Discrete time LPV I/O and State Space Representations,
Differences of Behavior and Pitfalls of Interpolation

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Abstract— A common approach for modeling LPV systems is to interpolate between local LTI models, often obtained by system identification methods. We study the results of interpolating in different domains, the so called I/O domain and the state-space domain. It is shown that significant differences can occur between the interpolated models, due to differences in time propagation of the (scheduling) parameter. We introduce canonical representations for LPV state-space realizations similar to the LTV framework and derive exact formulas for the connection between I/O and state-space based LPV models.

Index Terms— LPV, realization theory, canonical, model interpolation.

I. INTRODUCTION

During the past 15 years, intensive research has been carried out on *Linear Parameter Varying* (LPV) systems [1]. The main reason for this interest is that this framework provides a powerful modeling tool for a wide class of nonlinear systems, in particular found in servo-mechanical applications. Moreover, it is commonly accepted that the possibilities of classical control techniques, as developed for *Linear Time Invariant* (LTI) models, are often too limited to cope with the increasing industrial performance requirements. Although gain scheduling using LPV synthesis techniques is a promising control approach, as shown by a wide range of applied LPV control solutions on aerospace systems [2], induction motors [3], or CD players [4], it still remains a problem how to develop LPV models in a systematic fashion. Commonly, LPV models are produced by following the basic gain-scheduling idea: for a given nonlinear system $\mathcal{N}$, take locally valid LTI models of $\mathcal{N}$ at significant operation points and then interpolate between these local models by a smooth *scheduling function* [5], [6]. In the LTI case it is shown [7], that for any I/O model, there exist an equivalent *State-Space* (SS) based model and vice versa. This equivalence may be the reason why in the LPV framework often no care is taken about the domain where the interpolation between the local LTI models is carried out (i.e. in the state space or in the I/O operator domain). However, though the resulting LPV-I/O and LPV-SS models are equivalent in the interpolation points, their behavior can be significantly different for other operation points or during a dynamic transient between the interpolation points, even if the same interpolation function is used in both domains. Before illustrating the difference with an example, the following definitions are introduced.

Definition 1: (LPV-I/O model) The I/O model of a SISO, discrete time LPV system $\mathcal{S}$ is denoted by $\mathfrak{R}_{I/O}(\mathcal{S}, p)$, where $\mathfrak{R}_{I/O}(\mathcal{S}, p)$ is dependent on the scheduling parameter $p(k) \in \mathbb{P}$ with $\mathbb{P} \subset \mathbb{R}^{n_p}$ a compact set and it is defined as:

$$
\begin{aligned}
y(k) &= -\sum_{i=1}^{n_a} a_i(p(k)) y(k-i) + \quad (1) \\
&\quad + \sum_{j=0}^{n_b} b_j(p(k)) u(k-j),
\end{aligned}
$$

where $n_a \geq n_b > 0$, $u(k) \in \mathbb{R}$ is the system input and $y(k) \in \mathbb{R}$ is the system output. It is assumed, that each varying parameter in (1) is a smooth and continuous function of p. □

Definition 2: (LPV-SS model) The p dependent SS model $\mathfrak{R}_{SS}(\mathcal{S}, p)$ of $\mathcal{S}$ is defined as follows:

$$
\begin{aligned}
x(k+1) &= A(p(k)) x(k) + B(p(k)) u(k), \quad (2) \\
y(k) &= C(p(k)) x(k) + D(p(k)) u(k), \quad (3)
\end{aligned}
$$

where $x(k) \in \mathbb{R}^{n_x}$ is the state vector of $\mathfrak{R}_{SS}(\mathcal{S}, p)$, $n_x \in \mathbb{N}$ is the model order, and

$$
\left[\begin{array}{c|c} A(p) & B(p) \\ \hline C(p) & D(p) \end{array} \right] : \mathbb{P} \to \left[\begin{array}{c|c} \mathbb{R}^{n_x \times n_x} & \mathbb{R}^{n_x \times 1} \\ \hline \mathbb{R}^{1 \times n_x} & \mathbb{R} \end{array} \right],
$$

represents the parameter varying state-space matrices of $\mathfrak{R}_{SS}(\mathcal{S}, p)$. Again it is assumed, that all functions of p are smooth and continuous. □

Definition 3: (I/O Equivalence) LPV-I/O or LPV-SS models with identical I/O behavior are called equivalent. □

Consider the following example to illustrate the basic difference in the I/O behavior of the two representations.

Example 4: (Difference in interpolation) Assume that $\mathcal{N}$ has two local LTI-SS representations

$$
\mathcal{N} = \begin{cases} \left[\begin{array}{c|c} \alpha_0 & \beta_0 \\ \hline 1 & 0 \end{array} \right] & \text{if } p = 0, \\[2em] \left[\begin{array}{c|c} \alpha_1 & \beta_1 \\ \hline 1 & 0 \end{array} \right] & \text{if } p = 1, \end{cases}
$$

where $p \in [0, 1]$. If $\mathcal{S}$ represents the LPV approximation of $\mathcal{N}$, then a trivial $\mathfrak{R}_{SS}(\mathcal{S}, p)$ can be formulated as

$$
\left[\begin{array}{c|c} A(p) & B(p) \\ \hline C(p) & D(p) \end{array} \right] = \sum_{i=0}^{1} \left[\begin{array}{c|c} \delta_i(p) \alpha_i & \delta_i(p) \beta_i \\ \hline 0.5 & 0 \end{array} \right], \quad (4)
$$

where $\delta_0(p(k)) = 1 - p(k)$ and $\delta_1(p(k)) = p(k)$ are the linear interpolation functions. The two local models can also

be interpolated based on their I/O equivalent. By using the same interpolation functions, the resulting $\mathfrak{R}_{I/O}(\mathcal{S}, p)$ is

$$y_{I/O}(k) = a_1(p(k)) y(k-1) + b_1(p(k)) u(k-1), \quad (5)$$

where
$$a_1(p(k)) = \sum_{i=0}^{1} \delta_i(p(k)) \alpha_i,$$
$$b_1(p(k)) = \sum_{i=0}^{1} \delta_i(p(k)) \beta_i.$$

However by computing the I/O behavior of $\mathfrak{R}_{SS}(\mathcal{S}, p)$, one can conclude that

$$y_{SS}(k+1) = a_1(p(k)) y(k) + b_1(p(k)) u(k), \quad (6)$$

which is clearly not equal to (5). The reason for the difference lies in the propagation of the variation of p through the state evolution, which in this one-dimensional example only causes a one-sample delay in the coefficient variation (compare 5 and 6), but in higher dimensional cases more complicated effects can occur (see Section II). □

Note that in Example 4, $\mathfrak{R}_{I/O}(\mathcal{S}, p)$ is also the direct realization of $\mathfrak{R}_{SS}(\mathcal{S}, p)$ in the I/O domain utilizing the concepts of the LTI theory, therefore it is a trivial conclusion that the class of LPV-I/O and LPV-SS systems defined through the frequently used Definition 1 and 2 of the literature [8], [9] are not equivalent. Similar result holds in case of *Linear Time Varying* (LTV) systems as pointed out by [10]. Then it can be proved (see Section II), that the following problem holds:

Problem 5: The equivalence classes of system definitions 1 and 2 are distinct as there exists no $p(k)$ dependent (static) transformation between them. □

To establish equivalence transformations, we need to extend Definition 1 and 2, such that we allow representations of $\mathcal{S}$ to depend on different scheduling functions. By introducing $\phi(k) = p(k)$ and $\psi(k) = p(k+1)$, it holds true that $\mathfrak{R}_{SS}(\mathcal{S}, \phi)$ and $\mathfrak{R}_{I/O}(\mathcal{S}, \psi)$ in Example 4 are equivalent.

