20)	(2,20,4)	(12,6,10)*320
8.9	(17,34)	(1,17,2)	(28,28,0)
$8.10^{\mathrm{SD}\uparrow 8.3}$	(18,18)	(2,18,2)	(0,28,0)
$8.11^{\mathrm{SD}\uparrow 8.4}$	(32,32)	(1,16,1)	(28,28,28)

E

Irreducible reflexive platonic maps

LEGEND. All irreducible reflexive platonic maps (cf. Section 1.6) of genus $2 \leq g \leq 101$. The standard map presentation of a map (or presentations of tuplet members) can be computed from the knowledge of the isomorphism type of $\mathrm{Aut}^+(\mathbf{R})$.

Remark. The group Alt_6 has outer automorphism group $\mathbb{Z}_2^2$, generated by the action of an element from Sym_6 and an exceptional outer automorphism. For $\mathbf{R}_{91.39}$, $\mathbf{R}_{100.26}$, and $\mathbf{R}_{100.27}$ the group $\mathrm{Aut}^+(\mathbf{R}) \cong \mathrm{Alt}_6 \rtimes \mathbb{Z}_2$ is the subgroup of $\mathrm{Aut}(\mathrm{Alt}_6)$ obtained by adjoining to Alt_6 an exceptional outer automorphism. For $\mathbf{R}_{46.4}$, $\mathbf{R}_{46.5}$, $\mathbf{R}_{91.36}$, $\mathbf{R}_{91.37}$, and $\mathbf{R}_{91.38}$ the group $\mathrm{Aut}^+(\mathbf{R}) \cong \mathrm{Alt}_6 \rtimes \mathbb{Z}_2$ is the subgroup of $\mathrm{Aut}(\mathrm{Alt}_6)$ obtained by adjoining to Alt_6 the product of an outer automorphism in Sym_6 and an exceptional outer automorphism.

Map	Type	(v, e, f)	Isotype $\mathrm{Aut}^+(\mathbf{R})$	Isotype $\mathrm{Aut}(\mathbf{R})$
$\mathbf{R}_{3.1}$	(3,7)	(24,84,56)	$\mathrm{PSL}(2,7)$	$\mathrm{PGL}(2,7)$
$\mathbf{R}_{4.6}$	(5,5)	(12,30,12)	Alt_5	$\mathrm{Alt}_5 \times \mathbb{Z}_2$
$\mathbf{R}_{5.9}$	(5,5)	(16,40,16)	$\mathbb{Z}_2^4 \rtimes \mathbb{Z}_5$	$(\mathbb{Z}_2^4 \rtimes \mathbb{Z}_5) \rtimes \mathbb{Z}_2$
$\mathbf{R}_{6.2}$	(4,6)	(20,60,30)	Sym_5	$\mathrm{Sym}_5 \times \mathbb{Z}_2$
$\mathbf{R}_{7.1}$	(3,7)	(72,252,168)	$\mathrm{PSL}(2,8)$	$\mathrm{PSL}(2,8) \times \mathbb{Z}_2$
$\mathbf{R}_{10.6}$	(4,5)	(72,180,90)	$\mathrm{PSL}(2,9)$	$\mathrm{PGL}(2,9)$
$\mathbf{R}_{10.9}$	(4,7)	(24,84,42)	$\mathrm{PSL}(2,7)$	$\mathrm{PGL}(2,7)$
$\mathbf{R}_{11.5}$	(6,6)	(20,60,20)	Sym_5	$\mathrm{Sym}_5 \times \mathbb{Z}_2$
$\mathbf{R}_{14.1}$	(3,7)	(156,546,364)	$\mathrm{PSL}(2,13)$	$\mathrm{PGL}(2,13)$
$\mathbf{R}_{14.2}$	(3,7)	(156,546,364)	$\mathrm{PSL}(2,13)$	$\mathrm{PSL}(2,13) \times \mathbb{Z}_2$
$\mathbf{R}_{14.3}$	(3,7)	(156,546,364)	$\mathrm{PSL}(2,13)$	$\mathrm{PGL}(2,13)$
$\mathbf{R}_{15.1}$	(3,9)	(56,252,168)	$\mathrm{PSL}(2,8)$	$\mathrm{PSL}(2,8) \times \mathbb{Z}_2$
$\mathbf{R}_{19.13}$	(5,5)	(72,180,72)	$\mathrm{PSL}(2,9)$	$\mathrm{PGL}(2,9)$
$\mathbf{R}_{19.23}$	(7,7)	(24,84,24)	$\mathrm{PSL}(2,7)$	$\mathrm{PGL}(2,7)$
$\mathbf{R}_{26.2}$	(3,11)	(60,330,220)	$\mathrm{PSL}(2,11)$	$\mathrm{PGL}(2,11)$

