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PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE TECHNISCHE WETENSCHAPPEN AAN DE TECHNISCHE HOGESCHOOL EINDHOVEN, OP GEZAG VAN DE RECTOR MAGNIFICUS, PROF. DR. IR. G. VOSSERS, VOOR EEN COMMISSIE AANGEWEZEN DOOR HET COLLEGE VAN DEKANEN IN HET OPENBAAR TE VERDEDIGEN OP VRIJDAG 7 NOVEMBER 1975 TE 16.00 UUR.

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EN

LECTOR DR.IR.P.P.J.M. SCHRAM

dedicated to the memory of

L.V.K,

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SUMMARY

1. Introduction

This dissertation concerns itself with the nature of turbulence in a medium with large electrical conductivity. By and large, the matter in the universe, except in peculiar conditions like on the surface of the earth, is in an ionised state and as we see and interpret it, the existence of large scale magnetic fields and their impact on various dynamical phenomena in cosmic scales are a verified experimental fact. The question of the origin of these magnetic fields has been a matter for considerable scientific speculation and curiosity, ever since a systematic analysis of the astrophysical phenomena was started. (Cf. the review articles by L. Mestel[1] and E.N. Parker[2].)

Representing the various trends in cosmology and astrophysics, there have been two distinctly different approaches to the explanation of these fields, from the very start. One view which is closely related to the Big Bang Theory, tries to produce the magnetic field - almost simultaneously with the Big Bang and does not make any attempts to analyse its origin. This leaves the problem of the initial magnetic field to be explained by the cosmologist. Though as a point of view it is not disputable, it is aesthetically not appealing, since it tries to evade the question rather than answer it. This is generally referred to as "the Fossil Theory".

An alternate view is one of "Dynamo Action". A very general definition of dynamo action is the transformation of kinetic energy of mass motion into electrodynamic and consequently into magnetic energy. The equations of motion for the fluid and the magnetic field in a highly conducting medium, under the simplifying assumption of incompressibility can be written

$$\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u}\cdot\underset{\sim}{\nabla}\, \underset{\sim}{u} = -\underset{\sim}{\nabla} p + \nu \nabla^2 \underset{\sim}{u} + (\underset{\sim}{\nabla} \times \underset{\sim}{b}) \times \underset{\sim}{b} + \underset{\sim}{F}$$

$$\frac{\partial \underset{\sim}{b}}{\partial t} + \underset{\sim}{u}\cdot\underset{\sim}{\nabla}\, \underset{\sim}{b} = \underset{\sim}{b}\cdot\nabla \underset{\sim}{u} + \lambda \nabla^2 \underset{\sim}{b}$$

$$\nabla\cdot \underset{\sim}{u} = 0 \qquad \nabla\cdot \underset{\sim}{b} = 0 \qquad 1.1$$

where $\underset{\sim}{u}(\underset{\sim}{x}, t)$ is the local hydrodynamical velocity of the fluid and $b\,(4\pi\mu)^{1/2}$ is the local magnetic induction μ = magnetic permeability, ν = kinematic viscosity and λ = magnetic diffusivity of the medium, F (x,t) refers to all other types of body force which are responsible for the velocity field.

The question boils down to constructing a pattern of motions which can support a pattern of magnetic fields. It is a matter of ultimate consistency, to feed back the generated fields into the equations for balance of momentum and check that at the steady-state, the Lorentz force balances the sum of all other forces.

In a celebrated theorem[3], Cowling proved the impossibility of having a stationary axisymmetric homogeneous dynamo, supported by fluid motions. This had a considerable influence on thinking on Stellar Dynamos, ever since. Bullard and Gellmann[4], Herzenberg[5] and Backus[6] tried to look at this problem by relaxing the conditions of stationarity, homogeneity and axial symmetry, one at a time.

But in all these questions, the non-linear dynamical equations for the velocity and magnetic field were in spirit treated in a quasi-linear way. Further, the basic "seed" field, with which the system starts in a non-stationary situation was never replenished and when its sources were switched off, the whole field structure which depended on it as an initial value, died down too. This is an inherent difficulty, with all approaches in which the so-called Dynamo equation for the magnetic field is treated as an initial value problem with a given velocity field which is independent of the magnetic field. Because of the linearity of this equation, the formal solution to this has the structure in time of a Green's operator subject to the boundary values and the velocity field. Attempts to get rid of this difficulty will have to borrow on some non-linear aspects of the problem. (This seems to be a necessary requirement, irrespective of the level of looking at this question: whether in a strictly stationary state or a statistically-steady state. We will have an occasion to come back to this, when we discuss our results.)

The question of the amplification of the "seed" field, using the features of the turbulence in the medium was first considered by Batchelor[7] and Biermann and Schlüter[8]. Their analysis depended considerably on the accepted understanding of the nature of turbulence in hydrodynamics, based on the ideas of Kolmogorov at that time. In Section 2 we will review the ideas of Kolmogorov and the developments by Batchelor and Biermann and Schlüter, in Section 3.

This is where the analysis reported in this dissertation started. We took the point of view that, since the basic feed back to the seed field from hydrodynamical sources will have to depend on non-linear analysis, thus it is necessary to consider a dynamical approach to the evolution of turbulence in an electrically conducting medium, in the presence of electrical currents. We started our analysis using the Direct Interaction Approximation of Kraichnan[9], because this was (and still is) the only theory which is adequate to fit the needs of the analysis. In Section 4, we will review some of the salient points of this theory. Since a number of reviews exist at the moment (10, 11, 12) we will restrict our analysis of the theory considerably to points, which concern the extension to hydromagnetics of these ideas. In Section 5 we will review some of the salient alterations, in the hydromagnetic context and our method of attack. Section 6 will review some of the results of the analysis, with a special stress on limitations. In Section 7 we will try to bring the situation in the subject upto date, with respect to contemporary literature and attempt an evaluation of the prospects for the future in this field.

2. Kolmogorov's Hypotheses.

To develop the ideas of Kolmogorov (13a,b) we will recast the hydrodynamic equations, in a Fourier-transformed representation as

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) u_i(\underline{k},t) = M_{ijm}(\underline{k}) \sum_{\underline{k}=\underline{p}+\underline{q}} u_j(\underline{p}) u_m(\underline{q}) + f_i(\underline{k})$$

where

$$M_{ijm}(\underline{k}) = -\frac{i}{2}\left[k_m P_{ij}(\underline{k}) + k_j P_{im}(\underline{k})\right]$$

where

$$P_{ij}(\underline{k}) = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \qquad 2.1$$

In the case of homogeneous isotropic turbulence, since all mean motions vanish, the local pressure fluctuations are all dynamically determined by the Reynolds stresses and so these are eliminated, in writing 2 in terms of the velocity fluctuations. This is standard practice in turbulence theory (See for e.g. Leslie[12] page 4)

This model representation enables one to visualise the the effect of the non-linear term in the equation for the mode with wave number $\underline{k}$ in terms of dynamical interaction between modes with numbers $\underline{p}$ and $\underline{q}$ such that $\underline{k} = \underline{p} + \underline{q}$. Thus energy is being transferred from mode to mode because of this interaction.

If in equation 2.1, we put $\nu = 0$ and $f_i(k) \equiv 0$ and calculate

$$\frac{1}{2}\frac{d}{dt} \sum_{k} u_i^*(k,t)\, u_i(k:t) = \sum_{k} M_{ijm}(k) \sum_{k=p+q} u_i^*(k)\, u_j(p)\, u_m(q)$$

we can easily see using the properties of $M_{ijm}(k)$ with respect to symmetry that the right hand side vanishes. This is the so-called "conservative property of the non-linear interaction", which is a prime mover in turbulence theory. Thus the non-linear term neither creates nor destroys energy.

(This, one could have checked directly from the original equations (1.1) as well). The non-linear term only shuffles energy around from mode to mode. The interaction between modes is persistent in time. We see here itself a divergence from traditional concepts of collisions in statistical mechanics.

But if one can grade the modes according to spatial size and see whether there exists a region of 'mode space' in which the energy input from macroscopic boundary dependent sources and the viscous drain into microscopic motion of the fluid can be neglected, it may be useful. This region of modes would be completely dynamically determined by the non-linear shuffling between modes. One can make this requirement rigorous by asserting that for this region the net input of energy from either macroscopic sources or from modes from other regions exactly balances the net output of energy to other regions of mode space and by viscous dissipation. Such a situation can produce a state of statistical equilibrium amongst the modes.

For this concept to be really useful, one will have to see what it means in terms of the observational features of turbulence in the first place. The observational part can best be summarised in a rhyme due to L.F.Richardson:
"Big whorls have little whorls, which feed on their velocity;
Little whorls have smaller whorls, and so on unto viscosity."

If one looks at a turbulent fluid, suddenly a whirl makes its appearance and as one follows its path through the fluid, it seems to get smaller and finally disappear. This is qualitatively what one calls an "Eddy" in turbulence. It is a localised disturbance in the fluid, which propagates through the fluid and in so doing ultimately disappears. It is important to realise that localised wave packets of disturbances are quite different in character from the ordered pattern of wave motions, which are typified by the Fourier modes in equation (2.1).

Traditional stability analysis of hydrodynamical flows and the theoretical insight which one obtains into the nature of the instability have all been carried out in the normal mode representation for the particular geometry of flows considered. The general underlying moral that one learns from this can be summarised thus:

When the basic primary flow becomes unstable for large Reynolds number typified by the scale of the flow and its velocity, superposed on this flow grow a secondary pattern of motions typified by a size of the order of the most unstable disturbance on the primary flow. Conceivably, this secondary motion grows to a finite intensity, which depends on the amount of "instability energy" available from the primary flow. If this energy is sufficiently large, so that the typical Reynolds number for this secondary motion is large enough this pattern of motions also becomes unstable. On this grows a tertiary pattern of motions and so on. Thus in a fully turbulent fluid, there exist a complex superposition of motions of various scales linked to a previous (or a larger) scale for energy input and to a later (or a smaller) scale for energy drain. Further, in the limit of homogeneous isotropic turbulence, each of these secondary and higher order pattern of motions will have to be replaced by a continuous range of wave numbers, rather than a discrete set.

This dynamical information which one pieces together from stability analysis cannot still be effectively used to decide what happened to a whirl or an eddy in a straight forward fashion.

Kolmogorov's analysis is a subtle fusing together of these stability results with the general considerations of mode space, which we put forward earlier in this section. Since a clear review of Kolmogorov's arguments seems necessary for the hydromagnetic situation, we shall here quote part of his arguments in full:

The first observation of Kolmogorov hinges on the fact that the concept of isotropy as introduced by Taylor[15] and later developed by others cannot be used to separate various regions of mode space, as sequentially connected (with a constant stationary energy flow from region to region). To do this one will have to identify eddies or vortices, which are localised quantities in co-ordinate space, with modes which are collective coordinates.

To make this connection even closer, Kolmogorov argues that to remove the systematic larger scale motion away, when one considers motion of a certain scale, one should restrict oneself to the differential velocity between two neighbouring points separated by a distance characteristic of the same scale. This leads to the concept of Local Isotropy: that the probability distribution of these differential velocities is invariant wiht respect to translation, rotation and reflection of the system of coordinates.
To quote Kolmogorov:

" We shall denote by

$$\mathcal{U}_\alpha(P) = \mathcal{U}_\alpha(x_1, x_2, x_3, t) \quad : \quad \alpha = 1, 2, 3$$

the components of velocity at the moment t, at the point with cartesian coordinates x_1 , x_2 , x_3 ,.......Introduce in the four dimensional space (x_1 , x_2 , x_3, t) new coordinates

$$y_\alpha = x_\alpha - x_\alpha^{(0)} - \mathcal{U}_\alpha(P^{(0)}).(t - t^{(0)}) \qquad 1$$

$$s = t - t^{(0)}$$

where

$$P^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, t^{(0)})$$

is a certain fixed point in the four dimensional domain G....
The velocity components in the coordinates are

$$w_\alpha(P) = u_\alpha(P) - u_\alpha(P^{(0)})$$

Suppose for some fixed values of u_α ($P^{(0)}$) the points $p^{(k)}$ k = 1,2,........n having in the coordinate system (1), the coordinates $y_\alpha^{(k)}$ and $s^{(k)}$, are situated in the domain G. Then we may define a 3n-dimensional distribution law of probabilities F_n for the quantities

$$w_\alpha^{(k)} = w_\alpha(P^{(k)})\ ;\ \alpha=1,2,3\ ;\ k=1,2\cdots n$$

where $u_\alpha(P^{(0)}) = u_\alpha^{(0)}$ are given. Generally speaking, the distribution law F_n depends on the parameters

$$x_\alpha^{(0)},\ t^{(0)},\ u_\alpha^{(0)},\ y_\alpha^{(k)},\ s^{(k)}$$

Definition 1. The turbulence is called locally homogeneous in the domain G, if for every fixed n, $y_\alpha^{(k)}$ and $s^{(k)}$, the distribution law F_n is independent of $x_\alpha^{(0)}$, $t^{(0)}$ and $u_\alpha^{(0)}$, as long as the points p(k) are all situated in G.

Definition 2. The turbulence is called locally isotropic in the domain G, if it is homogeneous and if, besides, the distribution laws mentioned in definition 1, are invariant with respect to rotations and reflections of the original system of coordinate axes (x_1, x_2, x_3).

In comparison with the notion of isotropic turbulence, introduced by Taylor, this definition of locally isotropic turbulence is narrower in the sense that one demands the independence of the distribution law F_n from $t^{(o)}$, i.e. steadiness in time, and is wider in the sense that restrictions are imposed only on the distribution laws of differences of velocities and not of the velocities themselves."

At this point, in his paper, Kolmogorov digresses to offer some general considerations, in favour of the hypothesis, in a footnote:

" For very large R (Reynolds Number) the turbulent flow may be thought of in the following way: on the averaged flow characterised by the mathematical expectations $\bar{u}_\alpha$ are superposed the pulsations of the first order consisting of disorderly displacements of separate fluid volumes, one with respect to another of diameters of the order of magnitude $\ell^{(1)} = \ell$ (where ℓ is the Prandtl's mixing length); the order of magnitude of these relative velocities, we denote by $v^{(1)}$. The pulsations of the first order are for very large R, in their turn unsteady and on them are superposed the pulsations of the second order with mixing length $\ell^{(2)} < \ell^{(1)}$ and relative velocities such a process of successive refinement of turbulent pulsations may be carried through, until for some pulsations of sufficiently large order n, the Reynolds number

$$R^{(n)} = (\ell^{(n)} v^{(n)})/\nu$$

becomes so small that the effect of viscosity on the pulsations of the order n finally prevents the formations of pulsations of order n + 1.

" From the energetic point of view, it is natural to imagine the process of turbulent mixing in the following way: the pulsations of the first order absorb the energy of the motion and pass it over successively to pulsations of higher orders. The energy of the finest pulsations is dispersed in the energy of heat due to viscosity.

" In virtue of the chaotic mechanism of the translation of motion from the pulsations of the lower orders to the pulsations of higher orders, it is natural to assume that in the domains of the space, whose dimensions are small compared with $\ell^{(1)}$, the fine pulsations of the higher orders are subjected to approximately space-isotropic statistical regime. Within small time intervals, it is natural to consider this regime approximately steady even when the flow on the whole is not steady.

Since for very large R, the differences

$$w_\alpha(P) = u_\alpha(P) - u_\alpha(P^{(0)})$$

of the velocity components in neighbouring points P and $P^{(0)}$ of the four-dimensional space ($x_1, x_2, x_3; t$) are determined nearly exclusively by pulsations of higher orders, the scheme presented leads us to the hypothesis of local isotropy in small domains G, in the sense of definitions 1 and 2 ...."

Further arguments of Kolmogorov go through easily. These refer to the pulsations of order n where $\ell^{(n)} \gg$ or $\sim$ $\ell^{(\eta)}$ where

$\ell^{(n)}$ is the scale of the finest pulsations, whose energy is directly dispersed into heat by viscosity.*

We quoted in full some of the arguments of Kolmogorov, since we found that they offer considerable depth of vision and insight, which were missed by many of the readers for a number of years. Further, the concept of invariance with respect to random Galilian transfermations, which one tries to impose on the Eulerian solutions in turbulence theory - has its origin in these arguments. The concept of independent evolution of intermediate pulsations, free of viscosity on the one hand and free of the larger scale (or lower order) pulsations is now translated in terms of differences of velocities between two neighbouring space-time points in a domain, which is embedded in larger domains, which are moving randomly and which, in its own turn contains domains which are embedded in it, in a similar fashion. Further, there is an underlying hypothesis that the intermediate pulsations, effectively transmit the energy they receive from larger pulsations, down to the smaller ones, without loss or gain, so that the scheme of energy-transfer is in a sense of stationary energy flow-across the region of " Equivalent mode space ".

We raise this latter point, because this is an important feature, which plays a vital role in the generalisation to hydromagnetics of Kolmogorov's ideas. Before one is able to make use of it, it is necessary that a steady stationary pattern of energy transfer amongst the modes be set up. The net input of energy from macroscopic (or $\ell^{(0)}$) pulsations must balance the net out flow of energy from the finest pulsations (or $\ell^{(n)}$) into viscous losses.

3. Early Developments In Hydromagnetic Turbulence.

The first question to be considered in this field was whether a weak random excitation in the magnetic spectrum of a certain scale will grow or decay when left to interact with a steady homogeneous and isotropic turbulent velocity field (Batchelor, Biermann[7] and Schlüter[8]). Even to transcribe literally some of the arguments of Kolmogorov - about the nature of equilibrium between modes in a neighbourhood of mode space is rather difficult in this case. Firstly, the magnetic mode space is unexcited; further one has to distinguish between flow of energy within the magnetic mode space and the flow of energy between the magnetic and velocity mode spaces. In the hydrodynamic case, the concept of statistical independence of modes from different and distant regions of mode space was substantiated by Kolmogorov, as seen in the previous section, by two requirements. First there were regions of mode space, where there was a statistical equilibrium between net input of energy from larger scales of motions and net output of energy to smaller scales. Secondly, a steady larger scale motion can be seen only to bodily convect smaller scales of motion, without distorting them. This idea was made rigorous with arguments of Local Isotropy and used to draw conclusions about independence of distant regions of mode space or the Localness of Transfer in mode space.

Both of these arguments are inapplicable to the question posed above. In the magnetic mode space, in the initial condition specified above, equilibrium has not been set up. For translating the arguments about Local Isotropy to hydromagnetics, one must be able to transfer attention to differential magnetic intensities, rather than absolute magnetic intensities. This feature in the case of fluid velocity, naturally led to a local Galilean transfermation for a certain order of pulsations $\ell^{(n)}$ such that the effect of the lower order pulsations can be subtracted out by a choice of local coordinates. But such a choice for the magnetic case is not possible. The simple physical reasoning for this failure lies in the possibility of Alfvén mode coupling between different scales.

From both these considerations, it seems clear that to decide the fate of a random magnetic excitation in a turbulent medium further dynamical analysis is required. We carried out such an analysis. Our dynamical study was based on a model representation for turbulence, which will be described in the next section.

4. The Direct - Interaction Approximation.

Kraichnan[9] expounded his closure procedure for the problem of homogeneous isotropic fully developed turbulence, in two papers, in 1958 and 1959. Though the main basic schematics of the procedure have remained invariant, the significance and interpretations of the various steps have changed over the years. Further, as we mentioned earlier, there are quite a number of critical reviews of the theory, both with respect to foundations and with respect to validity in the context of turbulence (11, 12, 16, 17, 18)

We will try to put forward a physically motivated "derivation" of the Direct-Interaction Approximation. We will not attempt to justify the procedure, but we will try to indicate how one tries to bridge the gaps between analytical generalities and practical reasonableness.

To illustrate the method, we will start with a model equation, which has a structure very similar to the Fourier-mode representation of Navier-Stokes equations. Treatements of this type of the Direct-Interaction Procedure abound in literature (10,11, 12,19)

$$\left(\frac{d}{dt} + \nu_i\right) A_i = \sum_{j,k} M_{ijk} A_j A_k + F_i \qquad 4.1$$

Here A_i are the dynamical modes, F_i is a random source term, the statistical properties of which are given completely. In practice, the various turbulent modes draw their energy from macroscopic boundary dependent sources. For a viscous fluid, there is a loss of energy from the dynamical modes and this takes away the energy from the turbulent modes into the thermal energy of the fluid. In the idealisation of turbulence, through the symmetry conditions of isotropy and homogeneity we have eliminated the input of energy. We are retaining the viscosity still, since we are interested in the region of modes, where viscosity also plays an important role. Thus from energetic considerations, we include here a random source term. Though, in general the nature of the statistical equilibrium among the modes will be a function of these forces, we will try to arrange matters such that their dependence is global, rather than in detail. There is yet another reason to deal with these F_i . This has to do with the arguments of dynamical damping among the modes We will talk of this later.

We are interested in constructing all possible information about the statistical structure of the A_i's , when they are in a statistically-steady state. The number of modes is considered large, so that the dynamical effect of the coupling with other modes to a given mode is appreciable. The coupling matrix M_{ijk} is a known algebraic function of its indices. Let us look at a particular triad (a,b,c) of the modes, which are such that the coupling coefficients M_{abc}, M_{bca}, M_{cab} are not all trivially zero. We rewrite the equation of motion of the a,b,c modes as

$$\left(\frac{d}{dt} + \nu_a\right) A_a = M_{abc} A_b A_c + \sum_{j,k \neq b,c} M_{ajk} A_j A_k + F_a \qquad 4.2$$

$$\left(\frac{d}{dt} + \nu_b\right) A_b = M_{bca} A_c A_a + \sum_{j,k \neq c,a} M_{bjk} A_j A_k + F_b \qquad 4.3$$

$$\left(\frac{d}{dt} + \nu_c\right) A_c = M_{cab} A_a A_b + \sum_{j,k \neq a,b} M_{cjk} A_j A_k + F_c \qquad 4.4$$

At a certain time t_* , we switch off the interaction between the three specific modes a, b and c. The three modes are still interacting with an infinity of modes and indirectly through these infinity, with others. Since the number of such contributions to the RHS of any of these equations is rather large, the effect of the switch-off of one triad interaction does not change matters very much. In other words, around the state of equilibrium, attained through the non-linear interaction of an infinity of modes the dynamical change in the behaviour of any specific mode due to interaction with a specific set of two modes is small. This is what Kraichnan calls the Weak Dependence Principle. This is an exact statement, which has its origins in the assumptions of homogeneity of the turbulence, both with respect to boundary conditions and driving force-structure. The Direct Interaction Approximation sets up an elaborate scheme-with which to exploit this perturbation basis. But this intuitive scheme of separating the contribution **from** a finite subset of modes, in contrast with the rest of the modes and saying that their difference is dynamically small cannot be formulated in terms of a small parameter theory. Kraichnan's point of view was to assert that the non-linear contributions to a given mode play two different roles, depending on whwether we include in their contribution a finite subset of modes or an infinity of modes. Within a finite subset of n modes it is the non-linear interaction which builds correlations and ends up generating non-vanishing correlation $\langle A_i A_j A_k A_l A_m \ldots\ldots\ldots\ldots\ldots\ldots \rangle$ upto the order n . (For e.g. in the case of a strongly interacting " gas " of N particles (the range of interaction being infinite) the significant correlation that would be necessary to typify the state would involve all the N particles. Any argument based on any type of hierarchy truncation would not make sense.) But any finite set of modes never end up with such a correlation because during the same time, the non-linear interaction by each of the modes of the subset with the rest of the modes tends to decorrelate them. Thus it is important to distinguish the role played by the non-linear terms in building up the correlations and again in breaking them up.

The choice of a triad as the fundamental brick of interaction from which the building-up of the correlation within a finite subset arises is made first by the equations of motion themselves. Since the third cumulant (or what is the same in homogeneous turbulence, the third moment $\langle A_i A_j A_k \rangle$) must play an important role in deciding the energy transfer between modes and as the structure of the equation always couples this to the interaction between three modes, it would always involve an irreducible triad. (An irreducible triad is one in which all three modes are always simultaneously interacting.) The general tree of interactions between more modes can be built up in terms of multiples of triads. So it was Kraichnan's rationale that the first non-trivial Direct Interaction prescription between a finite set of modes will have to start with a triad.

Further Kraichnan makes another novel assertion which is again very difficult to find fault with or to justify. This is the so-called Maximal Randomness Condition. This asserts that there are no prferred modes in the system; in other words in the equations of motion of a typical mode, the various coupling coefficients with different modes are all of the same order. This coupled to the fact that we have envisaged a statistically steady state of homogeneous turbulence would imply that there are no preferred modes. The statistical dependence among the modes will be induced completely by the non-linear interactions and not at all by any boundary conditions or external forces.

The various requirements that we have listed so far to define the Direct Interaction Approximation, like the existence of a large number of modes and the concept of Maximum Randomness will all hold at large Reynolds numbers: but this in itself says nothing about the utility of our approximation procedure for large Reynolds numbers. (Necessary but not sufficient!)

We will now try to illustrate next the notion of an Impulse-Response Tensor, which plays a crucial role in Kraichnan's theory. We will denote the turbulent system in short hand as a sum of a triad and the rest of the modes $\left((a,b,c)+\sum_{i,j,k}(i,j,k)\right)$ At a certain time t_0 we can specify their dynamical state by a set of values $\tilde{A}_a(t_0)$, $\tilde{A}_b(t_0)$, $\tilde{A}_c(t_0)$, $\left(\tilde{A}_i(t_0), i \neq a,b,c\right)$ We introduce in the equation of motion of A_a an infinitismal driving force δF_a . This will produce an alteration in the amplitude of A_a from $\tilde{A}_a$ to $\tilde{A}_a + \delta A_a$. This δA_a at all later times $t > t_0$ will depend ofcourse on the values of $A_a(t)$ $A_b(t)$, $A_c(t)$ and $A_i(t)$ for all i for all $t > t_0$ and on the change δF_a . If the state of the modes when this δF_a was introduced can be considered as a state of statistical equilibrium, this change would depend on the entire spectral features of the $\tilde{A}(t_0)$. The formal " Green's " operator which relates the change in the amplitude of $A_a(t)$ at a time t, due to change in the driving force $\delta F_b(t')$ at time t' , is the impulse response tensor

$$\delta A_a(t) = \int_{t_0}^{t} G_{ab}(t, t')\, \delta F_b(t')\, dt' \qquad 4.5$$

From intuitive considerations, one infers that because of the randomness in the system, a change in the amplitude of one mode, will be correlated to itself only for a finite time. Alternately, the response of the mode will also be correlated to the disturbance which produces it only for a finite time. Formally averaging over an ensemble of disturbances δF , we can generate the averaged impulse-response tensor. But in equations 4.2, 4.3 and 4.4, we replace the total contribution due to the c, d, k -sum as $-\lambda_a(t)$, $-\lambda_b(t)$ and $-\lambda_c(t)$ .We see that these λ's are generally random but are ofcourse functions of the state of the modes c, d, k . If we vary the state of the modes c, d, k around value $\tilde{A}_c(t)$, $\tilde{A}_d(t)$, $\tilde{A}_k(t)$ at t_0 we will be varying $\lambda(t)$ around a value $\tilde{\lambda}(t_0)$ by an amount $\delta\lambda$. The response of the system to this $\delta\lambda$ will be defined by our definition of G as above: but now our averaging over an ensemble of $\delta\lambda$ cannot be done independent of the G. This crucial interchange of arguments between the random source and the effective force due to the non-linear coupling among modes is the second reason for our introduction of the random force. The argument here is quite reminiscent of the introduction of an effective field in

Hartree-Fock type of calculations. Instead of an effective field, we talk of an effective dynamical damping, which a mode sees due to its coupling to the rest of the modes.

Realising that this relaxation time is real and finite one can formally represent it by the eigenvalue of an undefined operator $\lambda_a(t)$ acting on $A_a(t)$. In particular $\lambda_a(t)$ may be a non-linear functional of the state of the modes at all previous times $t' < t$ and it will in general have an integral structure in time. Incorporating these one can give a formal definition of G as

$$\left(\frac{d}{dt} + \nu_a + \lambda_a(t) \right) G_{aa}(t,t') = \delta(t-t') \qquad 4.6$$

We hope to include in $\lambda_a(t)$ all or significant parts of the relaxation due to non-linear interaction (by this we imply the coupling of the (a,b,c) modes to (i,j,k) modes). Using this definition, one can generate the integration forward in time of the modes A_a, A_b, A_c . Their interaction with the rest of the modes being typified by the $G_{aa}(t,t')$, $G_{bb}(t,t')$, $G_{cc}(t,t')$ etc.

These arguments are quite reminiscent of the ideas of Prandtl and Heisenberg in introducing the notion of eddy-viscosity (a review of these can be found in Beran). We rewrite the equation of motion for a typical mode A_a as

$$\left(\frac{d}{dt} + \nu_a + \lambda_a(t) \right) A_a(t) = M_{abc} A_b(t) A_c(t) + F_a(t) \qquad 4.7$$

Here the Direct Interaction of the triad (a,b,c) is isolated on the R H S and the relaxation due to the rest of the modes is contained in the operator $\lambda_a(t)$. Now, we assert that these two contributions are not completely independent, but the consistent treatment of Direct Interaction, with the background relaxation due to the indirect interaction should determine $\lambda_a(t)$ completely. What we are offering here is not an exact justification, but a motivation.