This phenomenon leads to much confusion in the literature. In the area of LPV system identification, recently several (global) methods were proposed to extract an LPV model from measurement data. This concerns among others, methods based on subspace techniques [11], [12], basis functions [13], *Linear Matrix Inequalities* (LMI's) based optimization [14], simple *Least Means Square* (LMS) approaches [8], and on parameter estimation based gradient searches [9]. All of these methods can be categorized in the way that the obtained models are either SS or I/O operator based. After producing an LPV model, some authors convert it from one domain to the other, as in the LTI case, relying only on the local validity of the conversion. The error caused by this conversion, due to the propagation effect illustrated in Example 4, is often misinterpreted or overlooked. See for instance [6], [15]. In [15], a global I/O model of a compressor is obtained by identifying SISO LPV-I/O models on each input-output channel of the device. These channel models are then assembled to a canonical MIMO LPV-SS representation using the LTI realization theory locally. When

the MIMO model is validated, the gain and phase lags of the outputs are explained as the error of the approximation. However, such lags are commonly produced by the above mentioned problem of transformations between the domains (see Section IV for details). Though the LPV-SS model in [15] seems to correspond to the true system because of the slow variations of the scheduling parameter, it is not investigated what happens when the scheduling parameter changes rapidly, in which case the above mentioned phenomenon would show its full deviance.

There is a second problem in the conversion between LPV-I/O and LPV-SS systems that is often overlooked. It is well known that a SS representation is not unique and can only be identified up to an unknown state transformation. For a single system, this problem is generally unimportant if one is only interested in the general I/O behavior, which is left unaltered by state transformations. However, when assembling an LPV model by interpolating between different local LTI models, difficulties may occur. If all of the local LTI models are identified independently from the others, then their representations will generally be in different state space bases. This problem is recognized in [16] and solved in the case of piecewise linear models, a particular subclass of LPV models. In papers dealing with more general systems, such as in [6] this problem is neglected. The choice of writing all the local systems in a canonical form and then interpolating the parameters of these canonical representations seems attractive, but is not motivated theoretically, because of the following fact:

Problem 6: Given an LPV-SS system as defined by Equations (2-3). The use of *local* state transformation matrices $T(p(k))$ dependent only on the local value $p(k)$ to reach a canonical representation will produce SS-matrices $\tilde{A}(p(k))) = T(p(k))^{-1} A(p(k)) T(p(k))$ etc., which do not constitute a canonical form for the *global* LPV system. □

Problem 6 is verified by the computation of the impulse response of the locally converted system, resulting in:

$$\left[D(p_0), \quad C(p_1) T(p_1) T^{-1}(p_0) B(p_0), \quad \cdots \right]$$

for a $\{p_k\}$ sequence. Here it is obvious that the transformation matrices are not canceling each other like in the LTI case, completely altering the I/O behavior of the original system. As will follow from Section II, the correct approach will lead to related matrices of the form $\tilde{A}(\phi(k)) = T(\bar{p}(k+1))^{-1} A(p(k)) T(\bar{p}(k))$ etc., where T is a function of a sequence of parameters $\bar{p}$ and ϕ is constructed from $\bar{p}$. This is directly related to canonical forms for LTV systems [17]. A consequence of the local approach is that the states at consecutive parameter points $p(k), p(k+1)$, etc. are not related correctly. Even when the parameter variation is small the resulting induced error can be substantial, as shown in Example 24 (Section IV).

It is important that in Definitions 1 and 2 the dependency on p can be any continuous function. However, a particulary interesting class of LPV systems are affine LPV systems:

Definition 7: Any $\mathfrak{R}(\mathcal{S}, p)$ SS or I/O model of $\mathcal{S}$ has affine dependency on p, if every $f(p(k))$ varying parameter

(A, B, C, D, a, b) of $\mathfrak{R}(\mathcal{S}, p)$ can be written as

$$f(p(k)) = \sum_{i=1}^{n_p} p_i(k) f_i, \qquad (7)$$

where p_i denotes the ith element of p and $\{f_i\}_{i=1}^{n_p}$ are constants. $\square$

Then, the following theorem holds true:

Theorem 8: For any $\mathfrak{R}_{SS}(\mathcal{S}, p)$ or $\mathfrak{R}_{I/O}(\mathcal{S}, p)$ models, there exists an equivalent $\mathfrak{R}_{SS}(\mathcal{S}, \phi)$ or $\mathfrak{R}_{I/O}(\mathcal{S}, \psi)$ representation with affine dependency on $\phi(k) \in \mathbb{P}_\phi \subset \mathbb{P}^{n_\phi}$ and $\psi(k) \in \mathbb{P}_\psi \subset \mathbb{P}^{n_\psi}$, where $0 < n_\phi \leq n_x^2 + 2n_x + 1$ and $0 < n_\psi \leq n_a + n_b + 1$. $\square$

Proof: Define $\phi_i(k)$ or $\psi_i(k)$ to substitute any $f(p(k))$ parameter dependency which is not affine. For $\mathfrak{R}_{SS}(\mathcal{S}, p)$ at most $n_x^2 + 2n_x + 1$ and for $\mathfrak{R}_{I/O}(\mathcal{S}, p)$ at most $n_a + n_b + 1$ such substitutions are needed. By preserving the relation of each ϕ_i and ψ_i to p equivalence is preserved. $\blacksquare$

Theorem 8 states that any LPV system can be represented by an affine representation, which also resembles how nonlinear dependencies are usually eliminated from LPV systems in order to reach a representation which is suitable for LPV optimal control or LPV identification. In terms of analysis it is more important, that in order to establish equivalence relations and canonical forms, uniqueness of dependency over the scheduling parameter is required. Problem 5 shows that it is not trivial how to introduce the new parameters to compensate the propagation effects. Therefore in the sequel, to preserve mathematical validity, we denote by $\mathfrak{R}_{SS}(\mathcal{S}, \phi)$ and $\mathfrak{R}_{I/O}(\mathcal{S}, \psi)$ the affine representations of the LPV system $\mathcal{S}$ and define equivalence classes and transformations between affine systems. By knowing the exact functional dependence of ϕ or ψ over p, one can always reach nonlinear or even time lagged dependencies related equivalent representations of $\mathcal{S}$. However, these cases are not covered by Definition 1 and 2. The use of affine dependency is only a mathematical necessity and does not restrict the validity of the presented theory for general LPV systems. In the following, only the case of SISO systems is considered because of the highly complicated notation of the MIMO case. However, the results are easily generalizable to MIMO systems.