$\mathbf{R}_{29.9}$	(6,6)	(56,168,56)	$\mathrm{PGL}(2,7)$	$\mathrm{PGL}(2,7) \times \mathbb{Z}_2$
$\mathbf{R}_{34.6}$	(5,5)	(132,330,132)	$\mathrm{PSL}(2,11)$	$\mathrm{PGL}(2,11)$
$\mathbf{R}_{34.7}$	(5,5)	(132,330,132)	$\mathrm{PSL}(2,11)$	$\mathrm{PSL}(2,11) \times \mathbb{Z}_2$
$\mathbf{R}_{36.9}$	(6,8)	(42,168,56)	$\mathrm{PGL}(2,7)$	$\mathrm{PGL}(2,7) \times \mathbb{Z}_2$
$\mathbf{R}_{36.10}$	(6,8)	(42,168,56)	$\mathrm{PGL}(2,7)$	$\mathrm{PGL}(2,7) \times \mathbb{Z}_2$
$\mathbf{R}_{43.14}$	(8,8)	(42,168,42)	$\mathrm{PGL}(2,7)$	$\mathrm{PGL}(2,7) \times \mathbb{Z}_2$
$\mathbf{R}_{43.15}$	(8,8)	(42,168,42)	$\mathrm{PGL}(2,7)$	$\mathrm{PGL}(2,7) \times \mathbb{Z}_2$
$\mathbf{R}_{45.12}$	(5,6)	(110,330,132)	$\mathrm{PSL}(2,11)$	$\mathrm{PGL}(2,11)$
$\mathbf{R}_{45.13}$	(5,6)	(110,330,132)	$\mathrm{PSL}(2,11)$	$\mathrm{PSL}(2,11) \times \mathbb{Z}_2$
$\mathbf{R}_{46.4}$	(4,8)	(90,360,180)	$\mathrm{Alt}_6 \rtimes \mathbb{Z}_2$	$\mathrm{Aut}(\mathrm{Alt}_6)$
$\mathbf{R}_{46.5}$	(4,8)	(90,360,180)	$\mathrm{Alt}_6 \rtimes \mathbb{Z}_2$	$\mathrm{Aut}(\mathrm{Alt}_6)$
$\mathbf{R}_{49.57}$	(7,7)	(64,224,64)	$\mathbb{Z}_2^6 \rtimes \mathbb{Z}_7$	$(\mathbb{Z}_2^6 \rtimes \mathbb{Z}_7) \rtimes \mathbb{Z}_2$
$\mathbf{R}_{50.1}$	(3,13)	(84,546,364)	$\mathrm{PSL}(2,13)$	$\mathrm{PSL}(2,13) \times \mathbb{Z}_2$
$\mathbf{R}_{52.1}$	(3,8)	(306,1224,816)	$\mathrm{PSL}(2,17)$	$\mathrm{PGL}(2,17)$
$\mathbf{R}_{52.2}$	(3,8)	(306,1224,816)	$\mathrm{PSL}(2,17)$	$\mathrm{PGL}(2,17)$
$\mathbf{R}_{55.32}$	(7,7)	(72,252,72)	$\mathrm{PSL}(2,8)$	$\mathrm{PSL}(2,8) \times \mathbb{Z}_2$
$\mathbf{R}_{56.6}$	(6,6)	(110,330,110)	$\mathrm{PSL}(2,11)$	$\mathrm{PSL}(2,11) \times \mathbb{Z}_2$
$\mathbf{R}_{61.14}$	(6,6)	(120,360,120)	Sym_6	$\mathrm{Sym}_6 \times \mathbb{Z}_2$
$\mathbf{R}_{63.5}$	(7,9)	(56,252,72)	$\mathrm{PSL}(2,8)$	$\mathrm{PSL}(2,8) \times \mathbb{Z}_2$
$\mathbf{R}_{63.6}$	(7,9)	(56,252,72)	$\mathrm{PSL}(2,8)$	$\mathrm{PSL}(2,8) \times \mathbb{Z}_2$
$\mathbf{R}_{63.7}$	(7,9)	(56,252,72)	$\mathrm{PSL}(2,8)$	$\mathrm{PSL}(2,8) \times \mathbb{Z}_2$