If we introduce a formal variation of one of the amplitudes $A_a(t)$ at time t_0 , around a value $\widetilde{A}_a(t_0)$, the equation of motion of this $\delta A_a(t)$ can be written

$$\left(\frac{d}{dt} + \nu_a + \lambda_a(t)\right) \delta A_a(t) = M_{abc}\, \delta\left(A_b(t)\, A_c(t)\right) + \delta F_a(t) \tag{4.8}$$

In the equations of motion of the modes b and c, there will be terms due to this variation

$$\left(\frac{d}{dt} + \nu_b + \lambda_b(t)\right) \delta A_b(t) = M_{bca}\, \delta\left(A_c(t)\, A_a(t)\right) + \delta F_b(t) \tag{4.9}$$

$$\left(\frac{d}{dt} + \nu_c + \lambda_c(t)\right) \delta A_c(t) = M_{cab}\, \delta\left(A_a(t)\, A_b(t)\right) + \delta F_c(t) \tag{4.10}$$

We can use the response tensors of the modes a, b and c and rewrite

$$\delta A_b(t) = \int_{t_0}^{t} G_{bb}(t,t') \left\{ M_{bca}\, \delta\left(A_c(t')\, A_a(t')\right) + \delta F_b(t') \right\} dt' \tag{4.11}$$

$$\delta A_c(t) = \int_{t_0}^{t} G_{cc}(t,t') \left\{ M_{cab}\, \delta\left(A_a(t')\, A_b(t')\right) + \delta F_c(t') \right\} dt' \tag{4.12}$$

In writing equations (4.11) and (4.12), we have used the interchangeability of the effective force on a mode by the mode-coupling terms as well. Further the use of the concept of Maximal

Randomness reduces the G's to diagonal terms in the modes. A thorough justification of this on statistical grounds is attempted in a paper by Kraichnan[19]. We will omit this consideration here. Hereafter, we will drop one of the suffixes on G's and use only one. But we only want to point out that this step is quite general and does not still involve the Direct Interaction Approximation itself. Incorporating these into equation for $\delta A_a(t)$ we can write

$$\left[\frac{d}{dt} + \nu_a + \lambda_a(t)\right] \delta A_a(t) = M_{abc} A_c(t) \cdot \left\{ \int_{t_0}^{t} G_b(t,t') \left[M_{bca} \delta(A_c(t') A_a(t') + \delta F_b(t')\right] dt' \right\}$$
$$+ M_{abc} A_b(t) \left\{ \int_{t_0}^{t} dt' \, G_c(t,t') \cdot \left[M_{cab} \delta(A_a(t') A_b(t') + \delta F_c(t') \right] \right\} + \delta F_a(t) \quad 4.13$$

Dividing through by $\delta F_a(t')$ and averaging over an ensemble of realisations of the variation δ ,we can rewrite

$$\left[\frac{d}{dt} + \nu_a + \lambda_a(t)\right] \left\langle \frac{\delta A_a(t)}{\delta F_a(t'')} \right\rangle$$
$$= M_{abc} M_{bca} \int_{t_0}^{t} dt' \left\langle G_b(t,t') \, A_c(t) \, A_c(t') \, \frac{\delta A_a(t')}{\delta F_a(t'')} \right\rangle$$
$$+ M_{abc} M_{cab} \int_{t_0}^{t} dt' \left\langle G_c(t,t') \, A_b(t) \, A_b(t') \, \frac{\delta A_a(t')}{\delta F_a(t'')} \right\rangle + \delta(t-t') \quad 4.14$$

The other terms vanish using the arguments of Maximal Randomness and Weak Dependence both resulting from the assumption of homogeneity. $\left\langle \frac{\delta A_a(t)}{\delta F_a(t')} \right\rangle$ is nothing but the averaged impulse response tensor of the mode A_a defined around a neighbourhood of values of the modes $\tilde{A}$. A typical term on the R H S of equation 4.14 hasthe form

$$\langle G_b(t,t')\; A_c(t)\; A_c(t')\; G_a(t',t'') \rangle \tag{4.15}$$

This involves the total modal response of the two interacting modes a and b, during the time they are interacting with the mode c. This is an elementary triad interaction, which builds the non-linear mode coupling, we have no justification to neglect these terms, even in the limit of a weak contribution by one triad in contrast with all the rest of the triads. Formally carrying out the averaging around a statistical equilibrium, with fluctuations around it, we can rewrite this term as

$$\begin{aligned} &\bar{G}_b(t,t') \langle A_c(t)\, A_c(t') \rangle \bar{G}_a(t',t'') \\ &+ \langle G'_b(t,t')\; A_c(t)\; A_c(t')\; G'_a(t',t'') \rangle \\ &(\text{where } \langle G \rangle = \bar{G} \;;\; G' = G - \bar{G}) \end{aligned} \tag{4.16}$$

(Here we are introducing an idea of fluctuations around the statistical equilibrium, envisaged in mode space, by an argument similar to Kolmogorov's in the last section. These fluctuations should not be confused with fluctuations in the modes A, which are the basic dynamical variables, we are considering in the analysis. The fluctuations, which typify the response of a mode are related to fluctuations in the parameters, which determine the equilibrium form of the spectrum. These are related to the macroscopic input of energy which builds the stationary energy flow and the total microscopic loss of energy through viscous losses, which limit this flow. At the present stage of this argument to develop a closure scheme for turbulence, there seems to be no justification to neglect the second term of (4.16). This question is discussed by the author in a separate paper[23]).

Restricting one's attention to the average propagator alone, equation 4.15 reduces to

$$\int_{t_0}^{t} G_b(t,t') \langle A_c(t) A_c(t') \rangle G_a(t', t'') \, dt' \qquad 4.17$$

Now at this stage it is an assertion of Kraichnan that the total relaxational contribution due to all the modes $\lambda_a(t)$ is a sum over elimentary triad contributions on the R H S of the above equation. For turbulence, around a defined statistical equilibrium, with little fluctuations around it, it seems to be a reasonable assumption. A formal justification of this is not offerd here, since our aim is to motivate the derivation only. Thus one can write for the $\lambda_a(t)$

$$\lambda_a(t) \, G_a(t, t'') = \sum_{l,m} M_{alm} \int_{t_0}^{t} G_l(t, t') \langle A_m(t) A_m(t') \rangle G_a(t', t'') \, dt' \qquad 4.18$$

The main achievement of this argument is that every modulation of the amplitude of any mode is propagating in time with a response function determined by a sum of triad contributions. Further this contribution is effectively a relaxation. The formal closure problem in defining the response is achieved by operating around a neighbourhood of values in the function-space of amplitudes of modes. This is not necessarily a unique way of prescribing this relaxation. This question has not received the attention, it deserves. For example, Edwards[20] considers the eigenvalue of the relaxation operator to be defined by considerations of generalised entropy and approach to equilibrium, from an arbitrary deviation.

More recently, Kraichnan and Herring, in a series of papers have tried to compare the various approaches and try to generalise them. Accounts of these can be found in Leslie's book. The main point in our argument above is to show that explicit appeal to direct interaction is not necessary in defining the equation for G.

This generalised response tensor is made use of in evaluating a typical higher order correlation between modes. For example, if there is a non-vanishing contribution to a third order moment $\langle A_a(t)\, A_b(t')\, A_c(t')\rangle$ which satisfies all the restrictions to symmetries, we try to evaluate it using the G. The correlation between the modes is built up by the direct interaction between the three modes. While the three modes are interacting, their interaction with the rest tends effectively to suppress their amplitudes and thus reduce the direct interaction. The building up of the correlation is thus restricted by the finite memory time of any particular mode, about what happened to itself in the past, due to the existence of effective relaxation.

$$\langle A_a(t)\, A_b(t')\, A_c(t')\rangle = \sum_{\ell,m} \langle A_a(t)\, A_b(t') \int_{t_0}^{t'} G_c(t',t'')\, dt'' \cdot M_{c\ell m}\, A_\ell(t'')\, A_m(t'')\rangle \qquad 4.19$$

In writing this we neglect the effect of viscosity and the driving force, in consonance with ideas of Maximal Randomness. This term can be rewritten as

$$\sum_{\ell,m} M_{c\ell m} \int_{t_0}^{t'} dt''\, G_c(t',t'')\, \langle A_a(t)\, A_b(t')\, A_\ell(t'')\, A_m(t'')\rangle \qquad 4.20$$

It looks at this stage that we have achieved nothing, since we have ended up with a fourth order moment here. This moment can be rewritten

$$\langle A_a(t)\, A_b(t')\, A_l(t'')\, A_m(t'')\rangle = \langle A_a(t)\, A_a(t'')\rangle \langle A_b(t')\, A_b(t'')\rangle\, \delta_{al}\, \delta_{bm} + \langle A_a(t)\, A_b(t')\, A_l(t'')\, A_m(t'')\rangle_{a,b \neq l,m} \qquad 4.21$$

The second term is an irreducible fourth order moment, which is a correlation between four modes. The assertion of direct interaction states that the correlation between any subset of modes is built up in terms of triads of interactions. Thus we neglect this term. In fact, the number of terms in this category is large compared to the number included in the factored category. But the assertion is that these include another infinity of intermediate modes, which are summed over and their contributions would be random and average to zero.

At face value, this approximation does not seem to be any different from the usual cumulant discard procedure, incorporated at the level of the fourth order moment. As has been pointed by Leslie in his book (loc.cit.) the difference lies in the use of the relaxation features of the $G_a(t,t')$. This actually introduces into the evaluation of the cumulant of any order a part of the contribution from every higher order cumulant, but <u>only</u> a part. Thus we effect closure. The attractive aspects of the Direct

Interaction Procedure lie in some other features which we will just mention. One is the energetically consistent way of dealing with the non-linear interaction. This ensures that the approximation does not violate any of the energy conservation requirements and/or the requirements of positive definiteness of the probability distribution of the modes. These and other questions are considered in great detail in Leslie's book.

For the application to hydromagnetic turbulence, which we report here, the essential point about the Direct Interaction Approximation is its energetic consistency and the information, one derives from it, about the correlation and relaxation times of fluctuations, from a dynamical point of view. Since we are interested in investigating the possibility of transfer of energy from the velocity fluctuations to the magnetic field fluctuations, we would like to base our arguments on a theory which is manifestly energy-conserving.

When Kraichnan tried to solve the closed equations for the energy spectrum and the impulse response function, he discovered that there was still a gap to fill in the logic, before one could look for agreement with Kolmogorov's asymptotic analysis in the inertial range. As we saw in the previous section, Kolmogorov's argument implied the existence of a unique time scale (through the existence of a unique typical differential velocity associated with a certain scale.) In the arguments of Kraichnan, the dynamics provides equations for two typical time scales for a given mode, the correlation and relaxation times and it is not a priori clear that they are equal. Part of this confusion arises because of the fact that the correlation one talks of in the Direct Interaction Approximation are Eulerian Correlation times and Kolmogorov's arguments of Local Isotropy imply a Lagrangian frame work. Secondly the Eulerian analysis introduces transfer of energy between distant modes in mode space, as a steady balancing flow. The net transfer of energy across a wave number may be essentially local,but the distant wave number coupling gives rise to a large inflow into the region, balanced by an equally large outflow from the region. This is connected with a divergence in the steady state energy transport scheme (See Edwards[20] and also Leslie loc.cit.) This can be corrected by a rigorous Lagrangian formulation, as has been done by Kraichnan in a series of papers. Equally, they can be remedied by considering a truncation in the mode-mode coupling terms, in the relaxation function equations.

A rigorous justification for this procedure can be provided in terms of the Lagrangian History formulation. A simple intuitive justification was provided by Kadomtsev. Borrowing from traditional arguments of Landau damping in wave-particle coupling in plasma physics, he argued that two neighbouring "physical" eddies, which are really localised wave packets, which are close together and have phase velocities which are nearly equal, interact persistently for a long time and transfer sufficient energy. This is the so-called resonant coupling. At the same time two dissimilar wave packets with distinctly different wave lengths just pass through one another without much distortion of one or the other. This so-called adiabatic coupling is overestimated by the Direct Interaction. A correct remedy can be provided by introducing a " Coherence time " or a " Coherence length " for a wave packet of a certain scale and incorporating interaction only with modes within that range to determine the effective relaxation of the modes due to non-linear interaction.

We try to follow this simpler scheme of incorporating the " Locally Isotropic View " of turbulence. The choice is partially for simplicity. Further in the hydromagnetic context, as we shall see in the next section, the arguments of Local Isotropy are themselves suspect, so much so it is not clear whether all the elaborate effort of Lagrangian History formulation is worth it.

5. Hydromagnetic Turbulence and Kolmogorovian Arguments.

In Section 2, we elucidated the arguments of Kolmogorov to postulate the existence of a range of intermediate pulsations of velocity, which are determined completely by pulsations of neighbouring orders and not by the largest or the smallest pulsations. The main thrust of this argument came from the assertion that the differential velocity between two neighbouring points separated by a distance of the order of intermediate scales is determined completely by pulsations of the same order. The coupling with too large or too small scales, which (borrowing a term from Kadomtsev) can be called adiabatic interaction effectively produces neglegible effects. The larger fluctuations essentially convect the intermediate scales, without distortion and the intermediate scales in their turn convect the smaller scales without distortion . One can eliminate this convection-without distortion by systematically formulating the whole scheme in terms of differential velocities of fluid elements. Kraichnan[22], Kadomtsev and later Edwards (loc.cit.) independently discovered that the flaw in Kraichnan's original argument to construct a proper inertial range lay in the improper handling of a divergence, and a singularity in the response equation.

But the question is how good the assertions of Kolmogorov are in the context of hydromagnetics. The simple argument about transferring to local differential velocities and gauging away larger scale motions cannot be carried out with magnetic intensities. Absolute magnetic intensities play a crucial role in determining the dynamics of even very small magnetic disturbances, in the spirit of Alfvén wave coupling. Thus the different regions of the magnetic mode space never become even statistically independent. An absolute and thorough analysis of this question, for a turbulent system in the presence of an external homogeneous and constant magnetic field is still lacking. This should clarify some of the fundamental ideas.

We visualise a simpler situation in a system in which primarily the turbulence starts off in hydrodynamical modes. It reaches a steady state, with an energy-containing range, an inertial range and a dissipative range. This can be described by the Direct Interaction Procedure of Kraichnan, with suitable modifications to take care of the importance of the resonance interaction rather than adiabatic interaction. Now we introduce a randomised disturbance in the magnetic modes in the form of a localised spectral excitation well within the scales of the inertial range, with wave number and frequency widths compatible with elementary ideas of Kolmogorov [A unique spatial scale implies a unique time scale. This implicit equality of all relevent frequencies of interest for a given scale of motion implies a definite dispersion relation for coherent motion and a relation between fluctuation and dissipation processes for incoherent motions. For want of a simple shorthand notation for it, we refer to it as the Kolmogorov Fluctuation-Dissipation Relation (KFDR)]. As the interaction builds up within the magnetic spectrum between different modes, this simple K.F.D.R will not persist. Slowly the magnetic spectrum will start building up long range dependences in mode space and the K.F.D.R will be modified to include effects of excitation in other ranges. In our second paper, we try to find the modified FDR for the hydromagnetic case assuming that the K.F.D.R is unaltered for the hydrodynamical part.

Apart from the particular questions of relevence to astrophysics, we feel that this isolation of the lack of generality of Kolmogorov's assertions in the hydromagnetic context and its logical implications are the main and significant aspects of our results. We want to stress this, since the basic underlying arguments here are independent of a particular dynamical scheme to deal with turbulence, though our results in detail would be modulated by the success and pitfalls of the model.

In our study of the relationship between correlation and relaxation features of the fluctuations in the magnetic modes we isolate two separate assumptions to be equivalent to the analysis of Kolmogorov. First, which we have discussed in great detail, arises from the lack of applicability of the Local Isotropy ideas, in an unaltered form to the magnetic modes. This questions the validity of asserting that the coupling in mode space is completely local. The second is the assertion that the correlation and relaxation times are equal.

In the pure hydrodynamic case, this has been tested by the Lagrangian History formulation of the Direct Interaction Approximation, by Kraichnan. But, from a general point of view, in the absence of a valid proof of the applicability of Local Isotropy ideas to hydromagnetics, this equality, which implies a transcription of a fluctuation-dissipation relation conceivably proved for <u>Lagrangian</u> correlation and relaxation times to Eulerian ones is unjustified. We carry out a so-called Reduced Lagrangian History Modification, in which we leave the magnetic modes, unaltered by Kolmogorovian prescriptions.

In the following section we will discuss our results and try to draw some general conclusions about their applicability to problems in nature.

6. Analysis of Results.

There are three distinct though inter-related questions that we ask in our three papers included in this dissertation. The first is the ultimate fate of a weak random excitation in the magnetic mode spectrum of a turbulent fluid. This question is discussed in the first paper. The main conclusion of this paper is to focus attention on the importance of dynamical analysis of the equations, rather than stochastic analysis. By this, we refer to the many approaches in which the turbulent velocity field is considered as a given stochastic driving term in the equations of the magnetic mode. Further simplifying assumptions about the auto-correlation times of the velocity fluctuations are made, such that the statistical history of the magnetic modes and the velocity modes are on different scales of time. This leads to a kind of Langevin point of view for the magnetic mode equations. These approaches are not justified for the magnetic modes; alsothe neglecting of the Lorentz force terms from the velocity equations cannot be uniformly justified for all scales of the velocity field, though it can be justified as an initial condition. This is a positive conclusion from our study. There is an overriding negative conclusion of the same investigation, viz: that the theories of turbulence, which are based on arguments of asymptotic equilibrium between various transfer machanisms in mode space are not accurate enough to resolve the delicate balance of transfer in magnetic mode space, which produce local amplification or transfer to distant wave numbers. This inadequacy is partly because of the lack of understanding of the fundamental non-equilibrium features of the energy balance which is responsible for cross-field and self-field transfers in mode space. Quite strangely, this question does not seem to have interested many people in the field of non-equilibrium statistical mechanics as it should have. It is our satisfaction that our study tries to focus attention on this question from a statistical dynamical point of view.

The second question that we pose for ourselves is to analyse the steady-state features of the spectra of the two fields and in particular to analyse it to the same level of completeness dynamically, as has been done by Kraichnan in the hydrodynamical case. In this, we find a reevaluation of the prescriptions of Kolmogorov about the equality of the correlation and relaxation times of a typical mode is required for the magnetic modes. The various modifications which we incorporate imply rather drastic assumptions about the time structure of the correlation-relaxation features. The results show profound effects in terms of wide variations in integral parameters. Also the power law of the spectra in the inertial range are altered significantly too. But the most persuasive result of the calculation is the detailed equipartition between the magnetic and velocity modes through out the extended inertial range. This is a significant result from general dynamical considerations. Each of the types of modes has a spectrum which is far from equilibrium. ($k^{-5/3}$ or $k^{-3/2}$ as against k^2 for equipartition) But for each wave number, the magnetic and velocity modes are in equipartition. This substantiates the conjectures of Biermann and Schlüter.

In the third paper we try to fill up the evolutionary gap between the initial and final state analyses of the first and second papers. Here our aim was two fold. First to study in detail the dynamical effect of the Lorentz force terms. This shows itself in limiting the growth of the highest wave numbers and producing equipartition.

The second question was a bit more subtle. This was to check whether an arbitrary localised magnetic spectral disturbance of weak intensity will tend to produce transfer of **energy to smaller wave numbers.** In traditional arguments of cascade of energy in turbulence one implicitly assumes that energy always cascades up the wave number spectrum. This underlies the logic of universal equilibrium. But when the form of the magnetic spectrum is different from the equilibrium form (both in terms of functional dependence on wave number and relative strength with respect to velocity spectrum) the transfer should take place to neutralise this difference even if it meant back-transfer in wave number space to smaller wave numbers. This is at best a guess, till one verifies it by an explicit calculation. In our third paper, we demonstrate this by a carefully planned model, which confirms our conjecture.

What can we say about the applicability of the models to concrete situations in the laboratory or in astrophysics? A realistic study should have started with a specific well-defined problem with particular boundary and initial values and proper and complete definitions of sources of input of energy into the turbulence if any. We chose to study an infinite system with no boundaries, to minimise the complications of algebra in dealing with complicated tensorial equations. Symmetry conditions of homogeneity and isotropy were introduced like this. Similarly assumptions of statistical stationarity in time were introduced. These limit the applications considerably.

The basic model of Direct Interaction Approximation itself involves explicitly onlyone of the symmetry assumptions listed above, that of homogeneity alone. Thus from a symmetry point of view, the D.I.A is less restrictive. We invoke from the very beginning a continuum point of view for the fluid and consider it incompressible. Thus all effects involving finite temperature and/or discrete particle structure of the medium are excluded.

We have already indicated the limitations of the D.I.A itself . (Further details can be found in the oft-quoted book of Leslie) Our study raises a serious doubt about incorporating assumptions of isotropy in turbulent systems with strong magnetic fields. As one starts building up sufficient energy in the smaller wave number components of the magnetic modes, they will have a profound effect on smaller scale fluctuations through intensity-coupling. The changes in the larger scale magnetic mode parameters will affect the spectral characteristics of the smaller scales, in terms of enhanced fluctuations of spectral parameters. This phenomenan of intermittency will be more and more pronounced as the range of the coupling in mode space becomes larger and larger.

This seems to be an unstable situation atleast, in this way of looking at things. As the dependence on larger scales becomes stronger and stronger, the requirements of Maximal Randomness and their isotropy may break down. This is a conjecture, which is yet to be substantiated.

7. Contemporary Developments and Future.

In our paper I of this dissertation we had reviewed the situation in the field of hydromagnetic turbulence upto 1967 rather extensively and thoroughly. The situation upto that moment was rather speculative, with little or no stress on dynamical analysis. It was the main contention of our paper that a thorough dynamical analysis was necessary, but almost impossible under the existing state of theories of turbulence, at that time, without extensive calculation.

Around the same period, in the field of turbulent dynamos, with a publication of a series of papers by Krause, Rädler and Steenbeck[24], a new interest was stirred up about the symmetry conditions of turbulence. Arguing purely from kinematic grounds, Steenbeck and Krause suggested that the lack of reflexional symmetry in turbulence, which is very often a common feature in astrophysical systems may be an important factor in the turbulent regeneration of magnetic fields. This idea has been analysed further by a number of authors in recent years. (Parker[25] Moffatt[26] See a review of this field by Roberts[27]). Much of this work was based on a double scale analysis, in which the scales of the magnetic spectrum were considered uniformly larger than the scales of the velocity spectrum. Thus the dynamical coupling between the magnetic and velocity fields were treated by a stochastic prescription, very much in the spirit of Langevin Processes. There have been attempts by Moffatt[28] to remedy this. This truncated dynamical treatment makes it very difficult to see the dynamical effects which we analyse in the strictly isotropic case, for the turbulence in a pseudo-isotropic case. Quite recently, Frisch, Pouquet, Leorat and Mazure[29] have tried to investigate the possibility of an inverse cascade of magnetic helicity in magnetohydrodynamic turbulence. Their study is strictly restricted to an inviscid ensemble of flows with cutoffs for the lower and higher wave numbers. Such ensembles are characterised by absolute equilibrium spectra giving the classical equipartition k^2 spectrum for each field and further equipartition of magnetic and kinetic energy. The utility of these results in interpreting realistic flows with finite dissipation and no cut-off in wave numbers is at the moment pedagogic. A dynamically-equivalent study of the pseudo-isotropic case with methods similar to ours will prove forbiddingly difficult and is perhaps not worth the effort, for reasons which we have discussed in the last section.

Notwithstanding these reservations, there have been attempts by Vainshtein[30,31] and by Vainshtein and Zel'dovich[32], to apply a theory of the D.I.A type with additional simplifying assumptions to explain large scale magnetic phenomena in astrophysics. There have also been attempts by Vainshtein[33] to study the generation of a magnetic field by acoustic turbulence.

In conclusion, there seems to be a considerable amount of work which reiterates the basic premise and faith of our calculations that the turbulence in the fluid does regenerate the magnetic energy. There have also been numerical calculations of model equations to check these conjectures[34,35]. They seem to bear out the general expectations.

Much as one would like some observational evidence regarding the nature of M.H.D.Turbulence in astrophysical systems, the information one has is very sketchy and highly indirect. It should be interesting to see whether measurements of interplanatory magnetic fields, information about which has become available through satellite data, in recent years, will be able to throw any light on these.

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Growth of Turbulent Magnetic Fields

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The evolution of a weak, random initial magnetic field in a highly conducting, isotropically turbulent fluid is discussed with the aid of the exact expression for initial growth of the magnetic energy spectrum. Equipartition arguments, the vorticity analogy, and the known turbulence approximations all are found inadequate for predicting whether the magnetic energy eventually dies away or grows exponentially. This is true for any ratio of magnetic diffusivity λ to kinematic viscosity ν. The possibilities of eventual growth and eventual decay are therefore both admitted, and, for each, the shape of the magnetic-energy spectrum in the case $\lambda \gg \nu$ is estimated by simple dynamical arguments. If there is growth, it is concluded that the magnetic spectrum below the Ohmic cut-off eventually reaches equipartition with the kinetic-energy spectrum roughly in the fashion predicted by Biermann and Schlüter, with the principal exceptions that the spectrum of kinetic energy in the equipartition inertial range evolves to the form $k^{-3/2}$ and that equipartition is maintained, with rapidly falling spectrum through part of the Ohmic dissipation range. The evolution of the magnetic spectrum in the weak-field $\lambda \gg \nu$ regime is also computed numerically with a simplified transfer approximation suggested by the Lagrangian-history direct-interaction equations. This calculation turns out to yield an eventual very weak exponential growth of magnetic energy.

1. SUMMARY OF PREVIOUS WORK

THE behavior of a weak, turbulent magnetic field in a highly conducting fluid has been considered by a number of authors.[1-6] The early treatments of Batchelor[1] and Biermann and Schlüter[2] differed substantially. Biermann and Schlüter noted that the turbulent velocity field should stretch lines of force and thereby increase magnetic energy at the expense of kinetic energy. They assumed that this would continue until equipartition of magnetic and kinetic energy was reached, whereupon the Lorentz forces would inhibit the stretching. They proposed that the characteristic e-folding time for intensification of magnetic loops of given size would be the order of the circulation time for the turbulent eddies of that size. This implied that equipartition would be reached first at the smallest scales in the inertial range, and then proceed down the spectrum to the hydrodynamic energy-containing range.

Batchelor observed that the equation of motion for weak magnetic fields is identical with that for vorticity (Lorentz forces neglected), if the magnetic diffusivity λ and kinematic viscosity ν are equal. In this case, he proposed that an initial magnetic spectrum identical in form to the vorticity spectrum would be in equilibrium, neither growing nor decaying on time scales short compared to the lifetime of the turbulence. If $\lambda > \nu$, Batchelor argued that Ohmic dissipation effects on the magnetic field would be stronger than viscous effects on the vorticity field and that, therefore, the magnetic field should eventually decay. If $\lambda < \nu$, he concluded that the magnetic field would grow until approximate equipartition was reached at the top of the inertial range. Lorentz forces would then inhibit further growth, and equipartition never would be reached at lower wavenumbers, the magnetic spectrum always resembling the vorticity spectrum, with intensity increasing with wavenumber up to the dissipative cut-off. If the initial magnetic excitation was introduced at low wavenumbers, Batchelor predicted that stretching of magnetic lines of force would at first increase the magnetic energy regardless of the ratio λ/ν, and there would be transfer of the magnetic energy to higher wavenumbers as it was amplified. In the case $\lambda/\nu > 1$, growth would cease when the dominant spatial scale of the magnetic field was reduced to a characteristic Ohmic dissipation length, and thereafter the magnetic energy would die down monotonically. In the case $\lambda/\nu < 1$, growth would continue until the equilibrium vorticity-like spectrum resulted.

Further developments and modifications of Batchelor's ideas have been presented by Moffatt.[3]

Related treatments have been given by Saffman,[4]

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Pao,[5] and Parker.[6] These authors use different methods, but all three conclude that, whatever the value of λ/ν, an initial weak field should die away after initial amplification or, at best,[5] should be amplified by a finite factor to a steady-state level where Lorentz forces are still unimportant.

All authors seem to agree that an initial low-wavenumber magnetic excitation is at first amplified. There is complete disagreement about what happens after that.

2. INITIAL GROWTH OF ENERGY TRANSFER

The standard incompressible hydromagnetic equations are[7]

$$(\partial/\partial t - \nu\nabla^2)\mathbf{u} = -(\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{b}\cdot\nabla)\mathbf{b} - \nabla p, \quad (2.1)$$

$$(\partial/\partial t - \lambda\nabla^2)\mathbf{b} = -(\mathbf{u}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{u}, \quad (2.2)$$

$$\nabla\cdot\mathbf{u} = 0, \qquad \nabla\cdot\mathbf{b} = 0, \quad (2.3)$$

where $\mathbf{u}(\mathbf{x}, t)$ and $(4\pi\mu\rho)^{1/2}\mathbf{b}(\mathbf{x}, t)$ are the velocity and magnetic induction fields, ν is kinematic viscosity, λ is magnetic diffusivity, μ is susceptibility, ρ is fluid density, and ρp is pressure.

The total energy $\frac{1}{2}\rho \int (|\mathbf{u}|^2 + |\mathbf{b}|^2)\, d^3x$, as well as $\int \mathbf{u}\cdot\mathbf{b}\, d^3x$, is conserved by the hydromagnetic interaction.