The analysis presented in this paper will provide a method to solve Problem 5 and 6 by obtaining a globally equivalent SS or I/O realization of an LPV model with a given parameter dependence. The question in the main focus of the following discussion is "How one should treat LPV models after identification to preserve validity." The paper is organized as follows: in Section II, LPV-SS canonical representations are introduced with the investigation of their equivalent LPV-I/O model; in Section III, it is investigated how the derived results affect the current identification approaches and future research of the identification field; in Section IV, several examples are given to show the validity of the presented theories and to give some insight into the depths of the underlying problems; in Section V, the main conclusions are summarized.

II. STATE-SPACE CANONICAL FORMS OF LPV MODELS

In the LTI case, canonical forms of SS models are very important as they provide the common ground of representation of SS models that differ in parameters but are equal in I/O behavior due to the fact, that their SS matrices can be related through a linear state transformation. Canonical forms provide a common state-basis of these models, therefore if the state-basis of two SS models are equivalent, then the systems are called equivalent up to a state transformation and belong to the same equivalence class, which is uniquely determined by a canonical SS model [7]. For general LPV-SS systems it is also essential to develop such representations in order to compare and uniquely represent the underlying dynamics or equivalence classes.

In the LTV framework, canonical forms of discrete-time SS models were recently developed and investigated by several authors [18], [17], [19]. Formally, the class of LTV models can be considered to be broader than the class of LPV systems, since – by substitution as in $\tilde{f}(k) = f(p(k))$ – an LPV model can be reparameterized as an LTV model. For instance, one can rewrite Equation (2) to obtain $\tilde{A}(k) = A(p(k))$ etc, leading to a general LTV formulation. A major difference is that the trajectory of the scheduling parameter for LPV systems is generally assumed to be unknown but measurable in real time, whereas the dependence on time for LTV systems is generally assumed to be known. Naturally, the two classes of systems inhabit a lot of similarities. Therefore, in the LPV framework, one can utilize many results developed for LTV systems, be it that always care has to be taken to validate the assumptions of these theories in the LPV case. Based on the LTV framework, the following concepts are utilizable for LPV systems to obtain canonical SS forms of affine equivalent realizations.

Definition 9: (State-observability matrix) The parameter varying *state-observability* matrix of $\mathfrak{R}_{SS}(\mathcal{S}, \phi)$ is defined as the function $\mathcal{O}_{n_x} : \mathbb{P}_\phi^{n_x} \to \mathbb{R}^{n_x \times n_x}$ with

$$\mathcal{O}_{n_x}(\bar{\phi}) = \begin{bmatrix} C(\phi_1) \\ C(\phi_2)A(\phi_1) \\ \vdots \\ C(\phi_{n_x}) \prod_{l=1}^{n_x-1} A(\phi_l), \end{bmatrix}, \qquad (8)$$

where $\bar{\phi} = [\phi_1, \cdots, \phi_{n_x}] \subset \mathbb{P}_\phi$ is an arbitrary sequence. $\square$

Definition 10: (State-reachability matrix) The parameter varying *state-reachability* matrix of $\mathfrak{R}_{SS}(\mathcal{S}, \phi)$ is defined as the function $\mathcal{R}_{n_x} : \mathbb{P}_\phi^{n_x} \to \mathbb{R}^{n_x \times n_x}$ with

$$\mathcal{R}_{n_x}(\bar{\phi}) = \begin{bmatrix} B^T(\phi_1) \\ B^T(\phi_2)A^T(\phi_1) \\ \vdots \\ B^T(\phi_{n_x}) \prod_{l=1}^{n_x-1} A^T(\phi_l) \end{bmatrix}. \qquad (9)$$

$\square$

Now denote

$$\begin{aligned} \bar{\phi}_+(k) &= [\phi(k), \phi(k+1), \cdots, \phi(k+n_x-1)], \\ \bar{\phi}_-(k) &= [\phi(k), \phi(k-1), \cdots, \phi(k-n_x+1)]. \end{aligned}$$

Then the following two theorems hold:

Theorem 11: (Complete observability, [19]) $\mathfrak{R}_{SS}(\mathcal{S}, \phi)$ of $\mathcal{S}$ is completely observable, iff $\operatorname{rank}\left[\mathcal{O}_{n_x}(\bar{\phi}_+(k))\right] = n_x$ for all $k \in \mathbb{Z}$. $\square$

Theorem 12: (Complete reachability, [19]) $\mathfrak{R}_{SS}(\mathcal{S}, \phi)$ of $\mathcal{S}$ is completely reachable, iff $\operatorname{rank}\left[\mathcal{R}_{n_x}(\bar{\phi}_-(k))\right] = n_x$ for all $k \in \mathbb{Z}$. $\square$

Corollary 13: If $\mathfrak{R}_{SS}(\mathcal{S}, \phi)$ is completely observable (reachable), then the rows of $\mathcal{O}_{n_x}(\bar{\phi}_+(k))$ (columns of $\mathcal{R}_{n_x}(\bar{\phi}_-(k))$) are linearly independent and $\mathcal{O}_{n_x}(\bar{\phi}_+(k))$ ($\mathcal{R}_{n_x}(\bar{\phi}_-(k))$) is invertible. $\square$

Definition 14: (Minimality) $\mathfrak{R}_{SS}(\mathcal{S}, \phi)$ is called minimal, if it is both completely reachable and observable. $\square$

Because of space limitations, definition of a canonical form will not be exactly given, see for instance [20] and [21] for more details. Roughly spoken, a canonical $\mathfrak{R}_{SS}(\mathcal{S}, \phi)$ belongs to a subset of the equivalence class of $\mathcal{S}$ and possesses the least possible number of meaningful parameters. Through the utilization of complete reachability and observability, parameter varying state-transformations can be deduced that provide canonical representations of $\mathcal{S}$.

A. Observability canonical form

Assume that $\mathfrak{R}_{SS}(\mathcal{S}, \phi)$ is completely observable. We introduce a new parameter varying state basis with the transformation matrix

$$T_o(\bar{\phi}_+(k)) = \mathcal{O}_{n_x}^{-1}(\bar{\phi}_+(k)), \qquad (10)$$

leading to new state variables $\bar{x}_o(k)$, such that

$$x(k) = T_o(\bar{\phi}_+(k))\bar{x}_o(k), \qquad (11)$$
$$x(k+1) = T_o(\bar{\phi}_+(k+1))\bar{x}_o(k+1). \qquad (12)$$

The state transformation $T_o(\bar{\phi}_+(\cdot))$ fulfilling (11) and (12) implies algebraic equivalence between x and $\bar{x}_o$. However, algebraic equivalence does not preserve stability [7]. For this it is necessary and sufficient to have topological equivalence [7]. The condition for $T_o(\bar{\phi}_+(\cdot))$ to imply topological equivalence lies in the concept of Lyapunov transformations.

