Simple distribution-free confidence intervals for a difference in location

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Aan de nagedachtenis van mijn Moeder

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CONTENTS

INTRODUCTION

The purpose of this investigation is to derive distribution-free confidence intervals for the difference of location between two populations with the same shape of their distributions. A procedure is called distribution-free if its validity does not depend on the form of the underlying distributions at all, provided that these distributions are continuous. In one case distribution-free confidence intervals are also derived for the ratio of scale parameters.

In many practical situations it may be important and useful to determine distribution-free confidence intervals for the difference of location. This may arise in practice in the comparison of two treatments, products, or factors in a simulation experiment, etc. Methods do exist for determining confidence intervals based on distribution-free rank tests (e.g. the tests of Wilcoxon or Van der Waerden). But in general the computation of confidence intervals based on rank tests is rather laborious. In practice, however, there is often a need for rapid statistical calculations. In this monograph confidence intervals for difference of location will be derived which are based on pairs of order statistics. The advantage of these confidence intervals is that the determination in practice requires only slight computations. As a matter of fact these distribution-free confidence intervals can be converted into distribution-free tests for the two-sample problem with the null hypothesis that there is no shift against the alternative hypothesis that there is a shift.

Chapter 1 is a survey of some literature about distribution-free tests which are often more or less related to the tests considered in this monograph.

In chapter 2 some properties of order statistics necessary for our methods are presented. The confidence intervals based on order statistics and the corresponding tests as well as the determination of confidence coefficients will be described in chapter 3.

Later on we shall see that the class of confidence intervals considered is very wide, with the result that several candidates are available for the same problem. Thus the statistician might be tempted to choose the procedure leading to the conclusion he favours. Of course, this difficulty can be avoided by choosing the procedure beforehand, namely before the results of the experiments are known. However, in chapter 4 a procedure will be described by which for various cases preferable procedures are selected. The selection criterion will be based on the power function of the corresponding test. The corresponding power functions have been examined for Lehmann alternatives, Normal, Uniform and Exponential translation alternatives and Exponential scale alternatives.

In chapter 5 some remarks on the use of Lehmann alternatives are made and in chapter 6 Pitman's asymptotic relative efficiency will be introduced for various cases. Finally in chapter 7 some remarks are made on the relation between some of our tests and the Wilcoxon (Mann–Whitney) distribution-free two-sample tests and our tests are compared with Student's two-sample t-test for Normal translation alternatives and with Wilcoxon's test for Lehmann alternatives. Moreover, the tests selected for Normal translation alternatives are compared with Wilcoxon's test when testing against Lehmann alternatives.

1. SURVEY OF SOME LITERATURE

The quick confidence intervals and tests for the difference between the medians of two populations, which are different only in location, as described in this monograph, can be used when simple analysis of data is desirable or necessary. Simplicity means practical transportability, the possibility for the statistician to carry the procedure anywhere. It may be necessary in certain situations to construct confidence intervals or to test a null hypothesis by heart. This may be the case either when the observations cannot be taken away and analyzed, for example when a conclusion is required at the place of investigation, or when a quick analysis is useful to get a rough idea of the situation before handing the data to a computer. Moreover, these confidence intervals can be used as a quick method of checking whether an analysis completed by a desk calculator or electronic computer is correct. In short, such quick confidence intervals and tests as described in this monograph are especially for use "as a footrule", "in the field", etc.

For the two-sample problem with translation alternatives there are various distribution-free tests available in the literature. Some of the simplest or, more or less, related tests will now be summarized.

When two samples are compared in order to test the difference between their means, it is possible, provided that the sample sizes are equal, to apply the following sign test procedure described by Duckworth and Wyatt (1958). The samples are paired off in random order and the number of times that one sample result is greater than the corresponding result in the other sample is compared with the number of times that the reverse event occurs. The two-sided 5 per cent probability level of the absolute value of this difference is approximately $2\sqrt{N}$, where N is the total number of differences different from zero.

Another procedure, this time again for arbitrary sample sizes, is described by Tukey (1959). If one sample contains the highest value and the other the lowest value, then we may choose (i) to count the number of values in the one group exceeding all values in the other, (ii) to count the number of values in the other group falling below all those in the one, and (iii) to sum these two counts (we require that neither count be zero). If the two samples have roughly the same size, then the critical values of the total count are roughly 7, 10 and 13, i.e. 7 for a two-sided 5 per cent level, 10 for a two-sided 1 per cent level, and 13 for a two-sided 0·1 per cent level.

To construct a confidence interval one need only find out which shifts of one sample do not result in rejection of the null hypothesis. Even in these cases the construction of a confidence interval is more or less a trial-and-error method. The construction of confidence intervals described in this monograph hardly requires any computation.

The Westenberg–Mood test (often described as a median test *)) consists in determining the median $\hat{m}_e$ of the combined sample and the number of observations of one sample that are smaller than $\hat{m}_e$ and tests the hypothesis that equal proportions of the x- and y-population lie below $\hat{m}_e$. This can be done by using the method of the 2×2-table with the x-sample and the y-sample as row categories and with column categories "smaller than $\hat{m}_e$" and "larger than or equal to $\hat{m}_e$". Some other forms of the test statistics are also employed. Of course, this median test can be used as a test for location difference of two populations with the same shape for their distributions. Pitman's asymptotic relative efficiency of this median test for location difference relative to Student's two-sample test in the case of two Normal distributions with equal variances, is equal to $2/\pi \approx \cdot 637$ (Mood (1954)). In Westenberg (1948, 1950, 1952) various tables are presented. The median test is asymptotically most powerful in the case of a density of double Exponential type.

In Dixon's paper (1954) the powers of four nonparametric tests: rank-sum, maximum deviation, median and total number of runs, for the difference in location of two Normal populations with equal variances are computed for equal sample sizes of three, four and five observations.

Chakravarti, Leone and Alanen (1962) have shown that the asymptotic relative efficiencies of Mood's test and Massey's test (see Massey (1951)) based on the first quartile and the median are zero, when these two tests are compared against the likelihood-ratio test appropriate for detecting a shift in location of an Exponential distribution. They found Massey's test to be about three times as efficient as Mood's test for Exponential distribution and that the same is true already for the test based on the first quartile alone. In their (1961) paper they have derived the exact power of Mood's and Massey's two-sample tests for testing against Exponential and Uniform shift and Uniform scale alternatives.

An interesting class of tests is the class of quantile-tests. The median test is a special case of a quantile-test. The quantile-test consists of pooling the x- and y-samples and arranging them in increasing order of magnitude. Then one divides the combined sample of size N into two classes G and L, say, consisting respectively of the observations larger than or equal to the sample p-quantile $\hat{q}_p$ and those smaller than $\hat{q}_p$.

In the 2×2-table procedure sketched above the column categories are now "smaller than $\hat{q}_p$" and "larger than or equal to $\hat{q}_p$". This procedure tests whether or not the two populations have the same pth quantile. Note that the definitions of $\hat{m}_e$ and $\hat{q}_p$ in the test procedures are not always the same in the literature; the same can be said about test procedures of median- and quan-

*) J. V. Bradley wrote in his book (1968): "it might be better described as a quasi-median test or as test for a common, probably more or less centrally located, quantile". This is a strange remark and cannot be understood, because this test is not a conditional test.

tile-tests. Hemelrijk (1950) has investigated in his thesis the quantile-tests in the case of an arbitrary underlying distribution function, continuous or discrete, and has given a generalization of the quantile-test, namely a "two-quantiles-test". In this case, roughly formulated, two percentiles $\hat{q}_{p_1}$ and $\hat{q}_{p_2}$ ($> \hat{q}_{p_1}$) of the combined sample (for the exact definition of $\hat{q}_{p_1}$ and $\hat{q}_{p_2}$, see Hemelrijk) are chosen and the number s_1 of observations of one sample smaller than $\hat{q}_{p_1}$ and the number s_2 of observations of this same sample larger than $\hat{q}_{p_2}$ are determined. The critical region of this test consists of pairs (s_1, s_2) with the smallest probabilities under H_0. Hemelrijk remarks that it is possible to generalize this "two-quantiles-test" to more than two quantiles but that the construction of a critical region is very tedious.

In Barton's paper (1957) the powers against change of centre of location of the quantile test T_b' proposed by David and Johnson (1956) (assuming Normally distributed variables) and the general quantile-test T_b (cf. Mood (1954)) are compared asymptotically, under the assumption that the densities are everywhere differentiable. The test proposed by David and Johnson consists of the difference of the sample ϱ-tiles ($0 < \varrho < 1$) of the two samples multiplied by $(m+n)^{1/2}$ (m and n are the two sample sizes) and divided by an interquantile estimate of the standard deviation of this difference. Barton proved in his paper that the power functions of T_b' and T_b (against slippage alternatives) are the same in the limit. Writing $m = P\,N$ and $n = Q\,N$ ($N = m + n$) and considering P and Q to remain fixed as N tends to infinity, then for $N \rightarrow \infty$ one can take as test statistic the classical 2×2-table statistic *)

$$\underline{T}_b' = (\underline{a} - \varrho\, m) \left(\frac{N-1}{m\, n\, \varrho\,(1-\varrho)} \right)^{1/2}.$$

which is standard Normal under H_0 and where $\underline{a}$ is the number of x-variables less than or equal to the ϱNth pooled variable. In the case of slippage this statistic has unit variance and mean

$$\delta f_\varrho \left(\frac{P\,Q}{\varrho\,(1-\varrho)} \right)^{1/2},$$

where the slippage equals $\delta/\sqrt{N}$ and f_ϱ is the value of the density of $\underline{x}$ in the population ϱ-tile. For one special case ($m = n = 9$, $\varrho = \frac{1}{2}$) he computed the power of T_b' against some Normal slippage alternatives at the 5 per cent level.

Another possibility of a simple test procedure is the test proposed by Mathisen (1943): observe the number $\underline{k}$ of observations of the second sample

*) Random variables are underlined.

of size $2n$ whose values are smaller than the median of the first sample of size $2m + 1$. A table with lower and upper ·01 and ·05 percentage points for the distribution of $\underline{k}$ can be found in his paper. For large sample sizes a Normal approximation can be used.

Bowker (1944) has shown that this test is not consistent with respect to certain alternatives. A test is called consistent if the probability of rejecting the null hypothesis (that the two samples are from populations with the same continuous distribution functions) when it is false tends to unity if the sample sizes tend to infinity. If their cumulative distribution functions are identical in the neighbourhood of their medians, the test is not consistent. It is clear that the test is consistent for the class of slippage alternatives: $G(y) \equiv F(y + \Delta)$, where $F(x)$ and $G(y)$ are the cumulative distribution functions of the two populations, for the practically important case that $f(x) = F'(x)$ exists and the set $\{x; f(x) \neq 0\}$ is an interval. Similar remarks can be made for all these kinds of quantile-tests.

Mathisen discussed also another method which makes use of the median and quartiles in the first sample. The general principle of the kind of test procedures to which this method belongs, can be summarized briefly as follows. The first sample is used to establish any desired number of intervals into which the observations of the second sample may fall. The proposed test criterion is based on the deviations of the numbers of observations of the second sample in the intervals from the corresponding expected values of the numbers.

Gart (1963) has investigated the theoretical statistical properties of the test devised in 1957 by Kimball, Burnett and Doherty for certain screening experiments. This test is the same as the first-mentioned test proposed by Mathisen (1943). In Gart's paper one can find the null distribution and the construction of an approximate chi-square test as a large sample version of this test. Considering the two samples $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_{2s+1}$ and $\underline{y}_1, \underline{y}_2, \ldots, \underline{y}_n$ as the first and second sample of populations with distribution functions $F(x)$ and $G(y)$, respectively, he shows that

$$\underline{\chi}^2 = \frac{(|2\underline{k} - n| - 1)^2 \, (2s + 3)}{n \, (2s + n + 2)}$$

is distributed as chi-square with one degree of freedom if $2s + 1$ and n become infinite. Further he derives the asymptotic non-null distribution. It can be shown that $\underline{k}$ and $\underline{x}_{(s+1)}$ have an asymptotic bivariate Normal distribution with asymptotic mean of $\underline{k}$:

$$E\,\underline{k} = n\,G(m_1)$$

and asymptotic variance of $\underline{k}$:

$$\operatorname{var} \underline{k} = n\left[G(m_1)\,\{1 - G(m_1)\} + \frac{n}{4\,(2s+1)}\right],$$

where m_1 is the median of the x-population.

In some cases this last test of Mathisen has an advantage compared to the Westenberg–Mood test. For instance, in comparing life test results, it is possible that the unknown joint median is very large, whereas the median of the x-sample is not so large, so that with the Westenberg–Mood test the experiment becomes inordinately long.

A widely discussed class of tests consists of test procedures based on the number of so-called exceedances. One determines, for instance, the number $\underline{b}$ of y-observations which are smaller than the rth order statistic $\underline{x}_{(r)}$. Under the null hypothesis the distribution of $\underline{b}$ equals

$$\Pr[\underline{b} = b] = \frac{m}{m+n}\,\frac{\binom{r-1+b}{r-1}\binom{m+n-r-b}{n-r}}{\binom{m+n-1}{m-1}},$$

where m is the size of the x-sample and n is the size of the y-sample. It is clear that the probability that $\underline{b}$ is smaller than or equal to the actual value, can serve as the probability level for a one-sided test. An indication of a test of this form can already be found in Thompson (1938). Exceedance tests have great versatility. It is easy to see that the various possible choices of r provide tests sensitive to differences of a great variety of population percentiles.

In the papers of Epstein (1954), Gumbel and Von Schelling (1950), Sarkadi (1957) and Harris (1952) various derivations can be found concerning null distributions, moments, asymptotic distributions and a relation with the Pólya-distribution.

Many tests described by various authors can be seen as special cases of this class. For instance, letting $m = 2r - 1$, so that $\underline{x}_{(r)}$ is the sample median of the x-observations, one arrives at the test described by Mathisen and later on by Gart.

Rosenbaum's test (1954) is based on the number of exceedances in the case of $r = 1$. This procedure maximizes economy in life testing because it requires a minimal number of observations if all the items are started at the same moment. A (one-sided) test based on the sum of the number of y's larger than $\underline{x}_{(m)}$ and the number of x's smaller than $\underline{y}_{(1)}$ was presented by Sidák-Vondrácek (1957) and, as already mentioned, by Tukey (1959).

Let $\underline{s}_x$ and $\underline{r}_x$ denote the number of x-observations larger than $\underline{y}_{(n)}$ and smaller than $\underline{y}_{(1)}$, respectively, and $\underline{s}_y$ and $\underline{r}_y$ denote the number of y-observations larger than $\underline{x}_{(m)}$ and smaller than $\underline{x}_{(1)}$, respectively. A test based on $(\underline{s}_x + \underline{r}_y) - (\underline{s}_y + \underline{r}_x)$ was proposed by Haga (1959/60). Hájek and Sidák propose in their book "Theory of rank tests" (1967) the test statistic

$\min(\underline{s}_x, \underline{r}_y) - \min(\underline{s}_y, \underline{r}_x)$. These last two test statistics may be used immediately against one-and two-sided alternatives. For other forms of statistics we refer to Hájek and Sidák (1967).

Epstein (1955) considered in his paper two Normal populations with equal variances. In order to test the null hypothesis of equal means, the relative merits of four non-parametric test procedures are studied experimentally on the basis of samples of equal size 10 to be drawn from each population. One of these tests is a special kind of an exceedance test for samples of equal size. Let $\underline{x}_{(r)}$ and $\underline{y}_{(r)}$ be respectively the rth smallest observation in each of the two samples and let $\underline{w}_r = \max(\underline{x}_{(r)}, \underline{y}_{(r)})$. If $w_r = x_{(r)}$ count the number of y's which are $\geqslant x_{(r)}$, if $w_r = y_{(r)}$ count the number of x's which are $\geqslant y_{(r)}$. The test statistic E_r is the number of exceedances. The study was limited to the cases $r = 1$, 2 and 3. The other tests are the rank-sum test of Wilcoxon, the run test and the maximum-deviation test (this is a truncated maximum-deviation test (cf. Tsao (1954)) with the truncation taking place at a time not later than $\underline{u}_r = \max(\underline{x}_{(r)}, \underline{y}_{(r)})$; r is decided upon in advance. In the following table the experimental results for 200 pairs of samples are reproduced (the results for different rows are based on the same samples).

TABLE 1-I

Observed probability of accepting $H_0\,(d = 0)$ based on 200 pairs of samples, each of size ten

$d = \left\lvert\frac{\mu_1 - \mu_2}{\sigma}\right\rvert$	rank sum	run	exceedance			maximum deviation		
			$r = 1$	$r = 2$	$r = 3$	$r = 3$	$r = 6$	$r = 10$
0	·935	·965	·95	·96	·96	·955	·945	·945
1	·485	·795	·655	·65	·60	·575	·555	·555
2	·015	·275	·16	·12	·10	·065	·045	·045
3	0	·02	·025	0	0	0	0	0

Nelson (1963) presented in his paper a life test procedure (also useful in other situations) to test whether two samples come from the same population which is based on the number $\underline{k}_1$ of observations in the sample yielding the smallest observation which precede the observation of rth rank in the other sample. This test is called a precedence test. It is mathematically equivalent to the exceedance test in which one counts the number $\underline{k}_2$ of observations in the sample yielding the first failure which exceed the observation of rth rank in the other sample. The tests are related by $\underline{k}_1 = n - \underline{k}_2$ for all r, where n is the size of the sample yielding the smallest observation. Tables with critical values of $\underline{k}_1$ for the precedence test with $r = 1$ are given for significance levels $\leqslant$ ·10, ·05, ·01 (two-sided) and $\leqslant$ ·05, ·025, ·005 (one-sided), for all combinations of sample sizes up to twenty.

Eilbott and Nadler (1965) investigate the life test procedure, based on the number of exceedances, under the assumption of underlying Exponential distributions, ($F(x) = 1 - \exp(-x/\theta_x)$ and $G(y) = 1 - \exp(-y/\theta_y)$), in order, as they formulate it, to provide further insight into its properties in situations where the underlying distributions are unknown. They give closed-form expressions for the power functions. They present the UMP (uniformly most powerful in the Neyman–Pearson sense) one-sided test of the hypothesis of equal mean lifetimes when, for instance, only the k smallest and the r smallest lifetimes are observed in the respective samples of a life time experiment and compare this test asymptotically with a precedence test when the underlying distributions are both Exponential. As a two-sided version of a precedence life test Eilbott and Nadler proposed to reject the null hypothesis if, and only if, k_1 items whose lifetimes follow $F(x)$ fail before r_1 items fail with lifetimes distributed according to $G(y)$ or k_2 values from $G(y)$ are observed before r_2 values from $F(x)$. If, in addition, the restriction: $\min(k_1, k_2) \geqslant \max(r_1, r_2)$ is imposed upon the test plan, then the power function is given by $\Pr[\underline{x}_{(k_1)} < \underline{y}_{(r_1)}] + \Pr[\underline{y}_{(k_2)} < x_{(r_2)}]$. For the special case $r_1 = r_2 = r$, $k_1 = k_2 = k$ and $m = n$, these restricted test plans are equivalent to the procedures investigated by Epstein (1955). On the other hand, whenever $r_1 = r_2 = 1$ does *not* hold, these restricted test plans differ from the general two-tailed tests proposed by Nelson (1963). His procedure is conditioned by the variety of the item giving rise to the first observed failure, whereas their procedure clearly is not.

Shorack (1967) showed that the expressions of the power function derived by Eilbott and Nadler are in fact valid for a large class of distributions which include the Exponential distribution, namely the class of distributions $F = \{(F,G) : G = 1 - (1 - F)^\delta,\ \delta > 0\}$. He showed that the power function in the case of Exponential distributions with difference in scale parameters is a function of $\lambda = \theta_y/\theta_x$ only.

2. SOME PROPERTIES OF ORDER STATISTICS

All random variables appearing throughout this monograph will be real-valued. In order to distinguish random variables from other variables their symbols will be underlined, e.g.

$$\underline{x}, \underline{y}, \underline{x}_i, \underline{x}_{(i)}, \underline{y}_j, \ldots .$$

In this way one can write e.g. Pr $[\underline{x} \leqslant x]$, which will be the probability that the random variable $\underline{x}$ assumes a value smaller than or equal to the number x.

Let $\underline{x}$ and $\underline{y}$ be two independent random variables with unknown continuous cumulative distribution functions $F(x)$ and $G(y)$, respectively, and densities, if they exist, $f(x)$ and $g(y)$, respectively. Thus

$$F(x) = \Pr[\underline{x} \leqslant x] \quad \text{for all} \quad x \in R^1 \tag{2.1}$$

and

$$G(y) = \Pr[\underline{y} \leqslant y] \quad \text{for all} \quad y \in R^1. \tag{2.2}$$

The expected value of a random variable is denoted by the symbol E, thus e.g.

$$E\{\underline{x}\} = \int_{R^1} x \, \mathrm{d}F(x). \tag{2.3}$$

We shall say that the expectation of a random variable $\underline{z}$ with continuous distribution function $H(z)$ exists if $\int_{R^1} |z| \, \mathrm{d}H(z) < \infty$.

The variance of a random variable is denoted by the symbol σ^2, thus e.g.

$$\begin{aligned} \sigma^2\{\underline{x}\} &= E\{(\underline{x} - E\{\underline{x}\})^2\} \\ &= \int_{R^1} (x - E\{\underline{x}\})^2 \, \mathrm{d}F(x). \end{aligned} \tag{2.4}$$

Now, suppose two independent random samples of independent observations of $\underline{x}$ and $\underline{y}$ are given, namely

$$\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m \tag{2.5}$$

and

$$\underline{y}_1, \underline{y}_2, \ldots, \underline{y}_n, \tag{2.6}$$

respectively.

Arranging the observations in increasing order of magnitude one gets the following two samples of order statistics:

$$\underline{x}_{(1)} \leqslant \underline{x}_{(2)} \leqslant \ldots \leqslant \underline{x}_{(m)} \tag{2.7}$$

and

$$\underline{y}_{(1)} \leqslant \underline{y}_{(2)} \leqslant \ldots \leqslant \underline{y}_{(n)}. \tag{2.8}$$

Since the distribution functions $F(x)$ and $G(y)$ are continuous, one has

$$\begin{aligned} &\Pr [\underline{x}_{(i)} = \underline{y}_{(j)}] = 0; \quad i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n, \\ &\Pr [\underline{x}_{(i)} = \underline{x}_{(k)}; i \neq k] = 0; \quad i, k = 1, 2, \ldots, m, \\ &\Pr [\underline{y}_{(j)} = \underline{y}_{(l)}; \quad j \neq l] = 0; \quad j, l = 1, 2, \ldots, n. \end{aligned} \tag{2.9}$$

Hence it is allowed to assume that the two samples of order statistics form the sequences $\{\underline{x}_{(i)}\}$ and $\{\underline{y}_{(j)}\}$ such that

$$\underline{x}_{(1)} < \underline{x}_{(2)} < \ldots < \underline{x}_{(m)} \tag{2.10}$$

and

$$\underline{y}_{(1)} < \underline{y}_{(2)} < \ldots < \underline{y}_{(n)} \tag{2.11}$$

with probability one. The variable $\underline{x}_{(i)}$ is called the ith order statistic of the x-sample and the variable $\underline{y}_{(j)}$ is called the jth order statistic of the y-sample ($i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$).

Use of order statistics for estimators and test statistics implies use of the order or rank of an observation as well as of its magnitude. It can be seen as a combination of the techniques used in classical statistics and those in non-parametric statistics, which consider only the relative rank of the observations.

For many statistical problems the use of order statistics resulted in highly efficient tests and estimators as well as in short-cut tests with smaller efficiency. The short-cut tests do not always have high efficiency but may be useful and preferable when simply and rapidly computable statistics are desirable. In the case of observations being relatively inexpensive, application of tests with smaller efficiency may be useful. Another useful application can be made in the field of life testing experiments where the observations arrive in the order of their magnitude and where, moreover, one has sometimes to analyze the observations before all the observations become available. When observations are censored use of order statistic techniques may, in general, be very useful.

The sampling theory of order statistics is fundamental for obtaining the distribution-free confidence intervals considered in this monograph. Basic results in the sampling theory of order statistics are given in Wilks (1962). More detailed results can be found in Fraser (1957), Gumbel (1954 and 1958), Kendall and Stuart (1963) and Sarhan and Greenberg (1962). An extensive bibliography of publications in this field has been published by Savage (1953). In this chapter only some basic properties are quoted.

It is easy to see that the probability elements of $(\underline{x}_{(1)}, \underline{x}_{(2)}, \ldots, \underline{x}_{(m)})$ and $(\underline{y}_{(1)}, \underline{y}_{(2)}, \ldots, \underline{y}_{(n)})$ are

$$\begin{aligned} &m!\, \mathrm{d}F(x_{(1)})\, \mathrm{d}F(x_{(2)}) \ldots \mathrm{d}F(x_{(m)}) \\ &\qquad \text{for} \quad -\infty < x_{(1)} < x_{(2)} < \ldots < x_{(m)} < \infty \\ &0 \qquad \text{elsewhere} \end{aligned} \tag{2.12}$$

and

$$n!\, \mathrm{d}G(y_{(1)})\, \mathrm{d}G(y_{(2)}) \dots \mathrm{d}G(y_{(n)})$$
$$\text{for} \quad -\infty < y_{(1)} < y_{(2)} < \dots < y_{(n)} < \infty \qquad (2.13)$$
$$0 \qquad \text{elsewhere,}$$

respectively.

The joint distribution function of $(\underline{x}_{(1)}, \underline{x}_{(2)}, \dots, \underline{x}_{(m)})$ equals

$$m!\, F(x_{(1)})\, F(x_{(2)}) \dots F(x_{(m)}) \quad \text{for} \quad -\infty < x_{(1)} < x_{(2)} < \dots < x_{(m)} < \infty$$
$$0 \qquad \text{otherwise} \qquad (2.14)$$

and similarly for $(\underline{y}_{(1)}, \underline{y}_{(2)}, \dots, \underline{y}_{(n)})$.

If one selects k integers $m_1, m_2, \dots, m_k$ $(1 \leqslant k \leqslant m)$ such that

$$1 \leqslant m_1 < m_2 < \dots < m_k \leqslant m$$

the probability element of the joint distribution function of the k order statistics

$$\underline{x}_{(m_1)}, \underline{x}_{(m_2)}, \dots, \underline{x}_{(m_k)}$$

is given by

$$\frac{m!}{\prod\limits_{i=1}^{k+1} (m_i - m_{i-1} - 1)!} \prod_{i=1}^{k+1} \{F(x_{(m_i)}) - F(x_{(m_{i-1})})\}^{m_i - m_{i-1} - 1} \times$$
$$\times\, \mathrm{d}F(x_{(m_1)})\, \mathrm{d}F(x_{(m_2)}) \dots \mathrm{d}F(x_{(m_k)})$$
$$\text{for } -\infty < x_{(m_1)} < x_{(m_2)} < \dots < x_{(m_k)} < \infty$$
$$0 \qquad \text{otherwise,} \qquad (2.15)$$

where

$$\begin{aligned} x_{(m_0)} &= -\infty, \quad x_{(m_{k+1})} = \infty, \\ m_0 &= 0 \quad \text{and} \quad m_{k+1} = m + 1. \end{aligned} \qquad (2.16)$$

In particular, for the distribution of the ith order statistic, we obtain

$$\frac{m!}{(i-1)!\,(m-i)!} F^{i-1}(x_{(i)})\, \{1 - F(x_{(i)})\}^{m-i}\, \mathrm{d}F(x_{(i)}). \qquad (2.17)$$

The expectation of the ith order statistic is

$$E\{\underline{x}_{(i)}\} = \frac{m!}{(i-1)!\,(m-i)!} \int_{-\infty}^{\infty} x_{(i)}\, F^{i-1}(x_{(i)})\, \{1 - F(x_{(i)})\}^{m-i}\, \mathrm{d}F(x_{(i)}) \quad (2.18)$$

and the variance is

$$\sigma^2\{\underline{x}_{(i)}\} = \frac{m!}{(i-1)!\,(m-i)!}\left[\int_{-\infty}^{\infty} x_{(i)}{}^2\, F^{i-1}(x_{(i)})\,\{1-F(x_{(i)})\}^{m-i}\,\mathrm{d}F(x_{(i)}) + \right. \tag{2.19}$$

$$\left. - \frac{m!}{(i-1)!\,(m-i)!}\left\{\int_{-\infty}^{\infty} x_{(i)}\, F^{i-1}(x_{(i)})\,\{1-F(x_{(i)})\}^{m-i}\,\mathrm{d}F(x_{(i)})\right\}^2\right].$$

It is known (Sarhan and Greenberg (1962)) that for m tending to infinity the distribution of $\underline{x}_{(m_1)}$, where

$$m_1 = [\delta\, m] + 1$$

for fixed δ between 0 and 1, tends to a Normal distribution with mean ξ, defined by

$$\int_{-\infty}^{\xi} \mathrm{d}F(t) = \delta \tag{2.20}$$

and variance

$$\frac{\delta\,(1-\delta)}{m f^2(\xi)}, \tag{2.21}$$

where the following assumptions have been made (cf. Mosteller (1946)): $F(x)$ has the density $f(x)$ which is continuous in the neighbourhood of $x = \xi$ and $f(\xi) > 0$.

3. THE CLASS C OF CONFIDENCE BOUNDS FOR SHIFT AND THE CORRESPONDING V TESTS

3.1. General case

Suppose $\underline{x}$ and $\underline{y}$ are two independent random variables with unknown continuous cumulative distribution functions $F(x)$ and $G(y)$, respectively. The median ν_1 of the distribution function $F(x)$ is defined as follows:

$$\nu_1 = \frac{\nu_1^- + \nu_1^+}{2}, \tag{3.1.1}$$

where

$$\nu_1^- = \min \{\nu_1 : F(\nu_1) = \tfrac{1}{2}\}$$

and

$$\nu_1^+ = \max \{\nu_1 : F(\nu_1) = \tfrac{1}{2}\}.$$

The median ν_2 of the distribution function $G(y)$ is defined in a similar manner.

We assume that the two distribution functions $F(x)$ and $G(y)$ have the same shape, so

$$F(x + \nu_1) \equiv G(x + \nu_2) \quad \text{for all} \quad x \in R^1. \tag{3.1.2}$$

We suppose from now on that the median of $F(x)$ is zero and the median of $G(y)$ is ν. Since we are interested in the difference of location there is no loss of generality. So we have

$$F(x) \equiv G(x + \nu) \quad \text{for all} \quad x \in R^1, \tag{3.1.3}$$

where ν is the unknown shift between $F(x)$ and $G(y)$ (cf. fig. 3.1.1).

The problem considered in this monograph is that of obtaining a distribution-free lower and upper confidence bound for the real-valued parameter ν.

The statistic

$$\underline{l} = l(\underline{x}_{(1)}, \underline{x}_{(2)}, \ldots, \underline{x}_{(m)}, \underline{y}_{(1)}, \underline{y}_{(2)}, \ldots, \underline{y}_{(n)}) \tag{3.1.4}$$

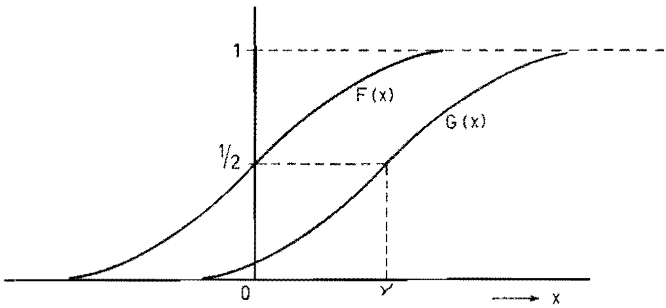

Fig. 3.1.1. The distribution functions $F(x)$ and $G(x)$.

is a lower confidence bound with confidence level $1 - \alpha_l$ $(0 < \alpha_l < 1)$ for the median difference ν if

$$\Pr [\underline{l} < \nu] \geqslant 1 - \alpha_l. \tag{3.1.5}$$

The confidence coefficient of this lower confidence bound is the probability $1 - \alpha_l^*$ defined by

$$\Pr [\underline{l} < \nu] = 1 - \alpha_l^*. \tag{3.1.6}$$

The statistic

$$\underline{r} = r(\underline{x}_{(1)}, \underline{x}_{(2)}, \ldots, \underline{x}_{(m)}, \underline{y}_{(1)}, \underline{y}_{(2)}, \ldots, \underline{y}_{(n)}) \tag{3.1.7}$$

is an upper confidence bound with confidence level $1 - \alpha_r$ $(0 < \alpha_r < 1)$ for the median difference ν if

$$\Pr [\underline{r} > \nu] \geqslant 1 - \alpha_r. \tag{3.1.8}$$

The confidence coefficient is the probability $1 - \alpha_r^*$ defined by

$$\Pr [\underline{r} > \nu] = 1 - \alpha_r^*. \tag{3.1.9}$$

Suppose that $\underline{l}$ and $\underline{r}$ are lower and upper confidence bounds for the median difference ν with confidence levels $1 - \alpha_l$ and $1 - \alpha_r$, respectively, then a two-sided confidence interval with confidence level $1 - \alpha = 1 - \alpha_l - \alpha_r$ is directly obtainable from these lower and upper confidence bounds. One gets

$$\begin{aligned} \Pr [\underline{l} < \nu < \underline{r}] &= \Pr [\underline{l} < \nu] + \Pr [\underline{r} > \nu] - \Pr [\underline{l} < \nu \vee \underline{r} > \nu] \\ &\geqslant 1 - \alpha_l + 1 - \alpha_r - 1 = 1 - \alpha. \end{aligned} \tag{3.1.10}$$

The confidence coefficient is the probability $1 - \alpha^*$ defined by

$$\Pr [\underline{l} < \nu < \underline{r}] = 1 - \alpha^*, \tag{3.1.11}$$

where $\alpha^* \leqslant \alpha_l^* + \alpha_r^*$. If $\underline{l} < \underline{r}$ with probability one then $\alpha^* = \alpha_l^* + \alpha_r^*$.

In the general case, i.e. without the restriction $\Pr [\underline{l} < \underline{r}] = 1$, it is possible that the confidence bounds become l and r with $l > r$. Then it can be seen immediately that the outcomes of the bounds are wrong and that the confidence interval (l, r) cannot contain the true value of the median difference ν.

In obtaining confidence bounds for the median difference ν based on probability relations of the order statistics of the x- and y-sample, we shall restrict ourselves in this monograph to lower confidence bounds for ν based on probability statements of the following form:

$$\Pr [\beta\text{th largest of } \underline{y}_{(j_r)} - \underline{x}_{(i_r)} < \nu; \quad r = 1, 2, \ldots, R] \tag{3.1.12}$$

and to upper confidence bounds for ν based on probability statements of the form

$$\Pr [\gamma\text{th largest of } \underline{y}_{(j_r)} - \underline{x}_{(i_r)} > \nu; \quad r = 1, 2, \ldots, R], \tag{3.1.13}$$

where the integers β, γ, R, i_r and j_r have to satisfy the following inequality relations:

$$1 \leqslant R \leqslant m\,n,$$
$$1 \leqslant \beta,\ \gamma \leqslant R,$$
$$1 \leqslant i_1, i_2, \ldots, i_R \leqslant m,$$
$$1 \leqslant j_1, j_2, \ldots, j_R \leqslant n.$$

The values of R, β, γ, i_r and j_r $(r = 1, 2, \ldots, R)$ can be chosen arbitrarily, often for the lower and upper confidence bounds separately, but will be integers.

Now we give two examples of the form (3.1.12):

Example 3.1.1. If $\beta = 1$ and $R = 1$, then a lower confidence bound for ν is e.g.

$$\underline{y}_{(3)} - \underline{x}_{(6)} < \nu,$$

provided $m \geqslant 6$ and $n \geqslant 3$.

Example 3.1.2. If $\beta = 2$ and $R = 2$, then a lower confidence bound for ν is e.g.

$$\min\{(\underline{y}_{(1)} - \underline{x}_{(7)}), (\underline{y}_{(3)} - \underline{x}_{(8)})\} < \nu,$$

provided $m \geqslant 8$ and $n \geqslant 3$.

Until further notice we shall consider lower confidence bounds for ν only. First of all we shall try to give an expression of

$$\Pr\,[\beta\text{th largest of } \underline{y}_{(j_r)} - \underline{x}_{(i_r)} < \nu; \quad r = 1, 2, \ldots, R]$$

in terms of probabilities of events which can be calculated directly.

For that purpose we introduce the following notation. For each selection of i and j, the symbol

$$\{j, i\} \quad (1 \leqslant i \leqslant m, \quad 1 \leqslant j \leqslant n). \tag{3.1.14}$$

denotes an arbitrary but fixed selection of one or both of the inequality signs $<$ and $>$. The selection of both inequality signs, denoted by $\lessgtr$, has the interpretation that no inequality relationship is specified between the two factors, so that for instance

$$\underline{y}_{(j)} - \underline{x}_{(i)} \lessgtr \nu$$

is identical to

$$-\infty < \underline{y}_{(j)} - \underline{x}_{(i)} < \infty.$$

This is a useful notation because it is possible to express each event of the form of

$$\{\beta\text{th largest of } \underline{y}_{(j_r)} - \underline{x}_{(i_r)} < \nu; \quad r = 1, 2, \ldots, R\} \tag{3.1.15}$$

by events of the form of

$$\{\underline{y}_{(j)} - \nu\{j, i\}\, \underline{x}_{(i)};\quad i = 1, 2, \ldots, m,\quad j = 1, 2, \ldots, n\} \qquad (3.1.16)$$

if the symbols $\{j, i\}$ are defined correctly.

From the fact that the $\underline{x}_{(i)}$ and $\underline{y}_{(j)}$ are ordered variables it follows that the symbols $\{j, i\}$ can never be chosen arbitrarily but have to satisfy certain consistency requirements.

Now we give two examples.

Example 3.1.3. Consider two independent samples of order statistics

$$\underline{x}_{(1)}, \underline{x}_{(2)}, \underline{x}_{(3)} \quad \text{and} \quad \underline{y}_{(1)}, \underline{y}_{(2)}, \underline{y}_{(3)}.$$

Then the form

$$\Pr\,[\text{largest of } \{\underline{y}_{(2)} - \underline{x}_{(1)},\ \ \underline{y}_{(3)} - \underline{x}_{(2)}\} < \nu]$$

can be written as

$$\Pr\,[\text{largest of } \{\underline{y}_{(2)} - \nu - \underline{x}_{(1)}, \underline{y}_{(3)} - \nu - \underline{x}_{(2)}\} < 0]$$
$$= \Pr\,[\text{largest of } \{\underline{y}'_{(2)} - \underline{x}_{(1)}, \underline{y}'_{(3)} - \underline{x}_{(2)}\} < 0],$$

with

$$\underline{y}'_{(j)} \overset{\text{def}}{=} \underline{y}_{(j)} - \nu \quad (j = 1, 2, 3).$$

This is equal to

$$\Pr\,[\underline{y}'_{(2)} < \underline{x}_{(1)} \wedge \underline{y}'_{(3)} < \underline{x}_{(2)}]$$
$$= \Pr\,[\bigcap_{\substack{i=1,2,3\\ j=1,2,3}} \{\underline{y}'_{(j)}\ \{j, i\}\ \underline{x}_{(i)}\}]$$

with $\quad \{j, i\} \equiv \lessgtr \quad$ for $\quad i = 1 \quad$ and $\quad j = 3,$

$\qquad\ \ \{j, i\} \equiv < \quad$ otherwise,

which can easily be derived from the consistency requirements.

Example 3.1.4. Consider two independent samples of order statistics

$$\underline{x}_{(1)}, \underline{x}_{(2)}, \underline{x}_{(3)}, \underline{x}_{(4)} \quad \text{and} \quad \underline{y}_{(1)}, \underline{y}_{(2)}, \underline{y}_{(3)}.$$

Then

$$\Pr\,[\text{2nd largest of } \{\underline{y}_{(1)} - \underline{x}_{(1)}, \underline{y}_{(2)} - \underline{x}_{(2)}, \underline{y}_{(3)} - \underline{x}_{(3)}\} < \nu]$$
$$= \Pr\,[\text{2nd largest of } \{\underline{y}'_{(1)} - \underline{x}_{(1)}, \underline{y}'_{(2)} - \underline{x}_{(2)}, \underline{y}'_{(3)} - \underline{x}_{(3)} < 0]$$
$$\text{with } \underline{y}'_{(j)} = \underline{y}_{(j)} - \nu \quad (j = 1, 2, 3)$$
$$= \Pr\,[\underline{y}'_{(1)} < \underline{x}_{(1)} \wedge \underline{y}'_{(2)} < \underline{x}_{(2)} \wedge \underline{y}'_{(3)} < \underline{x}_{(3)}] +$$
$$+ \Pr\,[\underline{y}'_{(1)} < \underline{x}_{(1)} \wedge \underline{y}'_{(2)} < \underline{x}_{(2)} \wedge \underline{y}'_{(3)} > \underline{x}_{(3)}] +$$
$$+ \Pr\,[\underline{y}'_{(1)} < \underline{x}_{(1)} \wedge \underline{y}'_{(2)} > \underline{x}_{(2)} \wedge \underline{y}'_{(3)} < \underline{x}_{(3)}] +$$
$$+ \Pr\,[\underline{y}'_{(1)} > \underline{x}_{(1)} \wedge \underline{y}'_{(2)} < \underline{x}_{(2)} \wedge \underline{y}'_{(3)} < \underline{x}_{(3)}].$$

Each of these four given events can be written as

$$\bigcap_{\substack{i=1,2,3,4\\ j=1,2,3}} \{\underline{y}'_{(j)}\ \{j, i\}\ \underline{x}_{(i)}\},$$

where the symbols $\{j, i\}$ are in these cases defined as

$$\begin{cases} \lesseqgtr & \text{for} \quad i=1,\ j=2,3 \quad \text{and} \quad i=2,\ j=3 \\ < & \text{otherwise,} \end{cases}$$

$$\begin{cases} \lesseqgtr & \text{for} \quad i=1,\ j=2 \quad \text{and} \quad i=4,\ j=3 \\ > & \text{for} \quad i=1,2,3,\ j=3 \\ < & \text{otherwise,} \end{cases}$$

$$\begin{cases} > & \text{for} \quad i=1,2,\ j=2,3 \\ < & \text{otherwise,} \end{cases}$$

and

$$\begin{cases} \lesseqgtr & \text{for} \quad i=2,\ j=3 \\ > & \text{for} \quad i=1,\ j=1,2,3 \\ < & \text{otherwise,} \end{cases}$$

respectively.

In general we have the following theorem.

Theorem 3.1.1. Let

$$\underline{x}_{(1)} < \underline{x}_{(2)} < \ldots < \underline{x}_{(m)} \quad \text{and} \quad \underline{y}_{(1)} < \underline{y}_{(2)} < \ldots < \underline{y}_{(n)}$$

be independent samples of order statistics from populations with continuous distribution functions $F(x)$ and $G(y) \equiv F(y - \nu)$, respectively. Let $\underline{y}'_{(j)} = \underline{y}_{(j)} - \nu$ $(j = 1, 2, \ldots, n)$. Then

$$\Pr\,[\beta\text{th largest of } \underline{y}_{(j_r)} - \underline{x}_{(i_r)} < \nu; \quad r = 1, 2, \ldots, R] =$$

$$= \sum_{b=0}^{\beta-1} \sum_{a=1}^{\binom{R}{b}} \Pr\,[E_a^b],$$

where E_a^b is the ath of the $\binom{R}{b}$ events that exactly b out of

$$\{(\underline{y}'_{(j_1)} - \underline{x}_{(i_1)}),\ (\underline{y}'_{(j_2)} - \underline{x}_{(i_2)}),\ \ldots,\ (\underline{y}'_{(j_R)} - \underline{x}_{(i_R)})\}$$

are larger than zero and the other differences are smaller than zero.

The proof of this theorem is straightforward and will be omitted.

It is easy to see that each probability

$$\Pr\,[E_a^b]$$

can be written as

$$\Pr\,[\bigcap_{\substack{i=1,\ldots,m\\ j=1,\ldots,n}} \{\underline{y}'_{(j)}\ \{j, i\}\ \underline{x}_{(i)}\}].$$

The next step consists in showing that each probability Pr $[E_a^b]$ is independent of the distribution function $F(x)$ and in providing a general method of determining this value.

Theorem 3.1.2. The probability Pr $[E_a^b] = \Pr[\bigcap_{\substack{i=1,\dots,m\\ j=1,\dots,n}} \{\underline{y}'_{(j)}\ \{j, i\}\ \underline{x}_{(i)}\}]$ is independent of the distribution functions $F(x)$ and $G(y)$ of $\underline{x}$ and $\underline{y}$, respectively ($G(y) \equiv F(y-\nu)$).

Proof:

$$\Pr[E_a^b] = \Pr[\bigcap_{\substack{i=1,\dots,m\\ j=1,\dots,n}} \{\underline{y}'_{(j)}\ \{j, i\}\ \underline{x}_{(i)}\}]$$

$$= \Pr[\bigcap_{\substack{i=1,\dots,m\\ j=1,\dots,n}} \{\underline{w}_{(j)}\ \{j, i\}\ \underline{v}_{(i)}\}], \tag{3.1.17}$$

where

$$\underline{v}_{(i)} \overset{\text{def}}{=} F(\underline{x}_{(i)}) \quad (i = 1, 2, \dots, m), \tag{3.1.18}$$

$$\underline{w}_{(j)} \overset{\text{def}}{=} F(\underline{y}'_{(j)}) \quad (j = 1, 2, \dots, n). \tag{3.1.19}$$

This relation is true, also when the function $F(x)$ is not (strictly) increasing.

From the definition of $\underline{v}_{(i)}$ and $\underline{w}_{(j)}$ it follows that

$$\underline{v}_{(1)} < \underline{v}_{(2)} < \dots < \underline{v}_{(m)} \tag{3.1.20}$$

are the order statistics of a sample of m independent observations from the standard Uniform distribution on the interval (0, 1) and that

$$\underline{w}_{(1)} < \underline{w}_{(2)} < \dots < \underline{w}_{(n)} \tag{3.1.21}$$

are the order statistics of an independent sample of n independent observations from the same standard Uniform distribution on (0,1). Therefore Pr $[E_a^b]$ and thus Pr [βth largest of $\underline{y}_{(j_r)} - \underline{x}_{(i_r)} < \nu$; $r = 1, 2, \dots, R$] are independent of the distribution function $F(x)$.

Theorem 3.1.2 provides a method of determining Pr $[E_a^b]$, since (see for the notation (3.1.18) and (3.1.19))

$$\Pr[\bigcap_{\substack{i=1,\dots,m\\ j=1,\dots,n}} \{\underline{w}_{(j)}\ \{j, i\}\ \underline{v}_{(i)}\}]$$

can be calculated directly.

First we shall sketch a computing-algorithm for the determination of Pr $[E_a^b]$. This algorithm supplies a rather simple method of computing this probability. Before giving the computing-algorithm we remark for clearness that the relations $\underline{w}_{(j)}\ \{j, i\}\ \underline{v}_{(i)}$ can be presented schematically in a diagram (fig. 3.1.2) with border points $j = 0, 1, 2, \dots, n$ along the horizontal axis and

$i = 0, 1, 2, \ldots, m$ along the vertical axis. Each square in the diagram will be indexed by the coordinates of its upper right corner point (j, i). A zero in the square (j, i) is equivalent to $\{j, i\} = \lessgtr$. A plus one (+1) or minus one (—1) in the square (j, i) is equivalent to $\{j, i\} = <$ and $\{j, i\} = >$, respectively.

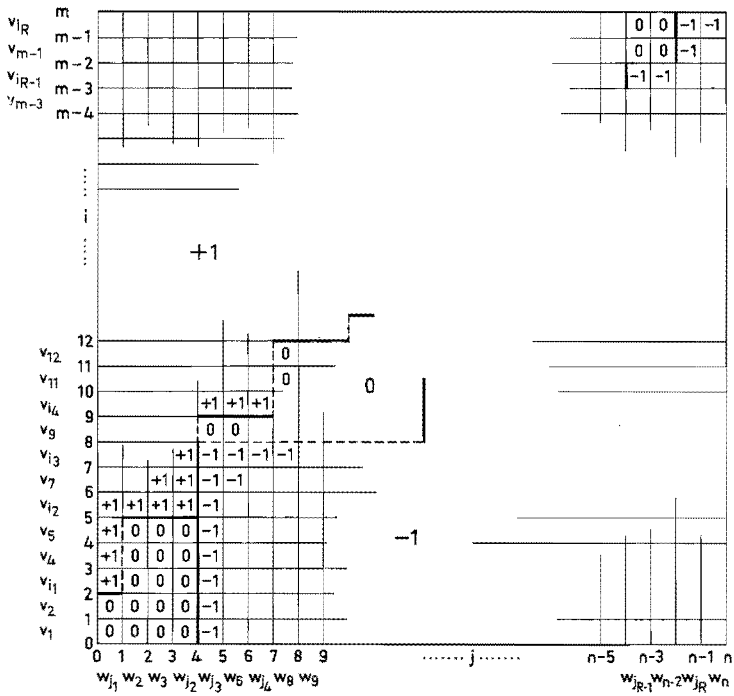

Fig. 3.1.2. Scheme of the inequalities between the order statistics of two samples (the order statistics $v_{(i)}$ and $w_{(j)}$ are denoted bij v_i and w_j, respectively).

In fig. 3.1.2 the three regions with values —1, 0 and +1, are indicated by way of an example according to the relations:

$$
\begin{array}{lll}
\{j_1, i_1\} = < & j_1 = 1 & i_1 = 3, \\
\{j_2, i_2\} = < & j_2 = 4 & i_2 = 6, \\
\{j_3, i_3\} = > & j_3 = 5 & i_3 = 8, \\
\{j_4, i_4\} = < & j_4 = 7 & i_4 = 10, \\
\cdot & & \\
\cdot & & \\
\cdot & & \\
\{j_{R-1}, i_{R-1}\} = > & j_{R-1} = n-3 & i_{R-1} = m-2, \\
\{j_R \;\;, i_R \;\;\} = > & j_R \;\; = n-1 & i_R \;\; = m.
\end{array}
$$

The three regions are separated from each other by the graphs of two non-decreasing step functions defined by $\{j, i\}$ and vice versa. There is a one to one correspondence between step functions and the configurations of the combined sample.

Now we confine our attention to the event E_a^b. It is easily seen that

$$\Pr\,[E_a^b] = \Pr\,[\bigcap_{r=1,\ldots,R} \{\underline{w}_{(j_r)}\,\{j_r, i_r\}\,\underline{v}_{(i_r)}\}] = \frac{r_{m,n}}{\binom{m+n}{m}}, \tag{3.1.22}$$

where $r_{m,n}$ is defined recursively as follows:

$$r_{00} = 1$$

$$r_{i0} = 1 \quad i = 1, 2, \ldots, i_{r_1} - 1 \quad \text{where} \quad i_{r_1} = \min_r \{i_r;\ \{j_r, i_r\} = <\}$$

$$r_{0j} = 1 \quad j = 1, 2, \ldots, j_{r_2} - 1 \quad \text{where} \quad j_{r_2} = \min_r \{j_r;\ \{j_r, i_r\} = >\}$$

while for $i, j \geqslant 1$:

$$\begin{aligned} r_{ij} &= r_{i-1,j} + r_{i,j-1} \\ &= r_{i,j-1} \quad \text{if} \quad w_{(j)} > v_{(i)} \\ &= r_{i-1,j} \quad \text{if} \quad w_{(j)} < v_{(i)}. \end{aligned}$$

In fact, only the r_{ij} within the zero domain (boundaries included) must be inserted in the computations. The proof of the correctness of this algorithm is simple, because $r_{m,n}$ is the number of equally probable arrangements of the two samples of order statistics satisfying the given inequalities, while the total number of possible, equally probable arrangements is equal to $\binom{m+n}{m}$.

Further there exists a direct method of determining $\Pr\,[E_a^b]$. It is possible to express $\Pr\,[E_a^b]$ in terms of

$$\Pr\,[\bigcap_{s=1,\ldots,S} \{\underline{w}_{(j_s)}\,\{j_s, i_s\}\,\underline{v}_{(i_s)}\}], \tag{3.1.23}$$

where $\{j_s, i_s\} = <$ only, i.e. for $s = 1, 2, \ldots, S$. This is easily seen by repeated application of the following rule for the events A and B:

$$\Pr\,[A \wedge \bar{B}] = \Pr\,[A] - \Pr\,[A \wedge B].$$

In a similar way as has been sketched for the algorithm, the conditions

$$\underline{w}_{(j_s)} < \underline{v}_{(i_s)} \quad (s = 1, 2, \ldots, S)$$

can be given schematically in a diagram like that in fig. 3.1.2. Then we have

$$\Pr\,[\underline{w}_{(j_s)} < \underline{v}_{(i_s)};\quad s = 1, 2, \ldots, S] = \frac{A_n}{\binom{m+n}{m}}, \tag{3.1.24}$$

where A_n is the total number of possible ways in which the zeros in such a diagram can be altered in plus ones in a consistent way, that is to say the numbers in the rows and columns must form a monotonic non-increasing and a monotonic non-decreasing row, respectively. To each of these completions there corresponds in a one-to-one way a monotonic non-decreasing step function within the domain of zeros (boundaries included). This step function can only take on integer values and jumps can only take place in the points $j = 0, 1, 2, \ldots, n$. So A_n is the total number of possible step functions which are below or on the boundary determined by the given conditions.

Now we introduce the following variables f_i $(i = 0, 1, 2, \ldots, n)$:

$$
\begin{aligned}
f_0 &= i_1 \\
f_1 &= i_1 \\
&\vdots \\
f_{j1} &= i_2 \\
&\vdots \\
f_{j2} &= i_3 \\
&\vdots \\
f_{jS} &= m + 1 \\
&\vdots \\
f_n &= m + 1.
\end{aligned}
$$

Defining A_k as the total number of possible step functions to $(k, f_k - 1)$ which are below or on the boundary determined by the given conditions, then we have (see Göbel (1963)):

$$A_n = \sum_{c(n)} (-1)^{n-k(c)} \binom{f_0}{c_1} \binom{f_{c_1}}{c_2} \binom{f_{c_1+c_2}}{c_3} \ldots, \tag{3.1.25}$$

where

$c(n)$ are the compositions of n (a composition of n is an ordered partition of n in positive integers),

$k(c)$ is the number of terms of the composition,

$c_1, c_2, \ldots$ are the terms of the composition in this order.

The total number of compositions of n is 2^{n-1} (Example: The 8 compositions of 4 are: 1111, 112, 121, 211, 22, 13, 31, 4).

In the next section we shall consider three subclasses C_1, C_2 and C_3 of the class C of lower confidence bounds for the median difference ν based on

$$\Pr\left[\beta\text{th largest of } \underline{y}_{(j_r)} - \underline{x}_{(i_r)} < \nu;\ r = 1, 2, \ldots, R\right];$$

the three cases that we shall consider are (i) $C_1 : R = 1$, (ii) $C_2 : R = 2$ and (iii) $C_3 : R = 3$.

3.2. Subclass $C_1 : R = 1$

The case $R = 1$ is the simplest one in obtaining confidence intervals for the median difference ν. The lower confidence bound for ν is then given by

$$\underline{y}_{(j)} - \underline{x}_{(i)} < \nu. \tag{3.2.1}$$

This form, the only possible one for the case $R = 1$, is mentioned and its confidence coefficient is given in Mood and Graybill (1963). The subclass C_1 contains $m\,n$ lower confidence bounds. The determination of the confidence coefficient of this lower confidence bound runs as follows:

$$\begin{aligned} \Pr\,[\underline{y}_{(j)} - \underline{x}_{(i)} < \nu] &= \Pr\,[\underline{y}_{(j)} - \nu < \underline{x}_{(i)}] \\ &= \Pr\,[\underline{w}_{(j)} < \underline{v}_{(i)}]. \end{aligned}$$

The event $\underline{w}_{(j)} < \underline{v}_{(i)}$ can be seen as the sum of events $(r = j, j+1, \ldots, n)$ for which among the first $i - 1 + r$ order statistics of the combined sample there must be exactly r order statistics of the w-sample and the $(i + r)$th order statistic is an order statistic of the v-sample. So we have

$$\Pr\,[\underline{y}_{(j)} - \underline{x}_{(i)} < \nu] = \frac{1}{\binom{m+n}{m}} \sum_{r=j}^{n} \binom{i-1+r}{r} \binom{m+n-i-r}{n-r}. \tag{3.2.2}$$

As a matter of fact this probability can also be determined in a more analytical way. Defining $\underline{r}$ as the number of w-order statistics smaller than $\underline{v}_{(i)}$, one gets:

$$\Pr\,[\underline{r} = r \,|\, \underline{v}_{(i)} = v_{(i)}] = \binom{n}{r} v_{(i)}{}^{r} (1 - v_{(i)})^{n-r}.$$

The probability density function of $\underline{v}_{(i)}$ equals

$$m\binom{m-1}{i-1} v_{(i)}{}^{i-1} (1 - v_{(i)})^{m-i}.$$

This results in the following unconditional probability:

$$\Pr\,[\underline{r} = r] = m\binom{n}{r}\binom{m-1}{i-1} \int_0^1 v_{(i)}{}^{r+i-1} (1 - v_{(i)})^{m+n-r-i}\, dv_{(i)}.$$

Using the properties of Beta-functions we find:

$$\Pr\,[\underline{r} = r] = m\binom{n}{r}\binom{m-1}{i-1} \frac{(r+i-1)!\,(m+n-i-r)!}{(m+n)!} = \frac{\binom{r+i-1}{r}\binom{m+n-i-r}{m-i}}{\binom{m+n}{m}}.$$

From this it follows that

$$\Pr[\underline{w}_{(j)} < \underline{v}_{(i)}] = \Pr[\underline{r} \geq j]$$

$$= \frac{1}{\binom{m+n}{m}} \sum_{r=j}^{n} \binom{r+i-1}{r} \binom{m+n-i-r}{m-i}$$

as before. For computational purposes one can also use the formulae

$$\Pr[\underline{y}_{(j)} - \underline{x}_{(i)} < v] = 1 - \frac{1}{\binom{m+n}{m}} \sum_{r=0}^{j-1} \binom{r+i-1}{r} \binom{m+n-i-r}{m-i} \qquad (3.2.3)$$

(useful when j is small)

$$= \frac{1}{\binom{m+n}{m}} \sum_{r=0}^{i-1} \binom{r+j-1}{r} \binom{m+n-j-r}{n-j} \qquad (3.2.4)$$

(useful when i is small)

$$= 1 - \frac{1}{\binom{m+n}{m}} \sum_{r=i}^{m} \binom{r+j-1}{r} \binom{m+n-j-r}{n-j} \qquad (3.2.5)$$

(useful when i is not small).