$\mathbf{R}_{69.1}$	(3,9)	(272,1224,816)	$\mathrm{PSL}(2,17)$	$\mathrm{PGL}(2,17)$
$\mathbf{R}_{69.2}$	(3,9)	(272,1224,816)	$\mathrm{PSL}(2,17)$	$\mathrm{PGL}(2,17)$
$\mathbf{R}_{69.3}$	(3,9)	(272,1224,816)	$\mathrm{PSL}(2,17)$	$\mathrm{PSL}(2,17) \times \mathbb{Z}_2$
$\mathbf{R}_{70.3}$	(5,11)	(60,330,132)	$\mathrm{PSL}(2,11)$	$\mathrm{PGL}(2,11)$
$\mathbf{R}_{70.4}$	(5,11)	(60,330,132)	$\mathrm{PSL}(2,11)$	$\mathrm{PGL}(2,11)$
$\mathbf{R}_{71.15}$	(9,9)	(56,252,56)	$\mathrm{PSL}(2,8)$	$\mathrm{PSL}(2,8) \times \mathbb{Z}_2$
$\mathbf{R}_{81.62}$	(6,11)	(60,330,110)	$\mathrm{PSL}(2,11)$	$\mathrm{PGL}(2,11)$
$\mathbf{R}_{81.125}$	(9,9)	(64,288,64)	$\mathbb{Z}_2^6 \rtimes \mathbb{Z}_9$	$(\mathbb{Z}_2^6 \rtimes \mathbb{Z}_9) \rtimes \mathbb{Z}_2$
$\mathbf{R}_{91.36}$	(8,8)	(90,360,90)	$\mathrm{Alt}_6 \rtimes \mathbb{Z}_2$	$\mathrm{Aut}(\mathrm{Alt}_6)$
$\mathbf{R}_{91.37}$	(8,8)	(90,360,90)	$\mathrm{Alt}_6 \rtimes \mathbb{Z}_2$	$\mathrm{Aut}(\mathrm{Alt}_6)$
$\mathbf{R}_{91.38}$	(8,8)	(90,360,90)	$\mathrm{Alt}_6 \rtimes \mathbb{Z}_2$	$\mathrm{Aut}(\mathrm{Alt}_6)$
$\mathbf{R}_{91.39}$	(8,8)	(90,360,90)	$\mathrm{Alt}_6 \rtimes \mathbb{Z}_2$	$\mathrm{Alt}_6 \rtimes \mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathbf{R}_{92.6}$	(6,6)	(182,546,182)	$\mathrm{PSL}(2,13)$	$\mathrm{PGL}(2,13)$
$\mathbf{R}_{96.1}$	(3,9)	(380,1710,1140)	$\mathrm{PSL}(2,19)$	$\mathrm{PGL}(2,19)$
$\mathbf{R}_{96.2}$	(3,9)	(380,1710,1140)	$\mathrm{PSL}(2,19)$	$\mathrm{PSL}(2,19) \times \mathbb{Z}_2$
$\mathbf{R}_{96.3}$	(3,9)	(380,1710,1140)	$\mathrm{PSL}(2,19)$	$\mathrm{PSL}(2,19) \times \mathbb{Z}_2$
$\mathbf{R}_{100.2}$	(4,10)	(132,660,330)	$\mathrm{PGL}(2,11)$	$\mathrm{PGL}(2,11) \times \mathbb{Z}_2$
$\mathbf{R}_{100.3}$	(4,10)	(132,660,330)	$\mathrm{PGL}(2,11)$	$\mathrm{PGL}(2,11) \times \mathbb{Z}_2$
$\mathbf{R}_{100.26}$	(8,10)	(72,360,90)	$\mathrm{Alt}_6 \rtimes \mathbb{Z}_2$	$(\mathrm{Alt}_6 \rtimes \mathbb{Z}_2) \times \mathbb{Z}_2$
$\mathbf{R}_{100.27}$	(8,10)	(72,360,90)	$\mathrm{Alt}_6 \rtimes \mathbb{Z}_2$	$(\mathrm{Alt}_6 \rtimes \mathbb{Z}_2) \times \mathbb{Z}_2$