Definition 15: (Lyapunov transformation, [22]) Let $n \geq 0$. A parameter varying SS transformation $T: \mathbb{P}_\phi^n \to \mathbb{R}^{n_x \times n_x}$ is a Lyapunov transformation, if it is full rank and there exists a finite constant $\varepsilon \in \mathbb{R}_0^+$, such that for any $\bar{\phi} \in \mathbb{P}_\phi^n$

$$\left\|T(\bar{\phi})\right\| \leq \varepsilon, \qquad (13)$$
$$\left\|T^{-1}(\bar{\phi})\right\| \leq \varepsilon. \qquad (14)$$

where $\|\cdot\|$ is the euclidian (arbitrary) norm. $\square$

Theorem 16: A Lyapunov transformation always preserves global stability of the system. Furthermore, the trajectories of the pole functionals, the parameter dependent local pole locations are also preserved [7]. $\square$

It turns out that the transformation defined by (10) fulfills the conditions:

Lemma 17: The observability transformation $T_o(\bar{\phi}_+(\cdot))$ is a Lyapunov transformation. $\square$

Proof: The compactness of $\mathbb{P}_\phi$ and the continuous dependence of the full rank $\mathcal{O}_{n_x}$ on $\bar{\phi}_+$ implies that $T_o(\bar{\phi}_+(\cdot))$ exists, both $T_o(\bar{\phi}_+(\cdot))$ and $T_o^{-1}(\bar{\phi}_+(\cdot))$ are bounded, and

$$\operatorname{rank}\left(T_o(\bar{\phi}_+(k))\right) = \operatorname{rank}\left(T_o^{-1}(\bar{\phi}_+(k))\right) = n_x, \qquad (15)$$

for any $k \in \mathbb{Z}$. ∎

Because T_o implies algebraic equivalence (preserving local equivalence) and it is a Lyapunov transformation (preserving global stability), therefore T_o implies topological equivalence which guarantees the preservation of the original I/O behavior. For proof see [7]. The transformed matrices for the new state-basis are given by [17]:

$$
\begin{aligned}
A_o(\phi_o(k)) &= T_o^{-1}(\bar{\phi}_+(k+1))A(\phi(k))T_o(\bar{\phi}_+(k)) = \\
&= \begin{bmatrix} 0 & \dots & 0 & -\alpha_{n_x}^{(o)}(\phi_o(k)) \\ 1 & \ddots & \vdots & -\alpha_{n_x-1}^{(o)}(\phi_o(k)) \\ \vdots & \ddots & 0 & \vdots \\ 0 & \dots & 1 & -\alpha_1^{(o)}(\phi_o(k)) \end{bmatrix}^T, \\
B_o(\phi_o(k)) &= T_o^{-1}(\bar{\phi}_+(k+1))B(\phi(k)) = \\
&= \begin{bmatrix} \beta_1^{(o)}(\phi_o(k)) & \dots & \beta_{n_x}^{(o)}(\phi_o(k)) \end{bmatrix}^T, \\
C_o(\phi_o(k)) &= \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}, \\
D_o(\phi_o(k)) &= D(\phi(k)) = \beta_0^{(o)}(\phi_o(k)),
\end{aligned}
$$

where $\phi_o(k)$ is a new scheduling vector, function of $[\phi(k), \dots, \phi(k+n_x)]$ such that $\{\alpha_i^{(o)}(\phi_o(k))\}_{i=1}^{n_x}$ and $\{\beta_j^{(o)}(\phi_o(k))\}_{j=0}^{n_x}$ has affine dependency on $\phi_o(k)$. Then,

$$\mathfrak{R}_{SS}^{\mathcal{O}}(\mathcal{S}, \phi_o) = \left[\begin{array}{c|c} A_o(\phi_o) & B_o(\phi_o) \\ \hline C_o(\phi_o) & D_o(\phi_o) \end{array}\right], \qquad (16)$$

is called the *observability canonical state space representation* of $\mathcal{S}$ and it is equivalent with $\mathfrak{R}_{SS}(\mathcal{S}, \phi)$. Proof of the results provided by the above given matrix operations can be found in [19].

For clarity purposes, we will use two symbols, q^{-1} and w^{-1} for the backward time shift operator, where q is used for the signals u and y, while w will only be used for variables that depend directly on the scheduling parameter. In this respect it has to be understood, that in the notation

$$w^s \beta(\phi) q^{-t} u = \beta(\phi(k+s))u(k-t),$$

the operator w only works on the scheduling parameter dependent function $\beta(\phi(\cdot))$ and q only on the function $u(\cdot)$.

Theorem 18: $\mathfrak{R}_{SS}^{\mathcal{O}}(\mathcal{S}, \phi_o)$ is completely observable [19]. $\square$

Based on the structure of $\mathfrak{R}_{SS}^{\mathcal{O}}(\mathcal{S}, \phi_o)$,

$$
\begin{aligned}
\bar{x}_1^{(o)} &= y - \beta_0^{(o)}(\phi_o)u, \\
\bar{x}_2^{(o)} &= qy - w\beta_0^{(o)}(\phi_o)qu - \beta_1^{(o)}(\phi_o)u, \\
&\vdots \\
\bar{x}_{n_x}^{(o)} &= q^{n_x-1}y - \sum_{t=0}^{n_x-1} w^t \beta_{n_x-1-t}^{(o)}(\phi_o)q^t u.
\end{aligned}
$$

Here time k is omitted from the expressions. Then, it follows:

$$\bar{x}^{(o)} = Q_o y - W_o(\phi_o) Q_o u, \qquad (17)$$

where

$$Q_o = \begin{bmatrix} 1 & q & \dots & q^{n_x-1} \end{bmatrix}^T,$$

and $W_o(\phi_o)$ is defined as

$$\begin{bmatrix} \beta_0^{(o)}(\phi_o) & 0 & \dots & 0 \\ \beta_1^{(o)}(\phi_o) & w\beta_0^{(o)}(\phi_o) & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \beta_{n_x-1}^{(o)}(\phi_o) & w\beta_{n_x-2}^{(o)}(\phi_o) & \dots & w^{n_x-1}\beta_0^{(o)}(\phi_o) \end{bmatrix}.$$