Let the fields obey cyclic boundary conditions on a cubical box of side L. The Fourier amplitudes defined by

$$\mathbf{u}(\mathbf{x}, t) = \sum_{\mathbf{k}} \mathbf{u}(\mathbf{k}, t)e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \mathbf{b}(\mathbf{x}, t) = \sum_{\mathbf{k}} \mathbf{b}(\mathbf{k}, t)e^{i\mathbf{k}\cdot\mathbf{x}} \quad (2.4)$$

satisfy

$$(\partial/\partial t + \nu k^2)u_i(\mathbf{k}) = ik_m P_{ij}(\mathbf{k}) \sum_{\mathbf{k}'} [b_j(\mathbf{k}')b_m(\mathbf{k} - \mathbf{k}') - u_j(\mathbf{k}')u_m(\mathbf{k} - \mathbf{k}')], \quad (2.5)$$

$$(\partial/\partial t + \lambda k^2)b_i(\mathbf{k}) = ik_m \sum_{\mathbf{k}'} [u_i(\mathbf{k} - \mathbf{k}')b_m(\mathbf{k}') - u_m(\mathbf{k} - \mathbf{k}')b_i(\mathbf{k}')], \quad (2.6)$$

where

$$P_{ij}(\mathbf{k}) = \delta_{ij} - k_ik_j/k^2$$

and the argument t is omitted. The sums in (2.4)–(2.6) are over all wavenumbers allowed by the cyclic boundary conditions.

In the case of isotropic turbulence (which requires $L \to \infty$), scalar energy-spectrum functions may be defined by

$$(L/2\pi)^3\langle u_i(\mathbf{k}, t)u_j^*(\mathbf{k}, t)\rangle = (4\pi k^2)^{-1}P_{ij}(\mathbf{k})E(k, t),$$
$$(L/2\pi)^3\langle b_i(\mathbf{k}, t)b_j^*(\mathbf{k}, t)\rangle = (4\pi k^2)^{-1}P_{ij}(\mathbf{k})F(k, t), \quad (2.7)$$

[7] T. G. Cowling, *Magnetohydrodynamics* (Interscience Publishers, Inc., New York, 1957).

where $\langle\ \rangle$ denotes averaging over an isotropic ensemble of flow realizations. These functions satisfy

$$E(t) = \int_0^\infty [E(k, t) + F(k, t)]\, dk, \quad (2.8)$$

where $E(t)$ is the total energy per unit mass. In addition to the overall conservation of $E(t)$ by the nonlinear interaction, the interaction of each triad of wave vectors $\pm\mathbf{k}$, $\pm\mathbf{k}'$, and $\pm(\mathbf{k} - \mathbf{k}')$ is individually conservative.

If (2.5) and (2.6) are multiplied by $u_i^*(\mathbf{k})$ and $b_i^*(\mathbf{k})$, respectively, the ensemble averages of the real parts of the results yield the energy-balance equations

$$(\partial/\partial t + 2\nu k^2)E(k, t) = T(k, t) + L(k, t), \quad (2.9)$$

$$(\partial/\partial t + 2\lambda k^2)F(k, t) = J(k, t), \quad (2.10)$$

where

$$T(k, t) = 4\pi k^2(L/2\pi)^3 \cdot \mathrm{Im}\, \{k_m \sum_{\mathbf{k}'} \langle u_i(\mathbf{k}', t)u_m(\mathbf{k} - \mathbf{k}', t)u_i^*(\mathbf{k}, t)\rangle\}, \quad (2.11)$$

$$L(k, t) = -4\pi k^2(L/2\pi)^3 \cdot \mathrm{Im}\, \{k_m \sum_{\mathbf{k}'} \langle b_i(\mathbf{k}', t)b_m(\mathbf{k} - \mathbf{k}', t)u_i^*(\mathbf{k}, t)\rangle\}, \quad (2.12)$$

$$J(k, t) = 4\pi k^2(L/2\pi)^3 \cdot \mathrm{Im}\, \{k_m \sum_{\mathbf{k}'} \langle [b_i(\mathbf{k}', t)u_m(\mathbf{k} - \mathbf{k}', t) - b_m(\mathbf{k}', t)u_i(\mathbf{k} - \mathbf{k}', t)]b_i^*(\mathbf{k}, t)\rangle\}. \quad (2.13)$$

Here $T(k, t)$ is the ordinary hydrodynamic transfer function, while $L(k, t)$ and $J(k, t)$ describe the exchange of energy between kinetic and magnetic degrees of freedom. The conservation properties give

$$\int_0^\infty T(k, t)\, dk = 0, \qquad \int_0^\infty [L(k, t) + J(k, t)]\, dk = 0. \quad (2.14)$$

Suppose that the magnetic and velocity fields are statistically independent and isotropic at the initial time $t = 0$, and that the magnetic field is multivariate normal at $t = 0$. These conditions are appropriate to the introduction of a randomly phased magnetic seed field into pre-existing turbulence. The velocity field is *not* assumed to be normally distributed at $t = 0$.

This ensemble gives

$$L(k, 0) = J(k, 0) = 0$$

and the functions $[\partial L(k, t)/\partial t]_{t=0}$ and $[\partial J(k, t)/\partial t]_{t=0}$ which describe the initial build-up of energy transfer can be determined exactly. The results are valuable for illustrating the various transfer mechanisms, which continue to operate at later times. To obtain, differentiate (2.12) and (2.13), use (2.5) and (2.6) to express the time-derivatives, and use (2.7) to reduce the products of covariances which result from the independence and normality of the initial magnetic field. The final equations are

$$[\partial L(k,t)/\partial t]_{t=0} = k \iint_\Delta [k^2 a_{kpq} F(p,0) - p^2 c_{kpq} E(k,0)] F(q,0) \frac{dp\,dq}{pq}, \quad (2.15)$$

$$[\partial J(k,t)/\partial t]_{t=0} = k \iint_\Delta [k^2 d_{kpq} F(p,0) - p^2 h_{kpq} F(k,0)] E(q,0) \frac{dp\,dq}{pq} - k \iint_\Delta p^2 j_{kpq} F(k,0) F(q,0) \frac{dp\,dq}{pq}. \quad (2.16)$$

Here $\iint_\Delta$ denotes integration over all p, q such that k, p, and q can form a triangle, and

$$a_{kpq} = \tfrac{1}{2}(1 - xyz - 2y^2z^2), \quad c_{kpq} = pk^{-1}z(1-y^2),$$
$$d_{kpq} = 1 + xyz, \quad h_{kpq} = pk^{-1}(z + xy) = 1 - y^2,$$
$$j_{kpq} = pk^{-1}z(1 - x^2), \quad (2.17)$$

where x, y, z are the cosines of the interior angles opposite the triangle sides k, p, q, respectively.[8]

The conservation properties are associated with trigonometric identities among the geometrical coefficients:

$$a_{kpq} = a_{kqp} \geq 0, \quad d_{kpq} = d_{pqk} = d_{qkp} \geq 0, \quad h_{kpq} \geq 0,$$
$$k^2 j_{kpq} = p^2 c_{pkq}, \quad k^2 h_{kpq} = p^2 h_{pkq}, \quad c_{kpq} + c_{kqp} = 2a_{kpq},$$
$$h_{kpq} + j_{kqp} = d_{kpq}. \quad (2.18)$$

In (2.15) and (2.16), the terms linear in F and bilinear in F separately conserve the total energy. This can be verified from (2.18), and is also clear from the fact that conservation of energy holds whatever the ratio of magnetic to kinetic energy. If the initial magnetic field is sufficiently weak in the sense

$$F(k) \ll E(k) \quad (\text{all } k), \quad (2.19)$$

the terms bilinear in F can be consistently neglected in (2.15) and (2.16), leaving

$$[\partial L(k,t)/\partial t]_{t=0} = -kE(k,0) \iint_\Delta p^2 c_{kpq} F(q,0) \frac{dp\,dq}{pq}, \quad (2.20)$$

$$[\partial J(k,t)/\partial t]_{t=0} = k \iint_\Delta [k^2 d_{kpq} F(p,0) - p^2 h_{kpq} F(k,0)] E(q,0) \frac{dp\,dq}{pq}. \quad (2.21)$$

Equation (2.21) can be obtained directly by omitting the $(\mathbf{b}\cdot\nabla)\mathbf{b}$ term at the outset. However (2.20), which describes the small reaction of magnetic field on velocity field, requires the $(\mathbf{b}\cdot\nabla)\mathbf{b}$ term. The normality assumption on the initial magnetic field is not needed to get (2.20) and (2.21); it affects only the transfer terms bilinear in F.

The coefficients d_{kpq} and h_{kpq} in (2.21) are never negative, since $|x|$, $|y|$, $|z|$ are always ≤ 1. Consequently, the d term always represents a positive flow of energy *into* magnetic wavenumber k due to interaction with magnetic wavenumber p and velocity wavenumber q, while the h term always represents a flow of energy *out* of magnetic wavenumber k. The h term is $\propto F(k, 0)$, so that it can be interpreted as a dynamical damping analogous to the $2\lambda k^2 F(k, 0)$ term in the energy-balance equation.

3. INABILITY TO PREDICT WHETHER DYNAMOS EXIST

3.1. Equipartition Arguments

Lee[9] has shown that when $\lambda = \nu = 0$, (2.5) and (2.6) yield Liouville's theorem, if the real and imaginary parts of the Fourier amplitudes are taken as phase-space coordinates. Since the energy is a simple sum of squares of these coordinates, an immediate consequence is that there are formal equilibrium ensembles with equipartition of energy over all the degrees of freedom. Since the density of modes is $\propto k^2$, these ensembles have

$$E(k) = F(k) \propto k^2. \quad (3.1)$$

Do the nondissipative equipartition-equilibrium properties lead to valid inferences about the growth or decay of weak initial magnetic fields when λ and ν do not vanish? The usual arguments of statistical mechanics suggest that, in the absence of contrary constraints, the dynamical interaction should act to carry a nonequilibrium initial state toward equipartition. This can happen in two ways in the weak-field hydromagnetic problem. The $(\mathbf{b}\cdot\nabla)\mathbf{u}$ term in (2.2) can transfer energy from the strongly excited velocity field to the magnetic field. The

[8] These geometrical coefficients are identical with those obtained in the direct-interaction closure approximation for hydromagnetic turbulence [R. H. Kraichnan, Phys. Rev. 109, 1407 (1958)]. Some algebraic errors in this reference are corrected in (2.17).

[9] T. D. Lee, Quart. Appl. Math. 10, 69 (1952).

$(\mathbf{u}\cdot\nabla)\mathbf{b}$ term cannot do this, but it can spread the existing magnetic energy out over k space.

Both processes can be demonstrated from the weak-field initial-transfer formulas. First, consider the contribution to (2.21) from $p \approx k$, which involves the factor

$$k^2 d_{kpq} - p^2 h_{kpq} \approx k^2(d_{kpq} - h_{kpq}) = k^2(xyz + y^2).$$

Since $p \approx k$ implies $x \approx y$, then

$$xyz + y^2 \approx y^2(1 + z),$$

which is non-negative because $|z| < 1$. Therefore, *the interaction of the magnetic modes within any sufficiently narrow wavenumber band gives a positive flow of energy into that band from the velocity field. This is true regardless of which modes contain the kinetic energy.*

Next, suppose that k is within the range of magnetic wavenumbers initially excited, and k' without. Then $F(k', 0) = 0$, and the non-negativity of the coefficients shows that the contribution to $[\partial J(k, t)/\partial t]_{t=0}$ from $p \approx k'$ is negative, while the contribution to $[\partial J(k', t)/\partial t]_{t=0}$ from $p \approx k$ is positive. Therefore, *energy is lost from the excited wavenumber and gained by the unexcited one, so that the net effect is a spread of magnetic energy in k space. There is transfer into the unexcited wavenumber whether it lies above or below the excited one, and regardless of which wavenumbers contain the kinetic energy.*

Absolute equilibrium is never achieved in actual turbulence, because of dissipation and the related fact that energy never reaches very high wavenumbers. In this case, it seems impossible to tell what will eventually happen from the facts so far developed. Suppose that at some time t there is a crucial band of wavenumbers which contains most of the magnetic energy. The spreading process will sweep the energy out of the band, principally to higher wavenumbers, while the local enhancement process will pump energy into the band from the velocity field. If the rate of sweeping out exceeds the rate of local enhancement, the magnetic energy in the band will decay, in the absence of a supporting reservoir of magnetic energy at lower wavenumbers.

The Gibbs statistical mechanics does not deal with rates and cannot resolve whether local enhancement of sweeping out wins the competition. The ingredients of that theory are only the form of the constant of motion and Liouville's theorem. The detailed structure of the equations of motion, which determines the rates of competing processes, is not used. In fact it is easy, with the aid of the projection operator $P_{ij}(\mathbf{k})$, to alter (2.5) and (2.6) so that the ratio of the contributions from the $(\mathbf{b}\cdot\nabla)\mathbf{u}$ and $(\mathbf{b}\cdot\nabla)\mathbf{b}$ terms to that from the $(\mathbf{u}\cdot\nabla)\mathbf{b}$ term has any desired value, but neither the form of the energy nor Liouville's theorem are changed. Thus the ratio of sweeping-out to local enhancement can be made anything desired without affecting the Gibbs equilibrium.

In absolute statistical equilibrium there is no competition among different processes and the process rates do not affect the equilibrium. The property of detailed balance states that each triad interaction is individually in equilibrium and gives no net energy transfer in or out of any degree of freedom. This is an exact property: the energy is a sum of squares so that the canonical ensemble is Gaussian and all the triple moments in (2.11)–(2.13) vanish.

3.2. The Vorticity Analogy

Equation (2.2) for $\mathbf{b}$ is identical with the equation of motion for the vorticity, if $\lambda = \nu$ and the $(\mathbf{b}\cdot\nabla)\mathbf{b}$ term in (2.1) is neglected.[1] This suggests that vorticity creation by stretching of vortex tubes should have a magnetic counterpart. Furthermore, it yields an immediate particular solution for the weak-field magnetic spectrum when $\lambda = \nu$. The identity of the equations means that there is an ensemble of solutions

$$\mathbf{b}(\mathbf{x}, t) \propto \nabla \times \mathbf{u}(\mathbf{x}, t), \tag{3.2}$$

which implies

$$F(k, t) \propto k^2 E(k, t). \tag{3.3}$$

If (3.2) is satisfied for all $\mathbf{x}$ at any t, it is preserved by (2.1) and (2.2) with the $(\mathbf{b}\cdot\nabla)\mathbf{b}$ term omitted. Finally, the neglect of Lorentz forces should, reasonably, have a negligible effect on the evolution of $\mathbf{b}(\mathbf{x}, t)$, if (2.19) is satisfied with sufficient strength, so that the neglect of $(\mathbf{b}\cdot\nabla)\mathbf{b}$ is justified.

When $\lambda \neq \nu$, the vorticity analogy is less sharp, as has been stressed by Cowling[7] and others. However, a crucial trouble arises already in the case $\lambda = \nu$. Equation (3.2) prescribes a definite phase relation between magnetic and velocity field everywhere in every realization. This phasing is not required by (2.2) and is not satisfied by a random ensemble of magnetic fields with the spectrum (3.3).

One implication of the artificial phase constraint is that, by (3.2) and (2.14),

$$\int_0^\infty k^{-2} J(k, t)\, dk = 0. \tag{3.4}$$

In other words, the quantity

$$Q(t) = \int_0^\infty Q(k, t)\, dk = \int_0^\infty k^{-2} F(k, t)\, dk \tag{3.5}$$

is conserved by the interaction, in precise analogy to the conservation of kinetic energy. $Q(t)$ and $Q(k, t)$ can be interpreted as the mean-square and spectrum function of the vector potential.[3]

Equation (3.4) requires that $J(k, t)$ be negative for some k. In contrast, it can be shown that, when (3.2) is not imposed, the phases of the magnetic field can be chosen to yield

$$J(k, t) \geq 0 \quad (\text{all } k)$$

at some instant t, whatever the functions $E(k, t)$ and $F(k, t)$ are. Moreover, if the initial magnetic field has random phases, the initial growth of $J(k, t)$ always makes the left-hand side of (3.4) >0. Equation (2.21), with (2.18), the sine and cosine laws for a plane triangle, and a renaming of variables, yields

$$\int_0^\infty k^{-2}[\partial J(k, t)/\partial t]_{t=0}\, dk = \int_0^\infty dp \int_0^\infty dq\, I(p, q)Q(p, 0)E(q, 0), \qquad (3.6)$$

where

$$I(p, q) = pq \int_{-1}^{1} [p^2(1 - x^2)x/(p^2 + q^2 - 2pqx)]\, dx. \qquad (3.7)$$

The numerator of the integrand of $I(p, q)$ is antisymmetric in x and ≥ 0 for $0 \leq x \leq 1$. The denominator is ≥ 0 for $-1 \leq x \leq 1$ and decreases monotonically as x increases. Hence, $I(p, q) > 0$ for all p and q.

Clearly (3.2) represents a severe artificial constraint on the growth of the magnetic field. It imposes a conservation property which is unidirectionally violated in the mean if the initial magnetic seed fields are random. This fact limits the inferences which validly can be made from the vorticity analogy. The growth of vorticity in one region of the spectrum must always be accompanied by loss of vorticity in another region. If (3.4) were true, the growth of magnetic energy would suffer the same constraint. Magnetic energy at high wavenumbers could be sustained against Ohmic loss only by withdrawals of magnetic energy from lower wavenumbers. After the reservoir at low wavenumbers were exhausted, the magnetic spectrum would necessarily decay at all wavenumbers.[10] Since (3.4) does not hold if the phases of the seed field are random, the possibility is open that $J(k, t)$ evolves to simultaneously positive values at all k, so that the magnetic field exhibits true spontaneous growth and increases indefinitely, until the weak-field condition is violated. A corollary is that the vorticity analogy does not indicate what ratio λ/ν, if any, marks the division between decay and spontaneous growth.

[10] Cf. L. Mestel, in *Stellar and Solar Magnetic Fields* (North-Holland Publishing Company, Amsterdam, Netherlands, 1965), p. 424.

3.3. Inadequacy of Approximate Turbulence Theories

No general principle so far expounded appears to determine whether turbulent dynamos exist. It should be stressed that the failures of the equipartition considerations and the vorticity analogy are not on points of rigor but because crucial physical questions are untouched. In order to decide whether dynamos exist, it seems necessary to treat the detailed dynamics of the turbulence *quantitatively* so as to determine whether the local energy-enhancing or the sweeping-out processes are stronger. The e-folding times associated with both kinds of process are plausibly of the same order of magnitude: the eddy circulation time of some crucial band of wavenumbers that dominates the magnetic spectrum. If so, it is necessary to find the numerical ratio of two effective e-folding times which have the same functional dependence on the basic flow parameters.

This kind of task seems beyond the capabilities of the kinds of approximate turbulence theories which are now available. The approximations may include the principal dynamical processes and estimate their orders of magnitude correctly. But it is impossible to obtain numerical bounds on errors and therefore impossible to obtain reliable balances between competing processes whose strengths are comparable and whose outcome is not controlled by a helpful conservation law. To put it another way, the *magnitude* of the asymptotic growth rate of the magnetic spectrum can (hopefully) be determined approximately, but not the *sign* of the growth rate. The dynamo problem seems to pose a uniquely difficult challenge to theorists. Unless some valid way of looking at the problem is uncovered which eliminates the need for detailed dynamical knowledge, attack by direct computer experiment may be required.

The conflicting predictions of eventual growth or decay reached in previous published work appear to arise from neglecting or denying either local-enhancement or sweeping-out processes in critical wavenumber regions. Biermann and Schlüter[2] predict exponential growth for any λ/ν, but ignore completely the transfer of magnetic energy between different k bands. Moffatt[3] infers decay for $\lambda \gg \nu$, in the absence of steady input. But he considers only one-way transfer outward in k space, accompanied by amplification. The possibility of domination by local processes like those demonstrated in Sec. 3.1, where *both* the higher and the lower of the pair of magnetic wavenumbers gain energy, is not admitted. Saffman[4] predicts decay for $\lambda \ll \nu$, in the absence of steady input. He examines the region far above the viscous cut-off of the velocity field and concludes that these wavenumbers are stable to magnetic disturbances (see also Moffatt[11]). But the possibility that the magnetic spectrum is supported by local-enhancement processes at lower wavenumbers is denied by an appeal to the vorticity analogy. Pao[5] examines the same region as Saffman with different conclusions (amplification by a finite factor instead of eventual decay), but also omits the possibility of spontaneous growth at lower wavenumbers. Parker[6] considers only one-way transfer of magnetic energy outward in k space, which precludes a self-supported dynamo at the outset.

The three sections to follow give estimates of the behavior of the magnetic energy spectrum for each of the two possibilities, eventual growth or eventual decay, in the case where $\lambda \gg \nu$. No attempt is made to guess which possibility prevails. The estimates are made by simple dynamical reasoning based on the theory of the Kolmogorov inertial range. To a considerable extent, they reproduce or overlap conclusions by previous workers, but, taken in entirety, they do not agree with any of the authors. For the possibility of eventual growth, estimates are made of what happens when the magnetic field becomes strong enough that reaction on the velocity field is no longer negligible.

Following the qualitative analysis, the growth in the weak-field regime is calculated numerically using a simplified closure approximation suggested by the Lagrangian-history direct-interaction equations.[12] The result is eventual weak exponential growth of the magnetic spectrum at all wavenumbers. This is not evidence that actual turbulence behaves similarly. Instead, the results reinforce the conclusion that the balance between local enhancement and sweeping-out is too close to be resolved by approximate theories like any now available.

4. GROWTH RATE AND SPECTRUM SHAPE IF THERE IS EXPONENTIAL GROWTH

Suppose that the Reynolds number is large and that, prior to introduction of a magnetic seed field, the turbulence is isotropic and exhibits a Kolmogorov inertial range in which the kinetic-energy spectrum is

$$E(k) = C\epsilon^{2/3}k^{-5/3}, \tag{4.1}$$

where C is a number of order one and ϵ is the rate of dissipation of kinetic energy by viscosity, per unit mass. Equation (4.1) holds for

$$k_0 \ll k \ll k_s, \tag{4.2}$$

where k_0 is a typical energy-range wavenumber and $k_s = (\epsilon/\nu^3)^{1/4}$ is the Kolmogorov dissipation wavenumber. For $k > k_s$, $E(k)$ is a rapidly decreasing function of k. The latter wavenumber range will not be considered in this paper.

Now let a weak, statistically-isotropic, and randomly-phased magnetic excitation be introduced. Weak means that the Lorentz forces produce changes in the velocity field so small that they can be neglected in determining what the velocity field does to the magnetic field.

If k is in the inertial range, the typical circulation time for the eddies of size k^{-1} is $(\epsilon k^2)^{-1/3}$. This is plausibly the characteristic time for distortion of the magnetic field on the scale k^{-1}, provided that Ohmic dissipation effects are negligible. The Ohmic decay time at this scale is $(\lambda k^2)^{-1}$, so that $k_m = (\epsilon/\lambda^3)^{1/4}$ is the wavenumber at which distortion and dissipation effects on the magnetic field can be expected to be comparable. Assume that the magnetic Reynolds number is large; that is,

$$k_0 \ll k_m \ll k_s, \tag{4.3}$$

since $\lambda \gg \nu$ has been taken.

If $k_0 \ll k \ll k_m$, the characteristic e-folding time for processes which act to produce local equipartition between kinetic and magnetic energy and the characteristic time for removal of magnetic energy from the neighborhood of k by sweeping-outward processes should each be of the order of the local eddy circulation time. There does not appear to be another relevant time. If local enhancement overpowers sweeping out, the growth rate for net increase of the magnetic spectrum $F(k)$ should then be of order $(\epsilon k^2)^{1/3}$. Since this rate increases with k, the fastest growth can be expected in the region $k \sim k_m$. For $k > k_m$, the Ohmic damping overpowers the local enhancement processes. Thus the region $k \sim k_m$ should eventually dominate the spectrum, whatever the shape of the initial spectrum, and the total magnetic energy should then increase exponentially with a growth rate

$$\alpha = K(\epsilon k_m^2)^{1/3} = K(\epsilon/\lambda)^{1/2}, \tag{4.4}$$

[11] K. Moffatt, J. Fluid Mech. 17, 225 (1963).
[12] R. H. Kraichnan, Phys. Fluids 8, 575 (1965); 9, 1728, 1884, 1937 (1966).

where K is a numerical constant. If the eventual net balance between local enhancement, sweeping-outward, and local dissipation is very close at wavenumbers near the spectrum maximum, K could be small compared to one.

The form of the asymptotic spectrum for $k \ll k_m$ and $k \gg k_m$ is easily estimated. In the exponential-growth regime, the principal magnetic excitation should be associated with irregularly twisted and elongated current loops whose transverse dimension is $\sim k_m^{-1}$. The loops represent a dipole field. Consequently their spectrum should exhibit an excitation per wave-vector mode which is $\propto k^2$ for $k \ll k_m$. Since the density of modes in wavenumber space is $\propto k^2$, this implies

$$F(k) \propto k^4 \qquad (k \ll k_m). \tag{4.5}$$

For $k_m \ll k \ll k_s$, $F(k)$ should fall off rapidly, so that eddies of size k^{-1} act on magnetic fields which are nearly uniform across an eddy. The action of these eddies against the strong Ohmic damping then produces slight wiggling and stretching of the lines of force. This situation has been treated by Golitsyn[13] and Moffatt.[3] Their analyses agree, and give

$$F(k) \propto k^{-11/3} \qquad (k_m \ll k \ll k_s). \tag{4.6}$$

A simple physical argument leads to (4.6): The rate-of-strain associated with eddies of size $\sim k^{-1}$ is $\sim [k^3E(k)]^{1/2}$. The amplitude of the magnetic field excited at scales $\sim k^{-1}$ should be proportional to this rate-of-strain, proportional to the underlying magnetic field at lower wavenumbers, and inversely proportional to the damping rate λk^2. Since $kF(k)$ measures the mean square of the amplitude, this implies

$$F(k) \sim (\lambda k^2)^{-2} b_0^2 k^2 E(k), \tag{4.7}$$

where $3b_0^2/2$ is the total magnetic energy per unit mass at lower wavenumbers. Equations (4.7) and (4.1) yield (4.6). Equation (4.7) further implies that $F(k)$ falls off rapidly for $k > k_s$.

An idealized representation of the asymptotic $F(k)$ in the exponential-growth weak-field regime is given in Fig. 1. All the features inferred above are supported by the more detailed analytical and numerical results of Secs. 7 and 8.

5. APPROACH TO EQUIPARTITION IF GROWTH PREVAILS

If the magnetic energy grows exponentially as described in Sec. 4, the Lorentz forces eventually

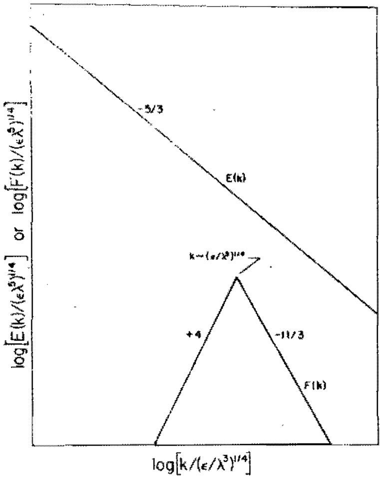

FIG. 1. Idealized magnetic energy spectrum $F(k)$ and kinetic energy spectrum $E(k)$ in the exponential-growth weak-field regime.

become significant and finally some equilibrium partition of energy between magnetic and velocity field should be reached. It has been argued[14] that the equilibrium inertial range of hydromagnetic turbulence exhibits exact equipartition between magnetic and kinetic energy $[F(k) = E(k)]$ and a spectrum law $E(k) \propto k^{-3/2}$ in place of the Kolmogorov law (4.1). The change in spectrum law is associated with a transformation of the physical character of the inertial-range motion. If the magnetic energy at wavenumbers below an inertial-range wavenumber k is larger than the kinetic energy in wavenumbers $\gtrsim k$, then the tension of the effectively uniform lines of force associated with the low wavenumbers changes the eddy motion at wavenumber k into Alfvén waves propagating along the lines of force.

The Alfvén period $(b_0k)^{-1}$, where $3b_0^2/2$ is the total contemporaneous magnetic energy per unit mass, is the characteristic time for exchange of energy between magnetic and kinetic modes at the same k, while the energy cascade up the spectrum is associated with the scattering between the waves travelling in opposite directions along the lines of force. This scattering is weak, in the sense that little of a wave's energy is scattered out in one Alfvén period.[14] As a result, the efficiency of cascade is

[13] G. S. Golitsyn, Dokl. Akad. Nauk SSSR 132, 315 (1960) [English transl.: Soviet Phys.—Doklady 5, 536 (1960)].

[14] R. H. Kraichnan, Phys. Fluids 8, 1385 (1965).

reduced, relative to the pure hydrodynamic case, and there is a pile-up of the energy sent up from below, which raises the spectrum from the $(k/k_0)^{-5/3}$ level to the $(k/k_0)^{-3/2}$ level. The equilibrium spectrum is $F(k) = E(k) \sim (\epsilon b_0)^{1/2} k^{-3/2}$, while ϵ is the same order of magnitude as in a pure hydrodynamic flow with the same energy-range parameters.

The effects just described imply an approach to equipartition agreeing in important respects with that proposed by Biermann and Schlüter.[2] Equipartition should be reached first at $k \sim k_m$, where the weak-field spectrum peaks, and should then spread down the spectrum until the energy range is reached. While this process is going on, the spectrum law for kinetic energy is $-5/3$ below the equipartition region and $-3/2$ within the region, up to wavenumbers where dissipation becomes important.