Some numerical calculations

For some values of m and n the confidence coefficients of the lower confidence bounds for v are given in tables 3.2-I and 3.2-II.

TABLE 3.2-I

Confidence coefficients of some lower confidence bounds $\underline{y}_{(j)} - \underline{x}_{(i)}$; $m = n = 6$

j \ i	1	2	3	4	5	6
1	·500	·773	·909	·970	·992	·999
2		·500	·727	·879	·960	·992
3			·500	·716	·879	·970

TABLE 3.2-II

Confidence coefficients of some lower confidence bounds $\underline{y}_{(j)}-\underline{x}_{(i)}$; $m=n=10$

j \ i	1	2	3	4	5	6	7	8	9
1	·500	·763	·895	·957	·984	·995	·998		
2		·500	·709	·848	·930	·971	·990	·997	
3			·500	·686	·825	·915	·965	·988	·997

3.3. Subclass $C_2 : R = 2$

In the case $R = 2$ there are two possible forms of lower confidence bounds for the median difference ν, namely those based on

$$\begin{aligned}&\Pr\,[\text{1st largest of } \{(\underline{y}_{(j)} - \underline{x}_{(i)}), (\underline{y}_{(l)} - \underline{x}_{(k)})\} < \nu]\\ &= \Pr\,[\max \{(\underline{y}_{(j)} - \underline{x}_{(i)}), (\underline{y}_{(l)} - \underline{x}_{(k)})\} < \nu] \end{aligned} \tag{3.3.1}$$

and those based on

$$\begin{aligned}&\Pr\,[\text{2nd largest of } \{(\underline{y}_{(j)} - \underline{x}_{(i)}), (\underline{y}_{(l)} - \underline{x}_{(k)})\} < \nu]\\ &= \Pr\,[\min \{(\underline{y}_{(j)} - \underline{x}_{(i)}), (\underline{y}_{(l)} - \underline{x}_{(k)})\} < \nu]. \end{aligned} \tag{3.3.2}$$

For lower confidence bounds for ν based on the first form, we have

$$\begin{aligned}&\Pr\,[\max \{(\underline{y}_{(j)} - \underline{x}_{(i)}), (\underline{y}_{(l)} - \underline{x}_{(k)})\} < \nu]\\ &= \Pr\,[(\underline{y}_{(j)} - \underline{x}_{(i)} < \nu) \wedge (\underline{y}_{(l)} - \underline{x}_{(k)} < \nu)]\\ &= \Pr\,[\underline{w}_{(j)} < \underline{v}_{(i)} \wedge \underline{w}_{(l)} < \underline{v}_{(k)}]. \end{aligned} \tag{3.3.3}$$

Without loss of generality we can assume that $j < l$ *). Now two possibilities can be distinguished, namely:

(i) $i > k$,

(ii) $i < k$.

In the first case we have

$$\Pr\,[\underline{w}_{(j)} < \underline{v}_{(i)} \wedge \underline{w}_{(l)} < \underline{v}_{(k)}] = \Pr\,[\underline{w}_{(l)} < \underline{v}_{(k)}].$$

Thus this lower confidence bound belongs already to C_1.

In order to determine $\Pr\,[\underline{w}_{(j)} < \underline{v}_{(i)} \wedge \underline{w}_{(l)} < \underline{v}_{(k)}]$ in the second case the following derivation can be given. The total number of possible, equally probable, rankings of the two samples of v- and w-order statistics is equal to $\binom{m+n}{m}$. A ranking of the two samples of order statistics is favourable when there are r ($j \leqslant r \leqslant n$) w-order statistics among the first $(i - 1 + r)$ order statistics, the $(i + r)$th order statistic is an element of the v-sample, there

*) For the case $j = l$ we refer to sec. 3.2.

are $(s-r)$ $(l \leqslant s \leqslant n)$ w-order statistics among the next $(k-1-i-r+s)$ order statistics, and the $(k+s)$th order statistic is an element of the v-sample. The number of favourable rankings is therefore

$$\sum_{r=j}^{n} \sum_{s=l}^{n} \binom{i-1+r}{r} \binom{k-1-i+s-r}{s-r} \binom{m+n-k-s}{n-s}.$$

Thus the following holds:

$$\Pr\left[\max\left\{(\underline{y}_{(j)} - \underline{x}_{(i)}), (\underline{y}_{(l)} - \underline{x}_{(k)})\right\} < v\right]$$

$$= \Pr\left[\underline{w}_{(j)} < \underline{v}_{(i)} \wedge \underline{w}_{(l)} < \underline{v}_{(k)}\right]$$

$$= \frac{1}{\binom{m+n}{m}} \sum_{r=j}^{n} \sum_{s=l}^{n} \binom{i-1+r}{r} \binom{k-1-i+s-r}{s-r} \binom{m+n-k-s}{n-s}. \tag{3.3.4}$$

For m and n large and j and l small one may prefer the following expression:

$$\Pr\left[\max\left\{(\underline{y}_{(j)} - \underline{x}_{(i)}), (\underline{y}_{(l)} - \underline{x}_{(k)})\right\} < v\right]$$

$$= 1 - \frac{1}{\binom{m+n}{m}} \sum_{r=0}^{n} \sum_{s=0}^{n} \binom{i-1+r}{r} \binom{k-1-i-r+s}{s-r} \binom{m+n-k-s}{n-s} +$$

$$+ \frac{1}{\binom{m+n}{m}} \sum_{r=j}^{n} \sum_{s=l}^{n} \binom{i-1+r}{r} \binom{k-1-i-r+s}{s-r} \binom{m+n-k-s}{n-s}$$

$$= 1 - \frac{1}{\binom{m+n}{m}} \sum_{r=j}^{n} \sum_{s=0}^{l-1} \binom{i-1+r}{r} \binom{k-1-i-r+s}{s-r} \binom{m+n-k-s}{n-s} +$$

$$- \frac{1}{\binom{m+n}{m}} \sum_{r=0}^{j-1} \sum_{s=0}^{n} \binom{i-1+r}{r} \binom{k-1-i-r+s}{s-r} \binom{m+n-k-s}{n-s}. \tag{3.3.5}$$

Formula (3.3.4) can also be derived analytically as well. Let $\underline{r}$ be the number of w-order statistics which are smaller than $\underline{v}_{(i)}$ and $\underline{s}$ the number of w-order statistics which are larger than $\underline{v}_{(i)}$ but smaller than $\underline{v}_{(k)}$. Then we have

$$\Pr\left[\underline{r} = r, \underline{s} = s \mid \underline{v}_{(i)} = v_{(i)}, \underline{v}_{(k)} = v_{(k)}\right]$$

$$= \frac{n!}{r!\, s!\, (n-r-s)!}\, v_{(i)}{}^{r}\, (v_{(k)} - v_{(i)})^{s}\, (1 - v_{(k)})^{n-r-s}.$$

The joint probability density function of $\underline{v}_{(i)}$ and $\underline{v}_{(k)}$ is

$$\frac{m!}{(i-1)!\,(k-i-1)!\,(m-k)!}\, v_{(i)}{}^{i-1}\,(v_{(k)}-v_{(i)})^{k-i-1}\,(1-v_{(k)})^{m-k},$$

so we get for the joint probability density of $\underline{r}$ and $\underline{s}$:

$\Pr[\underline{r}=r, \underline{s}=s] =$

$$\frac{m!\,n!}{r!\,s!\,(n-r-s)!\,(i-1)!\,(k-i-1)!\,(m-k)!}\int_0^1\int_0^{v_{(k)}} v_{(i)}{}^{r+i-1}\,(v_{(k)}-v_{(i)})^{s+k-i-1}\times$$
$$\times\,(1-v_{(k)})^{m+n-r-s-k}\,\mathrm{d}v_{(i)}\,\mathrm{d}v_{(k)}$$

$$=\frac{m!\,n!\,(r+i-1)!\,(s+k-i-1)!}{r!\,s!\,(n-r-s)!\,(i-1)!\,(k-i-1)!\,(m-k)!\,(r+s+k-1)!}\int_0^1 v_{(k)}{}^{r+s+k-1}\times$$
$$\times\,(1-v_{(k)})^{m+n-r-s-k}\,\mathrm{d}v_{(k)}$$

$$=\frac{m!\,n!\,(r+i-1)!\,(s+k-i-1)!\,(m+n-r-s-k)!}{r!\,s!\,(n-r-s)!\,(i-1)!\,(k-i-1)!\,(m-k)!\,(m+n)!}$$

$$=\frac{\binom{r+i-1}{r}\binom{s+k-i-1}{s}\binom{m+n-r-s-k}{m-k}}{\binom{m+n}{m}}.$$

From this it follows that

$$\Pr[\underline{w}_{(j)}<\underline{v}_{(i)}\wedge\underline{w}_{(l)}<\underline{v}_{(k)}]$$

$$=\sum_{r=j}^{n}\sum_{s=l-r}^{n-r}\Pr[\underline{r}=r,\underline{s}=s]$$

$$=\frac{1}{\binom{m+n}{m}}\sum_{r=j}^{n}\sum_{s=l-r}^{n-r}\binom{r+i-1}{r}\binom{s+k-i-1}{s}\binom{m+n-r-s-k}{m-k}$$

$$=\frac{1}{\binom{m+n}{m}}\sum_{r=j}^{n}\sum_{s=l}^{n}\binom{r+i-1}{r}\binom{s+k-r-i-1}{s-r}\binom{m+n-s-k}{m-k}$$

as before.

Notice that for $j=l$ we have $l-r\leqslant 0$ and thus

$$\sum_{s=l-r}^{n-r}\binom{s+k-i-1}{s}\binom{m+n-r-s-k}{m-k}=\sum_{s=0}^{n-r}\binom{s+k-i-1}{s}\binom{m+n-r-s-k}{n-r-s}=\binom{m+n-i-r}{n-r}$$

(cf. Feller, Ch. 2, sec. 12, problem 14). This equality inserted into (3.3.4) gives formula (3.2.2).

For lower confidence bounds for the median difference ν based on the second form (3.3.2), we have

$$\begin{aligned}
&\Pr\left[\min\{(\underline{y}_{(j)}-\underline{x}_{(i)}), (\underline{y}_{(l)}-\underline{x}_{(k)})\} < \nu\right] \\
&= \Pr\left[(\underline{y}_{(j)}-\underline{x}_{(i)} < \nu) \vee (\underline{y}_{(l)}-\underline{x}_{(k)} < \nu)\right] \\
&= \Pr\left[\underline{w}_{(j)} < \underline{v}_{(i)} \vee \underline{w}_{(l)} < \underline{v}_{(k)}\right] \\
&= \Pr\left[\underline{w}_{(j)} < \underline{v}_{(i)}\right] + \Pr\left[\underline{w}_{(l)} < \underline{v}_{(k)}\right] - \Pr\left[\underline{w}_{(j)} < \underline{v}_{(i)} \wedge \underline{w}_{(l)} < \underline{v}_{(k)}\right] \\
&= \frac{1}{\binom{m+n}{m}}\Bigg\{\sum_{r=j}^{n}\binom{i-1+r}{i-1}\binom{m+n-i-r}{m-i} + \sum_{s=l}^{n}\binom{k-1+s}{k-1}\binom{m+n-k-s}{m-k} + \\
&\qquad -\sum_{r=j}^{n}\sum_{s=l}^{n}\binom{r+i-1}{r}\binom{s+k-r-i-1}{s-r}\binom{m+n-s-k}{m-k}\Bigg\}.
\end{aligned} \tag{3.3.6}$$

Notice that (3.3.2) can also be written as follows:

$$\begin{aligned}
&\Pr\left[\min\{(\underline{y}_{(j)}-\underline{x}_{(i)}), (\underline{y}_{(l)}-\underline{x}_{(k)})\} < \nu\right] \\
&= \Pr\left[\min\{(\underline{w}_{(j)}-\underline{v}_{(i)}), (\underline{w}_{(l)}-\underline{v}_{(k)})\} < 0\right] \\
&= \Pr\left[\max\{(\underline{v}_{(i)}-\underline{w}_{(j)}), (\underline{v}_{(k)}-\underline{w}_{(l)})\} > 0\right] \\
&= 1-\Pr\left[\max\{(\underline{v}_{(i)}-\underline{w}_{(j)}), (\underline{v}_{(k)}-\underline{w}_{(l)})\} < 0\right].
\end{aligned}$$

TABLE 3.3-I

Confidence coefficients of some lower confidence bounds in C_2

$m = n = 5$

j	i	l	k	max	min
2	4	3	5	·857	·956
1	4	3	5	·905	·988
1	4	2	5	·960	·992

$m = n = 8$

j	i	l	k	max	min
5	7	6	8	·823	·936
4	7	5	8	·924	·978
3	7	4	8	·973	·994

$m = n = 10$

j	i	l	k	max	min
7	9	8	10	·813	·930
6	9	7	10	·913	·973
5	9	6	10	·964	·991

Thus this probability can also be derived from (3.3.5) by interchanging m and n, i and j, and k and l.

Some numerical calculations

In table 3.3-I some numerical results are given. The two forms of lower confidence bounds are denoted by "max" and "min".

3.4. Subclass $C_3 : R = 3$

In the case $R = 3$ there are three possible forms for lower confidence bounds for the median difference ν, namely based on:

(i) $\Pr\,[\text{1st largest of } \{(\underline{y}_{(h)} - \underline{x}_{(g)}), (\underline{y}_{(j)} - \underline{x}_{(i)}), (\underline{y}_{(l)} - \underline{x}_{(k)})\} < \nu]$
$$= \Pr\,[\max \{(\underline{y}_{(h)} - \underline{x}_{(g)}), (\underline{y}_{(j)} - \underline{x}_{(i)}), (\underline{y}_{(l)} - \underline{x}_{(k)})\} < \nu], \tag{3.4.1}$$

(ii) $$\Pr\,[\text{2nd largest of } \{(\underline{y}_{(h)} - \underline{x}_{(g)}), (\underline{y}_{(j)} - \underline{x}_{(i)}), (\underline{y}_{(l)} - \underline{x}_{(k)})\} < \nu], \tag{3.4.2}$$

(iii) $\Pr\,[\text{3rd largest of } \{(\underline{y}_{(h)} - \underline{x}_{(g)}), (\underline{y}_{(j)} - \underline{x}_{(i)}), (\underline{y}_{(l)} - \underline{x}_{(k)})\} < \nu]$
$$= \Pr\,[\min \{(\underline{y}_{(h)} - \underline{x}_{(g)}), (\underline{y}_{(j)} - \underline{x}_{(i)}), (\underline{y}_{(l)} - \underline{x}_{(k)})\} < \nu]. \tag{3.4.3}$$

First let us consider lower confidence bounds based on (3.4.1). Then

$$\begin{aligned}
&\Pr\,[\max \{(\underline{y}_{(h)} - \underline{x}_{(g)}), (\underline{y}_{(j)} - \underline{x}_{(i)}), (\underline{y}_{(l)} - \underline{x}_{(k)})\} < \nu] \\
&= \Pr\,[(\underline{y}_{(h)} - \underline{x}_{(g)} < \nu) \wedge (\underline{y}_{(j)} - \underline{x}_{(i)} < \nu) \wedge (\underline{y}_{(l)} - \underline{x}_{(k)} < \nu)] \\
&= \Pr\,[\underline{w}_{(h)} < \underline{v}_{(g)} \wedge \underline{w}_{(j)} < \underline{v}_{(i)} \wedge \underline{w}_{(l)} < \underline{v}_{(k)}].
\end{aligned}$$

Without loss of generality it can be supposed that

$$h \leqslant j \leqslant l.$$

It is easy to see that the only case where we have to make additional computations is the one for which

$$g < i < k.$$

We shall only give an analytical determination of confidence coefficient (3.4.1). With the definition

$\underline{r}$: is the total number of w-order statistics that are smaller than $\underline{v}_{(g)}$,

$\underline{s}$: is the total number of w-order statistics that are smaller than $\underline{v}_{(i)}$ but larger than $\underline{v}_{(g)}$,

$\underline{t}$: is the total number of w-order statistics that are smaller than $\underline{v}_{(k)}$ but larger than $\underline{v}_{(i)}$,

we get:

$$\Pr\,[\underline{r} = r, \underline{s} = s, \underline{t} = t \mid \underline{v}_{(g)} = v_{(g)}, \underline{v}_{(i)} = v_{(i)}, \underline{v}_{(k)} = v_{(k)}] =$$

$$\frac{n!}{r!\,s!\,t!\,(n-r-s-t)!}\, v_{(g)}{}^{r}\, (v_{(i)} - v_{(g)})^{s}\, (v_{(k)} - v_{(i)})^{t}\, (1 - v_{(k)})^{n-r-s-t}.$$

Further we have for the probability density of $\underline{v}_{(g)}$, $\underline{v}_{(i)}$ and $\underline{v}_{(k)}$:

$$\frac{m!}{(g-1)!(i-1-g)!(k-1-i)!(m-k)!} v_{(g)}^{g-1} (v_{(i)} - v_{(g)})^{i-1-g} (v_{(k)} - v_{(i)})^{k-1-i} \times \times (1 - v_{(k)})^{m-k},$$

so we get:

$\Pr [\underline{r} = r, \underline{s} = s, \underline{t} = t] =$

$$c^* \int_0^1 \int_0^{v_{(k)}} \int_0^{v_{(i)}} v_{(g)}^{r+g-1} (v_{(i)} - v_{(g)})^{s+i-g-1} (v_{(k)} - v_{(i)})^{t+k-i-1} \times \times (1 - v_{(k)})^{m+n-r-s-t-k} \, dv_{(g)} \, dv_{(i)} \, dv_{(k)}$$

(where $c^* =$
$= m!n! / \{(g-1)!\,(i-1-g)!\,(k-1-i)!\,(m-k)!\,r!\,s!\,t!\,(n-r-s-t)!\})$

$$= c^* \frac{(r+g-1)!\,(s+i-g-1)!}{(r+s+i-1)!} \times$$

$$\times \int_0^1 \int_0^{v_{(k)}} v_{(i)}^{r+s+i-1} (v_{(k)} - v_{(i)})^{t+k-i-1} (1 - v_{(k)})^{m+n-r-s-t-k} \, dv_{(i)} \, dv_{(k)}$$

$$= c^* \frac{(r+g-1)!\,(s+i-g-1)!\,(t+k-i-1)!}{(r+s+t+k-1)!} \times$$

$$\times \int_0^1 v_{(k)}^{r+s+t+k-1} (1 - v_{(k)})^{m+n-r-s-t-k} \, dv_{(k)}$$

$$= \frac{c^* (r+g-1)!\,(s+i-g-1)!\,(t+k-i-1)!\,(m+n-r-s-t-k)!}{(m+n)!}$$

$$= \binom{r+g-1}{r} \binom{s+i-g-1}{s} \binom{t+k-i-1}{t} \binom{m+n-r-s-t-k}{n-r-s-t} \Big/ \binom{m+n}{m}. \qquad (3.4.4)$$

Finally we get:

$\Pr [\underline{w}_{(h)} < \underline{v}_{(g)}, \underline{w}_{(j)} < \underline{v}_{(i)}, \underline{w}_{(l)} < \underline{v}_{(k)}] =$

$$= \frac{1}{\binom{m+n}{m}} \sum_{r=h}^{n} \sum_{s=j-r}^{n-r} \sum_{t=l-r-s}^{n-r-s} \binom{r+g-1}{r} \binom{s+i-g-1}{s} \binom{t+k-i-1}{t} \binom{m+n-r-s-t-k}{n-r-s-t} \qquad (3.4.5)$$

or

$$= \frac{1}{\binom{m+n}{m}} \sum_{r=h}^{n} \sum_{s=j}^{n} \sum_{t=l}^{n} \binom{r+g-1}{r} \binom{s+i-r-g-1}{s-r} \binom{t+k-r-s-i-1}{t-r-s} \binom{m+n-t-k}{n-t}.$$

Secondly we consider lower confidence bounds for the median difference ν based on form (3.4.2). Then

$$\Pr\,[\text{2nd largest of } \{\underline{y}_{(h)}-\underline{x}_{(g)}, \underline{y}_{(j)}-\underline{x}_{(i)}, \underline{y}_{(l)}-\underline{x}_{(k)}\} < \nu]$$
$$= \Pr\,[\underline{y}_{(h)}-\underline{x}_{(g)} < \nu, \underline{y}_{(j)}-\underline{x}_{(i)} < \nu, \underline{y}_{(l)}-\underline{x}_{(k)} < \nu] +$$
$$+ \Pr\,[\underline{y}_{(h)}-\underline{x}_{(g)} < \nu, \underline{y}_{(j)}-\underline{x}_{(i)} < \nu, \underline{y}_{(l)}-\underline{x}_{(k)} > \nu] +$$
$$+ \Pr\,[\underline{y}_{(h)}-\underline{x}_{(g)} < \nu, \underline{y}_{(j)}-\underline{x}_{(i)} > \nu, \underline{y}_{(l)}-\underline{x}_{(k)} < \nu] +$$
$$+ \Pr\,[\underline{y}_{(h)}-\underline{x}_{(g)} > \nu, \underline{y}_{(j)}-\underline{x}_{(i)} < \nu, \underline{y}_{(l)}-\underline{x}_{(k)} < \nu]$$
$$= \Pr\,[\underline{w}_{(h)} < \underline{v}_{(g)}, \underline{w}_{(j)} < \underline{v}_{(i)}, \underline{w}_{(l)} < \underline{v}_{(k)}] +$$
$$+ \Pr\,[\underline{w}_{(h)} < \underline{v}_{(g)}, \underline{w}_{(j)} < \underline{v}_{(i)}, \underline{w}_{(l)} > \underline{v}_{(k)}] +$$
$$+ \Pr\,[\underline{w}_{(h)} < \underline{v}_{(g)}, \underline{w}_{(j)} > \underline{v}_{(i)}, \underline{w}_{(l)} < \underline{v}_{(k)}] +$$
$$+ \Pr\,[\underline{w}_{(h)} > \underline{v}_{(g)}, \underline{w}_{(j)} < \underline{v}_{(i)}, \underline{w}_{(l)} < \underline{v}_{(k)}]$$
$$= \Pr\,[\underline{w}_{(h)} < \underline{v}_{(g)}, \underline{w}_{(j)} < \underline{v}_{(i)}] + \Pr\,[\underline{w}_{(h)} < \underline{v}_{(g)}, \underline{w}_{(l)} < \underline{v}_{(k)}] +$$
$$+ \Pr\,[\underline{w}_{(j)} < \underline{v}_{(i)}, \underline{w}_{(l)} < \underline{v}_{(k)}] - 2\Pr\,[\underline{w}_{(h)} < \underline{v}_{(g)}, \underline{w}_{(j)} < \underline{v}_{(i)}, \underline{w}_{(l)} < \underline{v}_{(k)}]$$
$$= \frac{1}{\binom{m+n}{m}} \Bigg\{ \sum_{r=h}^{n} \sum_{s=j}^{n} \binom{r+g-1}{r} \binom{s+i-r-g-1}{s-r} \binom{m+n-s-i}{m-i} +$$
$$+ \sum_{r=h}^{n} \sum_{s=l}^{n} \binom{r+g-1}{r} \binom{s+k-r-g-1}{s-r} \binom{m+n-s-k}{m-k} +$$
$$+ \sum_{r=j}^{n} \sum_{s=l}^{n} \binom{r+i-1}{r} \binom{s+k-r-i-1}{s-r} \binom{m+n-s-k}{m-k} +$$
$$-2 \sum_{r=h}^{n} \sum_{s=j}^{n} \sum_{t=l}^{n} \binom{r+g-1}{r} \binom{s+i-r-g-1}{s-r} \binom{t+k-r-s-i-1}{t-r-s} \binom{m+n-t-k}{n-t} \Bigg\}. \qquad (3.4.6)$$

Thirdly we consider lower confidence bounds for ν based on probability relations of the form (3.4.3). Then

$$\Pr\,[\min \{\underline{y}_{(h)}-\underline{x}_{(g)}, \underline{y}_{(j)}-\underline{x}_{(i)}, \underline{y}_{(l)}-\underline{x}_{(k)}\} < \nu]$$
$$= \Pr\,[\min \{\underline{w}_{(h)}-\underline{v}_{(g)}, \underline{w}_{(j)}-\underline{v}_{(i)}, \underline{w}_{(l)}-\underline{v}_{(k)}\} < 0]$$
$$= 1 - \Pr\,[\min \{\underline{w}_{(h)}-\underline{v}_{(g)}, \underline{w}_{(j)}-\underline{v}_{(i)}, \underline{w}_{(l)}-\underline{v}_{(k)}\} > 0]$$
$$= 1 - \Pr\,[\underline{w}_{(h)}-\underline{v}_{(g)} > 0, \underline{w}_{(j)}-\underline{v}_{(i)} > 0, \underline{w}_{(l)}-\underline{v}_{(k)} > 0]$$
$$= 1 - \Pr\,[\underline{w}_{(h)} > \underline{v}_{(g)}, \underline{w}_{(j)} > \underline{v}_{(i)}, \underline{w}_{(l)} > \underline{v}_{(k)}]$$
$$= 1 - \frac{1}{\binom{m+n}{m}} \sum_{r=g}^{m} \sum_{s=i-r}^{m-r} \sum_{t=k-r-s}^{m-r-s} \binom{r+h-1}{r} \binom{s+j-h-1}{s} \binom{t+l-j-1}{t} \binom{m+n-r-s-t-l}{m-r-s-t} \qquad (3.4.7)$$

or

$$= 1 - \frac{1}{\binom{m+n}{m}} \sum_{r=g}^{m} \sum_{s=l}^{m} \sum_{t=k}^{m} \binom{r+h-1}{r} \binom{s+j-r-h-1}{s-r} \binom{t+l-r-s-j-1}{t-r-s} \binom{m+n-t-l}{m-t}. \tag{3.4.8}$$

3.5. The class of V tests

The lower confidence bounds as well as the upper and two-sided confidence bounds for ν can be converted into families of tests. In the case of lower confidence bounds it is possible to convert these bounds into tests to test the null hypothesis

$$H_0 : \nu = 0 \tag{3.5.1}$$

against the alternative hypothesis

$$H_1 : \nu > 0. \tag{3.5.2}$$

Note that the null hypothesis

$$H_0{}^* : \nu = \nu_0 \tag{3.5.3}$$

can be transformed into $H_0 : \nu = 0$ by considering $\underline{y} - \nu_0$ instead of $\underline{y}$.

In the case of lower confidence bounds from the subclass C_1 one has the class V_1 of the following test statistics:

$$\underline{V}_{ji} = \underline{y}_{(j)} - \underline{x}_{(i)} \tag{3.5.4}$$

and it rejects the null hypothesis H_0 when $V_{ji} > 0$. It is clear that the class V_1 is equivalent to the class of exceedance tests.

In the case of lower confidence bounds from C_2 one has the class V_2 of the following test statistics:

$$\underline{V}_{lkji}{}^{(1)} = \max \{\underline{y}_{(l)} - \underline{x}_{(k)}, \underline{y}_{(j)} - \underline{x}_{(i)}\} \tag{3.5.5}$$

and

$$\underline{V}_{lkji}{}^{(2)} = \min \{\underline{y}_{(l)} - \underline{x}_{(k)}, \underline{y}_{(j)} - \underline{x}_{(i)}\} \tag{3.5.6}$$

and they will reject H_0 when $V_{lkji}{}^{(1)}$ and $V_{lkji}{}^{(2)}$, respectively, are larger than zero.

For testing the null hypothesis H_0 against

$$H_1 : \nu = \nu_1 > 0 \tag{3.5.7}$$

the test based on the statistic $\underline{V}_{ji}$ has a consistency property described in the next theorem.

Theorem 3.5.1. $F(x)$ has a density $f(x)$, which is continuous in the neighbourhood of ξ_λ, defined by $F(\xi_\lambda) = \lambda$, and $f(\xi_\lambda) \neq 0$ $(0 < \lambda < 1)$. If m, n, i and j tend to infinity such that $i/m \to \lambda$ and $j/n \to \lambda$, then Pr $[\underline{V}_{ji} > 0 \mid H_1]$ tends to unity.

Proof. Under the given conditions $\underline{y}_{(j)}$ and $\underline{x}_{(i)}$ converge in probability to η_λ and ξ_λ, respectively, where

$$F(\xi_\lambda) = \lambda \quad \text{and} \quad G(\eta_\lambda) = \lambda.$$

So for each $\varepsilon > 0$ and $0 < \delta < 1$ one can find $N_{\varepsilon,\delta}$ such that for m and n larger than $N_{\varepsilon,\delta}$ the following relations hold:

$$\Pr\left[\,|\underline{y}_{(j)} - \eta_\lambda| < \frac{\varepsilon}{3}\right] > 1 - \delta$$

and

$$\Pr\left[\,|\underline{x}_{(i)} - \xi_\lambda| < \frac{\varepsilon}{3}\right] > 1 - \delta.$$

Now under the alternative hypothesis H_1: $\eta_\lambda > \xi_\lambda$. Taking

$$\varepsilon = (\eta_\lambda - \xi_\lambda) = \nu_1,$$

one gets under H_1:

$$\begin{aligned} \Pr\,[\underline{V}_{ji} > 0] &= \Pr\,[\underline{y}_{(j)} - \underline{x}_{(i)} > 0] \\ &\geqslant \Pr\left[\,|\underline{y}_{(j)} - \eta_\lambda| < \frac{\varepsilon}{3} \wedge |\underline{x}_{(i)} - \xi_\lambda| < \frac{\varepsilon}{3}\right] \\ &= \Pr\left[\,|\underline{y}_{(j)} - \eta_\lambda| < \frac{\varepsilon}{3}\right] \Pr\left[\,|\underline{x}_{(i)} - \xi_\lambda| < \frac{\varepsilon}{3}\right] > (1-\delta)^2. \end{aligned}$$

From this inequality it follows that $\Pr\,[\underline{V}_{ji} > 0 \mid H_1]$ tends to unity under the given conditions.

It is of interest to note the advantages of the test procedures (and confidence intervals) described in this monograph. These advantages are valid in particular for the class of V_{ji} tests. The first thing worth mentioning is the opportunity provided by these tests to inspect the data in a rapid and easy manner. A second advantage is the fact that these test procedures and confidence bounds can be applied to life testing experiments in a very easy manner. It is a characteristic feature of life tests that the results of the experiments often become available in order of magnitude. Thus it becomes very natural to apply tests which are based on order statistics. It is clear that in cases where life tests are started at the same time it is often possible to shorten the time of experimentation and also the number of items destroyed in order to be able to decide whether or not one population of items is "better" than another population.

The V_{ji} tests can also be applied with success in other cases where the observations become available in order of magnitude. The subscripts j and i must be chosen beforehand, namely before the results of the experiment are known.

Experimentation can be stopped as soon as the specified order statistics have been observed. Thus the use of the criterion whether $V_{ji} > 0$ with this experimental design permits early termination of the experiment, sometimes after relatively few lifetimes have been observed. This is a desirable property since testing-time is often expensive.

Another interesting point is the fact that the V_{ji} tests, equivalent to the tests based on exceedances, are also equivalent to a certain general formulation of the quantile-tests. This relation, not mentioned in the literature to my knowledge, can be seen as follows:

$$\begin{aligned}\Pr\,[\underline{V}_{ji} > 0] &= \Pr\,[\underline{y}_{(j)} - \underline{x}_{(i)} > 0] \\ &= \Pr\,[\underline{y}_{(j)} > \underline{x}_{(i)}] \\ &= \Pr\,[\{\text{number of } y\text{'s smaller than } \underline{x}_{(i)}\} < j] \\ &= \Pr\,[\{\text{number of } x\text{'s smaller than } \underline{z}_{(i+j)}\} \geqslant i], \quad (3.5.8)\end{aligned}$$

where $\underline{z}_{(i+j)}$ is the $(i+j)$th smallest in the combined sample. Defining $i + j = \varrho(m+n)$, it can be seen from the last-formulated event that the V_{ji} test is nothing other than a ϱ-tile test, where the ϱ must be chosen correctly.

From expression (3.5.8) it follows immediately that $\Pr\,[\underline{y}_{(j)} - \underline{x}_{(i)} < 0 \mid H_0]$ can also be written as a sum of hypergeometric probabilities. One gets

$$\Pr\,[\underline{y}_{(j)} - \underline{x}_{(i)} < 0 \mid H_0] = \frac{1}{\binom{m+n}{m}} \sum_{r \geqslant j} \binom{i+j-1}{r} \binom{m+n-i-j+1}{n-r}, \qquad (3.5.9)$$

where the summation extends to all realistic values of $r \geqslant j$. On a similar line of reasoning one finds for the general case, with $1 \leqslant i_1 < i_2 < \ldots < i_R \leqslant m$ and $1 \leqslant j_1 < j_2 < \ldots < j_R \leqslant n$:

$$\Pr\,[\bigcap_{r=1,2,\ldots,R} \{\underline{y}_{(j_r)} - \underline{x}_{(i_r)} < 0\} \mid H_0]$$

$$= \frac{1}{\binom{m+n}{n}} \sum_{\substack{s_r \geqslant j_r \\ r=1,2,\ldots,R}} \binom{i_1+j_1-1}{s_1} \binom{i_2+j_2-i_1-j_1}{s_2-s_1} \ldots \binom{m+n-i_R-j_R+1}{n-s_R}. \qquad (3.5.10)$$

4. SELECTION OF CONFIDENCE BOUNDS AND POWER INVESTIGATIONS OF THE V_1 TESTS

4.1. Introduction

In this chapter a procedure will be presented by which lower confidence bounds from the class C_1 have been selected. At the end of this chapter some remarks will be made on upper and two-sided confidence bounds.

This selection of confidence bounds is based on the power of the corresponding tests. Power investigations have been carried out for Normal, Uniform and Exponential translation alternatives, Lehmann alternatives and Exponential scale alternatives.

4.2. The selection procedure

For the selection of tests from the class V_1 of tests corresponding to C_1 and the selection of corresponding lower confidence bounds for the median difference ν, from all level α tests available for each pair of sample sizes (m,n), the idea of most stringent tests has been applied in the following manner.

For any pair of sample sizes (m,n) and a given level of significance α the test with maximal size among all level α tests V_{ji} available is determined *). Denote the power function of this test by $\beta_{\max\ \text{size}}(\nu;\alpha)$. Then $\nu'_{\cdot 50}$ is determined for which $\beta_{\max\ \text{size}}(\nu'_{\cdot 50};\alpha) = \cdot 50$. Next among all level α tests V_{ji} the one with maximal power in $\nu'_{\cdot 50}$ is determined and its power function is denoted by $\beta^*(\nu)$. Then $\nu^*_{\cdot 25}$, $\nu^*_{\cdot 50}$ and $\nu^*_{\cdot 75}$ are calculated by $\beta^*(\nu^*_{\cdot 25}) = \cdot 25$, $\beta^*(\nu^*_{\cdot 50}) = \cdot 50$ and $\beta^*(\nu^*_{\cdot 75}) = \cdot 75$. These computations are carried out in order to find the region in which the power functions of the better tests have interesting values. Now for all level α tests the power is determined in these three points $\nu^*_{\cdot 25}$, $\nu^*_{\cdot 50}$ and $\nu^*_{\cdot 75}$. In each of these three points the maximal powers $b_{\cdot 25}$, $b_{\cdot 50}$ and $b_{\cdot 75}$, respectively, among all level α tests are determined. Then the test being selected is the one (with power function $\beta(\nu)$) among all level α tests for which

$$(b_{\cdot 25} - \beta(\nu^*_{\cdot 25})) + (b_{\cdot 50} - \beta(\nu^*_{\cdot 50})) + (b_{\cdot 75} - \beta(\nu^*_{\cdot 75})) \qquad (4.1.1)$$

is minimal, i.e. minimizes the average shortcoming over three interesting points. Thus, roughly speaking, this selected test has, on average, optimal power among all level α tests.

This selection of tests, and consequently of lower confidence bounds, has been performed for various alternatives and for sample sizes ($\geqslant 3$) up to and including $m = 15$ and $n = 15$ and for six significance levels, namely

$$\cdot 001,\ \cdot 005,\ \cdot 010,\ \cdot 025,\ \cdot 05 \text{ and } \cdot 10. \qquad (4.1.2)$$

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*) In some cases there is more than one test with maximal size. In these cases the test with maximal i has been taken.

4.2.1. *Power against Normal translation alternatives and selected lower confidence bounds*

In this section the power function of the test V_{ji} against Normal shift alternatives will be determined. Suppose the random variables $\underline{x}$ and $\underline{y}$ have Normal distributions with equal variances σ^2. We introduce the following notation:

$$\underline{x}^*_{(i)} = \frac{\underline{x}_{(i)}}{\sigma} \quad \text{and} \quad \underline{y}^*_{(j)} = \frac{\underline{y}_{(j)} - \nu}{\sigma}. \tag{4.2.1.1}$$

Thus $\underline{x}^*_{(i)}$ and $\underline{y}_{(j)}^*$ are the ith and jth order statistics of standardized Normal random variables.

In testing the null hypothesis

$$H_0 : \nu = 0$$

against the alternative hypothesis:

$$H_1 : \nu = \nu^* > 0$$

one gets for the power function $\beta(\delta)$ of the test V_{ji}, as function of $\delta = \nu^*/\sigma$:

$$\begin{aligned} \beta(\delta) &= \Pr\,[\underline{y}_{(j)} - \underline{x}_{(i)} > 0 \mid H_1] \\ &= \Pr\left[\frac{1}{\sigma}(\underline{y}_{(j)} - \nu^*) - \frac{1}{\sigma}\underline{x}_{(i)} > \frac{-\nu^*}{\sigma} \,\middle|\, H_1\right] \\ &= \Pr\,[\underline{y}^*_{(j)} - \underline{x}^*_{(i)} > -\delta]. \end{aligned} \tag{4.2.1.2}$$

Denoting the cumulative distribution functions of the random variables $\underline{x}^*$ and $\underline{y}^*$ by $\Phi(x^*)$ and $\Phi(y^*)$, respectively, where

$$\Phi(t) = \frac{1}{(2\,\pi)^{1/2}} \int_{-\infty}^{t} \exp\,(-x^2/2)\,\mathrm{d}x \tag{4.2.1.3}$$

and the probability elements of $\underline{x}^*_{(i)}$ and $\underline{y}^*_{(j)}$ by $\mathrm{d}G(x^*_{(i)})$ and $\mathrm{d}H(y^*_{(j)})$, respectively, then one finds:

$$\begin{aligned} \beta(\delta) &= \Pr\,[\underline{y}^*_{(j)} - \underline{x}^*_{(i)} > -\delta] \\ &= \int\int_{y^*_{(j)} - x^*_{(i)} > -\delta} \mathrm{d}G(x^*_{(i)})\,\mathrm{d}H(y^*_{(j)}) \\ &= \int_{-\infty}^{\infty}\int_{x^*_{(i)}-\delta}^{\infty} j\tbinom{n}{j}\,\Phi^{j-1}(y^*_{(j)})\,\{1 - \Phi(y^*_{(j)})\}^{n-j}\,\mathrm{d}\Phi(y^*_{(j)})\,\mathrm{d}G(x^*_{(i)}). \end{aligned}$$

Now one has

$$\int_{x^*_{(i)}-\delta}^{\infty} \Phi^{j-1}(y^*_{(j)}) \{1-\Phi(y^*_{(j)})\}^{n-j} \, d\Phi(y^*_{(j)})$$

$$= \int_{\Phi(x^*_{(i)}-\delta)}^{1} t^{j-1}(1-t)^{n-j} \, dt$$

$$= -\frac{1}{n-j+1} \int_{\Phi(x^*_{(i)}-\delta)}^{1} t^{j-1} \, d(1-t)^{n-j+1}$$

$$= \frac{1}{n-j+1} \Phi^{j-1}(x^*_{(i)}-\delta) \{1-\Phi(x^*_{(i)}-\delta)\}^{n-j+1} +$$

$$+ \frac{j-1}{n-j+1} \int_{\Phi(x^*_{(i)}-\delta)}^{1} t^{j-2}(1-t)^{n-j+1} \, dt$$

$$= (j-1)!\,(n-j)! \sum_{r_1=1}^{j} \frac{\Phi^{j-r_1}(x^*_{(i)}-\delta)\{1-\Phi(x^*_{(i)}-\delta)\}^{n-j+r_1}}{(j-r_1)!\,(n-j+r_1)!}\,.$$

Hence

$$\beta(\delta) = i\,n!\binom{m}{i} \sum_{r_1=1}^{j} \frac{1}{(j-r_1)!\,(n-j+r_1)!} \times$$

$$\times \int_{-\infty}^{\infty} \Phi^{i-1}(x^*_{(i)}) \{1-\Phi(x^*_{(i)})\}^{m-i} \Phi^{j-r_1}(x^*_{(i)}-\delta) \times$$

$$\times \{1-\Phi(x^*_{(i)}-\delta)\}^{n-j+r_1} \, d\Phi(x^*_{(i)})$$

$$= i\binom{m}{i} \sum_{r_1=1}^{j} \binom{n}{j-r_1} \int_{-\infty}^{\infty} \Phi^{i-1}(x^*_{(i)}) \{1-\Phi(x^*_{(i)})\}^{m-i} \Phi^{j-r_1}(x^*_{(i)}-\delta) \times$$

$$\times \{1-\Phi(x^*_{(i)}-\delta)\}^{n-j+r_1} \, d\Phi(x^*_{(i)})$$

$$= \int_{-\infty}^{\infty} i\binom{m}{i} \Phi^{i-1}(x^*_{(i)}) \{1-\Phi(x^*_{(i)})\}^{m-i} \times$$

$$\times \Big[\sum_{r_1=1}^{j} \binom{n}{j-r_1} \Phi^{j-r_1}(x^*_{(i)}-\delta) \{1-\Phi(x^*_{(i)}-\delta)\}^{n-j+r_1}\Big] \varphi(x^*_{(i)}) \, dx^*_{(i)}, \qquad (4.2.1.4)$$

where

$$\varphi(t) = \frac{d\Phi(t)}{dt}\,.$$

This expansion has been chosen in order to get a sum of only positive terms, which improves the computational accuracy. The approximation of the integral (4.2.1.4) must be carried out with great care. For this purpose the Normal distribution function in the integrand,

$$i\binom{m}{i}\,\Phi(x)^{i-1}\,[1-\Phi(x)]^{m-i}\,\varphi(x)\sum_{r_1=1}^{j}\left\{\binom{n}{j-r_1}\,\Phi(x-\delta)^{j-r_1}\,[1-\Phi(x-\delta)]^{n-j+r_1}\right\},$$

defined by

$$\Phi(x)=\frac{1}{2}\left\{1+\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right\},$$

$$1-\Phi(x)=\frac{1}{2}\left\{1-\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right\},$$

where

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x \exp -t^2\,dt,$$

has been approximated with maximal error 10^{-9} by

for $|x|\leqslant 2:\operatorname{erf}(x)=x\sum_{k=0}^{10} a_k\,t^k$ with $t=\dfrac{x^2}{4}$ and

$a_0 = +1{\cdot}1283791667,$ $a_6 = +0{\cdot}48500467188,$
$a_1 = -1{\cdot}5045054502,$ $a_7 = -0{\cdot}22680671298,$
$a_2 = +1{\cdot}8054023026,$ $a_8 = +0{\cdot}08401703770,$
$a_3 = -1{\cdot}7193645760,$ $a_9 = -0{\cdot}02146900482,$
$a_4 = +1{\cdot}3367541722,$ $a_{10} = +0{\cdot}0027473965898$
$a_5 = -0{\cdot}87249787074,$

and

for $x>2:\operatorname{erf}(x)=G$
$x<-2:\operatorname{erf}(x)=G-2$ with

$G=1-\dfrac{\exp -x^2}{x}\sum_{k=0}^{10} b_k\,t^k$ with $t=\dfrac{4}{x^2}$ and

$b_0 = +0{\cdot}56418958337,$ $b_6 = +0{\cdot}013943300602,$
$b_1 = -0{\cdot}070523649818,$ $b_7 = -0{\cdot}011419727332,$
$b_2 = +0{\cdot}026444214057,$ $b_8 = +0{\cdot}0067277663670,$
$b_3 = -0{\cdot}016490136845,$ $b_9 = -0{\cdot}0024377794392,$
$b_4 = +0{\cdot}014095687080,$ $b_{10} = +0{\cdot}0004011738252.$
$b_5 = -0{\cdot}014139079071,$

Now starting from the interval [—10, +10] the smallest interval [A,B], where A and B are integers, is found such that the maximum of the integrand, for which it is checked that it has only one maximum in the interval [—10, +10], is attained in [A,B] and the integrand values in A and B are both smaller than 10^{-6}. Next using Simpson's integration method with self-regulating step, the integral is determined with maximal error smaller than 10^{-4}.

According to the selection procedure described in sec. 4.2 one-sided tests to the right and consequently lower confidence bounds have been selected. The selected lower confidence bounds are given in the tables 4.2.1-I–4.2.1-VI.

Notice that if $\underline{y}_{(j)} - \underline{x}_{(i)}$ is the selected test for the pair of sample sizes m and n, then $\underline{y}_{(m-i+1)} - \underline{x}_{(n-j+1)}$ is the selected test for the pair of sample sizes n and m. Thus computations were only necessary in approximately half of the possible cases.

If $m = n$, then it is clear that there are often two lower confidence bounds which are statistically equivalent under H_0 and H_1. This remark can also be made for the Uniform distribution.

Now we present some examples.

Example 4.2.1.1.

Assume the following two independent samples $x_1, x_2, \ldots, x_{10}$ and $y_1, y_2, \ldots, y_{10}$ are drawn from two Normal populations with equal variances:

x-sample	and	y-sample
2·08		3·92
3·64		3·29
2·25		3·31
1·95		4·34
1·49		3·23
3·69		2·84
2·07		3·69
2·34		4·31
·09		2·53
2·55		4·59

Now one finds after some tiresome computations (taking about 13 minutes on an electric desk calculator) as a lower confidence bound with confidence level ·95 for the location difference based on the t-distribution:

$$\bar{y} - \bar{x} - t_{18}(\cdot 95)\left\{\sum_{i=1}^{10}(x_i - \bar{x})^2 + \sum_{j=1}^{10}(y_j - \bar{y})^2\right\}^{1/2}\sqrt{\frac{20}{1800}} = \cdot 71.$$

Using table 4.2.1-V one finds immediately as a distribution-free lower confidence bound with confidence level ·95 (confidence coefficient ·965) for the location difference:

$$y_{(3)} - x_{(7)} = 3{\cdot}23 - 2{\cdot}34 = {\cdot}89.$$

The lower bound based on the t-distribution becomes ·64 when we take the same confidence coefficient ·965.

Example 4.2.1.2.

The following two samples of size 10 are drawn from two Normal populations with equal variances:

x-sample	and	y-sample
3·66		5·53
4·43		3·32
2·82		5·70
2·93		5·40
2·59		4·17
3·51		2·02
3·97		3·39
1·80		5·03
2·90		4·66
2·44		3·38

As lower confidence bound with confidence level ·95 (confidence coefficient ·965) for the location difference one finds from table 4.2.1-V:

$$y_{(3)} - x_{(7)} = -{\cdot}13.$$

Using the t-distribution one finds:

·36 with confidence coefficient ·95,
·27 with confidence coefficient ·965.

Example 4.2.1.3.