By substitution of (17) into (2) and (3), it follows that the I/O operator form of $\mathfrak{R}_{SS}^{\mathcal{O}}(\mathcal{S}, \phi_o)$ is

$$[\Lambda(\phi_o) Q_o] y = [\Lambda(\phi_o) W_o(\phi_o) Q_o + \qquad (18)$$
$$+ w^{-1} B_o(\phi_o)] u,$$

where $\Lambda(\phi_o) = (Iq - w^{-1}A_o(\phi_o))$. Relation (18) can be given in a more simplified form of (1), defining $\mathfrak{R}_{I/O}(\mathcal{S}, \psi)$ as the I/O representation of $\mathcal{S}$, where

$$a_i(\psi) = w^{-n_x}\alpha_i^{(o)}(\phi_o), \qquad (19)$$
$$b_j(\psi) = w^{-j}\beta_j^{(o)}(\phi_o) + \qquad (20)$$
$$+ \sum_{t=1}^{j} \left[w^{-n_x}\alpha_t^{(o)}(\phi_o) \right] \left[w^{-j}\beta_{j-t}^{(o)}(\phi_o) \right],$$

with $i \in \mathbb{I}_1^{n_x}, j \in \mathbb{I}_0^{n_x}$, $n_a = n_b = n_x$, and $\psi(k)$ a new scheduling vector, function of $[\phi_o(k), \dots, \phi_o(k-n_x)]$, such that each $a_i(\psi)$ and $b_j(\psi)$ has affine dependency on ψ. Here $\mathbb{I}_s^t = \{s, s+1, \dots, t\} \subset \mathbb{Z}$ is the index set.

Theorem 19: (Equivalence transformation) For any given completely observable and affine $\mathfrak{R}_{SS}(\mathcal{S}, \phi)$, there exists an affine, unique, and equivalent $\mathfrak{R}_{I/O}(\mathcal{S}, \psi)$ in the form of (1) with coefficients given by (19) and (20). For any given affine $\mathfrak{R}_{I/O}(\mathcal{S}, \psi)$, there exists an affine, unique, equivalent, and minimal $\mathfrak{R}_{SS}^{\mathcal{O}}(\mathcal{S}, \phi_o)$ with $n_x = n_a$ and coefficients

$$\alpha_i^{(o)}(\phi_o) = w^{n_x} a_i(\psi), \qquad (21)$$
$$\beta_j^{(o)}(\phi_o) = w^j b_j(\psi) - \sum_{t=1}^{j} \left[w^j a_t(\psi) \right] \beta_{j-t}^{(o)}(\phi_o), \quad (22)$$

where $i \in \mathbb{I}_1^{n_a}, j \in \mathbb{I}_0^{n_a}$, all dependence is affine, and if $j > n_b$, then $b_j(\psi) \equiv 0$. $\square$

B. Reachability canonical form

Assume that $\mathfrak{R}_{SS}(\mathcal{S}, \phi)$ is completely reachable. We introduce a new state basis, using

$$T_r(\bar{\phi}_-(k)) = \mathcal{R}_{n_x}(\bar{\phi}_-(k)), \qquad (23)$$

with a new state variable $\bar{x}_r(k)$ such that

$$x(k) = T_r(\bar{\phi}_-(k-1)) \bar{x}_r(k), \qquad (24)$$
$$x(k+1) = T_r(\bar{\phi}_-(k)) \bar{x}_r(k+1). \qquad (25)$$

Again topological equivalence is pursued through T_r.

Lemma 20: The reachability transformation $T_r(\bar{\phi}_-(\cdot))$ is a Lyapunov transformation. $\square$

Proof: Analogous to the proof of Lemma 17. ∎

Since $T_r(\bar{\phi}_-)$ implies topological equivalence, therefore the state transformation will not change the I/O behavior of the system. Proof of the I/O equivalence can be deduced from [7] and the transformed matrices are given as [18]:

$$A_r(\phi_r(k)) = T_r^{-1}(\bar{\phi}_-(k)) A(\phi(k)) T_r(\bar{\phi}_-(k-1)) =$$
$$= \begin{bmatrix} 0 & \dots & 0 & -\alpha_{n_x}^{(r)}(\phi_r(k)) \\ 1 & \ddots & \vdots & -\alpha_{n_x-1}^{(r)}(\phi_r(k)) \\ \vdots & \ddots & 0 & \vdots \\ 0 & \dots & 1 & -\alpha_1^{(r)}(\phi_r(k)) \end{bmatrix},$$

$$B_r(\phi_r(k)) = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T,$$
$$C_r(\phi_r(k)) = C(\phi(k)) T_r(\bar{\phi}_-(k-1)) =$$
$$= \left[\beta_1^{(r)}(\phi_r(k)), \dots, \beta_{n_x}^{(r)}(\phi_r(k)) \right],$$
$$D_r(\phi_r(k)) = D(\phi(k)) = \beta_0^{(r)}(\phi_r(k)).$$

where $\phi_r(k)$ is a new scheduling vector, function of $[\phi(k), \dots, \phi(k-n_x)]$, such that $\{\alpha_i^{(o)}(\phi_r(k))\}_{i=1}^{n_x}$ and $\{\beta_j^{(o)}(\phi_r(k))\}_{j=0}^{n_x}$ has affine dependency on $\phi_r(k)$. Then,

$$\mathfrak{R}_{SS}^{\mathcal{R}}(\mathcal{S}, \phi_r) = \left[\begin{array}{c|c} A_r(\phi_r) & B_r(\phi_r) \\ \hline C_r(\phi_r) & D_r(\phi_r) \end{array} \right], \qquad (26)$$

is called the *reachability canonical state space representation* of $\mathcal{S}$. For a proof of the results provided by the above given matrix operations see [19].

Theorem 21: $\mathfrak{R}_{SS}^{\mathcal{R}}(\mathcal{S}, \phi_r)$ is completely reachable [19]. $\square$
Then, because of the structure of $\mathfrak{R}_{SS}^{\mathcal{R}}(\mathcal{S}, \phi_r)$,

$$\bar{x}_1^{(r)} = q^{-1}u - w^{-1}\alpha_{n_x}^{(r)}(\phi_r) q^{-1}\bar{x}_{n_x}^{(r)},$$
$$\bar{x}_2^{(r)} = q^{-2}u - w^{-2}\alpha_{n_x}^{(r)}(\phi_r) q^{-2}\bar{x}_{n_x}^{(r)} -$$
$$- w^{-1}\alpha_{n_x-1}^{(r)}(\phi_r) q^{-1}\bar{x}_{n_x}^{(r)},$$
$$\vdots$$
$$\bar{x}_{n_x}^{(r)} = q^{-n_x}u - \sum_{l=0}^{n_x-1} w^{l-n_x}\alpha_{n_x-l}^{(r)}(\phi_r) q^{l-n_x}\bar{x}_{n_x}^{(r)},$$

from which it follows that

$$\bar{x}^{(r)} = Q_r u - W_r(\phi_r) Q_r \bar{x}_{n_x}^{(r)}, \qquad (27)$$

where

$$Q_r = \begin{bmatrix} q^{-1} & q^{-2} & \dots & q^{-n_x} \end{bmatrix}^T,$$

$$W_r(\phi_r) = \begin{bmatrix} w^{-1}\alpha_{n_x}^{(r)}(\phi_r) & 0 & & \\ w^{-1}\alpha_{n_x-1}^{(r)}(\phi_r) & w^{-2}\alpha_{n_x}^{(r)}(\phi_r) & & \\ \vdots & & \ddots & \\ w^{-1}\alpha_1^{(r)}(\phi_r) & \dots & & \\ & \dots & & 0 \\ & \ddots & & \vdots \\ & \ddots & & 0 \\ w^{1-n_x}\alpha_{n_x-1}^{(r)}(\phi_r) & w^{-n_x}\alpha_{n_x}^{(r)}(\phi_r) \end{bmatrix}.$$