The effects of Ohmic dissipation to be expected during the spread of equipartition are more complex than in the weak-field regime. For $k < b_0/\lambda$, the Alfvén frequency $b_0 k$ exceeds λk^2, which implies that equipartition between $F(k)$ and $E(k)$ is maintained in the face of the Ohmic loss. If the wavenumber k_d at which the $-3/2$ region cuts off is $< b_0/\lambda$, it can then be determined by equating the rate of total

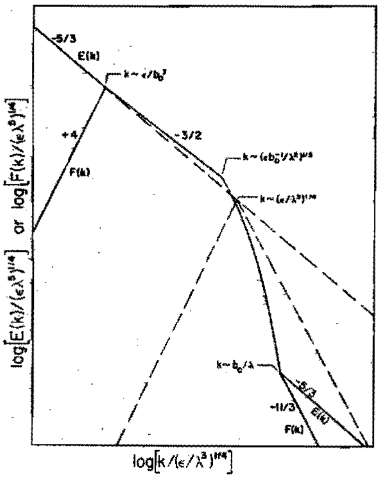

FIG. 2. Idealized spectra after an extensive equipartition range has been established. The curved line-segment is the equipartition Ohmic-dissipation subrange. The total contemporaneous magnetic energy is $3b_0^2/2$. The dashed lines show $F(k)$ and $E(k)$ at the earlier time when equipartition was first attained at k_m. Note that both $F(k)$ and $E(k)$ have decreased at high k since that time.

energy cascade ϵ to the Ohmic dissipation, taking $k \sim \epsilon/b_0^3$ as the low end of the $-3/2$ range:

$$\lambda(\epsilon b_0)^{1/2} k_d^{3/2} \sim \epsilon, \qquad k_d \sim (\epsilon b_0^{-1}/\lambda^2)^{1/3}, \tag{5.1}$$

where numerical factors are neglected.

In the region $k_d < k < b_0/\lambda$, equipartition is maintained and the competition between rising Ohmic damping and weakening cascade should give a rapid fall-off of the spectrum. Analogy to a pure hydrodynamic dissipation range suggests that the fall-off is exponential in character. For $k \gg b_0/\lambda$, equipartition cannot be maintained against Ohmic loss, and the asymptotic dynamics should be those of a weakly conducting fluid. If so, the $-5/3$ law for $E(k)$ should be re-established, but with a cascade rate much smaller than ϵ if the equipartition dissipation range is extensive, while $F(k)$ should behave like $k^{-11/3}$.

When equipartition is first reached,

$$b_0 \sim [k_m E(k_m)]^{1/2} \sim (\epsilon\lambda)^{1/4},$$

so that k_m, k_d, and b_0/λ are all about the same. As the equipartition region grows, k_d moves downward from k_m and b_0/λ moves upward, since b_0 increases. Thus the preceding discussion is self-consistent. The parameter ϵ is expected to stay roughly constant during the spread of equipartition, because it represents the conservative cascade of total energy up from the energy containing wavenumbers. After the equipartition region is extensive, this cascade is balanced principally by Ohmic dissipation. The spectrum structure during the spread of equipartition is shown in idealized form in Fig. 2.

When equipartition has reached down to the energy-containing wavenumbers k_0, then $b_0 \sim v_0$, where $3v_0^2/2$ is the kinetic energy per unit mass. At that stage, the preceding formulas give

$$k_d \sim R_{\text{mag}}^{2/3} k_0, \qquad b_0/\lambda \sim R_{\text{mag}} k_0, \tag{5.2}$$

where

$$R_{\text{mag}} = v_0/(\lambda k_0) \tag{5.3}$$

is the magnetic Reynolds number of the turbulent motion. For comparison,

$$k_m \sim R_{\text{mag}}^{3/4} k_0 \tag{5.4}$$

since $\epsilon \sim v_0^3 k_0$ is an accepted consequence of the energy-range dynamics.

It can be objected that the continued approach to equipartition at lower wavenumbers does not follow logically from the supposition of exponential growth under weak-field conditions. Perhaps after equipartition is reached in the neighborhood of k_m, the magnetic spectrum attains a steady form that falls off toward lower wavenumbers. Such behavior has been suggested by Batchelor,[1] although for $\lambda < \nu$. This possibility seems unlikely. The supposition of exponential growth implies that local enhancement processes at $k \sim k_m$ prevail over sweeping-outward processes and local Ohmic losses. At $k \ll k_m$, the Ohmic loss in one eddy-circulation time (the e-folding time for local enhancement) is less than at $k \sim k_m$. Also, a spectrum which rises with k tends to inhibit the sweeping-outward processes. The net transfer between different wavenumber regions is the resultant of a *two-way* exchange, and the back-flow from high to low wavenumbers is roughly proportional to the excitation at the higher wavenumbers (cf. Sec. 2). Both considerations suggest that if local enhancement processes can give exponential growth at $k \sim k_m$, where the spectrum bends over, they surely can do so at lower wavenumbers, where the specturm is rising with wavenumber. Of course, a longer e-folding time is expected at the lower wavenumbers because the eddy-circulation time is longer.

6. STEADY-STATE SPECTRUM IF THERE IS NOT SPONTANEOUS GROWTH

If an initial weak magnetic field eventually dies away, contrary to the supposition of Secs. 4 and 5, then a statistically steady supply of magnetic energy should produce a steady-state spectrum. An attempt will now be made to estimate the form of this spectrum if the input is in a band of wavenumbers $k_{in} \ll k_m$.

Moffatt[3] has treated the problem on the basis of the vorticity analogy and finds $F(k) \propto k^{1/3}$ for $k_{in} \ll k \ll k_m$. It is important to note that derivations of the $\epsilon^{2/3}k^{1/3}$ vorticity spectrum in the Kolmogorov inertial range are possible only because there *is* the parameter ϵ which measures the conservative cascade of kinetic energy up the spectrum. In the magnetic-field-vorticity analogy, the quantity analogous to kinetic energy is $Q(t)$, the mean square of the vector potential.[3] However, Q is *not* conserved if the phase of the initial magnetic seed field is random. As shown in Sec. 3.2, the initial growth of Q is always positive. Thus the $k^{1/3}$ law for $F(k)$ cannot validly be inferred.

If it is meaningful at all to speak of a cascade of magnetic energy analogous to the cascade of vorticity, then the tendency of Q to grow implies that the parameter analogous to ϵ is not a constant, but instead increases with k. The increase is associated with the local enhancement processes, which add to Q at each step of the cascade. Similarity considerations suggest that the amplification of Q during each, say, doubling of wavenumber in the cascade should be constant throughout the region $k_{in} \ll k \ll k_m$. Comparison with the vorticity result then gives $F(k) \propto k^n$, with $n > 1/3$.

However, there is an upper limit on n. The excitation at wavenumbers $\sim k_m$ is dipole in character so that it contributes a low-wavenumber tail $\propto k^4$ to $F(k)$, according to an argument of Sec. 4. Then if n were >4, there would be a driven excitation in the neighborhood of k_{in} that exceeded the input excitation and the system would be regenerative, contradicting the present supposition that the magnetic field cannot maintain itself. Thus the final result is

$$F(k) \propto k^n, \quad \tfrac{1}{3} < n \leq 4 \quad (k_{in} \ll k \ll k_m). \tag{6.1}$$

It should be stressed that the value of the exponent n in (6.1) depends on the numerical value of the effective amplification per cascade step. In the absence of a relevant conservation law, it does not seem possible to determine n by general considerations. In order to sharpen (6.1), it seems necessary to make detailed dynamical calculations.

For $k \gg k_m$, the arguments of Sec. 4 should apply equally well under the present assumptions, so that the spectrum should obey (4.6), as concluded by Moffatt.

7. AN APPROXIMATE LONG-TIME TRANSFER FORMULA

Consider the approximate long-time transfer formulas

$$L(k, t) = -kE(k, t) \iint_\Delta p^2 c_{kpq} F(q, t)\theta_{pkq} \frac{dp\,dq}{pq}, \tag{7.1}$$

$$J(k, t) = k \iint_\Delta [k^2 d_{kpq} F(p, t)\theta_{kpq} - p^2 h_{kpq} F(k, t)\theta_{pkq}] E(q, t) \frac{dp\,dq}{pq}, \tag{7.2}$$

where the quantity θ_{kpq} is an effective memory time for the interaction of wavenumbers k, p, q. If the θ factors were removed from (7.1) and (7.2), the right-hand sides would be identical with the exact expressions (2.20) and (2.21) for the initial time derivatives of $L(k, t)$ and $J(k, t)$ in the weak-field regime. Thus (7.1) and (7.2) may be interpreted as follows. They incorporate all the processes which contribute to the initial development of energy transfer, and with the same geometrical coefficients. However, they recognize that the initial growth of phase correlation between magnetic and velocity field (growth of triple moments) cannot continue forever; if it did, $L(k, t)$ and $J(k, t)$ would become infinite. Instead, the phase correlation among modes k, p, q should level off at some relevant relaxation or memory time.

Equations (7.1) and (7.2) may be derived by the Lagrangian-history direct-interaction approximation,[12] which also yields integrodifferential equations that determine θ_{kpq}. This fact will not be used in the present application. Equations (7.1) and (7.2) will be taken on their merits, and the θ_{kpq} will be approximated according to simple ideas. The discussion will make clear that more refined approximation and a detailed derivation would add little to the persuasiveness of the final results.

Since a cosine never exceeds one in absolute value, the coefficients $\bar{a}_{kpq}$ and h_{kpq} are never negative. Thus the first term on the right-hand side of (7.2) represents a positive input of energy at magnetic wavenumber k that is proportional to both $F(p, t)$ and $E(q, t)$. In other words, it represents a driving of wavenumber k by magnetic wavenumber p and kinetic wavenumber q. There are two types of characteristic times which should determine the effective memory time θ_{kpq} for the driving process. First, the memory time is limited by the effective correlation times of the driving amplitudes. Second, it is limited by the effective damping time of the driven mode. If k is sufficiently large, the latter time is just the Ohmic damping time $(\lambda k^2)^{-1}$. For lower k, there will also be effective damping by eddy processes, expressible by an eddy diffusivity.

A simple form incorporating these ideas is

$$\theta_{kpq} = [(\lambda k^2)^2 + \eta_k^2 + \xi_p^2 + \zeta_q^2]^{-1/2}, \qquad (7.3)$$

where η_k, ξ_p, and ζ_q are, respectively, the effective reciprocal times for eddy damping of magnetic mode k, correlation of magnetic mode p, and correlation of velocity mode q. The particular functional form is chosen for convenience and has no deeper justification.

The second term on the right-hand side of (7.2) is always negative, representing a loss of energy from magnetic mode k. Since the interaction of magnetic and velocity fields conserves the total energy, this loss must show up as a net gain in magnetic mode p and velocity mode q. The conservation requirement imposes a relation among the θ factors in the two terms of (7.2) and in (7.1) which has already been incorporated in writing the equations: the three factors are all the same, except for a permutation of indices. It is easily verified that, whatever the form of θ_{kpq} and of the spectra $F(k, t)$ and $E(k, t)$, (7.1), (7.2), and (2.18) yield the second conservation law (2.14). In the weak-field regime, $L(k, t)$ is too small to affect the evolution of $E(k, t)$ appreciably, but it is important to the logical consistency of the transfer approximation that the detailed conservation properties of the exact equations of motion survive.

When the velocity spectrum has the form (4.1) at all wavenumbers of interest, the ideas underlying the Kolmogorov theory suggest that η_k, ξ_k, and ζ_k should all be proportional to the local reciprocal eddy-circulation time $(\epsilon k^2)^{1/3}$, with numerical coefficients of order one. The final form of θ_{kpq} in this case is then

$$\theta_{kpq} = [(\lambda k^2)^2 + \epsilon^{2/3}(A_1^2 k^{4/3} + A_2^2 p^{4/3} + A_3^2 q^{4/3})]^{-1/2}, \qquad (7.4)$$

where A_1, A_2, and A_3 are the numerical coefficients.

8. NUMERICAL RESULTS AND THEIR INTERPRETATION

The magnetic spectrum growth was calculated numerically in the weak-field regime using the approximate transfer formulas of Sec. 7. The constants in (4.1) and (7.4) were assigned the values

$$C = 1.5, \quad A_1 = 1.5, \quad A_2 = A_3 = 1.0, \qquad (8.1)$$

and the initial magnetic-energy spectrum was given the form

$$F(k, 0) \propto k^4 \exp[-2(k/k_{in})^2], \qquad (8.2)$$

which has a maximum at $k = k_{in}$. The value $k_{in} = 0.0027\, k_m$ was taken as an arbitrary choice satisfying $k_{in} \ll k_m$.

The equality of effective magnetic and kinetic modal correlation times assumed in (8.1) was chosen as the simplest possibility. The eddy-damping time was taken shorter than the correlation times because inertial-range calculations for hydrodynamic turbulence indicated such behavior for eddy damping of velocity modes.[12] The particular choice 1.5 for the ratio of the times is arbitrary. It should be noted that only the ratios of the A's can affect the nature of the transfer. Changing the value of the Kolmogorov constant C or scaling the A's by a constant factor multiplies $J(k, t)$ by the same factor for all k and is equivalent to a rescaling of time and wavenumber units.

The results of the calculation are displayed in Figs. 3 to 6. The growth of total magnetic energy was monotonic. The initial value of the exponential growth rate for total energy was about $0.4(\epsilon/\lambda)^{1/2}$, and the growth rate decreased monotonically to an asymptotic value $2.3 \times 10^{-3}(\epsilon/\lambda)^{1/2}$ which was achieved by $t \sim 10^3(\lambda/\epsilon)^{1/2}$.

The evolution of the spectrum (Fig. 4) fell into three stages. First, there was a rapidly established increase in energy at high wavenumbers accompanied by a slowly growing and enormously smaller loss in the region around k_{in}. Next, the increase at high wavenumbers continued, while the rate of loss in the region of initial excitation decreased. Finally, the energy grew at all wavenumbers and the spectrum evolved toward an asymptotic equilibrium shape with the intensity increasing at the same exponential growth rate everywhere. The maximum loss of energy at k_{in} was 27%, at $t \sim 300(\lambda/\epsilon)^{1/2}$. By the end of the calculation $[t \sim 4.6 \times 10^3(\lambda/\epsilon)^{1/2}]$, the total energy was amplified by 8.8×10^7 and the spectrum level at k_{in} was 3.8×10^3 times its initial value.

The energy maximum in the asymptotic spectrum was at $k \sim 0.12\, k_m$. Throughout the evolution, most of the positive contributions to $\int_0^\infty J(k, t)\, dk$ were at wavenumbers near and above this asymptotic maximum, with a small positive contribution at wavenumbers below the region of peak initial excitation. The form of $J(k, t)$ during the first and third stages of evolution is shown in Figs. 5 and 6.

The shape of the asymptotic spectrum at wavenumbers below and above the maximum is consistent

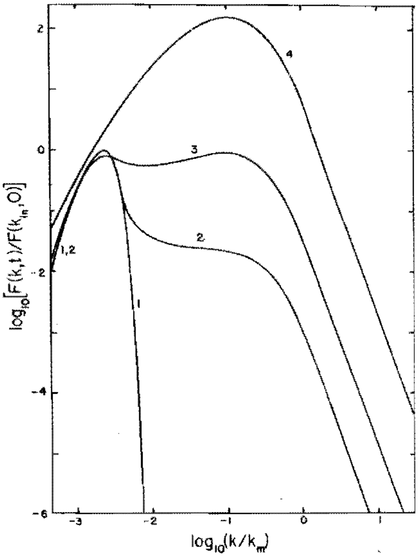

Fig. 4. Evolution of $F(k, t)$ computed with the approximate transfer function. Curve 1, $t = 0$; curve 2, $t = 11.4$; curve 3, $t = 114$; curve 4, $t = 1140$. These times are measured in the unit $(\lambda/\epsilon)^{1/2}$.

with (4.5) and (4.6). The latter results are also easily obtained analytically from (7.2)–(7.4) under the assumption that most of the energy is at wavenumbers the order of $(\epsilon/\lambda^3)^{1/4}$ and is growing with time.

A feature of particular interest is that the energy increase was dominated by high wavenumbers even at the earliest times of evolution, when insufficient excitation had developed at intermediate wavenumbers to support a cascade. At these times, there was strong *direct* energy coupling over a jump of two decades in wavenumber. This phenomenon has a simple physical interpretation and appears not to be an artifact of the approximation. The generation of small-scale magnetic excitation by small eddies depends on the *strength* of the magnetic field rather than the *gradient* of the field, since this excitation can be visualized as a wiggling of the lines of force. Thus the rate of energy transfer to high wavenumbers depends principally upon the total magnetic energy in lower wavenumbers and is largely independent of how low in wavenumber that energy may be. The absolute rate of energy input to the high wavenumbers (as opposed to the exponential growth rate) increases with time during the initial

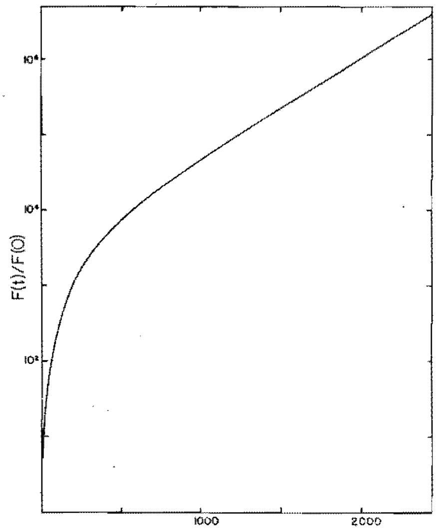

Fig. 3. Growth of total magnetic energy $F(t)$ computed with the approximate transfer function.

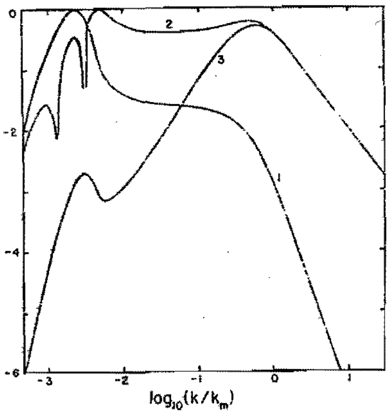

FIG. 5. Magnetic energy, transfer, and dissipation spectra at $t = 11.4(\lambda/\epsilon)^{1/2}$. Curve 1, $\log_{10} F(k, t)$; curve 2, $\log_{10}|J(k, t)|$; curve 3, $\log_{10} [2\lambda k^2 F(k, t)]$. $F(k, t)$ is normalized by its peak value at time t, while $J(k, t)$ and $2\lambda k^2 F(k, t)$ both are normalized by the peak value of $J(k, t)$. The breaks in curve 2 mark changes of sign of $J(k, t)$; the transfer is negative in the middle segment.

stage of evolution because the total magnetic energy at lower wavenumbers increases.

The transfer mechanism for the magnetic field differs profoundly from that for a convected passive scalar field. In the latter case, generation of small-scale excitation depends on the magnitude of the gradients on which the small-scale eddies can act, and vanishes if the initial excitation is spatially uniform.

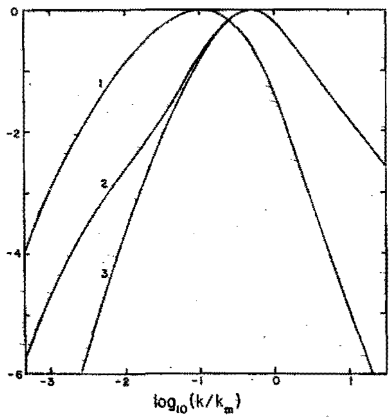

FIG. 6. Magnetic energy, transfer, and dissipation spectra after the asymptotic exponential growth rate has been attained [$t = 4580(\lambda/\epsilon)_0^{1/2}$]. The curves are labeled as in Fig. 5.

At the maximum of the asymptotic energy spectrum ($k \sim 0.12\, k_m$), the reciprocal eddy circulation time $(\epsilon k^2)^{1/3}$ is $\sim 0.25(\epsilon/\lambda)^{1/2}$, while the Ohmic decay rate λk^2 is $\sim 0.014(\epsilon/\lambda)^{1/2}$. The total contribution to $J(k, t)$ at this wavenumber can be divided into positive contributions (from triad interactions yielding local enhancement) and negative contributions (from triad interactions associated with sweeping-out processes). It is fairly clear, and verified by the calculation, that either the positive or the negative contributions, taken alone, would give a local growth rate with absolute value the order of the reciprocal eddy circulation time. Since the actual asymptotic growth rate is only $2.3 \times 10^{-3}(\epsilon/\lambda)^{1/2}$, the local enhancement and sweeping-out processes are very nearly in balance. They differ, to first approximation, only by the relatively small Ohmic dissipation rate.

This close balance is the most important result of the calculation. The prediction of eventual growth rather than decay cannot be deemed persuasive, because it may be reversible by a change in the form of the transfer approximation, or even by a change in the ratios of the A's. But it is significant that a physically plausible transfer function, which includes consistently all the processes found in the exact initial transfer formula, gives a neck-and-neck race between enhancement and sweeping outward. This adds substance to the conclusion that approximate turbulence theories cannot be relied upon to predict whether turbulent dynamos exist. However, if eventual growth does prevail in nature, the predictions of (7.1) and (7.2) may be valid in considerable detail.

In some applications it may turn out not to be crucial whether there are self-sustaining dynamos. If the inertial range is very extensive, and the input magnetic excitation is on a large enough spatial scale and sufficiently persistent, the turbulence may produce large amplification of the magnetic energy at times of interest whether the eventual fate is growth or decay. It remains possible that well-constructed turbulence approximations can be useful in predicting the earlier stages of growth, whatever happens later.

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STEADY-STATE HYDROMAGNETIC TURBULENCE

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ABSTRACT

The direct interaction approximation for hydromagnetic turbulence maintained by stationary, isotropic, random stirring forces is formulated in the wave-number-frequency domain. Simplifying assumptions are introduced about the functional forms of correlation and response functions of the velocity and magnetic fields and a closed set of six nonlinear integral equations are derived for them. These are solved through an iterative procedure, for prescribed spectra and frequencies of the random stirring forces. Kolmogorov's ideas of local isotropy and their relevence to the hydromagnetic situation are reviewed with the special view to study the Galilean non-invariance of the hydromagnetic equations to a random constant magnetic field transformation.

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Lagrangian implications of this are discussed and a recipe through which this non-invariance can be taken care of is suggested and exploited. Solutions to the steady-state equations under these limitations display an unequivocal almost-exact and detailed equipartition between magnetic and velocity modes in the inertial range. The peculiar hydromagnetic non-invariance tends only to accentuate this.

KEY WORDS: Turbulence, Non-equilibrium Statistical Mechanics, Generalised relaxation processes, Magnetohydrodynamics, Plasma turbulence, Astrophysics, Turbulent Dynamo.

1. Introduction

In an earlier paper[1], an attempt was made to predict unequivocally whether in a highly conducting, isotropically turbulent fluid, a random, weak initial magnetic field would grow and, if so, to determine the kinematic conditions under which this would ensue. It was concluded that the present day dynamical theories of turbulence were inadequate in resolving the question of growth or decay uniquely. But, granting that one or the other alternative wins, one could predict the ultimate magnetic energy spectrum. In this paper, we explore the question of ultimate evolutionary steady-state assuming that the growth wins.

In Section 2, we derive the direct-interaction approximation equations in the isotropic, homogeneous and stationary hydromagnetic case, which bring the magnetic situation to the same level of completeness as the hydrodynamic situation[2,3]. In Section 3, we develop this formalism, in the wave number-frequency domain, for the stationary situation. In Section 4, this model is further developed with an idea of trying to explore what minimal information about the time structure of the correlation and response functions would be needed to determine the steady-state spectra of the velocity and magnetic modes. Assuming a functional form for these two functions, the complicated non-linear dispersion-theoretic formalism connecting the two Green's functions, spectral functions and correlation functions is reduced to six integral equations.

In Section 5, a not-too-detailed survey of the Kolmogorov ideas of local isotropy and their relevence in the hydromagnetic context is given. It is argued that, in fact, the usual discrepency between the unmodified direct interaction inertial-range solutions, which display an energy range-inertial-range coupling and the Kolmogorov spectrum, which is strictly local in wave number space, arises from the confusion between Lagrangian and Eulerian correlation times[4,5]. These are connected with the invariance of the Navier-Stokes equations under random Galilean Transformations. But the hydromagnetic equations display an asymmetry, in so far as a constant magnetic field cannot be gauged out, in the same way as a constant velocity field can, in a co-moving coordinate system.

In Section 6, an iterative procedure, through which these equations can be solved numerically is described. Various modifications to the iterative procedure, through which the various Lagrangian and quasi-Lagrangian history behaviour can be approximately taken care of are delineated. Finally, a partial Lagrangian modification through which the Galilean non-invariance of the hydromagnetic equations in relation to magnetic terms is explicitly taken care of is described.

In Section 7, the results of this study are compared critically with other investigations. It is concluded that the really convincing, invariant result of the analysis, which remains common in all modifications is that in the steady-state, there is an almost-exact and detailed equipartition between magnetic and velocity modes in the inertial range. Further, the reduced Lagrangian history hydromagnetic modification also shows that the peculiar non-invariance tends to increase the energy-level in the magnetic spectrum.

2. Equations in the Stationary Hydromagnetic Case

We will start with a recapitulation of the direct interaction approximation for hydromagnetic turbulence, which is homogeneous, isotropic and stationary. The standard incompressible hydromagnetic equations are[6]

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\underset{\sim}{u}(\underline{x};t) = -\underset{\sim}{\nabla}p + (\underset{\sim}{w}\cdot\underset{\sim}{\nabla})\underset{\sim}{w} - (\underset{\sim}{u}\cdot\underset{\sim}{\nabla})\underset{\sim}{u} + \underset{\sim}{f}$$

$$\left(\frac{\partial}{\partial t} - \lambda\nabla^2\right)\underset{\sim}{w}(\underline{x};t) = -(\underset{\sim}{u}\cdot\underset{\sim}{\nabla})\underset{\sim}{w} + (\underset{\sim}{w}\cdot\underset{\sim}{\nabla})\underset{\sim}{u}$$

$$(\nabla\cdot\underset{\sim}{u}) = 0 \qquad ; \quad (\underset{\sim}{\nabla}\cdot\underset{\sim}{w}) = 0 \tag{1}$$

where $\underset{\sim}{u}(\underline{x};t)$ and $(4\pi\mu\rho)^{1/2}\,\underset{\sim}{w}(\underline{x};t)$ are the velocity and magnetic induction fields, $\underset{\sim}{f}$ is the random solenoidal driving force, ρp is the pressure and ν, λ, μ, ρ are the kinematic viscosity, magnetic diffusivity, susceptibility and density of the fluid, respectively.