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Index

Platonic maps of low genus
Summary

In what ways can one tile a surface such that the tiling has a large measure of symmetry? This question lies at the basis of the research area with which this dissertation is concerned. To make the question more exact, we suppose we have a closed orientable surface with a connected finite graph with non-empty vertex set and non-empty edge set embedded into it, such that the complement of the graph consists of a union of two-dimensional disks. This puts a cell structure on the surface, with graph vertices, graph edges and disks being cells of dimension 0, 1, and 2, respectively. We can identify a disk together with its boundary as a polygon, where the edges on the disk boundary become sides of the polygon, and vertices on the boundary vertices of the polygon. A surface together with a cell division is called a *map*.

A homeomorphism of the surface is called cellular with respect to a map if it sends cells to cells. The symmetry we demand of a map in this thesis is the existence of a group of cellular homeomorphisms that acts transitively on the set of oriented $(0, 1)$-flags of the graph, that is, on pairs of a vertex and an incident directed edge. This property entails that all disks have the same number of sides, when viewed as polygons, and all vertices of the graph have the same number of incident edges, but is in general stronger than these two conditions. We call a map on a surface satisfying our condition a *platonic map*. If there is also a cellular homeomorphism that preserves an oriented $(0, 1)$-flag but is not the identity, then we say the platonic map is *reflexive*, and call such a homeomorphism a reflection. Classical examples of reflexive platonic maps are the platonic solids (the tetrahedron, cube, octahedron, dodecahedron and icosahedron), whence their name. The platonic solids and their symmetry groups have long been of interest to mathematicians.

The group of cellular homeomorphisms of a map $\mathbf{M}$ induces a subgroup of the mapping class group of the surface, which is the group formed by the equivalence classes of homeomorphisms with respect to isotopy. This subgroup is called the *automorphism group* of the map, $\mathrm{Aut}(\mathbf{M})$. One can in fact realize $\mathrm{Aut}(\mathbf{M})$ as an actual group of homeomorphisms of the surface. If the map is platonic, the subgroup $\mathrm{Aut}^+(\mathbf{M})$ of orientation preserving map automorphisms can be generated by two elements R and S, which are primitive counterclockwise rotations around a polygon (disk) and an incident vertex, respectively. A presentation of $\mathrm{Aut}^+(\mathbf{M})$ in R and S is called a *standard map presentation* of $\mathbf{M}$. A standard map presentation already contains all the information of a platonic map.

A fascinating fact is that on each surface of genus $g \geq 2$ there are only a finite number of platonic maps. With the aid of group theory one can enumerate all possibile standard map

presentations for a given genus, and this has indeed been done in [CD2001] up to genus 15. This list has been the starting point of the investigation into platonic maps described in this thesis.

We develop two fundamental tools to categorize and order platonic maps, namely polynomial families and diagonal maps. A *polynomial family* is a parametrized group-theoretic recipe for constructing an infinite series of platonic maps in a controlled way. *Diagonal maps* are the result of a standard construction applied, under certain conditions, to a map to yield a new map of the same genus. These tools, along with covering theory of platonic maps, are used throughout the thesis. We determine those reflexive platonic maps whose number of vertices is an odd prime. Also, we classify all reflexive platonic maps whose *density* is higher than $\frac{1}{2}$, with the stellar roles played by the tetrahedron and the Fermat maps.

A remarkable property of a platonic map, a hidden gem lying dormant, is that it uniquely determines a compact Riemann surface on which the map can be realized by a graph with geodesic edges, and such that $\mathrm{Aut}(\mathbf{M})$ acts by isometries. Furthermore, there is a correspondence between compact Riemann surfaces and smooth complex algebraic curves. Both types of objects and the correspondence are of major importance in mathematics, and have guided a great deal of research, starting with Klein's quartic curve. In general, this correspondence is not effective. For platonic maps, it is! We undertake the task to find algebraic curves for as many platonic maps as possible, so that other researchers may utilize them and study their properties further. We have succeeded for all platonic maps of genus at most 8, and for various other ones of higher genus, among which the members of the *first Hurwitz triplet*. In a separate chapter, we study the properties of this triplet and try to compute its Weierstraß points, answering a question by Kay Magaard and Helmut Völklein.

Curriculum Vitae

Maxim Hendriks was born on the 9th of May 1980 in Leiden, The Netherlands. In 1998, after completing his secondary education at the Stedelijk Gymnasium in his place of birth, he started his studies in Mathematics at the University of Leiden. Thoroughly enjoying the learning process, he spent the year of 2003 at the University of Cologne in Germany, afterwards to return to Leiden and pass his Doctoraalexamen (the old Dutch equivalent of a Master's exam) cum laude in January 2007. His *afstudeerscriptie* (Master's thesis) was entitled

"Surface automorphisms and the Nielsen realization problem"

and written under supervision of Prof.dr. Hansjörg Geiges and Prof.dr. Bas Edixhoven. After graduating, he participated in a short joint project at the Free University of Amsterdam and the Radboud University in Nijmegen on the teaching of logic to undergraduate students aided by computer. Then, in October 2007, he started research as a PhD candidate in the Discrete Algebra and Geometry group at Eindhoven University of Technology, under the supervision of Prof.dr. Arjeh M. Cohen. The result of that research is the PhD thesis

"Platonic maps of low genus",

now lying before you. Besides this research he also partook in the Inter2Geo project, which was a part of the EU eContent*plus* program and focused on the computer-aided teaching of geometry in secondary schools. This project resulted in a web platform for dynamic geometry resources:

http://i2geo.net/