Given are the two samples

x-sample	and	y-sample
2·48		2·91
4·18		4·61
1·78		4·83
4·27		3·56
3·70		4·14
2·32		4·35
2·13		3·69
2·41		4·90
2·11		5·41
1·61		4·46

TABLE 4.2.1-I

Distribution-free lower confidence bounds $\underline{y}_{(j)} - \underline{x}_{(i)}$, denoted by $j - i$, for shift (selection based on Normal shift alternatives) with confidence level $1 - \alpha = {\cdot}999$ and the values of: 1-confidence coefficient

$$\Pr[\underline{y}_{(j)} - \underline{x}_{(i)} < \nu] \geqslant {\cdot}999$$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3													
4								1—4 ·00100	1—4 ·00073	1—4 ·00055	1—4 ·00042	1—4 ·00033	1—4 ·00026
5						1—5 ·00078	1—5 ·00050	1—5 ·00033	1—5 ·00023	2—5 ·00097	2—5 ·00070	2—5 ·00052	2—5 ·00039
6					1—6 ·00058	1—6 ·00033	1—6 ·00020	2—6 ·00087	1—5 ·00097	1—5 ·00070	1—5 ·00052	3—6 ·00072	3—6 ·00052
7				1—7 ·00058	1—7 ·00029	1—7 ·00016	1—6 ·00087	1—6 ·00057	1—6 ·00038	3—7 ·00071	3—7 ·00046	2—6 ·00085	2—6 ·00062
8			1—8 ·00078	1—8 ·00033	1—8 ·00016	1—7 ·00070	1—7 ·00041	1—7 ·00025	3—8 ·00060	2—7 ·00077	4—8 ·00081	2—7 ·00035	2—7 ·00025
9			1—9 ·00050	1—9 ·00020	2—9 ·00087	2—9 ·00041	1—8 ·00021	2—8 ·00099	2—8 ·00060	4—9 ·00075	4—9 ·00044	3—8 ·00073	2—7 ·00075
10		1—10 ·00100	1—10 ·00033	1—9 ·00087	2—10 ·00057	2—10 ·00025	2—9 ·00099	2—9 ·00055	4—10 ·00081	3—9 ·00095	2—8 ·00073	2—8 ·00049	4—9 ·00090
11		1—11 ·00073	1—11 ·00023	2—11 ·00097	2—11 ·00038	1—9 ·00060	2—10 ·00060	1—8 ·00081	2—9 ·00095	2—9 ·00059	1—7 ·00095	4—10 ·00079	3—9 ·00077
12		1—12 ·00055	1—11 ·00097	2—12 ·00070	1—10 ·00071	2—11 ·00077	1—9 ·00075	2—10 ·00095	3—11 ·00059	2—10 ·00032	2—9 ·00090	3—10 ·00064	5—11 ·00097
13		1—13 ·00042	1—12 ·00070	2—13 ·00052	1—11 ·00046	1—10 ·00081	1—10 ·00044	3—12 ·00073	5—13 ·00095	4—12 ·00090	3—11 ·00060	3—11 ·00036	4—11 ·00092
14		1—14 ·00033	1—13 ·00052	1—12 ·00072	2—13 ·00085	2—13 ·00035	2—12 ·00073	3—13 ·00049	2—11 ·00079	3—12 ·00064	3—12 ·00036	4—12 ·00092	3—11 ·00058
15		1—15 ·00026	1—14 ·00039	1—13 ·00052	2—14 ·00062	2—14 ·00025	3—14 ·00075	2—12 ·00090	3—13 ·00077	2—11 ·00097	3—12 ·00092	4—13 ·00058	2—10 ·00085

TABLE 4.2.1-II

Distribution-free lower confidence bounds $\underline{y}_{(j)} - \underline{x}_{(i)}$, denoted by $j - i$, for shift (selection based on Normal shift alternatives) with confidence level $1 - \alpha = {\cdot}995$ and the values of : 1-confidence coefficient

$$\Pr\left[\underline{y}_{(j)} - \underline{x}_{(i)} < \nu\right] \geqslant {\cdot}995$$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3							1—3 ·00455	1—3 ·00350	1—3 ·00275	1—3 ·00220	1—3 ·00179	1—3 ·00147	2—3 ·00490
4				1—4 ·00476	1—4 ·00303	1—4 ·00202	1—4 ·00140	2—4 ·00500	2—4 ·00366	2—4 ·00275	2—4 ·00210	3—4 ·00490	3—4 ·00387
5			1—5 ·00397	1—5 ·00216	1—5 ·00126	2—5 ·00466	1—4 ·00500	1—4 ·00366	3—5 ·00481	3—5 ·00339	3—5 ·00245	4—5 ·00482	2—4 ·00490
6		1—6 ·00476	1—6 ·00216	1—6 ·00108	1—5 ·00466	1—5 ·00300	1—5 ·00200	3—6 ·00350	3—6 ·00226	2—5 ·00393	2—5 ·00291	2—5 ·00219	5—6 ·00387
7		1—7 ·00303	1—7 ·00126	2—7 ·00466	1—6 ·00233	1—6 ·00140	1—5 ·00481	2—6 ·00365	4—7 ·00377	2—6 ·00169	3—6 ·00444	3—6 ·00320	2—5 ·00432
8		1—8 ·00202	1—7 ·00466	2—8 ·00300	2—8 ·00140	1—6 ·00350	2—7 ·00300	4—8 ·00377	3—7 ·00488	2—6 ·00444	2—6 ·00320	4—7 ·00464	4—7 ·00327
9	1—9 ·00455	1—9 ·00140	2—9 ·00500	2—9 ·00200	3—9 ·00481	2—8 ·00300	1—6 ·00452	2—7 ·00488	2—7 ·00322	2—7 ·00217	4—8 ·00376	3—7 ·00416	5—8 ·00471
10	1—10 ·00350	1—9 ·00500	2—10 ·00366	1—8 ·00350	2—9 ·00365	1—7 ·00377	3—9 ·00488	2—8 ·00274	1—6 ·00387	3—8 ·00478	3—8 ·00318	5—9 ·00416	4—8 ·00483
11	1—11 ·00275	1—10 ·00366	1—9 ·00481	1—9 ·00226	1—8 ·00377	2—9 ·00488	3—10 ·00322	5—11 ·00387	3—9 ·00446	3—9 ·00278	5—10 ·00416	3—8 ·00483	6—10 ·00445
12	1—12 ·00220	1—11 ·00275	1—10 ·00339	2—11 ·00393	2—11 ·00169	3—11 ·00444	3—11 ·00217	3—10 ·00478	3—10 ·00278	2—8 ·00471	3—9 ·00404	6—11 ·00471	5—10 ·00477
13	1—13 ·00179	1—12 ·00210	1—11 ·00245	2—12 ·00291	2—11 ·00444	3—12 ·00320	2—10 ·00376	3—11 ·00318	2—9 ·00416	4—11 ·00404	3—10 ·00242	5—11 ·00477	4—10 ·00370
14	1—14 ·00147	1—12 ·00490	1—11 ·00482	2—13 ·00219	2—12 ·00320	2—11 ·00464	3—12 ·00416	2—10 ·00416	4—12 ·00483	2—9 ·00471	3—10 ·00477	4—11 ·00351	2—8 ·00470
15	1—14 ·00490	1—13 ·00387	2—14 ·00490	1—11 ·00387	3—14 ·00432	2—12 ·00327	2—11 ·00471	3—12 ·00483	2—10 ·00445	3—11 ·00477	4—12 ·00370	7—14 ·00470	5—12 ·00461

TABLE 4.2.1-III

Distribution-free lower confidence bounds $\underline{y}_{(j)} - \underline{x}_{(i)}$, denoted by $j-i$, for shift (selection based on Normal shift alternatives) with confidence level $1-\alpha = {\cdot}99$ and the values of : 1-confidence coefficient

$$\Pr\,[\underline{y}_{(j)} - \underline{x}_{(i)} < \nu] \geqslant {\cdot}99$$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3					1—3 ·00833	1—3 ·00606	1—3 ·00455	1—3 ·00350	1—3 ·00275	2—3 ·00879	2—3 ·00714	2—3 ·00588	2—3 ·00490
4			1—4 ·00794	1—4 ·00476	1—4 ·00303	1—4 ·00202	2—4 ·00699	2—4 ·00500	1—3 ·00879	3—4 ·00824	3—4 ·00630	3—4 ·00490	4—4 ·00903
5		1—5 ·00794	1—5 ·00397	1—5 ·00216	2—5 ·00758	1—4 ·00699	1—4 ·00500	3—5 ·00699	3—5 ·00481	2—4 ·00986	2—4 ·00770	2—4 ·00611	5—5 ·00813
6		1—6 ·00476	1—6 ·00216	1—5 ·00758	1—5 ·00466	3—6 ·00932	3—6 ·00559	2—5 ·00762	2—5 ·00541	2—5 ·00393	3—5 ·00955	3—5 ·00722	6—6 ·00851
7	1—7 ·00833	1—7 ·00303	1—6 ·00758	2—7 ·00466	1—6 ·00233	2—6 ·00886	2—6 ·00559	4—7 ·00617	3—6 ·00905	2—5 ·00955	2—5 ·00722	4—6 ·00898	4—6 ·00661
8	1—8 ·00606	1—8 ·00202	2—8 ·00699	1—6 ·00932	2—7 ·00886	2—7 ·00505	1—5 ·00905	2—6 ·00905	2—6 ·00627	4—7 ·00988	4—7 ·00671	3—6 ·00832	5—7 ·00841
9	1—9 ·00455	1—8 ·00699	2—9 ·00500	1—7 ·00559	2—8 ·00559	4—9 ·00905	2—7 ·00761	2—7 ·00488	4—8 ·00917	3—7 ·00845	3—7 ·00588	5—8 ·00693	4—7 ·00884
10	1—10 ·00350	1—9 ·00500	1—8 ·00699	2—9 ·00762	1—7 ·00617	3—9 ·00905	3—9 ·00488	2—7 ·00988	3—8 ·00733	3—8 ·00478	5—9 ·00650	3—7 ·00884	3—7 ·00650
11	1—11 ·00275	2—11 ·00879	1—9 ·00481	2—10 ·00541	2—9 ·00905	3—10 ·00627	2—8 ·00917	3—9 ·00733	3—9 ·00446	4—9 ·00950	3—8 ·00693	6—10 ·00704	5—9 ·00771
12	1—11 ·00879	1—10 ·00824	2—11 ·00986	2—11 ·00393	3—11 ·00955	2—9 ·00988	3—10 ·00845	3—10 ·00478	3—9 ·00950	4—10 ·00614	2—7 ·00998	4—9 ·00893	4—9 ·00617
13	1—12 ·00714	1—11 ·00630	2—12 ·00770	2—11 ·00955	3—12 ·00722	2—10 ·00671	3—11 ·00588	2—9 ·00650	4—11 ·00693	6—12 ·00998	4—10 ·00847	7—12 ·00876	6—11 ·00830
14	1—13 ·00588	1—12 ·00490	2—13 ·00611	2—12 ·00722	2—11 ·00898	3—12 ·00832	2—10 ·00693	4—12 ·00884	2—9 ·00704	4—11 ·00893	2—8 ·00876	3—9 ·00915	4—10 ·00729
15	1—14 ·00490	1—12 ·00903	1—11 ·00813	1—10 ·00851	2—12 ·00661	2—11 ·00841	3—12 ·00884	4—13 ·00650	3—11 ·00771	4—12 ·00617	3—10 ·00830	5—12 ·00729	5—12 ·00461

TABLE 4.2.1-IV

Distribution-free lower confidence bounds $\underline{y}_{(j)} - \underline{x}_{(i)}$, denoted by $j - i$, for shift (selection based on Normal shift alternatives) with confidence level $1 - \alpha = \cdot 975$ and the values of : 1-confidence coefficient

$$\Pr[\underline{y}_{(j)} - \underline{x}_{(i)} < \nu] \geqslant \cdot 975$$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3			1—3 ·01786	1—3 ·01190	1—3 ·00833	2—3 ·02424	2—3 ·01818	2—3 ·01399	2—3 ·01099	3—3 ·02198	1—2 ·02500	1—2 ·02206	4—3 ·02451
4		1—4 ·01429	1—4 ·00794	2—4 ·02381	1—3 ·02424	1—3 ·01818	3—4 ·02098	3—4 ·01499	3—4 ·01099	4—4 ·01923	2—3 ·02227	5—4 ·02288	5—4 ·01806
5	1—5 ·01786	1—5 ·00794	1—4 ·02381	1—4 ·01515	1—4 ·01010	3—5 ·01632	2—4 ·02298	2—4 ·01698	2—4 ·01282	5—5 ·02036	3—4 ·02171	3—4 ·01729	3—4 ·01393
6	1—6 ·01190	1—5 ·02381	2—6 ·01515	1—5 ·00758	1—4 ·02098	2—5 ·01632	4—6 ·01678	3—5 ·02448	3—5 ·01753	2—4 ·02171	4—5 ·02374	4—5 ·01806	4—5 ·01393
7	1—7 ·00833	2—7 ·02424	2—7 ·01010	3—7 ·02098	2—6 ·01457	4—7 ·01865	2—5 ·02448	2—5 ·01753	4—6 ·02489	4—6 ·01739	3—5 ·02232	5—6 ·02090	2—4 ·02073
8	1—7 ·02424	2—8 ·01818	1—6 ·01632	2—7 ·01632	1—5 ·01865	2—6 ·02028	2—6 ·01337	4—7 ·02297	3—6 ·02155	5—7 ·02489	5—7 ·01703	4—6 ·02198	6—7 ·01878
9	1—8 ·01818	1—7 ·02098	2—8 ·02298	1—6 ·01678	3—8 ·02448	3—8 ·01337	2—6 ·02489	3—7 ·01852	5—8 ·02489	4—7 ·02417	6—8 ·02472	3—6 ·01664	5—7 ·02137
10	1—9 ·01399	1—8 ·01499	2—9 ·01698	2—8 ·02448	3—9 ·01753	2—7 ·02297	3—8 ·01852	3—8 ·01151	3—7 ·02417	3—7 ·01703	6—9 ·01662	4—7 ·02443	3—6 ·02209
11	1—10 ·01099	1—9 ·01099	2—10 ·01282	2—9 ·01753	2—8 ·02489	3—9 ·02155	2—7 ·02489	4—9 ·02417	3—8 ·01499	2—6 ·02396	4—8 ·02068	4—8 ·01463	8—10 ·02426
12	1—10 ·02198	1—9 ·01923	1—8 ·02036	3—11 ·02171	2—9 ·01739	2—8 ·02489	3—9 ·02417	4—10 ·01703	6—11 ·02396	4—9 ·01956	7—11 ·02023	5—9 ·02359	4—8 ·01912
13	2—13 ·02500	2—12 ·02227	2—11 ·02171	2—10 ·02374	3—11 ·02232	2—9 ·01703	2—8 ·02472	2—8 ·01662	4—10 ·02068	2—7 ·02023	4—9 ·02359	4—9 ·01653	7—11 ·02006
14	2—14 ·02206	1—10 ·02288	2—12 ·01729	2—11 ·01806	2—10 ·02090	3—11 ·02198	4—12 ·01664	4—11 ·02443	4—11 ·01463	4—10 ·02359	5—11 ·01653	3—8 ·02304	5—10 ·01988
15	1—12 ·02451	1—11 ·01806	2—13 ·01393	2—12 ·01393	4—14 ·02073	2—10 ·01878	3—11 ·02137	5—13 ·02209	2—8 ·02426	5—12 ·01912	3—9 ·02006	5—11 ·01988	5—11 ·01342

TABLE 4.2.1-V

Distribution-free lower confidence bounds $y_{(j)} - x_{(i)}$, denoted by $j - i$, for shift (selection based on Normal shift alternatives) with confidence level $1 - \alpha = \cdot 95$ and the values of : 1-confidence coefficient

$$\Pr [y_{(j)} - x_{(i)} < \nu] \geqslant \cdot 95$$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3	1—3	1—3	1—3	2—3	2—3	2—3	3—3	3—3	3—3	4—3	4—3	4—3	5—3
	·05000	·02857	·01786	·04762	·03333	·02424	·04545	·03497	·02747	·04396	·03571	·02941	·04289
4	1—4	1—4	1—3	1—3	3—4	3—4	4—4	2—3	2—3	1—2	1—2	3—3	3—3
	·02857	·01429	·04762	·03333	·04545	·03030	·04895	·04096	·03297	·05000	·04412	·04412	·03741
5	1—5	2—5	1—4	3—5	2—4	4—5	2—4	3—4	3—4	6—5	4—4	4—4	4—4
	·01786	·04762	·02381	·04545	·04545	·04351	·02298	·04695	·03571	·04072	·04739	·03793	·03070
6	1—5	2—6	1—4	2—5	4—6	4—6	2—4	2—4	4—5	7—6	5—5	5—5	3—4
	·04762	·03333	·04545	·04004	·04895	·02797	·04695	·03571	·04299	·04977	·04954	·03793	·03070
7	1—6	1—5	2—6	1—4	4—7	2—5	2—5	4—6	2—4	5—6	5—6	4—5	4—5
	·03333	·04545	·04545	·04895	·03497	·03497	·02448	·03640	·04739	·03989	·02864	·04076	·03223
8	1—7	1—6	1—5	1—5	3—7	1—4	3—6	3—6	5—7	4—6	3—5	5—6	5—6
	·02424	·03030	·04351	·02797	·03497	·03846	·04447	·03065	·03711	·03989	·04076	·04799	·03668
9	1—7	1—6	2—8	3—8	3—8	3—7	3—7	2—5	3—6	6—8	5—7	4—6	6—7
	·04545	·04895	·02298	·04695	·02448	·04447	·02834	·04954	·03989	·03746	·04025	·04158	·04469
10	1—8	2—9	2—8	3—9	2—7	3—8	5—9	3—7	6—9	4—7	3—6	6—8	5—7
	·03497	·04096	·04695	·03571	·03640	·03065	·04954	·03489	·04257	·04561	·03668	·04025	·04158
11	1—9	2—10	2—9	2—8	4—10	2—7	4—9	2—6	4—8	7—10	5—8	4—7	9—10
	·02747	·03297	·03571	·04299	·04739	·03711	·03989	·04257	·04305	·04469	·04977	·04158	·04943
12	1—9	3—12	1—7	1—6	2—8	3—9	2—7	4—9	2—6	5—9	4—8	7—10	4—7
	·04396	·05000	·04072	·04977	·03989	·03989	·03746	·04561	·04469	·04977	·03581	·04202	·04928
13	1—10	3—13	2—10	2—9	2—9	4—11	3—9	5—11	4—9	5—10	7—11	5—9	8—11
	·03571	·04412	·04739	·04954	·02864	·04076	·04025	·03668	·04977	·03581	·04842	·04123	·04343
14	1—11	2—12	2—11	2—10	3—11	3—10	4—11	3—9	5—11	3—8	5—10	5—10	5—9
	·02941	·04412	·03793	·03793	·04076	·04799	·04158	·04025	·04158	·04202	·04123	·02849	·04756
15	1—11	2—13	2—12	3—13	3—12	3—11	3—10	4—11	2—7	6—12	3—8	6—11	5—10
	·04289	·03741	·03070	·03070	·03223	·03668	·04469	·04158	·04943	·04928	·04343	·04756	·03280

TABLE 4.2.1-VI

Distribution-free lower confidence bounds $\underline{y}_{(j)} - \underline{x}_{(i)}$, denoted by $j - i$, for shift (selection based on Normal shift alternatives) with confidence level $1 - \alpha = \cdot 90$ and the values of : 1-confidence coefficient

$$\Pr [\underline{y}_{(j)} - \underline{x}_{(i)} < \nu] \geqslant \cdot 90$$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3	1—3	1—3	2—3	1—2	3—3	3—3	4—3	4—3	5—3	2—2	2—2	6—3	6—3
	·05000	·02857	·07143	·08333	·08333	·06061	·09091	·06993	·09615	·08132	·07143	·08235	·06863
4	1—4	1—3	1—3	3—4	2—3	1—2	5—4	3—3	6—4	3—3	4—3	4—3	2—2
	·02857	·07143	·04762	·07143	·08788	·09091	·09790	·09491	·09231	·06319	·09874	·08333	·09701
5	1—4	2—5	1—3	2—4	4—5	3—4	2—3	2—3	4—4	4—4	3—3	3—3	6—4
	·07143	·04762	·08333	·06710	·07071	·08625	·09491	·07692	·07692	·05995	·09874	·08437	·09752
6	2—6	1—4	2—5	1—3	2—4	2—4	4—5	3—4	2—3	2—3	4—4	4—4	7—5
	·08333	·07143	·06710	·09091	·08625	·06294	·08392	·09191	·09874	·08333	·09469	·07766	·09365
7	1—5	2—6	1—4	3—6	1—3	4—6	3—5	5—6	4—5	3—4	3—4	5—5	5—5
	·08333	·08788	·07071	·08625	·09615	·08392	·07168	·08176	·08824	·09469	·07766	·08050	·06424
8	1—6	3—8	2—6	3—7	2—5	3—6	5—7	3—5	6—7	5—6	4—5	6—6	3—4
	·06061	·09091	·08625	·06294	·08392	·06597	·08824	·08824	·07990	·08490	·08996	·09133	·08141
9	1—6	1—5	3—8	2—6	3—7	2—5	3—6	6—8	4—6	3—5	6—7	5—6	4—5
	·09091	·09790	·09491	·08392	·07168	·08824	·07672	·09133	·09490	·08050	·08204	·08568	·09069
10	1—7	2—8	3—9	3—8	2—6	4—8	2—5	4—7	7—9	5—7	4—6	9—9	6—7
	·06993	·09491	·07692	·09191	·08176	·08824	·09133	·08945	·09365	·09919	·08568	·09668	·08200
11	1—7	1—6	2—8	4—10	3—8	2—6	4—8	2—5	4—7	8—10	7—9	4—6	4—6
	·09615	·09231	·07692	·09874	·08824	·07990	·09490	·09365	·09919	·09546	·08356	·09817	·07918
12	2—11	2—10	2—9	4—11	4—10	3—8	5—10	4—8	2—5	3—6	5—8	8—10	6—8
	·08132	·06319	·05995	·08333	·09469	·08490	·08050	·09919	·09546	·09651	·08118	·08467	·09064
13	2—12	2—10	3—11	3—10	4—11	4—10	3—8	5—10	3—7	5—9	2—5	5—8	9—11
	·07143	·09874	·09874	·09469	·07766	·08996	·08204	·08568	·08356	·08118	·08012	·09064	·08551
14	1—9	2—11	3—12	3—11	3—10	3—9	4—10	2—6	6—11	3—7	6—10	5—9	6—9
	·08235	·08333	·08437	·07766	·08050	·09133	·08568	·09668	·09817	·08467	·09064	·06417	·09739
15	1—10	3—14	2—10	2—9	3—11	5—13	5—12	4—10	6—12	5—10	3—7	6—10	6—10
	·06863	·09701	·09752	·09365	·06424	·08141	·09069	·08200	·07918	·09064	·08551	·09739	·07156

from Normal populations with equal variance. As lower confidence bound with confidence level ·95 (confidence coefficient ·965) one finds from table 4.2.1-V:

$$y_{(3)} - x_{(7)} = 1{\cdot}21.$$

Using the t-distribution one finds:

·92 with confidence coefficient ·95,
·84 with confidence coefficient ·965.

Computation for large m and n

Even for very small m and n the evaluation of the integral (4.2.1.4) is extremely tedious and the computation must be carried out carefully because the integrand is a complicated function and has often steep, exponential, features. So it is easy to understand that certainly for $m,n > 15$ the necessary computations carried out in the manner sketched above are nearly impossible, even with a big electronic computer. An easily applied Normal approximation is available for m,n large and i/m, j/n not near zero or unity. Under the alternative hypothesis, $\underline{y}^*_{(j)} - \underline{x}^*_{(i)}$ is approximately Normal distributed with expectation $\xi_2 - \xi_1$ and variance

$$2\pi \left[\frac{\delta_1 (1 - \delta_1) \exp \xi_1{}^2}{m} + \frac{\delta_2 (1 - \delta_2) \exp \xi_2{}^2}{n} \right], \qquad (4.2.1.5)$$

where ξ_1, ξ_2, δ_1 and δ_2 are defined by

$$i = [\delta_1 m] + 1 \quad \text{and} \quad \int_{-\infty}^{\xi_1} \mathrm{d}\Phi(t) = \delta_1,$$

$$j = [\delta_2 n] + 1 \quad \text{and} \quad \int_{-\infty}^{\xi_2} \mathrm{d}\Phi(t) = \delta_2,$$

where $\Phi(t)$ is the standard Normal cumulative distribution function (cf. ch. 2).

Thus under the above indicated conditions, evaluating $\beta(\delta)$ should not be difficult. In chapter 6 another possible asymptotical approach, indicated by Mood and Graybill (1963), will be given.

4.2.2. *Power against Lehmann alternatives and selected lower confidence bounds*

If, in general, one wishes to test the null hypothesis

$$H_0 : F(x) \equiv G(x) \qquad (4.2.2.1)$$

in the two-sample problem, where $F(x)$ and $G(y)$ are continuous distribution functions of the random variables $\underline{x}$ and $\underline{y}$, respectively, against the alternative hypothesis

$$H_1 : G(x) \not\equiv F(x), \qquad (4.2.2.2)$$

in particular against shift alternatives, there are various two-sample rank tests available. For instance, one can use the distribution-free two-sample tests of Wilcoxon (Mann–Whitney), Van der Waerden or Terry (Fisher–Yates).

It is of interest to quote Kendall and Stuart (1963, II, p. 472):
"The search for distribution-free procedures is motivated by the desire to broaden the range of validity of our inferences. We cannot expect to make great gains in generality without some loss of efficiency in particular circumstances; that is to say, we cannot expect a distribution-free test, chosen in ignorance of the form of the parent distribution, to be as efficient as the test we would have used had we known the parental form. But to use this as an argument against distribution-free procedures is manifestly mistaken: it is precisely the absence of information as to parental form which leads us to choose a distribution-free method. The only "fair" standard of efficiency for a distribution-free test is that provided by other distribution-free tests. We should naturally choose the most efficient such test available. But in what sense are we to judge efficiency? Even in the parametric case, UMP tests are rare, and we cannot hope to find distribution-free tests which are most powerful against all possible alternatives.

We are thus led to examine the power of distribution-free tests against parametric alternatives to the non-parametric hypothesis tested. Despite its paradoxical sound, there is nothing contradictory about this, and the procedure has one great practical virtue. If we examine power against the alternatives considered in *Normal* distribution theory, we obtain a measure of how much we can lose by using a distribution-free test if the assumptions of Normal theory really are valid (though, of course, we would not know this in practice). If this loss is small, we are encouraged to sacrifice the little extra efficiency of the standard Normal theory methods for the extended range of validity attached to the use of the distribution-free test.

We may take this comparison of Normal theory tests with distribution-free tests a stage further. In certain cases, it is possible to examine the relative efficiency of the two methods for a wide range of underlying parent distributions; and it should be particularly noted that we have no reason to expect the Normal theory method to maintain its efficiency advantages over the distribution-free method when the parent distribution is not truly Normal. In fact, we might hazard a guess that distribution-free methods should suffer less from the falsity of the Normality assumption than do the Normal theory methods which depend upon that assumption. Such few investigations as have been carried out seem on the whole to support this guess".

However, for most distribution functions the computation of the power functions of the tests mentioned before is a mathematically difficult problem. To simplify power computations for the two-sample rank tests Lehmann proposed to use the so-called Lehmann alternatives (see Lehmann (1953)).

More generally one can discern four types of Lehmann alternatives, namely of the form

(i) $G(y) \equiv F^k(y) \quad (k > 1),$ (4.2.2.3)
which, for small k, can be considered as a rough approximation for a shift to the right;

(ii) $G(y) \equiv 1 - \{1 - F(y)\}^k \quad (k > 1),$ (4.2.2.4)
which, for small k, can be considered as a rough approximation for a shift to the left;

(iii) $G(y) \equiv F^{1/k}(y) \quad (k > 1),$ (4.2.2.5)
which, for small k, can be considered as a rough approximation for a shift to the left,

(iv) $G(y) \equiv 1 - \{1 - F(y)\}^{1/k} \quad (k > 1),$ (4.2.2.6)
which, for small k, can be considered as a rough approximation for a shift to the right.

It is easy to see that in the cases (i) and (iv) the median is shifted to the right, whereas in the cases (ii) and (iii) the median is shifted to the left. Some remarks on the behaviour of moments using Lehmann alternatives will be made in chapter 5. Readers not familiar with Lehmann alternatives are advised to read chapter 5 first.

Of course these Lehmann alternatives do not come primarily to mind for practical purposes. Lehmann himself remarks: "We do not, of course, claim that these are the alternatives that actually prevail when the hypothesis is not true. Rather, it seems that where nonparametric methods are appropriate, one usually does not have a very precise knowledge of the alternatives". In practice one usually expresses the alternative hypothesis in location displacements. However, the Lehmann alternatives can serve as a rough approximation to location-displacement alternatives, at least for k not too far removed from 1. In practice one will usually try to choose values of k which correspond roughly to given translations.

Now the power function of the test V_{ji} will be derived for each case.

(i) $G(y) \equiv F^k(y)$

The probability density functions of the random variables $\underline{y}_{(j)}$ and $\underline{x}_{(i)}$ are under the alternative hypothesis $H_1 : G(y) = F^k(y)$ $(k > 1)$:

$$\frac{n!}{(j-1)!\,(n-j)!} F^{k(j-1)}(y_{(j)})\,\{1 - F^k(y_{(j)})\}^{n-j}\,\mathrm{d}F^k(y_{(j)}) \qquad (4.2.2.7)$$

and

$$\frac{m!}{(i-1)!\,(m-i)!} F^{i-1}(x_{(i)})\,\{1 - F(x_{(i)})\}^{m-i}\,\mathrm{d}F(x_{(i)}), \qquad (4.2.2.8)$$

respectively.

For the power function $\beta(k)$ as a function of k one can write

$$\beta(k) = \Pr\,[\underline{y}_{(j)} - \underline{x}_{(i)} > 0 \mid H_1]$$

$$= \frac{m!\,n!}{(i-1)!\,(m-i)!\,(j-1)!\,(n-j)!} \int_{-\infty}^{\infty} \int_{x(i)}^{\infty} F^{i-1}(x_{(i)})\,\{1 - F(x_{(i)})\}^{m-i} \times$$

$$\times F^{k(j-1)}(y_{(j)})\,\{1 - F^k(y_{(j)})\}^{n-j}\,\mathrm{d}F^k(y_{(j)})\,\mathrm{d}F(x_{(i)}).$$

With partial integration one finds

$$\int_{x(i)}^{\infty} F^{k(j-1)}(y_{(j)})\,\{1 - F^k(y_{(j)})\}^{n-j}\,\mathrm{d}F^k(y_{(j)})$$

$$= \int_{F^k(x_{(i)})}^{1} z^{j-1}\,(1-z)^{n-j}\,\mathrm{d}z$$

$$= \sum_{r_1=0}^{j-1} \frac{1}{j}\,\frac{j\,(j-1)\,(j-2)\ldots(r_1+1)}{(n-j+1)\,(n-j+2)\ldots(n-r_1)}\,F^{r_1 k}(x_{(i)})\,\{1 - F^k(x_{(i)})\}^{n-r_1}. \tag{4.2.2.9}$$

Using this result one gets for the power function:

$$\beta(k) = \frac{m!\,n!}{(i-1)!\,(m-i)!\,(j-1)!\,(n-j)!} \int_{-\infty}^{\infty} \frac{1}{j} \sum_{r_1=0}^{j-1} \frac{j\,(j-1)\,(j-2)\ldots(r_1+1)}{(n-j+1)\ldots(n-r_1)} \times$$

$$\times\, F^{i-1+r_1 k}(x_{(i)})\,\{1 - F(x_{(i)})\}^{m-i}\,\{1 - F^k(x_{(i)})\}^{n-r_1}\,\mathrm{d}F(x_{(i)})$$

$$= i\binom{m}{i} \sum_{r_1=0}^{j-1} \binom{n}{r_1} \int_0^1 z^{i-1+r_1 k}\,(1-z)^{m-i}\,(1-z^k)^{n-r_1}\,\mathrm{d}z$$

$$= i\binom{m}{i} \sum_{r_1=0}^{j-1} \binom{n}{r_1} \sum_{r_2=0}^{n-r_1} (-1)^{r_2} \binom{n-r_1}{r_2} \int_0^1 z^{i-1+(r_1+r_2)k}\,(1-z)^{m-i}\,\mathrm{d}z$$

$$= i\binom{m}{i} \sum_{r_1=0}^{j-1} \binom{n}{r_1} \sum_{r_2=0}^{n-r_1} (-1)^{r_2} \binom{n-r_1}{r_2} B(i+[r_1+r_2]k;\, m-i+1),$$

where $B(p,q)$ is the Beta-function with parameters p and q. One can write

$$\beta(k) = i\binom{m}{i} \sum_{r_1=0}^{j-1} \binom{n}{r_1} \sum_{r_2=0}^{n-r_1} (-1)^{r_2} \binom{n-r_1}{r_2} \times$$

$$\times \frac{(m-i)!}{\{(r_1+r_2)\,k+i\}\,\{(r_1+r_2)\,k+i+1\}\ldots\{(r_1+r_2)\,k+m\}}$$

$$= \frac{m!}{(i-1)!} \sum_{r_1=0}^{j-1} \binom{n}{r_1} \sum_{r_2=0}^{n-r_1} (-1)^{r_2} \binom{n-r_1}{r_2} \times$$

$$\times \ [\{(r_1+r_2)\,k+i\}\,\{(r_1+r_2)\,k+i+1\} \ldots \{(r_1+r_2)\,k+m\}]^{-1}. \quad (4.2.2.10)$$

To obtain some idea of the local efficiency of the tests V_{ji} one can determine the first derivative of the power function in the point $k = 1$. One finds

$$\frac{\mathrm{d}\beta(k)}{\mathrm{d}k} = \frac{m!}{(i-1)!} \sum_{r_1=0}^{j-1} \binom{n}{r_1} \sum_{r_2=0}^{n-r_1} (-1)^{r_2} \binom{n-r_1}{r_2} \times$$

$$\times \frac{-(\mathrm{d}/\mathrm{d}k)\,[\{(r_1+r_2)\,k+i\} \ldots \{(r_1+r_2)\,k+m\}]}{[\{(r_1+r_2)\,k+i\} \ldots \{(r_1+r_2)\,k+m\}]^2}$$

$$= \frac{m!}{(i-1)!} \sum_{r_1=0}^{j-1} \binom{n}{r_1} \sum_{r_2=0}^{n-r_1} (-1)^{r_2+1} \binom{n-r_1}{r_2} \times$$

$$\times \frac{(r_1+r_2)}{\{(r_1+r_2)\,k+i\} \ldots \{(r_1+r_2)\,k+m\}} \left[\frac{1}{(r_1+r_2)\,k+i} + \ldots + \frac{1}{(r_1+r_2)k+m}\right]$$

and thus

$$\left[\frac{\mathrm{d}\beta(k)}{\mathrm{d}k}\right]_{k=1} = \frac{m!}{(i-1)!} \sum_{r_1=0}^{j-1} \binom{n}{r_1} \sum_{r_2=0}^{n-r_1} (-1)^{r_2+1} \binom{n-r_1}{r_2} \times$$

$$\times \frac{r_1+r_2}{(i+r_1+r_2)\,(i+1+r_1+r_2) \ldots (m+r_1+r_2)} \left[\frac{1}{i+r_1+r_2} + \ldots + \frac{1}{m+r_1+r_2}\right]. \quad (4.2.2.11)$$

(ii) $G(y) \equiv 1 - \{1 - F(y)\}^k$

In this section the power of the tests V_{ji} against alternatives of the following form will be determined:

$$H_2 : G(y) = 1 - \{1 - F(y)\}^k \quad (k > 1). \quad (4.2.2.12)$$

The median of the distribution $G(y)$ is a (strictly) decreasing function of k. New computations are not necessary.

In fact we have for the power function of the test V_{ji} against H_2:

$$\begin{aligned}\beta(k) &= \Pr\,[\underline{y}_{(j)} - \underline{x}_{(i)} < 0 \mid G(y) = 1 - \{1 - F(y)\}^k] \\ &= \Pr\,[\underline{y}_{(j)} - \underline{x}_{(i)} < 0 \mid 1 - G(y) = \{1 - F(y)\}^k] \\ &= \Pr\,[(-\underline{y}_{(j)}) - (-\underline{x}_{(i)}) > 0 \mid 1 - G(y) = \{1 - F(y)\}^k].\end{aligned}$$

Defining $\underline{t} = -\underline{y}$ and $\underline{s} = -\underline{x}$ with the distribution functions $G^*(t)$ and $F^*(s)$ we get:

$$\Pr\,[\underline{t}_{(n-j+1)} - \underline{s}_{(m-i+1)} > 0 \mid G^*(t) = \{F^*(t)\}^k], \tag{4.2.2.13}$$

where $\underline{t}_{(n-j+1)}$ is the $(n-j+1)$th order statistic of a sample of size n from a population with distribution function $G^*(t)$ and $\underline{s}_{(m-i+1)}$ is the $(m-i+1)$th order statistic of a sample of size m from a population with distribution function $F^*(s)$. Notice that

$$\begin{aligned} G^*(t) &= \Pr\,[\underline{t} < t] = \Pr\,[-\underline{y} < t] \\ &= \Pr\,[\underline{y} > -t] = 1 - G(-t) \\ &= \{1 - F(-t)\}^k = \{\Pr\,[\underline{s} < t]\}^k = \{F^*(t)\}^k. \end{aligned}$$

Hence a simple transformation of the subscripts of the selected tests in the case that the alternative hypotheses have the form $H_1 : G(y) = F^k(y)$ $(k > 1)$ supplies the "optimal" test in the case that the alternative hypotheses are of type H_2.

(iii) $G(y) \equiv F^{1/k}(y)$

For the power function of the test against alternatives of the form

$$H_3 : G(y) = F^{1/k}(y) \qquad (k > 1) \tag{4.2.2.14}$$

we have

$$\begin{aligned} \beta(k) &= \Pr\,[\underline{V}_{ji} < 0 \mid G(y) = F^{1/k}(y)] \\ &= \Pr\,[\underline{y}_{(j)} - \underline{x}_{(i)} < 0 \mid G(y) = F^{1/k}(y)] \\ &= \Pr\,[\underline{x}_{(i)} - \underline{y}_{(j)} > 0 \mid F(y) = G^k(y)]. \end{aligned} \tag{4.2.2.15}$$

So it is easy to see that the power function can be derived from formula (4.2.2.10) and that if $V_{j'i'}$ for sample sizes m' and n' is the selected test in the case of the alternatives $G(y) = F^k(y)$, then V_{ji} with $j = i'$ and $i = j'$ is the selected test in the case of the alternatives $G(y) = F^{1/k}(y)$ for sample sizes $m = n'$ and $n = m'$.

(iv) $G(y) \equiv 1 - \{1 - F(y)\}^{1/k}$

For the power function of the test against alternatives of the form

$$H_4 : G(y) = 1 - \{1 - F(y)\}^{1/k} \qquad (k > 1) \tag{4.2.2.16}$$

we have

$$\begin{aligned} \beta(k) &= \Pr\,[\underline{y}_{(j)} - \underline{x}_{(i)} > 0 \mid G(y) = 1 - \{1 - F(y)\}^{1/k}] \\ &= \Pr\,[\underline{x}_{(i)} - \underline{y}_{(j)} < 0 \mid F(y) = 1 - \{1 - G(y)\}^k]. \end{aligned} \tag{4.2.2.17}$$

So the power function can be derived from the power against H_2. In a similar way as in case (iii), one sees that if $V_{j'i'}$ for sample sizes m' and n' is the selected test in the case of alternatives $G(y) = 1 - \{1 - F(y)\}^k$, then V_{ji} with $j = i'$

TABLE 4.2.2-I

Distribution-free lower confidence bounds $\underline{y}_{(j)} - \underline{x}_{(i)}$, denoted by $j - i$, for shift (selection based on Lehmann alternatives: $G(x) = F^k(x)$) with confidence level $1 - \alpha = \cdot 999$ and the values of : 1-confidence coefficient

$$\Pr [\underline{y}_{(j)} - \underline{x}_{(i)} < \nu] \geqslant \cdot 999$$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3													
4								1—4 ·00100	1—4 ·00073	1—4 ·00055	1—4 ·00042	1—4 ·00033	1—4 ·00026
5						1—5 ·00078	1—5 ·00050	1—5 ·00033	1—5 ·00023	2—5 ·00097	2—5 ·00070	2—5 ·00052	2—5 ·00039
6					1—6 ·00058	1—6 ·00033	1—6 ·00020	2—6 ·00087	1—5 ·00097	1—5 ·00070	1—5 ·00052	1—5 ·00039	1—5 ·00029
7				1—7 ·00058	1—7 ·00029	1—7 ·00016	1—6 ·00087	1—6 ·00057	1—6 ·00038	1—6 ·00026	1—6 ·00018	2—6 ·00085	1—5 ·00080
8			1—8 ·00078	1—8 ·00033	1—8 ·00016	1—7 ·00070	1—7 ·00041	1—7 ·00025	1—7 ·00016	1—6 ·00072	1—6 ·00052	1—6 ·00038	1—6 ·00028
9			1—9 ·00050	1—9 ·00020	1—8 ·00070	1—8 ·00037	1—8 ·00021	1—7 ·00071	1—7 ·00046	1—7 ·00031	1—7 ·00021	1—6 ·00083	1—6 ·00062
10		1—10 ·00100	1—10 ·00033	1—9 ·00087	1—9 ·00041	1—9 ·00021	1—8 ·00060	1—8 ·00036	1—8 ·00022	1—7 ·00070	1—7 ·00049	1—7 ·00035	1—7 ·00025
11		1—11 ·00073	1—11 ·00023	1—10 ·00057	1—10 ·00025	1—9 ·00060	1—9 ·00033	1—8 ·00081	2—9 ·00095	1—8 ·00034	1—7 ·00095	1—7 ·00069	2—8 ·00081
12		1—12 ·00055	1—11 ·00097	1—11 ·00038	1—10 ·00071	1—10 ·00036	1—9 ·00075	2—10 ·00095	2—10 ·00055	1—8 ·00067	2—9 ·00090	2—9 ·00059	1—7 ·00089
13		1—13 ·00042	1—12 ·00070	1—12 ·00026	1—11 ·00046	1—10 ·00081	1—10 ·00044	1—9 ·00087	1—9 ·00055	2—10 ·00079	1—8 ·00082	1—8 ·00058	2—9 ·00084
14		1—14 ·00033	1—13 ·00052	1—12 ·00072	1—12 ·00031	1—11 ·00052	1—10 ·00087	1—10 ·00051	1—9 ·00098	1—9 ·00064	1—9 ·00043	1—8 ·00097	1—8 ·00070
15		1—15 ·00026	1—14 ·00039	1—13 ·00052	1—12 ·00070	1—12 ·00034	1—11 ·00055	1—10 ·00092	1—10 ·00057	2—11 ·00097	1—9 ·00072	1—9 ·00050	2—10 ·00085

TABLE 4.2.2-II

Distribution-free lower confidence bounds $\underline{y}_{(j)} - \underline{x}_{(i)}$, denoted by $j - i$, for shift (selection based on Lehmann alternatives: $G(x) = F^k(x)$) with confidence level $1 - \alpha = {\cdot}995$ and the values of : 1-confidence coefficient

$$\Pr[\underline{y}_{(j)} - \underline{x}_{(i)} < \nu] \geqslant {\cdot}995$$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3							1—3 ·00455	1—3 ·00350	1—3 ·00275	1—3 ·00220	1—3 ·00179	1—3 ·00147	2—3 ·00490
4				1—4 ·00476	1—4 ·00303	1—4 ·00202	1—4 ·00140	2—4 ·00500	2—4 ·00366	2—4 ·00275	2—4 ·00210	1—3 ·00490	1—3 ·00413
5			1—5 ·00397	1—5 ·00216	1—5 ·00126	2—5 ·00466	1—4 ·00500	1—4 ·00366	1—4 ·00275	1—4 ·00210	1—4 ·00163	1—4 ·00129	2—4 ·00490
6		1—6 ·00476	1—6 ·00216	1—6 ·00108	1—5 ·00466	1—5 ·00300	1—5 ·00200	1—5 ·00137	1—5 ·00097	1—4 ·00490	1—4 ·00387	1—4 ·00310	1—4 ·00251
7		1—7 ·00303	1—7 ·00126	1—6 ·00408	1—6 ·00233	1—6 ·00140	1—5 ·00481	1—5 ·00339	1—5 ·00245	1—5 ·00181	3—6 ·00444	3—6 ·00320	1—4 ·00478
8		1—8 ·00202	1—7 ·00466	1—7 ·00233	1—7 ·00124	1—6 ·00350	1—6 ·00226	1—6 ·00151	1—5 ·00482	1—5 ·00361	1—5 ·00275	1—5 ·00213	1—5 ·00166
9	1—9 ·00455	1—9 ·00140	1—8 ·00300	1—8 ·00140	1—7 ·00315	1—7 ·00185	1—6 ·00452	2—7 ·00488	1—6 ·00217	1—6 ·00155	1—5 ·00478	2—6 ·00494	2—6 ·00374
10	1—10 ·00350	1—9 ·00500	1—9 ·00200	1—8 ·00350	1—8 ·00185	1—7 ·00377	2—8 ·00449	2—8 ·00274	1—6 ·00387	2—7 ·00464	2—7 ·00327	2—7 ·00235	1—5 ·00474
11	1—11 ·00275	1—10 ·00366	1—9 ·00481	1—9 ·00226	1—8 ·00377	2—9 ·00488	1—7 ·00426	1—7 ·00284	2—8 ·00376	1—6 ·00458	1—6 ·00343	2—7 ·00442	2—7 ·00327
12	1—12 ·00220	1—11 ·00275	1—10 ·00339	1—9 ·00452	1—9 ·00238	1—8 ·00393	1—8 ·00243	1—7 ·00464	1—7 ·00323	2—8 ·00471	2—8 ·00326	1—6 ·00401	1—6 ·00312
13	1—13 ·00179	1—12 ·00210	1—11 ·00245	1—10 ·00310	1—9 ·00426	1—9 ·00243	1—8 ·00402	1—8 ·00262	1—7 ·00496	1—7 ·00357	1—7 ·00261	2—8 ·00400	1—6 ·00455
14	1—14 ·00147	1—12 ·00490	1—11 ·00482	1—11 ·00217	1—10 ·00284	1—9 ·00402	1—9 ·00245	1—8 ·00408	1—8 ·00278	2—9 ·00471	1—7 ·00386	1—7 ·00290	2—8 ·00470
15	1—14 ·00490	1—13 ·00387	1—12 ·00361	1—11 ·00387	1—10 ·00464	1—10 ·00262	1—9 ·00383	1—9 ·00245	1—8 ·00412	1—8 ·00290	1—8 ·00207	1—7 ·00412	1—7 ·00316

TABLE 4.2.2-III

Distribution-free lower confidence bounds $\underline{y}_{(j)} - \underline{x}_{(i)}$, denoted by $j - i$, for shift (selection based on Lehmann alternatives: $G(x) = F^k(x)$) with confidence level $1 - \alpha = \cdot 99$ and the values of : 1-confidence coefficient

$$\Pr\,[\underline{y}_{(j)} - \underline{x}_{(i)} < v] \geqslant \cdot 99$$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3					1—3 ·00833	1—3 ·00606	1—3 ·00455	1—3 ·00350	1—3 ·00275	2—3 ·00879	2—3 ·00714	2—3 ·00588	2—3 ·00490
4			1—4 ·00794	1—4 ·00476	1—4 ·00303	1—4 ·00202	2—4 ·00699	2—4 ·00500	1—3 ·00879	1—3 ·00714	1—3 ·00588	1—3 ·00490	1—3 ·00413
5		1—5 ·00794	1—5 ·00397	1—5 ·00216	2—5 ·00758	1—4 ·00699	1—4 ·00500	1—4 ·00366	1—4 ·00275	2—4 ·00986	2—4 ·00770	2—4 ·00611	1—3 ·00877
6		1—6 ·00476	1—6 ·00216	1—5 ·00758	1—5 ·00466	1—5 ·00300	1—5 ·00200	1—4 ·00824	1—4 ·00630	1—4 ·00490	3—5 ·00955	3—5 ·00722	3—5 ·00555
7	1—7 ·00833	1—7 ·00303	1—6 ·00758	1—6 ·00408	1—6 ·00233	1—5 ·00699	1—5 ·00481	1—5 ·00339	3—6 ·00905	1—4 ·00903	1—4 ·00722	1—4 ·00585	1—4 ·00478
8	1—8 ·00606	1—8 ·00202	1—7 ·00466	1—6 ·00932	1—6 ·00559	1—6 ·00350	1—5 ·00905	1—5 ·00654	1—5 ·00482	1—5 ·00361	1—5 ·00275	1—4 ·00957	1—4 ·00791
9	1—9 ·00455	1—8 ·00699	1—8 ·00300	1—7 ·00559	1—7 ·00315	1—6 ·00679	2—7 ·00761	2—7 ·00488	1—5 ·00813	2—6 ·00898	2—6 ·00661	2—6 ·00494	4—7 ·00884
10	1—10 ·00350	1—9 ·00500	1—8 ·00699	1—8 ·00350	1—7 ·00617	2—8 ·00761	1—6 ·00774	2—7 ·00988	2—7 ·00671	1—5 ·00957	1—5 ·00749	2—6 ·00884	2—6 ·00680
11	1—11 ·00275	1—10 ·00366	1—9 ·00481	1—8 ·00679	2—9 ·00905	1—7 ·00655	2—8 ·00917	1—6 ·00851	1—6 ·00619	2—7 ·00841	2—7 ·00606	1—5 ·00870	1—5 ·00702
12	1—11 ·00879	1—10 ·00824	1—9 ·00905	1—9 ·00452	1—8 ·00655	2—9 ·00988	1—7 ·00681	1—7 ·00464	1—6 ·00915	1—6 ·00686	2—7 ·00998	2—7 ·00741	1—5 ·00981
13	1—12 ·00714	1—11 ·00630	1—10 ·00654	1—9 ·00774	1—9 ·00426	1—8 ·00632	1—8 ·00402	1—7 ·00700	1—7 ·00496	1—6 ·00969	1—6 ·00745	1—6 ·00580	2—7 ·00870
14	1—13 ·00588	1—12 ·00490	1—11 ·00482	1—10 ·00542	1—9 ·00681	1—8 ·00939	1—8 ·00612	1—7 ·00992	1—7 ·00714	1—7 ·00522	2—8 ·00876	1—6 ·00797	1—6 ·00632
15	1—14 ·00490	1—12 ·00903	1—11 ·00813	1—10 ·00851	1—10 ·00464	1—9 ·00612	1—8 ·00875	1—8 ·00595	1—7 ·00978	1—7 ·00725	3—10 ·00830	2—8 ·00950	1—6 ·00843

TABLE 4.2.2-IV

Distribution-free lower confidence bounds $\underline{y}_{(j)} - \underline{x}_{(i)}$, denoted by $j-i$, for shift (selection based on Lehmann alternatives: $G(x) = F^k(x)$) with confidence level $1 - \alpha = \cdot 975$ and the values of : 1-confidence coefficient

$$\Pr\left[\underline{y}_{(j)} - \underline{x}_{(i)} < v\right] \geqslant \cdot 975$$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3			1—3	1—3	1—3	2—3	2—3	2—3	2—3	3—3	1—2	1—2	1—2
			·01786	·01190	·00833	·02424	·01818	·01399	·01099	·02198	·02500	·02206	·01961
4		1—4	1—4	2—4	1—3	1—3	1—3	1—3	1—3	1—3	2—3	2—3	2—3
		·01429	·00794	·02381	·02424	·01818	·01399	·01099	·00879	·00714	·02227	·01863	·01574
5	1—5	1—5	1—4	1—4	1—4	1—4	2—4	1—3	1—3	1—3	3—4	1—3	1—3
	·01786	·00794	·02381	·01515	·01010	·00699	·02298	·02198	·01786	·01471	·02171	·01032	·00877
6	1—6	1—5	1—5	1—5	1—4	1—4	1—4	3—5	3—5	1—3	1—3	1—3	1—3
	·01190	·02381	·01299	·00758	·02098	·01499	·01099	·02448	·01753	·02451	·02064	·01754	·01504
7	1—7	1—6	1—6	1—5	1—5	1—5	1—4	1—4	1—4	1—4	3—5	3—5	1—3
	·00833	·01515	·00758	·01632	·01049	·00699	·01923	·01471	·01144	·00903	·02232	·01729	·02273
8	1—7	1—7	1—6	1—6	1—5	2—6	2—6	1—4	2—5	2—5	2—5	4—6	4—6
	·02424	·01010	·01632	·00932	·01865	·02028	·01337	·02288	·02374	·01806	·01393	·02198	·01664
9	1—8	1—7	2—8	1—6	2—7	1—5	2—6	2—6	2—6	1—4	2—5	2—5	2—5
	·01818	·02098	·02298	·01678	·02028	·02036	·02489	·01739	·01238	·02105	·02308	·01831	·01467
10	1—9	1—8	1—7	2—8	1—6	2—7	1—5	1—5	2—6	2—6	1—4	1—4	2—5
	·01399	·01499	·01865	·02448	·01697	·02297	·02167	·01625	·02090	·01548	·02372	·01976	·02253
11	1—10	1—9	1—8	1—7	1—6	1—6	2—7	1—5	1—5	2—6	2—6	2—6	1—4
	·01099	·01099	·01282	·01697	·02489	·01703	·02489	·02270	·01754	·02396	·01831	·01414	·02207
12	1—10	1—9	1—8	1—7	1—7	1—6	1—6	1—6	1—5	1—5	1—5	2—6	3—7
	·02198	·01923	·02036	·02489	·01572	·02384	·01703	·01238	·02354	·01863	·01491	·02087	·01937
13	1—11	1—10	1—9	1—8	1—7	1—7	1—6	1—6	1—6	1—5	1—5	1—5	2—6
	·01786	·01471	·01471	·01703	·02214	·01476	·02300	·01700	·01275	·02422	·01957	·01594	·02319
14	1—12	1—10	1—9	1—8	2—10	1—7	1—7	2—8	1—6	3—9	1—5	3—8	1—5
	·01471	·02288	·02167	·02384	·02090	·02012	·01400	·02450	·01696	·01870	·02480	·02304	·01686
15	1—12	1—11	1—10	1—9	1—8	2—10	2—9	2—9	2—8	2—8	3—9	2—7	1—5
	·02451	·01806	·01625	·01703	·02012	·01878	·02450	·01623	·02426	·01754	·02006	·02249	·02107

TABLE 4.2.2-V

Distribution-free lower confidence bounds $\underline{y}_{(j)} - \underline{x}_{(i)}$, denoted by $j - i$, for shift (selection based on Lehmann alternatives: $G(x) = F^k(x)$) with confidence level $1 - \alpha = \cdot 95$ and the values of : 1-confidence coefficient

$$\Pr\ [\underline{y}_{(j)} - \underline{x}_{(i)} < \nu] \geqslant \cdot 95$$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3	1—3 ·05000	1—3 ·02857	1—3 ·01786	2—3 ·04762	2—3 ·03333	2—3 ·02424	1—2 ·04545	1—2 ·03846	1—2 ·03297	1—2 ·02857	1—2 ·02500	1—2 ·02206	1—2 ·01961
4	1—4 ·02857	1—4 ·01429	1—3 ·04762	1—3 ·03333	1—3 ·02424	1—3 ·01818	1—3 ·01399	2—3 ·04096	2—3 ·03297	1—2 ·05000	1—2 ·04412	1—2 ·03922	1—2 ·03509
5	1—5 ·01786	1—4 ·03968	1—4 ·02381	1—4 ·01515	1—3 ·04545	1—3 ·03497	1—3 ·02747	3—4 ·04695	3—4 ·03571	3—4 ·02763	2—3 ·04412	2—3 ·03741	2—3 ·03199
6	1—5 ·04762	1—5 ·02381	1—4 ·04545	1—4 ·03030	1—4 ·02098	1—4 ·01499	1—3 ·04396	1—3 ·03571	1—3 ·02941	1—3 ·02451	3—4 ·04603	3—4 ·03741	3—4 ·03070
7	1—6 ·03333	1—5 ·04545	2—6 ·04545	1—4 ·04895	1—4 ·03497	2—5 ·03497	1—4 ·01923	1—4 ·01471	2—4 ·04739	1—3 ·03612	1—3 ·03070	4—5 ·04076	1—3 ·02273
8	1—7 ·02424	1—6 ·03030	1—5 ·04351	1—5 ·02797	2—6 ·03170	1—4 ·03846	2—5 ·04299	2—5 ·03167	2—5 ·02374	1—3 ·04912	2—4 ·04747	2—4 ·03934	2—4 ·03287
9	1—7 ·04545	1—6 ·04895	1—6 ·02797	1—5 ·04196	1—5 ·02885	2—6 ·03640	1—4 ·04118	2—5 ·04954	2—5 ·03793	2—5 ·02941	2—5 ·02308	1—3 ·04743	2—4 ·04743
10	1—8 ·03497	1—7 ·03497	1—6 ·04196	1—6 ·02622	1—5 ·04072	1—5 ·02941	2—6 ·03989	1—4 ·04334	1—4 ·03509	2—5 ·04334	2—5 ·03453	2—5 ·02777	5—7 ·04158
11	1—9 ·02747	1—8 ·02564	1—7 ·02885	1—6 ·03733	1—6 ·02489	1—5 ·03973	1—5 ·02980	2—6 ·04257	1—4 ·04511	1—4 ·03727	2—5 ·04805	2—5 ·03913	2—5 ·03211
12	1—9 ·04396	1—8 ·03846	1—7 ·04072	1—6 ·04977	1—6 ·03406	1—6 ·02384	1—5 ·03892	1—5 ·03008	2—6 ·04469	1—4 ·04658	1—4 ·03913	1—4 ·03311	3—6 ·04928
13	1—10 ·03571	1—9 ·02941	2—10 ·04739	2—9 ·04954	1—6 ·04427	2—1 ·03746	1—5 ·04887	1—5 ·03825	3—8 ·04025	2—6 ·04641	1—4 ·04783	1—4 ·04074	1—4 ·03492
14	1—11 ·02941	1—9 ·04118	1—8 ·03973	1—7 ·04427	2—9 ·04257	1—6 ·04025	2—8 ·03552	1—5 ·04710	2—7 ·03768	3—8 ·04202	2—6 ·04783	1—4 ·04889	3—7 ·04068
15	1—11 ·04289	1—10 ·03251	1—9 ·02980	1—8 ·03162	1—7 ·03773	1—6 ·04958	2—8 ·04812	2—8 ·03395	2—7 ·04943	2—7 ·03768	3—8 ·04343	2—6 ·04902	1—4 ·04981

TABLE 4.2.2-VI

Distribution-free lower confidence bounds $\underline{y}_{(j)} - \underline{x}_{(i)}$, denoted by $j - i$, for shift (selection based on Lehmann alternatives: $G(x) = F^k(x)$) with confidence level $1 - \alpha = \cdot 90$ and the values of : 1-confidence coefficient

$$\Pr\,[\underline{y}_{(j)} - \underline{x}_{(i)} < \nu] \geqslant \cdot 90$$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3	1—3	1—3	2—3	1—2	1—2	1—2	1—2	1—2	2—2	2—2	2—2	2—2	2—2
	·05000	·02857	·07143	·08333	·06667	·05455	·04545	·03846	·09341	·08132	·07143	·06324	·05637
4	1—4	1—3	1—3	1—3	2—3	1—2	1—2	3—3	1—2	1—2	4—3	4—3	2—2
	·02857	·07143	·04762	·03333	·08788	·09091	·07692	·09491	·05714	·05000	·09874	·08333	·09701
5	1—4	1—4	1—3	1—3	1—3	3—4	2—3	1—2	1—2	1—2	3—3	3—3	3—3
	·07143	·03968	·08333	·06061	·04545	·08625	·09491	·09524	·08333	·07353	·09874	·08437	·07263
6	1—5	1—4	1—4	1—3	2—4	1—3	1—3	3—4	2—3	1—2	1—2	1—2	1—2
	·04762	·07143	·04545	·09091	·08625	·05495	·04396	·09191	·09874	·09804	·08772	·07895	·07143
7	1—5	2—6	1—4	2—5	1—3	1—3	2—4	2—4	4—5	3—4	3—4	2—3	1—2
	·08333	·08788	·07071	·07751	·09615	·07692	·07692	·05995	·08824	·09469	·07766	·08772	·09091
8	1—6	1—5	1—4	1—4	2—5	2—5	1—3	2—4	2—4	2—4	4—5	3—4	3—4
	·06061	·07071	·09790	·06993	·08392	·05944	·08235	·08824	·07104	·05779	·08996	·09626	·08141
9	1—6	1—5	1—5	1—4	1—4	2—5	2—5	1—3	2—4	2—4	2—4	5—6	4—5
	·09091	·09790	·06294	·09231	·06923	·08824	·06561	·08669	·09752	·08050	·06699	·08568	·09069
10	1—7	1—6	1—5	1—5	1—4	1—4	2—5	3—6	1—3	1—3	2—4	2—4	2—4
	·06993	·06993	·08392	·05769	·08824	·06863	·09133	·08490	·09023	·07792	·08862	·07510	·06403
11	1—7	1—6	1—6	1—5	3—8	1—4	1—4	2—5	3—6	1—3	1—3	2—4	2—4
	·09615	·09231	·05769	·07466	·08824	·08514	·06811	·09365	·09133	·09317	·08152	·09565	·08227
12	1—8	1—7	1—6	1—5	2—7	3—8	1—4	4—8	2—5	3—6	1—3	1—3	1—3
	·07692	·06923	·07466	·09244	·07990	·08490	·08271	·09919	·09546	·09651	·09565	·08462	·07521
13	1—9	1—7	1—6	2—8	1—5	2—7	1—4	1—4	3—7	2—5	2—5	1—3	1—3
	·06250	·08824	·09244	·09133	·08301	·07379	·09774	·08075	·08356	·09689	·08012	·09778	·08730
14	1—9	1—8	1—7	1—6	1—5	2—7	2—7	2—6	1—4	3—7	2—5	2—5	1—3
	·08235	·06863	·06811	·07748	·09838	·09525	·06912	·09668	·07913	·08467	·09807	·08237	·09962
15	1—10	1—8	1—7	1—6	1—6	1—5	2—7	2—7	2—6	2—6	3—7	2—5	2—5
	·06863	·08514	·08301	·09223	·06708	·08924	·08749	·06545	·09348	·07488	·08551	·09904	·08429

and $i = j'$ is the selected test for sample sizes $m = n'$ and $n = m'$ in the case of alternatives $G(y) = 1 - \{1 - F(y)\}^{1/k}$.

According to the selection procedure of sec. 4.2 one-sided tests to the right and thus lower confidence bounds have been selected. The selected lower confidence bounds are given in the tables 4.2.2-I–4.2.2-VI.

In order to get a better idea of the criterion of selection for the tests in the case of Lehmann alternatives $G(y) \equiv F^k(y)$ we have counted how many of the selected tests have at the same time maximum size (α max.), maximum slope in $k = 1$ ($\beta'(1)$ max.), and (or) maximum power in the point where the test with maximal size has a power of ·50 ($\beta_{ms.,\cdot 50}$ max.), respectively. The results are presented in table 4.2.2-VII.

TABLE 4.2.2-VII

Comparison of selection criteria

	$\beta_{ms.,\cdot 50}$ max.		$\beta_{ms.,\cdot 50}$ not max.		
	$\beta'(1)$ max.	$\beta'(1)$ not max.	$\beta'(1)$ max.	$\beta'(1)$ not max.	
α max.	500	2	0	0	502
α not max.	174	260	5	6	445
	674	262	5	6	947
	936		11		

In percentages:

	$\beta_{ms.,\cdot 50}$ max.		$\beta_{ms.,\cdot 50}$ not max.		
	$\beta'(1)$ max.	$\beta'(1)$ not max.	$\beta'(1)$ max.	$\beta'(1)$ not max.	
α max.	52·8 %	·2 %	0 %	0 %	53 %
α not max.	18·4 %	27·5 %	·5 %	·6 %	47 %
	71 %	28 %	·5 %	·6 %	100 %
	99 %		1 %		

From the results it can be concluded that in an appreciable number of cases the criterion "$\beta_{ms.,\cdot 50}$ maximal" leads to the same test as the selection criterion used.