Then it is possible to write the outputs as

$$y = \check{C}_r\left(\phi_r\right)\bar{x}^{(r)} + \beta_{n_x}^{(r)}\left(\phi_r\right)\bar{x}_{n_x}^{(r)} + \beta_0^{(r)}\left(\phi_r\right)u, \quad (28)$$

where

$$\check{C}_r\left(\phi_r\right) = \left[\begin{array}{cccc} \beta_1^{(r)}\left(\phi_r\right) & \dots & \beta_{n_x-1}^{(r)}\left(\phi_r\right) & 0 \end{array}\right].$$

After some algebraic manipulation based on the non-singularity of $C_r\left(\phi_r\right)$ and substitution by (27), the equation (28) can be written in the form of (1) where

$$a_i\left(\psi\right) = \left[\omega^{-1}\alpha_i^{(r)}\left(\phi_r\right)\right]\frac{\beta_{n_x-i+1}^{(r)}\left(\phi_r\right)}{\left[\omega^{-i}\beta_{n_x-i+1}^{(r)}\left(\phi_r\right)\right]}, \quad (29)$$

$$b_j\left(\psi\right) = \beta_j^{(r)}\left(\phi_r\right) + \sum_{t=1}^{j}\alpha_t\left(\phi_r\right)\left[w^{-t}\beta_{j-t}^{(r)}\left(\phi_r\right)\right] \quad (30)$$

with $i \in \mathbb{I}_1^{n_x}, j \in \mathbb{I}_0^{n_x}$, $n_a = n_b = n_x$, and $\psi(k)$ a new scheduling vector, function of $\left[\phi_r\left(k\right), \dots, \phi_r\left(k - n_x\right)\right]$, such that each $a_i\left(\psi\right)$ and $b_j\left(\psi\right)$ has affine dependency on ψ.

Remark 22: The division in (29) is always accomplishable if none of the elements of $\beta^{(r)}(\phi_r)$ is zero which is guaranteed if the system is well defined. If $\beta^{(r)}\left(\phi_r\right)$ can have zero values, then for the specific time instance of their occurrence the (29) and (30) formulas change. As the formulas in these cases are completely case dependent (index and number of $\beta^{(r)}\left(\phi_r(k)\right) = 0$ elements), no general derivation is possible.

Theorem 23: (Equivalence transformation) For any given completely reachable and affine $\mathfrak{R}_{SS}\left(\mathcal{S}, \phi\right)$, there exists an unique, equivalent, and affine $\mathfrak{R}_{I/O}\left(\mathcal{S}, \psi\right)$ in the form of (1) and with coefficients given by (29) and (30). For any given affine $\mathfrak{R}_{I/O}\left(\mathcal{S}, \psi\right)$, there exists an affine unique, equivalent, and minimal $\mathfrak{R}_{SS}^{\mathcal{R}}\left(\mathcal{S}, \phi_r\right)$ with $n_x = n_a$ and coefficients

$$\alpha_i^{(r)}\left(\phi_r\right) = \left[wa_i\left(\psi\right)\right]\frac{\left[w^{-(i-1)}\beta_{n_x-i+1}^{(r)}\left(\phi_r\right)\right]}{\left[w\beta_{n_x-i+1}^{(r)}\left(\phi_r\right)\right]}, \quad (31)$$

$$\beta_j^{(r)}\left(\phi_r\right) = b_j\left(\psi\right) - \sum_{t=1}^{j}a_t\left(\psi\right)\left[w^{-t}\beta_{j-t}^{(r)}\left(\phi_r\right)\right], \quad (32)$$

where $i \in \mathbb{I}_1^{n_a}, j \in \mathbb{I}_0^{n_a}$, all dependence is affine, and if $j > n_b$, then $b_j\left(\psi\right) \equiv 0$. $\square$

Recall from LTI theory, that the observability and reachability canonical form can be given in an other so called companion form [23]. This also holds for the LPV case, but it is omitted here because of space limitations.

It is also important to point out, that for a given SISO LTI system $\mathcal{F}$, it holds that

$$\mathfrak{R}_{SS}\left(\mathcal{F}\right) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right],$$

$$\mathfrak{R}_{SS}^T\left(\mathcal{F}\right) = \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D \end{array}\right],$$

have equivalent I/O behaviors [24]. Here $\mathfrak{R}_{SS}^T\left(\mathcal{F}\right)$ is called the adjoint of $\mathfrak{R}_{SS}\left(\mathcal{F}\right)$. However, in the LPV case, the I/O behavior of $\mathfrak{R}_{SS}\left(\mathcal{S}, \phi\right)$ is strictly not equal to $\mathfrak{R}_{SS}^T\left(\mathcal{S}, \phi\right)$.

Equivalence only holds if $\mathfrak{R}_{SS}^T\left(\mathcal{S}, \phi\right)$ is simulated backward in time [7]. This means that for the LPV-SS case

$$\mathfrak{R}_{SS}^{\mathcal{O}}\left(\mathcal{S}, \phi_o\right) \neq \left[\mathfrak{R}_{SS}^{\mathcal{R}}\left(\mathcal{S}, \phi_r\right)\right]^T, \quad (33)$$

$$\mathfrak{R}_{SS}^{\mathcal{R}}\left(\mathcal{S}, \phi_r\right) \neq \left[\mathfrak{R}_{SS}^{\mathcal{O}}\left(\mathcal{S}, \phi_o\right)\right]^T, \quad (34)$$

as due to the forward time flow, it is not guaranteed for this transposed form that it is observable, controllable or has any connection with the non-transposed form between the local points (see Section IV). Only the canonical forms, defined before, guarantee validity and observability or reachability for any variation of ϕ globally [17], [18].

III. IDENTIFICATION OF LPV SYSTEMS

In the previous section the relation between LPV models in the I/O and SS domains was established. Through the introduced equivalence transformation, the I/O behavior of the system is preserved during a conversion between the domains, eliminating problems of the propagation gain and phase lags (Problem 5 in Section II). Moreover, through the state transformations used to achieve canonical forms the problems of unknown state basis (Problem 6 in Section II) also do not apply. However, the price to be paid for the *equivalence* is in the increased complexity of the resulting models (growing dimension of $\mathbb{P}$).