The direct interaction closure procedure of Kraichnan, when applied to these set of equations yields in the stationary situation[2,6]

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) W^V(k;t) = \iint_\Delta \pi k p q \, dp \, dq \int_{-\infty}^{\infty} ds \,\cdot$$

$$\cdot \Big[a(k,p,q)\, G^V(k;s) \big\{ W^V(p;t+s)\, W^V(q;t+s) + W^M(p;t+s)\, W^M(q;t+s) \big\}$$

$$- \big\{ b(k,p,q)\, G^V(p;t+s)\, W^V(q;t+s) + c(k,p,q)\, G^M(p;t+s)\, W^M(q;t+s) \big\} W^V(k;s) \Big]$$

$$+ Z(k;t) \qquad (2)$$

$$\left(\frac{\partial}{\partial t} + \lambda k^2\right) W^M(k;t) = \iint_\Delta \pi k p q \, dp \, dq \int_{-\infty}^{\infty} ds \,\cdot$$

$$\cdot \Big[d(k,p,q)\, G^M(k;s)\, W^M(p;t+s)\, W^V(q;t+s)$$

$$- \big\{ h(k,p,q)\, G^M(p;t+s)\, W^V(q;t+s) + j(k,p,q)\, G^M(p;t+s)\, W^M(q;t+s) \big\} W^M(k;s) \Big] \qquad (3)$$

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) G^V(k;t) = \delta(t) - \iint_\Delta \pi k p q \, dp \, dq \int_{-\infty}^{\infty} ds \,\cdot$$

$$\cdot \Big[\big\{ b(k,p,q)\, G^V(p;t-s)\, W^V(q;t-s) + c(k,p,q)\, G^M(p;t-s)\, W^M(q;t-s) \big\} G^V(k;s) \Big] \qquad (4)$$

$$\left(\frac{\partial}{\partial t} + \lambda k^2\right) G^M(k;t) = \delta(t) - \iint_\Delta \pi k p q \, dp \, dq \int_{-\infty}^{\infty} ds \,\cdot$$

$$\cdot \Big[\big\{ h(k,p,q)\, G^M(p;t-s)\, W^V(q;t-s) + j(k,p,q)\, G^V(p;t-s)\, W^M(q;t-s) \big\} G^M(k;s) \Big] \qquad (5)$$

Here $\iint_D$ denotes integration over all $\underline{p}$, $\underline{q}$ such that $\underline{k}$, $\underline{p}$ and $\underline{q}$ can form a triangle

$$W^V(\underline{k};t) = (2\pi)^{-3} \int d^3(\underline{x}-\underline{y}) \langle \underset{\sim}{u}(\underline{x};t)\cdot \underset{\sim}{u}(\underline{y};0)\rangle e^{i\underline{k}\cdot(\underline{x}-\underline{y})}$$

$$W^M(\underline{k};t) = (2\pi)^{-3} \int d^3(\underline{x}-\underline{y}) \langle \underset{\sim}{w}(\underline{x};t)\cdot \underset{\sim}{w}(\underline{y};0)\rangle e^{i\underline{k}\cdot(\underline{x}-\underline{y})}$$

$$F(\underline{k}:t) = (2\pi)^{-3} \int d^3(\underline{x}-\underline{y}) \langle \underset{\sim}{f}(\underline{x}:t)\cdot \underset{\sim}{f}(\underline{y}:0)\rangle e^{i\underline{k}\cdot(\underline{x}-\underline{y})}$$

and

$$Z(\underline{k};t) = \int_{-\infty}^{\infty} G^V(\underline{k};s)\, F(\underline{k};t+s)\, ds$$

$G^V(\underline{k};t-t')$ and $G^M(\underline{k};t-t')$ are the average Green's functions which give the mean response of the amplitude of mode k(V or M) at time t to an infinitesimal perturbation of that mode at time t′

$$G^{\{\}}(\underline{k};0) = 1$$

$$G^{\{\}}(\underline{k};t) = 0 \qquad t < 0 \tag{6}$$

The geometrical coefficients in the integrals on the right hand sides of (2) to (5) are given by

$$a(\underline{k},p,q) = \tfrac{1}{2}(1 - xyz - 2y^2z^2) \qquad d(\underline{k},p,q) = 1 + xyz$$

$$b(\underline{k},p,q) = (p/k)(xy + z^3) \qquad h(\underline{k},p,q) = (p/k)(xy + z)$$

$$c(\underline{k},p,q) = (p/k)\, z(1 - y^2) \qquad j(\underline{k},p,q) = (p/k)\, z(1 - x^2)$$

where x,y and z are the cosines of the interior angles opposite to the triangle sides k,p and q respectively. There are some relations between these geometrical coefficients, which illustrate the overall conservation properties of the non-linear interaction and the conservation-preservation feature of the direct interaction approximation. They are

$$2a(k,p,q) = b(k,p,q) + b(k,q,p)$$
$$= c(k,p,q) + c(k,q,p)$$

$$d(k,p,q) = h(k,p,q) + j(k,q,p)$$

$$a(k,p,q) = a(k,q,p) \geq 0$$

$$d(k,p,q) = d(k,q,p) = d(p,k,q) \geq 0$$

$$k^2 b(k,p,q) = p^2 b(p,k,q)$$

$$k^2 h(k,p,q) = p^2 h(p,k,q)$$

$$k^2 c(k,p,q) = p^2 (j(p,k,q))$$

The energy-spectrum functions $E^v(k)$, $E^m(k)$ and the modal correlation functions $R^v(k;t)$, $R^m(k;t)$ are defined according to

$$\frac{1}{2} W^v(k,t) = (4\pi k^2)^{-1} E^v(k) R^v(k,t)$$

$$\frac{1}{2} W^M(k,t) = (4\pi k^2)^{-1} E^M(k) R^M(k,t)$$

Similarly for the force field

$$F(k,t) = (4\pi k^2)^{-1} F(k) M(k,t)$$

with the initial conditions

$$R^V(k,0) = 1 \quad ; \quad R^M(k,0) = 1 \quad ; \quad M(k,0) = 1$$

3. Wave number - frequency domain

Consider the frequency domain functions given by

$$\tilde{W}^a(k:\omega) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i\omega t} W^a(k:t)\, dt$$

$$\tilde{F}(k:\omega) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i\omega t} F(k;t)\, dt$$

$$\tilde{G}^a(k:\omega) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i\omega t} G^a(k,t)\, dt$$

$$\tilde{R}^a(k:\omega) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i\omega t} R^a(k,t)\, dt$$

$$\tilde{M}(k:\omega) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i\omega t} M(k:t)\, dt \qquad (7)$$

In all these a = V or M.

From the reality, stationarity and isotropy of the $\underset{\sim}{u}$, $\underset{\sim}{\omega}$ and $\underset{\sim}{f}$, it follows that

$$\tilde{W}^a(k:\omega) = \left| \tilde{W}^a(k:\omega) \right| = \tilde{W}^a(k;-\omega)$$

$$\tilde{R}^a(k:\omega) = \left| \tilde{R}^a(k:\omega) \right| = \tilde{R}^a(k;-\omega)$$

$$\tilde{F}(k:\omega) = \left| \tilde{F}(k:\omega) \right| = \tilde{F}(k:-\omega)$$

$$\tilde{M}(k:\omega) = |\tilde{M}(k:\omega)| = \tilde{M}(k;-\omega) \tag{8}$$

From the reality, stationarity and isotropy properties and the causal definitions of the Green's functions follow the dispersion relations

$$\tilde{G}^a(k:\omega) = \rho^a(k:\omega) + i\pi^{-1} P\left\{\int_{-\infty}^{\infty} \rho^a(k:\omega')(\omega-\omega')^{-1} d\omega'\right\} \tag{9}$$

in which $\rho^a(k:\omega) = Re\{\tilde{G}^a(k:\omega)\}$

with the identification $P\{\ \}$ is the principal part and $Re\{\ \}$ the real part. (a = V or M)

The transforms of (2-5) are

$$\left[-i\omega + \nu k^2 + \Sigma(k;\omega) + \Sigma^M(k:\omega)\right]\tilde{G}^V(k:\omega) = 1 \tag{10}$$

$$\left[-i\omega + \lambda k^2 + \Theta(k:\omega) + \Theta^M(k:\omega)\right]\tilde{G}^M(k:\omega) = 1 \tag{11}$$

$$\left[-i\omega + \nu k^2 + \Sigma(k:\omega) + \Sigma^M(k:\omega)\right]\tilde{W}^V(k:\omega)$$

$$= \left[\Gamma(k:\omega) + \Gamma^M(k:\omega)\right]\tilde{G}^{V*}(k:\omega) + \tilde{Z}^*(k:\omega) \tag{12}$$

$$\left[-i\omega + \lambda k^2 + \Theta(k:\omega) + \Theta^M(k:\omega)\right]\tilde{W}^M(k:\omega) = \lambda(k:\omega)\tilde{G}^{M*}(k:\omega) \tag{13}$$

with the following identifications:

$$\Sigma(k:\omega) = \iint_\Delta \pi kpq\,dp\,dq\; b(k,p,q)\int_{-\infty}^{\infty} d\omega'\, \tilde{G}^V(p;\omega')\,\tilde{W}^V(q;\omega-\omega') \tag{14}$$

$$\Sigma^M(k:\omega) = \iint_\Delta \pi kpq\,dp\,dq\; c(k,p,q)\int_{-\infty}^{\infty} d\omega'\, \tilde{G}^M(p;\omega')\,\tilde{W}^M(q;\omega-\omega') \tag{15}$$

$$\textcircled{H}(k:\omega) = \iint_{\Delta} \pi k p q \, dp \, dq \; h(k,p,q) \int_{-\infty}^{\infty} d\omega' \, \tilde{G}^{m}(p;\omega') \, \tilde{W}^{v}(q;\omega-\omega') \tag{16}$$

$$\textcircled{H}^{m}(k;\omega) = \iint_{\Delta} \pi k p q \, dp \, dq \; j(k,p,q) \int_{-\infty}^{\infty} d\omega' \, \tilde{G}^{v}(p;\omega') \, \tilde{W}^{m}(q;\omega-\omega') \tag{17}$$

$$\Gamma(k:\omega) = \iint_{\Delta} \pi k p q \, dp \, dq \; a(k,p,q) \int_{-\infty}^{\infty} d\omega' \, \tilde{W}^{v}(p:\omega') \, \tilde{W}^{v}(q;\omega-\omega') \tag{18}$$

$$\Gamma^{m}(k:\omega) = \iint_{\Delta} \pi k p q \, dp \, dq \; a(k,p,q) \int_{-\infty}^{\infty} d\omega' \, \tilde{W}^{m}(p,\omega') \, \tilde{W}^{m}(q;\omega-\omega') \tag{19}$$

$$\lambda(k:\omega) = \iint_{\Delta} \pi k p q \, dp \, dq \; d(k,p,q) \int_{-\infty}^{\infty} d\omega' \, \tilde{W}^{m}(p;\omega') \, \tilde{W}^{v}(q;\omega-\omega') \tag{20}$$

$$\tilde{Z}(k:\omega) = \tilde{G}^{v*}(k:\omega) \, \tilde{F}(k:\omega) \tag{21}$$

From the definitions of $\tilde{W}^{a}(k:\omega)$ and the coeffecients $a(k,p,q)$ and $d(k,p,q)$, it follows that $\Gamma(k:\omega)$, $\Gamma^{M}(k:\omega)$ and $\lambda(k:\omega)$ are positive and real.

From (10) and (11) we get

$$\tilde{G}^{v}(k:\omega) = \left[-i\omega + \nu k^{2} + \Sigma(k:\omega) + \Sigma^{m}(k:\omega)\right]^{-1} \tag{22}$$

$$\tilde{G}^{m}(k:\omega) = \left[-i\omega + \lambda k^{2} + \textcircled{H}(k:\omega) + \textcircled{H}^{m}(k:\omega)\right]^{-1} \tag{23}$$

From (12), (13), (22) and (23) we get

$$\tilde{W}^{v}(k:\omega) = \left[\Gamma(k:\omega) + \Gamma^{m}(k:\omega) + F(k:\omega)\right] \cdot \left|G^{v}(k:\omega)\right|^{2} \tag{24}$$

$$\tilde{W}^{m}(k:\omega) = \lambda(k:\omega) \cdot \left|\tilde{G}^{m}(k:\omega)\right|^{2} \tag{25}$$

Equations (22) to (25) constitute a complete set of four integral equations for determining the Green's functions and covariances of the velocity and magnetic field, given the form of the forcing field covariance.

4. Characteristic Frequency Approximation.

In (2) and (3), if we get t = 0, we get the energy balance equations

$$
\begin{aligned}
\nu k^2 E^V(k) = \iint_\Delta \frac{k}{2pq}\, dp\, dq \Big[\Big\{ & k^2 a(k,p,q)\, E^V(p)\, E^V(q)\, \theta(VVV; k,p,q) \\
& - p^2 b(k,p,q)\, E^V(k)\, E^V(q)\, \theta(VVV; p,q,k) \Big\} \\
+ \Big\{ & k^2 a(k,p,q)\, E^M(p)\, E^M(q)\, \theta(VMM; k,p,q) \\
& - p^2 c(k,p,q)\, E^V(k)\, E^M(q)\, \theta(MMV; p,q,k) \Big\} \Big] \\
& + \frac{1}{2} F(k)\, \theta(k)
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
\lambda k^2 E^M(k) = \iint_\Delta \frac{k}{2pq}\, dp\, dq \Big[\Big\{ & k^2 d(k,p,q)\, E^M(p)\, E^M(q)\, \theta(MMV; k,p,q) \\
& - p^2 h(k,p,q)\, E^M(k)\, E^V(q)\, \theta(MVM; p,q,k) \\
& - p^2 j(k,p,q)\, E^M(k)\, E^M(q)\, \theta(VMM; p,q,k) \Big\} \Big]
\end{aligned}
\tag{27}
$$

where

$$
\theta(a,b,c\,; l,m,n) = \int_{-\infty}^{\infty} ds\, G^a(l,s)\, R^b(m,s)\, R^c(n,s) \qquad (a,b,c = V \text{ or } M) \tag{28}
$$

and

$$
\theta(k) = \int_{-\infty}^{\infty} ds\, G^V(k,s)\, M(k,s) \tag{29}
$$

These two equations involving $E^v(k)$ and $E^m(k)$ can be solved if we could provide at least integral information regarding the G's and R's, in the form of the Θ-functions defined above. In sofar as steady-state energy balance relations are involved, no more than this much is needed. Keeping this in mind, we will explore the frequency relationship, which we derived in the last section and try to find the minimal overall information about the time structure of the Green's functions and correlation functions, (or what amounts to the same, their frequency structure), that will be necessary to solve for the spectral functions.

Let us define correlation frequencies $\zeta_v(k)$, $\zeta_m(k)$ and response frequencies $\eta_v(k)$ and $\eta_m(k)$ for mode k in the velocity and magnetic modes respectively by

$$\left[\zeta_v(k)\right]^{-1} = \int_0^\infty R^v(k,t)\,dt \quad ; \quad \left[\zeta_m(k)\right]^{-1} = \int_0^\infty R^v(k,t)\,dt$$

$$\left[\eta_v(k)\right]^{-1} = \int_0^\infty G^v(k,t)\,dt \quad ; \quad \left[\eta_m(k)\right]^{-1} = \int_0^\infty G^M(k,t)\,dt \qquad (30)$$

and the characteristic frequency of the stirring forces by

$$\left[\zeta(k)\right]^{-1} = \int_0^\infty M(k,t)\,dt \qquad (31)$$

Relations between these frequencies and the spectral functions can be obtained by integrating (3), (4), (5) and (6) over the time t from zero to infinity. But remembering that the time average of a function is equal to its zero frequency Fourier component, we can get just the same from (22), (23), (24) and (25) by putting $\omega = 0$. With these substitutions and a few manipulations, these yield

$$\eta_v(k) = \nu k^2 + \iint_\Delta \frac{kp}{2q}\,dp\,dq \{ b(k,p,q)\, E^v(q)\, \theta_1(VV;p,q)$$

$$+ c(k,p,q)\, E^m(q)\, \theta_1(MM;p,q) \} \qquad (32)$$

$$\eta_m(k) = \lambda k^2 + \iint_\Delta \frac{kp}{2q}\,dp\,dq \{ h(k,p,q) E^v(q)\, \theta_1(MV;p,q)$$

$$+ j(k,p,q)\, E^m(q)\, \theta_1(VM;p,q) \} \qquad (33)$$

$$[\xi_v(k)]^{-1} = [\eta_v(k)]^{-2} \cdot \left[\tfrac{1}{2} F(k) \{ E^v(k)\, \theta(k) \}^{-1} \right.$$

$$+ \iint_\Delta (k^3/2pq)\,dp\,dq\; a(k,p,q) \left\{ \frac{E^v(p)\, E^v(q)}{E^v(k)}\, \theta_2(VV;p,q) \right.$$

$$\left.\left. + \frac{E^m(p)\, E^m(q)}{E^v(k)}\, \theta_2(MM;p,q) \right\} \right] \qquad (34)$$

$$[\xi_m(k)]^{-1} = [\eta_m(k)]^{-2} \left[\iint_\Delta \frac{k^3}{2pq}\,dp\,dq\; d(k,p,q) \cdot \right.$$

$$\left. \cdot \frac{E^m(p)\, E^v(q)}{E^m(k)}\, \theta_2(MV;p,q) \right] \qquad (35)$$

Rearranging (26) and (27), we get

$$E^v(k) = \frac{\left[\frac{F(k)\,\theta(k)}{2} + \iint_\Delta \frac{k^3}{2pq}\,dp\,dq\; a(k,p,q) \{ E^v(p) E^v(q)\, \theta(VVV;k,p,q) + E^m(p) E^m(q)\, \theta(VMM;kpq) \} \right]}{\left[\nu k^2 + \iint_\Delta \frac{kp}{2q}\,dp\,dq \{ b(k,p,q)\, E^v(q)\, \theta(VVV;p,q,k) + c(k,p,q)\, E^m(q)\, \theta(MMV;p,q,k) \} \right]} \qquad (36)$$

$$E^m(k) = \frac{\left[\iint_\Delta \frac{k^3}{2pq}\, dp\, dq\, d(k,p,q)\, E^m(p)\, E^v(q)\, \theta(M M v;\, k,p,q) \right]}{\left[\lambda k^2 + \iint_\Delta \frac{kp}{2q}\, dp\, dq \left\{ h(k,p,q)\, E^v(q)\, \theta(M v M;\, p,q,k) + j(k,p,q)\, E^m(q)\, \theta(v M M;\, p,q,k) \right\} \right]} \tag{37}$$

where

$$\theta_1(a, b;\, p, q) = \int_0^\infty G^a(p,s)\, R^b(q;s)\, ds = \int_{-\infty}^{\infty} d\omega\, \tilde{G}^a(p;\omega)\, \tilde{R}^b(q;\omega) \tag{38}$$

$$\theta_2(a\, b;\, p, q) = \int_0^\infty ds\, R^a(p,s)\, R^b(q;s) = \int_{-\infty}^{\infty} d\omega\, \tilde{R}^a(p,\omega)\, \tilde{R}^b(q;\omega) \tag{39}$$

where in writing (38) and (39), use is made of the fact that R - functions are even functions of time.

(32) to (37) along with (38), (39) and (28) for the constitute a complete set. We have to know some more about the G and R - functions apart from their characteristic frequencies to be able to solve these. In particular, the overlap integrals defined by the θ's , between the G and R - functions taken two or three at a time are important in deciding the internal correlation-relaxation features of the non-linear interaction[2].

We can try to approximate these overlap integrals by assuming some suitable functional forms for the G and R - functions subject to equations which restrict their initial values, symmetry with respect to time reversal and integral features. In fact any general form $A\exp\{-\lambda(t)\}$ where $\lambda(t)$ is a positive definite polynomial in t with all coefficients positive would be a sufficient though not a necessary choice. This includes the usual Fokker-Plank type of relaxation $\sim e^{-\lambda_1 t}$ as the first member. The statistical model of turbulence based on the Generalised Liouville Equations of Edwards[7] would follow if we make this approximation, along with some further simplifications. But the basic phase relaxation process, which tends to produce correlation as well as de-correlation in the turbulent situation is strictly non-linear and as can clearly be seen from the studies of Kraichnan[2,3], non-Markovian. Asymptotic considerations that lead to an inertial range spectrum also indicate that the asymptotic equations to the Green's functions have the form

$$\dot{G}_{\lambda}(\tau) = \sum_{\mu,\nu} A(\lambda,\mu,\nu)\int_0^{\tau} G_{\mu}(\tau-s)\, G_{\nu}(s)\, ds \qquad (40($$

We can consider this as typical of the eddy relaxation processes in turbulence and as such it is very dissimilar to the usual molecular friction and dynamical friction terms which arise in a Fokker-Plank type of approach. Further experimental observations regarding the correlation analysis of velocity also tend to bear out more an $e^{-\lambda^2 t^2}$ rather than an $e^{-\lambda_1 t}$ type of relaxation[2,3,8]. Kraichnan[9] has compared the results of the complete direct interaction equations, with the approximate equations, involving functional forms of the exponential and gaussian type in the steady state for hydrodynamic turbulence. He found that the agreement was better with the gaussian form. Keeping these in mind we will take a gaussian functional form for the functions G and R

$$
\begin{aligned}
R^{V}(k,t) &= \exp\left\{-\tfrac{1}{4}\pi(\zeta_{v}(k)t)^{2}\right\} \\
R^{M}(k:t) &= \exp\left\{-\tfrac{1}{4}\pi(\zeta_{M}(k)t)^{2}\right\} \\
G^{V}(k,t) &= \exp\left\{-\tfrac{1}{4}\pi(\eta_{v}(k)t)^{2}\right\} \\
G^{M}(k;t) &= \exp\left\{-\tfrac{1}{4}\pi(\eta_{M}(k)t)^{2}\right\} \\
M(k,t) &= \exp\left\{-\tfrac{1}{4}\pi(\xi(k)t)^{2}\right\}
\end{aligned}
\tag{41}
$$

With this identification, one gets

$$
\begin{aligned}
\theta(a,b,c;k,p,q) &= \left\{[\eta_{a}(k)]^{2}+[\zeta_{b}(p)]^{2}+[\zeta_{c}(q)]^{2}\right\}^{-1/2} \\
\theta_{1}(a,b;p,q) &= \left\{[\eta_{a}(p)]^{2}+[\zeta_{b}(q)]^{2}\right\}^{-1/2} \\
\theta_{2}(a,b;p,q) &= \left\{[\zeta_{a}(p)]^{2}+[\zeta_{b}(q)]^{2}\right\}^{-1/2} \\
\theta(k) &= \left\{[\eta_{v}(k)]^{2}+[\xi(k)]^{2}\right\}^{-1/2}
\end{aligned}
\tag{42}
$$

Thus (42) along with (32) to (37) complete our requirements. An iterative method for their solution will be described in Section 6.

5. Kolmogorov's Hypotheses and Hydromagnetic Turbulence.

The basic arguments that lead to the direct interaction approximation and their compatibility with the ideas of

Kolmogorov, for hydrodynamic turbulence have been discussed in great detail by Kraichnan[3,4,5]. The basic reasoning that leads to the Kolmogorov concept of scaling- that large scale motions should carry small eddies about, without distorting them,is essentially Lagrangian in spirit. Thus the Eulerian history correlations which are the starting points in the direct interaction approximation, must be suitably modified to correct for Lagrangian history.

As has been discussed by Kraichnan[4,5], the convection without distortion of the small scale motions by the large scale motions, which is the basic assumption underlying Kolmogorov's ideas, owes its justification to an exact invariance property of the Navier-Stokes equations. A constant homogeneous velocity field can be gauged away by transforming to a comoving coordinate system. This Galilean invariance, which is an exact property of the Navier-Stokes equations,when formulated in terms of Lagrangian velocities and correlations, rather than Eulerian velocities and correlations would make the Kolmogorov assumption more plausible. One could get to the same result, at least in so far as steady-state energy-balance information is concerned through systematic procedures of modifying the Navier-Stokes equations so as to eliminate the convection of a given spatial scale by much larger scales (by a prescribed ratio). These modifications have been considered by Kraichnan[3,4]. He has also compared them with a systematic Lagrangian History formulation of correlations.

But the hydromagnetic equations do not show such an universal invariance. The coupling between different scales now can take place through velocity as well as magnetic field elements. The velocity field shows the Galilean invariance. But the magnetic field does not. This is because a magnetic field of a certain scale is coupled to every magnetic loop and eddy of smaller scales, through the possibility of an Alfven wave excitation. Thus the coupling between different scales is changed profoundly with the introduction of a large scale magnetic field. The local-isotropy concept of Kolmogorov, which implies that the detailed information about low wave number structure is degraded through transfer of energy in the wave number space, has no a priori validity in the hydromagnetic case. Thus it is more plausible that the ideas[10], that lead to an inertial range in the unmodified direct interaction scheme, where the energy-containing range excitation explicitly appears in the inertial range spectrum. An unequivocal answer to this will be available - only when a Lagrangian history study of the hydromagnetic equations is taken up in all its completeness as in the hydrodynamical case by Kraichnan. A similar investigation is underway.

From the experience gained in the construction of quasi-Lagrangian solutions with the direct interaction scheme in the hydrodynamic case, a number of modifications suggest themselves. We shall in the following section discuss some of these modifications and their utility in the construction of the solutions in the hydromagnetic case.

6. Iterative Solutions For The Spectra And Characteristic Frequencies.

Despite all the simplifications that we have introduced in section 4, the final set (32) to (37) (along with (42)) is still formidable. They are a set of six coupled nonlinear integral equations for the six quantities $E^v(k)$, $E^m(k)$, $\zeta_v(k)$, $\zeta_m(k)$, $\eta_v(k)$ and $\eta_m(k)$. These are formally solvable, if the spectrum F(k) and the characteristic frequency $\zeta(k)$ of the external driving force are given. Explicit analytical solutions are ruled out and only numerical solution suggests itself. But,even so, the double integral over wave number space and the number of equations to be solved and functions to be determined make it a huge and monstrous calculation even in modern digital machines. We resort to a procedure of iteration. If one can start with a set of trial or guess values for the η's , ζ's and E's for a given spectrum F(k) and characteristic frequency $\zeta(k)$ of the random force, we can substitute these in the kernel functions of the integral equations and evaluate a new set of values for η, ζ & E's . These new sets of values may be suitably mixed with the old set and a new iteration started with this set. This iterative procedure converges unequivocally and efficiently because of the non-linearity (One of the very few occasions in life, when non-linearity is a help!). So much is easier said than done. To be able to reduce this to a tractable numerical problem, one has to replace the infinite range of wave numbers by a discrete set of wave numbers. The two integrations over p and q in k-space with a restriction that $\underline{k} = \underline{p} + \underline{q}$ make a reduction of the double integral to a straightforward single integral impossible. In turbulence, most of the interaction in wave number space is very local and so the discretisation of the wave number space must be sufficiently smooth to include a number of possible triads of modes in a region, to take care of the effective contribution to transfer or relaxation. At the same time, we require a fairly long chain of wave numbers going over a considerable number of octave intervals, in any calculation, as the one we envisage here, where we want to include a meaningful division into an energy-containing range, inertial range and a dissipative range for the velocity and magnetic modes. And this becomes not an acdemic question but a crucial one, since we have a set of six integral equations which we want to iterate and solve. Since each equation has a double integral structure, which is basically irreducible, the memory requirements increase and $6 \times N^2$ as a function of the number N of discrete steps in k, that we allow.

We choose a set of twenty-five half-octave steps in k. We use a discretisation for the weight factors in the k integrals, which is very well described elsewhere[11]. The external force has a flat spectrum which is non-zero for the first four modes of the velocity and zero for the rest. In a set of preliminary calculation, both the number of modes for which the force is non-zero and the relative value in the non-zero region were kept variable to check reliability of the various numerical procedures. The particular choice was made so that for the discrete wave number range, which is allowed by the limitations of the computer, a meaningful energy containing range, inertial range and dissipative tail are possible. The iterations were performed on an IBM 7094 and a CDC 3600.

First making a suitable choice of the kinematic parameters, viscosity ν and resistivity λ for the system, we iterate the complete Unmodified Direct Interaction Scheme and converge on a set of solutions. Then we decrease either ν or λ by a fixed ratio and again iterate to get a new converged set, with the old converged set as a starting point. In this way, we construct a set of solutions, wherein ν and λ run through a range of values such that their ratio changes from one-tenth to ten. In these results, the Eulerian relaxations are allowed to include energy-range mixing. This leads to two distinct characteristic frequencies for each spectrum, for a given scale k . Both these show modulation by energy-containing range parameters. The physical interpretation of this is given by Kraichnan, in the hydrodynamical context[2].

Next we impose a less restricted detailed-balance condition than Kolmogorov's[9]. We require that the relaxation and correlation frequencies are equal in the inertial range, but still leaving them with arbitrary energy range mixing. Thus we replace (34) and (35) for the ζ's with

$$\zeta_v(k) = \left[(\eta_v(k))^2 - (\nu k^2)^2\right]^{1/2} \qquad (43)$$

$$\zeta_m(k) = \left[(\eta_m(k))^2 - (\lambda k^2)^2\right]^{1/2} \qquad (44)$$

This leads to equality of the correlation and relaxation frequencies in the inertial range and $\zeta < \eta$ in the dissipative range[5].

With these modifications, we iterate the set to get a new set of converged spectra. The direct interaction scheme is otherwise left unaltered, in so far as the convection of small eddies by large scale motions is concerned.

To get the complete Kolmogorov scaling without worrying about the 'Galilean non-invariance' of the hydromagnetic equations with respect to a constant random magnetic field, we make an alternate and more restrictive modification. In (36) and (37) we replace the $\theta(abc\,; k, p, q)$ - factors by $[E^a(k)k]^{1/2}$ This assumption implies that, for every scale of motion, (or magnetic field) there exists only one unique scale of time and that it is decided completely by the local value of the spectrum at that scale. An equivalent way of introducing this assumption would be to leave (36) and (37) unaltered but in (32) to (35) to restrict the p, q integrations by requiring that $k < \alpha p$, $k < \alpha q$ where α is a cut-off parameter. The cut-off parameter α is so chosen that the ζ's calculated from (34) and (35) are equal to the ζ's calculated from (43) and (44). We also check to see whether the main integral parameters and features obtained this way agree with the solutions, obtained by the other Kolmogorov modification. In fact the cut-off parameter is varied to achieve this. We find that for our half-octave discrete k set, a choice $\alpha = 8$ does this reasonably.

We then construct what we call a reduced Lagrangian history direct interaction hydromagnetic modification. In this we impose the Kolmogorov modification (with restriction in the p , q integration) only on the velocity terms. This leaves the magnetic eddy-damping times and correlation times to be modulated, by energy-containing range parameters.

For each of these modifications, a variety of integral parameters, which typify the structure and characteristic of the energy-containing, inertial and dissipative ranges of both the velocity and magnetic are calculated.

Let us define for our discrete k - space

$$E_V = \sum_k E^V(k)\Delta k \; ; \; E_M = \sum_k E^M(k)\Delta k \; ; \; E_T = E_V + E_M$$

E_T is the total energy density in the system and E_V and E_M are the energy-densities in the velocity and magnetic modes of turbulence. From these, one can define root-mean square "velocities" of excitation in the velocity, magnetic and total modes, which characterise the energy containing range

$$v_0 = \{2E_V/3\}^{1/2} \; ; \; h_0 = \{2E_M/3\}^{1/2} \; ; \; \alpha_0 = \{2E_T/3\}^{1/2}$$

The total dissipations in the velocity, magnetic and the combined systems are defined as

$$\epsilon_V = \nu \sum_k E^V(k)\, k^2 \Delta k \; ; \; \epsilon_M = \lambda \sum_k E^M(k) \Delta k \; ; \; \epsilon_T = \epsilon_V + \epsilon_M$$

Following Batchelor[12] we can define characteristic lengths L_V, L_M, L_T and λ_V, λ_M, λ_T which typify the 'integral scale' and the Tayler micro-scale for each of these modes and their sum. The integral scale typifies that region of the spectrum which contributes dominantly to the energy in the particular mode. The Tayler micro-scales on the other hand show the dispersion of energy in wave-number space and thus typify the characteristic lengths of the dissipation spectra.

$$L_V = \frac{\left[\frac{3\pi}{4} \sum_k \frac{E^V(k)}{k} \Delta k\right]}{E_V} \; ; \; L_M = \frac{\left[\frac{3\pi}{4} \sum_k \frac{E^M(k)}{k} \Delta k\right]}{E_M}$$

$$\lambda_V = \left[\frac{5 E_V \nu}{\epsilon_V}\right]^{1/2} \; ; \; \lambda_M = \left[\frac{5 E_M \lambda}{\epsilon_M}\right]^{1/2}$$

$$L_T = \frac{\left[\frac{3\pi}{4} \sum_k \frac{E^T(k)}{k} \Delta k\right]}{E^T} \; ; \; \lambda_T = \left[\frac{5 E_T (\nu + \lambda)}{\epsilon_T}\right]^{1/2}$$

With the definitions of these lengths and "velocities" we are in a position to define the effective Reynolds numbers in each of these ranges for the respective systems.

$$R_L^V = \upsilon_0 L_V \nu^{-1} \; ; \; R_L^M = h_0 L_m \lambda^{-1} \; ; \; R_L^T = \alpha_0 L_T (\nu + \lambda)^{-1}$$

$$R_\lambda^V = \upsilon_0 \lambda_V \nu^{-1} \; ; \; R_\lambda^M = h_0 \lambda_m \lambda^{-1} \; ; \; R_\lambda^T = \alpha_0 \lambda_T (\nu + \lambda)^{-1}$$

Following Kraichnan[3], we can also define Reynolds numbers, which characterise the energy-range in each of these modes. These are

$$R_0^{v} = v_0^{4}(\epsilon_v \nu)^{-1} \; ; \; R_0^{M} = R_0^{4}(\epsilon_M \lambda)^{-1} ; \; R_0^{T} = \alpha_0^{4}[\epsilon_T (\nu+\lambda)]^{-1}$$

In tables I - III, we give the values of these integral parameters for the various modifications and choice of ν and λ .