As an illustration the derivation of the power function against Lehmann alternatives for tests from V_2 will be given. This will make clear that, even for Lehmann alternatives, computations for cases where $R > 1$ become rather tedious. We shall determine the power function for the test rejecting H_0 when

$$\min \{(\underline{y}_{(j_1)} - \underline{x}_{(i_1)}), (\underline{y}_{(j_2)} - \underline{x}_{(i_2)})\}$$

is larger than zero ($i_1 < i_2$, $j_1 < j_2$). Then we have

$$\beta(k) = \Pr [\min \{\underline{y}_{(j_1)} - \underline{x}_{(i_1)}, \underline{y}_{(j_2)} - \underline{x}_{(i_2)}\} > 0 \mid G = F^k]$$

$$= \Pr [\underline{y}_{(j_1)} > \underline{x}_{(i_1)} \wedge \underline{y}_{(j_2)} > \underline{x}_{(i_2)} \mid G = F^k]$$

$$= \frac{m!}{(i_1-1)!\,(i_2-i_1-1)!\,(m-i_2)!} \frac{n!}{(j_1-1)!\,(j_2-j_1-1)!\,(n-j_2)!} \times$$

$$\times \underset{\substack{x(i_1) < x(i_2) \\ y(j_1) < y(j_2) \\ x(i_1) < y(j_1) \\ x(i_2) < y(j_2)}}{\int\int\int\int} F^{i_1-1}(x_{(i_1)}) \{F(x_{(i_2)}) - F(x_{(i_1)})\}^{i_2-i_1-1} \{1 - F(x_{(i_2)})\}^{m-i_2} \times$$

$$\times F^{k(j_1-1)}(y_{(j_1)}) \{F^k(y_{(j_2)}) - F^k(y_{(j_1)})\}^{j_2-j_1-1} \{1 - F^k(y_{(j_2)})\}^{n-j_2} \times$$

$$\times \mathrm{d}F(x_{(i_1)})\, \mathrm{d}F(x_{(i_2)})\, \mathrm{d}F^k(y_{(j_1)})\, \mathrm{d}F^k(y_{(j_2)})$$

$$= \frac{m!}{(i_1-1)!\,(i_2-i_1-1)!\,(m-i_2)!} \frac{n!}{(j_1-1)!\,(j_2-j_1-1)!\,(n-j_2)!} \times$$

$$\times \int\limits_{t=0}^{1} \int\limits_{s=0}^{t} \int\limits_{v=t^k}^{1} \int\limits_{u=s^k}^{v} s^{i_1-1} (t-s)^{i_2-i_1-1} \times$$

$$\times (1-t)^{m-i_2}\, u^{j_1-1}\, (v-u)^{j_2-j_1-1}\, (1-v)^{n-j_2}\, \mathrm{d}u\, \mathrm{d}v\, \mathrm{d}s\, \mathrm{d}t.$$

Now we have, using partial integration:

$$\frac{1}{(j_1-1)!\,(j_2-j_1-1)!} \int\limits_{s^k}^{v} u^{j_1-1} (v-u)^{j_2-j_1-1}\, \mathrm{d}u$$

$$= \sum_{r_1=0}^{j_1-1} \frac{s^{k(j_1-1-r_1)} (v-s^k)^{j_2-j_1+r_1}}{(j_1-1-r_1)!\,(j_2-j_1+r_1)!},$$

$$\frac{1}{(n-j_2)!\,(j_2-j_1+r_1)!} \int\limits_{t^k}^{1} (v-s^k)^{j_2-j_1+r_1} (1-v)^{n-j_2}\, \mathrm{d}v$$

$$= \sum_{r_2=0}^{j_2-j_1+r_1} \frac{(1-t^k)^{n-j_2+1+r_2} (t^k-s^k)^{j_2-j_1+r_1-r_2}}{(n-j_2+1+r_2)!\,(j_2-j_1+r_1-r_2)!}.$$

Further we have

$$\int_0^t s^{i_1-1+k(j_1-1-r_1)}\,(t-s)^{i_2-i_1-1}\,(t^k-s^k)^{j_2-j_1+r_1-r_2}\,ds$$

$$= t^{i_2+k(j_2-r_2-1)-1}\int_0^1 s^{i_1-1+k(j_1-1-r_1)}\,(1-s)^{i_2-i_1-1}\,(1-s^k)^{j_2-j_1+r_1-r_2}\,ds$$

$$= t^{i_2+k(j_2-r_2-1)-1}\sum_{r_3=0}^{j_2-j_1+r_1-r_2}(-1)^{r_3}\binom{j_2-j_1+r_1-r_2}{r_3}\int_0^1 s^{i_1-1+k(j_1-1-r_1)+kr_3}\times \times\,(1-s)^{i_2-i_1-1}\,ds$$

$$= t^{i_2+k(j_2-r_2-1)-1}\sum_{r_3=0}^{j_2-j_1+r_1-r_2}(-1)^{r_3}\binom{j_2-j_1+r_1-r_2}{r_3}\times \times\, B(i_1+k\,[j_1-1-r_1+r_3],\,i_2-i_1)$$

and finally:

$$\int_0^1 t^{i_2+k(j_2-r_2-1)-1}\,(1-t^k)^{n-j_2+1+r_2}\,(1-t)^{m-i_2}\,dt$$

$$= \sum_{r_4=0}^{n-j_2+1+r_2}(-1)^{r_4}\binom{n-j_2+1+r_2}{r_4}B(i_2+k\,[j_2-r_2-1]+kr_4,\,m-i_2+1).$$

Thus
$$\beta(k)=\frac{m!\,n!}{(i_1-1)!\,(i_2-i_1-1)!\,(m-i_2)!}\sum_{r_1=0}^{j_1-1}\ \sum_{r_2=0}^{j_2-j_1+r_1}\ \sum_{r_3=0}^{j_2-j_1+r_1-r_2}\ \sum_{r_4=0}^{n-j_2+1+r_2}$$

$$(-1)^{r_3+r_4}\frac{\binom{j_2-j_1+r_1-r_2}{r_3}\binom{n-j_2+1+r_2}{r_4}}{(j_1-1-r_1)!\,(n-j_2+1+r_2)!\,(j_2-j_1+r_1-r_2)!}\times$$

$$\times\, B(i_1+k\,[j_1-1-r_1+r_3],\,i_2-i_1)\,B(i_2+k\,[j_2-r_2-1]+kr_4,\,m-i_2+1)$$

$$=\frac{m!\,n!}{(i_1-1)!}\sum_{r_1=0}^{j_1-1}\ \sum_{r_2=0}^{j_2-j_1+r_1}\ \sum_{r_3=0}^{j_2-j_1+r_1-r_2}\ \sum_{r_4=0}^{n-j_2+1+r_2}(-1)^{r_3+r_4}\times$$

$$\times\,\{i_1+k\,(j_1-1-r_1+r_3)-1\}!\,\{i_2+k\,(j_2-r_2-1)+kr_4-1\}!\times$$

$$\times\,[r_3!\,r_4!\,(j_2-j_1+r_1-r_2-r_3)!\,(n-j_2+1+r_2-r_4)!\,(j_1-1-r_1)!\times$$

$$\times\,\{i_2+k\,(j_1-1-r_1+r_3)-1\}!\,\{m+k\,(j_2-r_2-1)+kr_4\}!]^{-1}.$$

Notice that for the power function against Lehmann alternatives of the second test from V_2 rejecting H_0 when

$$\max\{(y_{(j_1)}-x_{(i_1)}),\,(y_{(j_2)}-x_{(i_2)})\}>0\qquad (j_1<j_2,\ i_1<i_2)$$

can be written

$$\begin{aligned}
&\Pr\left[\max\{(\underline{y}_{(j_1)}-\underline{x}_{(i_1)}),(\underline{y}_{(j_2)}-\underline{x}_{(i_2)})\}>0 \mid G=F^k\right]\\
&=\Pr\left[\min\{(\underline{x}_{(i_1)}-\underline{y}_{(j_1)}),(\underline{x}_{(i_2)}-\underline{y}_{(j_2)})\}<0 \mid G=F^k\right]\\
&=1-\Pr\left[\min\{(\underline{x}_{(i_1)}-\underline{y}_{(j_1)}),(\underline{x}_{(i_2)}-\underline{y}_{(j_2)})\}>0 \mid F=G^{1/k}\right].
\end{aligned}$$

So the power of this test can be derived from (4.2.2.18) by replacing m, n, i_1, j_1, i_2, j_2 and k by n, m, j_1, i_1, j_2, i_2 and k^{-1}, respectively.

4.2.3. *Power against difference in scale parameter of the Exponential distribution and selected lower confidence bounds for scale ratio*

Assume the two samples are drawn from populations with distribution functions

$$F(x)=\begin{cases}0 & \text{for } x<0\\ 1-\exp -x & \text{for } 0\leqslant x\end{cases} \tag{4.2.3.1}$$

and

$$G(y)=\begin{cases}0 & \text{for } y<0\\ 1-\exp(-y/\lambda) & \text{for } 0\leqslant y \quad (0<\lambda)\end{cases} \tag{4.2.3.2}$$

(without loss of generality it can be assumed that the scale parameter of $F(x)$ is equal to 1).

If we want to construct lower confidence bounds for λ of the following form:

$$\underline{R}_{ji}\equiv\frac{\underline{y}_{(j)}}{\underline{x}_{(i)}}<\lambda, \tag{4.2.3.3}$$

then we can use the following relation:

$$\begin{aligned}
\Pr[\underline{R}_{ji}<\lambda]&=\Pr\left[\frac{\underline{y}_{(j)}}{\underline{x}_{(i)}}<\lambda\right]\\
&=\Pr\left[\frac{\underline{y}_{(j)}}{\lambda}-\underline{x}_{(i)}<0\right]\\
&=\Pr\left[\underline{\tilde{y}}_{(j)}-\underline{x}_{(i)}<0\right],
\end{aligned}$$

where $\underline{\tilde{y}}_{(j)}=\underline{y}_{(j)}/\lambda$ is the jth order statistic of a sample of size n from a population with cumulative distribution function

$$F(\tilde{y})=\begin{cases}0 & \text{for } \tilde{y}<0\\ 1-\exp-\tilde{y} & \text{for } 0\leqslant\tilde{y}.\end{cases}$$

Thus we can use the computations of sec. 3.2 to find confidence coefficients of the various possible lower confidence bounds. For the selection of the "best" lower confidence bound from all possible confidence bounds with confidence coefficients larger than or equal to $1-\alpha$, we again use as selection characteristic the power function of the corresponding test to which each lower confidence bound can be converted. With the corresponding test one tests the null hypothesis

$$H_0 : \lambda = 1 \tag{4.2.3.4}$$

against the one-sided alternative hypothesis

$$H_1 : \lambda > 1. \tag{4.2.3.5}$$

One rejects H_0 if $R_{ji} > 1$. Noticing that under H_1

$$\begin{aligned} G(y) &= 1 - \exp(-y/\lambda) \\ &= 1 - \{1-(1-\exp -y)\}^{1/\lambda} \\ &= 1 - \{1-F(y)\}^{1/\lambda} \end{aligned} \tag{4.2.3.6}$$

we see that H_1 is a Lehmann alternative of the form $1-(1-F)^{1/k}$ $(k > 1)$. Hence we can use the selected confidence bounds for these alternatives. These selected confidence bounds can be found from the tables 4.2.2-I–4.2.2-VI after a simple transformation of the indices. As ultimate result we have: if $V_{j'i'}$ is the selected test in the case of sample sizes m' and n' for the alternative $G = F^k$, then R_{ji} with $j = m' - i' + 1$ and $i = n' - j' + 1$ is the selected test in the case of sample sizes $m = n'$ and $n = m'$ for the alternative considered.

In the general case that $F(x)$ and $G(y)$ are two arbitrary continuous distribution functions with $G(y) \equiv F(y/\lambda)$, then R_{ji} is also a lower confidence bound for λ.

It is easy to see that alternatives of the form

$$H_1 : \lambda < 1$$

can be dealt with in a similar manner.

4.2.4. *Power against Exponential translation alternatives*

In this section we shall consider the power of the test V_{ji} against Exponential translation alternatives. Let us assume that the random variables $\underline{x}$ and $\underline{y}$ are Exponentially distributed according to the following distribution functions:

$$F(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 1-\exp -x & \text{for} \quad 0 \leqslant x \end{cases} \tag{4.2.4.1}$$

and

$$G(y) = \begin{cases} 0 & \text{for} \quad x < \delta' \\ 1-\exp[-(y-\delta')] & \text{for} \quad \delta' \leqslant x, \end{cases} \tag{4.2.4.2}$$

respectively. Testing

$$H_0 : \delta' = 0 \tag{4.2.4.3}$$

against

$$H_1 : \delta' = \delta > 0 \tag{4.2.4.4}$$

the test V_{ji} has the following power function (cf. eq. (4.2.1.4)):

$$\beta(\delta) =$$

$$i \binom{m}{i} \sum_{r_1=1}^{j} \binom{n}{j-r_1} \int_{-\infty}^{\infty} F^{i-1}(s) \{1-F(s)\}^{m-i} F^{j-r_1}(s-\delta) \{1-F(s-\delta)\}^{n-j+r_1} \mathrm{d}F(s)$$

$$= i \binom{m}{i} \{ \int_0^{\delta} (1-\exp-s)^{i-1} \exp[-(m-i)s] \,\mathrm{d}(1-\exp-s) + \sum_{r_1=1}^{j} \binom{n}{j-r_1} \times$$

$$\times \int_{\delta}^{\infty} (1-\exp-s)^{i-1} \exp[-(m-i)s] \{1-\exp-(s-\delta)\}^{j-r_1} \exp[-(n-j+r_1)(s-\delta)] \times$$

$$\times \,\mathrm{d}(1-\exp-s)\}.$$

For the first integral in this expression one finds by partial integration:

$$\int_0^{\delta} (1-\exp-s)^{i-1} \exp[-(m-i)s] \,\mathrm{d}(1-\exp-s)$$

$$= \int_0^{1-\exp-\delta} t^{i-1} (1-t)^{m-i} \,\mathrm{d}t$$

$$= \frac{(i-1)!\,(m-i)!}{m!} \sum_{r_2=i}^{m} \binom{m}{r_2} \exp[-(m-r_2)\delta] (1-\exp-\delta)^{r_2}$$

and similarly for the second integral:

$$\int_{\delta}^{\infty} (1-\exp-s)^{i-1} \exp[-(m-i)s] \{1-\exp-(s-\delta)\}^{j-r_1} \exp[-(n-j+r_1)(s-\delta)] \times$$

$$\times \,\mathrm{d}(1-\exp-s)$$

$$= \exp[(n-j+r_1)\delta] \int_{1-\exp-\delta}^{1} t^{i-1} (1-t)^{m-i} (1-\exp\delta + t\exp\delta)^{j-r_1} (1-t)^{n-j+r_1} \,\mathrm{d}t$$

$$= \exp(n\delta) \int_{1-\exp-\delta}^{1} t^{i-1} (1-t)^{m+n-i-j+r_1} \{t-(1-\exp-\delta)\}^{j-r_1} \,\mathrm{d}t$$

$$= \exp(n\delta) \sum_{r_3=0}^{i-1} \binom{i-1}{r_3} (1-\exp-\delta)^{r_3} \int_0^{\exp-\delta} s^{i+j-r_1-r_3-1} [(\exp-\delta)-s]^{m+n-i-j+r_1} \,\mathrm{d}s,$$

where the last equality can be seen by the substitution

$t = s + 1 - \exp -\delta$ and the expansion

$$\{s + (1-\exp-\delta)\}^{i-1} = \sum_{r_3=0}^{i-1} \binom{i-1}{r_3} s^{i-1-r_3} (1-\exp-\delta)^{r_3} .$$

Now, substituting $s = (\exp -\delta)\, u$, one finds:

$$\exp(n\delta) \sum_{r_3=0}^{i-1} \binom{i-1}{r_3} (1-\exp-\delta)^{r_3} \exp[-\delta(m+n-r_3)] \int_0^1 u^{i+j-r_1-r_3-1} \times (1-u)^{m+n-i-j+r_1}\, du$$

$$= \exp(n\delta) \sum_{r_3=0}^{i-1} \binom{i-1}{r_3} (1-\exp-\delta)^{r_3} \exp[-\delta(m+n-r_3)] \times \frac{(i+j-r_1-r_3-1)!\,(m+n-i-j+r_1)!}{(m+n-r_3)!} .$$

So we have:

$$\beta(\delta) = i\binom{m}{i}\left[\frac{(i-1)!\,(m-i)!}{m!} \sum_{r_2=i}^{m} \binom{m}{r_2} \exp[-\delta(m-r_2)]\,(1-\exp-\delta)^{r_2} + \sum_{r_1=1}^{j} \binom{n}{j-r_1} \sum_{r_3=0}^{i-1} \binom{i-1}{r_3} (1-\exp-\delta)^{r_3} \exp[-\delta(m-r_3)] \times \frac{(i+j-r_1-r_3-1)!\,(m+n-i-j+r_1)!}{(m+n-r_3)!}\right]$$

$$= \sum_{r_2=i}^{m} \binom{m}{r_2} \exp[-\delta(m-r_2)]\,(1-\exp-\delta)^{r_2} + \frac{1}{\binom{m+n}{m}} \sum_{r_1=1}^{j} \binom{m+n-i-j+r_1}{m-i} \times \sum_{r_3=0}^{i-1} \binom{i+j-r_1-r_3-1}{j-r_1} \binom{m+n}{r_3} \exp[-\delta(m-r_3)]\,(1-\exp-\delta)^{r_3}. \qquad (4.2.4.5)$$

Since the alternatives (4.2.4.2) may be expected to be of minor practical importance, no numerical computations of the power function and selection of lower confidence bounds have been carried out.

4.2.5. *Power against Uniform translation alternatives and selected lower confidence bounds*

Suppose the random variables $\underline{x}$ and $\underline{y}$ are Uniformly distributed according to the following cumulative distribution functions:

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leqslant x < 1 \\ 1 & \text{if } 1 \leqslant x \end{cases} \qquad (4.2.5.1)$$

and

$$G(y) = \begin{cases} 0 & \text{if } y < \delta' \\ y & \text{if } \delta' \leqslant y < 1 + \delta' \\ 1 & \text{if } 1 + \delta' \leqslant y, \end{cases} \qquad (4.2.5.2)$$

respectively. Testing the null hypothesis

$$H_0 : \delta' = 0 \qquad (4.2.5.3)$$

against

$$H_1 : \delta' = \delta > 0 \qquad (4.2.5.4)$$

the test V_{ji} has the following power function (see sec. 4.2.1):

$$\beta(\delta) = i\binom{m}{i} \sum_{r_1=1}^{j} \binom{n}{j-r_1} \int_{-\infty}^{\infty} F^{i-1}(s)\,\{1 - F(s)\}^{m-i}\, F^{j-r_1}(s-\delta) \times$$
$$\times \{1 - F(s-\delta)\}^{n-j+r_1}\, \mathrm{d}F(s)$$

$$= i\binom{m}{i} [\int_{-\infty}^{\infty} F^{i-1}(s)\,\{1 - F(s)\}^{m-i}\,\{1 - F(s-\delta)\}^{n}\,\mathrm{d}F(s) +$$

$$+ \sum_{r_1=1}^{j-1} \binom{n}{j-r_1} \int_{-\infty}^{\infty} F^{i-1}(s)\{1-F(s)\}^{m-i}F^{j-r_1}(s-\delta)\{1-F(s-\delta)\}^{n-j+r_1}\mathrm{d}F(s)]$$

$$= i\binom{m}{i} [\int_{0}^{\delta} s^{i-1}\,(1-s)^{m-i}\,\mathrm{d}s + \int_{\delta}^{1} s^{i-1}\,(1-s)^{m-i}\,(1-s+\delta)^{n}\,\mathrm{d}s +$$

$$+ \sum_{r_1=1}^{j-1} \binom{n}{j-r_1} \int_{\delta}^{1} s^{i-1}\,(1-s)^{m-i}\,(s-\delta)^{j-r_1}\,(1-s+\delta)^{n-j+r_1}\,\mathrm{d}s]$$

$$= i\binom{m}{i} [\int_{0}^{\delta} s^{i-1}\,(1-s)^{m-i}\,\mathrm{d}s + \sum_{r_1=1}^{j} \binom{n}{j-r_1} \int_{\delta}^{1} s^{i-1}\,(1-s)^{m-i}\,(s-\delta)^{j-r_1} \times$$
$$\times (1-s+\delta)^{n-j+r_1}\,\mathrm{d}s].$$

The first integral can be reduced by partial integration:

$$\int_0^\delta s^{i-1}(1-s)^{m-i}\,\mathrm{d}s = \frac{1}{i}\int_0^\delta (1-s)^{m-i}\,\mathrm{d}s^i$$

$$= \frac{1}{i}\,\delta^i(1-\delta)^{m-i} + \frac{m-i}{i}\int_0^\delta s^i(1-s)^{m-i-1}\,\mathrm{d}s$$

$$= (i-1)!\,(m-i)!\sum_{r_2=i}^{m}\frac{\delta^{r_2}(1-\delta)^{m-r_2}}{r_2!\,(m-r_2)!}. \qquad (4.2.5.5)$$

For the second integral we have

$$\int_\delta^1 s^{i-1}(1-s)^{m-i}(s-\delta)^{j-r_1}(1-s+\delta)^{n-j+r_1}\,\mathrm{d}s$$
$$= \sum_{r_3=0}^{n-j+r_1}\binom{n-j+r_1}{r_3}\delta^{r_3}\int_\delta^1 s^{i-1}(1-s)^{m+n-i-j+r_1-r_3}(s-\delta)^{j-r_1}\,\mathrm{d}s.$$

Inserting $s = t + \delta$ we see that this integral equals

$$\sum_{r_3=0}^{n-j+r_1}\binom{n-j+r_1}{r_3}\delta^{r_3}\int_0^{1-\delta}(t+\delta)^{i-1}(1-\delta-t)^{m+n-i-j+r_1-r_3}\,t^{j-r_1}\,\mathrm{d}t$$

$$= \sum_{r_3=0}^{n-j+r_1}\binom{n-j+r_1}{r_3}\delta^{r_3}\sum_{r_4=0}^{i-1}\binom{i-1}{r_4}\delta^{r_4}\int_0^{1-\delta}t^{i+j-r_1-r_4-1}(1-\delta-t)^{m+n-i-j+r_1-r_3}\,\mathrm{d}t,$$

and with $t = (1-\delta)\,u$ one gets

$$\sum_{r_3=0}^{n-j+r_1}\binom{n-j+r_1}{r_3}\delta^{r_3}\sum_{r_4=0}^{i-1}\binom{i-1}{r_4}\delta^{r_4}(1-\delta)^{m+n-r_3-r_4}\int_0^1 u^{i+j-r_1-r_4-1}\times$$
$$\times(1-u)^{m+n-i-j+r_1-r_3}\,\mathrm{d}u$$

$$= \sum_{r_3=0}^{n-j+r_1}\binom{n-j+r_1}{r_3}\delta^{r_3}\sum_{r_4=0}^{i-1}\binom{i-1}{r_4}\delta^{r_4}(1-\delta)^{m+n-r_3-r_4}\times$$

$$\times\frac{(i+j-r_1-r_4-1)!\,(m+n-i-j+r_1-r_3)!}{(m+n-r_3-r_4)!}. \qquad (4.2.5.6)$$

Thus the power function $\beta(\delta)$ is equal to

$$\beta(\delta) = i\binom{m}{i}\left\{(i-1)!\,(m-i)! \sum_{r_2=i}^{m} \frac{\delta^{r_2}(1-\delta)^{m-r_2}}{r_2!\,(m-r_2)!} + \right.$$

$$+ \sum_{r_1=1}^{j} \binom{n}{j-r_1} \sum_{r_3=0}^{n-j+r_1} \binom{n-j+r_1}{r_3} \delta^{r_3} \sum_{r_4=0}^{i-1} \binom{i-1}{r_4} \delta^{r_4} (1-\delta)^{m+n-r_3-r_4} \times$$

$$\times \frac{(i+j-r_1-r_4-1)!\,(m+n-i-j+r_1-r_3)!}{(m+n-r_3-r_4)!}$$

$$= \sum_{r_2=i}^{m} \binom{m}{r_2} \delta^{r_2} (1-\delta)^{m-r_2} +$$

$$+ \frac{1}{\binom{m+n}{m}} \sum_{r_1=1}^{j} \sum_{r_3=0}^{n-j+r_1} \binom{m+n-i-j+r_1-r_3}{m-i} \sum_{r_4=0}^{i-1} \binom{i+j-r_1-r_4-1}{j-r_1} \binom{r_3+r_4}{r_3} \binom{m+n}{r_3+r_4} \times$$

$$\times \delta^{r_3+r_4} (1-\delta)^{m+n-r_3-r_4}. \qquad (4.2.5.7)$$

According to the selection procedure described in sec. 4.2 one-sided tests to the right and consequently lower confidence bounds have been selected. The selected lower confidence bounds are given in the tables 4.2.5-I–4.2.5-VI.

4.3. Upper confidence bounds and two-sided confidence bounds

When the distribution of $\underline{x}$ is symmetric, we have, considering translation alternatives,

$$\begin{aligned}\Pr[\underline{y}_{(j)} - \underline{x}_{(i)} > \nu] &= \Pr[(\underline{y}_{(j)} - \nu) - \underline{x}_{(i)} > 0] \\ &= \Pr[\underline{y}^*_{(j)} - \underline{x}_{(i)} > 0] \\ &= \Pr[\underline{y}^*_{(n-j+1)} - \underline{x}_{(m-i+1)} < 0] \\ &= \Pr[\underline{y}_{(n-j+1)} - \underline{x}_{(m-i+1)} < \nu], \end{aligned} \qquad (4.3.1)$$

where

$$\underline{y}^*_{(j)} = \underline{y}_{(j)} - \nu \qquad (j = 1, 2, \ldots, n).$$

Further we have under $H_1 : G(y) = F(y+\delta)$ $(\delta > 0)$:

$$\begin{aligned}\beta(\delta) &= \Pr[\underline{y}_{(j)} - \underline{x}_{(i)} < 0 \mid H_1 : G(y) = F(y+\delta)] \\ &= \Pr[\underline{y}_{(n-j+1)} - \underline{x}_{(m-i+1)} > 0 \mid H_1 : G(y) = F(y-\delta)]. \end{aligned} \quad (4.3.2)$$

From this it is clear that if $\underline{y}_{(j)} - \underline{x}_{(i)}$ is the selected lower confidence bound for ν then $\underline{y}_{(n-j+1)} - \underline{x}_{(m-i+1)}$ is the selected upper confidence bound for ν.

TABLE 4.2.5-I

Distribution-free lower confidence bounds $\underline{y}_{(j)} - \underline{x}_{(i)}$, denoted by $j - i$, for shift (selection based on Uniform shift alternatives) with confidence level $1 - \alpha = \cdot 999$ and the values of : 1-confidence coefficient

$$\Pr[\underline{y}_{(j)} - \underline{x}_{(i)} < \nu] \geqslant \cdot 999$$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3													
4								1—4 ·00100	1—4 ·00073	1—4 ·00055	1—4 ·00042	1—4 ·00033	1—4 ·00026
5						1—5 ·00078	1—5 ·00050	1—5 ·00033	1—5 ·00023	2—5 ·00097	2—5 ·00070	2—5 ·00052	2—5 ·00039
6					1—6 ·00058	1—6 ·00033	1—6 ·00020	2—6 ·00087	1—5 ·00097	1—5 ·00070	1—5 ·00052	1—5 ·00039	1—5 ·00029
7				1—7 ·00058	1—7 ·00029	1—7 ·00016	1—6 ·00087	1—6 ·00057	1—6 ·00038	3—7 ·00071	3—7 ·00046	2—6 ·00085	1—5 ·00080
8			1—8 ·00078	1—8 ·00033	1—8 ·00016	1—7 ·00070	1—7 ·00041	1—7 ·00025	3—8 ·00060	1—6 ·00072	1—6 ·00052	1—6 ·00038	1—6 ·00028
9			1—9 ·00050	1—9 ·00020	2—9 ·00087	2—9 ·00041	1—8 ·00021	1—7 ·00071	1—7 ·00046	4—9 ·00075	4—9 ·00044	1—6 ·00083	1—6 ·00062
10		1—10 ·00100	1—10 ·00033	1—9 ·00087	2—10 ·00057	2—10 ·00025	3—10 ·00071	1—8 ·00036	4—10 ·00081	1—7 ·00070	5—10 ·00087	1—7 ·00035	6—10 ·00092
11		1—11 ·00073	1—11 ·00023	2—11 ·00097	2—11 ·00038	1—9 ·00060	3—11 ·00046	1—8 ·00081	1—8 ·00052	1—8 ·00034	1—7 ·00095	1—7 ·00069	1—7 ·00050
12		1—12 ·00055	1—11 ·00097	2—12 ·00070	1—10 ·00071	3—12 ·00072	1—9 ·00075	4—12 ·00070	4—12 ·00034	1—8 ·00067	1—8 ·00046	6—12 ·00064	1—7 ·00089
13		1—13 ·00042	1—12 ·00070	2—13 ·00052	1—11 ·00046	3—13 ·00052	1—10 ·00044	1—9 ·00087	5—13 ·00095	5—13 ·00046	1—8 ·00082	1—8 ·00058	7—13 ·00072
14		1—14 ·00033	1—13 ·00052	2—14 ·00039	2—13 ·00085	3—14 ·00038	4—14 ·00083	4—14 ·00035	5—14 ·00069	1—9 ·00064	6—14 ·00058	1—8 ·00097	1—8 ·00070
15		1—15 ·00026	1—14 ·00039	2—15 ·00029	3—15 ·00080	3—15 ·00028	4—15 ·00062	1—10 ·00092	5—15 ·00050	6—15 ·00089	1—9 ·00072	7—15 ·00070	1—9 ·00035

TABLE 4.2.5-II

Distribution-free lower confidence bounds $\underline{y}_{(j)} - \underline{x}_{(i)}$, denoted by $j - i$, for shift (selection based on Uniform shift alternatives) with confidence level $1 - \alpha = \cdot 995$ and the values of : 1-confidence coefficient

$$\Pr [\underline{y}_{(j)} - \underline{x}_{(i)} < \nu] \geqslant \cdot 995$$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3							1—3 ·00455	1—3 ·00350	1—3 ·00275	1—3 ·00220	1—3 ·00179	1—3 ·00147	2—3 ·00490
4				1—4 ·00476	1—4 ·00303	1—4 ·00202	1—4 ·00140	2—4 ·00500	2—4 ·00366	2—4 ·00275	2—4 ·00210	1—3 ·00490	1—3 ·00413
5			1—5 ·00397	1—5 ·00216	1—5 ·00126	2—5 ·00466	1—4 ·00500	1—4 ·00366	1—4 ·00275	1—4 ·00210	1—4 ·00163	4—5 ·00482	2—4 ·00490
6		1—6 ·00476	1—6 ·00216	1—6 ·00108	1—5 ·00466	1—5 ·00300	1—5 ·00200	3—6 ·00350	3—6 ·00226	1—4 ·00490	1—4 ·00387	1—4 ·00310	1—4 ·00251
7		1—7 ·00303	1—7 ·00126	2—7 ·00466	1—6 ·00233	1—6 ·00140	1—5 ·00481	1—5 ·00339	1—5 ·00245	1—5 ·00181	5—7 ·00426	5—7 ·00284	1—4 ·00478
8		1—8 ·00202	1—7 ·00466	2—8 ·00300	2—8 ·00140	1—6 ·00350	1—6 ·00226	4—8 ·00377	1—5 ·00482	1—5 ·00361	1—5 ·00275	1—5 ·00213	1—5 ·00166
9	1—9 ·00455	1—9 ·00140	2—9 ·00500	2—9 ·00200	3—9 ·00481	3—9 ·00226	1—6 ·00452	1—6 ·00310	5—9 ·00426	5—9 ·00243	1—5 ·00478	1—5 ·00374	1—5 ·00296
10	1—10 ·00350	1—9 ·00500	2—10 ·00366	1—8 ·00350	3—10 ·00339	1—7 ·00377	4—10 ·00310	1—7 ·00155	1—6 ·00387	6—10 ·00464	1—6 ·00208	7—10 ·00408	1—5 ·00474
11	1—11 ·00275	1—10 ·00366	2—11 ·00275	1—9 ·00226	3—11 ·00245	4—11 ·00482	1—7 ·00426	5—11 ·00387	1—7 ·00193	1—6 ·00458	7—11 ·00496	1—6 ·00261	8—11 ·00412
12	1—12 ·00220	1—11 ·00275	2—12 ·00210	3—12 ·00490	3—12 ·00181	4—12 ·00361	1—8 ·00243	1—7 ·00464	6—12 ·00458	1—7 ·00229	7—12 ·00357	1—6 ·00401	1—6 ·00312
13	1—13 ·00179	1—12 ·00210	2—13 ·00163	3—13 ·00387	1—9 ·00426	4—13 ·00275	5—13 ·00478	5—13 ·00208	1—7 ·00496	1—7 ·00357	1—7 ·00261	8—13 ·00386	1—6 ·00455
14	1—14 ·00147	2—14 ·00490	1—11 ·00482	3—14 ·00310	1—10 ·00284	4—14 ·00213	5—14 ·00374	1—8 ·00408	6—14 ·00261	7—14 ·00401	1—7 ·00386	1—7 ·00290	9—14 ·00412
15	1—14 ·00490	2—15 ·00413	2—14 ·00490	3—15 ·00251	4—15 ·00478	4—15 ·00166	5—15 ·00296	6—15 ·00474	1—8 ·00412	7—15 ·00312	8—15 ·00455	1—7 ·00412	1—7 ·00316

TABLE 4.2.5-III

Distribution-free lower confidence bounds $\underline{y}_{(j)} - \underline{x}_{(i)}$,denoted by $j-i$, for shift (selection based on Uniform shift alternatives) with confidence level $1-\alpha = \cdot 99$ and the values of : 1-confidence coefficient

$$\Pr\left[\underline{y}_{(j)} - \underline{x}_{(i)} < \nu\right] \geqslant \cdot 99$$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3					1—3	1—3	1—3	1—3	1—3	2—3	2—3	2—3	2—3
					·00833	·00606	·00455	·00350	·00275	·00879	·00714	·00588	·00490
4			1—4	1—4	1—4	1—4	2—4	2—4	1—3	1—3	1—3	1—3	1—3
			·00794	·00476	·00303	·00202	·00699	·00500	·00879	·00714	·00588	·00490	·00413
5		1—5	1—5	1—5	2—5	1—4	1—4	3—5	1—4	2—4	2—4	2—4	1—3
		·00794	·00397	·00216	·00758	·00699	·00500	·00699	·00275	·00986	·00770	·00611	·00877
6		1—6	1—6	1—5	1—5	3—6	3—6	1—4	1—4	1—4	1—4	1—4	6—6
		·00476	·00216	·00758	·00466	·00932	·00559	·00824	·00630	·00490	·00387	·00310	·00851
7	1—7	1—7	1—6	2—7	1—6	1—5	1—5	4—7	3—6	1—4	1—4	1—4	1—4
	·00833	·00303	·00758	·00466	·00233	·00699	·00481	·00617	·00905	·00903	·00722	·00585	·00478
8	1—8	1—8	2—8	1—6	3—8	1—6	1—5	1—5	1—5	4—7	6—8	1—4	1—4
	·00606	·00202	·00699	·00932	·00699	·00350	·00905	·00654	·00482	·00988	·00632	·00957	·00791
9	1—9	1—8	2—9	1—7	3—9	4—9	1—6	5—9	1—5	1—5	1—5	1—5	8—9
	·00455	·00699	·00500	·00559	·00481	·00905	·00452	·00774	·00813	·00619	·00478	·00374	·00875
10	1—10	1—9	1—8	3—10	1—7	4—10	1—6	1—6	6—10	1—5	1—5	8—10	1—5
	·00350	·00500	·00699	·00824	·00617	·00654	·00774	·00542	·00851	·00957	·00749	·00992	·00474
11	1—11	2—11	2—11	3—11	2—9	4—11	5—11	1—6	1—6	7—11	7—11	1—5	1—5
	·00275	·00879	·00275	·00630	·00905	·00482	·00813	·00851	·00619	·00915	·00496	·00870	·00702
12	1—11	2—12	2—11	3—12	4—12	2—9	5—12	6—12	1—6	1—6	8—12	8—12	1—5
	·00879	·00714	·00986	·00490	·00903	·00988	·00619	·00957	·00915	·00686	·00969	·00522	·00981
13	1—12	2—13	2—12	3—13	4—13	1—8	5—13	6—13	1—7	1—6	1—6	1—6	1—6
	·00714	·00588	·00770	·00387	·00722	·00632	·00478	·00749	·00496	·00969	·00745	·00580	·00455
14	1—13	2—14	2—13	3—14	4—14	5—14	5—14	1—7	7—14	1—7	8—14	1—6	1—6
	·00588	·00490	·00611	·00310	·00585	·00957	·00374	·00992	·00870	·00522	·00580	·00797	·00632
15	1—14	2—15	3—15	1—10	4—15	5—15	1—8	6—15	7—15	8—15	8—15	9—15	1—6
	·00490	·00413	·00877	·00851	·00478	·00791	·00875	·00474	·00702	·00981	·00455	·00632	·00843

TABLE 4.2.5-IV

Distribution-free lower confidence bounds $\underline{y}_{(j)} - \underline{x}_{(i)}$, denoted by $j - i$, for shift (selection based on Uniform shift alternatives) with confidence level $1 - \alpha = \cdot 975$ and the values of : 1-confidence coefficient

$$\Pr [\underline{y}_{(j)} - \underline{x}_{(i)} < \nu] \geqslant \cdot 975$$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3			1—3	1—3	1—3	2—3	2—3	2—3	2—3	3—3	1—2	1—2	1—2
			·01786	·01190	·00833	·02424	·01818	·01399	·01099	·02198	·02500	·02206	·01961
4		1—4	1—4	2—4	1—3	1—3	1—3	1—3	1—3	4—4	2—3	2—3	2—3
		·01429	·00794	·02381	·02424	·01818	·01399	·01099	·00879	·01923	·02227	·01863	·01574
5	1—5	1—5	1—4	1—4	1—4	3—5	2—4	1—3	1—3	1—3	1—3	1—3	1—3
	·01786	·00794	·02381	·01515	·01010	·01632	·02298	·02198	·01786	·01471	·01225	·01032	·00877
6	1—6	1—5	2—6	1—5	1—4	1—4	4—6	3—5	5—6	1—3	1—3	1—3	1—3
	·01190	·02381	·01515	·00758	·02098	·01499	·01678	·02448	·01697	·02451	·02064	·01754	·01504
7	1—7	2—7	2—7	3—7	1—5	4—7	1—4	1—4	6—7	6—7	7—7	7—7	1—3
	·00833	·02424	·01010	·02098	·01049	·01865	·01923	·01471	·02489	·01572	·02214	·01476	·02273
8	1—7	2—8	1—6	3—8	1—5	1—5	5—8	1—4	1—4	7—8	1—4	8—8	1—4
	·02424	·01818	·01632	·01499	·01865	·01282	·02036	·02288	·01806	·02384	·01170	·02012	·00791
9	1—8	2—9	2—8	1—6	4—9	1—5	1—5	6—9	5—8	1—4	1—4	1—4	1—4
	·01818	·01399	·02298	·01678	·01923	·02036	·01471	·02167	·02489	·02105	·01722	·01423	·01186
10	1—9	2—10	3—10	2—8	4—10	5—10	1—5	1—5	7—10	7—10	1—4	1—4	1—4
	·01399	·01099	·02198	·02448	·01471	·02288	·02167	·01625	·02270	·01238	·02372	·01976	·01660
11	1—10	2—11	3—11	1—7	1—6	5—11	2—7	1—5	1—5	8—11	1—5	9—11	1—4
	·01099	·00879	·01786	·01697	·02489	·01806	·02489	·02270	·01754	·02354	·01087	·01696	·02207
12	1—10	1—9	3—12	4—12	1—7	1—6	6—12	1—6	1—5	1—5	9—12	1—5	10—12
	·02198	·01923	·01471	·02451	·01572	·02384	·02105	·01238	·02354	·01863	·02422	·01204	·01691
13	2—13	2—12	3—13	4—13	1—7	5—13	6—13	7—13	7—13	1—5	1—5	10—13	1—5
	·02500	·02227	·01225	·02064	·02214	·01170	·01722	·02372	·01087	·02422	·01957	·02480	·01310
14	2—14	2—13	3—14	4—14	1—8	1—7	6—14	7—14	1—6	8—14	1—5	1—5	1—5
	·02206	·01863	·01032	·01754	·01476	·02012	·01423	·01976	·01696	·01204	·02480	·02037	·01686
15	2—15	2—14	3—15	4—15	5—15	5—15	6—15	7—15	8—15	1—6	9—15	10—15	1—5
	·01961	·01574	·00877	·01504	·02273	·00791	·01186	·01660	·02207	·01691	·01310	·01686	·02107

TABLE 4.2.5-V

Distribution-free lower confidence bounds $\underline{y}_{(j)} - \underline{x}_{(i)}$, denoted by $j - i$, for shift (selection based on Uniform shift alternatives) with confidence level $1 - \alpha = {\cdot}95$ and the values of : 1-confidence coefficient

$$\Pr[\underline{y}_{(j)} - \underline{x}_{(i)} < \nu] \geqslant {\cdot}95$$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3	1—3	1—3	1—3	2—3	2—3	2—3	1—2	1—2	1—2	1—2	1—2	1—2	1—2
	·05000	·02857	·01786	·04762	·03333	·02424	·04545	·03846	·03297	·02857	·02500	·02206	·01961
4	1—4	1—4	1—3	1—3	3—4	3—4	4—4	2—3	2—3	1—2	1—2	1—2	1—2
	·02857	·01429	·04762	·03333	·04545	·03030	·04895	·04096	·03297	·05000	·04412	·03922	·03509
5	1—5	2—5	1—4	3—5	1—3	1—3	1—3	5—5	1—3	6—5	2—3	2—3	2—3
	·01786	·04762	·02381	·04545	·04545	·03497	·02747	·04196	·01786	·04072	·04412	·03741	·03199
6	1—5	2—6	1—4	1—4	4—6	4—6	1—3	1—3	1—3	7—6	1—3	8—6	1—3
	·04762	·03333	·04545	·03030	·04895	·02797	·04396	·03571	·02941	·04977	·02064	·04427	·01504
7	1—6	1—5	3—7	1—4	1—4	1—4	5—7	6—7	1—3	1—3	1—3	1—3	1—3
	·03333	·04545	·04545	·04895	·03497	·02564	·02885	·04072	·04289	·03612	·03070	·02632	·02273
8	1—7	1—6	3—8	1—5	4—8	1—4	1—4	6—8	7—8	1—3	1—3	1—3	1—3
	·02424	·03030	·03497	·02797	·02564	·03846	·02941	·02941	·03973	·04912	·04211	·03636	·03162
9	2—9	1—6	3—9	4—9	1—5	5—9	1—4	1—4	1—4	8—9	9—9	1—3	1—3
	·04545	·04895	·02747	·04396	·02885	·02941	·04118	·03251	·02601	·03892	·04887	·04743	·04150
10	2—10	2—9	1—6	4—10	1—5	1—5	6—10	1—4	1—4	1—4	9—10	10—10	10—10
	·03846	·04096	·04196	·03571	·04072	·02941	·03251	·04334	·03509	·02871	·03825	·04710	·02826
11	2—11	2—10	3—11	4—11	5—11	1—5	6—11	7—11	1—4	1—4	1—4	10—11	11—11
	·03297	·03297	·01786	·02941	·04289	·03973	·02601	·03509	·04511	·03727	·03106	·03768	·04565
12	2—12	3—12	1—7	1—6	5—12	6—12	1—5	7—12	8—12	1—4	1—4	1—4	11—12
	·02857	·05000	·04072	·04977	·03612	·04912	·03892	·02871	·03727	·04658	·03913	·03311	·03720
13	2—13	3—13	3—12	4—13	5—13	6—13	1—5	1—5	8—13	9—13	1—4	1—4	1—4
	·02500	·04412	·04412	·02064	·03070	·04211	·04887	·03825	·03106	·03913	·04783	·04074	·03492
14	2—14	3—14	3—13	1—7	5—14	6—14	7—14	1—5	1—5	9—14	10—14	1—4	1—4
	·02206	·03922	·03741	·04427	·02632	·03636	·04743	·04710	·03768	·03311	·04074	·04889	·04215
15	2—15	3—15	3—14	4—15	5—15	6—15	7—15	1—6	1—5	1—5	10—15	11—15	1—4
	·01961	·03509	·03199	·01504	·02273	·03162	·04150	·02826	·04565	·03720	·03492	·04215	·04981

TABLE 4.2.5-VI

Distribution-free lower confidence bounds $\underline{y}_{(j)} - \underline{x}_{(i)}$, denoted by $j - i$, for shift (selection based on Uniform shift alternatives) with confidence level $1 - \alpha = \cdot 90$ and the values of : 1-confidence coefficient

$$\Pr[\underline{y}_{(j)} - \underline{x}_{(i)} < \nu] \geqslant \cdot 90$$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3	1—3	1—3	2—3	1—2	1—2	1—2	4—3	4—3	2—2	2—2	2—2	6—3	2—2
	·05000	·02857	·07143	·08333	·06667	·05455	·09091	·06993	·09341	·08132	·07143	·08235	·05637
4	1—4	1—3	1—3	3—4	2—3	1—2	1—2	1—2	1—2	1—2	7—4	1—2	2—2
	·02857	·07143	·04762	·07143	·08788	·09091	·07692	·06593	·05714	·05000	·08824	·03922	·09701
5	1—4	2—5	1—3	1—3	4—5	5—5	2—3	1—2	1—2	1—2	1—2	1—2	1—2
	·07143	·04762	·08333	·06061	·07071	·09790	·09491	·09524	·08333	·07353	·06536	·05848	·05263
6	2—6	1—4	3—6	1—3	1—3	5—6	6—6	3—4	2—3	1—2	1—2	1—2	1—2
	·08333	·07143	·06061	·09091	·06993	·06993	·09231	·09191	·09874	·09804	·08772	·07895	·07143
7	2—7	2—6	1—4	4—7	1—3	1—3	1—3	7—7	1—3	8—7	9—7	10—7	1—2
	·06667	·08788	·07071	·06993	·09615	·07692	·06250	·08824	·04289	·06811	·08301	·09838	·09091
8	2—8	3—8	1—4	1—4	5—8	1—4	1—3	1—3	8—8	1—3	9—8	10—8	11—8
	·05455	·09091	·09790	·06993	·07692	·03846	·08235	·06863	·08514	·04912	·06325	·07602	·08924
9	1—6	3—9	3—8	1—4	5—9	6—9	1—4	1—3	1—3	9—9	10—9	1—3	11—9
	·09091	·07692	·09491	·09231	·06250	·08235	·04118	·08669	·07368	·08271	·09774	·04743	·07065
10	1—7	3—10	4—10	3—8	1—4	6—10	7—10	1—4	1—3	1—3	10—10	11—10	1—3
	·06993	·06593	·09524	·09191	·08824	·06863	·08669	·04334	·09023	·07792	·08075	·09420	·05217
11	2—10	3—11	4—11	4—10	5—11	1—4	7—11	8—11	1—4	1—3	1—3	1—3	12—11
	·09341	·05714	·08333	·09874	·04289	·08514	·07368	·09023	·04511	·09317	·08152	·07174	·09130
12	2—11	3—12	4—12	5—12	1—5	6—12	1—4	8—12	9—12	1—4	1—3	1—3	1—3
	·08132	·05000	·07353	·09804	·06811	·04912	·08271	·07792	·09317	·04658	·09565	·08462	·07521
13	2—12	1—7	4—13	5—13	1—5	1—5	1—4	1—4	9—13	10—13	1—4	1—3	1—3
	·07143	·08824	·06536	·08772	·08301	·06325	·09774	·08075	·08152	·09565	·04783	·09778	·08730
14	1—9	3—14	4—14	5—14	1—5	1—5	7—14	1—4	9—14	10—14	11—14	1—4	1—3
	·08235	·03922	·05848	·07895	·09838	·07602	·04743	·09420	·07174	·08462	·09778	·04889	·09962
15	2—14	3—14	4—15	5—15	6—15	1—5	1—5	8—15	1—4	10—15	11—15	12—15	1—4
	·05637	·09701	·05263	·07143	·09091	·08924	·07065	·05217	·09130	·07521	·08730	·09962	·04981

Thus a simple transformation of the subscripts gives the best upper confidence bounds for ν.

Example 4.3.1. Uniform distribution, $m = 14$, $n = 15$, confidence level $1 - \alpha = \cdot 90$. In table 4.2.5-VI we find as the best lower confidence bound $\underline{y}_{(1)} - \underline{x}_{(3)}$. Thus the best upper confidence bound is: $\underline{y}_{(15)} - \underline{x}_{(12)}$.

Further two-sided confidence bounds will be obtained in these cases as the combination of two one-sided confidence bounds (see sec. 3.1).

4.4. Some concluding remarks

On closer examination of the tables 4.2.1-I–4.2.1-VI, 4.2.2-I–4.2.2-VI and 4.2.5-I–4.2.5-VI containing the selected tests (and consequently the lower confidence bounds) for Normal shift, Lehmann and Uniform shift alternatives, it is seen that the selected tests are in general unfortunately not the same in these three cases.

It is intuitively clear that for the Normal and Uniform shift alternatives the selected tests would in many cases be different: in the case of Uniform shift one would expect that in most cases either $\underline{x}_{(i)}$ or $\underline{y}_{(j)}$ is equal to the minimum or maximum of the relevant sample, whereas in the case of Normal shift one would not expect this as a rule. In table 4.4-I the numbers of selected lower confidence bounds which are the same for Normal shift and Lehmann alternatives for some significance levels are given.

TABLE 4.4.-I

Numbers of selected lower confidence bounds which are the same for Normal shift and Lehmann alternatives

$\alpha \rightarrow$	·01	·05	·10
number of identical lower confidence bounds	69	53	59
total number of lower confidence bounds	161	169	169

In table 4.4-II the numbers of lower confidence bounds which are the same for Uniform shift and Lehmann alternatives are given for some values of the significance level.

TABLE 4.4-II

Numbers of selected lower confidence bounds which are the same for Uniform shift and Lehmann alternatives

$\alpha \rightarrow$	·01	.05	·10
number of identical lower confidence bounds	95	74	66
total number of lower confidence bounds	161	169	169

Finally, the following remark can be made. Often it is permissible to assume that the underlying distributions are approximately Normal, so that, in my opinion, the confidence bounds with selection based on Normal shift alternatives can be recommended for many practical two-sample problems.

5. LEHMANN ALTERNATIVES

5.1. Introduction

In the preceding chapter we have seen that the use of Lehmann alternatives simplifies the power computations for the tests V_{ji}. In this chapter we shall make first some general remarks concerning Lehmann alternatives and after that properties of the characteristics of the distribution function $F^k(x)$ $(k > 0)$ will be presented.

In the special case where $G(x) = F^k(x)$ we have that $G(x)$ is stochastically larger than $F(x)$ if $k > 1$ and $F(x)$ is stochastically larger than $G(x)$ if $(0 <)\, k < 1$. In the special case where $G(x) = 1 - \{1 - F(x)\}^k$ we have that $F(x)$ is stochastically larger than $G(x)$ if $k > 1$ and $G(x)$ is stochastically larger than $F(x)$ if $(0 <)\, k < 1$. Further it can be remarked that, supposing that k is a positive integer, $F^k(x)$ is the cumulative distribution function of the maximum of k independently distributed random variables with common cumulative distribution function $F(x)$ and $1 - \{1 - F(x)\}^k$ is the cumulative distribution function of the minimum of k independently distributed random variables with common cumulative distribution function $F(x)$. These alternatives may be of practical interest in life testing, reliability testing, extreme-value distribution investigations and related fields.

Notice that for the extreme-value distribution

$$F(x) = \exp(-e^{-x/\beta}) \qquad (\beta > 0; -\infty < x < \infty) \tag{5.1.1}$$

the Lehmann alternative

$$F^k(x) = \exp(-k\, e^{-x/\beta}) \tag{5.1.2}$$

is equivalent to a displacement alternative

$$F(x - \delta) = \exp(-e^{-(x-\delta)/\beta}), \tag{5.1.3}$$

where

$$k = e^{\delta/\beta} \qquad \text{or} \qquad \delta = \beta \ln k.$$

For different possibilities of extreme value distributions see Kendall and Stuart (1963, I, p. 330). Another well-known fact (see Lloyd and Lipow (1962)) is that the failure rate of a random variable with cumulative distribution function $F(x)$,

$$h_F(x) = \frac{f(x)}{1 - F(x)}, \tag{5.1.4}$$

equals k^{-1} times the failure rate $h_G(x)$ of a random variable with cumulative distribution function $G(x)$ if and only if

$$G(x) = 1 - \{1 - F(x)\}^k.$$

This can easily be seen from the fact that

$$F(x) = 1 - \exp\,[- \int_{-\infty}^{x} h_F(t)\,\mathrm{d}t]. \tag{5.1.5}$$

Shorack (1967) remarks that if $F(x)$ and $G(y)$ are Weibull distribution functions differing only in scale parameters:

$$F(x) = 1 - \exp\left\{-\left(\frac{x}{\theta_1}\right)^{\delta}\right\} \qquad (x > 0;\ \theta_0 > 0,\ \delta > 0) \tag{5.1.6}$$

and

$$G(y) = 1 - \exp\left\{-\left(\frac{y}{\theta_1}\right)^{\delta}\right\} \qquad (y > 0;\ \theta_1 > 0), \tag{5.1.7}$$

then

$$G(y) = 1 - \{1 - F(y)\}^k$$

with

$$k = \left(\frac{\theta_0}{\theta_1}\right)^{\delta} \qquad \text{or} \qquad \delta = \frac{\ln k}{\ln \theta_0 - \ln \theta_1}.$$

From Shorack's article we quote the following interesting remarks:
"Examples where these distributions occur abound in life testing, reliability testing, and related fields. However, the vast majority of life testing procedures in current use assume that failure times follow the Exponential distribution, even though one rarely has sufficient data to verify whether or not this is actually the case. The tacit assumptions of Exponential theory which do occur will not seriously degrade the performance of procedures derived under them. However, Zelen and Dannemiller (1961) show that several common procedures derived under Exponential theory perform quite poorly if the underlying distributions are in fact Weibull distributions with shape parameters other than 1. To avoid such degradation which can occur when a parametric procedure is inappropriately applied one might wish to use a distribution-free approach".

Using Lehmann alternatives of the form $G(x) = F^k(x)$ one gets besides displacement of the median of the distribution also alterations of the moments. In particular if $F(x)$ is symmetric, then the cumulative distribution function $G(x)$ will be skewed relative to $F(x)$. In the following sections some properties of the cumulative distribution function $F^k(x)$ will be derived.

5.2. Unimodality and mode

Throughout this section we start from the following assumptions. Let $F(x)$ be a cumulative distribution function and $f(x)$ its density such that the set

$\{x; f(x) \neq 0\}$ is an open interval $I = (a,b)$ $(-\infty \leqslant a < b \leqslant \infty)$. Further let $f(x)$ be differentiable on I.

Definition 5.2.1. A density is called unimodal if one of the following three conditions (A), (B) or (C) holds:

(A) $\exists\, x_0$ in (a,b) such that $f(x)$ is non-decreasing on (a,x_0) and non-increasing on (x_0,b);
(B) $f(x)$ is non-increasing on (a,b); it follows that $a > -\infty$;
(C) $f(x)$ is non-decreasing on (a,b); it follows that $b < \infty$.

Definition 5.2.2. A density is called strongly unimodal if $-\ln f(x)$ is convex on I.

The following lemma can easily be verified.

Lemma 5.2.1. If $f(x)$ is strongly unimodal then $f(x)$ is unimodal.

For the study of Lehmann alternatives the following theorems are of interest.

Theorem 5.2.1. If $f(x)$ is strongly unimodal then the density $k\,F^{k-1}(x)\,f(x)$ of the distribution function $F^k(x)$ $(k > 1)$ is strongly unimodal.

Proof *). From the fact that $-\ln f(x)$ is convex on $I=(a,b)$ and that $f'(x)/f(x)$ is consequently a non-increasing function, it follows that

$$\frac{f'(x)}{f(x)} \leqslant \frac{\int_a^x f'(t)\,\mathrm{d}t}{\int_a^x f(t)\,\mathrm{d}t} = \frac{f(x)-f(a)}{F(x)} \leqslant \frac{f(x)}{F(x)}.$$

From this it follows that

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\{-\ln F(x)\} = \frac{f^2(x)-f'(x)\,F(x)}{F^2(x)} \geqslant 0$$

or $-\ln F(x)$ is convex. Thus

$$-\ln\{k\,F^{k-1}(x)\,f(x)\} = -\ln k - (k-1)\ln F(x) - \ln f(x)$$

is convex.

Theorem 5.2.2. If $f(x)$ is strongly unimodal, then the density $k\,\{1-F(x)\}^{k-1} f(x)$ of the distribution function $1-\{1-F(x)\}^k$ $(k > 1)$ is strongly unimodal.

---

*) This theorem and the next one can also be proved using the theorem that a density $f(x)$ is strongly unimodal if and only if $f(x)$ is a Pólya frequency function of order 2 (PF_2) and the well-known fact that if $f(x)$ is PF_2 then $F(x)$ and $1-F(x)$ are PF_2, and consequently $-\ln F(x)$ and $-\ln\{1-F(x)\}$ are convex (see Karlin (1968)).

Proof. Notice that if the random variables $\underline{x}$ and $\underline{y}$ have distribution functions $F(x)$ and $G(y) = 1 - \{1 - F(y)\}^k$, respectively, and if $\underline{v} = -\underline{y}$ and $\underline{u} = -\underline{x}$, then, denoting the distribution functions of $\underline{v}$ and $\underline{u}$ by $G^*(v)$ and $F^*(u)$, respectively, we have:

$$G^*(v) = 1 - G(-v), \quad F^*(u) = 1 - F(-u)$$

and

$$G^*(v) = \{F^*(v)\}^k.$$

So it is easy to see that the proof follows from theorem 5.2.1.

It can be remarked that the assumption of the Normal distribution being standard in the following theorem can be made without loss of generality.

Theorem 5.2.3. If $\Phi(x)$ is the (standard) Normal distribution function with density $\varphi(x)$, then the density $k\, \Phi^{k-1}(x)\, \varphi(x)$ of $\Phi^k(x)$ $(0 < k < \infty)$ is strongly unimodal.