The consequences are threefold. First, current identification methods are only focusing on LPV systems that have static dependence on $p(k)$. Therefore, if the physical system $\mathcal{S}$ to be identified has an underlying $\mathfrak{R}_{SS}(\mathcal{S}, p)$ or $\mathfrak{R}_{I/O}(\mathcal{S}, p)$ model completely defining the I/O behavior of $\mathcal{S}$, then only those methods will be capable of efficient LPV system identification of $\mathcal{S}$ with static $p(k)$ dependence, which are based on the specific domain of $\mathcal{S}$. Methods of the other domain will result only in a poor approximation in terms of output error, as they are not able to cope with the time-lagged dependency on p, which is needed to describe the equivalent model in their domain (see Section II). This means, that for example sub-space methods will never be able to fully identify an LPV-I/O model if they operate with static parameter dependence only. On the other hand, if the system has an underlying $\mathfrak{R}_{SS}(\mathcal{S}, p)$ model, then the identification of a $\mathfrak{R}_{I/O}(\mathcal{S}, p)$ is impossible. Moreover, even if the equivalent $\mathfrak{R}_{SS}(\mathcal{S}, \psi)$ is given, then finding back the specific unknown state-basis where the dependence on $p(k)$ is static is as hard as the identification problem itself.

The other consequence is that the interpolation of local models to reproduce a global LPV or NL behavior is a very delicate operation. Basically after identification of the local models, the interpolation on the coefficients must be carried out in the domain where the produced model is going to be used. If the global model was produced in an undesirable domain, then transformation to the other is still possible through the presented equivalence transformations but with extreme caution on the increasing scheduling parameter dimensions and the possibly increasing model variance. Moreover, for systems that have a well fitting first principle model described by a nonlinear difference equation, the local

observations of the parameters must be interpolated in the I/O domain, otherwise, even if the interpolation function is 100% correct, the locally equivalent LPV-I/O model of the system will not be able to reproduce the same output. This phenomena is true vice-versa.

The third consequence is purely related to the usability of the presented equivalence transformations. As it can be observed, causality of the model with respect to $p(k)$ can be seriously violated during the conversions between the domains. Therefore, the reachability canonical form should be distinguishably used in any conversions of SS to I/O systems as it always guarantees causality of the equivalent model. Also by converting I/O models to SS systems the reachability canonical form only requires one sample advance of the scheduling parameter in comparison with the required n_x sample advance of the observability form.

These consequences suggest, that parting the LPV-I/O and LPV-SS identifiable systems is strongly needed to improve the efficiency of the current identification methods and enable the application of the produced models for control.

IV. EXAMPLES

In the following some simple examples are going to be presented to visualize the theorems of Section II.

Example 24: (Canonical forms) In the first example consider the following LPV system

$$\mathfrak{R}_{SS}(\mathcal{S}_1, p) = \left[\begin{array}{cc|c} p(k) & 1 & 1 \\ 0 & p(k) & p(k) \\ \hline 1 & p(k)^{-1} & p(k) \end{array} \right]$$

with $\mathbb{P} = \left[\begin{array}{cc} \frac{1}{4} & \frac{3}{4} \end{array} \right]$. It is trivial that the above system is affine for $\phi(k) = \left[p(k), p^{-1}(k) \right]$ and completely controllable and reachable for any $p(k)$ sequence (the rank of $\mathcal{O}_{n_x}\left(\bar{\phi}_+\right)$ and $\mathcal{R}_{n_x}\left(\bar{\phi}_-\right)$ never drops below 2 on $\mathbb{P}$). By calculating the $\mathfrak{R}_{SS}^{\mathcal{O}}(\mathcal{S}_1, \phi_o)$ observability and $\mathfrak{R}_{SS}^{\mathcal{R}}(\mathcal{S}_1, \phi_r)$ reachability canonical forms defined in Section II, one can conclude that

$$\begin{aligned}
\alpha_1^{(o)}(\phi_o(k)) &= [\phi_o(k)]_1 = p(k)\frac{p^2(k+1)}{p(k+2)}, \\
\alpha_2^{(o)}(\phi_o(k)) &= [\phi_o(k)]_2 = p(k+1) + \frac{p^2(k+1)}{p(k+2)}, \\
\beta_1^{(o)}(\phi_o(k)) &= [\phi_o(k)]_3 = 1 + \frac{p(k)}{p(k+1)}, \\
\beta_2^{(o)}(\phi_o(k)) &= [\phi_o(k)]_4 = \frac{p(k+1)p(k)}{p(k+2)} + p(k) + \\
&\quad + p(k+1),
\end{aligned}$$

and

$$\begin{aligned}
\alpha_1^{(r)}(\phi_r(k)) &= [\phi_r(k)]_1 = p(k-1) + p(k-2), \\
\alpha_2^{(r)}(\phi_r(k)) &= [\phi_r(k)]_2 = p^2(k-1), \\
\beta_1^{(r)}(\phi_r(k)) &= [\phi_r(k)]_3 = 1 + \frac{p(k-1)}{p(k)}, \\
\beta_2^{(r)}(\phi_r(k)) &= [\phi_r(k)]_4 = \frac{p(k-1)p(k-2)}{p(k)} + \\
&\quad + p(k-2) + p(k-1).
\end{aligned}$$

There are similarities between the two set of coefficients, but it remains obvious that the two LPV systems are not the transpose of each other. However, if we freeze $p(k) = p \in \mathbb{P}$ (applying constant scheduling), then the two representations become at the constant p point to be each others transpose and therefore local equivalence is trivial. Now compute what would result from putting the original system into reachability and observability canonical forms at every local point based on LTI theory. This very intuitive but wrong approach produces the following LPV systems:

$$\begin{aligned}
\check{\mathfrak{R}}_{SS}^{\mathcal{O}}(\mathcal{S}_1, p) &= \left[\begin{array}{cc|c} 0 & 1 & 2 \\ -p^2(k) & 2p(k) & 3p(k) \\ \hline 1 & 0 & p(k) \end{array} \right] \\
\check{\mathfrak{R}}_{SS}^{\mathcal{R}}(\mathcal{S}_1, p) &= \left(\check{\mathfrak{R}}_{SS}^{\mathcal{O}}(\mathcal{S}_1, p) \right)^T
\end{aligned}$$

Generally in the literature some follow this approach [6]. It is important to note that the two sets of LPV systems coincide for constant parameter trajectories, but they are unequal globally. To show this phenomenon, try what the outputs of these system are for $u(k) = \sin\left(\frac{1}{4}k + \frac{\pi}{6}\right)$, $p(k) = \frac{1}{2} + \frac{1}{4}\sin\left(\frac{1}{4}k + \frac{\pi}{2}\right)$ and assuming initial rest at $k = 0$. Here k is in radial for the computation of the $\sin(.)$.

As it can be seen in Figure 1, the two canonical forms $\mathfrak{R}_{SS}^{\mathcal{O}}(\mathcal{S}_1, \phi_o)$ and $\mathfrak{R}_{SS}^{\mathcal{R}}(\mathcal{S}_1, \phi_r)$ are completely reproducing the original output with zero error. However, $\check{\mathfrak{R}}_{SS}^{\mathcal{O}}(\mathcal{S}_1, p)$ and $\check{\mathfrak{R}}_{SS}^{\mathcal{R}}(\mathcal{S}_1, p)$ have a relatively huge representation error in the magnitude of 35%, which mainly comes from a parameter dependent phase and gain lag with respect to y.