7. Discussion of the Results.

In a previous paper[1], a detailed survey of the various kinematic approaches to the problem of the growth of a weak random magnetic excitation in a turbulent fluid were given. All these approaches try to look at the growth characteristics of the magnetic field, by direct analogous extensions of the arguments that lead to the asymptotic Kolmogorov spectrum in the pure hydrodynamic case. Arguments along these lines, exploiting the analogy between the dynamo equation and the vorticity equation were put forward by Batchelor[13] and his ideas have been developed further by Moffatt[14]. Similar arguments based on the rates of strain, assuming that the dynamical equilibrium character of the transfer in the universal range is unaffected by the Lorentz forces, have been given by Saffman[15], Pao[16] and Parker[17]. A very subtle but unavoidable prerequisite for all such models is the concept of unidirectional cascade of energy in the wave number spectrum. The velocity field in the pure hydrodynamic case is in statistical equilibrium and the *net* transport of energy proceeds towards larger wave numbers. In the weak magnetic case, which is not even in statistical equilibrium in the first place, there is no *a priori* justification to believe that the cascade should be unidirectional. In fact, the nature of the dynamical couplings between the magnetic and velocity terms do not justify this assumption. Further the Galilean non-invariance of the hydromagnetic equations, with respect to the magnetic terms make the Kolmogorovian requirements of localness of cascade questionable. Thus, from two different considerations, there is reason to expect that the structure of the transfer in the hydromagnetic case will be profoundly different from its pure hydrodynamical counterpart.

There have been attempts by Chandrasekhar[18], Roberts and Tatsumi[19], Tatsumi[20], Betchov[21] and Deissler[22] to construct theories for magnetohydrodynamical turbulence, based on dynamical approaches. Their procedures depend either on discarding the fourth order cumulants or neglecting the non-linear cross field terms completely, thus restricting their applicability to either weak fields or the final state of decay.

In our earlier paper, for the weak field case, we attempted to check the balance between the local-enhancement and sweeping-out processes. In the absence of any quantitative knowledge about the characteristic times of relaxation and correlation for the two fields and because of the closeness of the balance between the two competing processes, we concluded that the net result of growth or decay cannot be deemed conclusive. Already in the weak field case, the nonlocalness of the cascade, which is a direc consequence of the impossibility of convection without distortion in the magnetic case, manifested itself even at the earlier times of evolution, as a direct enrgy coupling over a jump of two decades in wave number. Thus the generation of the small scale magnetic excitation by small eddies depends on the strength of the field rather than the gradient of the field. This focuses one's attention on the inadequacy of the usual analogy between magnetic fields and a convected passive scalar field[23].

We have here tried to construct a steady-state theory, in which the local relaxation times are now treated as internal parameters and are determined consistently along with the spectrum. Thus many of the arbitrary assumptions of the earlier paper removed. But simultaneously, mathematical simplicity and convenience have forced us to make assumptions about the functional form of the correlation and response functions. Further, we have only partially been able to take care of the Lagrangian modification. Thus our study fills an important gap in our earlier work; the relative and absolute values of the relaxation times are determined together with the spectra.

The main results of this calculation are presented in figures I - XI . In each of these, we plot the spectral functions E^v (k) and E^m (k) as well as the spectra of the vorticity $G(k) = k^2 E^v(k)$ and the current $J(k) = k^2 E^m(k)$

Figs. I - III and first curve of figure IV give the spectra for the unmodified direct interaction approximation, when no attempt has been made to correct for Lagrangian History. One would expect from asymptotic inertial range considerations[10] (see footnote in reference 10) to get a $k^{-3/2}$ power law in the inertial range. This is found to be true. Figure II gives the inertial range for one of these curves in an enlarged scale, to accentuate the point. Further in the inertial range detailed equipartition is a striking result.

Figs. IV ii, V and VI give the spectra, when we make the restricted detailed balance assumption of $\zeta = \eta$ in the inertial range. Qualitatively, this already depresses the energy range mixing, which is a feature of the unmodified direct interaction equations. In this sense they are intermediary to the complete Kolmogorov adaptation and the complete Eulerian results. The overshooting of the magnetic spectrum in the energy-containing range is an indication of the effective coupling distance in the wave number space. The overshooting distance is typically the range over which equilibrium is reached in the energy range.

Figs. VII and VIII display the spectra when we make complete Kolmogorov modifications for both the magnetic and velocity relaxations. The depression of the magnetic spectrum for low wave numbers and the longer wave number interval necessary before $E^m(k)$ builds up to equipartition are noteworthy.

Figs. IX and X display the spectra when we make the reduced Lagrangian history modification. Fig. XI presents the magnetic and velocity spectra for the inertial range in these cases in an enlarged scale. This clearly shows a tendency to a two-piece inertial range for magnetic spectrum : a $k^{-5/3}$ region in the low inertial range and a $k^{-3/2}$ region in the high inertial range. In the $k^{-3/2}$ region, the magnetic spectrum overshoots above the velocity.[10]

The qualitative nature of the curves does not charge very profoundly, when we make such drastic assumptions about the time-structure of the correlation-relaxation features. They all feature equipartition in the inertial range and the behaviour in the dissipative and far-dissipative ranges, when $\nu \ll \lambda$ is compatible with the results of Moffatt[14], Golitsyn[24] and Saffman[15], from equilibrium considerations. The various modifications, which either take an extreme Kolmogorovian or Eulerian point of view change the total energy and the coupling between the ranges profoundly, as displayed by the effect on the partition ratio, i . e . the ratio of the magnetic to the velocity spectrum in the normal range.

Thus the main persuasive result of this calculation is that in the steady-state the total energies in the magnetic and velocity modes are comparable and that there is a detailed equipartition between the two modes in the extended inertial range. By and large, in magnitude and detail, this corroborates the conjectures of Biermann and Schlüter.[25]

This result coupled with the results of the previous paper can be taken as a really compelling demonstration of the possibility of the existence of a turbulent dynamo. Since our scheme of calculations are based on a detailed dynamical theory, it is possible to extend these results to the non-stationary case or to situations with further complications, as are likely in the astrophysical conditions.

In two recent papers,[26] Parker has investigated the back-transfer to low wave numbers from large wave number excitations of the magnetic spectrum, through a prescribed stationary random velocity field, when its correlation time is short. The main question, which Parker raises in his paper is whether in a realistic turbulent situation when the characteristic times of growth in any scale are of the order of the eddy-circulation time in that scale, there would be a back-transfer in wave number space. Further, there is a long-range coupling in the magnetic spectrum, through the convective terms, which lead to the additional non-Galilean features. These questions will be dealt with in a subsequent paper.

The skewness of the distribution of the derivatives of velocity and magnetic field and the cross-field skewness factors are also evaluated in these calculations, though not explicitly. These and the related quantities like rate of vorticity production and current production will be discussed in a forthcoming paper.

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TABLE CAPTIONS

Table I. Values of net energy and dissipation for the velocity, magnetic and total modes, for the various modifications and different choice of ν and λ

Table II. The RMS excitations and the corresponding Reynolds numbers for the velocity, magnetic and total modes, for the various modifications and different choice of ν and λ

Table III. The Reynolds numbers characterising the integral scale and the dissipative scale for the velocity, magnetic and total modes for the various modifications and differnt choice of ν and λ .

TABLE I

TYPE	ν/λ	ν	λ	E_ν	E_m	E_T	ϵ_ν x10	ϵ_M x10	ϵ_T x10
$R^{-3/2}$ $\zeta \neq \eta \neq 0$	1/10	5×10^{-5}	5×10^{-4}	14.574	8.890	23.474	0.4081	3.964	4.372
	1	5×10^{-5}	5×10^{-5}	15.083	9.518	24.602	2.511	2.562	5.072
	10	5×10^{-4}	5×10^{-5}	14.492	9.019	23.511	3.959	0.4382	4.397
$R^{-3/2}$ $\zeta = \eta$	1/10	5×10^{-5}	5×10^{-4}	18.489	9.021	27.511	0.4714	4.660	5.131
	1	5×10^{-5}	5×10^{-5}	19.132	9.853	28.984	3.008	3.110	6.118
	10	5×10^{-4}	5×10^{-5}	18.353	9.229	27.582	4.652	0.5326	5.184
KOLMOGOROV $\zeta = \eta$	1/10	5×10^{-5}	5×10^{-4}	17.086	4.046	21.132	0.2975	2.883	3.181
	1	5×10^{-5}	5×10^{-5}	13.639	3.634	17.273	2.452	2.492	4.944
	10	5×10^{-4}	5×10^{-5}	13.438	3.594	17.032	3.666	0.6346	4.301
REDUCED LAGRANGIAN HISTORY	1/10	5×10^{-5}	5×10^{-4}	13.869	3.916	17.784	0.8134	1.110	1.923
	1	5×10^{-5}	5×10^{-5}	14.046	4.418	18.464	1.467	0.6148	0.2081
	10	5×10^{-4}	5×10^{-5}	13.698	4.461	18.159	1.628	0.3338	1.962

TABLE II

TYPE	ν/λ	ν	λ	σ_0	h_0	α_0	R_0^V $\times 10^4$	R_0^M $\times 10^4$	R_0^T $\times 10^4$
$k^{-3/2}$ $\zeta \neq \eta \neq 0$	1/10	5×10^{-5}	5×10^{-4}	3.117	2.436	3.996	46.260	0.1776	2.037
	1	5×10^{-5}	5×10^{-5}	3.171	2.519	4.050	8.056	3.143	10.610
	10	5×10^{-4}	5×10^{-5}	3.108	2.452	3.959	0.4716	16.500	2.032
$k^{-3/2}$ $\zeta = \eta$	1/10	5×10^{-5}	5×10^{-4}	3.510	2.452	4.283	64.460	0.1552	2.384
	1	5×10^{-5}	5×10^{-5}	3.571	2.563	4.396	10.820	2.774	12.210
	10	5×10^{-4}	5×10^{-5}	3.498	2.480	4.288	0.6436	14.220	2.372
KOLMOGOROV $\zeta = \eta$	1/10	5×10^{-5}	5×10^{-4}	3.375	1.642	1.704	87.210	0.05047	2.269
	1	5×10^{-5}	5×10^{-5}	3.019	1.556	3.393	6.743	0.4711	5.304
	10	5×10^{-4}	5×10^{-5}	2.993	1.548	3.370	0.4378	1.809	1.090
REDUCED LAGRANGIAN HISTORY	1/10	5×10^{-5}	5×10^{-4}	3.041	1.616	3.443	21.020	0.1228	2.658
	1	5×10^{-5}	5×10^{-5}	3.060	1.716	3.508	11.960	2.822	14.2560
	10	5×10^{-4}	5×10^{-5}	3.022	1.725	3.479	1.024	5.299	2.716

TABLE III

TYPE	ν/λ	ν	λ	R_L^V $\times 10^4$	R_L^M $\times 10^3$	R_L^T $\times 10^4$	R_λ^V $\times 10^3$	R_λ^M $\times 10^2$	R_λ^T $\times 10^2$
$R^{-3/2}$ $\zeta \neq \eta \neq 0$	1/10	5×10^{-5}	5×10^{-4}	5.373	1.856	0.9776	2.634	1.632	5.494
	1	5×10^{-5}	5×10^{-5}	5.275	18.020	5.251	1.099	6.867	12.610
	10	5×10^{-4}	5×10^{-5}	0.5375	18.540	0.9760	0.2666	15.730	5.405
$R^{-3/2}$ $\zeta = \eta$	1/10	5×10^{-5}	5×10^{-4}	6.016	1.229	1.025	3.109	1.526	5.965
	1	5×10^{-5}	5×10^{-5}	5.906	11.860	5.490	1.274	6.451	13.530
	10	5×10^{-4}	5×10^{-5}	0.6017	12.330	1.022	0.3107	14.600	5.797
KOLMOGOROV $\zeta = \eta$	1/10	5×10^{-5}	5×10^{-4}	5.690	0.5598	0.9748	6.617	0.8701	5.796
	1	5×10^{-5}	5×10^{-5}	5.947	6.120	5.565	1.006	2.658	8.970
	10	5×10^{-4}	5×10^{-5}	0.5981	6.277	1.018	0.2563	5.210	3.574
REDUCED LAGRANGIAN HISTORY	1/10	5×10^{-5}	5×10^{-4}	5.820	0.9543	1.016	1.776	1.357	3.884
	1	5×10^{-5}	5×10^{-5}	5.778	9.115	5.486	1.339	6.506	14.780
	10	5×10^{-4}	5×10^{-5}	0.5848	9.119	1.006	0.3920	8.916	5.410

FIGURE CAPTIONS

Fig. 1. Unmodified direct interaction spectra for the case $\nu = 5\times10^{-5}$ and $\lambda = 5\times10^{-4}$

Fig. 2. Same as Fig. 1 but exploded to show fine structure in the inertial and dissipative ranges. The power law in the inertial range is unmistakably $-3/2$

Fig. 3. Unmodified direct interaction spectra for $\nu = 5\times10^{-4}$ $\lambda = 5\times10^{-5}$

Fig. 4. Unmodified directinteraction spectra with $\nu = \lambda = 5\times10^{-5}$ at the top. The bottom curves are for the same set of ν and λ but with $\zeta = \eta$ modification in the inertial range. The insensitivity of the functional forms of the spectra to the modification is striking.

Fig. 5. Direct interaction spectra with $\zeta = \eta$ modification in the inertial range for $\nu = 5\times10^{-5}$ and $\lambda = 5\times10^{-4}$

Fig. 6. Direct interaction spectra with $\zeta = \eta$ in the inertial range for $\nu = 5\times10^{-4}$ and $\lambda = 5\times10^{-5}$

Fig. 7. Spectra with $\nu = 5\times10^{-5}$, $\lambda = 5\times10^{-4}$ for Kolmogorov modifications for both the magnetic and velocity relaxations.

Fig. 8. Spectra with $\nu = 5\times10^{-4}$ $\lambda = 5\times10^{-5}$ at the top and $\nu = \lambda = 5\times10^{-5}$ at the bottom, both with complete Kolmogorov modifications for the magnetic and velocity relaxations. The more or less identical structure in the energy-containing range and the difference in level, in the dissipative range between the three sets of Kolmogorov modifications is noteworthy.

Fig. 9. Reduced Lagrangian History spectra for $\nu = 5\times10^{-4}$ $\lambda = 5\times10^{-5}$

Fig. 10. Reduced Lagrangian History spectra for $\nu = 5\times10^{-5}$ $\lambda = 5\times10^{-4}$ at the top and $\nu = \lambda = 5\times10^{-5}$ (bottom)

Fig. 11. Reduced Lagrangian History $E^{M}(k)$ for various values of ν λ in an enlarged scale to show the Lagrangian History features.

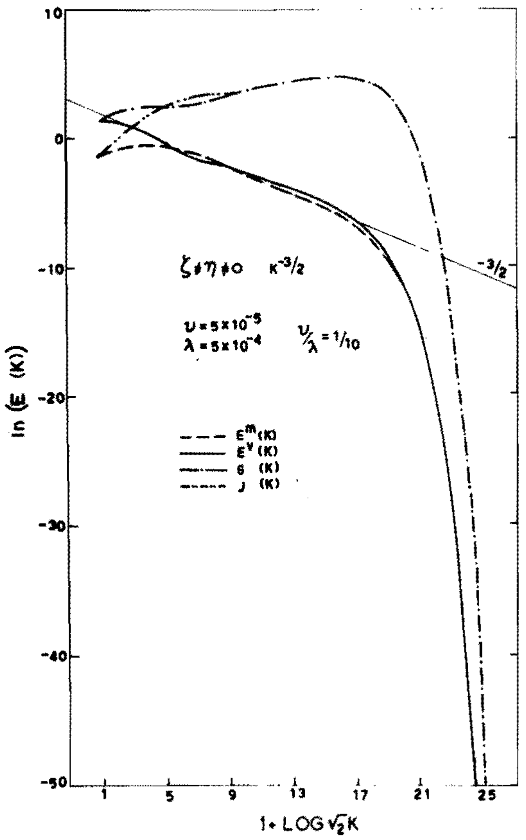

ζ≠η≠0 κ^-3/2
ν=5x10^-5
λ=5x10^-4
ν/λ = 1/10
E^m(K)
E^v(K)
G (K)
J (K)
-3/2
ln (E (K))
1+ LOG √2 K
10
0
-10
-20
-30
-40
-50
1
5
9
13
17
21
25

Fig. 1.

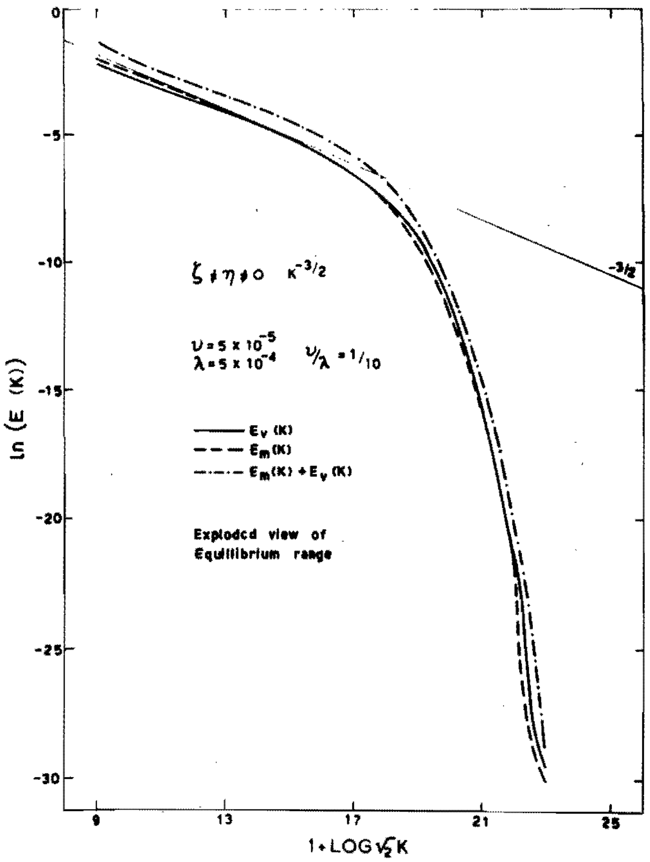

ζ≠η≠0 κ^-3/2
ν=5 x 10^-5
λ=5 x 10^-4
ν/λ = 1/10
E_v (K)
E_m (K)
E_m (K) + E_v (K)
Exploded view of
Equilibrium range
-3/2
ln (E (K))
1+ LOG √2 K
0
-5
-10
-15
-20
-25
-30
9
13
17
21
25

Fig. 2.

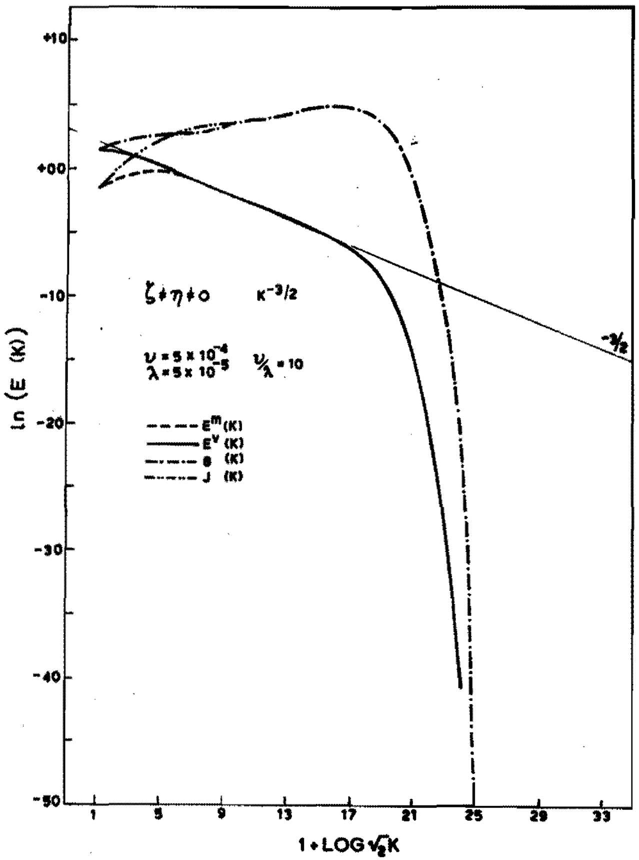

+10
+00
-10
-20
-30
-40
-50
ln (E (K))
$\zeta \neq \eta \neq 0$ $K^{-3/2}$
$\nu = 5 \times 10^{-4}$
$\lambda = 5 \times 10^{-5}$ $\nu/\lambda = 10$
E^m (K)
E^v (K)
G (K)
J (K)
-3/2
1 5 9 13 17 21 25 29 33
$1 + \mathrm{LOG}\,\sqrt{2}K$

Fig. 3.

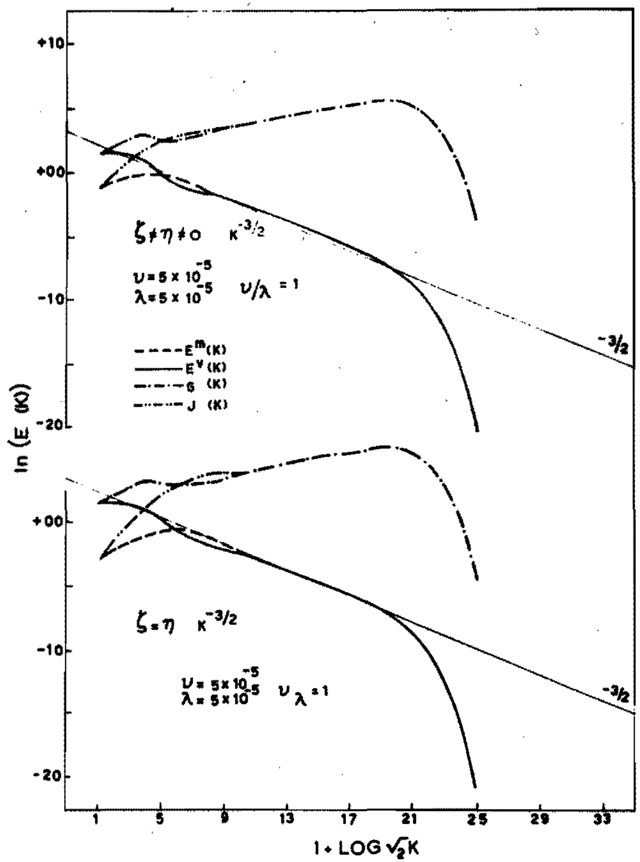
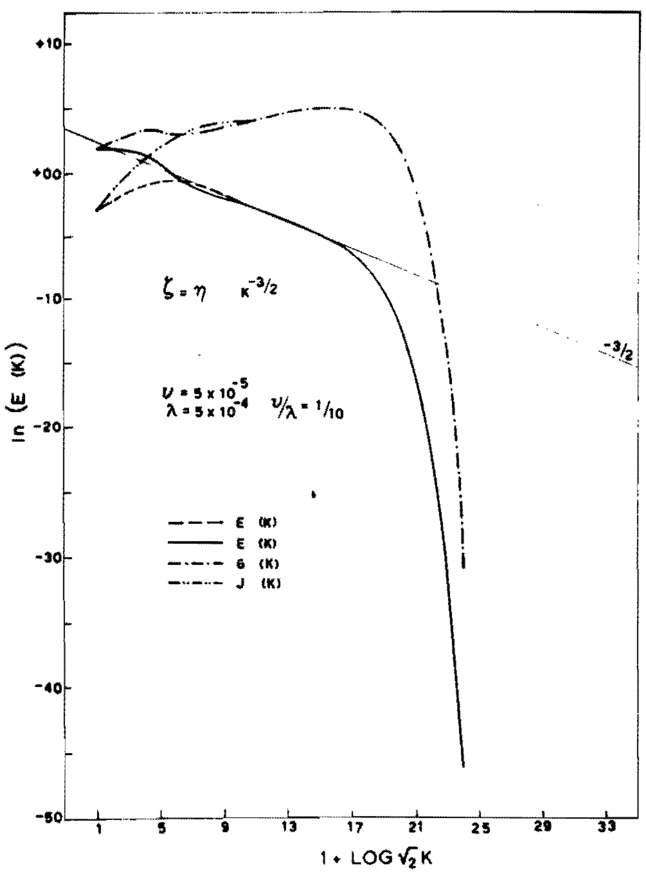
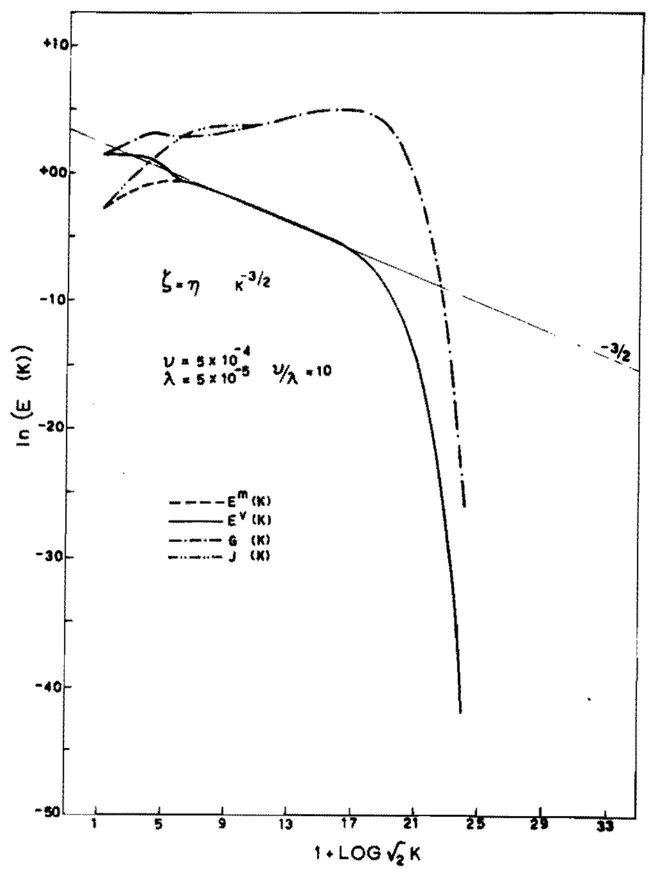

+10
+00
-10
-20
ln (E (K))
$\zeta \neq \eta \neq 0$ $K^{-3/2}$
$\nu = 5 \times 10^{-5}$
$\lambda = 5 \times 10^{-5}$ $\nu/\lambda = 1$
E^m (K)
E^v (K)
G (K)
J (K)
-3/2
+00
-10
-20
$\zeta = \eta$ $K^{-3/2}$
$\nu = 5 \times 10^{-5}$
$\lambda = 5 \times 10^{-5}$ $\nu/\lambda = 1$
-3/2
1 5 9 13 17 21 25 29 33
$1 + \mathrm{LOG}\,\sqrt{2}K$

Fig. 4.



ln (E (K))
+10
+00
-10
-20
-30
-40
-50
1 5 9 13 17 21 25 29 33
1 + LOG √2 K
ζ = η κ^-3/2
ν = 5 x 10^-5
λ = 5 x 10^-4
ν/λ = 1/10
-3/2
E (K)
E (K)
G (K)
J (K)

Fig. 5.



ln (E (K))
+10
+00
-10
-20
-30
-40
-50
1 5 9 13 17 21 25 29 33
1 + LOG √2 K
ζ = η κ^-3/2
ν = 5 x 10^-4
λ = 5 x 10^-5
ν/λ = 10
-3/2
E^m (K)
E^v (K)
G (K)
J (K)

Fig. 6.