Proof. It is clear that the proof for $1 < k < \infty$ follows immediately from theorem 5.2.1.

Now let $0 < k < 1$. We must prove that

$$\begin{aligned}\psi(x) &= -\ln \Phi^{-c}(x)\, \varphi(x)\\ &= c \ln \Phi(x) - \ln \varphi(x)\end{aligned}$$

is convex for all $c = 1 - k$ with $0 < k < 1$, or that the second derivative

$$\psi''(x) = 1 - c\, \frac{x\, \varphi(x)\, \Phi(x) + \varphi^2(x)}{\Phi^2(x)}$$

is non-negative. We shall prove the stronger condition that $\psi''(x)$ is positive. We define

$$\begin{aligned}\chi(x) &= \Phi^2(x)\, \psi''(x)\\ &= \Phi^2(x) - c\, x\, \varphi(x)\, \Phi(x) - c\, \varphi^2(x).\end{aligned}$$

Then

$$\chi'(x) = \varphi(x)\, \{c\, x\, \varphi(x) + (2 - c + c\, x^2)\, \Phi(x)\}.$$

Define

$$h(x) = \frac{\chi'(x)}{\varphi(x)}.$$

Then

$$h'(x) = 2\, \varphi(x) + 2\, c\, x\, \Phi(x).$$

It is clear that $h'(x) > 0$ for $x \geqslant 0$. In order to prove that $h'(x) > 0$ for $x < 0$, we note that

$$\frac{\Phi(x)}{\varphi(x)} < -\frac{x^2 + 2}{x^3 + 3x} \qquad (x < 0)$$

(Shenton (1954)). It is easy to see that for all $0 < c < 1$:

$$-\frac{x^2+2}{x^3+3x} < -\frac{1}{cx} \qquad (x<0),$$

thus

$$\frac{\Phi(x)}{\varphi(x)} < -\frac{1}{cx} \qquad (x<0).$$

From this it follows that $h'(x) > 0$ for all $x < 0$. Thus $h(x)$ is a (strictly) increasing function of x. From the fact that $h(-\infty) = 0$ it follows that $h(x)$ is positive on $(-\infty, \infty)$. Thus $\chi(x)$ is a (strictly) increasing function of x. From the fact that $\chi(-\infty) = 0$ it follows that $\chi(x)$ and therefore $\psi''(x)$ is positive on $(-\infty, \infty)$.

Theorem 5.2.4. Let $\Phi(x)$ be the cumulative (standard) Normal distribution function and $x_m(k)$ be the mode of $\Phi^k(x)$, then $x_m(k)$ is a (strictly) increasing function of k $(0 < k < \infty)$ and its second derivative is negative.
Proof. The density corresponding to $\Phi^k(x)$ is equal to $k\,\Phi^{k-1}(x)\,\varphi(x)$, where $\varphi(x)$ is the density of $\Phi(x)$; so the mode $x_m(k)$ satisfies the following equation:

$$k\,(k-1)\,\Phi^{k-2}(x_m)\,\varphi^2(x_m) - k\,x_m\,\Phi^{k-1}(x_m)\,\varphi(x_m) = 0 \qquad (5.2.1)$$

and thus

$$\frac{x_m\,\Phi(x_m)}{\varphi(x_m)} = k-1. \qquad (5.2.2)$$

First consider $k \geqslant 1$. It is easy to see that the equation (5.2.2) cannot be satisfied for $x_m < 0$. Noticing that $x\,\Phi(x)/\varphi(x)$ is a (strictly) increasing function of x for $x \geqslant 0$ (and is equal to zero for $x = 0$), one can conclude that, for $k \geqslant 1$, the mode $x_m(k)$ is a (strictly) increasing function of k. From the fact that the second derivative $2 + x^2 + x\,\{\Phi(x)/\varphi(x)\}\,(3 + x^2)$ of $x\,\Phi(x)/\varphi(x)$ is positive for $x \geqslant 0$, it is easily verified that the second derivative of $x_m(k)$ with respect to k is negative for $k \geqslant 1$.

For $k < 1$ the proof runs as follows. Now it is sufficient to consider only negative x. It is clear that $x\,\Phi(x)/\varphi(x)$ is equal to zero for $x = 0$ while

$$\lim_{x \to -\infty} \frac{x\,\Phi(x)}{\varphi(x)} = -1.$$

The latter fact can be derived from the asymptotic expansion for Mills' ratio:

$$\frac{1-\Phi(x)}{\varphi(x)} \sim \frac{1}{x} - \frac{1}{x^3} \qquad (x > 0 \quad \text{and} \quad x \to \infty). \qquad (5.2.3)$$

Now define

$$\psi(x) = \frac{x\,\Phi(x)}{\varphi(x)}.$$

Then

$$\psi'(x) = \frac{\Phi(x) + x\,\varphi(x) + x^2\,\Phi(x)}{\varphi(x)}.$$

Define

$$\chi(x) = (1 + x^2)\,\Phi(x) + x\,\varphi(x).$$

Then

$$\chi'(x) = 2\,x\,\Phi(x) + 2\,\varphi(x)$$

and

$$\chi''(x) = 2\,\Phi(x).$$

From the fact that $\chi''(x) > 0$ and from $\chi'(-\infty) = 0$ it follows that $\chi'(x) > 0$ on $(-\infty, 0)$. Now it follows from the fact that $\chi(-\infty) = 0$ that $\chi(x)$ and therefore $\psi'(x)$ is larger than zero on $(-\infty, 0)$. From this it follows that $\psi(x)$ is a (strictly) increasing function of x on $(-\infty, 0)$. So it can be concluded that also for $0 < k < 1$, $x_m(k)$ is a (strictly) increasing function of k with values between $-\infty$ and 0.

In order to prove that the second derivative of $x_m(k)$ is negative, for $0 < k < 1$, it is again sufficient to consider the second derivative of $x\,\Phi(x)/\varphi(x)$ for $x < 0$. We have

$$\frac{\partial^2}{\partial x^2}\left\{\frac{x\,\Phi(x)}{\varphi(x)}\right\} = 2 + x^2 + x\,(3 + x^2)\,\frac{\Phi(x)}{\varphi(x)}$$

$$> 2 + x^2 + x\,(3 + x^2)\,\frac{2 + x^2}{3\,|x| + |x|^3} = 0.$$

The inequality follows from the well-known (Shenton (1954)) inequality

$$\frac{1 - \Phi(x)}{\varphi(x)} < \frac{x^2 + 2}{x^3 + 3\,x} \qquad (x > 0). \tag{5.2.4}$$

This completes the proof.

An interesting point is that from the fact that $\psi'(x) > 0$ for all x, it follows that

$$\Phi(x) + x\,\varphi(x) + x^2\,\Phi(x) > 0 \tag{5.2.5}$$

and thus

$$\frac{x}{1+x^2} > -\frac{\Phi(x)}{\varphi(x)}$$

or, replacing x by $-x$,

$$\frac{1-\Phi(x)}{\varphi(x)} > \frac{x}{1+x^2} \qquad (-\infty < x < \infty). \tag{5.2.6}$$

Notice that

$$\frac{x}{1+x^2} > \frac{1}{x} - \frac{1}{x^3} \qquad (x > 0).$$

Formula (5.2.6) is a well-known inequality for Mills' ratio which can also be determined directly by Laplace's continued fraction for

$$R(x) = \frac{1-\Phi(x)}{\varphi(x)},$$

namely (cf. Kendall and Stuart, I (1963)):

$$R(x) = \frac{1}{x+}\,\frac{1}{x+}\,\frac{2}{x+}\,\frac{3}{x+}\cdots\frac{n}{x+}\cdots {}^{*)} \qquad (x > 0).$$

For various distributions (e.g. Logistic, Normal, Lognormal and Gamma) a well-known fact is that for natural k, $x_m(k)$ tends asymptotically to u_k with

$$F(u_k) = 1 - \frac{1}{k}$$

(cf. Sarhan and Greenberg (1962)).

5.3. Some computations concerning expectation and variance of $\Phi^k(x)$

For some small integer values of k it is possible to determine exactly the expectation $E_k\{\underline{x}\}$ and the variance $\sigma_k^2\{\underline{x}\}$ of $\Phi^k(x)$, where $\Phi(x)$ is the cumulative standard Normal distribution function. In table 5.3-I the results are given.

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*) Another continued fraction which converges for small $x > 0$ much more rapidly than Laplace's has been given by Shenton (1954):

$$R^*(x) = \frac{1}{2\,\varphi(x)} - R(x) = \frac{x}{1-}\,\frac{x^2}{3+}\,\frac{2\,x^2}{5-}\,\frac{3\,x^2}{7+}\cdots \qquad (x \geqslant 0).$$

TABLE 5.3-I

The values of $E_k\{\underline{x}\}$, $E_k\{\underline{x}^2\}$ and $\sigma_k{}^2\{\underline{x}\}$ for some values of k

	$k = 1$	$k = 2$	$k = 3$	$k = 4$
$E_k\{\underline{x}\}$	0	$1/\sqrt{\pi}$	$3/(2\sqrt{\pi})$	
$E_k\{\underline{x}^2\}$	1	1	$1 + (1/2\pi)\sqrt{3}$	$1 + (1/\pi)\sqrt{3}$
$\sigma_k{}^2\{\underline{x}\}$	1	$1 - 1/\pi$	$1 - (1/4\pi)(9 - 2\sqrt{3})$	

Now we shall give the proof of these results. For the expectation $E_k\{\underline{x}\}$ one can write

$$\begin{aligned} E_k\{\underline{x}\} &= \int_{-\infty}^{\infty} x \, \mathrm{d}\Phi^k(x) \\ &= k \int_{-\infty}^{\infty} x \, \Phi^{k-1}(x) \, \varphi(x) \, \mathrm{d}x \\ &= -k \int_{-\infty}^{\infty} \Phi^{k-1}(x) \, \varphi'(x) \, \mathrm{d}x \\ &= -k \int_{-\infty}^{\infty} \Phi^{k-1}(x) \, \mathrm{d}\varphi(x) \end{aligned}$$

and with partial integration one finds

$$E_k\{\underline{x}\} = k\,(k-1) \int_{-\infty}^{\infty} \varphi^2(x) \, \Phi^{k-2}(x) \, \mathrm{d}x. \tag{5.3.1}$$

For $k = 2$ and 3 the integration can be carried out. One gets

$$\begin{aligned} E_2\{\underline{x}\} &= 2 \int_{-\infty}^{\infty} \varphi^2(x) \, \mathrm{d}x \\ &= 2 \int_{-\infty}^{\infty} \frac{1}{2\,\pi} \exp(-x^2) \, \mathrm{d}x \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{(2\,\pi)^{1/2}} \exp(-x^2/2) \, \mathrm{d}x = \frac{1}{\sqrt{\pi}} \end{aligned} \tag{5.3.2}$$

and

$$E_3\{\underline{x}\} = 6 \int_{-\infty}^{\infty} \varphi^2(x)\, \Phi(x)\, \mathrm{d}x$$

$$= \frac{6}{(2\,\pi)^{1/2}} \int_{-\infty}^{\infty} \frac{1}{(2\,\pi)^{1/2}} \exp(-x^2)\, \Phi(x)\, \mathrm{d}x$$

$$= \frac{3}{\sqrt{\pi}} \int_{-\infty}^{\infty} \varphi(x)\, \Phi\left(\frac{x}{\sqrt{2}}\right) \mathrm{d}x$$

$$= \frac{3}{\sqrt{\pi}} \int_{-\infty}^{\infty} \Phi\left(\frac{x}{\sqrt{2}}\right) \mathrm{d}\Phi(x) = \frac{3}{2\sqrt{\pi}}. \qquad (5.3.3)$$

Further one finds for the expectation of $\underline{x}^2$, $E_k\{\underline{x}^2\}$, the following expression:

$$E_k\{\underline{x}^2\} = \int_{-\infty}^{\infty} x^2\, \mathrm{d}\Phi^k(x)$$

$$= k \int_{-\infty}^{\infty} x^2\, \Phi^{k-1}(x)\, \varphi(x)\, \mathrm{d}x$$

$$= -k \int_{-\infty}^{\infty} x\, \Phi^{k-1}(x)\, \mathrm{d}\varphi(x)$$

and with partial integration this reduces, for $k > 2$, to

$$E_k\{\underline{x}^2\} = k \int_{-\infty}^{\infty} \varphi(x)\, \Phi^{k-1}(x)\, \mathrm{d}x + k\,(k-1) \int_{-\infty}^{\infty} x\, \varphi^2(x)\, \Phi^{k-2}(x)\, \mathrm{d}x$$

$$= 1 - k\,(k-1) \int_{-\infty}^{\infty} \varphi(x)\, \varphi'(x)\, \Phi^{k-2}(x)\, \mathrm{d}x \qquad (5.3.4)$$

$$= 1 - \tfrac{1}{2}\, k\,(k-1) \int_{-\infty}^{\infty} \Phi^{k-2}(x)\, \mathrm{d}\varphi^2(x)$$

$$= 1 + \tfrac{1}{2}\, k\,(k-1)\,(k-2) \int_{-\infty}^{\infty} \varphi^3(x)\, \Phi^{k-3}(x)\, \mathrm{d}x. \qquad (5.3.5)$$

Hence

$$E_2\{\underline{x}^2\} = 1 = E_1\{\underline{x}^2\} \quad \text{(from (5.3.4))}, \qquad (5.3.6)$$

$$E_3\{\underline{x}^2\} = 1 + 3 \int_{-\infty}^{\infty} \varphi^3(x)\,\mathrm{d}x$$

$$= 1 + \frac{3}{2\,\pi} \int_{-\infty}^{\infty} \frac{1}{(2\,\pi)^{1/2}} \exp(-3\,x^2/2)\,\mathrm{d}x$$

$$= 1 + \frac{\sqrt{3}}{2\,\pi} \int_{-\infty}^{\infty} \frac{1}{(2\,\pi)^{1/2}} \exp(-x^2/2)\,\mathrm{d}x$$

$$= 1 + \frac{1}{2\,\pi} \sqrt{3} \qquad (5.3.7)$$

and

$$E_4\{\underline{x}^2\} = 1 + 12 \int_{-\infty}^{\infty} \varphi^3(x)\,\Phi(x)\,\mathrm{d}x$$

$$= 1 + \frac{6}{\pi} \int_{-\infty}^{\infty} \frac{1}{(2\,\pi)^{1/2}} \exp(-3\,x^2/2)\,\Phi(x)\,\mathrm{d}x$$

$$= 1 + \frac{2\sqrt{3}}{\pi} \int_{-\infty}^{\infty} \Phi\left(\frac{x}{\sqrt{3}}\right) \mathrm{d}\Phi(x)$$

$$= 1 + \frac{1}{\pi} \sqrt{3}. \qquad (5.3.8)$$

For large natural k the formula of the asymptotic distribution of the largest value of a sample of size k from a standard Normal distribution applies (see Sarhan and Greenberg (1962)). The asymptotic mean and variance are

$$u_k + \frac{\gamma}{\alpha_k} \qquad \text{and} \qquad \frac{\pi^2}{6\,\alpha_k{}^2},$$

respectively, where $\gamma = {\cdot}57722$ stands for Euler's number and while u_k and α_k are defined by

$$\Phi(u_k) = 1 - \frac{1}{k}$$

and

$$\alpha_k = k\,\varphi(u_k).$$

It is known that $k\,\varphi(u_k)$ increases with k, thus the variance of the asymptotic distribution decreases with k.

Needless to say that similar computations can be made for $1 - \{1 - \Phi(x)\}^k$ (cf. method of proof of theorem 5.2.2). One finds for the expectation $-E_k\{\underline{x}\}$, while the second moment and variance have the same value as for $\Phi^k(x)$.

5.4. Some properties of moments

In this section we shall present certain properties of the moments of the random variable $\underline{y}$ with continuous cumulative distribution function $G(y) = F^k(y)$, the kth power of a continuous cumulative distribution function $F(y)$. At the end of this chapter some numerical results will be given. In particular, for the Normal, Exponential, Gamma and Beta distributions (these last two only for some values of the parameters) tables with numerical values of the first and second moment, the variance and standard deviation will be given for various values of k.

A family of cumulative distribution functions $F(y;\theta)$ on the real line, depending on a real parameter θ, is said to be stochastically increasing (and the same term is applied to random variables possessing these distributions) if the distribution functions are distinct and if $\theta < \theta'$ implies $F(y;\theta) \geqslant F(y;\theta')$ for all y. If then $\underline{y}$ and $\underline{y}'$ have distributions $F(y;\theta)$ and $F(y;\theta')$, respectively, it follows that $\Pr[\underline{y} > y] \leqslant \Pr[\underline{y}' > y]$ for all y, with strict inequality for at least one value of y, so that $\underline{y}'$ tends to have larger values than $\underline{y}$. In that case the variable $\underline{y}'$ is said to be stochastically larger than $\underline{y}$.

In the case of Lehmann alternatives it can easily be verified that $G(y) = F(y;k) = F^k(y)$ is a family of stochastically increasing cumulative distribution functions.

First of all some definitions will be given.

The rth non-central moments of $\underline{y}$ will be denoted, if they exist (i.e. if y^r is integrable over $(-\infty, \infty)$ with respect to $G(y)$), by $\mu_r(k)$ $(r = 1, 2, \ldots)$, thus

$$\mu_r(k) = E\{\underline{y}^r\} = \int_{R^1} y^r \,\mathrm{d}G(y) = \int_{R^1} y^r \,\mathrm{d}F^k(y) \qquad (0 < k < \infty). \tag{5.4.1}$$

Further the function $F(y)$ is called a symmetric distribution function if there exists some y_0 with the following property:

$$F(y_0 + y) = 1 - F(y_0 - y) \quad \text{for all} \quad y \in R^1. \tag{5.4.2}$$

Finally we suppose from now on that $y^{R-1}\,\{1 - F^{\Delta}(y) + F^{\Delta}(-y)\}$ (for R integer and $\geqslant 1$) is integrable over the positive part R^1_+ of the real axis R^1 for some Δ satisfying $0 < \Delta < 1$. From this assumption it follows that the Rth moment $\mu_R(k)$ $(k \geqslant \Delta)$ exists, and so do $\mu_1(k), \mu_2(k), \ldots, \mu_{R-1}(k)$. This will be proved in the following theorem.

Theorem 5.4.1. The moments $\mu_r(k)$ exist for all $0 < \Delta \leqslant k < \infty$ ($r = 1, 2, \ldots, R$).

Proof. From the assumption made above we have $y^{R-1}\{1 - F^{\Delta}(y) + F^{\Delta}(-y)\}$ is integrable over R^1_+, and so are $y^{R-1}\{1 - F^{\Delta}(y)\}$ and $y^{R-1}F^{\Delta}(-y)$. From this it follows that $y^{R-1}F^{\Delta+\delta}(-y)$ is integrable over R^1_+ for any positive constant δ.

Next we have

$$\int_0^\infty y^{R-1}\{1 - F^{\Delta+\delta}(y)\}\,\mathrm{d}y$$

$$= \int_0^\infty y^{R-1}[\{1 - F^{\Delta}(y)\}\{1 + F^{\delta}(y)\} - \{F^{\delta}(y) - F^{\Delta}(y)\}]\,\mathrm{d}y$$

$$= \int_0^\infty y^{R-1}\{1 - F^{\Delta}(y)\}\{1 + F^{\delta}(y)\}\,\mathrm{d}y - \int_0^\infty y^{R-1}\{F^{\delta}(y) - F^{\Delta}(y)\}\,\mathrm{d}y,$$

with $$\int_0^\infty y^{R-1}\{1 - F^{\Delta}(y)\}\{1 + F^{\delta}(y)\}\,\mathrm{d}y \leqslant 2\int_0^\infty y^{R-1}\{1 - F^{\Delta}(y)\}\,\mathrm{d}y < \infty$$

and $$\int_0^\infty y^{R-1}\{F^{\delta}(y) - F^{\Delta}(y)\}\,\mathrm{d}y \leqslant \int_0^\infty y^{R-1}\{1 - F^{\Delta}(y)\}\,\mathrm{d}y < \infty.$$

Thus all moments $\mu_r(k)$ ($r = 1, 2, \ldots, R$ and $\Delta \leqslant k < \infty$) do exist.

Theorem 5.4.2. $\mu_{2r-1}(k)$ is a (strictly) increasing function of k

$$\left(0 < \Delta \leqslant k < \infty;\quad r = 1, 2, \ldots, \left[\frac{R+1}{2}\right]\right).$$

Proof. This theorem is a consequence of the well-known theorem that if $F_1(y)$ and $F_2(y)$ are two cumulative distribution functions with $F_1(y) \geqslant F_2(y)$, then for each increasing function $\psi(y)$ the following holds:

$$\int_{R^1} \psi(y)\,\mathrm{d}F_1(y) \leqslant \int_{R^1} \psi(y)\,\mathrm{d}F_2(y) \tag{5.4.3}$$

(cf. Lehmann (1959) p. 112), assuming both integrals exist.

From now on we suppose that $R \geqslant 2$.

Theorem 5.4.3. If $F(y)$ is a symmetric continuous cumulative distribution function with expectation equal to zero, i.e.

$$\int_{R^1} y\,\mathrm{d}F(y) = 0,$$

then

$$\mu_{2r}(2) = \mu_{2r}(1) \qquad \left(r = 1, 2, \ldots, \left[\frac{R}{2}\right]\right).$$

Proof. We can write

$$\begin{aligned}\mu_{2r}(2) &= \int_{R^1} y^{2r}\, \mathrm{d}F^2(y)\\ &= 2 \int_{R^1} y^{2r}\, F(y)\, \mathrm{d}F(y)\\ &= 2 \int_{R^1} y^{2r}\, \{F(y) - \tfrac{1}{2}\}\, \mathrm{d}F(y) + \int_{R^1} y^{2r}\, \mathrm{d}F(y).\end{aligned} \tag{5.4.4}$$

The function $H(y) = y^{2r}\{F(y) - \frac{1}{2}\}$, being the integrand of the first integral, is an anti-symmetric function with respect to zero, i.e. $H(y) = -H(-y)$, so the first integral vanishes and the proof is complete.

Theorem 5.4.4. If $F(y)$ is a symmetric continuous cumulative distribution function with expectation equal to zero, i.e.

$$\int_{R^1} y\, \mathrm{d}F(y) = 0,$$

then for $r = 1, 2, \ldots, [R/2]$ the following holds:

(i) $\mu_{2r}(k)$ is a (strictly) decreasing function of k on the closed interval $0 < \Delta \leqslant k \leqslant 1$;

(ii) $\mu_{2r}(k)$ is a (strictly) increasing function of k for $2 \leqslant k < \infty$.

Proof.

(i) Choose an ε with $0 < \varepsilon < 1$. Then one can write

$$\begin{aligned}\mu_{2r}(k+\varepsilon) - \mu_{2r}(k) &\\ &= \int_{R^1} y^{2r}\, \mathrm{d}F^{k+\varepsilon}(y) - \int_{R^1} y^{2r}\, \mathrm{d}F^{k}(y)\\ &= \int_{R^1} y^{2r}\, \mathrm{d}\{F^{k+\varepsilon}(y) - F^{k}(y)\}\\ &= 2\, r \int_{R^1} y^{2r-1}\, \{F^{k}(y) - F^{k+\varepsilon}(y)\}\, \mathrm{d}y,\end{aligned} \tag{5.4.5}$$

using partial integration, which is allowed because

$$\lim_{y \to \pm\infty} y^{2r}\, \{F^{k}(y) - F^{k+\varepsilon}(y)\} = 0. \tag{5.4.6}$$

Thus a sufficient condition for $\mu_{2r}(k)$ being a (strictly) decreasing function of k $(0 < \Delta \leqslant k \leqslant 1)$ is:

$$\begin{aligned}F^{k}(y) - F^{k+\varepsilon}(y) &\leqslant F^{k}(-y) - F^{k+\varepsilon}(-y)\\ &= \{1 - F(y)\}^{k} - \{1 - F(y)\}^{k+\varepsilon}\end{aligned} \tag{5.4.7}$$

for all $y > 0$ and with strict inequality for $y \in I$ with I some interval in $(0, \infty)$ where $F(y) < 1$, or:

$$\left\{\frac{1-F(y)}{F(y)}\right\}^{1-k} \frac{F(y)}{1-F(y)} \frac{1-F^{\varepsilon}(y)}{1-\{1-F(y)\}^{\varepsilon}} \leqslant 1 \tag{5.4.8}$$

for all $y > 0$ with $(\frac{1}{2} <)\ F(y) < 1$ and with strict inequality for $y \in I$ with I some interval in $(0, \infty)$ where $F(y) < 1$.

Consider the function $\psi(z)$ defined by

$$\begin{aligned}\psi(z) &= z\,(1-z^{\varepsilon}) - (1-z)\,\{1-(1-z)^{\varepsilon}\} \\ &= 2\,z - 1 - z^{1+\varepsilon} + (1-z)^{1+\varepsilon}.\end{aligned} \tag{5.4.9}$$

Then

$$\psi(\tfrac{1}{2}) = 0 \quad \text{and} \quad \psi(1) = 0.$$

From the fact that

$$\psi''(z) = -\varepsilon\,(1+\varepsilon)\,\{z^{\varepsilon-1} - (1-z)^{\varepsilon-1}\} > 0 \quad \text{for} \quad \tfrac{1}{2} < z < 1 \tag{5.4.10}$$

it follows that

$$\psi(z) < 0 \quad \text{for} \quad \tfrac{1}{2} < z < 1.$$

From this property of $\psi(z)$ and from the fact that the form

$$\left\{\frac{1-F(y)}{F(y)}\right\}^{1-k}$$

is smaller than or equal to unity for $0 < \Delta \leqslant k \leqslant 1$ and all $y \geqslant 0$, this part of the theorem follows.

(ii) Again choose an ε with $0 < \varepsilon < 1$. A sufficient condition for (ii) is that for all $k \geqslant 2$

$$F^{k}(y) - F^{k+\varepsilon}(y) \geqslant \{1-F(y)\}^{k} - \{1-F(y)\}^{k+\varepsilon}$$

for all $y > 0$ with $(\frac{1}{2} <)\ F(y) < 1$ and with strict inequality for $y \in I$ with I some interval in $(0, \infty)$ where $F(y) < 1$, or:

$$\left\{\frac{F(y)}{1-F(y)}\right\}^{k} \frac{1-F^{\varepsilon}(y)}{1-\{1-F(y)\}^{\varepsilon}} \geqslant 1$$

for all $y > 0$ with $(\frac{1}{2} <)\ F(y) < 1$ and with strict inequality for $y \in I$ with I some interval in $(0, \infty)$ where $F(y) < 1$.

Analogously to part (i) one can prove that

$$\frac{F^{2}(y)}{\{1-F(y)\}^{2}} \frac{1-F^{\varepsilon}(y)}{1-\{1-F(y)\}^{\varepsilon}} > 1 \quad \text{for} \quad \tfrac{1}{2} < F(y) < 1.$$

From this fact and from the fact that the form

$$\left\{\frac{F(y)}{1-F(y)}\right\}^{k-2}$$

is larger than or equal to unity for $k \geqslant 2$ and all $y \geqslant 0$, the second part of the theorem follows.

Using the fundamental theory of total positivity (see Karlin (1968)) one can give an alternative proof of a similar theorem, namely that $\mu_{2r}(k)$ is first non-increasing and then non-decreasing with k, assuming $F(x)$ has a density $f(x)$. Using the theory of totally positive functions one can prove this without the requirement of symmetry and zero expectation. First we shall review some definitions and properties.

A real function $h(x,y)$, $x \in X$, $y \in Y$, $X \subset R^1$, $Y \subset R^1$, is said to be totally positive of order r (abbreviated TP_r) if for all m and all $x_1 < x_2 < \ldots < x_m$, $y_1 < y_2 < \ldots < y_m$ ($x_i \in X$, $y_j \in Y$, $i,j = 1, 2, \ldots, m$; $1 \leqslant m \leqslant r$):

$$\begin{vmatrix} h(x_1,y_1) & h(x_1,y_2) & \ldots & h(x_1,y_m) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ h(x_m,y_1) & h(x_m,y_2) & \ldots & h(x_m,y_m) \end{vmatrix} \geqslant 0.$$

If strict inequality holds, then we say that $h(x,y)$ is strictly totally positive of order r. Usually, X and Y are either intervals or countable sets of discrete values, such as the sets of all integers or the sets of non-negative integers.

Exponential families of densities, e.g. the class of densities $k\,F^{k-1}(y)\,f(y) = k\,f(y)\exp\{(k-1)\ln F(y)\}$ of two variables y and k, are totally positive of arbitrary order (TP_∞). The function $y^{2r} - c$ has for all real c at most two changes of sign as follows: positive→negative→positive. From the theorem of the variation-diminishing property it follows that

$$\mu_{2r}(k) - c = \int_{R^1} (y^{2r} - c)\,k\,F^{k-1}(y)\,f(y)\,\mathrm{d}y$$

has as a function of y for all c at most two changes of sign as follows: positive→negative→positive. Thus $\mu_{2r}(k)$ is first non-increasing and then non-decreasing with k on the interval I_1 where it is defined. If $f(y) > 0$ on I_1, then $k\,F^{k-1}(y)\,f(y)$ is strictly TP_∞ and consequently $\mu_{2r}(k)$ is first (strictly) decreasing and then (strictly) increasing.

If $f(y)$ is symmetric with

$$\int_{R^1} y\,\mathrm{d}F(y) = 0$$

then one can conclude, using theorem 5.4.3, that the minimum is reached in the interval (1,2).

In general we have for $\varepsilon > 0$:

$$\mu_{2r}(k+\varepsilon) = \left(1 + \frac{\varepsilon}{k}\right) \int_{R^1} y^{2r} F^{\varepsilon}(y)\, \mathrm{d}F^k(y)$$

$$< \mu_{2r}(k) + \frac{\varepsilon}{k}\, \mu_{2r}(k). \tag{5.4.11}$$

The following theorem gives some properties concerning the variance:

$$\sigma^2(k) \overset{\text{def}}{=} \int_{R^1} y^2\, \mathrm{d}F^k(y) - \{\int_{R^1} y\, \mathrm{d}F^k(y)\}^2. \tag{5.4.12}$$

Theorem 5.4.5. Concerning the variance $\sigma^2(k)$ we have:
(i) $\sigma^2(k+\varepsilon) < \sigma^2(k) + (\varepsilon/k)\,\sigma^2(k)$ for all $k \geqslant \Delta$ and all $\varepsilon > 0$,
(ii) If $F(y)$ is a symmetric continuous distribution function, then

$$\sigma^2(2) < \sigma^2(1)$$

and if $\mu_1(1) = 0$ and $f(y)$ exists and is positive on the interval I where it is defined then $\sigma^2(k)$ is a (strictly) decreasing function of k for $1 \leqslant k \leqslant k_0$ where k_0 is the value of k for which $\mu_2(k)$ is minimal.

Proof:
(i) We shall give the proof for a certain k ($\geqslant \Delta$). Without loss of generality it can be assumed that $\mu_1(k) = 0$. Then we have:

$$\begin{aligned}\sigma^2(k+\varepsilon) &= \mu_2(k+\varepsilon) - \mu_1{}^2(k+\varepsilon)\\ &< \mu_2(k+\varepsilon)\\ &< \mu_2(k) + \frac{\varepsilon}{k}\,\mu_2(k)\\ &= \sigma^2(k) + \frac{\varepsilon}{k}\,\sigma^2(k).\end{aligned}$$

(ii) Without loss of generality it can be assumed that $\mu_1(1) = 0$. Using theorem 5.4.3 one gets:

$$\begin{aligned}\sigma^2(1) - \sigma^2(2) &= \int_{R^1} y^2\, \mathrm{d}F(y) - \int_{R^1} y^2\, \mathrm{d}F^2(y) + \{\int_{R^1} y\, \mathrm{d}F^2(y)\}^2\\ &= \{\int_{R^1} y\, \mathrm{d}F^2(y)\}^2 > 0.\end{aligned}$$

The proof of the second part is straightforward.

Notice that e.g. for the non-symmetrical Exponential distribution with cumulative distribution function ($0 < \lambda < \infty$):

$$F(y) = \begin{cases} 0 & \text{for} \quad y < 0 \\ 1 - \exp(-y/\lambda) & \text{for} \quad 0 \leqslant y \end{cases}$$

we have

$$\sigma^2(1) = \lambda^2 \qquad \text{and} \qquad \sigma^2(2) = \tfrac{5}{4}\lambda^2$$

and thus:

$$\sigma^2(2) > \sigma^2(1) \qquad \text{for all } \lambda.$$

Another remark worthy of note is the following.

If $F(y)$ is a symmetric (continuous) distribution function and if

$$y^s F^{k-1-\delta}(y) \ln F(y) \qquad (s = 1, 2)$$

is integrable with respect to F for an arbitrarily small $\delta > 0$ then $\sigma^2(k)$ is differentiable with respect to k ($0 < \Delta < k < \infty$) and

$$\left[\frac{\partial \sigma^2(k)}{\partial k}\right]_{k=1} < 0.$$

The first part of this statement follows from the bounded convergence theorem of Lebesgue. It is sufficient to prove that there exists a function $h(y)$ integrable with respect to F, such that

$$\left| y^s \frac{F^{k-1+\varepsilon}(y) - F^{k-1}(y)}{\varepsilon} \right| \leqslant h(y)$$

for all y and all sufficiently small ε.

For $\varepsilon > 0$ one can write:

$$\left| y^s \frac{F^{k-1+\varepsilon}(y) - F^{k-1}(y)}{\varepsilon} \right| = |y^s| F^{k-1}(y) \left| \frac{F^{\varepsilon}(y) - 1}{\varepsilon \ln F(y)} \right| |\ln F(y)|$$

and for $\varepsilon = -\varepsilon' < 0$ one has:

$$\left| y^s \frac{F^{k-1-\varepsilon'}(y) - F^{k-1}(y)}{\varepsilon'} \right| = |y^s| F^{k-1-\varepsilon'}(y) \left| \frac{1 - F^{\varepsilon'}(y)}{\varepsilon' \ln F(y)} \right| |\ln F(y)|.$$

It is easy to see that for every positive ε,

$$\left|\frac{1 - F^{\varepsilon}(y)}{\varepsilon \ln F(y)}\right| \leqslant 1,$$

thus

$$\frac{\partial}{\partial k}\int_{R^1} y^s F^{k-1}(y)\, \mathrm{d}F(y) = \int_{R^1} y^s F^{k-1}(y) \ln F(y)\, \mathrm{d}F(y)$$

if $y^s F^{k-1-\delta}(y) \ln F(y)$ is integrable with respect to $F(y)$ for an arbitrarily small $\delta > 0$ $(s = 1, 2)$.

The proof of the second part of the statement runs as follows. Without loss of generality it can be assumed that $\mu_1(1) = 0$. Then we have:

$$\left[\frac{\partial}{\partial k}\sigma^2(k)\right]_{k=1} = \left[\frac{\partial}{\partial k}\int_{R^1} y^2\, \mathrm{d}F^k(y)\right]_{k=1} - 2\left\{\int_{R^1} y\, \mathrm{d}F(y)\right\}\left\{\int_{R^1} y\, \mathrm{d}F(y) + \right.$$

$$\left. + \int_{R^1} y \ln F(y)\, \mathrm{d}F(y)\right\} = \left[\frac{\partial}{\partial k}\int_{R^1} y^2\, \mathrm{d}F^k(y)\right]_{k=1} < 0,$$

which follows from the proof of part (i) of theorem 5.4.4.

If we define $\Delta(k;\alpha)$, with $0 < \alpha < \frac{1}{2}$, as the difference between the $(1 - \alpha)$-point $\xi_{1-\alpha}(k)$, thus

$$\int_{-\infty}^{\xi_{1-\alpha}(k)} \mathrm{d}F^k(y) = 1 - \alpha,$$

and the α-point $\xi_\alpha(k)$ of a strictly monotonic distribution function $F^k(y)$, then the following theorem can be proved.

Theorem 5.4.6. If $F(y)$ is the Logistic distribution function $(1 + \exp -y)^{-1}$ then $\Delta(k;\alpha)$ is a (strictly) decreasing function of k.

Proof. The inverse of the function

$$F^k(y) = (1 + \exp -y)^{-k}$$

equals

$$\psi(F^k) = \ln \frac{(F^k)^{1/k}}{1 - (F^k)^{1/k}},$$

so

$$\begin{aligned}\Delta(k;\alpha) &= \ln \frac{(1-\alpha)^{1/k}}{1-(1-\alpha)^{1/k}} - \ln \frac{\alpha^{1/k}}{1-\alpha^{1/k}} \\ &= \ln \left\{ \frac{(1-\alpha)^{1/k}}{1-(1-\alpha)^{1/k}} \, \frac{1-\alpha^{1/k}}{\alpha^{1/k}} \right\} \\ &= \ln \frac{(1/\alpha)^{1/k}-1}{\{1/(1-\alpha)\}^{1/k}-1} \\ &= \ln \frac{\beta^t - 1}{\{\beta/(\beta-1)\}^t - 1},\end{aligned}$$

where

$$t = \frac{1}{k} > 0, \quad \beta = \frac{1}{\alpha} > 2 \quad \text{and} \quad (0 <) \frac{\beta}{\beta-1} = \frac{1}{1-\alpha} < 2.$$

Now it can easily be proved that

$$h(t) = \frac{\beta^t - 1}{\{\beta/(\beta-1)\}^t - 1}$$

is a (strictly) increasing function of t. For

$$\left(\frac{\beta-1}{\beta}\right)^t \left\{\left(\frac{\beta}{\beta-1}\right)^t - 1\right\}^2 h'(t) = \ln \beta - (\beta-1)^t \ln \beta + (\beta^t - 1) \ln (\beta - 1)$$

and it is simple to verify that the function

$$\ln \beta - (\beta-1)^t \ln \beta + (\beta^t - 1) \ln (\beta-1)$$

is equal to zero if $t = 0$ and that its first derivative to t equals

$$\{\beta^t - (\beta-1)^t\} \ln \beta \ln (\beta-1),$$

which is larger than zero for all $t > 0$. From this it follows that $h(t)$ is a (strictly) increasing function of t, and so is $\ln h(t)$. Thus $\Delta(k;\alpha)$ is a (strictly) decreasing function of k.

Theorem 5.4.7. If $F(y)$ is a strictly monotonic distribution function with density function $f(x)$ for which, for a certain fixed α with $0 < \alpha < \frac{1}{2}$,

$$\frac{\alpha \ln \alpha}{f(\xi_\alpha(1))} < \frac{(1-\alpha)\ln(1-\alpha)}{f(\xi_{1-\alpha}(1))}, \tag{5.4.13}$$

then

$$\left[\frac{\partial \Delta(k;\alpha)}{\partial k}\right]_{k=1} < 0. \tag{5.4.14}$$

Remark. This theorem holds for each α with $0 < \alpha < \frac{1}{2}$. Notice that if the distribution function $F(y)$ is symmetric then condition (5.4.13) is satisfied. This follows from the fact that $\alpha \ln \alpha = (1-\alpha)\ln(1-\alpha)$ for $\alpha = 0, \frac{1}{2}$ and that $(d/d\alpha)\{(1-\alpha)\ln(1-\alpha) - \alpha \ln \alpha\}$ is larger than zero for $0 \leqslant \alpha < \frac{1}{2}\{1-(1-4/e^2)^{1/2}\}$ and smaller than zero for $\frac{1}{2}\{1-(1-4/e^2)^{1/2}\} < \alpha \leqslant \frac{1}{2}$.
Proof. Denote the inverse function of $F(y)$ by $G(z)$, then we have $(0 < z < 1)$

$$F(G(z)) = z.$$

From this it follows that

$$f(G(z))\, G'(z) = 1$$

or

$$G'(z) = \frac{1}{f(G(z))}.$$

The inverse function of $F^k(y)$ will be denoted by $G_k(z)$, thus

$$G_k(F^k(y)) = y = G(F(y)) \qquad (0 < F(y) < 1)$$

or

$$G_k(z) = G(z^{1/k}).$$

Now we have

$$\begin{aligned} \Delta(k;\alpha) &= G_k(1-\alpha) - G_k(\alpha) \\ &= G((1-\alpha)^{1/k}) - G(\alpha^{1/k}). \end{aligned}$$

Thus

$$\frac{\partial}{\partial k}\Delta(k;\alpha) = G'((1-\alpha)^{1/k})\left\{-\frac{1}{k^2}(1-\alpha)^{1/k}\ln(1-\alpha)\right\} + \\ - G'(\alpha^{1/k})\left\{-\frac{1}{k^2}\alpha^{1/k}\ln\alpha\right\}$$

and

$$\left[\frac{\partial}{\partial k}\Delta(k;\alpha)\right]_{k=1} = \frac{\alpha\ln\alpha}{f(G(\alpha))} - \frac{(1-\alpha)\ln(1-\alpha)}{f(G(1-\alpha))} < 0.$$

5.5. Some numerical results

Tables with numerical values of $\mu_1(k)$, $\mu_2(k)$, $\sigma^2(k)$ and $\sigma(k)$ for some well-known distributions are given below for

$$k = \cdot 1\ (\cdot 1)\ 1\ (\cdot 5)\ 5\ (1)\ 10.$$

Normal distribution

In table 5.5-I the results for the Normal distribution are presented. For these computations (by Simpson's rule) the approximation formula

$$\Phi^*(y) = 1 - (a_1\,\eta + a_2\,\eta^2 + a_3\,\eta^3 + a_4\,\eta^4 + a_5\,\eta^5)\,\Phi_0'(y), \quad (5.5.1)$$

where

$$\eta = \frac{1}{1+py}, \quad p = \cdot 3275911 \quad \text{and}$$

$$a_1 = \cdot 2258\,36846,$$

$$a_2 = -\cdot 2521\,28668,$$

$$a_3 = 1{\cdot}2596\,95130,$$

$$a_4 = -1{\cdot}2878\,22453,$$

$$a_5 = \cdot 9406\,46070$$

for

$$\Phi_0(y) = \frac{2}{\sqrt{\pi}} \int_0^y \exp(-t^2)\,dt \qquad (0 \leqslant y < \infty)$$

TABLE 5.5-I

Some moments of $\Phi^k(y)$ where $\Phi(y)$ is the standard Normal cumulative distribution function

k	$\mu_1(k)$	$\mu_2(k)$	$\sigma^2(k)$	$\sigma(k)$
·1	−3·264	16·048	5·395	2·323
·2	−1·956	6·881	3·056	1·748
·3	−1·347	4·056	2·241	1·497
·4	− ·970	2·760	1·819	1·349
·5	− ·704	2·053	1·557	1·248
·6	− ·503	1·631	1·378	1·174
·7	− ·342	1·363	1·247	1·117
·8	− ·209	1·189	1·146	1·070
·9	− ·097	1·075	1·066	1·032
1	0	1	1	1
1·5	·344	·912	·794	·891
2	·564	1	·682	·826
2·5	·723	1·133	·610	·781
3	·846	1·276	·559	·748
3·5	·946	1·416	·522	·722
4	1·029	1·551	·492	·701
4·5	1·101	1·679	·468	·684
5	1·163	1·800	·448	·669
6	1·267	2·022	·416	·645
7	1·352	2·220	·392	·626
8	1·424	2·400	·373	·611
9	1·485	2·563	·357	·598
10	1·539	2·712	·344	·587

(Hastings (1955)), and the asymptotic expansion:

$$\frac{1}{(2\,\pi)^{1/2}}\int_{y}^{\infty}\exp(-t^2/2)\,\mathrm{d}t \sim \frac{1}{(2\,\pi)^{1/2}}\,\frac{\exp(-y^2/2)}{y}\left\{1+\sum_{i=1}^{\infty}(-1)^i\,\frac{1.3...(2i-1)}{y^{2i}}\right\}$$

have been used. The last expression, up to and including the term with $i = 6$, has been used for $y \geqslant 4$.

For purposes of illustration we present an additional table 5.5-II of the median $\nu(k)$ and interquartile distance $\tau(k)$ for $\Phi^k(y)$, where $\Phi(y)$ is the cumulative standard Normal distribution function. This gives a rough idea of the changes of location and variation. One has

$$\nu(k) = \Phi^{-1}((\tfrac{1}{2})^{1/k})$$
$$\tau(k) = \Phi^{-1}((\tfrac{3}{4})^{1/k}) - \Phi^{-1}((\tfrac{1}{4})^{1/k}),$$

where $\Phi^{-1}(.)$ is the inverse function of $\Phi(.)$. For these computations an approximation formula for $\Phi^{-1}(.)$ has been used (Hastings (1955), sheet 68). The accuracy is $\pm$ 1 in the last decimal.

TABLE 5.5-II

Median $\nu(k)$ and interquartile distance $\tau(k)$ of $\Phi^k(y)$

k	$\nu(k)$	$\tau(k)$	k	$\nu(k)$	$\tau(k)$
·05	—4·763	4·317	2·5	·699	1·047
·10	—3·098	3·176	3	·819	1·001
·15	—2·333	2·677	3·5	·917	·965
·20	—1·863	2·383	4	·998	·936
·25	—1·534	2·183	4·5	1·068	·912
·30	—1·286	2·036	5	1·129	·891
·35	—1·089	1·923	6	1·231	·858
·40	— ·928	1·831	7	1·315	·831
·45	— ·791	1·755	8	1·385	·810
·50	— ·674	1·691	9	1·446	·792
·55	— ·572	1·637	10	1·499	·777
·60	— ·481	1·589	15	1·694	·724
·65	— ·400	1·547	20	1·825	·691
·70	— ·327	1·510	30	1·999	·650
·75	— ·261	1·476	40	2·116	·625
·80	— ·200	1·446	50	2·204	·607
·85	— ·144	1·418	60	2·274	·593
·90	— ·093	1·393	70	2·332	·582
·95	— ·045	1·370	80	2·382	·573
1	0	1·348	90	2·425	·565
1·5	·331	1·197	100	2·462	·558
2	·545	1·108	1000	3·198	·451

Beta distribution

In the tables 5.5-III–5.5-VII some moments of the kth power of the Beta

TABLE 5.5-III

Some moments of $B_{1,3}{}^{k}(y)$

k	$\mu_1(k)$	$\mu_2(k)$	$\sigma^2(k)$	$\sigma(k)$
·1	·041	·013	·011	·105
·2	·076	·025	·019	·137
·3	·107	·036	·024	·156
·4	·134	·047	·028	·169
·5	·159	·057	·031	·177
·6	·180	·066	·034	·183
·7	·200	·075	·035	·187
·8	·218	·084	·036	·190
·9	·235	·092	·037	·192
1	·250	·100	·038	·194
1·5	·312	·135	·038	·195
2	·357	·164	·037	·192
2·5	·393	·189	·035	·187
3	·421	·211	·033	·183
3·5	·445	·230	·032	·178
4	·466	·247	·030	·174
4·5	·484	·263	·029	·170
5	·499	·277	·028	·167
6	·526	·302	·026	·160
7	·547	·323	·024	·154
8	·565	·342	·022	·149
9	·581	·358	·021	·145
10	·594	·373	·020	·141

TABLE 5.5-IV

Some moments of $B_{2,2}{}^{k}(y)$

k	$\mu_1(k)$	$\mu_2(k)$	$\sigma^2(k)$	$\sigma(k)$
·1	·115	·049	·036	·189
·2	·199	·091	·051	·227
·3	·265	·128	·058	·241
·4	·317	·161	·060	·245
·5	·361	·190	·060	·244
·6	·397	·216	·058	·242
·7	·428	·240	·057	·238
·8	·455	·262	·054	·233
·9	·479	·282	·052	·228
1	·500	·300	·050	·224
1·5	·578	·374	·040	·201
2	·629	·429	·033	·183
2·5	·665	·471	·028	·168
3	·693	·505	·024	·157
3·5	·715	·533	·022	·147
4	·733	·556	·019	·138
4·5	·748	·577	·017	·131
5	·761	·594	·016	·125
6	·782	·624	·013	·115
7	·798	·648	·011	·107
8	·811	·668	·010	·100
9	·822	·685	·009	·094
10	·831	·699	·008	·090

TABLE 5.5-V

Some moments of $B_{2,5}{}^{k}(y)$

k	$\mu_1(k)$	$\mu_2(k)$	$\sigma^2(k)$	$\sigma(k)$
·1	·060	·016	·012	·110
·2	·106	·030	·019	·136
·3	·143	·043	·022	·149
·4	·173	·054	·024	·155
·5	·199	·065	·025	·159
·6	·221	·074	·026	·160
·7	·240	·083	·026	·161
·8	·257	·092	·026	·161
·9	·272	·100	·026	·160
1	·286	·107	·026	·160
1·5	·339	·139	·024	·155
2	·376	·163	·022	·150
2·5	·404	·184	·021	·144
3	·426	·201	·020	·140
3·5	·445	·216	·018	·136
4	·460	·229	·018	·133
4·5	·474	·241	·017	·130
5	·485	·252	·016	·127
6	·505	·270	·015	·122
7	·522	·286	·014	·118
8	·535	·300	·013	·115
9	·547	·312	·013	·112
10	·557	·323	·012	·109

TABLE 5.5-VI

Some moments of $B_{6,2}{}^{k}(y)$

k	$\mu_1(k)$	$\mu_2(k)$	$\sigma^2(k)$	$\sigma(k)$
·1	·303	·160	·068	·261
·2	·449	·266	·064	·254
·3	·536	·342	·055	·234
·4	·595	·400	·046	·214
·5	·638	·446	·039	·198
·6	·670	·483	·034	·184
·7	·696	·514	·029	·171
·8	·717	·541	·026	·161
·9	·735	·563	·023	·152
1	·750	·583	·021	·144
1.5	·801	·655	·014	·116
2	·831	·700	·010	·099
2·5	·851	·732	·008	·087
3	·866	·755	·006	·078
3·5	·877	·774	·005	·072
4	·886	·789	·004	·066
4·5	·893	·801	·004	·062
5	·899	·812	·003	·058
6	·909	·829	·003	·052
7	·916	·842	·002	·048
8	·922	·853	·002	·044
9	·927	·862	·002	·041
10	·931	·869	·002	·039

TABLE 5.5-VII

Some moments of $B_{1/2,1/2}{}^{k}(y)$

k	$\mu_1(k)$	$\mu_2(k)$	$\sigma^2(k)$	$\sigma(k)$
·1	·078	·050	·044	·210
·2	·147	·097	·075	·274
·3	·208	·140	·097	·311
·4	·263	·181	·111	·334
·5	·313	·219	·121	·347
·6	·358	·254	·126	·355
·7	·399	·287	·128	·358
·8	·435	·318	·129	·359
·9	·469	·348	·127	·357
1	·500	·375	·125	·354
1·5	·621	·490	·105	·325
2	·703	·578	·084	·290
2·5	·761	·645	·066	·257
3	·804	·698	·052	·227
3·5	·836	·740	·041	·202
4	·862	·775	·032	·180
4·5	·881	·803	·026	·161
5	·897	·826	·021	·145
6	·921	·862	·014	·118
7	·937	·888	·010	·098
8	·949	·908	·007	·083
9	·958	·922	·005	·071
10	·965	·934	·004	·061

distribution are presented. The cumulative distribution function of the Beta distribution is defined as

$$B_{p,q}(y) = \begin{cases} 0 & \text{for} \quad y < 0 \\ \dfrac{\Gamma(p+q)}{\Gamma(p)\,\Gamma(q)} \displaystyle\int_0^y t^{p-1}\,(1-t)^{q-1}\,\mathrm{d}t & \text{for} \quad 0 \leqslant y \leqslant 1 \\ 1 & \text{for} \quad 1 < y. \end{cases} \tag{5.5.2}$$

Gamma distribution

In the tables 5.5-VIII–5.5.-XI some moments of the kth power of the Gamma distribution $\Gamma_p(y)$, $\Gamma_p{}^k(y)$, are given for different values of k. Here $\Gamma_p(y)$ is defined as the cumulative distribution function

$$\Gamma_p(y) = \begin{cases} 0 & y < 0 \\ \dfrac{1}{\Gamma(p)} \displaystyle\int_0^y \exp(-t)\, t^{p-1}\,\mathrm{d}t & 0 \leqslant y \\ & (p > 0) \end{cases} \tag{5.5.3}$$

with

$$\Gamma(p) = \int_0^\infty \exp(-t)\, t^{p-1}\,\mathrm{d}t.$$

TABLE 5.5-VIII

Some moments of $\Gamma_1{}^k(y)$, the kth power of the cumulative Exponential distribution function

k	$\mu_1(k)$	$\mu_2(k)$	$\sigma^2(k)$	$\sigma(k)$
·1	·153	·235	·212	·460
·2	·288	·461	·378	·614
·3	·408	·677	·511	·715
·4	·516	·886	·620	·787
·5	·614	1·087	·710	·843
·6	·703	1·281	·787	·887
·7	·786	1·469	·852	·923
·8	·862	1·651	·908	·953
·9	·933	1·828	·957	·978
1	1	2	1	1
1·5	1·280	2·794	1·155	1·075
2	1·500	3·500	1·250	1·118
2·5	1·680	4·138	1·315	1·147
3	1·833	4·722	1·361	1·167
3·5	1·966	5·262	1·396	1·182
4	2·083	5·764	1·424	1·193
4·5	2·188	6·234	1·446	1·202
5	2·283	6·677	1·464	1·210
6	2·450	7·494	1·491	1·221
7	2·593	8·235	1·512	1·230
8	2·718	8·914	1·527	1·236
9	2·829	9·543	1·540	1·241
10	2·929	10·129	1·550	1·245

TABLE 5.5-IX

Some moments of $\Gamma_2{}^k(y)$

k	$\mu_1(k)$	$\mu_2(k)$	$\sigma^2(k)$	$\sigma(k)$
·1	·391	·806	·653	·808
·2	·698	1·545	1·057	1·028
·3	·949	2·228	1·323	1·152
·4	1·161	2·865	1·516	1·231
·5	1·344	3·462	1·655	1·287
·6	1·505	4·024	1·760	1·327
·7	1·647	4·555	1·841	1·357
·8	1·776	5·060	1·906	1·381
·9	1·893	5·541	1·958	1·399
1	2	6	2	1·414
1·5	2·431	8·036	2·128	1·459
2	2·750	9·750	2·188	1·479
2·5	3·003	11·238	2·219	1·489
3	3·213	12·559	2·236	1·495
3·5	3·392	13·748	2·245	1·498
4	3·547	14·832	2·250	1·500
4·5	3·685	15·830	2·252	1·501
5	3·808	16·756	2·253	1·501
6	4·022	18·430	2·251	1·500
7	4·204	19·916	2·247	1·499
8	4·361	21·256	2·242	1·497
9	4·499	22·478	2·237	1·496
10	4·623	23·603	2·231	1·494

TABLE 5.5-X

Some moments of $\Gamma_3{}^k(y)$

k	$\mu_1(k)$	$\mu_2(k)$	$\sigma^2(k)$	$\sigma(k)$
·1	·690	1·796	1·320	1·149
·2	1·184	3·376	1·974	1·405
·3	1·564	4·788	2·342	1·530
·4	1·871	6·068	2·569	1·603
·5	2·127	7·241	2·716	1·648
·6	2·347	8·324	2·816	1·678
·7	2·539	9·332	2·887	1·699
·8	2·709	10·276	2·937	1·714
·9	2·862	11·162	2·973	1·724
1	3	12	3	1·732
1·5	3·544	15·617	3·056	1·748
2	3·938	18·562	3·059	1·749
2·5	4·244	21·061	3·045	1·745
3	4·496	23·238	3·027	1·740
3·5	4·708	25·171	3·007	1·734
4	4·891	26·914	2·988	1·728
4·5	5·053	28·503	2·969	1·723
5	5·197	29·964	2·951	1·718
6	5·446	32·581	2·920	1·709
7	5·656	34·878	2·892	1·701
8	5·836	36·930	2·867	1·693
9	5·995	38·787	2·845	1·687
10	6·137	40·484	2·826	1·681

TABLE 5.5-XI

Some moments of $\Gamma_4{}^k(y)$

k	$\mu_1(k)$	$\mu_2(k)$	$\sigma^2(k)$	$\sigma(k)$
·1	1·040	3·281	2·200	1·483
·2	1·725	6·050	3·074	1·753
·3	2·227	8·454	3·493	1·869
·4	2·620	10·584	3·718	1·928
·5	2·942	12·499	3·845	1·961
·6	3·213	14·242	3·919	1·980
·7	3·447	15·844	3·962	1·991
·8	3·652	17·326	3·986	1·996
·9	3·835	18·707	3·997	1·999
1	4	20	4	2
1·5	4·639	25·482	3·961	1·990
2	5·094	29·844	3·897	1·974
2·5	5·445	33·484	3·835	1·958
3	5·731	36·619	3·779	1·944
3·5	5·971	39·378	3·728	1·931
4	6·178	41·846	3·684	1·919
4·5	6·359	44·081	3·644	1·909
5	6·521	46·126	3·608	1·899
6	6·798	49·762	3·546	1·883
7	7·031	52·931	3·494	1·869
8	7·231	55·745	3·450	1·858
9	7·407	58·277	3·412	1·847
10	7·563	60·582	3·379	1·838

Finally, for purposes of illustration, in table 5.5-XII the mode of $k F^{k-1}(y) f(y)$ is presented as function of k, where $F(y)$ is the Weibull cumulative distribution function

$$F(y) = \begin{cases} 0 & \text{for } \; y < 0 \\ 1 - \exp(-y^{\alpha}) & \text{for } \; 0 \leqslant y \end{cases} \tag{5.5.4}$$

and with density

$$f(y) = \begin{cases} 0 & \text{for } \; y < 0 \\ \alpha\, y^{\alpha-1} \exp(-y^{\alpha}) & \text{for } \; 0 \leqslant y. \end{cases} \tag{5.5.5}$$

The density of $F^k(y)$ has the following form:

$$\alpha\, k\, y^{\alpha-1} \exp(-y^{\alpha}) \{1 - (\exp -y^{\alpha})\}^{k-1} \qquad (0 \leqslant y). \tag{5.5.6}$$

Defining $z = y^{\alpha}$ one gets for the first derivative of (5.5.6) with respect to y

$$k\, e^{-z} (1 - e^{-z})^{k-2} \{(k-1) e^{-z} (z')^2 - (1 - e^{-z}) (z')^2 + (1 - e^{-z}) z''\}. \tag{5.5.7}$$

Considering only cases where the top is reached in a point larger than zero, thus $\alpha k > 1$, then the top of the density function of $F^k(y)$ is reached in the point $y_0 = (1 - 1/\alpha)^{1/\alpha}$ for $k = 1$, whereas for $k \neq 1$ the point y_0 has been solved from the equation

$$(\exp z_0)\left(1 - \frac{1}{\alpha} - z_0\right) + k\, z_0 - 1 + \frac{1}{\alpha} = 0, \tag{5.5.8}$$

where

$$z_0 = y_0^{\alpha},$$

using Regula Falsi.