Example 25: (SS form of I/O representations) Now we investigate the representation of an LPV-I/O model in a SS form. The following $\mathfrak{R}_{I/O}(\mathcal{S}_2, p)$ of an LPV system $\mathcal{S}_2$ is defined

$$\begin{aligned}
y(k) &= -p(k)y(k-1) - p(k)y(k-2) \quad (35) \\
&\quad + p(k)u(k-1),
\end{aligned}$$

with $\mathbb{P} = \left[\begin{array}{cc} \frac{1}{4} & \frac{3}{4} \end{array} \right]$. It is trivial that $\mathfrak{R}_{I/O}(\mathcal{S}_2, p)$ has affine dependency on $\psi(k) = p(k)$. Now develop the SS canonical representation of this I/O system as it is given in Section II. The resulting $\mathfrak{R}_{SS}^{\mathcal{O}}(\mathcal{S}_2, \phi_o)$ and $\mathfrak{R}_{SS}^{\mathcal{R}}(\mathcal{S}_2, \phi_r)$ are defined by

$$\begin{aligned}
\alpha_i^{(o)}(\phi_o(k)) &= [\phi_o(k)]_1 = p(k+2), \quad i = 1, 2 \\
\beta_1^{(o)}(\phi_o(k)) &= [\phi_o(k)]_2 = p(k+1), \\
\beta_2^{(o)}(\phi_o(k)) &= [\phi_o(k)]_3 = -p(k+2)p(k+1),
\end{aligned}$$

and

$$\begin{aligned}
\alpha_i^{(r)}(\phi_r(k)) &= [\phi_r(k)]_1 = p(k-1), \quad i = 1, 2 \\
\beta_1^{(r)}(\phi_r(k)) &= [\phi_r(k)]_2 = p(k), \\
\beta_2^{(r)}(\phi_r(k)) &= [\phi_r(k)]_3 = -p(k)p(k-1).
\end{aligned}$$

The I/O system can also be converted to a reachable and observable canonical form locally at every scheduling point based on the LTI theory. This would be the same as what is used in the literature by obtaining I/O models, then
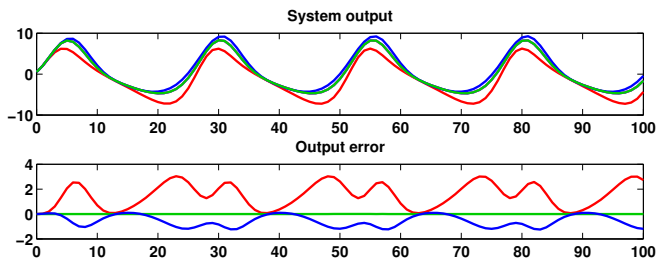
Fig. 1. Comparison of canonical representations of the LPV system $\mathcal{S}_1$. $\breve{\mathfrak{R}}_{SS}^{\mathcal{O}}(\mathcal{S}_1, p)$ (blue), $\breve{\mathfrak{R}}_{SS}^{\mathcal{R}}(\mathcal{S}_1, p)$ (red), $\mathfrak{R}_{SS}^{\mathcal{O}}(\mathcal{S}_1, \phi_o)$ (green), $\mathfrak{R}_{SS}^{\mathcal{R}}(\mathcal{S}_1, \phi_r)$ (identical to green), $\mathfrak{R}_{SS}(\mathcal{S}_1, p)$ (identical to green).

Fig. 2. Comparison of canonical representations of the LPV system $\mathcal{S}_2$. $\breve{\mathfrak{R}}_{SS}^{\mathcal{O}}(\mathcal{S}_2, p)$ (blue), $\breve{\mathfrak{R}}_{SS}^{\mathcal{R}}(\mathcal{S}_2, p)$ (red), $\mathfrak{R}_{SS}^{\mathcal{O}}(\mathcal{S}_2, \phi_o)$ (green), $\mathfrak{R}_{SS}^{\mathcal{R}}(\mathcal{S}_2, \phi_r)$ (identical to green), $\mathfrak{R}_{SS}(\mathcal{S}_2, p)$ (identical to green).
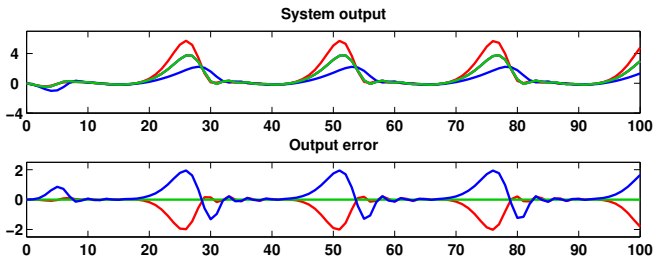
interpolating between the coefficients, and then putting them into a controllability form, see [15]:

$$
\begin{aligned}
\breve{\mathfrak{R}}_{SS}^{\mathcal{O}}(\mathcal{S}_2, p) &= \left[\begin{array}{cc|c} 0 & 1 & p(k) \\ -p(k) & -p(k) & -p^2(k) \\ \hline 1 & 0 & 0 \end{array} \right] \\
\breve{\mathfrak{R}}_{SS}^{\mathcal{R}}(\mathcal{S}_2, p) &= \left(\breve{\mathfrak{R}}_{SS}^{\mathcal{O}}(\mathcal{S}_2, p) \right)^T
\end{aligned}
$$

Again, this is an intuitive approach, but completely destroys the structure. To prove it, consider the similar p and u functions as $u(k) = \sin\left(\frac{1}{4}k\right)$, $p(k) = \frac{1}{2} + \frac{1}{4}\cos\left(\frac{1}{4}k\right) - \sin\left(\frac{1}{4}k + \frac{\pi}{2}\right)$. As it can bee seen on Figure 2, the pragmatic approach produces 35% difference while the transformation proposed in Section II retains the correct output.

V. Conclusion

In this paper an investigation of canonical SS and I/O representations of LPV systems was presented, exploring the connections and the differences of the two representations. It was also clearly pictured how interpolation between local models in one of the (SS or I/O) domains relate to the interpolation of the locally equivalent models in the other domain. Through affine equivalent representations, direct equivalence transformations between general LPV-SS and LPV-I/O systems were introduced. Due to the dynamic nature of these transformations, conclusions were drawn about the identifiability of LPV systems based on static dependence on the scheduling parameter. Based on this conclusion, one can point out that future research in the area must make a distinction between models that are identifiable in the SS or in the I/O domain, because it might happen, that there exists no LPV-SS or I/O model with equivalent I/O behavior

depending on the non-shifted sequence of p, reducing greatly the effectiveness of all methods and approaches of the identification field.

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