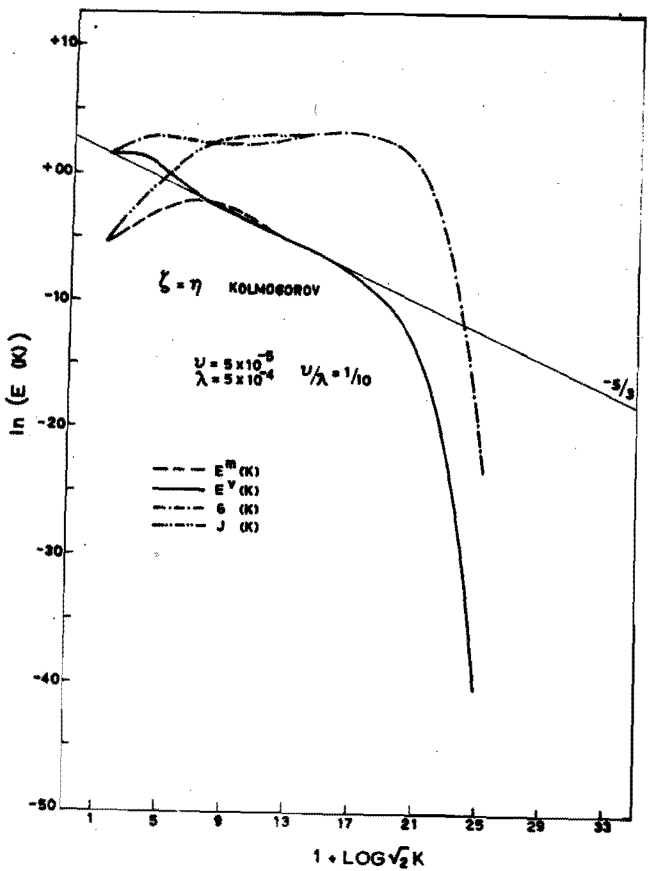

+10
+00
-10
-20
-30
-40
-50
ln (E (K))
ζ = η KOLMOGOROV
ν = 5x10-5
λ = 5x10-4
ν/λ = 1/10
Em (K)
EV (K)
G (K)
J (K)
-5/3
1
5
9
13
17
21
25
29
33
1 + LOG√2 K

Fig. 7.

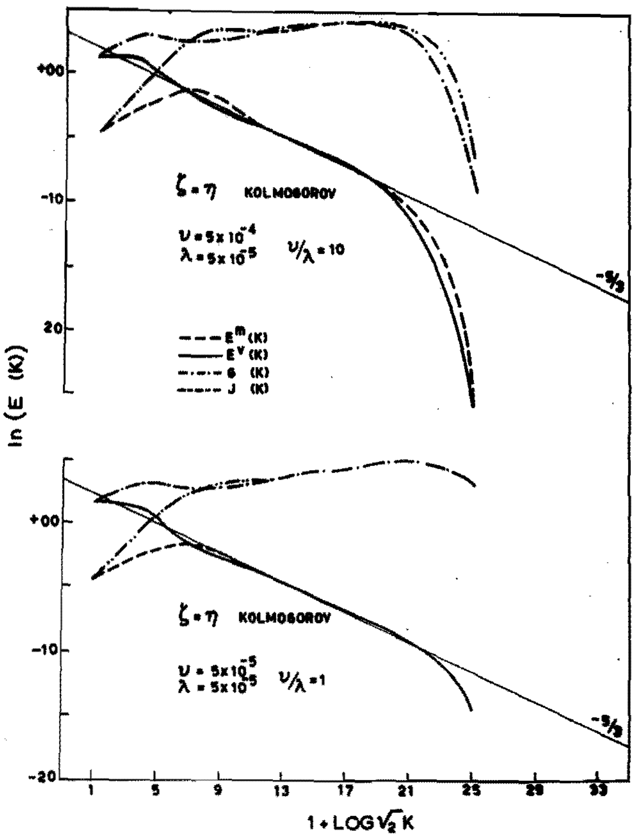

+00
-10
20
ln (E (K))
ζ = η KOLMOGOROV
ν = 5x10-4
λ = 5x10-5
ν/λ = 10
Em (K)
EV (K)
G (K)
J (K)
-5/3
+00
-10
-20
ζ = η KOLMOGOROV
ν = 5x10-5
λ = 5x10-5
ν/λ = 1
-5/3
1
5
9
13
17
21
25
29
33
1 + LOG√2 K

Fig. 8.

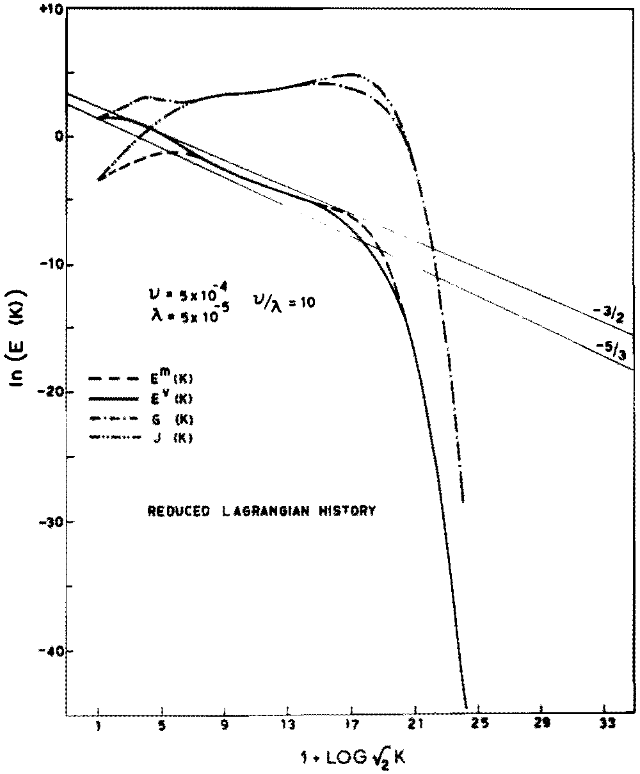

ln (E (K))
+10
0
-10
-20
-30
-40
1
5
9
13
17
21
25
29
33
1 + LOG $\sqrt{2}$ K
$\nu = 5 \times 10^{-4}$
$\lambda = 5 \times 10^{-5}$
$\nu/\lambda = 10$
-3/2
-5/3
E^m (K)
E^v (K)
G (K)
J (K)
REDUCED LAGRANGIAN HISTORY

Fig. 9.

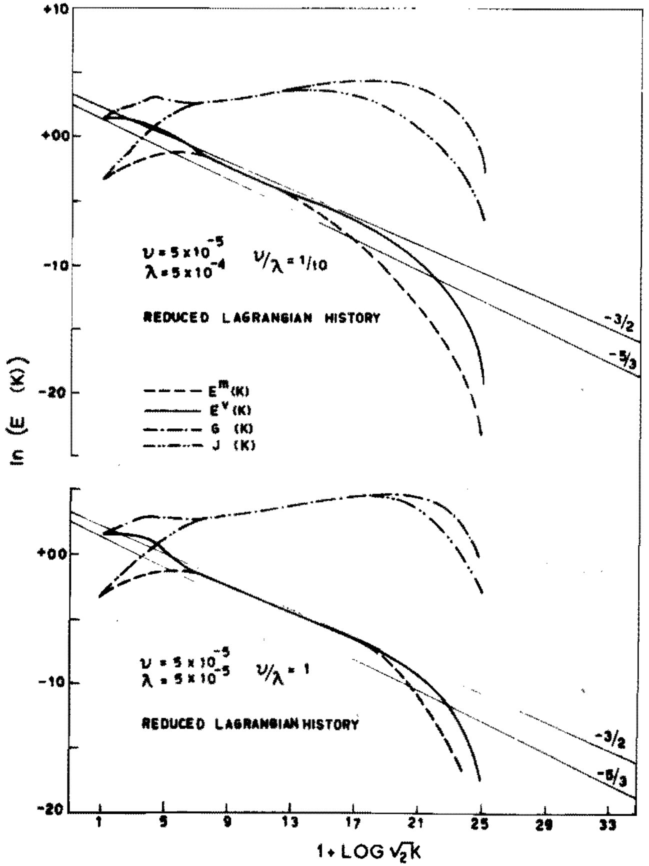

ln (E (K))
+10
+00
-10
-20
$\nu = 5 \times 10^{-5}$
$\lambda = 5 \times 10^{-4}$
$\nu/\lambda = 1/10$
REDUCED LAGRANGIAN HISTORY
-3/2
-5/3
E^m (K)
E^v (K)
G (K)
J (K)
+00
-10
-20
$\nu = 5 \times 10^{-5}$
$\lambda = 5 \times 10^{-5}$
$\nu/\lambda = 1$
REDUCED LAGRANGIAN HISTORY
-3/2
-5/3
1
5
9
13
17
21
25
29
33
1 + LOG $\sqrt{2}$ K

Fig. 10.

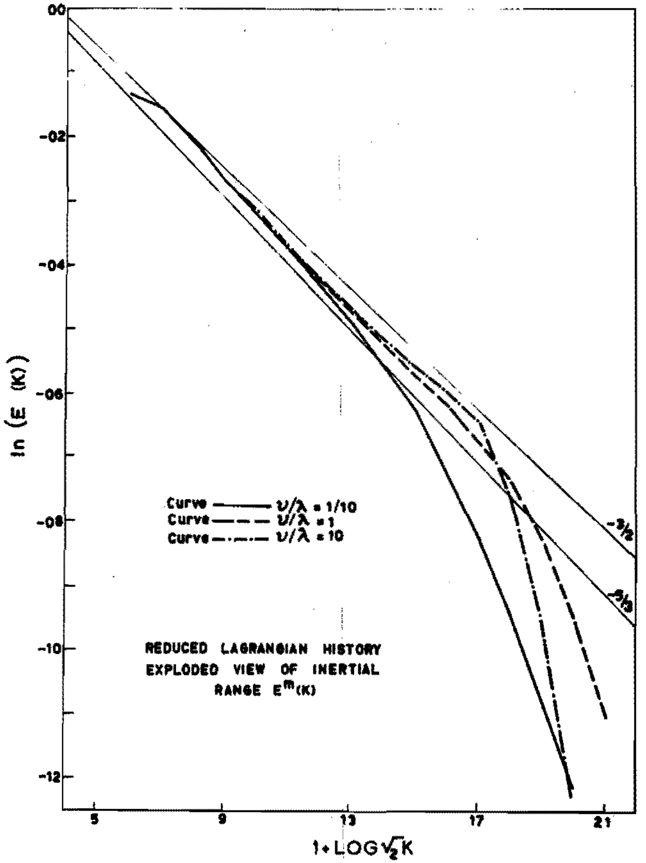

00
-02
-04
-06
-08
-10
-12
ln (E (K))
5
9
13
17
21
1+LOG√2K
Curve —— ν/λ = 1/10
Curve — — ν/λ = 1
Curve —·— ν/λ = 10
-3/2
-5/3
REDUCED LAGRANGIAN HISTORY
EXPLODED VIEW OF INERTIAL
RANGE E^m(K)

Fig. 11.

EVOLUTION OF TURBULENT MAGNETIC FIELDS – APPROACH TO A STEADY STATE

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Université Libre de Bruxelles, Brussels, Belgium

Abstract. The dynamical evolution of a weak, random, magnetic excitation in a turbulent electrically-conducting fluid is examined under varying kinematic conditions. It is found that the results of an earlier paper (Kraichnan and Nagarajan, 1967) can be reliably extended to a stage of evolution wherein the magnetic spectrum has reached local equipartition with the velocity. The transfer of the magnetic energy to smaller wavenumbers (larger scales) is considerable and significant. This result is highly pertinent to the *turbulent dynamo* question, which has been variously investigated recently. The relevance of the coupling of the rms magnetic field to the magnetic modes of all scales in deciding the efficiency of this transfer is discussed.

1. Introduction and Review

In a number of recent investigations, (Parker, 1970; Moffatt, 1970; Parker, 1969; Krause, 1968; Rädler, 1968; Steenbeck *et al.*, 1966; Steenbeck and Krause, 1966, 1967; Krause and Rädler, 1971; Fitremann and Frisch, 1969; Vainshtein, 1970), the question of regeneration of a magnetic field, by turbulent motions has been reconsidered, under a variety of kinematic assumptions about the turbulence. In an earlier paper (Kraichnan and Nagarajan, 1967), we have reviewed the previous work on this subject in great detail and found that simple intuitive statistical arguments like equipartition, or analogical and heuristic kinematic considerations like the vorticity analogy are highly inadequate in resolving this question. In a recent paper, Kraichnan (1970) has considered the analogous question of the growth and propagation of the deviations between the point-to-point velocity fields in two flow systems, which are statistically identical. Here again, one finds that the ultimate evolution depends on the quantitative competition between the local-enhancement and sweeping-away processes in the wave-number domain. One needs a considerable amount of knowledge of the internal dynamics and characteristic times, and assertions of kinematic nature based on universal equilibrium hypotheses are highly inadequate.

In our paper referred to earlier, we could not carry our calculations very much forward in time, because we had no reliable information about the internal time structure of the combined fields of velocity and magnetic field, at that time. In a more recent paper (Nagarajan, 1971), we have investigated the internal structure of the steady state spectra on the basis of a detailed *dynamical* theory. In this, we have also reviewed the relevance of the ideas of Kolmogorov to the hydromagnetic case, keeping in mind the Galilean non-invariance of the hydromagnetic equations to a random constant magnetic field transformation. The cascade of energy in the hydromagnetic case is not strictly local in the wave number domain. A large scale rms magnetic field presents the possibility of Alfvén wave propagation along it and thus provides a significant dynamical coupling between magnetic fields of large and small scales. Our steady state considerations provide us with the necessary information about the local internal relaxation features and their relative magnitudes, so much so we plan to extend our earlier study of evolution of weak magnetic fields – to a stage in which the spectrum of the magnetic field has evolved sufficiently to a point of dynamical feedback to the velocity field and consequently a statistical steady-state.

And since we are basing our calculations on a well-considered dynamical theory of turbulence, we will be able to throw some light on the nature of the transfer of energy in the magnetic spectrum: in particular, without using either oversimplifications or idealisations of the characteristic length and time scales of the magnetic field and turbulence as have been done by Moffatt (1970), Parker (1969), Fitremann and Frisch (1969) or Vainshtein (1970).

2. The Dynamical Model

We start with a steady turbulence with an extended inertial range. The choice of the kinematic parameters and the wave number range is made suitably, so that we can talk of an extended equilibrium range, without worrying about the sources of input of energy into the system from the geometric range. Further, there exists a sufficiently noticeable dissipative tail to the spectrum at the high wave number end. The form of the spectrum and parameters are chosen so as to be compatible with the asymptotic requirements of the direct interaction approximation of Kraichnan (1958, 1959, 1965, 1966), with suitable modifications to reproduce Kolmogorov scaling.

A disturbance which is localized in the wave number range of the magnetic spectrum is introduced at time $t=0$.

Following the notations of our earlier papers (Kraichnan, 1958; Kraichnan and Nagarajan, 1967; Nagarajan, 1971), we can write the equation for the secular evolution of the two spectra for times >0 as

$$\left(\frac{\partial}{\partial T} + 2\nu k^2\right) E^V(k; T) = \iint \frac{k}{2pq}\, dp\, dq\, [\{k^2 a_{kpq} E^V(p; T)\, E^V(q; T)\, \theta^{VVV}_{kpq} - p^2 b_{kpq} E^V(k; T)\, E^V(q; T)\, \theta^{VVV}_{pqk}\} + \{k^2 a_{kpq} E^M(p; T)\, E^M(q; T)\, \theta^{VMM}_{kpq} - p^2 C_{kpq} E^V(k; T)\, E^M(q; T)\, \theta^{MMV}_{pqk}\}] \tag{1}$$

$$\left(\frac{\partial}{\partial T} + 2\lambda k^2\right) E^M(k; T) = \iint_{\Delta} \frac{k}{2pq}\, dp\, dq\, [k^2 d_{kpq} E^M(p; T)\, E^V(q; T)\, \theta^{MMV}_{kpq} - p^2 h_{kpq} E^M(k; T)\, E^V(q; T)\, \theta^{MVM}_{pqk} - p^2 j_{kpq} E^M(k; T)\, E^M(q; T)\, \theta^{VMM}_{pqk}] \tag{2}$$

The spectral functions $E^V(k;T)$ and $E^M(k;T)$ are connected to the velocity and magnetic fields as follows:

$$W^V(k;t,t') = (2\pi)^{-3} \int d^3(\mathbf{x}-\mathbf{y}) \langle \mathbf{U}(\mathbf{x};t)\cdot\mathbf{U}(\mathbf{y};t')\rangle e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}$$

$$W^M(k;t,t') = (2\pi)^{-3} \int d^3(\mathbf{x}-\mathbf{y}) \langle \mathbf{W}(\mathbf{x};t)\cdot\mathbf{W}(\mathbf{y};t')\rangle e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}$$

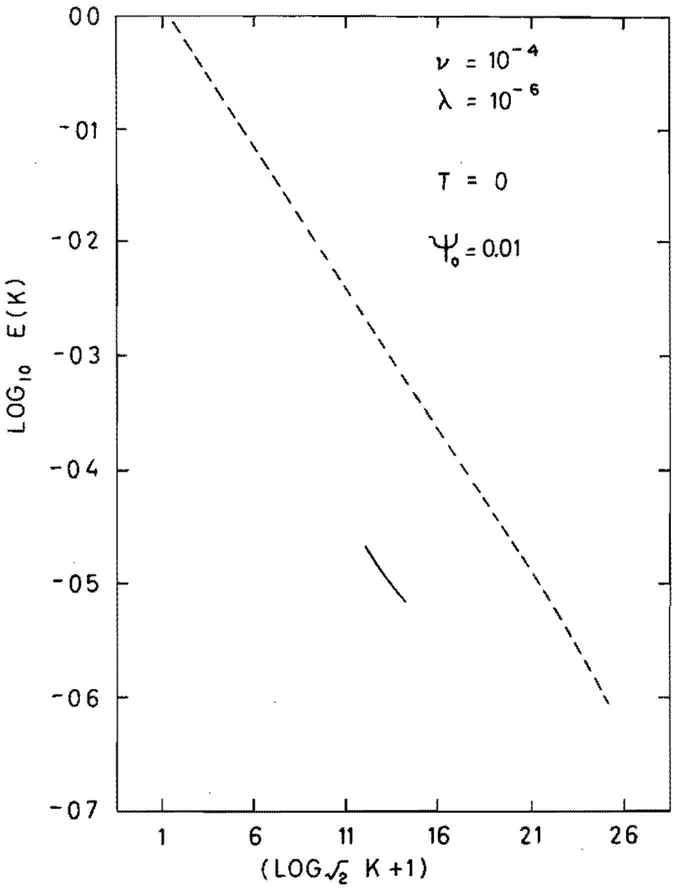


Fig. 1.

where $\mathbf{U}(\mathbf{x}, t)$ is the fluid velocity and $(4\pi\mu\varrho)^{1/2}$ $\mathbf{W}(\mathbf{x}, t)$ is the magnetic induction field, ϱ is the fluid density, μ the magnetic susceptibility of the fluid, ν and λ are the kinematic viscosity and magnetic diffusivity respectively.

We assume the turbulence to be homogeneous and isotropic

$$\tfrac{1}{2}W^V(k;t,t') = (4\pi k^2)^{-1}\, E^V\left(k;\frac{t+t'}{2}\right) R^V(k;t-t').$$

$$\tfrac{1}{2}W^M(k;t,t') = (4\pi k^2)^{-1}\, E^M\left(k;\frac{t+t'}{2}\right) R^M(k;t-t')$$

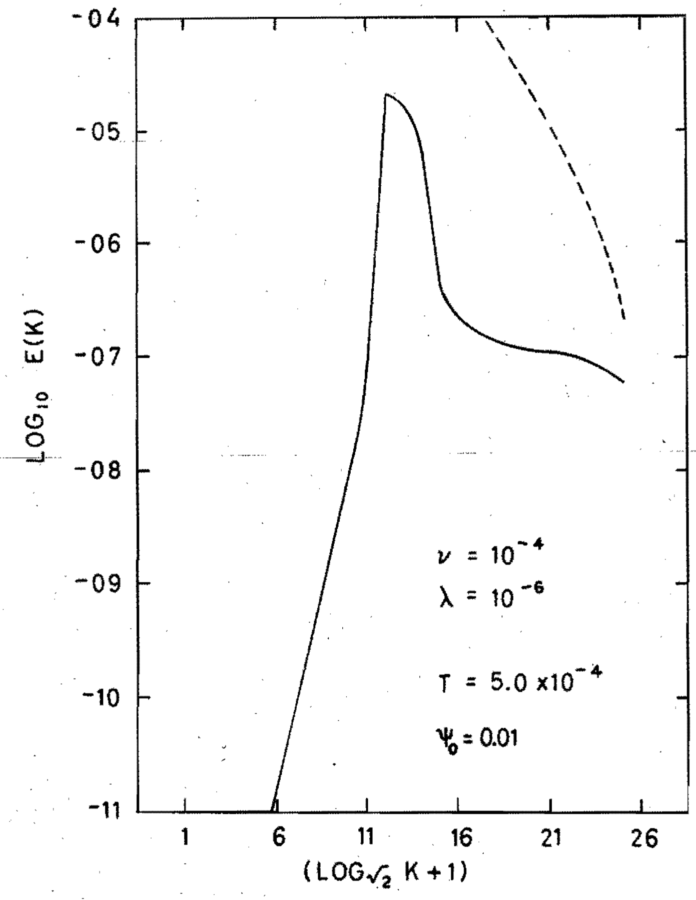


Fig. 2.

where $W^V\{\ \}$ and $W^M\{\ \}$ are energy functions and $R^V\{\ \}$ and $R^M\{\ \}$ are modal correlation functions.

The θ'-s which appear in Equations (1) and (2) are the effective memory times of the interaction between the three respective wave numbers. They are given by

$$\theta^{abc}_{lmn}(T) = \int_{-\infty}^{\infty} G^a_l(T-s)\, R^b_m(T+s)\, R^c_n(T+s)\, ds$$

(where a, b, $c=V$ or M) and $G^V(k;T)$ and $G^M(k;T)$ are the averaged response functions of the velocity and magnetic fields for the given wave number respectively.

In a general turbulent system in which a weak macroscopic (i.e. geometric range) disturbance in the magnetic spectrum is introduced at time $t=0$, the θ'-s will be very complicated functions of the correlation and response features of the turbulence

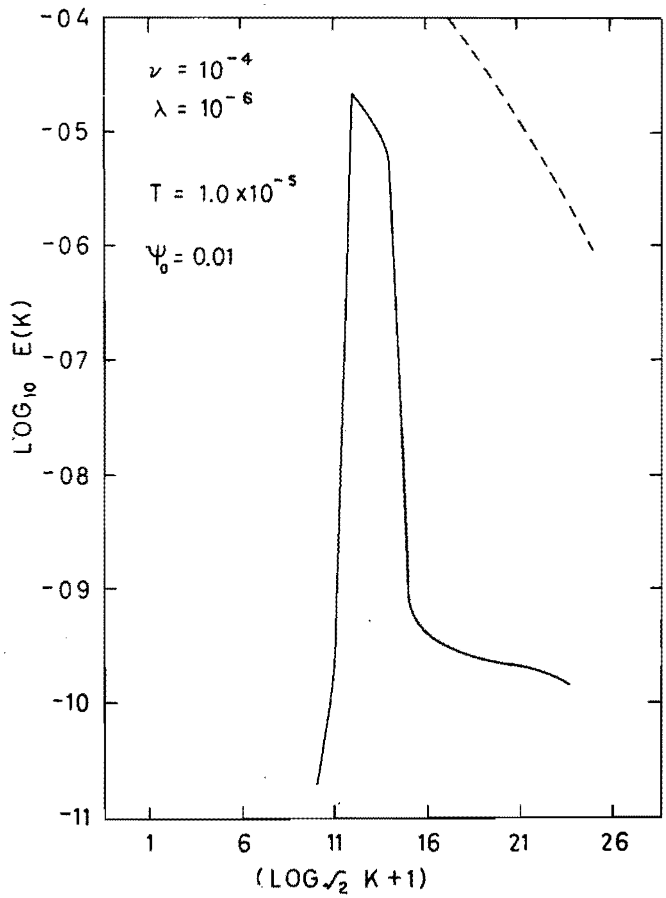


Fig. 3.

and initial magnetic field. But if we assume that the weak magnetic excitation is sufficiently localized in the inertial range, the secular time dependence of the θ'-s can be ignored. This point has been discussed in detail by Kraichnan (1959) in the hydrodynamic context. In the magnetic situation also much of the argument goes through unaltered.

We choose a form for the correlation and relaxation functions and the θ'-s from Nagarajan, 1971.

$$R^a(k;t) = \exp\{-\tfrac{1}{4}\pi(\zeta_a(k)\,t)^2\}$$
$$G^a(k;T) = \exp\{-\tfrac{1}{4}\pi(\eta_a(k)\,t)^2\}$$

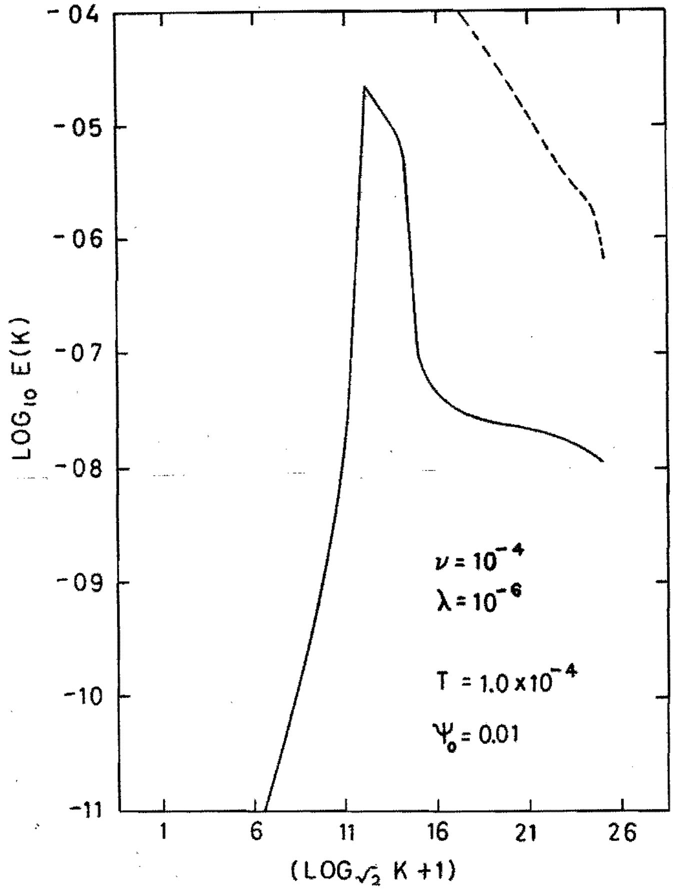


Fig. 4.

which gives for θ

$$\theta^{abc}_{kpq} = [\{\eta_a(k)\}^2 + \{\zeta_b(p)\}^2 + \{\zeta_c(q)\}^2]^{-1/2}.$$

Our elaborate study of the various extreme considerations of Galilean-invariance and Kolmogorov's arguments on the one hand and Galilean non-invariant Eulerian solutions on the other in the steady-state case (Nagarajan, 1971) convinces us that in so far as energy transfer information is concerned, the details of the internal correlation times are not very important. Using the results of this study, we evolve a quasi-Lagrangian scheme. We take the velocity correlations and relaxations to be Kolmogorovian i.e. decided by the local parameters of the position in the wave number spectrum. The magnetic terms are modulated by energy range parameters as in the unmodified direct interaction approximation of Kraichnan (1959, 1965). With

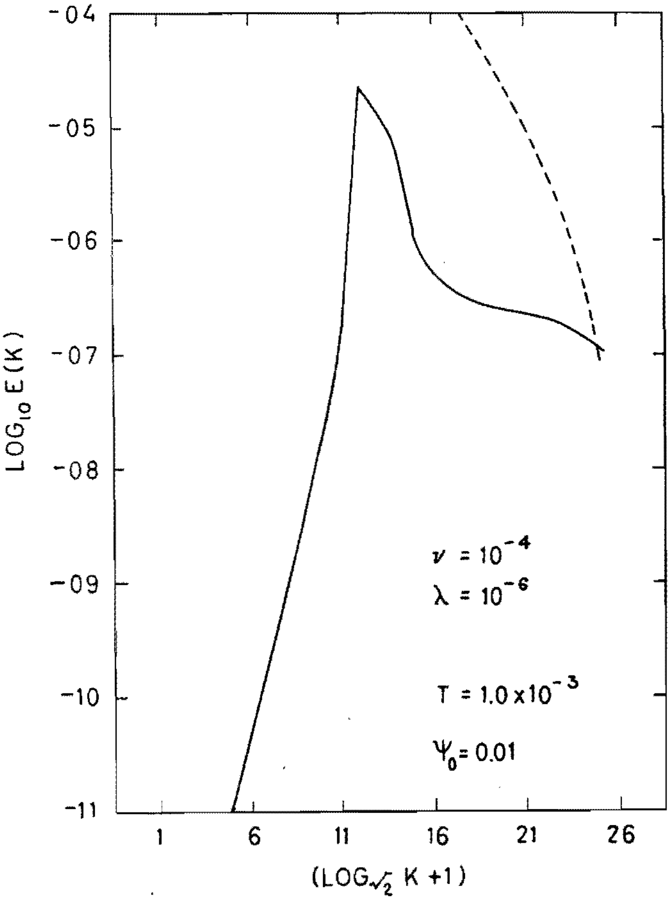


Fig. 5.

these preliminaries one can write

$$\zeta_V(k) = [E^V(k; T)\,k^3]^{1/2}$$
$$\eta_V(k) = [\{\zeta_V(k)\}^2 + (\nu k^2)^2]^{1/2}$$
$$\zeta_m(k) = (v_0 k)$$
$$\eta_m(k) = [\{\zeta_m(k)\}^2 + (\lambda\, k^2)^2]^{1/2}.$$

Here v_0 is the rms velocity in the energy range. (It will be apparent that this energy-range mixing was the reason why we chose the initial magnetic excitation to be localized in the inertial range. But for that the results of the hydrodynamic case or even the steady-state study will be inapplicable.) We choose a convenient unit of wave numbers and time scales such that $v_0 = 1$.