TABLE 5.5-XII

Modes of $F^k(y)$ where $F(y)$ is the Weibull cumulative distribution function

k \ α	·2	·5	·8	1	1·5	2	4	6
·2								·617
·4							·678	·830
·6						·353	·810	·900
·8					·279	·576	·882	·941
1					·481	·707	·931	·971
1·5			·135	·405	·775	·907	1·009	1·017
2			·436	·693	·958	1·030	1·058	1·047
4		·767	1·326	1·386	1·351	1·290	1·160	1·109
6	·0023	1·836	1·909	1·792	1·561	1·426	1·212	1·141
8	·216	2·840	2·343	2·079	1·703	1·516	1·246	1·161

In the figures 5.5.1–5.5.9 the densities $g_k(y) = k\, F^{k-1}(y)\, f(y)$ corresponding to $G_k(y) = F^k(y)$ are presented for some well-known distribution functions $F(y)$ with a selection of values for k. These diagrams give some idea about the changes of location, variance and skewness due to changes of k. The parent distributions and the corresponding densities are:

Standard Normal distribution $\Phi(y)$ with

$$g_k(y) = \frac{k}{(2\pi)^{1/2}} \exp(-y^2/2) \left(\frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{y} \exp(-t^2/2)\, \mathrm{d}t \right)^{k-1},$$

Beta distribution $B_{2,2}(y)$ with

$$g_k(y) = 6\, k\, y\, (1-y)\, \{y^3 + 3\, y^2\, (1-y)\}^{k-1},$$

Beta distribution $B_{2,5}(y)$ with

$$g_k(y) = 30\, k\, y\, (1-y)^4\, \{1-(1-y)^6 - 6\, y\, (1-y)^5\}^{k-1},$$

Exponential distribution $\Gamma_1(y)$ with

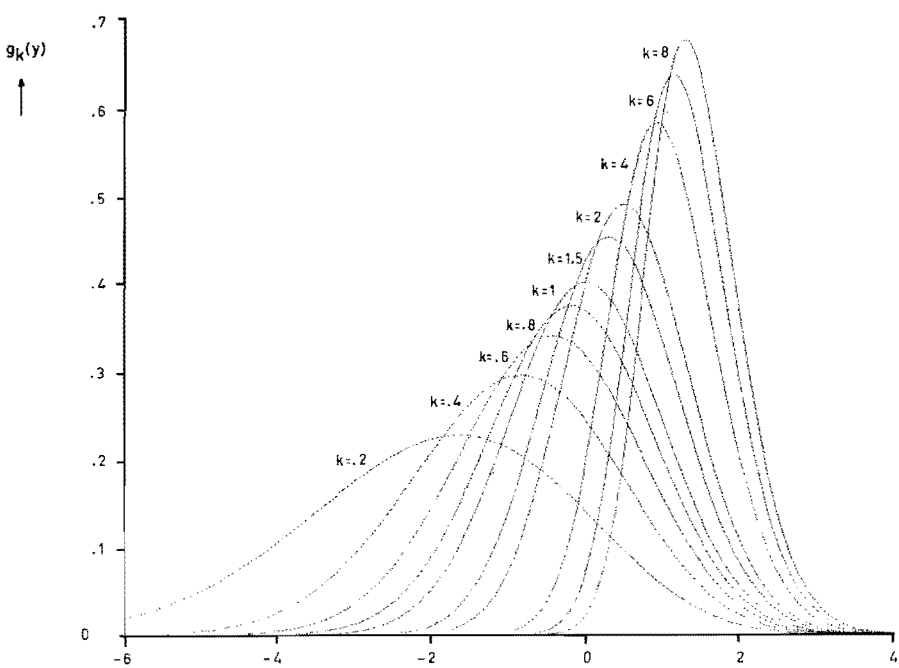


Fig. 5.5.1. The density $g_k(y) = \mathrm{d}F^k(y)/\mathrm{d}y = k\, F^{k-1}(y)\, f(y)$, where $F(y)$ is the standard Normal distribution function, for some values of k.

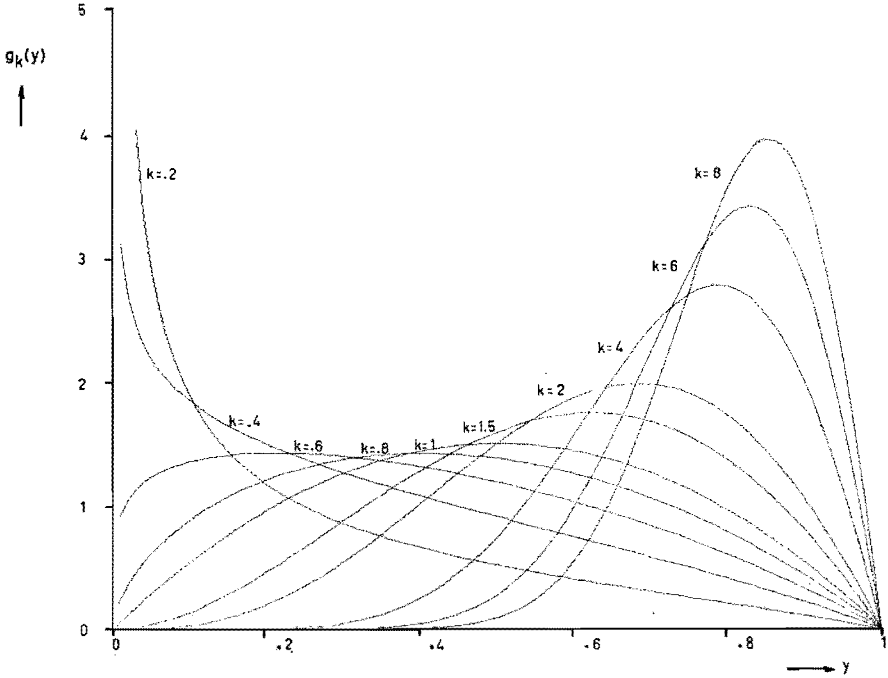


Fig. 5.5.2. The density $g_k(y) = \mathrm{d}F^k(y)/\mathrm{d}y = k\,F^{k-1}(y)\,f(y)$, where $F(y)$ is the Beta distribution function $B_{2,2}(y)$, for some values of k.

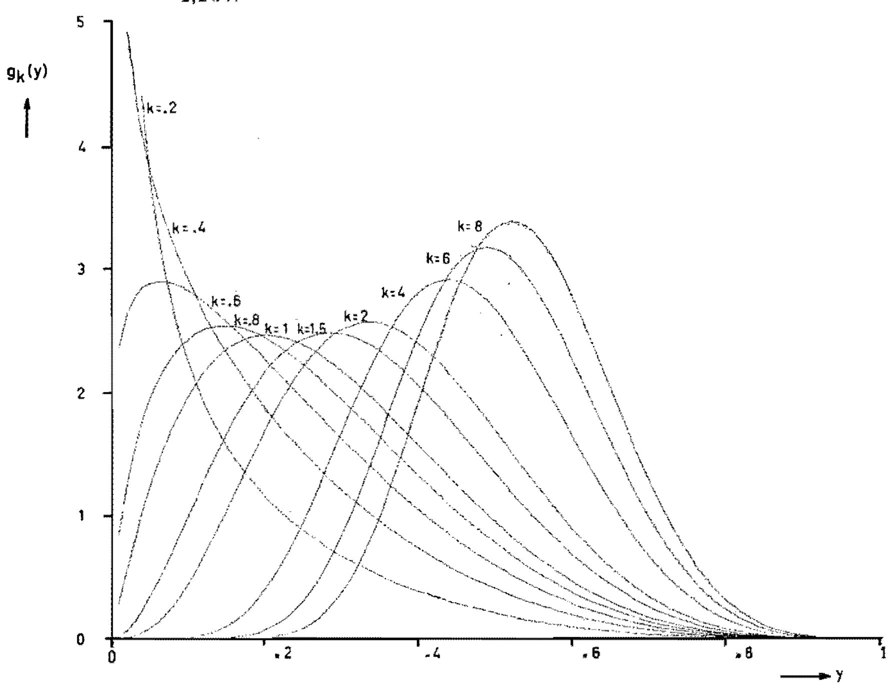


Fig. 5.5.3. The density $g_k(y) = \mathrm{d}F^k(y)/\mathrm{d}y = k\,F^{k-1}(y)\,f(y)$, where $F(y)$ is the Beta distribution function $B_{2,5}(y)$, for some values of k.

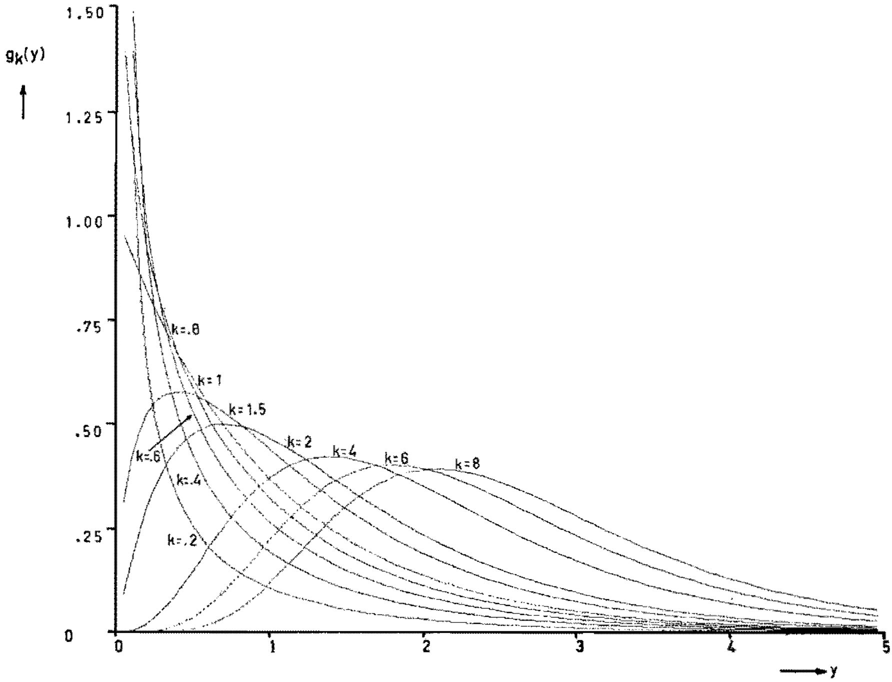


Fig. 5.5.4. The density $g_k(y) = \mathrm{d}F^k(y)/\mathrm{d}y = k\,F^{k-1}(y)\,f(y)$, where $F(y)$ is the Exponential distribution function $\Gamma_1(y)$, for some values of k.

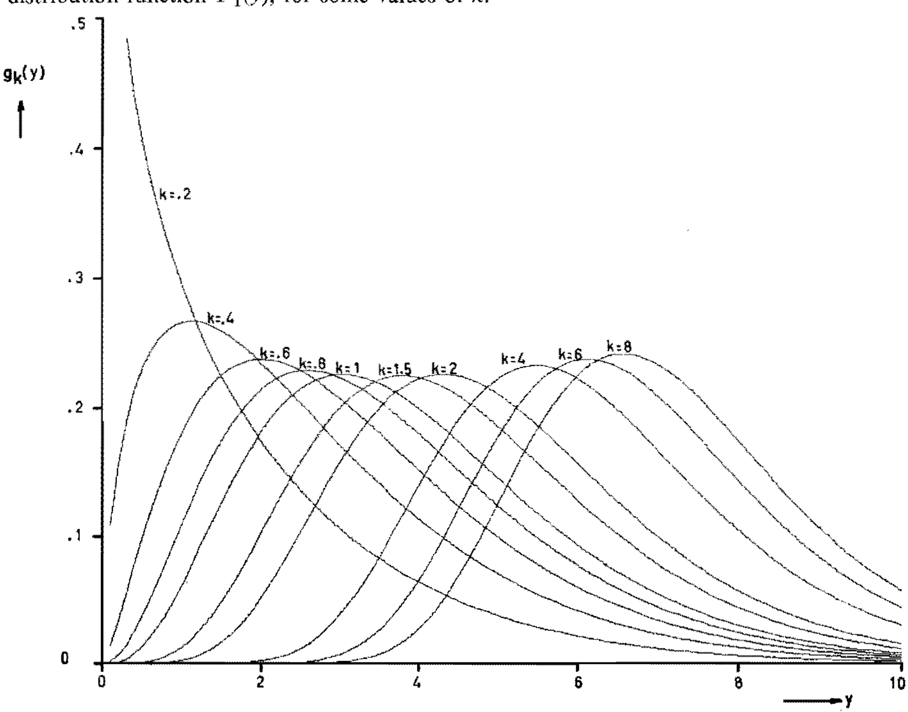


Fig. 5.5.5. The density $g_k(y) = \mathrm{d}F^k(y)/\mathrm{d}y = k\,F^{k-1}(y)\,f(y)$, where $F(y)$ is the Gamma distribution function $\Gamma_4(y)$, for some values of k.

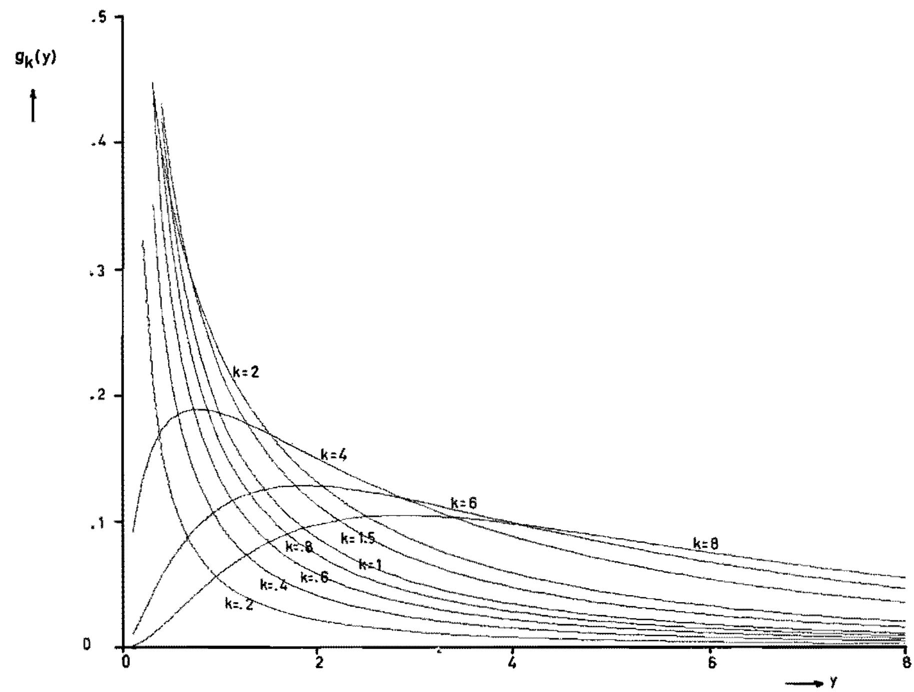


Fig. 5.5.6. The density $g_k(y) = \mathrm{d}F^k(y)/\mathrm{d}y = k\,F^{k-1}(y)\,f(y)$, where $F(y)$ is the Weibull distribution function with $\alpha = \cdot 5$, for some values of k.

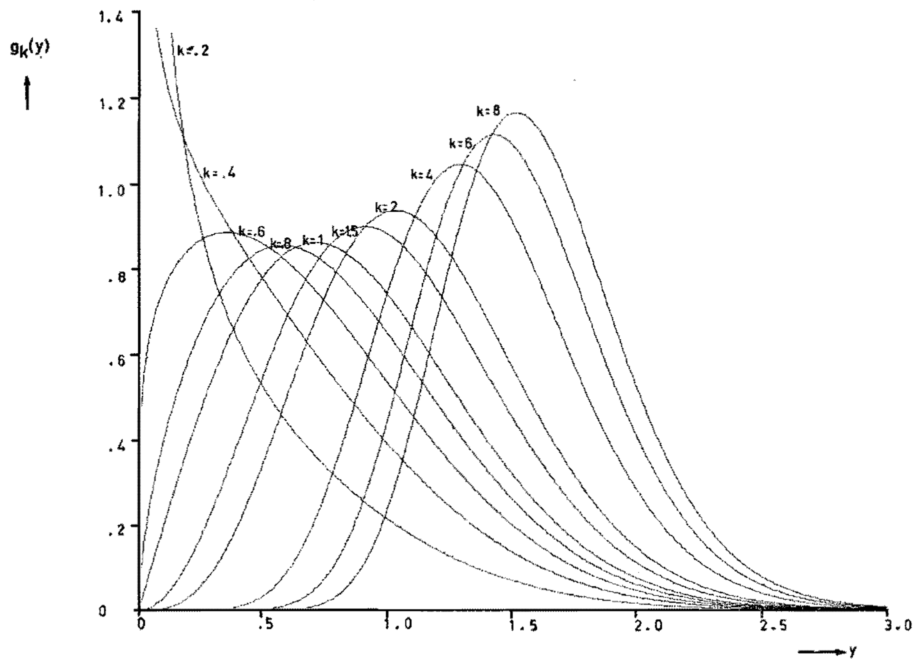
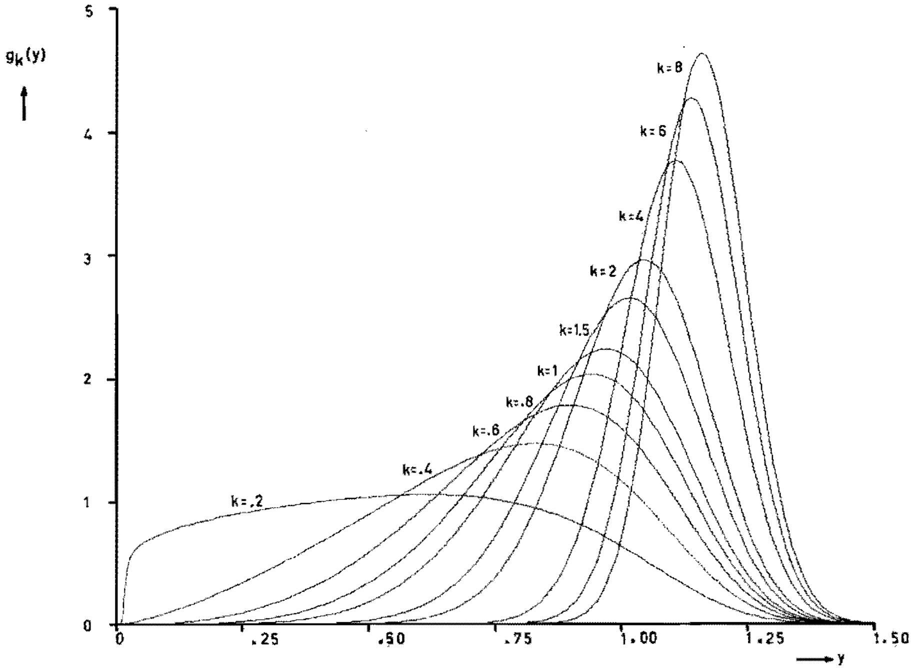


Fig. 5.5.7. The density $g_k(y) = \mathrm{d}F^k(y)/\mathrm{d}y = k\,F^{k-1}(y)\,f(y)$, where $F(y)$ is the Weibull distribution function with $\alpha = 2$, for some values of k.



Fi. 5.5.8. The density $g_k(y) = \mathrm{d}F^k(y)/\mathrm{d}y = k\,F^{k-1}(y)\,f(y)$, where $F(y)$ is the Weibull distribution function with $\alpha = 6$, for some values of k.

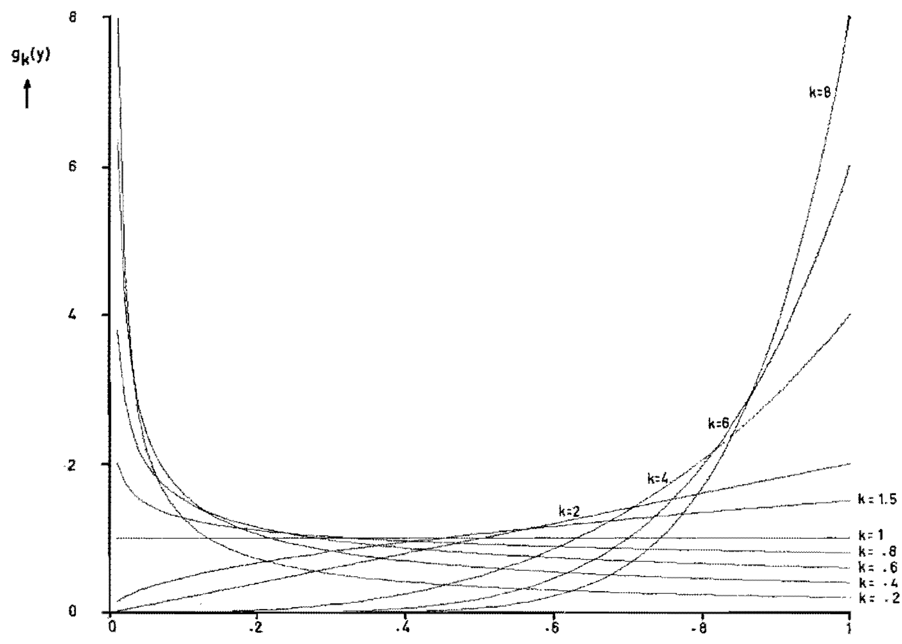


Fig. 5.5.9. The density $g_k(y) = \mathrm{d}F^k(y)/\mathrm{d}y = k\,F^{k-1}(y)\,f(y)$, where $F(y)$ is the Uniform distribution function, for some values of k.

$$g_k(y) = k\, e^{-y} (1 - e^{-y})^{k-1},$$

Gamma distribution $\Gamma_4(y)$ with

$$g_k(y) = \tfrac{1}{6} k\, y^3 e^{-y} (1 - e^{-y} - y\, e^{-y} - \tfrac{1}{2} y^2 e^{-y} - \tfrac{1}{6} y^3 e^{-y})^{k-1},$$

Weibull distribution with

$$\alpha = \cdot 5,\ 2,\ 6,$$

so that

$$g_k(y) = \alpha\, k\, y^{\alpha-1} \exp(-y^{\alpha}) \{1 - (\exp - y^{\alpha})\}^{k-1}$$

with $\alpha = \cdot 5$, 2 and 6,

Uniform distribution on (0, 1) with

$$g_k(y) = k\, y^{k-1}.$$

6. ASYMPTOTIC EFFICIENCY OF THE V_1 TESTS

It is of interest to investigate the efficiency of the V_1 tests as has been done for other two-sample tests like those of Wilcoxon, Van der Waerden, Terry, etc. One would surmise that they are less efficient than those of Wilcoxon or Van der Waerden in most cases. However, in chapter 7 we shall compare the V_1 tests with some other well-known tests in the small-sample case, and the result will be that in the case of Lehmann alternatives Wilcoxon's test is sometimes better and sometimes less efficient than the V_{jl} tests selected for Lehmann alternatives. In this chapter we shall determine Pitman's asymptotic relative efficiency of the V_1 tests relative to the two-sample tests of Student and Wilcoxon (Mann–Whitney), respectively.

A test T_1 has an efficiency e relative to a test T_2 (both with the same size) with respect to a certain alternative hypothesis, if the power of test T_1 using n observations is equal to the power of test T_2 using $e\,n$ observations. In other words, the relative efficiency of the first test with respect to the second test, both with the same size, the same alternative hypothesis and the same power, is given by the ratio of the sample sizes $e\,n$ and n of the second and first test, respectively.

Unfortunately, for many test procedures the asymptotic power is the only one available. Then one can often use the concept of Pitman's asymptotic local relative efficiency. Let $\underline{T}_n = T(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n)$ be a consistent test statistic for testing the statistical hypothesis that a certain parameter θ has the value θ_0. Let $E\{\underline{T}_n\} = \psi_n(\theta)$ and $\sigma^2\{\underline{T}_n\} = \sigma_n^2(\theta)$. Pitman called the quantity

$$H^2(n) = \frac{\{\psi_n'(\theta_0)\}^2}{\sigma_n^2(\theta_0)} \tag{6.1}$$

the efficacy of the test statistic $\underline{T}_n$ for testing the null hypothesis

$$H_0 : \theta = \theta_0 \tag{6.2}$$

against the alternative hypothesis

$$H_1 : \theta = \theta_n = \theta_0 + \frac{k}{\sqrt{n}} \qquad (k > 0). \tag{6.3}$$

Comparing two test statistics with efficacies H_1^2 and H_2^2 the asymptotic relative efficiency is defined as the limit of the ratio H_1^2/H_2^2 of the efficacies. The following conditions must be satisfied for the two righthand-sided tests:

(i) The tests are consistent.

(ii) The variances of both test statistics are finite.

(iii) $\psi_{in}'(\theta) = \partial\psi_{in}(\theta)/\partial\theta$ exists $(i = 1, 2)$, where $\psi_{in}(\theta) = E\{\underline{T}_{in}\}$ and $\theta_0 \leqslant \theta \leqslant \theta_0 + k/\sqrt{n}$ $(k > 0)$.

(iv) $\sigma_{in}(\theta_n)/\sigma_{in}(\theta_0) \to 1$ and $\psi_{in}'(\theta_n)/\psi_{in}'(\theta_0) \to 1$ where $\theta_n \to \theta_0$ as $n \to \infty$.
(v) $\{\underline{T}_{in} - \psi_{in}(\theta)\}/\sigma_{in}(\theta)$ has a distribution which tends to a standard Normal distribution, uniformly in θ, as $n \to \infty$.
(vi) $\psi_{in}'(\theta_0)/\sigma_n(\theta_0) \sim c\sqrt{n}$ $(c > 0)$.

For the investigation of the asymptotic power of V_{ji} tests with

$$\underline{V}_{ji} = \underline{y}_{(j)} - \underline{x}_{(i)}$$

one has to be specific about the way j and i tend to infinity.

Suppose $i/m \to F(t)$ with $0 < F(t) < 1$, and that $f(t) = F'(t)$ exists and is continuous in the neighbourhood of t and $f(t) \neq 0$, then

$$\sigma^2\{\underline{x}_{(i)}\} \sim \frac{F(t)\,\{1 - F(t)\}}{m f^2(t)}.$$

It is clear that j/n must tend to the same constant $F(t)$, thus

$$\sigma^2\{\underline{y}_{(j)}\} \sim \frac{F(t)\,\{1 - F(t)\}}{n f^2(t)}.$$

Then

$$\sigma^2\{\underline{V}_{ji}\} \sim \frac{m+n}{m\,n}\,\frac{F(t)\,\{1 - F(t)\}}{f^2(t)}.$$

Now it is of interest to answer the question for which t $\sigma^2\{\underline{V}_{ji}\}$ is asymptotically minimal. It is clear that this will give maximal Pitman efficiency. We shall restrict ourselves to the Normal distribution.

If $F(t)$ is the standard Normal distribution function $\Phi(t)$, then it can be proved that $\sigma^2\{\underline{V}_{ji}\}$ is minimal for $t = 0$. This can be proved by showing that the first derivative of $[\Phi(t)\,\{1 - \Phi(t)\}]/\varphi^2(t)$, which is equal to

$$-2\,\frac{\varphi'(t)}{\varphi(t)} + \frac{\varphi(t)}{\Phi(t)} - \frac{\varphi(t)}{1-\Phi(t)} = 2\,t + \frac{\varphi(t)}{\Phi(t)} - \frac{\varphi(t)}{1-\Phi(t)},$$

is negative for $t < 0$, equal to zero for $t = 0$ and positive for $t > 0$. Notice that this first derivative is equal to zero for $t = 0$ and is an anti-symmetric function, thus it is sufficient to prove that this first derivative is positive for $t > 0$.

Consider the function

$$\psi(t) = 2\,t\,\Phi(t)\,\{1 - \Phi(t)\} + \varphi(t)\,\{1 - 2\,\Phi(t)\}.$$

One finds, using the inequality of Shenton (5.2.4), for $t > 0$:

$$\psi(t) > 2\,t\,\Phi(t)\,\{1-\Phi(t)\} + \{1-\Phi(t)\}\,\{1-2\,\Phi(t)\}\frac{t^3+3\,t}{t^2+2}$$

$$= \frac{1-\Phi(t)}{t^2+2}\,[2\,t\,(t^2+2)\,\Phi(t) + (t^3+3\,t)\,\{1-2\,\Phi(t)\}]$$

$$= \frac{t}{t^2+2}\,\{1-\Phi(t)\}\,\{t^2+3-2\,\Phi(t)\} > 0.$$

It is of interest to note that this can also be proved as follows:

$$\psi'(t) = 2\,\Phi(t)\,\{1-\Phi(t)\} + t\,\varphi(t)\,\{1-2\,\Phi(t)\} - 2\,\varphi^2(t),$$

$$\psi''(t) = \varphi(t)\,[(3-t^2)\,\{1-2\,\Phi(t)\} + 2\,t\,\varphi(t)].$$

Defining

$$\chi(t) = \frac{\psi''(t)}{\varphi(t)}$$

one finds:

$$\chi'(t) = -2\,t\,\{1-2\,\Phi(t)\} - 4\,\varphi(t),$$

$$\chi''(t) = -2\,\{1-2\,\Phi(t)\} + 8\,t\,\varphi(t),$$

$$\chi'''(t) = 4\,(3-2\,t^2)\,\varphi(t).$$

It is easy to see that $\chi'''(t)$ is first positive and then negative on $[0, \infty)$. Thus $\chi''(t)$ is first increasing and then decreasing. From the fact that $\chi''(0) = 0$ and $\chi''(\infty) = +2$, it follows that $\chi''(t)$ is positive on $(0, \infty)$. Thus $\chi'(t)$ is increasing on $(0, \infty)$. From the fact that $\chi'(0) < 0$ and $\chi'(\infty) = +\infty$, it follows that $\chi'(t)$ is first negative and then positive on $[0, \infty)$. Thus $\chi(t)$ is first decreasing and then increasing on $[0, \infty)$. From the fact that $\chi(0) = 0$ and $\chi(\infty) = +\infty$, it follows that $\chi(t)$ is first negative and then positive on $(0, \infty)$, and so is $\psi''(t)$. Thus $\psi'(t)$ is first decreasing and then increasing on $(0, \infty)$. From this, and the fact that $\psi'(0) > 0$ and $\psi'(\infty) = 0$, it follows that $\psi'(t)$ is first positive and then negative on $[0, \infty)$. Thus $\psi(t)$ is first increasing and then decreasing on $[0, \infty)$. From the fact that $\psi(0) = 0$ and $\psi(\infty) = 0$ it follows that $\psi(t)$ is positive on $(0, \infty)$.

From this property of the Normal distribution it follows that asymptotically one must take for $\underline{x}_{(i)}$ and $\underline{y}_{(j)}$ the sample medians in order to get maximal Pitman efficiency.

In order to be able to choose j and i for m and n large one has to be more specific about the way j/n and i/m tend to $\frac{1}{2}$. One possibility is (cf. Mood and Graybill (1963)):

$$j \approx \frac{n}{2} - \tfrac{1}{2}\,\xi_{1-\alpha}\frac{\{n\,(m+n)\}^{1/2}}{\sqrt{m}+\sqrt{n}} \tag{6.4}$$

and

$$i \approx \frac{m}{2} + \tfrac{1}{2}\,\xi_{1-\alpha}\,\frac{\{m(m+n)\}^{1/2}}{\sqrt{m}+\sqrt{n}}, \tag{6.5}$$

where $\xi_{1-\alpha}$ is defined by

$$\Phi(\xi_{1-\alpha}) = 1 - \alpha. \tag{6.6}$$

Then the test V_{ji} has approximately a significance level α, as can be proved simply by approximating the terms in expression (3.2.4) by a Normal distribution. This choice may be justified, following Mood and Graybill, as follows. The consecutive terms in the expression for $\Pr[\underline{w}_{(j)} < \underline{v}_{(i)}]$ can be considered as probabilities of a random variable $\underline{r}$, taking on values 1, 2, ..., n. Mean and variance of $\underline{r}$ are:

$$E\{\underline{r}\} = \frac{jm}{n+1}, \tag{6.7}$$

$$\sigma^2\{\underline{r}\} = \frac{jm}{n+1}\left\{\frac{(j+1)(n+3)}{n+2} + \frac{(j+1)m}{n+2} - (2j+1) - \frac{jm}{n+1}\right\}. \tag{6.8}$$

Their approximate values for m and n large may be found by

$$m + n = N,\ m = \alpha_1 N,\ n = (1-\alpha_1)N \quad \text{and} \quad j = \beta_1 n = \beta_1(1-\alpha_1)N,$$

resulting into

$$E\{\underline{r}\} \approx \alpha_1 \beta_1 N, \tag{6.9}$$

$$\sigma^2\{\underline{r}\} \approx \alpha_1 \beta_1 N \frac{1-\beta_1}{1-\alpha_1}. \tag{6.10}$$

Let i and j asymptotically satisfy

$$\frac{i - m/2}{\sqrt{m}} = \frac{n/2 - j}{\sqrt{n}}. \tag{6.11}$$

Given j one finds i from

$$(i - 1 + \tfrac{1}{2}) - \alpha_1 \beta_1 N = \xi_{1-\alpha}\left(\alpha_1 \beta_1 N \frac{1-\beta_1}{1-\alpha_1}\right)^{1/2}. \tag{6.12}$$

From these equations (6.4) and (6.5) follow (approximately).

For $\alpha = {\cdot}01$ and ${\cdot}05$ and underlying Normal distributions we have compared for some values of m and n the subscripts j and i of the selected tests with the numbers j and i computed by (6.4) and (6.5), respectively. In table 6-I the results are given.

An equality (3,9) = (4,10), say, for $m = n = 12$, means that (3,9) is the result of the computation but that these two confidence bounds, namely $\underline{y}_{(3)} - \underline{x}_{(9)}$ and $\underline{y}_{(4)} - \underline{x}_{(10)}$, respectively, have the same statistical properties in the case of Normal distributions. This follows easily by considerations of symmetry.

TABLE 6-I

Comparison of the selected lower confidence bounds with the best asymptotic ones computed by (6.4) and (6.5)

	$\alpha = \cdot 01$		$\alpha = \cdot 05$	
	(j,i) selected	(j,i) computed by (6.4) and (6.5)	(j,i) selected	(j,i) computed by (6.4) and (6.5)
$m = n = 8$	(2,7)	(1,7)	(5,8)	(2,6) = (3,7)
$m = n = 9$	(2,7)	(2,7)	(3,7)	(3,7)
$m = n = 10$	(2,7)	(2,8)	(3,7)	(3,7)
$m = n = 11$	(3,9)	(2,8)	(4,8)	(3,8)
$m = n = 12$	(4,10)	(3,9) = (4,10)	(5,9)	(4,8) = (5,9)
$m = n = 13$	(4,10)	(3,10)	(7,11)	(4,9) = (5,10)
$m = n = 14$	(3,9)	(4,10)	(5,10)	(4,10)
$m = n = 15$	(5,12)	(4,11) = (5,12)	(5,10)	(5,10)
$m = 8, n = 12$	(4,7)	(3,7)	(4,6)	(4,6)
$m = 9, n = 13$	(3,7)	(3,7)	(5,7)	(4,6)
$m = 10, n = 14$	(3,7)	(3,8)	(6,8)	(4,7)
$m = 11, n = 15$	(5,9)	(4,9)	(9,10)	(5,8)

The rounding off of the solution of (6.4) and (6.5) was such that j became $[j] + 1$ if $j > [j] + \cdot 9$ and $[j]$ otherwise and i became $[i]$ if $i < [i] + \cdot 1$ and $[i] + 1$ otherwise. Since the agreement in table 6-I is not bad we have some confidence in showing the tables 6-II–6-VII with the confidence bounds computed by (6.4) and (6.5), for some values of α and for $12 \leqslant m,n \leqslant 30$.

The subscripts, denoted by j and i, are given to one decimal place. This provides a partial comparison with the results presented in the tables 4.2.1-I –4.2.1-VI. When using these results for determining confidence bounds a possible procedure might be to construct integers j^* and i^* as follows:

$$j^* = [j] \quad \text{and} \quad i^* = [i] + 1,$$

in order to be sure that

$$\Pr [\underline{y}_{(j^*)} - \underline{x}_{(i^*)} < v] \geqslant 1 - \alpha.$$

TABLE 6-II

Approximate lower confidence bounds $\underline{y}_{(j)} - \underline{x}_{(i)}$, denoted by j–i, for location difference using formula (6.4) and (6.5). The subscripts j and i are given to one decimal place. Confidence level ·999. It is possible to round off j and i downwards and upwards, respectively

m \ n	12	13	14	15	16
12	2·2– 9·8	2·6– 9·8	2·9– 9·8	3·3– 9·8	3·6– 9·8
13	2·2–10·4	2·6–10·4	2·9–10·4	3·3–10·4	3·6–10·4
14	2·2–11·1	2·6–11·1	2·9–11·1	3·3–11·1	3·6–11·1
15	2·2–11·7	2·6–11·7	2·9–11·7	3·3–11·7	3·6–11·7
16	2·2–12·4	2·6–12·4	2·9–12·4	3·3–12·4	3·6–12·4
17	2·2–13·0	2·6–13·0	2·9–13·0	3·3–13·0	3·6–13·0
18	2·2–13·7	2·5–13·7	2·9–13·6	3·3–13·6	3·6–13·6
19	2·2–14·3	2·5–14·3	2·9–14·3	3·3–14·3	3·6–14·3
20	2·2–14·9	2·5–14·9	2·9–14·9	3·3–14·9	3·6–14·9
21	2·2–15·6	2·5–15·5	2·9–15·5	3·3–15·5	3·6–15·5
22	2·2–16·2	2·5–16·2	2·9–16·2	3·2–16·1	3·6–16·1
23	2·2–16·8	2·5–16·8	2·9–16·8	3·2–16·8	3·6–16·8
24	2·2–17·4	2·5–17·4	2·9–17·4	3·2–17·4	3·6–17·4
25	2·2–18·1	2·5–18·0	2·9–18·0	3·2–18·0	3·6–18·0
26	2·1–18·7	2·5–18·7	2·9–18·6	3·2–18·6	3·6–18·6
27	2·1–19·3	2·5–19·3	2·9–19·3	3·2–19·2	3·6–19·2
28	2·1–19·9	2·5–19·9	2·9–19·9	3·2–19·9	3·6–19·8
29	2·1–20·5	2·5–20·5	2·8–20·5	3·2–20·5	3·6–20·4
30	2·1–21·1	2·5–21·1	2·8–21·1	3·2–21·1	3·6–21·1

m \ n	22	23	24	25	26
12	5·8– 9·8	6·2– 9·8	6·6– 9·8	6·9– 9·8	7·3– 9·9
13	5·8–10·5	6·2–10·5	6·6–10·5	7·0–10·5	7·3–10·5
14	5·8–11·1	6·2–11·1	6·6–11·1	7·0–11·1	7·4–11·1
15	5·9–11·8	6·2–11·8	6·6–11·8	7·0–11·8	7·4–11·8
16	5·9–12·4	6·2–12·4	6·6–12·4	7·0–12·4	7·4–12·4
17	5·9–13·0	6·2–13·0	6·6–13·0	7·0–13·0	7·4–13·0
18	5·9–13·6	6·3–13·6	6·6–13·6	7·0–13·7	7·4–13·7
19	5·9–14·3	6·3–14·3	6·6–14·3	7·0–14·3	7·4–14·3
20	5·9–14·9	6·3–14·9	6·6–14·9	7·0–14·9	7·4–14·9
21	5·9–15·5	6·3–15·5	6·6–15·5	7·0–15·5	7·4–15·5
22	5·9–16·1	6·3–16·1	6·6–16·1	7·0–16·1	7·4–16·1
23	5·9–16·7	6·3–16·7	6·6–16·7	7·0–16·7	7·4–16·7
24	5·9–17·4	6·3–17·4	6·6–17·4	7·0–17·4	7·4–17·4
25	5·9–18·0	6·3–18·0	6·6–18·0	7·0–18·0	7·4–18·0
26	5·9–18·6	6·3–18·6	6·6–18·6	7·0–18·6	7·4–18·6
27	5·9–19·2	6·3–19·2	6·6–19·2	7·0–19·2	7·4–19·2
28	5·9–19·8	6·3–19·8	6·6–19·8	7·0–19·8	7·4–19·8
29	5·9–20·4	6·3–20·4	6·6–20·4	7·0–20·4	7·4–20·4
30	5·9–21·0	6·2–21·0	6·6–21·0	7·0–21·0	7·4–21·0

17	18	19	20	21
4·0– 9·8	4·3– 9·8	4·7– 9·8	5·1– 9·8	5·4– 9·8
4·0–10·4	4·3–10·5	4·7–10·5	5·1–10·5	5·5–10·5
4·0–11·1	4·4–11·1	4·7–11·1	5·1–11·1	5·5–11·1
4·0–11·7	4·4–11·7	4·7–11·7	5·1–11·7	5·5–11·7
4·0–12·4	4·4–12·4	4·7–12·4	5·1–12·4	5·5–12·4
4·0–13·0	4·4–13·0	4·7–13·0	5·1–13·0	5·5–13·0
4·0–13·6	4·4–13·6	4·7–13·6	5·1–13·6	5·5–13·6
4·0–14·3	4·4–14·3	4·7–14·3	5·1–14·3	5·5–14·3
4·0–14·9	4·4–14·9	4·7–14·9	5·1–14·9	5·5–14·9
4·0–15·5	4·4–15·5	4·7–15·5	5·1–15·5	5·5–15·5
4·0–16·1	4·4–16·1	4·7–16·1	5·1–16·1	5·5–16·1
4·0–16·8	4·4–16·7	4·7–16·7	5·1–16·7	5·5–16·7
4·0–17·4	4·4–17·4	4·7–17·4	5·1–17·4	5·5–17·4
4·0–18·0	4·3–18·0	4·7–18·0	5·1–18·0	5·5–18·0
4·0–18·6	4·3–18·6	4·7–18·6	5·1–18·6	5·5–18·6
4·0–19·2	4·3–19·2	4·7–19·2	5·1–19·2	5·5–19·2
4·0–19·8	4·3–19·8	4·7–19·8	5·1–19·8	5·5–19·8
4·0–20·4	4·3–20·4	4·7–20·4	5·1–20·4	5·5–20·4
4·0–21·0	4·3–21·0	4·7–21·0	5·1–21·0	5·5–21·0

27	28	29	30	
7·7– 9·9	8·1– 9·9	8·5– 9·9	8·9– 9·9	
7·7–10·5	8·1–10·5	8·5–10·5	8·9–10·5	
7·7–11·1	8·1–11·1	8·5–11·2	8·9–11·2	
7·8–11·8	8·1–11·8	8·5–11·8	8·9–11·8	
7·8–12·4	8·2–12·4	8·6–12·4	8·9–12·4	
7·8–13·0	8·2–13·0	8·6–13·0	9·0–13·0	
7·8–13·7	8·2–13·7	8·6–13·7	9·0–13·7	
7·8–14·3	8·2–14·3	8·6–14·3	9·0–14·3	
7·8–14·9	8·2–14·9	8·6–14·9	9·0–14·9	
7·8–15·5	8·2–15·5	8·6–15·5	9·0–15·5	
7·8–16·1	8·2–16·1	8·6–16·1	9·0–16·1	
7·8–16·7	8·2–16·7	8·6–16·7	9·0–16·8	
7·8–17·4	8·2–17·4	8·6–17·4	9·0–17·4	
7·8–18·0	8·2–18·0	8·6–18·0	9·0–18·0	
7·8–18·6	8·2–18·6	8·6–18·6	9·0–18·6	
7·8–19·2	8·2–19·2	8·6–19·2	9·0–19·2	
7·8–19·8	8·2–19·8	8·6–19·8	9·0–19·8	
7·8–20·4	8·2–20·4	8·6–20·4	9·0–20·4	
7·8–21·0	8·2–21·0	8·6–21·0	9·0–21·0	

TABLE 6-III

Same data as in table 6-II: confidence level ·995

m \ n	12	13	14	15	16
12	2·8– 9·2	3·2– 9·2	3·6– 9·2	4·0– 9·2	4·3– 9·2
13	2·8– 9·8	3·2– 9·8	3·6– 9·8	4·0– 9·8	4·4– 9·8
14	2·8–10·4	3·2–10·4	3·6–10·4	4·0–10·4	4·4–10·4
15	2·8–11·0	3·2–11·0	3·6–11·0	4·0–11·0	4·4–11·0
16	2·8–11·7	3·2–11·6	3·6–11·6	4·0–11·6	4·4–11·6
17	2·8–12·3	3·2–12·3	3·6–12·3	4·0–12·3	4·4–12·3
18	2·8–12·9	3·2–12·9	3·6–12·9	4·0–12·9	4·4–12·9
19	2·8–13·5	3·2–13·5	3·6–13·5	4·0–13·5	4·4–13·5
20	2·8–14·1	3·2–14·1	3·6–14·1	4·0–14·1	4·4–14·1
21	2·8–14·7	3·2–14·7	3·6–14·7	4·0–14·7	4·3–14·7
22	2·8–15·3	3·2–15·3	3·6–15·3	4·0–15·3	4·3–15·3
23	2·8–15·9	3·2–15·9	3·6–15·9	4·0–15·9	4·3–15·9
24	2·8–16·5	3·2–16·5	3·6–16·5	3·9–16·5	4·3–16·5
25	2·8–17·1	3·2–17·1	3·6–17·1	3·9–17·1	4·3–17·1
26	2·8–17·7	3·2–17·7	3·6–17·7	3·9–17·7	4·3–17·7
27	2·8–18·3	3·2–18·3	3·5–18·3	3·9–18·3	4·3–18·3
28	2·8–18·9	3·2–18·9	3·5–18·9	3·9–18·9	4·3–18·9
29	2·8–19·5	3·2–19·5	3·5–19·5	3·9–19·5	4·3–19·5
30	2·8–20·1	3·1–20·1	3·5–20·1	3·9–20·1	4·3–20·0

m \ n	22	23	24	25	26
12	6·7– 9·2	7·1– 9·2	7·5– 9·2	7·9– 9·2	8·3– 9·2
13	6·7– 9·8	7·1– 9·8	7·5– 9·8	7·9– 9·8	8·3– 9·8
14	6·7–10·4	7·1–10·4	7·5–10·4	7·9–10·4	8·3–10·4
15	6·7–11·0	7·1–11·0	7·5–11·1	7·9–11·1	8·3–11·1
16	6·7–11·7	7·1–11·7	7·5–11·7	7·9–11·7	8·3–11·7
17	6·7–12·3	7·1–12·3	7·5–12·3	7·9–12·3	8·3–12·3
18	6·7–12·9	7·1–12·9	7·5–12·9	7·9–12·9	8·3–12·9
19	6·7–13·5	7·1–13·5	7·5–13·5	7·9–13·5	8·3–13·5
20	6·7–14·1	7·1–14·1	7·5–14·1	7·9–14·1	8·3–14·1
21	6·7–14·7	7·1–14·7	7·5–14·7	7·9–14·7	8·3–14·7
22	6·7–15·3	7·1–15·3	7·5–15·3	7·9–15·3	8·4–15·3
23	6·7–15·9	7·1–15·9	7·5–15·9	7·9–15·9	8·4–15·9
24	6·7–16·5	7·1–16·5	7·5–16·5	7·9–16·5	8·4–16·5
25	6·7–17·1	7·1–17·1	7·5–17·1	7·9–17·1	8·4–17·1
26	6·7–17·6	7·1–17·6	7·5–17·6	7·9–17·6	8·4–17·6
27	6·7–18·2	7·1–18·2	7·5–18·2	7·9–18·2	8·4–18·2
28	6·7–18·8	7·1–18·8	7·5–18·8	7·9–18·8	8·4–18·8
29	6·7–19·4	7·1–19·4	7·5–19·4	7·9–19·4	8·4–19·4
30	6·7–20·0	7·1–20·0	7·5–20·0	7·9–20·0	8·4–20·0

17	18	19	20	21
4·7– 9·2	5·1– 9·2	5·5– 9·2	5·9– 9·2	6·3– 9·2
4·7– 9·8	5·1– 9·8	5·5– 9·8	5·9– 9·8	6·3– 9·8
4·7–10·4	5·1–10·4	5·5–10·4	5·9–10·4	6·3–10·4
4·7–11·0	5·1–11·0	5·5–11·0	5·9–11·0	6·3–11·0
4·7–11·6	5·1–11·6	5·5–11·6	5·9–11·6	6·3–11·7
4·7–12·3	5·1–12·3	5·5–12·3	5·9–12·3	6·3–12·3
4·7–12·9	5·1–12·9	5·5–12·9	5·9–12·9	6·3–12·9
4·7–13·5	5·1–13·5	5·5–13·5	5·9–13·5	6·3–13·5
4·7–14·1	5·1–14·1	5·5–14·1	5·9–14·1	6·3–14·1
4·7–14·7	5·1–14·7	5·5–14·7	5·9–14·7	6·3–14·7
4·7–15·3	5·1–15·3	5·5–15·3	5·9–15·3	6·3–15·3
4·7–15·9	5·1–15·9	5·5–15·9	5·9–15·9	6·3–15·9
4·7–16·5	5·1–16·5	5·5–16·5	5·9–16·5	6·3–16·5
4·7–17·1	5·1–17·1	5·5–17·1	5·9–17·1	6·3–17·1
4·7–17·7	5·1–17·7	5·5–17·7	5·9–17·7	6·3–17·7
4·7–18·3	5·1–18·3	5·5–18·3	5·9–18·2	6·3–18·2
4·7–18·9	5·1–18·8	5·5–18·8	5·9–18·8	6·3–18·8
4·7–19·4	5·1–19·4	5·5–19·4	5·9–19·4	6·3–19·4
4·7–20·0	5·1–20·0	5·5–20·0	5·9–20·0	6·3–20·0

27	28	29	30	
8·7– 9·2	9·1– 9·2	9·5– 9·2	9·9– 9·2	
8·7– 9·8	9·1– 9·8	9·5– 9·8	9·9– 9·9	
8·7–10·5	9·1–10·5	9·5–10·5	9·9–10·5	
8·7–11·1	9·1–11·1	9·5–11·1	9·9–11·1	
8·7–11·7	9·1–11·7	9·5–11·7	10·0–11·7	
8·7–12·3	9·1–12·3	9·6–12·3	10·0–12·3	
8·7–12·9	9·2–12·9	9·6–12·9	10·0–12·9	
8·7–13·5	9·2–13·5	9·6–13·5	10·0–13·5	
8·8–14·1	9·2–14·1	9·6–14·1	10·0–14·1	
8·8–14·7	9·2–14·7	9·6–14·7	10·0–14·7	
8·8–15·3	9·2–15·3	9·6–15·3	10·0–15·3	
8·8–15·9	9·2–15·9	9·6–15·9	10·0–15·9	
8·8–16·5	9·2–16·5	9·6–16·5	10·0–16·5	
8·8–17·1	9·2–17·1	9·6–17·1	10·0–17·1	
8·8–17·6	9·2–17·6	9·6–17·6	10·0–17·6	
8·8–18·2	9·2–18·2	9·6–18·2	10·0–18·2	
8·8–18·8	9·2–18·8	9·6–18·8	10·0–18·8	
8·8–19·4	9·2–19·4	9·6–19·4	10·0–19·4	
8·8–20·0	9·2–20·0	9·6–20·0	10·0–20·0	

TABLE 6-IV

Same data as in table 6-II: confidence level ·99

m \ n	12	13	14	15	16
12	3·2– 8·8	3·5– 8·8	3·9– 8·9	4·3– 8·9	4·7– 8·9
13	3·2– 9·5	3·5– 9·5	3·9– 9·5	4·3– 9·5	4·7– 9·5
14	3·1–10·1	3·5–10·1	3·9–10·1	4·3–10·1	4·7–10·1
15	3·1–10·7	3·5–10·7	3·9–10·7	4·3–10·7	4·7–10·7
16	3·1–11·3	3·5–11·3	3·9–11·3	4·3–11·3	4·7–11·3
17	3·1–11·9	3·5–11·9	3·9–11·9	4·3–11·9	4·7–11·9
18	3·1–12·5	3·5–12·5	3·9–12·5	4·3–12·5	4·7–12·5
19	3·1–13·1	3·5–13·1	3·9–13·1	4·3–13·1	4·7–13·1
20	3·1–13·7	3·5–13·7	3·9–13·7	4·3–13·7	4·7–13·7
21	3·1–14·3	3·5–14·3	3·9–14·3	4·3–14·3	4·7–14·3
22	3·1–14·9	3·5–14·9	3·9–14·9	4·3–14·9	4·7–14·9
23	3·1–15·5	3·5–15·5	3·9–15·5	4·3–15·5	4·7–15·5
24	3·1–16·1	3·5–16·1	3·9–16·1	4·3–16·1	4·7–16·0
25	3·1–16·7	3·5–16·7	3·9–16·7	4·3–16·6	4·7–16·6
26	3·1–17·3	3·5–17·3	3·9–17·2	4·3–17·2	4·7–17·2
27	3·1–17·9	3·5–17·8	3·9–17·8	4·3–17·8	4·7–17·8
28	3·1–18·4	3·5–18·4	3·9–18·4	4·3–18·4	4·7–18·4
29	3·1–19·0	3·5–19·0	3·9–19·0	4·3–19·0	4·7–19·0
30	3·1–19·6	3·5–19·6	3·9–19·6	4·3–19·6	4·7–19·6

m \ n	22	23	24	25	26
12	7·1– 8·9	7·5– 8·9	7·9– 8·9	8·3– 8·9	8·7– 8·9
13	7·1– 9·5	7·5– 9·5	7·9– 9·5	8·3– 9·5	8·7– 9·5
14	7·1–10·1	7·5–10·1	7·9–10·1	8·3–10·1	8·8–10·1
15	7·1–10·7	7·5–10·7	7·9–10·7	8·4–10·7	8·8–10·7
16	7·1–11·3	7·5–11·3	8·0–11·3	8·4–11·3	8·8–11·3
17	7·1–11·9	7·5–11·9	8·0–11·9	8·4–11·9	8·8–11·9
18	7·1–12·5	7·5–12·5	8·0–12·5	8·4–12·5	8·8–12·5
19	7·1–13·1	7·6–13·1	8·0–13·1	8·4–13·1	8·8–13·1
20	7·1–13·7	7·6–13·7	8·0–13·7	8·4–13·7	8·8–13·7
21	7·1–14·3	7·6–14·3	8·0–14·3	8·4–14·3	8·8–14·3
22	7·1–14·9	7·6–14·9	8·0–14·9	8·4–14·9	8·8–14·9
23	7·1–15·4	7·6–15·4	8·0–15·4	8·4–15·4	8·8–15·4
24	7·1–16·0	7·6–16·0	8·0–16·0	8·4–16·0	8·8–16·0
25	7·1–16·6	7·6–16·6	8·0–16·6	8·4–16·6	8·8–16·6
26	7·1–17·2	7·6–17·2	8·0–17·2	8·4–17·2	8·8–17·2
27	7·1–17·8	7·6–17·8	8·0–17·8	8·4–17·8	8·8–17·8
28	7·1–18·4	7·6–18·4	8·0–18·4	8·4–18·4	8·8–18·4
29	7·1–18·9	7·5–18·9	8·0–18·9	8·4–18·9	8·8–18·9
30	7·1–19·5	7·5–19·5	8·0–19·5	8·4–19·5	8·8–19·5

17	18	19	20	21
5·1– 8·9	5·5– 8·9	5·9– 8·9	6·3– 8·9	6·7– 8·9
5·1– 9·5	5·5– 9·5	5·9– 9·5	6·3– 9·5	6·7– 9·5
5·1–10·1	5·5–10·1	5·9–10·1	6·3–10·1	6·7–10·1
5·1–10·7	5·5–10·7	5·9–10·7	6·3–10·7	6·7–10·7
5·1–11·3	5·5–11·3	5·9–11·3	6·3–11·3	6·7–11·3
5·1–11·9	5·5–11·9	5·9–11·9	6·3–11·9	6·7–11·9
5·1–12·5	5·5–12·5	5·9–12·5	6·3–12·5	6·7–12·5
5·1–13·1	5·5–13·1	5·9–13·1	6·3–13·1	6·7–13·1
5·1–13·7	5·5–13·7	5·9–13·7	6·3–13·7	6·7–13·7
5·1–14·3	5·5–14·3	5·9–14·3	6·3–14·3	6·7–14·3
5·1–14·9	5·5–14·9	5·9–14·9	6·3–14·9	6·7–14·9
5·1–15·5	5·5–15·5	5·9–15·4	6·3–15·4	6·7–15·4
5·1–16·0	5·5–16·0	5·9–16·0	6·3–16·0	6·7–16·0
5·1–16·6	5·5–16·6	5·9–16·6	6·3–16·6	6·7–16·6
5·1–17·2	5·5–17·2	5·9–17·2	6·3–17·2	6·7–17·2
5·1–17·8	5·5–17·8	5·9–17·8	6·3–17·8	6·7–17·8
5·1–18·4	5·5–18·4	5·9–18·4	6·3–18·4	6·7–18·4
5·1–19·0	5·5–19·0	5·9–19·0	6·3–18·9	6·7–18·9
5·1–19·5	5·5–19·5	5·9–19·5	6·3–19·5	6·7–19·5

27	28	29	30	
9·1– 8·9	9·6– 8·9	10·0– 8·9	10·4– 8·9	
9·2– 9·5	9·6– 9·5	10·0– 9·5	10·4– 9·5	
9·2–10·1	9·6–10·1	10·0–10·1	10·4–10·1	
9·2–10·7	9·6–10·7	10·0–10·7	10·4–10·7	
9·2–11·3	9·6–11·3	10·0–11·3	10·4–11·3	
9·2–11·9	9·6–11·9	10·0–11·9	10·5–11·9	
9·2–12·5	9·6–12·5	10·0–12·5	10·5–12·5	
9·2–13·1	9·6–13·1	10·0–13·1	10·5–13·1	
9·2–13·7	9·6–13·7	10·1–13·7	10·5–13·7	
9·2–14·3	9·6–14·3	10·1–14·3	10·5–14·3	
9·2–14·9	9·6–14·9	10·1–14·9	10·5–14·9	
9·2–15·4	9·6–15·4	10·1–15·5	10·5–15·5	
9·2–16·0	9·6–16·0	10·1–16·0	10·5–16·0	
9·2–16·6	9·6–16·6	10·1–16·6	10·5–16·6	
9·2–17·2	9·6–17·2	10·1–17·2	10·5–17·2	
9·2–17·8	9·6–17·8	10·1–17·8	10·5–17·8	
9·2–18·4	9·6–18·4	10·1–18·4	10·5–18·4	
9·2–18·9	9·6–18·9	10·1–18·9	10·5–18·9	
9·2–19·5	9·6–19·5	10·1–19·5	10·5–19·5	

TABLE 6-V

Same data as in table 6-II: confidence level ·975

m \ n	12	13	14	15	16
12	3·6– 8·4	4·0– 8·4	4·4– 8·4	4·8– 8·4	5·2– 8·4
13	3·6– 9·0	4·0– 9·0	4·4– 9·0	4·8– 9·0	5·2– 9·0
14	3·6– 9·6	4·0– 9·6	4·4– 9·6	4·8– 9·6	5·2– 9·6
15	3·6–10·2	4·0–10·2	4·4–10·2	4·8–10·2	5·2–10·2
16	3·6–10·8	4·0–10·8	4·4–10·8	4·8–10·8	5·2–10·8
17	3·6–11·4	4·0–11·4	4·4–11·4	4·8–11·4	5·2–11·4
18	3·6–12·0	4·0–11·9	4·4–11·9	4·8–11·9	5·2–11·9
19	3·6–12·5	4·0–12·5	4·4–12·5	4·8–12·5	5·2–12·5
20	3·6–13·1	4·0–13·1	4·4–13·1	4·8–13·1	5·2–13·1
21	3·6–13·7	4·0–13·7	4·4–13·7	4·8–13·7	5·2–13·7
22	3·6–14·3	4·0–14·3	4·4–14·3	4·8–14·3	5·2–14·3
23	3·6–14·9	4·0–14·9	4·4–14·8	4·8–14·8	5·2–14·8
24	3·6–15·4	4·0–15·4	4·4–15·4	4·8–15·4	5·2–15·4
25	3·6–16·0	4·0–16·0	4·4–16·0	4·8–16·0	5·2–16·0
26	3·6–16·6	4·0–16·6	4·4–16·6	4·8–16·6	5·2–16·6
27	3·6–17·2	4·0–17·2	4·4–17·1	4·8–17·1	5·2–17·1
28	3·5–17·7	4·0–17·7	4·4–17·7	4·8–17·7	5·2–17·7
29	3·5–18·3	4·0–18·3	4·4–18·3	4·8–18·3	5·2–18·3
30	3·5–18·9	3·9–18·9	4·4–18·9	4·8–18·9	5·2–18·8

m \ n	22	23	24	25	26
12	7·7– 8·4	8·1– 8·4	8·6– 8·4	9·0– 8·4	9·4– 8·4
13	7·7– 9·0	8·1– 9·0	8·6– 9·0	9·0– 9·0	9·4– 9·0
14	7·7– 9·6	8·2– 9·6	8·6– 9·6	9·0– 9·6	9·4– 9·6
15	7·7–10·2	8·2–10·2	8·6–10·2	9·0–10·2	9·4–10·2
16	7·7–10·8	8·2–10·8	8·6–10·8	9·0–10·8	9·4–10·8
17	7·7–11·4	8·2–11·4	8·6–11·4	9·0–11·4	9·4–11·4
18	7·7–11·9	8·2–11·9	8·6–11·9	9·0–11·9	9·5–12·0
19	7·7–12·5	8·2–12·5	8·6–12·5	9·0–12·5	9·5–12·5
20	7·7–13·1	8·2–13·1	8·6–13·1	9·0–13·1	9·5–13·1
21	7·7–13·7	8·2–13·7	8·6–13·7	9·0–13·7	9·5–13·7
22	7·7–14·3	8·2–14·3	8·6–14·3	9·0–14·3	9·5–14·3
23	7·7–14·8	8·2–14·8	8·6–14·8	9·0–14·8	9·5–14·8
24	7·7–15·4	8·2–15·4	8·6–15·4	9·0–15·4	9·5–15·4
25	7·7–16·0	8·2–16·0	8·6–16·0	9·0–16·0	9·5–16·0
26	7·7–16·5	8·2–16·5	8·6–16·5	9·0–16·5	9·5–16·5
27	7·7–17·1	8·2–17·1	8·6–17·1	9·0–17·1	9·5–17·1
28	7·7–17·7	8·2–17·7	8·6–17·7	9·0–17·7	9·5–17·7
29	7·7–18·2	8·2–18·2	8·6–18·2	9·0–18·2	9·5–18·2
30	7·7–18·8	8·2–18·8	8·6–18·8	9·0–18·8	9·5–18·8

17	18	19	20	21
5·6– 8·4	6·0– 8·4	6·5– 8·4	6·9– 8·4	7·3– 8·4
5·6– 9·0	6·1– 9·0	6·5– 9·0	6·9– 9·0	7·3– 9·0
5·6– 9·6	6·1– 9·6	6·5– 9·6	6·9– 9·6	7·3– 9·6
5·6–10·2	6·1–10·2	6·5–10·2	6·9–10·2	7·3–10·2
5·6–10·8	6·1–10·8	6·5–10·8	6·9–10·8	7·3–10·8
5·6–11·4	6·1–11·4	6·5–11·4	6·9–11·4	7·3–11·4
5·6–11·9	6·1–11·9	6·5–11·9	6·9–11·9	7·3–11·9
5·6–12·5	6·1–12·5	6·5–12·5	6·9–12·5	7·3–12·5
5·6–13·1	6·1–13·1	6·5–13·1	6·9–13·1	7·3–13·1
5·6–13·7	6·1–13·7	6·5–13·7	6·9–13·7	7·3–13·7
5·6–14·3	6·1–14·3	6·5–14·3	6·9–14·3	7·3–14·3
5·6–14·8	6·1–14·8	6·5–14·8	6·9–14·8	7·3–14·8
5·6–15·4	6·1–15·4	6·5–15·4	6·9–15·4	7·3–15·4
5·6–16·0	6·1–16·0	6·5–16·0	6·9–16·0	7·3–16·0
5·6–16·6	6·0–16·5	6·5–16·5	6·9–16·5	7·3–16·5
5·6–17·1	6·0–17·1	6·5–17·1	6·9–17·1	7·3–17·1
5·6–17·7	6·0–17·7	6·5–17·7	6·9–17·7	7·3–17·7
5·6–18·3	6·0–18·3	6·5–18·3	6·9–18·2	7·3–18·2
5·6–18·8	6·0–18·8	6·5–18·8	6·9–18·8	7·3–18·8

27	28	29	30	
9·8– 8·4	10·3– 8·5	10·7– 8·5	11·1– 8·5	
9·8– 9·0	10·3– 9·0	10·7– 9·0	11·1– 9·1	
9·9– 9·6	10·3– 9·6	10·7– 9·6	11·1– 9·6	
9·9–10·2	10·3–10·2	10·7–10·2	11·1–10·2	
9·9–10·8	10·3–10·8	10·7–10·8	11·2–10·8	
9·9–11·4	10·3–11·4	10·7–11·4	11·2–11·4	
9·9–12·0	10·3–12·0	10·7–12·0	11·2–12·0	
9·9–12·5	10·3–12·5	10·7–12·5	11·2–12·5	
9·9–13·1	10·3–13·1	10·8–13·1	11·2–13·1	
9·9–13·7	10·3–13·7	10·8–13·7	11·2–13·7	
9·9–14·3	10·3–14·3	10·8–14·3	11·2–14·3	
9·9–14·8	10·3–14·8	10·8–14·8	11·2–14·8	
9·9–15·4	10·3–15·4	10·8–15·4	11·2–15·4	
9·9–16·0	10·3–16·0	10·8–16·0	11·2–16·0	
9·9–16·5	10·3–16·5	10·8–16·5	11·2–16·5	
9·9–17·1	10·3–17·1	10·8–17·1	11·2–17·1	
9·9–17·7	10·3–17·7	10·8–17·7	11·2–17·7	
9·9–18·2	10·3–18·2	10·8–18·2	11·2–18·2	
9·9–18·8	10·3–18·8	10·8–18·8	11·2–18·8	

TABLE 6-VI

Same data as in table 6-II: confidence level ·95

m \ n	12	13	14	15	16
12	4·0– 8·0	4·4– 8·0	4·8– 8·0	5·2– 8·0	5·7– 8·0
13	4·0– 8·6	4·4– 8·6	4·8– 8·6	5·2– 8·6	5·7– 8·6
14	4·0– 9·2	4·4– 9·2	4·8– 9·2	5·2– 9·2	5·7– 9·2
15	4·0– 9·8	4·4– 9·8	4·8– 9·8	5·2– 9·8	5·7– 9·8
16	4·0–10·3	4·4–10·3	4·8–10·3	5·2–10·3	5·7–10·3
17	4·0–10·9	4·4–10·9	4·8–10·9	5·2–10·9	5·7–10·9
18	4·0–11·5	4·4–11·5	4·8–11·5	5·2–11·5	5·7–11·5
19	4·0–12·1	4·4–12·0	4·8–12·0	5·2–12·0	5·7–12·0
20	4·0–12·6	4·4–12·6	4·8–12·6	5·2–12·6	5·7–12·6
21	4·0–13·2	4·4–13·2	4·8–13·2	5·2–13·2	5·7–13·2
22	4·0–13·8	4·4–13·8	4·8–13·7	5·2–13·7	5·7–13·7
23	4·0–14·3	4·4–14·3	4·8–14·3	5·2–14·3	5·7–14·3
24	4·0–14·9	4·4–14·9	4·8–14·9	5·2–14·9	5·7–14·9
25	4·0–15·5	4·4–15·4	4·8–15·4	5·2–15·4	5·7–15·4
26	3·9–16·0	4·4–16·0	4·8–16·0	5·2–16·0	5·7–16·0
27	3·9–16·6	4·4–16·6	4·8–16·6	5·2–16·6	5·7–16·5
28	3·9–17·1	4·4–17·1	4·8–17·1	5·2–17·1	5·7–17·1
29	3·9–17·7	4·4–17·7	4·8–17·7	5·2–17·7	5·6–17·7
30	3·9–18·3	4·4–18·3	4·8–18·2	5·2–18·2	5·6–18·2

m \ n	22	23	24	25	26
12	8·2– 8·0	8·7– 8·0	9·1– 8·0	9·5– 8·0	10·0– 8·1
13	8·2– 8·6	8·7– 8·6	9·1– 8·6	9·6– 8·6	10·0– 8·6
14	8·3– 9·2	8·7– 9·2	9·1– 9·2	9·6– 9·2	10·0– 9·2
15	8·3– 9·8	8·7– 9·8	9·1– 9·8	9·6– 9·8	10·0– 9·8
16	8·3–10·3	8·7–10·3	9·1–10·3	9·6–10·3	10·0–10·3
17	8·3–10·9	8·7–10·9	9·1–10·9	9·6–10·9	10·0–10·9
18	8·3–11·5	8·7–11·5	9·1–11·5	9·6–11·5	10·0–11·5
19	8·3–12·0	8·7–12·0	9·1–12·0	9·6–12·0	10·0–12·0
20	8·3–12·6	8·7–12·6	9·1–12·6	9·6–12·6	10·0–12·6
21	8·3–13·2	8·7–13·2	9·1–13·2	9·6–13·2	10·0–13·2
22	8·3–13·7	8·7–13·7	9·2–13·7	9·6–13·7	10·0–13·7
23	8·3–14·3	8·7–14·3	9·2–14·3	9·6–14·3	10·0–14·3
24	8·3–14·8	8·7–14·8	9·2–14·8	9·6–14·8	10·0–14·8
25	8·3–15·4	8·7–15·4	9·2–15·4	9·6–15·4	10·0–15·4
26	8·3–16·0	8·7–16·0	9·2–16·0	9·6–16·0	10·0–16·0
27	8·3–16·5	8·7–16·5	9·1–16·5	9·6–16·5	10·0–16·5
28	8·3–17·1	8·7–17·1	9·1–17·1	9·6–17·1	10·0–17·1
29	8·3–17·6	8·7–17·6	9·1–17·6	9·6–17·6	10·0–17·6
30	8·3–18·2	8·7–18·2	9·1–18·2	9·6–18·2	10·0–18·2

17	18	19	20	21
6·1– 8·0	6·5– 8·0	6·9– 8·0	7·4– 8·0	7·8– 8·0
6·1– 8·6	6·5– 8·6	7·0– 8·6	7·4– 8·6	7·8– 8·6
6·1– 9·2	6·5– 9·2	7·0– 9·2	7·4– 9·2	7·8– 9·2
6·1– 9·8	6·5– 9·8	7·0– 9·8	7·4– 9·8	7·8– 9·8
6·1–10·3	6·5–10·3	7·0–10·3	7·4–10·3	7·8–10·3
6·1–10·9	6·5–10·9	7·0–10·9	7·4–10·9	7·8–10·9
6·1–11·5	6·5–11·5	7·0–11·5	7·4–11·5	7·8–11·5
6·1–12·0	6·5–12·0	7·0–12·0	7·4–12·0	7·8–12·0
6·1–12·6	6·5–12·6	7·0–12·6	7·4–12·6	7·8–12·6
6·1–13·2	6·5–13·2	7·0–13·2	7·4–13·2	7·8–13·2
6·1–13·7	6·5–13·7	7·0–13·7	7·4–13·7	7·8–13·7
6·1–14·3	6·5–14·3	7·0–14·3	7·4–14·3	7·8–14·3
6·1–14·9	6·5–14·9	7·0–14·9	7·4–14·9	7·8–14·9
6·1–15·4	6·5–15·4	7·0–15·4	7·4–15·4	7·8–15·4
6·1–16·0	6·5–16·0	7·0–16·0	7·4–16·0	7·8–16·0
6·1–16·5	6·5–16·5	7·0–16·5	7·4–16·5	7·8–16·5
6·1–17·1	6·5–17·1	7·0–17·1	7·4–17·1	7·8–17·1
6·1–17·7	6·5–17·7	7·0–17·6	7·4–17·6	7·8–17·6
6·1–18·2	6·5–18·2	6·9–18·2	7·4–18·2	7·8–18·2

27	28	29	30	
10·4– 8·1	10·9– 8·1	11·3– 8·1	11·7– 8·1	
10·4– 8·6	10·9– 8·6	11·3– 8·6	11·7– 8·6	
10·4– 9·2	10·9– 9·2	11·3– 9·2	11·8– 9·2	
10·4– 9·8	10·9– 9·8	11·3– 9·8	11·8– 9·8	
10·5–10·3	10·9–10·3	11·3–10·4	11·8–10·4	
10·5–10·9	10·9–10·9	11·3–10·9	11·8–10·9	
10·5–11·5	10·9–11·5	11·3–11·5	11·8–11·5	
10·5–12·0	10·9–12·0	11·4–12·0	11·8–12·1	
10·5–12·6	10·9–12·6	11·4–12·6	11·8–12·6	
10·5–13·2	10·9–13·2	11·4–13·2	11·8–13·2	
10·5–13·7	10·9–13·7	11·4–13·7	11·8–13·7	
10·5–14·3	10·9–14·3	11·4–14·3	11·8–14·3	
10·5–14·9	10·9–14·9	11·4–14·9	11·8–14·9	
10·5–15·4	10·9–15·4	11·4–15·4	11·8–15·4	
10·5–16·0	10·9–16·0	11·4–16·0	11·8–16·0	
10·5–16·5	10·9–16·5	11·4–16·5	11·8–16·5	
10·5–17·1	10·9–17·1	11·4–17·1	11·8–17·1	
10·5–17·6	10·9–17·6	11·4–17·6	11·8–17·6	
10·5–18·2	10·9–18·2	11·4–18·2	11·8–18·2	

TABLE 6-VII

Same data as in table 6-II: confidence level ·90

m \ n	12	13	14	15	16
12	4·4– 7·6	4·9– 7·6	5·3– 7·6	5·7– 7·6	6·2– 7·6
13	4·4– 8·1	4·9– 8·1	5·3– 8·1	5·7– 8·1	6·2– 8·1
14	4·4– 8·7	4·9– 8·7	5·3– 8·7	5·7– 8·7	6·2– 8·7
15	4·4– 9·3	4·9– 9·3	5·3– 9·3	5·7– 9·3	6·2– 9·3
16	4·4– 9·8	4·9– 9·8	5·3– 9·8	5·7– 9·8	6·2– 9·8
17	4·4–10·4	4·9–10·4	5·3–10·4	5·7–10·4	6·2–10·4
18	4·4–10·9	4·9–10·9	5·3–10·9	5·7–10·9	6·2–10·9
19	4·4–11·5	4·9–11·5	5·3–11·5	5·7–11·5	6·2–11·5
20	4·4–12·0	4·9–12·0	5·3–12·0	5·7–12·0	6·2–12·0
21	4·4–12·6	4·9–12·6	5·3–12·6	5·7–12·6	6·2–12·6
22	4·4–13·1	4·9–13·1	5·3–13·1	5·7–13·1	6·2–13·1
23	4·4–13·7	4·8–13·7	5·3–13·7	5·7–13·7	6·2–13·7
24	4·4–14·3	4·8–14·2	5·3–14·2	5·7–14·2	6·2–14·2
25	4·4–14·8	4·8–14·8	5·3–14·8	5·7–14·8	6·2–14·8
26	4·4–15·4	4·8–15·3	5·3–15·3	5·7–15·3	6·2–15·3
27	4·4–15·9	4·8–15·9	5·3–15·9	5·7–15·9	6·2–15·9
28	4·4–16·4	4·8–16·4	5·3–16·4	5·7–16·4	6·2–16·4
29	4·4–17·0	4·8–17·0	5·3–17·0	5·7–17·0	6·2–17·0
30	4·4–17·5	4·8–17·5	5·3–17·5	5·7–17·5	6·2–17·5

m \ n	22	23	24	25	26
12	8·9– 7·6	9·3– 7·6	9·7– 7·6	10·2 –7·6	10·6– 7·6
13	8·9– 8·1	9·3– 8·2	9·8– 8·2	10·2– 8·2	10·7– 8·2
14	8·9– 8·7	9·3– 8·7	9·8– 8·7	10·2– 8·7	10·7– 8·7
15	8·9– 9·3	9·3– 9·3	9·8– 9·3	10·2– 9·3	10·7– 9·3
16	8·9– 9·8	9·3– 9·8	9·8– 9·8	10·2– 9·8	10·7– 9·8
17	8·9–10·4	9·3–10·4	9·8–10·4	10·2–10·4	10·7–10·4
18	8·9–10·9	9·3–10·9	9·8–10·9	10·2–10·9	10·7–10·9
19	8·9–11·5	9·3–11·5	9·8–11·5	10·2–11·5	10·7–11·5
20	8·9–12·0	9·3–12·0	9·8–12·0	10·2–12·0	10·7–12·0
21	8·9–12·6	9·3–12·6	9·8–12·6	10·2–12·6	10·7–12·6
22	8·9–13·1	9·3–13·1	9·8–13·1	10·2–13·1	10·7–13·1
23	8·9–13·7	9·3–13·7	9·8–13·7	10·2–13·7	10·7–13·7
24	8·9–14·2	9·3–14·2	9·8–14·2	10·2–14·2	10·7–14·2
25	8·9–14·8	9·3–14·8	9·8–14·8	10·2–14·8	10·7–14·8
26	8·9–15·3	9·3–15·3	9·8–15·3	10·2–15·3	10·7–15·3
27	8·9–15·9	9·3–15·9	9·8–15·9	10·2–15·9	10·7–15·9
28	8·9–16·4	9·3–16·4	9·8–16·4	10·2–16·4	10·7–16·4
29	8·9–16·9	9·3–16·9	9·8–16·9	10·2–16·9	10·7–16·9
30	8·9–17·5	9·3–17·5	9·8–17·5	10·2–17·5	10·7–17·5

17	18	19	20	21
6·6– 7·6	7·1– 7·6	7·5 –7·6	8·0 –7·6	8·4– 7·6
6·6– 8·1	7·1– 8·1	7·5– 8·1	8·0– 8·1	8·4– 8·1
6·6– 8·7	7·1– 8·7	7·5– 8·7	8·0– 8·7	8·4– 8·7
6·6– 9·3	7·1– 9·3	7·5– 9·3	8·0– 9·3	8·4 –9·3
6·6– 9·8	7·1– 9·8	7·5– 9·8	8·0– 9·8	8·4– 9·8
6·6–10·4	7·1–10·4	7·5–10·4	8·0–10·4	8·4–10·4
6·6–10·9	7·1–10·9	7·5–10·9	8·0–10·9	8·4–10·9
6·6–11·5	7·1–11·5	7·5–11·5	8·0–11·5	8·4–11·5
6·6–12·0	7·1–12·0	7·5–12·0	8·0–12·0	8·4–12·0
6·6–12·6	7·1–12·6	7·5–12·6	8·0–12·6	8·4–12·6
6·6–13·1	7·1–13·1	7·5–13·1	8·0–13·1	8·4–13·1
6·6–13·7	7·1–13·7	7·5–13·7	8·0–13·7	8·4–13·7
6·6–14·2	7·1–14·2	7·5–14·2	8·0–14·2	8·4–14·2
6·6–14·8	7·1–14·8	7·5–14·8	8·0–14·8	8·4–14·8
6·6–15·3	7·1–15·3	7·5–15·3	8·0–15·3	8·4–15·3
6·6–15·9	7·1–15·9	7·5–15·9	8·0–15·9	8·4–15·9
6·6–16·4	7·1–16·4	7·5–16·4	8·0–16·4	8·4–16·4
6·6–17·0	7·1–17·0	7·5–17·0	8·0–17·0	8·4–16·9
6·6–17·5	7·1–17·5	7·5–17·5	8·0–17·5	8·4–17·5

27	28	29	30	
11·1– 7·6	11·6– 7·6	12·0– 7·6	12·5– 7·6	
11·1– 8·2	11·6– 8·2	12·0– 8·2	12·5– 8·2	
11·1– 8·7	11·6– 8·7	12·0– 8·7	12·5– 8·7	
11·1– 9·3	11·6– 9·3	12·0– 9·3	12·5– 9·3	
11·1– 9·8	11·6– 9·8	12·0– 9·8	12·5– 9·8	
11·1–10·4	11·6–10·4	12·0–10·4	12·5–10·4	
11·1–10·9	11·6–10·9	12·0–10·9	12·5–10·9	
11·1–11·5	11·6–11·5	12·0–11·5	12·5–11·5	
11·1–12·0	11·6–12·0	12·0–12·0	12·5–12·0	
11·1–12·6	11·6–12·6	12·1–12·6	12·5–12·6	
11·1–13·1	11·6–13·1	12·1–13·1	12·5–13·1	
11·1–13·7	11·6–13·7	12·1–13·7	12·5–13·7	
11·1–14·2	11·6–14·2	12·1–14·2	12·5–14·2	
11·1–14·8	11·6–14·8	12·1–14·8	12·5–14·8	
11·1–15·3	11·6–15·3	12·1–15·3	12·5–15·3	
11·1–15·9	11·6–15·9	12·1–15·9	12·5–15·9	
11·1–16·4	11·6–16·4	12·1–16·4	12·5–16·4	
11·1–16·9	11·6–16·9	12·1–16·9	12·5–16·9	
11·1–17·5	11·6–17·5	12·1–17·5	12·5–17·5	

In order to determine the asymptotic relative efficiency of Pitman of the test V_{ji} relative to other tests (testing median differences) we shall first determine the efficacy of $\underline{V}_{ji}$. Now one finds for the expectation of $\underline{V}_{ji}$:

$$E\{\underline{V}_{ji}\} = E\{\underline{y}_{(j)} - \underline{x}_{(i)}\}$$

$$= \frac{n!}{(j-1)!\,(n-j)!} \int_{-\infty}^{\infty} t\, F^{j-1}(t-v)\,\{1 - F(t-v)\}^{n-j}\, \mathrm{d}F(t-v) +$$

$$- \frac{m!}{(i-1)!\,(m-i)!} \int_{-\infty}^{\infty} t\, F^{i-1}(t)\,\{1 - F(t)\}^{m-i}\, \mathrm{d}F(t)$$

$$= v + h(F,m,n,i,j), \tag{6.13}$$

where the function $h(F,m,n,i,j)$ is independent of the median difference v. Then

$$\left(\frac{\partial E\{\underline{V}_{ji}\}}{\partial v}\right)_{v=0} = 1. \tag{6.14}$$

Further we have for m and n tending to infinity:

$$\sigma^2\{\underline{y}_{(j)}\} \sim \frac{1}{n}\left(\tfrac{1}{2} - \tfrac{1}{2}\,\frac{\xi_{1-\alpha}}{n}\,\frac{\{n\,(m+n)\}^{1/2}}{\sqrt{m}+\sqrt{n}}\right)\left(\tfrac{1}{2} + \tfrac{1}{2}\,\frac{\xi_{1-\alpha}}{n}\,\frac{\{n\,(m+n)\}^{1/2}}{\sqrt{m}+\sqrt{n}}\right)\Big/ f^2(y_{\lambda_1}), \tag{6.15}$$

where y_{λ_1} is defined by

$$\int_{-\infty}^{y_{\lambda_1}} f(t)\,\mathrm{d}t = \lambda_1 = \tfrac{1}{2} - \tfrac{1}{2}\,\frac{\xi_{1-\alpha}}{n}\,\frac{\{n\,(m+n)\}^{1/2}}{\sqrt{m}+\sqrt{n}}.$$

For n tending to infinity we have

$$\tfrac{1}{2}\,\frac{\xi_{1-\alpha}}{n}\,\frac{\{n\,(m+n)\}^{1/2}}{\sqrt{m}+\sqrt{n}} \to 0 \tag{6.16}$$

and thus

$$\lambda_1 \to \tfrac{1}{2}.$$

Similarly:

$$\sigma^2\{\underline{x}_{(i)}\} \sim \frac{1}{m}\left(\tfrac{1}{2} + \tfrac{1}{2}\,\frac{\xi_{1-\alpha}}{m}\,\frac{\{m(m+n)\}^{1/2}}{\sqrt{m}+\sqrt{n}}\right)\left(\tfrac{1}{2} - \tfrac{1}{2}\,\frac{\xi_{1-\alpha}}{m}\,\frac{\{m(m+n)\}^{1/2}}{\sqrt{m}+\sqrt{n}}\right)\Big/ f^2(x_{\lambda_2}), \tag{6.17}$$

where x_{λ_2} is defined by

$$\int_{-\infty}^{x_{\lambda_2}} f(t)\,\mathrm{d}t = \lambda_2 = \tfrac{1}{2} + \tfrac{1}{2}\frac{\xi_{1-\alpha}}{m}\frac{\{m(m+n)\}^{1/2}}{\sqrt{m}+\sqrt{n}}$$

and again we have

$$\lambda_2 \to \tfrac{1}{2}$$

for m tending to infinity.

From the asymptotic expressions for $\sigma^2\{\underline{x}_{(i)}\}$ and $\sigma^2\{\underline{y}_{(j)}\}$ it follows that

$$\sigma^2\{\underline{V}_{ji}\,|\,v=0\} \sim \frac{1}{4\,n f^2(0)} + \frac{1}{4\,m f^2(0)} \tag{6.18}$$

for m and n tending to infinity. The efficacy of $\underline{V}_{ji}$ is asymptotically equal to

$$\left\{\frac{1}{4\,n f^2(0)} + \frac{1}{4\,m f^2(0)}\right\}^{-1} = 4 f^2(0)\,\frac{m\,n}{m+n}, \tag{6.19}$$

where 0 is the population median of the x-variable.

It is well known that the efficacies of the tests of Wilcoxon and Student are

$$\frac{12\,m\,n}{m+n+1}\left\{\int_{-\infty}^{\infty} f^2(t)\,\mathrm{d}t\right\}^2 \tag{6.20}$$

and

$$\sim \frac{m\,n}{(m+n)\,\sigma^2}, \tag{6.21}$$

respectively.

So we have the following theorem:

Theorem 6.1. Pitman's asymptotic efficiency of the V_{ji} test relative to the two-sample test of Wilcoxon (Mann–Whitney) (for any continuous frequency function) is equal to

$$e_{v,w} = \tfrac{1}{3}\,\frac{f^2(0)}{\left\{\int_{-\infty}^{\infty} f^2(t)\,\mathrm{d}t\right\}^2}$$

and relative to the test of Student (for any continuous frequency function with variance σ^2):

$$e_{v,s} = 4\,\sigma^2 f^2(0).$$

Both expressions are scale-independent.

Note that

$$4\,\sigma^2 f^2(0) \Bigg/ \left[\tfrac{1}{3} \frac{f^2(0)}{\{\int_{-\infty}^{\infty} f^2(t)\,\mathrm{d}t\}^2} \right] = 12\,\sigma^2 \{ \int_{-\infty}^{\infty} f^2(t)\,\mathrm{d}t\}^2,$$

which is Pitman's asymptotic efficiency of Wilcoxon's test relative to Student's test, and that $e_{v,s}$ is equal to the asymptotic relative efficiency of the Sign test compared to the test of Student (cf. Pitman (1948)).

For the Normal distribution one gets

$$\int_{-\infty}^{\infty} \left(\frac{1}{(2\,\pi)^{1/2}} \exp\left(-t^2/2\right) \right)^2 \mathrm{d}t = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{(2\,\pi)^{1/2}} \exp\left(-s^2/2\right) \mathrm{d}s = \frac{1}{2\sqrt{\pi}},$$

thus

$$e_{v,w} = \tfrac{2}{3}.$$

For the Gamma distribution with distribution function

$$\Gamma(x;p) = \frac{1}{\Gamma(p)} \int_0^x (\exp -t)\, t^{p-1}\,\mathrm{d}t$$

one gets

$$e_{v,w} = \frac{1}{3\,\Gamma^2(p)} \exp\left(-2\nu_0\right) \nu_0^{2(p-1)} \Bigg/ \left\{ \frac{1}{\Gamma^2(p)} \int_0^{\infty} \exp\left(-2t\right) t^{2(p-1)}\,\mathrm{d}t \right\}^2$$

$$= \frac{2^{4p-2}\,\Gamma^2(p)}{3\,\Gamma^2(2p-1)} \exp\left(-2\nu_0\right) \nu_0^{2p-2},$$

where ν_0 is the median of the Gamma distribution. Notice that ν_0 is a function of p. In table 6-VIII this efficiency has been given for some values of p. It is clear that $e_{v,w}$ tends to $\frac{2}{3}$ as p tends to infinity.

For the Logistic distribution

$$f(x) = \frac{e^{-x}}{(1 + e^{-x})^2} \qquad (-\infty < x < \infty)$$

one gets

TABLE 6-VIII

Efficiencies of V_{ji} relative to Wilcoxon's test

distribution	efficiency $e_{v,w}$	Gamma distribution	efficiency $e_{v,w}$
Uniform	$\frac{1}{3} \approx \cdot 333$	$p = 2$	·524
Exponential	$\frac{1}{3} \approx \cdot 333$	$p = 3$	·577
Normal	$\frac{2}{3} \approx \cdot 667$	$p = 4$	·601
Logistic	$\frac{3}{4} = \cdot 750$	$p = 5$	·615
Cauchy	$\frac{4}{3} \approx 1 \cdot 333$	$p = 7$	·631
Double exponential	$\frac{4}{3} \approx 1 \cdot 333$	$p = 10$	·642
		$p = 15$	·650
		$p = 20$	·654
		$p = 30$	·659
		$p = 40$	·661

$$\int_{-\infty}^{\infty} f^2(x)\,\mathrm{d}x = 2\int_0^{\infty} \frac{\mathrm{e}^{-2x}}{(1+\mathrm{e}^{-x})^4}\,\mathrm{d}x$$

$$= -2\int_0^{\infty} \frac{\mathrm{e}^{-x}}{(1+\mathrm{e}^{-x})^4}\,\mathrm{d}\mathrm{e}^{-x}$$

$$= \quad 2\int_0^{1} \frac{t}{(1+t)^4}\,\mathrm{d}t$$

$$= \frac{1}{6}.$$

Thus

$$e_{v,w} = \frac{1}{3}\frac{(\frac{1}{4})^2}{(\frac{1}{6})^2} = \frac{3}{4}.$$

For the Cauchy distribution

$$f(x) = \frac{1}{\pi(1+x^2)} \qquad (-\infty < x < \infty)$$

one gets

$$\int_{-\infty}^{\infty} f^2(x)\,\mathrm{d}x = \frac{1}{\pi^2}\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(1+x^2)^2}$$

$$= \frac{1}{\pi^2}\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^2} + \frac{1}{2\,\pi^2}\int_{-\infty}^{\infty} x\,\mathrm{d}\frac{1}{1+x^2} = \frac{1}{2\,\pi}.$$

Thus

$$e_{v,w} = \frac{1}{3}\frac{(1/\pi)^2}{\{1/(2\pi)\}^2} = \frac{4}{3}.$$

In table 6-VIII efficiencies of the V_{ji} test relative to Wilcoxon's test are summarized for various distributions.

In table 6-IX efficiencies of the V_{ji} test relative to Student's test are summarized for various distributions.

TABLE 6-IX

Efficiencies of V_{ji} relative to Student's test

distribution	efficiency $e_{v,s}$	Gamma distribution	efficiency $e_{v,s}$
Uniform	$\frac{1}{3} \approx \cdot 333$	$p = 2$	·785
Normal	$2/\pi \approx \cdot 637$	$p = 3$	·730
Logistic	$\pi^2/12 \approx \cdot 822$	$p = 4$	·704
Exponential	1	$p = 5$	·690
Double exponential	2	$p = 7$	·674
		$p = 10$	·662
		$p = 15$	·654
		$p = 20$	·649
		$p = 30$	·645
		$p = 40$	·643

Note that if the distribution F is symmetric and is assumed to possess a unimodal density in the weak sense with the mode in ν_0 (this means that for $x_2' < x_1' \leqslant \nu_0 \leqslant x_1 < x_2$ the following holds: $f(x_2') \leqslant f(x_1')$ and $f(x_1) \geqslant f(x_2)$), then it is easily seen that $e_{v,s} \geqslant \frac{1}{3}$; this lower bound is attained in the case of a Uniform distribution. For let $\nu_0 = 0$ and $f(0) = 1$, without loss of generality, then we must minimize

$$\int x^2 f(x)\,\mathrm{d}x$$

subject to $0 \leqslant f(x) \leqslant 1$ and $\int f(x)\,\mathrm{d}x = 1$, and this is minimal when

$$f(x) = \begin{cases} 1 & \text{for } -\tfrac{1}{2} \leqslant x \leqslant \tfrac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

(cf. Hodges and Lehmann (1956) who give a more or less similar derivation for the Sign test).

It can be shown that Pitman's asymptotic relative efficiency of V_{it} relative to Student's t-test can be arbitrarily large. Consider the family of distributions due to Hodges and Lehmann (1961) defined by

for $x \geqslant 0$:

$$F_{\delta,\varepsilon}(x) = \begin{cases} \Phi(x) & \text{for } 0 \leqslant x \leqslant \varepsilon \\ \Phi(\varepsilon + \delta\,(x - \varepsilon)) & \text{for } \varepsilon < x \quad (0 < \delta) \end{cases} \tag{6.22}$$

and

for $x < 0$ by symmetry.

The density of this distribution is

$$f_{\delta,\varepsilon}(x) = \begin{cases} \varphi(x) & \text{for } |x| \leqslant \varepsilon \\ \delta\,\varphi(\varepsilon + \delta\,(x - \varepsilon)) & \text{for } \varepsilon < |x|. \end{cases} \tag{6.23}$$

For this distribution we have

$$4\,\sigma^2 f^2(0) = \frac{4}{\pi}\,[\int_0^{\varepsilon} x^2\,\varphi(x)\,\mathrm{d}x + \delta \int_{\varepsilon}^{\infty} x^2\,\varphi(\varepsilon + \delta\,(x - \varepsilon))\,\mathrm{d}x].$$

As ε tends to zero:

$$4\,\sigma^2 f^2(0) \to \frac{4}{\pi}\,\delta \int_0^{\infty} x^2\,\varphi(\delta\,x)\,\mathrm{d}x = \frac{2}{\pi\,\delta^2}. \tag{6.24}$$

From this it can be seen that $e_{v,s}$ can be made arbitrarily large by making δ sufficiently small.

For the same family of distributions it can easily be shown that $e_{v,w}$ can be made arbitrarily large by making δ sufficiently large. For as ε tends to zero the denominator of $e_{v,w}$ tends to the square of

$$\int_{-\infty}^{\infty} \varphi^2(\delta\,x)\,\mathrm{d}x = \frac{1}{\delta} \int_{-\infty}^{\infty} \varphi^2(x)\,\mathrm{d}x \tag{6.25}$$

and this tends to zero as δ tends to infinity.

From the fact that the asymptotic efficiency of the Normal Scores tests relative to Student's test is larger than or equal to 1, it follows that the asymptotic efficiency of V_{ji} relative to the Normal Scores tests is smaller than or equal to $4\,\sigma^2 f^2(0)$.

When applied to a Normal population the V_{ji} test (defined by (6.4) and (6.5)) as compared with Student's two-sample test, the most efficient test for such a population, is shown to have an asymptotic local efficiency of about 64 % in table 6-IX. In chapter 7 it will be shown that for small samples the efficiency is often larger.

7. COMPARISON OF THE V_1 TESTS WITH STUDENT'S AND WILCOXON'S TWO-SAMPLE TESTS FOR SMALL SAMPLE SIZES

7.1. Some general remarks

In this chapter some comparisons are made between the V_1 tests and Student's two-sample test for Normal shift alternatives and between the V_1 tests and Wilcoxon's two-sample test for Lehmann alternatives. Further the tests selected for Normal translation alternatives have been compared with Wilcoxon's test when testing against Lehmann alternatives of the form $G(y) = F^k(y)$ $(k > 1)$.

Before we describe the comparison and the results, we shall make some general remarks concerning the V tests. It is of interest to note the possibility of equivalence between Wilcoxon's two-sample test and one of the V_1 tests at a certain level of significance. In nearly all cases, however, these tests are completely different.

In particular the test V_{1m}, rejecting the null hypothesis if $y_{(1)} > x_{(m)}$, is identical with the two-sample test of Wilcoxon with size $1/\binom{m+n}{m}$. However, the test V_{1i} $(1 \leqslant i < m)$, rejecting the null hypothesis if $y_{(1)} > x_{(i)}$, will be shown not to be equivalent with the Wilcoxon test. Considering all possible configurations of the combined sample we see that the following configurations belong to the critical region of V_{1i}:

$$
\begin{array}{l}
x_{(1)} \ldots x_{(i)}\, x_{(i+1)} \ldots x_{(m)} \quad y_{(1)}\, y_{(2)} \ldots y_{(n)} \\
x_{(1)} \ldots x_{(i)}\, x_{(i+1)} \ldots y_{(1)} \quad x_{(m)}\, y_{(2)} \ldots y_{(n)} \\
\cdot \qquad \cdot \\
\cdot \qquad \cdot \\
\cdot \qquad \cdot \\
x_{(1)} \ldots x_{(i)}\, y_{(1)}\, y_{(2)} \ldots y_{(n)}\, x_{(i+1)} \ldots x_{(m)}.
\end{array}
$$

For this test V_{1i} to be identical with Wilcoxon's two-sample test (with certain size) the critical region should be of the form

$$U_{m,n} \geqslant U,$$

where $U_{m,n}$ is the number of y's which are larger than an x and U an integer between 0 and $m\,n$. It is easy to see that in case of equivalence U must be equal to $i\,n$. However, the configuration

$$x_{(1)} \ldots x_{(i-1)}\, y_{(1)}\, x_{(i)}\, x_{(i+1)} \ldots x_{(m)}\, y_{(2)} \ldots y_{(n)}, \tag{7.1.1}$$

which does not belong to the critical region of the test V_{1i}, is an element of the critical region of the Wilcoxon test ($U_{m,n} \geqslant i\,n$) if

$$i - 1 + (n-1)\, m \geqslant i\,n,$$

thus if

$$i \leqslant m - \frac{1}{n-1}, \tag{7.1.2}$$

which is always true for $n > 1$ (under the assumption made that $i < m$). Hence the test of Wilcoxon is never equivalent with the test V_{1i} ($1 \leqslant i < m$, $n > 1$). Supposing $m,n > 1$, one can easily show that the tests V_{ji} (excluding the case $j = 1$, $i = m$) are completely different from the Wilcoxon test.

For more complicated V tests some equivalence with Wilcoxon's test can be indicated. We shall give some examples.

The test based on the form

$$\min \{\underline{y}_{(1)} - \underline{x}_{(m-1)}, \underline{y}_{(2)} - \underline{x}_{(m)}\}$$

and rejecting H_0 if $\min \{y_{(1)} - x_{(m-1)}, y_{(2)} - x_{(m)}\} > 0$, is identical with the Wilcoxon test with size $2/\binom{m+n}{n}$.

Further the test based on the form

$$\text{3rd largest of } \{\underline{y}_{(1)} - \underline{x}_{(m-2)}, \underline{y}_{(3)} - \underline{x}_{(m)}, \underline{y}_{(1)} - \underline{x}_{(m-1)}, \underline{y}_{(2)} - \underline{x}_{(m)}\}$$

and rejecting H_0 when this form is larger than zero, is identical with the Wilcoxon test $U_{m,n} \geqslant mn - 2$ with size $4/\binom{m+n}{m}$ ($m,n \geqslant 3$). The last case we shall mention is the test based on the form

$$\text{5th largest of } \{\underline{y}_{(1)} - \underline{x}_{(m-3)}, \underline{y}_{(1)} - \underline{x}_{(m-2)}, \underline{y}_{(1)} - \underline{x}_{(m-1)}, \underline{y}_{(2)} - \underline{x}_{(m-1)}, \underline{y}_{(2)} - \underline{x}_{(m)},$$
$$\underline{y}_{(3)} - \underline{x}_{(m)}, \underline{y}_{(4)} - \underline{x}_{(m)}\} \tag{7.1.3}$$

and rejecting H_0 when this form is larger than zero. This test is equivalent with the Wilcoxon text $U_{m,n} \geqslant mn - 3$ with size $7/\binom{m+n}{m}$ ($m,n \geqslant 4$). It is rather tedious work to trace the special cases of equivalence between the V test and the Wilcoxon test. In general the two tests are different.

A particular relation with the Wilcoxon two-sample rank test can be described as follows. Defining the Wilcoxon test statistic $\underline{W}$ as the total number of pairs of order statistics $(\underline{y}_{(j)}, \underline{x}_{(i)})$ for which $\underline{y}_{(j)} - \underline{x}_{(i)}$ is larger than zero and considering the differences of all pairs of order statistics $\underline{y}_{(j)} - \underline{x}_{(i)}$ ($i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$) one gets:

$$\begin{aligned} \Pr [\underline{W} < \beta] &= \Pr [\beta\text{th largest of } \{\underline{y}_{(j)} - \underline{x}_{(i)}\} < 0 \mid H_0] \\ &= \Pr [\beta\text{th largest of } \{\underline{y}_{(j)} + \nu - \underline{x}_{(i)}\} < \nu \mid H_0] \\ &= \Pr [\beta\text{th largest of } \{\underline{y}_{(j)} - \underline{x}_{(i)}\} < \nu \mid H_1]. \end{aligned} \tag{7.1.4}$$

7.2. Comparison between V_1 tests and Wilcoxon's two-sample test for Lehmann alternatives

The power of Wilcoxon's two-sample test against Lehmann alternatives can be determined by the following recursive expression (cf. Shorack (1967)) for the distribution of $\underline{U}_{n,m}$ when $G(y) = F^k(y)$ and where $U_{n,m}$ is the number of times a y is larger than an x for the samples $x_1, x_2, \ldots, x_m$ from a population with (strictly) increasing (continuous) distribution function $F(x)$ and $y_1, y_2, \ldots, y_n$ from a population with distribution function $G(y)$ $(= F^k(y))$:

$$p_{m,n}(u) = \frac{m}{kn + m} p_{m-1,n}(u - n) + \frac{kn}{kn + m} p_{m,n-1}(u),$$

where

$$p_{m,n}(u) = \Pr\,[\underline{U}_{m,n} = u]$$

and

$$U_{m,n} = m\,n - U_{n,m}.$$

The power of the V_{ji} tests against alternatives of the form $G(y) = F^k(y)$ is given by (4.2.2.10).

The comparison of the power functions of the two tests took place in the following manner. For each (m,n)-combination the V_{ji} test with size α^* selected for Lehmann alternatives was to be compared with the Wilcoxon test of the same size. If the corresponding Wilcoxon test does not exist, which in general is the case, then two Wilcoxon tests have been taken, one with size $\alpha_W{}'$ smaller and one with size $\alpha_W{}''$ larger than α^* and as close as possible to α^*. Denoting the power functions of both tests against Lehmann alternative $H_1 : G(y) = F^k(y)$ $(1 < k)$ by

$$\beta_W{}'(k) \quad \text{and} \quad \beta_W{}''(k),$$

respectively, we computed the following values of the power:

$$\beta_W(k) = \beta_W{}'(k) + \frac{\alpha^* - \alpha_W{}'}{\alpha_W{}'' - \alpha_W{}'} \{\beta_W{}''(k) - \beta_W{}'(k)\}$$

of a randomized Wilcoxon test with size α^* for the three values of k where the power $\beta(k)$ of the V_{ji} test is (in good approximation) equal to

$$\cdot 25,\ \cdot 50 \text{ and } \cdot 75,$$

respectively. In the tables 7.2-I–7.2-VI the three differences (absolute error is smaller than ·002) between $\beta_W(k)$ and $\beta(k)$ for these three values of k have been given for various significance levels, namely ·001, ·005, ·01, ·025, ·05 and ·10, respectively. Between brackets the subscripts j and i, respectively, have been given. In my opinion, it is amazing that the difference in power between the V_{ji} tests and the much more complicated Wilcoxon test is rather small. Moreover, there are various cases where the V_{ji} test is even better than the Wilcoxon test.

TABLE 7.2-I

Power differences for Lehmann alternatives between Wilcoxon's test and the V_{ji} tests, denoted by j–i, in the three points where the power of the V_{ji} test is equal to ·25, ·50 and ·75, respectively ($\alpha = \cdot 001$)

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3													
4								1—4	1—4	1—4	1—4	1—4	1—4
								0	0	0	0	0	0
								0	0	0	0	0	0
								0	0	0	0	0	0
5						1—5	1—5	1—5	1—5	2—5	2—5	2—5	2—5
						0	0	0	0	·073	·071	·072	·070
						0	0	0	0	·095	·092	·091	·089
						0	0	0	0	·068	·065	·064	·062
6					1—6	1—6	1—6	2—6	1—5	1—5	1—5	1—5	1—5
					0	0	0	·094	—·004	—·007	—·007	—·009	—·012
					0	0	0	·121	—·034	—·039	—·041	—·044	—·048
					0	0	0	·086	—·069	—·074	—·077	—·080	—·085
7				1—7	1—7	1—7	1—6	1—6	1—6	1—6	1—6	2—6	1—5
				0	0	0	—·004	—·002	—·002	—·004	—·005	·030	—·012
				0	0	0	—·030	—·030	—·031	—·035	—·038	·022	—·049
				0	0	0	—·063	—·064	—·066	—·071	—·074	—·010	—·089
8			1—8	1—8	1—8	1—7	1—7	1—7	1—7	1—6	1—6	1—6	1—6
			0	0	0	—·001	—·002	—·000	—·001	—·005	—·006	—·008	—·009
			0	0	0	—·024	—·027	—·027	—·029	—·035	—·038	—·043	—·046
			0	0	0	—·056	—·060	—·062	—·064	—·070	—·076	—·082	—·087
9			1—9	1—9	1—8	1—8	1—8	1—7	1—7	1—7	1—7	1—6	1—6
			0	0	·001	—·000	·001	·001	—·002	—·002	—·006	·002	·000
			0	0	—·019	—·024	—·023	—·022	—·028	—·031	—·039	—·018	—·022
			0	0	—·049	—·055	—·057	—·054	—·063	—·068	—·077	—·045	—·050
10		1—10	1—10	1—9	1—9	1—9	1—8	1—8	1—8	1—7	1—7	1—7	1—7
		0	0	·004	·001	·004	·006	·007	·001	·008	·007	·008	·002
		0	0	—·012	—·019	—·018	—·013	—·014	—·025	—·008	—·011	—·012	—·020
		0	0	—·039	—·048	—·051	—·043	—·047	—·060	—·032	—·037	—·040	—·049
11		1—11	1—11	1—10	1—10	1—9	1—9	1—8	2—9	1—8	1—7	1—7	2—8
		0	0	·004	·001	·010	·008	·013	·062	·012	·014	·014	·036
		0	0	—·011	—·019	—·003	—·011	·004	·080	—·003	·005	·002	·038
		0	0	—·038	—·048	—·030	—·041	—·019	·055	—·028	—·018	—·022	·017
12		1—12	1—11	1—11	1—10	1—10	1—9	2—10	2—10	1—8	2—9	2—9	1—7
		0	·002	·005	·010	·012	·017	·074	·072	·017	·049	·045	·020
		0	—·012	—·010	·000	—·001	·011	·098	·091	·010	·060	·052	·014
		0	—·037	—·038	—·023	—·028	—·011	·071	·064	—·013	·038	·030	—·008
13		1—13	1—12	1—12	1—11	1—10	1—10	1—9	1—9	2—10	1—8	1—8	2—9
		0	·002	·005	·010	·018	·017	·021	·022	·059	·023	·023	·040
		0	—·013	—·010	·001	·015	·011	·019	·018	·075	·022	·019	·047
		0	—·037	—·037	—·022	—·005	—·011	—·001	—·004	·052	·001	—·002	·028
14		1—14	1—13	1—12	1—12	1—11	1—10	1—10	1—9	1—9	1—9	1—8	1—8
		0	·004	·014	·012	·020	·023	·023	·026	·026	·028	029	·029
		0	—·009	·010	·003	·018	·024	·022	·029	·027	·026	·033	·030
		0	—·034	—·008	—·020	—·003	·005	·001	·010	·007	·005	·014	·011
15		1—15	1—14	1—13	1—12	1—12	1—11	1—10	1—10	2—11	1—9	1—9	2—10
		0	·004	·016	·020	·023	·026	·029	·030	·060	·030	·032	·045
		0	—·009	·014	·021	·021	·028	·034	·033	·079	·035	·036	·056
		0	—·034	—·005	·003	·000	·008	·016	·014	·058	·017	·016	·038

TABLE 7.2-II

Same data as in table 7.2-I: $\alpha = \cdot 005$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3							1—3	1—3	1—3	1—3	1—3	1—3	2—3
							0	0	0	0	0	0	·041
							0	0	0	0	0	0	·056
							0	0	0	0	0	0	·042
4				1—4	1—4	1—4	1—4	2—4	2—4	2—4	2—4	1—3	1—3
				0	0	0	0	·059	·058	·057	·056	—·017	—·018
				0	0	0	0	·082	·079	·077	·075	—·054	—·056
				0	0	0	0	·061	·058	·056	·055	—·090	—·092
5			1—5	1—5	1—5	2—5	1—4	1—4	1—4	1—4	1—4	1—4	2—4
			0	0	0	·079	—·008	—·008	—·008	—·010	—·012	—·013	·006
			0	0	0	·110	—·036	—·038	—·039	—·043	—·047	—·049	—·013
			0	0	0	·081	—·069	—·072	—·073	—·078	—·082	—·085	—·042
6		1—6	1—6	1—6	1—5	1—5	1—5	1—5	1—5	1—4	1—4	1—4	1—4
		0	0	0	—·003	—·004	—·005	—·005	—·004	—·010	—·012	—·013	—·015
		0	0	0	—·025	—·029	—·032	—·034	—·034	—·040	—·046	—·048	—·053
		0	0	0	—·054	—·061	—·065	—·068	—·069	—·074	—·082	—·086	—·091
7		1—7	1—7	1—6	1—6	1—6	1—5	1—5	1—5	1—5	3—6	3—6	1—4
		0	0	·002	—·001	·070	—·002	—·003	—·006	—·007	·071	·066	—·004
		0	0	—·015	—·022	·059	—·024	—·027	—·034	—·038	·099	·088	—·026
		0	0	—·042	—·052	·011	—·052	—·058	—·067	—·073	·077	·064	—·051
8		1—8	1—7	1—7	1—7	1—6	1—6	1—6	1—5	1—5	1—5	1—5	1—5
		0	·000	·003	—·000	·004	·000	·000	·004	·003	·001	·000	—·002
		0	—·015	—·014	—·021	—·011	—·020	—·024	—·009	—·012	—·017	—·021	—·024
		0	—·040	—·041	—·051	—·037	—·049	—·055	—·031	—·036	—·042	—·046	—·051
9	1—9	1—9	1—8	1—8	1—7	1—7	1—6	2—7	1—6	1—6	1—5	2—6	2—6
	0	0	·001	·003	·007	·007	·008	·046	·007	·006	·008	·021	·019
	0	0	—·014	—·012	—·004	—·008	—·001	·062	—·006	—·010	—·001	·022	·018
	0	0	—·039	—·040	—·026	—·034	—·020	·044	—·028	—·034	—·022	·005	·000
10	1—10	1—9	1—9	1—8	1—8	1—7	2—8	2—8	1—6	2—7	2—7	2—7	1—5
	0	·000	·002	·010	·009	·011	·058	·056	·013	·032	·034	·030	·014
	0	—·011	—·013	·005	—·002	·006	·082	·075	·007	·043	·038	·034	·010
	0	—·032	—·038	—·013	—·024	—·012	·062	·054	—·012	·027	·021	·015	—·009
11	1—11	1—10	1—9	1—9	1—8	2—9	1—7	1—7	2—8	1—6	1—6	2—7	2—7
	0	·000	·010	·012	·014	·070	·015	·016	·044	·017	·017	·027	·026
	0	—·011	·009	·007	·012	·103	·014	·012	·060	·017	·015	·034	·030
	0	—·032	—·005	—·011	—·004	·081	—·002	—·006	·043	·000	—·003	·019	·014
12	1—12	1—11	1—10	1—9	1—9	1—8	1—8	1—7	1—7	2—8	2—8	1—6	1—6
	0	·001	·011	·015	·016	·018	·018	·019	·020	·037	·036	·022	·023
	0	—·010	·010	·016	·015	·020	·018	·023	·022	·051	·048	·026	·025
	0	—·031	—·004	·002	—·002	·004	·000	·008	·005	·036	·032	·009	·008
13	1—13	1—12	1—11	1—10	1—9	1—9	1—8	1—8	1—7	1—7	1—7	2—8	1—6
	0	·001	·012	·017	·020	·020	·021	·022	·023	·024	·025	·032	·027
	0	—·010	·011	·019	·025	·023	·027	·027	·031	·030	·030	·043	·035
	0	—·031	—·003	·004	·011	·007	·013	·011	·017	·016	·013	·029	·020
14	1—14	1—12	1—11	1—11	1—10	1—9	1—9	1—8	1—8	2—9	1—7	1—7	2—8
	0	·009	·015	·018	·022	·023	·024	·025	·026	·040	·029	·030	·030
	0	·010	·020	·021	·028	·031	·030	·034	·034	·058	·039	·039	·041
	0	·000	·008	·006	·014	·019	·016	·021	·020	·044	·025	·024	·028
15	1—14	1—13	1—12	1—11	1—10	1—10	1—9	1—9	1—8	1—8	1—8	1—7	1—7
	·003	·010	·016	·021	·022	·025	·026	·028	·029	·031	·032	·033	·034
	—·001	·011	·021	·029	·032	·034	·037	·038	·042	·042	·042	·047	·047
	—·016	·000	·009	·017	·022	·021	·025	·024	·030	·029	·027	·034	·033