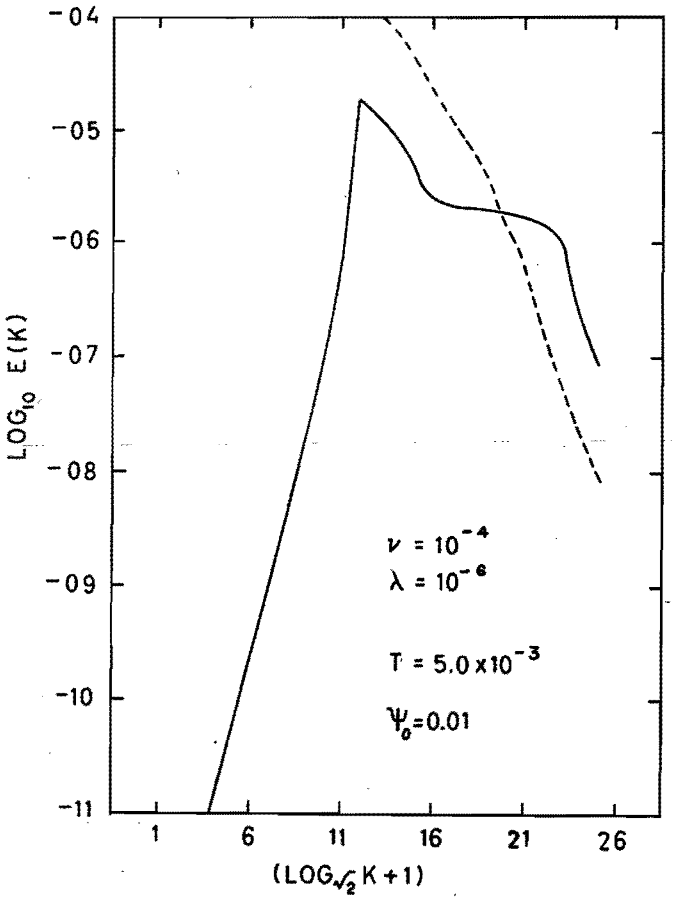


Fig. 6.

3. Evolution Study

Now that all the quantities in Equations (1) and (2) are completely defined, we integrate them forward in time. In time, they have the character of a set of non-linear coupled differential equations. But for each time value there is an integral to be performed over the contributions from various regions of wave number space. We discretise the wave number region into twenty-five logarithmic half-octave intervals.

The details of this procedure are much the same as in an earlier paper (Nagarajan, 1971). We perform the time integration using a fourth-order variable-step Runge-Kutta Scheme. The details of the numerical scheme are given elsewhere (Nagarajan,

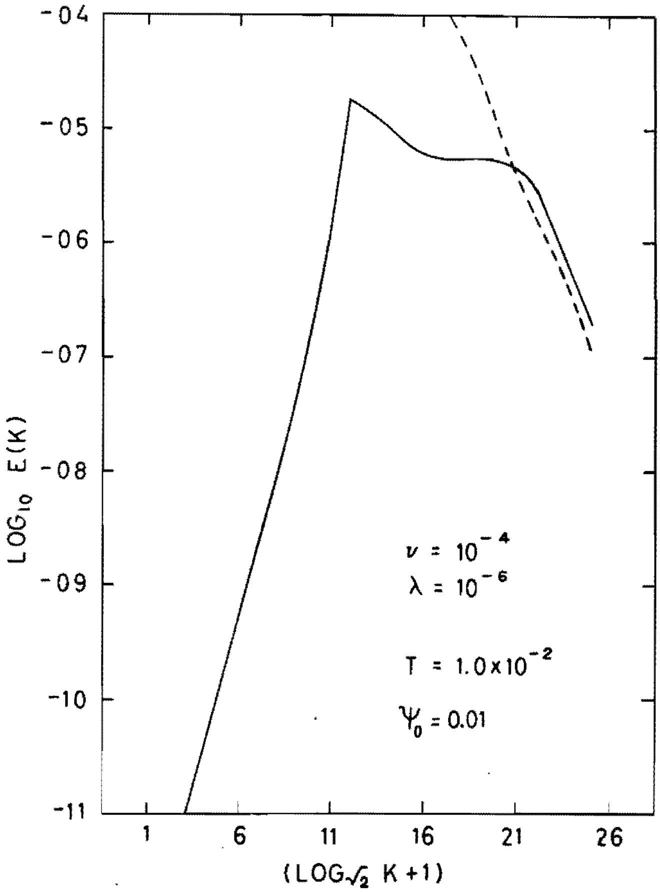


Fig. 7.

1970). We shall here consider only the results and their astrophysical implications.

Figure 1 shows the initial spectral disposition in one of the runs. The dotted line gives the velocity spectrum, and the continuous line, the magnetic disturbance. ψ_0 is the value of the initial ratio of the magnetic spectrum to the velocity spectrum at nonzero points, which is a parameter of the run. Though we are going to display here only initial disturbances which have the same spectral shape as the velocity and are localised in wave number space in a delta-function way, we had performed a number of runs with a variety of initial shapes $ak^n \exp(-bk^m)$ and initial ratio ψ_0. There was no pathological feature arising from the initial choice either numerically or otherwise.

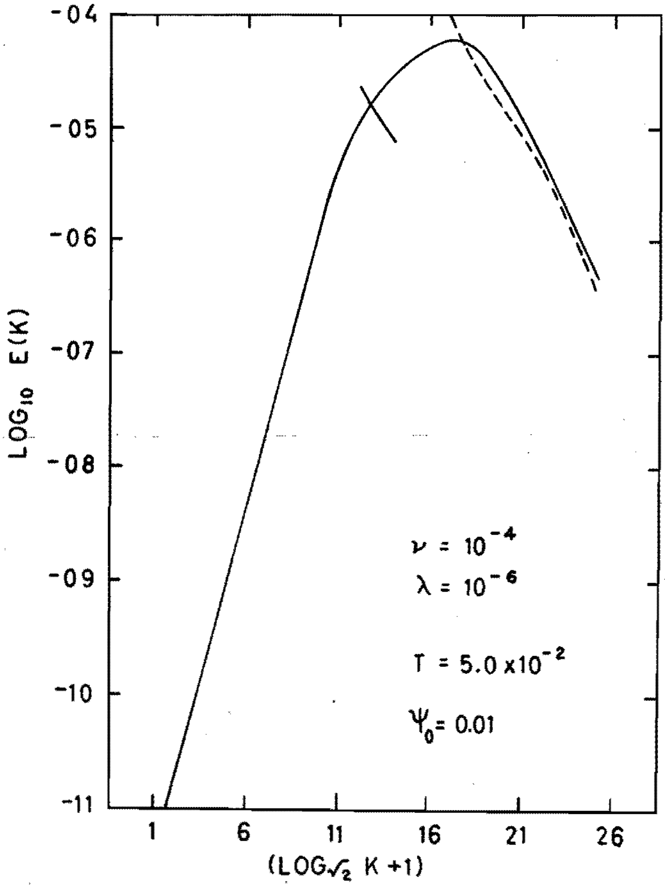


Fig. 8.

Figures 2 and 3 give the spectra at characteristic times $t = 1.0 \times 10^{-5}$ and $t = 1.0 \times 10^{-4}$. These time scales are so normalised that they are unity for the largest wave numbers in our system. The noteworthy feature of the curves is that the energy has now moved both to higher and lower wave numbers. The rate of transfer to lower wave numbers is essentially smaller than the rate of transfer to higher wave numbers, because the characteristic times of transfer are of the order of the internal times of the given scale.

Figures 4 and 5 give the spectra at $t=5.0\times10^{-4}$ and 1.0×10^{-3}. Already, within a time of the order of the local eddy-circulation time in the largest wave numbers, the magnetic spectrum has wrapped up sufficiently to almost equality with the velocity

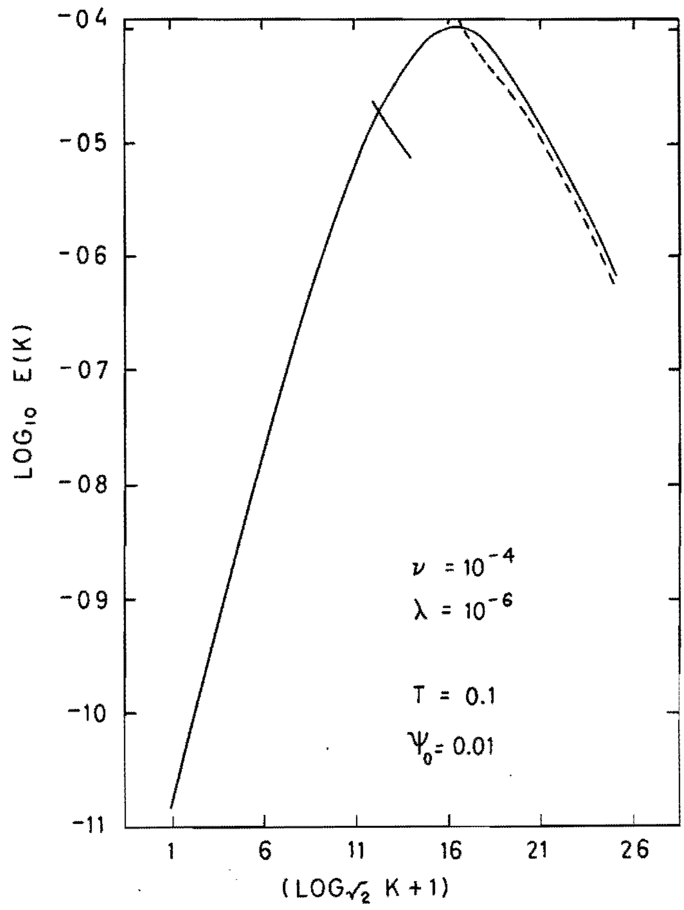


Fig. 9.

spectrum at the highest wave number. Figure 5 to some extent and Figure 6, in a more profound way show that the magnetic spectrum has overshot significantly above the velocity at lowest scales. This arises because of two reasons: (1) The choice of kinematic parameters ν and λ. In this run λ is very much smaller, so much so the magnetic spectrum has a *longer* dissipative tail. (2) The second reason for the overshooting is the fact that the form of the spectrum is still non-equilibrium so much so the approach to local equipartition is in an overstable way.

Figure 7 and more prominently Figure 8 show how the feature of equipartition is transferred to smaller wave numbers, much in the same way as argued by Biermann and Schlüter (1951). By now the evolution has reached a stage in which any peculiar

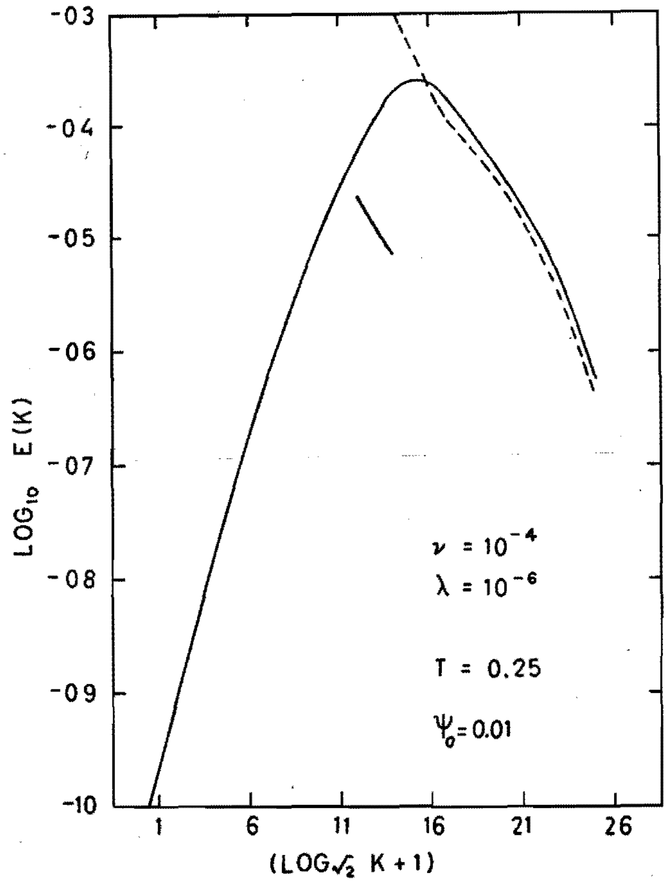


Fig. 10.

dependence on choice of initial form has been completely lost. Figures 9 and 10, which are for the same run for times $t=0.1$ and 0.25 show that by now the evolution has reached a stage when one can safely conclude about ultimate features. The numerical integration times involved at this stage are so large that one stops the calculations because no new features are likely to evolve from further evolution study.

Figures 11, 12 and 13 feature the final and initial spectra for a few other runs which start different initial ratios and kinematic parameters. These are meant for the purist to show that pathological features are not included in the choice of initial assumptions.

In all these runs, at a fairly advanced evolution, the spectral shape reaches an approximate form $A(t)\,k^4 \exp\{-B(t)\,k^2\}$. Thereafter the integral features of the spectrum evolve more or less without change of form.

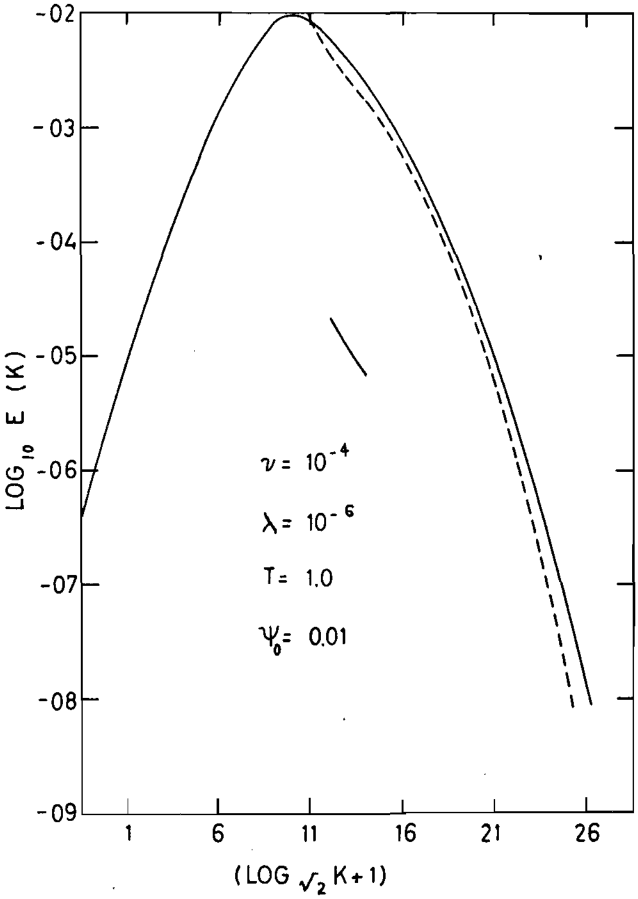


Fig. 11.

4. Conclusions

Apart from the fact that this evolution study fills many a gap in our earlier study, this proves more or less conclusively that there is no reason to expect, in evolving non-equilibrium hydromagnetic turbulence, that the transfer will take place only to larger wave numbers. In fact, the transfer to smaller wave numbers is significant and this can provide just the missing link in the *turbulent dynamo* problem. The regeneration of larger magnetic loops through a co-operative interaction of the velocity fluctuations of all scales and magnetic fluctuations of smaller scales is not only feasible but very significant. In our study, we find that this is facilitated by two dynamical requirements. Firstly, the non-equilibrium feature of the magnetic spectrum: the ultimate steady-

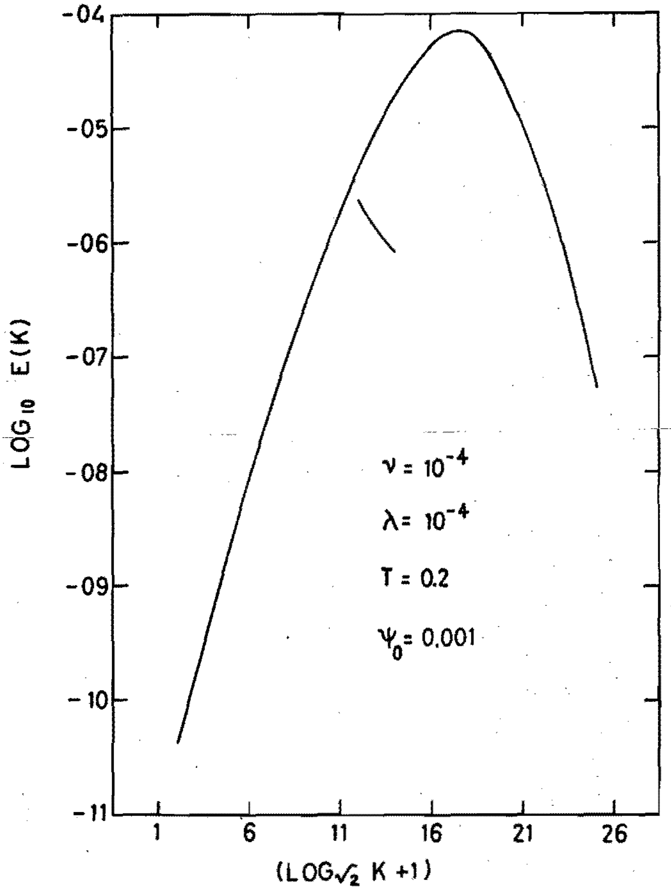


Fig. 12.

state magnetic spectrum will be in equipartition with the velocity in all scales other than the ones where either the inputs of energy from external sources of the train of energy through molecular dissipation depresses or raises either of them. Any other form of the spectral ratio is not an invariant form which will be left invariant by the non-linear interaction. The non-linear interaction will change the ratio to get into the equilibrium form. Secondly, the Galilean non-invariance: The fact that a magnetic field cannot be gauged out makes a profound modification in the internal dynamics. Here probably one can stretch our comparison a bit with other recent studies. Krause (1968), Rädler (1968), Steenbeck *et al.* (1966), Steenbeck and Krause (1966, 1967), Krause and Rädler (1971) and Moffatt (1970) have considered the α-effect of regenera-

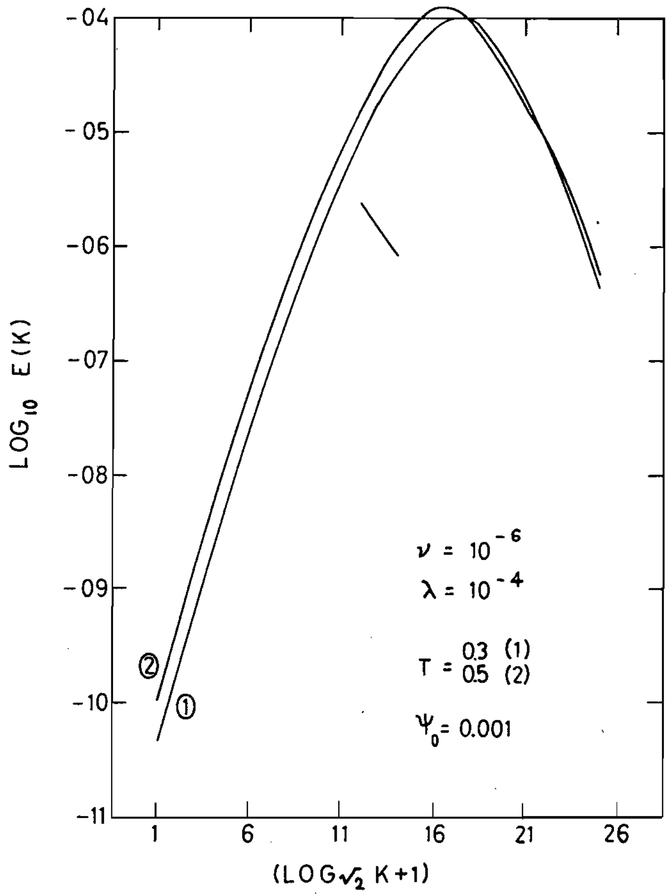
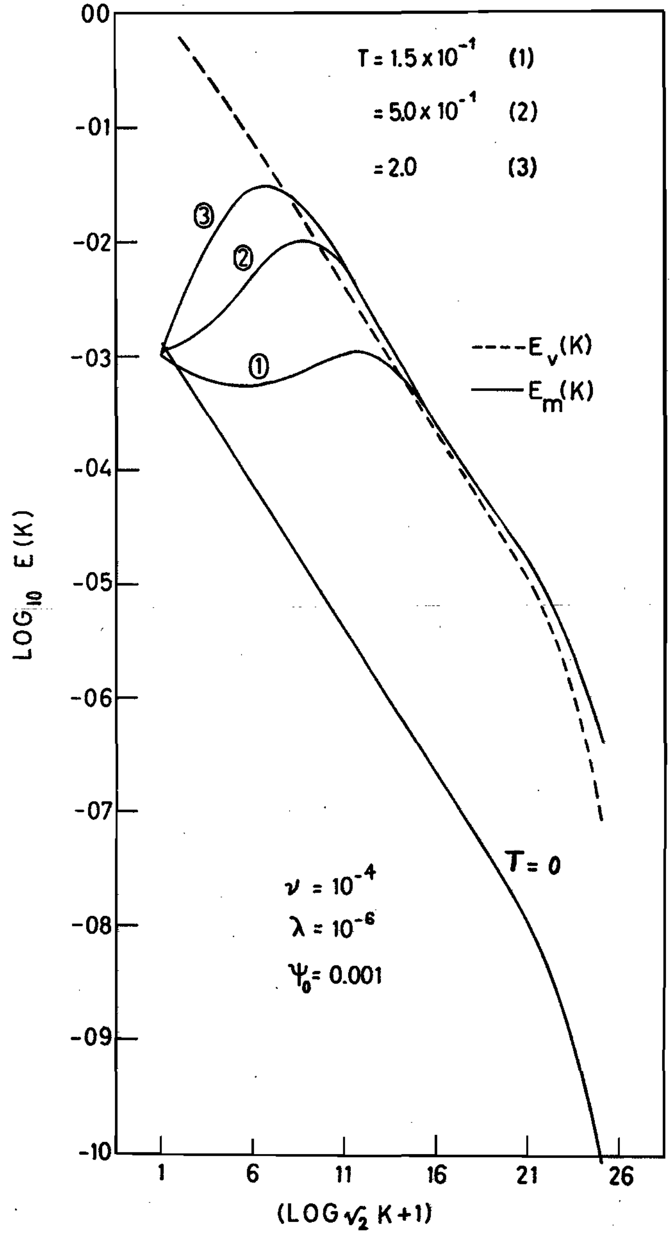


Fig. 13.

tion in great detail. A certain aspect of the α-effect is included in our Galilean non-invariance picture, because a larger magnetic loop, when it is impressed on a system of smaller magnetic and velocity fluctuations, introduces a condition of reflectional non-invariance. Beyond this point one cannot carry the analogies because their inferences about the values of the α-effect are based on equilibrium transfer theory, which as our study has clearly shown, are inapplicable.



T = 1.5 x 10^-1 (1)
= 5.0 x 10^-1 (2)
= 2.0 (3)
----E_v(K)
——E_m(K)
T = 0
ν = 10^-4
λ = 10^-6
Ψ_0 = 0.001
LOG_10 E(K)
(LOG √2 K+1)
00
-01
-02
-03
-04
-05
-06
-07
-08
-09
-10
1
6
11
16
21
26

Fig. 14.

Parker (1969) and Vainshtein (1970) have asked much the same question, as we have, but since they had to invoke some extreme idealisations to get their results, the physical validity of their conclusions is in doubt. Qualitatively, our results corroborate theirs.

Robinson and Rusbridge (1971), in a study of Plasma turbulence in the Zeta plasmas, have found that plasma turbulence seems to resemble fluid turbulence except that the turbulent elements are enlarged along the mean magnetic field to form rolls and suggest that an appropriate comparison would have to explain the existence of significant transfer to large scales from small-scales, as against isotopic hydrodynamic theory, which will not permit this. One hopes that it will not be too presumptuous to believe that the effect, they find is contained in our procedure. Further the importance of this to heat transfer in the presence of magnetic turbulence is also very tempting.

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Discussion

Nakagawa: What is your assumption concerning the initial velocity and magnetic field spectra?

Nagarajan: The initial velocity is in quasi-equilibrium with an extended inertial range. The magnetic spectrum is localized in the middle of the inertial range in all but one of the runs, with a level of excitation very much lower than the velocity.

Weiss: After equipartition has been achieved for intermediate wave numbers, is your steady energy spectrum maintained over periods comparable with the resistive decay time for the smallest wave numbers?

Nagarajan: Yes. We follow the time evolution until the initial form dependence is washed out. Essentially this turns out to be larger than the resistive time scale of the initial specimen. But after that time, the further buildup of the spectrum – even towards smaller wave numbers – takes energy from the velocity spectrum. This time-invariant self-preserving form with the tail in steady-state with the velocity, keeps growing in over-all energy and extent. This may look like a violation of simple physical and statistical requirements. But it is not.

Cowling: In many ways the assumptions made (nature of background fields, motions, statistical assumptions) appear to be as important in the theory of magnetohydrodynamic turbulence as the detailed theory.

Nagarajan: True: statistical description does not in any sense minimize the number of necessary assumptions. But the statistical theory has an advantage in that one requires only on-the-average features. So many of the phasing requirements are weakened. But the main feature of this investigation has been to show that the back-transfer in wave-number spectrum is significant, which can have truly deep conceptual consequences.

BIOGRAPHY

The author, born on April 11, 1936, had his school and university education in India. In 1956, after obtaining his Master's Degree from the University of Madras in India, he joined the Theoretical Physics group of the Tata Institute of Fundamental Research, in Bombay, India. During the years 1960-65 he was associated with the Division of Electromagnetic Research, of the Courant Institute of Mathematical Sciences, in New York. Late in 1965, he returned to the T.I.F.R., in Bombay. In 1968, he moved to the Institute of Mathematical Sciences, in Madras, India, which was followed by a shift to University Libre de Bruxelles, in Belgium, in 1970. In 1973, he moved to the Netherlands, first to the Institute of Theoretical Physics at the State University of Utrecht, then to the Technishe Hogeschool in Eindhoven.

STELLINGEN

1. Weak turbulence ordering used by Sagdeev and Galeev (and later by a number of authors) based on the occupation number representation for a steady set of interacting waves is invalid, because it does not distinguish in a proper manner between coherent waves (space-time coherence in the infinite domain)&fluctuations (localised random wave packets with finite life-times).

   Sagdeev R.Z. and Galeev A.A., Non-linear Plasma Theory (W.A.Benjamin Inc., 1969)

2. The energy balance equations for the spectral transport in plasma turbulent reactors derived by Tsytovich and further developed by ter Haar are inappropriate for the case of plasma turbulence, because of the use of a Detailed Balance Condition.

   See Tsytovich V.N. and Kaplan S.A., Plasma Astrophysics (Pergamon Press 1973) and Norman C.A. and ter Haar D., Plasma Turbulent Reactors; An Astrophysical Paradigm, Physics Reports 17c no:6 1975.

3. The use of the term, 'non-linear dispersion relation', to define the propagation and decay of fluctuations in a turbulent medium is misleading. It is further unjustified except in the case of complete isotropy and statistical

stationarity in time, even in a restricted sense.

Sagdeev and Galeev, loc. cit.,,Kadomtsev B.B., Plasma Turbulence Pergamon Press, 1965. see chapters 3 and 5 of the summary in this dissertation

4. Much of what has been derived for the case of wave propagation in a random medium cannot be applied to a turbulent medium, even under limiting conditions of separation of scales of turbulence, wave lengths of the waves and the depth of propagation in the medium. The interplay of various transfer phenomena to the fluctuating components of the wave from the medium will make the 'Freezing Approximation' of turbulence invalid, after the medium has been irradicated by the wave for a while.

See a review of Wave Propagation in Random Media by U.Frisch in Probabilistic Methods in Applied Mathematics, ed. by A.J. Bharucha-Reid Academic Press, (1968)

5. Despite over abundence in scientific activity and output, there is a lack of fundamental breakthroughs in science. This is an inescapable outcome of the economic attractions of the golden age of patronage of science, by a society which was frightened by it.

6. It seems fashionable these days to play down the intellectual aspects of scientific pursuits (Perhaps the dropping of the term natural philosophy to refer to science is an outcome

from this). Few scientists would call themselves intellectuals even in their own interpretation of the word 'intellectual'. This is a pity.

7. Much of what goes around in the literature in the name of "Non-linear Intuition" is the result of a carefully covered random linear analysis (This dissertation not completely excluded!).

   See for e.g., W.Heisenberg in "Topics in Non-linear Physics" ed. by Norman Zabusky, (Springer 1968)

8. Turbulence is a phenomenon which is appealed to in order to shroud all and sundry difficulties of measurements, observation, comprehension and interpretation of phenomena at large.

9. Weak turbulence is not necessarily easier to deal with than strong turbulence.

10. Modern operational approaches to scientific education convert scientists into walking "Encyclopaedias of Recipes" rather than perceptive creative thinkers.

11. A corollary of proposition 10;- A global neglect of teaching of chronological development and proper perspective of the history of science to serious students in science is responsible for the romantic Idolatry of concepts and people in science.

12. A true scientist is an Iconoclast.

13. To be totally consistent, a theory must be empty. ("To be completely consistent, a man must be a saint or a crook" v.Goethe).