TABLE 7.2-III

Same data as in table 7.2-I: $\alpha = \cdot 01$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3					1—3	1—3	1—3	1—3	1—3	2—3	2—3	2—3	2—3
					0	0	0	0	0	·042	·042	·042	·041
					0	0	0	0	0	·060	·059	·058	·056
					0	0	0	0	0	·045	·044	·043	·042
4			1—4	1—4	1—4	1—4	2—4	2—4	1—3	1—3	1—3	1—3	1—3
			0	0	0	0	·055	·059	—·013	—·015	—·016	—·017	—·018
			0	0	0	0	·080	·082	—·046	—·050	—·052	—·054	—·056
			0	0	0	0	·061	·061	—·080	—·084	—·087	—·090	—·092
5		1—5	1—5	1—5	2—5	1—4	1—4	1—4	1—4	2—4	2—4	2—4	1—3
		0	0	0	·081	—·007	—·008	—·008	—·008	·013	·010	·008	—·016
		0	0	0	·116	—·033	—·036	—·038	—·039	·004	—·003	—·008	—·054
		0	0	0	·087	—·064	—·069	—·072	—·073	—·022	—·030	—·037	—·091
6		1—6	1—6	1—5	1—5	1—5	1—5	1—4	1—4	1—4	3—5	3—5	3—5
		0	0	·000	—·003	—·004	—·005	—·006	—·009	—·010	·050	·046	·044
		0	0	—·017	—·025	—·029	—·032	—·031	—·037	—·040	·071	·062	·054
		0	0	—·044	—·054	—·061	—·065	—·061	—·069	—·074	·054	·042	·032
7	1—7	1—7	1—6	1—6	1—6	1—5	1—5	1—5	3—6	1—4	1—4	1—4	1—4
	0	0	·000	·002	—·001	·000	—·002	—·003	·077	·000	·000	—·002	—·004
	0	0	—·016	—·015	—·022	—·016	—·024	—·027	·115	—·015	—·018	—·022	—·026
	0	0	—·041	—·042	—·052	—·041	—·052	—·058	·093	—·037	—·041	—·046	—·051
8	1—8	1—8	1—7	1—6	1—6	1—6	1—5	1—5	1—5	1—5	1—5	1—4	1—4
	0	0	·000	·007	·005	·004	·005	·005	·004	·003	·001	·006	·006
	0	0	—·015	·000	—·007	—·011	—·004	—·007	—·009	—·012	—·017	—·005	—·006
	0	0	—·040	—·017	—·030	—·037	—·023	—·027	—·031	—·036	—·042	—·025	—·028
9	1—9	1—8	1—8	1—7	1—7	1—6	2—7	2—7	1—5	2—6	2—6	2—6	4—7
	0	—·006	·001	·009	·007	·009	·048	·046	·010	·025	·024	·021	·076
	0	—·020	—·014	·003	—·004	·003	·069	·062	·004	·032	·028	·022	·115
	0	—·040	—·039	—·015	—·026	—·015	·052	·044	—·014	·018	·012	·005	·094
10	1—10	1—9	1—8	1—8	1—7	2—8	1—6	2—7	2—7	1—5	1—5	2—6	2—6
	0	·000	·009	·010	·011	·060	·012	·036	·035	·014	·014	·020	·019
	0	—·011	·008	·005	·009	·090	·010	·053	·049	·014	·013	·025	·021
	0	—·032	—·006	—·013	—·008	·072	—·006	·040	·034	—·002	—·004	·011	·007
11	1—11	1—10	1—9	1—8	2—9	1—7	2—8	1—6	1—6	2—7	2—7	1—5	1—5
	0	·000	·010	·014	·072	·015	·046	·016	·017	·029	·028	·019	·019
	0	—·011	·009	·014	·110	·016	·070	·019	·018	·041	·038	·023	·022
	0	—·032	—·005	·000	·091	·002	·057	·005	·003	·028	·024	·008	·006
12	1—11	1—10	1—9	1—9	1—8	2—9	1—7	1—7	1—6	1—6	2—7	2—7	1—5
	·003	·008	·013	·015	·017	·056	·018	·019	·020	·021	·026	·026	·024
	—·002	·009	·017	·016	·021	·088	·023	·023	·027	·026	·038	·035	·032
	—·016	—·001	·005	·002	·009	·073	·010	·008	·015	·013	·026	·022	·020
13	1—12	1—11	1—10	1—9	1—9	1—8	1—8	1—7	1—7	1—6	1—6	1—6	2—7
	·003	·009	·014	·017	·020	·020	·021	·022	·023	·024	·025	·026	·024
	—·002	·010	·019	·024	·025	·028	·027	·030	·031	·035	·035	·036	·034
	—·016	—·001	·007	·013	·011	·016	·013	·019	·017	·024	·023	·022	·023
14	1—13	1—12	1—11	1—10	1—9	1—8	1—8	1—7	1—7	1—7	2—8	1—6	1—6
	·003	·009	·015	·019	·020	·021	·023	·023	·025	·027	·030	·029	·030
	—·001	·010	·020	·026	·029	·032	·033	·036	·038	·039	·045	·044	·044
	—·016	·000	·008	·015	·019	·023	·022	·027	·027	·026	·034	·032	·031
15	1—14	1—12	1—11	1—10	1—10	1—9	1—8	1—8	1—7	1—7	3—10	2—8	1—6
	·003	·012	·017	·019	·022	·024	·024	·027	·027	·029	·060	·029	·033
	—·001	·018	·026	·030	·032	·036	·039	·040	·043	·045	·092	·044	·052
	—·016	·010	·018	·022	·022	·026	·030	·030	·035	·035	·078	·034	·041

TABLE 7.2-IV

Same data as in table 7.2-I: $\alpha = \cdot 025$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3			1—3	1—3	1—3	2—3	2—3	2—3	2—3	3—3	1—2	1—2	1—2
			0	0	0	·044	·044	·043	·043	·050	—·018	—·018	—·018
			0	0	0	·069	·066	·064	·062	·077	—·054	—·056	—·056
			0	0	0	·054	·051	·049	·047	·060	—·089	—·091	—·092
4		1—4	1—4	2—4	1—3	1—3	1—3	1—3	1—3	1—3	2—3	2—3	2—3
		0	0	·062	—·009	—·011	—·013	—·013	—·013	—·015	·000	—·003	—·004
		0	0	·099	—·034	—·040	—·043	—·045	—·046	—·050	—·018	—·023	—·027
		0	0	·078	—·064	—·071	—·075	—·078	—·080	—·084	—·044	—·050	—·056
5	1—5	1—5	1—4	1—4	1—4	1—4	2—4	1—3	1—3	1—3	3—4	1—3	1—3
	0	0	—·003	—·001	—·005	—·007	·021	—·008	—·010	—·012	·030	—·015	—·016
	0	0	—·021	—·020	—·028	—·033	·025	—·034	—·038	—·042	·044	—·050	—·054
	0	0	—·046	—·048	—·058	—·064	·006	—·062	—·069	—·075	·029	—·086	—·091
6	1—6	1—5	1—5	1—5	1—4	1—4	1—4	3—5	3—5	1—3	1—3	1—3	1—3
	0	—·002	—·002	·000	—·002	—·002	—·005	·057	·056	—·001	—·002	—·003	—·004
	0	—·015	—·018	—·017	—·016	—·020	—·027	·096	·089	—·016	—·019	—·021	—·023
	0	—·036	—·043	—·044	—·037	—·044	—·055	·082	·075	—·036	—·040	—·044	—·047
7	1—7	1—6	1—6	1—5	1—5	1—5	1—4	1—4	1—4	1—4	3—5	3—5	1—3
	0	—·001	·000	·004	·002	·001	·002	·002	·001	·000	·034	·032	·004
	0	—·014	—·016	—·003	—·012	—·016	—·007	—·010	—·013	—·015	·053	·047	—·005
	0	—·035	—·041	—·020	—·033	—·041	—·025	—·029	—·034	—·037	·042	·035	—·025
8	1—7	1—7	1—6	1—6	1—5	2—6	2—6	1—4	2—5	2—5	2—5	4—6	4—6
	·002	·000	·007	·007	·006	·038	·037	·006	·018	·017	·015	·058	·057
	—·003	—·012	·004	·000	·001	·060	·055	·003	·025	·021	·017	·096	·091
	—·018	—·034	—·010	—·017	—·014	·048	·041	—·013	·014	·008	·002	·083	·077
9	1—8	1—7	2—8	1—6	2—7	1—5	2—6	2—6	2—6	1—4	2—5	2—5	2—5
	·002	·006	·086	·008	·050	·008	·028	·028	·027	·011	·014	·014	·013
	—·002	·006	·144	·007	·082	·008	·046	·042	·037	·012	·018	·016	·013
	—·017	—·005	·122	—·006	·070	—·005	·037	·031	·024	—·002	·008	·004	·000
10	1—9	1—8	1—7	2—8	1—6	2—7	1—5	1—5	2—6	2—6	1—4	1—4	2—5
	·002	·007	·010	·060	·011	·037	·012	·013	·022	·022	·015	·016	·013
	—·002	·007	·012	·102	·013	·062	·016	·015	·034	·032	·021	·021	·017
	—·017	—·003	·001	·090	·002	·054	·005	·003	·025	·020	·009	·008	·006
11	1—10	1—9	1—8	1—7	1—6	1—6	2—7	1—5	1—5	2—6	2—6	2—6	1—4
	·003	·008	·012	·012	·011	·014	·030	·015	·016	·020	·019	·019	·020
	—·002	·008	·015	·017	·018	·019	·050	·023	·023	·031	·028	·027	·030
	—·017	—·002	·003	·007	·010	·009	·043	·014	·013	·022	·019	·016	·019
12	1—10	1—9	1—8	1—7	1—7	1—6	1—6	1—6	1—5	1—5	1—5	2—6	3—7
	·005	·010	·012	·012	·014	·014	·017	·018	·018	·020	·022	·018	·031
	·007	·015	·019	·020	·022	·024	·026	·027	·030	·031	·032	·028	·050
	·001	·007	·012	·014	·013	·018	·017	·016	·022	·022	·022	·020	·042
13	1—11	1—10	1—9	1—8	1—7	1—7	1—6	1—6	1—6	1—5	1—5	1—5	2—6
	·006	·011	·014	·015	·015	·018	·017	·020	·022	·021	·024	·025	·018
	·008	·016	·022	·024	·026	·028	·030	·032	·034	·037	·039	·040	·029
	·001	·008	·014	·017	·020	·020	·024	·025	·025	·030	·031	·031	·021
14	1—12	1—10	1—9	1—8	2—10	1—7	1—7	2—8	1—6	3—9	1—5	3—8	1—5
	·006	·012	·014	·014	·051	·018	·021	·028	·023	·048	·024	·033	·029
	·008	·021	·025	·026	·088	·032	·035	·050	·039	·082	·044	·058	·048
	·002	·017	·021	·022	·079	·026	·027	·044	·033	·073	·038	·052	·040
15	1—12	1—11	1—10	1—9	1—8	2—10	2—9	2—9	2—8	2—8	3—9	2—7	1—5
	·007	·013	·016	·017	·018	·044	·033	·035	·026	·027	·042	·021	·030
	·012	·023	·028	·030	·032	·076	·059	·058	·045	·045	·071	·037	·054
	·008	·018	·023	·025	·028	·068	·054	·051	·041	·038	·063	·031	·047

TABLE 7.2-V

Same data as in table 7.2-I: $\alpha = \cdot 05$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3	1—3	1—3	1—3	2—3	2—3	2—3	1—2	1—2	1—2	1—2	1—2	1—2	1—2
	0	0	0	·043	·044	·044	—·014	—·013	—·015	—·017	—·018	—·018	—·018
	0	0	0	·076	·072	·069	—·046	—·046	—·049	—·052	—·054	—·056	—·056
	0	0	0	·062	·058	·054	—·078	—·079	—·083	—·087	—·089	—·091	—·092
4	1—4	1—4	1—3	1—3	1—3	1—3	1—3	2—3	2—3	1—2	1—2	1—2	1—2
	0	0	—·005	—·004	—·009	—·011	—·013	·006	·003	—·010	—·012	—·012	—·014
	0	0	—·025	—·025	—·034	—·040	—·043	—·002	—·009	—·037	—·041	—·044	—·047
	0	0	—·051	—·053	—·064	—·071	—·075	—·022	—·032	—·067	—·073	—·076	—·082
5	1—5	1—4	1—4	1—4	1—3	1—3	1—3	3—4	3—4	3—4	2—3	2—3	2—3
	0	—·002	—·003	—·001	—·004	—·005	—·007	·036	·035	·033	·002	·000	—·001
	0	—·017	—·021	—·020	—·019	—·023	—·029	·067	·061	·052	—·006	—·010	—·014
	0	—·039	—·046	—·048	—·038	—·046	—·055	·061	·053	·040	—·020	—·025	—·030
6	1—5	1—5	1—4	1—4	1—4	1—4	1—3	1—3	1—3	1—3	3—4	3—4	3—4
	·001	—·002	·001	·000	—·002	—·002	·001	·000	·000	—·001	·019	·019	·017
	—·004	—·015	—·004	—·009	—·016	—·020	—·008	—·010	—·013	—·016	·033	·029	·025
	—·020	—·036	—·017	—·026	—·037	—·044	—·024	—·029	—·032	—·036	·025	·020	·014
7	1—6	1—5	2—6	1—4	1—4	2—5	1—4	1—4	2—4	1—3	1—3	4—5	1—3
	·002	·004	·055	·003	·003	·027	·002	·002	·010	·004	·004	·039	·004
	—·004	·002	·104	—·001	—·002	·045	—·007	—·010	·013	—·001	—·003	·071	—·005
	—·019	—·009	·095	—·013	—·016	·036	—·025	—·029	·004	—·017	—·020	·064	—·025
8	1—7	1—6	1—5	1—5	2—6	1—4	2—5	2—5	2—5	1—3	2—4	2—4	2—4
	·002	·005	·004	·005	·038	·005	·019	·018	·018	·008	·008	·008	·007
	—·003	·004	·004	·003	·065	·004	·032	·029	·025	·011	·011	·009	·006
	—·018	—·006	—·006	—·010	·056	—·008	·025	·020	·014	·000	·001	—·002	—·005
9	1—7	1—6	1—6	1—5	1—5	2—6	1—4	2—5	2—5	2—5	2—5	1—3	2—4
	·004	·006	·008	·006	·008	·028	·008	·015	·015	·015	·014	·013	·008
	·005	·008	·009	·009	·009	·049	·012	·027	·024	·022	·018	·020	·011
	—·002	·001	—·002	·001	—·002	·043	·002	·020	·016	·013	·008	·011	·002
10	1—8	1—7	1—6	1—6	1—5	1—5	2—6	1—4	1—4	2—5	2—5	2—5	5—7
	·004	·007	·007	·009	·008	·010	·022	·011	·013	·013	·013	·013	·056
	·006	·011	·012	·013	·013	·015	·040	·020	·020	·022	·021	·019	·104
	·000	·004	·005	·004	·007	·006	·034	·012	·011	·015	·013	·010	·096
11	1—9	1—8	1—7	1—6	1—6	1—5	1—5	2—6	1—4	1—4	2—5	2—5	2—5
	·005	·009	·010	·009	·011	·011	·013	·019	·014	·016	·012	·012	·013
	·007	·013	·016	·016	·018	·020	·021	·034	·026	·028	·022	·021	·019
	·000	·005	·009	·010	·010	·014	·014	·030	·021	·020	·016	·014	·012
12	1—9	1—8	1—7	1—6	1—6	1—6	1—5	1—5	2—6	1—4	1—4	1—4	3—6
	·005	·009	·009	·009	·012	·014	·014	·016	·016	·017	·019	·021	·018
	·010	·018	·018	·019	·022	·024	·026	·028	·031	·034	·035	·037	·034
	·006	·014	·015	·016	·018	·018	·021	·022	·027	·029	·030	·030	·031
13	1—10	1—9	2—10	2—9	1—6	2—8	1—5	1—5	3—8	2—6	1—4	1—4	1—4
	·006	·011	·056	·039	·011	·030	·014	·017	·037	·015	·020	·022	·024
	·011	·020	·109	·078	·024	·056	·029	·033	·070	·030	·040	·043	·045
	·007	·016	·103	·075	·022	·053	·028	·029	·066	·026	·037	·038	·039
14	1—11	1—9	1—8	1—7	2—9	1—6	2—8	1—5	2—7	3—8	2—6	1—4	3—7
	·006	·010	·010	·011	·034	·014	·026	·016	·020	·031	·014	·022	·021
	·011	·020	·021	·024	·066	·030	·049	·036	·037	·060	·029	·046	·042
	·007	·018	·020	·023	·064	·028	·046	·034	·034	·057	·027	·045	·038
15	1—11	1—10	1—9	1—8	1—7	1—6	2—8	2—8	2—7	2—7	3—8	2—6	1—4
	·007	·011	·013	·014	·014	·014	·022	·024	·017	·019	·026	·015	·026
	·014	·022	·025	·027	·030	·031	·045	·046	·036	·037	·053	·031	·054
	·013	·020	·023	·026	·029	·032	·044	·042	·034	·034	·050	·027	·052

TABLE 7.2-VI

Same data as in table 7.2-I: $\alpha = \cdot 10$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3	1—3	1—3	2—3	1—2	1—2	1—2	1—2	1—2	2—2	2—2	2—2	2—2	2—2
	0	0	·041	—·010	—·012	—·013	—·014	—·013	—·004	—·006	—·007	—·009	—·010
	0	0	·080	—·035	—·041	—·044	—·046	—·046	—·023	—·027	—·030	—·034	—·038
	0	0	·069	—·063	—·071	—·075	—·078	—·079	—·046	—·052	—·056	—·061	—·066
4	1—4	1—3	1—3	1—3	2—3	1—2	1—2	3—3	1—2	1—2	4—3	4—3	2—2
	0	—·004	—·005	—·004	·011	—·005	—·006	·017	—·009	—·010	·019	·018	—·004
	0	—·020	—·025	—·025	·021	—·022	—·026	·038	—·034	—·037	·044	·039	—·016
	0	—·043	—·051	—·053	·012	—·043	—·049	·037	—·062	—·067	·044	·038	—·032
5	1—4	1—4	1—3	1—3	1—3	3—4	2—3	1—2	1—2	1—2	3—3	3—3	3—3
	·001	—·002	—·002	—·002	—·004	·034	·006	·000	·000	—·001	·008	·007	·007
	—·006	—·017	—·011	—·012	—·019	·077	·010	—·009	—·010	—·012	·015	·012	·009
	—·021	—·039	—·023	—·026	—·038	·076	·002	—·026	—·028	—·030	·008	·004	·000
6	1—5	1—4	1—4	1—3	2—4	1—3	1—3	3—4	2—3	1—2	1—2	1—2	1—2
	·001	·002	·001	·001	·015	·001	·001	·019	·003	·003	·004	·004	·004
	—·004	—·002	—·004	—·003	·032	—·006	—·008	·043	·004	·002	·001	·001	·000
	—·020	—·013	—·017	—·015	·028	—·022	—·024	·042	—·005	—·011	—·013	—·014	—·016
7	1—5	2—6	1—4	2—5	1—3	1—3	2—4	2—4	4—5	3—4	3—4	2—3	1—2
	·002	·048	·002	·024	·002	·003	·010	·010	·034	·012	·013	·003	·008
	·003	·111	·000	·052	·002	·002	·019	·017	·078	·028	·026	·004	·014
	—·004	·109	—·010	·050	—·007	—·008	·014	·009	·079	·026	·022	—·006	·003
8	1—6	1—5	1—4	1—4	2—5	2—5	1—3	2—4	2—4	2—4	4—5	3—4	3—4
	·003	·004	—·046	·004	·017	·018	·006	·008	·008	·008	·023	·009	·009
	·004	·005	—·064	·006	·037	·035	·010	·016	·015	·013	·054	·021	·019
	—·003	—·003	—·062	—·002	·036	·031	·002	·012	·008	·006	·055	·017	·014
9	1—6	1—5	1—5	1—4	1—4	2—5	2—5	1—3	2—4	2—4	2—4	5—6	4—5
	·003	·003	·004	·004	·005	·013	·014	·008	·006	·007	·008	·035	·017
	·006	·007	·007	·009	·010	·030	·028	·017	·016	·015	·013	·080	·040
	·002	·003	·001	·004	·005	·028	·024	·011	·011	·009	·006	·082	·040
10	1—7	1—6	1—5	1—5	1—4	1—4	2—5	3—6	1—3	1—3	2—4	2—4	2—4
	·004	·006	·004	·006	·006	·007	·010	·023	·010	·012	·006	·008	·008
	·008	·012	·010	·012	·013	·016	·026	·052	·024	·026	·015	·015	·015
	·003	·008	·007	·008	·010	·012	·024	·054	·020	·021	·012	·009	·008
11	1—7	1—6	1—6	1—5	3—8	1—4	1—4	2—5	3—6	1—3	1—3	2—4	2—4
	·004	·004	·007	·006	·044	·008	·010	·009	·017	·012	·014	·006	·007
	·008	·010	·014	·015	·105	·019	·022	·023	·043	·029	·032	·017	·017
	·008	·008	·011	·013	·105	·018	·019	·021	·044	·029	·030	·014	·012
12	1—8	1—7	1—6	1—5	2—7	3—8	1—4	4—8	2—5	3—6	1—3	1—3	1—3
	·005	·006	·005	·006	·019	·035	·010	·036	·008	·014	·014	·021	·018
	·011	·014	·014	·016	·045	·084	·025	·092	·022	·036	·036	·037	·041
	·010	·012	·013	·016	·046	·085	·025	·095	·022	·038	·036	·030	·040
13	1—9	1—7	1—6	2—8	1—5	2—7	1—4	1—4	3—7	2—5	2—5	1—3	1—3
	·005	·004	·005	·022	·008	·017	·009	·012	·020	·008	·009	·016	·018
	·012	·013	·014	·054	·021	·040	·027	·032	·048	·023	·022	·043	·045
	·011	·013	·015	·058	·022	·040	·029	·032	·049	·022	·021	·044	·045
14	1—9	1—8	1—7	1—6	1—5	2—7	2—7	2—6	1—4	3—7	2—5	2—5	1—3
	·005	·007	·007	·007	·007	·013	·016	·009	·014	·016	·008	·009	·018
	·014	·016	·018	·021	·023	·035	·036	·027	·037	·041	·024	·025	·049
	·014	·017	·019	·022	·026	·038	·036	·028	·036	·044	·024	·024	·052
15	1—10	1—8	1—7	1—6	1—6	1—5	2—7	2—7	2—6	2—6	3—7	2—5	2—5
	·006	·005	·006	·007	·010	·009	·012	·015	·009	·011	·014	·008	·009
	·015	·016	·018	·020	·026	·028	·033	·035	·027	·028	·038	·025	·027
	·016	·017	·020	·024	·028	·032	·035	·035	·028	·028	·039	·025	·026

7.3. Comparison between V_1 tests and Student's two-sample test for Normal shift alternatives

The V_{ji} tests selected for Normal shift alternatives have been compared with the two-sample test of Student in the case of Normal shift alternatives. For the computation of the power function of Student's t-test we have used an approximation $t^*(f,\delta,\beta)$ of the (1—β)-percentage point $t(f,\delta,\beta)$ of the non-central t-distribution with f degrees of freedom and non-centrality parameter δ (cf. C. van Eeden (1961)) with one additional term (cf. Veselá (1964)) for $t^*(f,0,\beta)$:

$$t^*(f,\delta,\beta) = u_\beta + \frac{u_\beta^3 + u_\beta}{4f} + \frac{5\,u_\beta^5 + 16\,u_\beta^3 + 3\,u_\beta}{96f^2} + \frac{3\,u_\beta^7 + 19\,u_\beta^5 + 17\,u_\beta^3 - 15\,u_\beta}{384f^3} + \delta\left\{1 + \frac{2\,u_\beta^2 + 1}{4f} + \delta\frac{u_\beta}{4f} + \frac{4\,u_\beta^4 + 12\,u_\beta^2 + 1}{32f^2} + \delta\frac{u_\beta^3 + 4\,u_\beta}{16f^2} - \delta^2\frac{u_\beta^2 - 1}{24f^2} - \delta^3\frac{u_\beta}{32f^2}\right\}, \tag{7.3.1}$$

where μ_β is the (1—β)-point of the standard Normal distribution. The determination of the power requires the solution of the equation

$$t^*(f,\delta,\beta) - t_\alpha(f) \equiv \psi(u_\beta) = 0, \tag{7.3.2}$$

where $t_\alpha(f)$ is the (1—α)-point of the (central) t-distribution with f degrees of freedom. The root of this equation has been calculated using the Newton–Raphson procedure.

The tables 7.3-I–7.3-VI show the results of the comparison between V_{ji} tests selected for Normal shift alternatives and Student's two-sample test. Only the upper right corners of the tables are given, because the lower left corners can be found by symmetry considerations (cf. sec. 4.2.1). For the method of comparison we refer to sec. 7.2.

From these tables it will be seen that for Student alternatives the power differences between the V_{ji} tests and Student's test are sometimes not small, as could be expected from the fact that Pitman's asymptotic relative efficiency is equal to ·64. Apart from its applicability to non-Normal populations where of course also the tests of Van der Waerden, Wilcoxon, etc. can be applied, the V_{ji} test is useful even for Normal populations as a rough test and a first procedure providing confidence bounds almost without any effort. One disadvantage of the V_1 test is a limited freedom of choice of a confidence coefficient for small sample sizes.

It is also of interest to compare the selected lower confidence bounds with the lower confidence bounds based on the t-distribution directly by means of the expected lengths of the one-sided confidence intervals. The length of a

TABLE 7.3-I

Power differences for Normal shift alternatives between Student's test and the V_{ji} tests, denoted by j–i, in the three points where the power of the V_{ji} test equals ·25, ·50 and ·75, respectively ($\alpha = \cdot 001$)

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3													
4								1—4 ·08 ·14 ·14	1—4 ·09 ·15 ·15	1—4 ·10 ·17 ·15	1—4 ·11 ·18 ·16	1—4 ·12 ·19 ·17	1—4 ·13 ·20 ·17
5						1—5 ·07 ·13 ·13	1—5 ·08 ·14 ·14	1—5 ·09 ·16 ·15	1—5 ·11 ·18 ·16	2—5 ·11 ·18 ·16	2—5 ·12 ·19 ·16	2—5 ·12 ·20 ·17	2—5 ·13 ·20 ·17
6					1—6 ·07 ·13 ·13	1—6 ·08 ·14 ·14	1—6 ·09 ·16 ·16	2—6 ·11 ·18 ·16	1—5 ·14 ·21 ·17	1—5 ·15 ·22 ·18	1—5 ·16 ·23 ·18	3—6 ·14 ·22 ·18	3—6 ·15 ·23 ·18
7					1—7 ·08 ·14 ·15	1—7 ·09 ·16 ·16	1—6 ·12 ·19 ·17	1—6 ·14 ·21 ·17	1—6 ·15 ·22 ·18	3—7 ·15 ·23 ·18	3—7 ·16 ·23 ·19	2—6 ·14 ·20 ·17	2—6 ·14 ·21 ·17
8						2—8 ·12 ·19 ·16	1—7 ·13 ·20 ·17	1—7 ·14 ·22 ·18	3—8 ·16 ·24 ·19	2—7 ·13 ·20 ·16	4—8 ·18 ·26 ·20	2—7 ·15 ·22 ·17	2—7 ·16 ·22 ·18
9							2—9 ·14 ·21 ·18	2—8 ·12 ·19 ·16	2—8 ·13 ·20 ·16	4—9 ·19 ·27 ·20	4—9 ·20 ·28 ·21	3—8 ·15 ·21 ·17	2—7 ·16 ·23 ·18
10								2—9 ·14 ·20 ·16	4—10 ·20 ·28 ·21	3—9 ·14 ·21 ·17	2—8 ·16 ·22 ·18	2—8 ·17 ·23 ·18	4—9 ·16 ·23 ·18
11									2—9 ·15 ·21 ·17	2—9 ·16 ·22 ·18	1—7 ·25 ·33 ·22	4—10 ·17 ·23 ·18	3—9 ·16 ·22 ·18
12										2—10 ·17 ·23 ·18	2—9 ·17 ·24 ·18	3—10 ·16 ·22 ·18	5—11 ·18 ·25 ·19
13											3—11 ·16 ·23 ·18	3—11 ·17 ·23 ·18	4—11 ·16 ·23 ·18
14												4—12 ·16 ·23 ·18	3—11 ·17 ·24 ·18
15													2—10 ·21 ·28 ·20

TABLE 7.3-II

Same data as in table 7.3-I: $\alpha = {\cdot}005$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3							1—3 ·05 ·10 ·11	1—3 ·06 ·12 ·12	1—3 ·07 ·13 ·12	1—3 ·08 ·14 ·13	1—3 ·09 ·15 ·14	1—3 ·09 ·16 ·14	2—3 ·08 ·14 ·13
4				1—4 ·04 ·08 ·09	1—4 ·05 ·09 ·10	1—4 ·06 ·11 ·12	1—4 ·07 ·12 ·13	2—4 ·08 ·14 ·13	2—4 ·08 ·14 ·14	2—4 ·09 ·15 ·14	2—4 ·10 ·16 ·15	3—4 ·10 ·16 ·15	3—4 ·10 ·17 ·15
5			1—5 ·04 ·08 ·09	1—5 ·05 ·09 ·10	1—5 ·06 ·11 ·12	2—5 ·08 ·14 ·13	1—4 ·10 ·17 ·15	1—4 ·11 ·18 ·16	3—5 ·10 ·18 ·16	3—5 ·11 ·18 ·16	3—5 ·12 ·19 ·16	4—5 ·12 ·20 ·17	2—4 ·11 ·18 ·15
6				1—6 ·06 ·11 ·12	1—5 ·09 ·15 ·14	1—5 ·10 ·17 ·15	1—5 ·11 ·18 ·16	3—6 ·12 ·19 ·17	3—6 ·12 ·20 ·17	2—5 ·11 ·17 ·15	2—5 ·12 ·18 ·15	2—5 ·12 ·19 ·16	5—6 ·15 ·24 ·19
7					2—7 ·10 ·16 ·15	1—6 ·11 ·18 ·16	1—5 ·14 ·22 ·18	2—6 ·10 ·17 ·15	4—7 ·15 ·23 ·19	2—6 ·12 ·18 ·16	3—6 ·11 ·18 ·15	3—6 ·12 ·18 ·16	2—5 ·14 ·21 ·17
8						3—8 ·13 ·21 ·18	2—7 ·11 ·17 ·15	4—8 ·16 ·24 ·19	3—7 ·11 ·18 ·15	2—6 ·13 ·19 ·16	2—6 ·13 ·20 ·17	4—7 ·12 ·19 ·16	4—7 ·13 ·20 16
9							4—9 ·16 ·25 ·20	2—7 ·12 ·18 ·16	2—7 ·13 ·20 ·16	2—7 ·14 ·20 ·17	4—8 ·13 ·20 ·16	3—7 ·13 ·19 ·16	5—8 ·14 ·21 ·17
10								2—8 ·13 ·19 ·16	1—6 ·21 ·29 ·21	3—8 ·12 ·19 ·16	3—8 ·13 ·20 ·16	5—9 ·14 ·22 ·18	4—8 ·13 ·19 ·16
11									3—9 ·12 ·19 ·16	3—9 ·13 ·20 ·16	5—10 ·15 ·22 ·18	3—8 ·13 ·20 ·16	6—10 ·16 ·23 ·18
12										2—8 ·15 ·23 ·18	3—9 ·13 ·20 ·16	6—11 ·16 ·24 ·19	5—10 ·14 ·21 ·17
13											3—10 ·14 ·21 ·17	5—11 ·14 ·21 ·17	4—10 ·14 ·21 ·17
14												4—11 ·14 ·21 ·17	2—8 ·19 ·27 ·20
15													5—12 ·14 ·21 ·17

TABLE 7.3-III

Same data as in table 7.3-I: $\alpha = \cdot 01$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3					1—3 ·04 ·08 ·09	1—3 ·04 ·09 ·10	1—3 ·05 ·10 ·11	1—3 ·06 ·12 ·12	1—3 ·07 ·13 ·12	2—3 ·06 ·12 ·12	2—3 ·07 ·12 ·12	2—3 ·07 ·13 ·13	2—3 ·08 ·14 ·13
4			1—4 ·03 ·06 ·08	1—4 ·04 ·08 ·09	1—4 ·05 ·09 ·10	1—4 ·06 ·11 ·12	2—4 ·07 ·13 ·12	2—4 ·08 ·14 ·13	1—3 ·11 ·18 ·16	3—4 ·09 ·15 ·14	3—4 ·09 ·16 ·15	3—4 ·10 ·16 ·15	4—4 ·10 ·17 ·15
5			1—5 ·04 ·08 ·09	1—5 ·05 ·09 ·10	2—5 ·07 ·13 ·13	1—4 ·09 ·16 ·14	1—4 ·10 ·17 ·15	3—5 ·10 ·17 ·15	3—5 ·10 ·18 ·16	2—4 ·09 ·15 ·14	2—4 ·10 ·16 ·14	2—4 ·10 ·17 ·15	5—5 ·12 ·20 ·18
6				2—6 ·08 ·14 ·13	1—5 ·09 ·15 ·14	3—6 ·10 ·18 ·16	3—6 ·11 ·18 ·16	2—5 ·09 ·15 ·14	2—5 ·10 ·16 ·14	2—5 ·11 ·17 ·15	3—5 ·10 ·16 ·14	3—5 ·10 ·17 ·15	6—6 ·14 ·23 ·19
7					1—6 ·10 ·16 ·15	2—6 ·09 ·15 ·13	2—6 ·10 ·16 ·14	4—7 ·14 ·22 ·18	3—6 ·10 ·16 ·14	2—5 ·11 ·18 ·16	2—5 ·12 ·19 ·16	4—6 ·10 ·17 ·15	4—6 ·11 ·18 ·16
8						2—7 ·10 ·16 ·14	1—5 ·15 ·24 ·19	2—6 ·10 ·17 ·15	2—6 ·12 ·18 ·16	4—7 ·11 ·18 ·15	4—7 ·12 ·18 ·16	3—6 ·11 ·18 ·15	5—7 ·12 ·19 ·16
9							3—8 ·10 ·17 ·15	2—7 ·12 ·18 ·16	4—8 ·11 ·18 ·16	3—7 ·11 ·18 ·15	3—7 ·12 ·18 ·16	5—8 ·13 ·20 ·17	4—7 ·11 ·18 ·15
10								4—9 ·12 ·19 ·16	3—8 ·11 ·18 ·15	3—8 ·12 ·19 ·16	5—9 ·13 ·21 ·17	3—7 ·12 ·19 ·16	3—7 ·13 ·20 ·16
11									3—9 ·12 ·19 ·16	4—9 ·11 ·18 ·15	3—8 ·12 ·19 ·16	6—10 ·15 ·22 ·18	5—9 ·12 ·19 ·16
12										4—10 ·12 ·19 ·16	2—7 ·15 ·23 ·19	4—9 ·12 ·19 ·16	4—9 ·13 ·20 ·16
13											4—10 ·12 ·19 ·16	7—12 ·16 ·25 ·19	6—11 ·13 ·21 ·17
14												3—9 ·13 ·21 ·17	4—10 ·13 ·20 ·16
15													5—12 ·14 ·21 ·17

TABLE 7.3-IV

Same data as in table 7.3-I: $\alpha = \cdot 025$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3			1—3 ·02 ·05 ·06	1—3 ·03 ·06 ·07	1—3 ·04 ·08 ·09	2—3 ·04 ·09 ·10	2—3 ·05 ·10 ·10	2—3 ·05 ·10 ·11	2—3 ·06 ·11 ·11	3—3 ·06 ·12 ·12	1—2 ·10 ·17 ·15	1—2 ·10 ·18 ·16	4—3 ·06 ·12 ·12
4		1—4 ·02 ·04 ·06	1—4 ·03 ·06 ·08	2—4 ·05 ·10 ·11	1—3 ·07 ·13 ·13	1—3 ·08 ·14 ·14	3—4 ·07 ·13 ·13	3—4 ·08 ·14 ·14	3—4 ·08 ·15 ·14	4—4 ·08 ·15 ·15	2—3 ·08 ·14 ·13	5—4 ·08 ·16 ·15	5—4 ·09 ·16 ·15
5			2—5 ·06 ·11 ·11	1—4 ·07 ·13 ·13	1—4 ·08 ·14 ·14	3—5 ·08 ·15 ·14	2—4 ·07 ·13 ·12	2—4 ·08 ·14 ·13	2—4 ·08 ·15 ·13	5—5 ·10 ·18 ·17	3—4 ·08 ·14 ·13	3—4 ·08 ·14 ·13	3—4 ·09 ·15 ·14
6				1—5 ·08 ·14 ·13	1—4 ·10 ·18 ·16	2—5 ·07 ·13 ·13	4—6 ·11 ·19 ·17	3—5 ·07 ·14 ·13	3—5 ·08 ·14 ·13	2—4 ·09 ·16 ·15	4—5 ·08 ·14 ·14	4—5 ·09 ·15 ·14	4—5 ·09 ·16 ·14
7					2—6 ·08 ·14 ·13	4—7 ·12 ·20 ·18	2—5 ·08 ·15 ·14	2—5 ·09 ·16 ·14	4—6 ·08 ·15 ·14	4—6 ·09 ·16 ·14	3—5 ·09 ·15 ·14	5—6 ·09 ·16 ·15	2—4 ·12 ·20 ·17
8						3—7 ·08 ·15 ·14	2—6 ·09 ·16 ·14	4—7 ·09 ·16 ·14	3—6 ·08 ·15 ·14	5—7 ·09 ·17 ·15	5—7 ·10 ·18 ·15	4—6 ·09 ·16 ·14	6—7 ·10 ·18 ·16
9							4—8 ·09 ·16 ·15	3—7 ·09 ·16 ·14	5—8 ·10 ·17 ·16	4—7 ·09 ·15 ·14	6—8 ·10 ·18 ·16	3—6 ·10 ·18 ·15	5—7 ·09 ·16 ·14
10								3—8 ·10 ·17 ·15	3—7 ·09 ·16 ·14	3—7 ·10 ·17 ·15	6—9 ·12 ·20 ·17	4—7 ·09 ·16 ·14	3—6 ·11 ·18 ·16
11									4—9 ·10 ·17 ·15	2—6 ·12 ·21 ·18	4—8 ·09 ·16 ·14	4—8 ·10 ·17 ·15	8—10 ·12 ·21 ·18
12										4—9 ·10 ·16 ·15	7—11 ·13 ·22 ·18	5—9 ·09 ·16 ·15	4—8 ·10 ·18 ·15
13											4—9 ·09 ·16 ·15	4—9 ·10 ·18 ·15	7—11 ·11 ·19 ·16
14												3—8 ·11 ·19 ·16	5—10 ·10 ·17 ·15
15													5—11 ·11 ·18 ·16

TABLE 7.3-V

Same data as in table 7.3-I: $\alpha = \cdot 05$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3	1—3	1—3	1—3	2—3	2—3	2—3	3—3	3—3	3—3	4—3	4—3	4—3	5—3
	·01	·01	·02	·03	·04	·04	·04	·05	·05	·05	·06	·06	·06
	·02	·03	·05	·08	·08	·09	·10	·10	·11	·11	·12	·12	·12
	·03	·04	·06	·09	·09	·10	·11	·11	·11	·12	·12	·12	·12
4		1—4	1—3	1—3	3—4	3—4	4—4	2—3	2—3	1—2	1—2	3—3	3—3
		·02	·05	·06	·06	·06	·06	·06	·06	·10	·11	·06	·06
		·04	·10	·12	·12	·13	·13	·12	·12	·20	·21	·12	·12
		·06	·11	·12	·12	·13	·14	·12	·12	·18	·18	·12	·12
5			1—4	3—5	2—4	4—5	2—4	3—4	3—4	6—5	4—4	4—4	4—4
			·06	·06	·05	·08	·07	·06	·06	·09	·06	·06	·07
			·11	·14	·11	·16	·13	·12	·12	·18	·12	·13	·13
			·11	·14	·11	·15	·12	·12	·12	·17	·12	·12	·13
6				2—5	4—6	4—6	2—4	2—4	4—5	7—6	5—5	5—5	3—4
				·05	·08	·10	·07	·08	·06	·10	·06	·07	·08
				·11	·17	·18	·13	·14	·13	·20	·13	·14	·15
				·11	·16	·16	·13	·14	·13	·18	·13	·13	·14
7					4—7	2—5	2—5	4—6	2—4	5—6	5—6	4—5	4—5
					·10	·07	·08	·07	·08	·07	·08	·07	·08
					·19	·14	·15	·14	·16	·15	·15	·14	·14
					·17	·13	·14	·14	·15	·14	·14	·13	·13
8						5—8	3—6	3—6	5—7	4—6	3—5	5—6	5—6
						·12	·06	·08	·08	·07	·08	·07	·08
						·22	·13	·14	·16	·14	·15	·14	·14
						·19	·13	·13	·15	·13	·14	·13	·14
9							3—7	2—5	3—6	6—8	5—7	4—6	6—7
							·08	·08	·08	·09	·07	·08	·07
							·14	·16	·14	·17	·14	·14	·14
							·13	·15	·14	·16	·14	·14	·14
10								3—7	6—9	4—7	3—6	6—8	5—7
								·08	·09	·07	·09	·08	·07
								·15	·18	·14	·16	·15	·14
								·14	·16	·13	·15	·14	·14
11									4—8	7—10	5—8	4—7	9—10
									·07	·10	·07	·08	·10
									·14	·19	·14	·15	·20
									·13	·17	·13	·14	·18
12										5—9	4—8	7—10	4—7
										·07	·08	·09	·08
										·14	·15	·16	·15
										·13	·14	·15	·14
13											7—11	5—9	8—11
											·08	·08	·09
											·17	·15	·17
											·15	·14	·16
14												5—10	5—9
												·09	·07
												·16	·15
												·14	·14
15													5—10
													·09
													·16
													·14

TABLE 7.3-VI

Same data as in table 7.3-I: $\alpha = \cdot 10$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3	1—3 ·01 ·02 ·03	1—3 ·01 ·03 ·05	2—3 ·03 ·07 ·08	1—2 ·04 ·10 ·11	3—3 ·03 ·09 ·10	3—3 ·04 ·09 ·10	4—3 ·04 ·09 ·11	4—3 ·04 ·10 ·11	5—3 ·04 ·10 ·11	2—2 ·04 ·10 ·11	2—2 ·05 ·11 ·11	6—3 ·04 ·11 ·12	6—3 ·05 ·11 ·12
4		2—4 ·03 ·08 ·10	1—3 ·05 ·10 ·11	3—4 ·05 ·11 ·12	2—3 ·04 ·09 ·10	1—2 ·06 ·15 ·15	5—4 ·05 ·12 ·14	3—3 ·04 ·09 ·10	6—4 ·05 ·13 ·14	3—3 ·05 ·10 ·11	4—3 ·04 ·09 ·10	4—3 ·04 ·10 ·10	2—2 ·06 ·14 ·14
5			3—5 ·05 ·12 ·13	2—4 ·04 ·10 ·10	4—5 ·06 ·14 ·15	3—4 ·04 ·10 ·10	2—3 ·04 ·11 ·12	2—3 ·05 ·12 ·13	4—4 ·04 ·11 ·11	4—4 ·05 ·11 ·12	3—3 ·04 ·11 ·12	3—3 ·05 ·12 ·12	6—4 ·04 ·10 ·11
6				4—6 ·06 ·15 ·15	2—4 ·04 ·11 ·11	2—4 ·06 ·12 ·12	4—5 ·04 ·11 ·12	3—4 ·04 ·11 ·11	2—3 ·05 ·14 ·14	2—3 ·06 ·15 ·14	4—4 ·04 ·10 ·11	4—4 ·05 ·11 ·12	7—5 ·05 ·12 ·12
7					1—3 ·07 ·18 ·17	4—6 ·05 ·12 ·12	3—5 ·05 ·11 ·12	5—6 ·05 ·12 ·13	4—5 ·04 ·11 ·11	3—4 ·05 ·12 ·12	3—4 ·06 ·13 ·13	5—5 ·05 ·11 ·12	5—5 ·06 ·12 ·12
8						3—6 ·05 ·12 ·12	5—7 ·05 ·13 ·13	3—5 ·05 ·12 ·12	6—7 ·06 ·14 ·14	5—6 ·05 ·11 ·12	4—5 ·05 ·12 ·12	6—6 ·05 ·11 ·12	3—4 ·06 ·14 ·14
9							4—7 ·05 ·12 ·12	6—8 ·06 ·14 ·14	4—6 ·04 ·11 ·12	3—5 ·06 ·13 ·13	6—7 ·05 ·12 ·12	5—6 ·05 ·12 ·12	4—5 ·05 ·12 ·13
10								4—7 ·05 ·11 ·12	7—9 ·06 ·15 ·15	5—7 ·04 ·11 ·12	4—6 ·05 ·12 ·12	9—9 ·06 ·16 ·16	6—7 ·05 ·12 ·12
11									5—8 ·04 ·11 ·12	8—10 ·07 ·17 ·16	7—9 ·06 ·14 ·14	4—6 ·05 ·12 ·13	4—6 ·06 ·14 ·13
12										3—6 ·05 ·14 ·14	5—8 ·05 ·12 ·12	8—10 ·06 ·14 ·14	6—8 ·05 ·12 ·12
13											2—5 ·09 ·20 ·18	5—8 ·05 ·12 ·12	9—11 ·06 ·15 ·15
14												5—9 ·06 ·13 ·13	6—9 ·05 ·12 ·12
15													6—10 ·06 ·13 ·13

one-sided confidence interval is defined in this context as the absolute value of the difference of ν and the confidence bound. In the case of Normal distributions with variance 1 (this assumption can be made without loss of generality) and confidence level $1-\alpha = \cdot 95$ the selected lower confidence bounds $\underline{y}_{(j)} - \underline{x}_{(i)}$ (with confidence coefficient $1-\alpha^*$) have been compared with the lower confidence bounds based on Student's two-sample test with the same confidence coefficient for each pair of sample sizes (smaller than or equal to 15) by determining the ratios of the expected lengths of the one-sided confidence intervals, namely:

$$E\{\nu - (\underline{y}_{(j)} - \underline{x}_{(i)})\} \times$$

$$\times \left\langle E\left\{\nu - \left[\underline{\bar{y}} - \underline{\bar{x}} - t_{1-\alpha^*}(m+n-2)\left(\frac{\sum_{i=1}^{m}(\underline{x}_i - \underline{\bar{x}})^2 + \sum_{j=1}^{n}(\underline{y}_j - \underline{\bar{y}})^2}{m\,n\,(m+n-2)\,(m+n)^{-1}}\right)^{1/2}\right]\right\}\right\rangle^{-1}, \quad (7.3.3)$$

where the $(1-\alpha^*)$-percentage points $t_{1-\alpha^*}(m+n-2)$ of the t-distribution with $(m+n-2)$ degrees of freedom and

$$E\left\{[(m+n-2)^{-1}\{\sum_{i=1}^{m}(\underline{x}_i - \underline{\bar{x}})^2 + \sum_{j=1}^{n}(\underline{y}_j - \underline{\bar{y}})^2\}]^{1/2}\right\}$$

have been approximated by some simple extra- and interpolations from tables in Pearson and Hartley (1958). In table 7.3-VII the results are presented. From this table it can be seen that for this case the loss is about 20 per cent, which is surprisingly good.

TABLE 7.3-VII

Values of the ratio of the expected lengths of the selected one-sided confidence intervals and the one-sided confidence intervals corresponding with Student's t-test in the case of Normal distributions with the same confidence coefficients (confidence level $1-\alpha = \cdot 95$)

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3	1·03												
4	1·05	1·06											
5	1·06	1·17	1·16										
6	1·13	1·18	1·22	1·17									
7	1·13	1·20	1·17	1·29	1·30								
8	1·13	1·19	1·26	1·27	1·20	1·35							
9	1·16	1·23	1·17	1·21	1·20	1·20	1·20						
10	1·16	1·18	1·18	1·21	1·21	1·20	1·27	1·21					
11	1·16	1·18	1·18	1·20	1·26	1·24	1·22	1·28	1·21				
12	1·18	1·33	1·29	1·35	1·22	1·21	1·26	1·22	1·31	1·22			
13	1·18	1·34	1·19	1·21	1·22	1·23	1·22	1·24	1·22	1·22	1·27		
14	1·18	1·18	1·19	1·21	1·21	1·21	1·22	1·23	1·23	1·26	1·22	1·22	
15	1·19	1·18	1·19	1·21	1·21	1·21	1·22	1·22	1·33	1·24	1·27	1·23	1·23

7.4. Comparison between the V_1 tests selected for Normal shift alternatives and Wilcoxon's two-sample test when testing against Lehmann alternatives

In this section the powers against Lehmann alternatives of the V_{ji} tests selected for Normal shift alternatives have been compared with the power against Lehmann alternatives of the two-sample test of Wilcoxon. The Lehmann alternatives considered are of the form

$$G(y) = F^k(y) \quad \text{with} \quad k > 1.$$

For the method of comparison, for instance the procedure of randomization and absolute error, we refer to the description in sec. 7.2. The comparison has been made in this case only for three values of the significance level α, namely

$$\cdot 01, \ \cdot 025 \text{ and } \cdot 05.$$

The results have been presented in the tables 7.4-I, 7.4-II and 7.4-III, respectively. It turns out, as could be expected, that they are less efficient than the V_{ji} tests specially selected for Lehmann alternatives.

TABLE 7.4-I

Power differences for Lehmann alternatives between Wilcoxon's test and the V_{ji} tests, denoted by j–i, selected for Normal translation alternatives in the three points where the power of the V_{ji} test is equal to ·25, ·50 and ·75, respectively (α = ·01)

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3					1—3	1—3	1—3	1—3	1—3	2—3	2—3	2—3	2—3
					0	0	0	0	0	·042	·042	·042	·041
					0	0	0	0	0	·060	·059	·058	·056
					0	0	0	0	0	·045	·043	·043	·042
4			1—4	1—4	1—4	1—4	2—4	2—4	1—3	3—4	3—4	3—4	4—4
			0	0	0	0	·060	·059	—·013	·078	·077	·076	·091
			0	0	0	0	·085	·082	—·046	·111	·107	·104	·127
			0	0	0	0	·064	·061	—·080	·082	·079	·076	·094
5		1—5	1—5	1—5	2—5	1—4	1—4	3—5	3—5	2—4	2—4	2—4	5—5
		0	0	0	·081	—·007	—·008	·111	·109	·013	·010	·008	·143
		0	0	0	·116	—·033	—·036	·153	·147	·004	—·003	—·008	·196
		0	0	0	·087	—·064	—·069	·112	·106	—·022	—·030	—·037	·141
6		1—6	1—6	1—5	1—5	3—6	3—6	2—5	2—5	2—5	3—5	3—5	6—6
		0	0	·000	—·003	·148	·144	·032	·028	·025	·050	·046	·208
		0	0	—·017	—·025	·207	·194	·036	·028	·020	·071	·062	·282
		0	0	—·044	—·054	·150	·138	·013	·002	—·008	·054	·042	·199
7	1—7	1—7	1—6	2—7	1—6	2—6	2—6	4—7	3—6	2—5	2—5	4—6	4—6
	0	0	—·000	·119	—·001	·053	·050	·218	·077	·020	·018	·090	·088
	0	0	—·016	·164	—·022	·077	·066	·288	·115	·022	·017	·134	·128
	0	0	—·041	·120	—·052	·058	·043	·200	·093	·006	·001	·108	·102
8	1—8	1—8	2—8	1—6	2—7	2—7	1—5	2—6	2—6	4—7	4—7	3—6	5—7
	0	0	·139	·007	·071	·069	·005	·035	·033	·118	·117	·046	·129
	0	0	·196	·000	·107	·098	—·004	·048	·042	·176	·169	·067	·188
	0	0	·144	—·017	·087	·077	—·023	·032	·026	·139	·132	·050	·146
9	1—9	1—8	2—9	1—7	2—8	4—9	2—7	2—7	4—8	3—7	3—7	5—8	4—7
	0	·000	·155	·009	·086	·290	·048	·046	·144	·063	·063	·159	·076
	0	—·012	·214	·003	·126	·361	·069	·062	·210	·098	·091	·224	·115
	0	—·033	·154	—·015	·100	·228	·052	·044	·161	·079	·072	·168	·094
10	1—10	1—9	1—8	2—9	1—7	3—9	3—9	2—7	3—8	3—8	5—9	3—7	3—7
	0	·000	·009	·101	·011	·143	·143	·036	·083	·081	·188	·047	·046
	0	—·011	·008	·151	·009	·210	·201	·053	·123	·117	·257	·070	·065
	0	—·032	—·006	·121	—·008	·161	·152	·040	·100	·092	·187	·056	·050
11	1—11	2—11	1—9	2—10	2—9	3—10	2—8	3—9	3—9	4—9	3—8	6—10	5—9
	0	·198	·010	·113	·072	·164	·046	·100	·099	·117	·061	·229	·128
	0	·283	·009	·165	·110	·231	·070	·149	·142	·175	·091	·302	·188
	0	·204	—·005	·130	·091	·172	·057	·120	·113	·141	·074	·208	·148
12	1—11	1—10	2—11	3—12	2—11	2—9	3—10	3—10	3—9	3—9	2—7	4—9	4—9
	·003	·008	·125	·123	·120	·056	·115	·115	·076	·075	·026	·087	·086
	—·002	·009	·190	·176	·164	·088	·173	·167	·117	·112	·038	·133	·129
	—·016	—·001	·151	·136	·125	·073	·138	·131	·098	·091	·026	·110	·105
13	1—12	1—11	2—12	2—11	3—12	2—10	3—11	2—9	4—11	6—12	4—10	7—12	6—11
	·003	·009	·135	·092	·201	·065	·131	·046	·154	·278	·102	·029	·181
	—·002	·010	·200	·143	·276	·099	·190	·069	·221	·352	·155	·363	·254
	—·016	—·001	·156	·118	·197	·082	·149	·055	·168	·228	·127	·230	·187
14	1—13	1—12	2—13	2—12	2—11	3—12	2—10	4—12	2—9	4—11	2—8	3—9	4—10
	·003	·009	·145	·102	·073	·145	·054	·170	·040	·117	·030	·050	·082
	—·001	·010	·211	·154	·114	·212	·083	·244	·061	·177	·045	·078	·124
	—·016	·000	·162	·126	·096	·164	·068	·182	·048	·142	·034	·065	·103
15	1—14	1—12	1—11	1—10	2—12	2—11	3—12	4—13	3—11	4—12	3—10	5—12	4—11
	·003	·012	·017	·019	·082	·061	·113	·189	·082	·134	·060	·146	·095
	—·001	·018	·026	·030	·125	·096	·172	·262	·126	·194	·092	·212	·140
	—·016	·010	·018	·022	·104	·082	·139	·190	·105	·152	·078	·164	·113

TABLE 7.4-II

Same data as in table 7.4-I: $\alpha = \cdot 025$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3			1—3	1—3	1—3	2—3	2—3	2—3	2—3	3—3	1—2	1—2	4—3
			0	0	0	·044	·044	·043	·043	·050	—·018	—·018	·055
			0	0	0	·069	·066	·064	·062	·077	—·054	—·056	·087
			0	0	0	·054	·051	·049	·047	·060	—·089	—·091	·068
4		1—4	1—4	2—4	1—3	1—3	3—4	3—4	3—4	4—4	2—3	5—4	5—4
		0	0	·062	—·009	—·011	·081	·080	·079	·094	—·000	·102	·101
		0	0	·099	—·034	—·040	·125	·120	·115	·141	—·018	·156	·151
		0	0	·078	—·064	—·071	·097	·091	·086	·108	—·044	·120	·114
5	1—5	1—5	1—4	1—4	1—4	3—5	2—4	2—4	2—4	5—5	3—4	3—4	3—4
	0	0	—·003	—·001	—·005	·115	·021	·018	·015	·148	·030	·028	·025
	0	0	—·021	—·020	—·028	·172	·025	·016	·010	·224	·044	·036	·029
	0	0	—·046	—·048	—·058	·129	·006	—·005	—·014	·168	·029	·019	·009
6	1—6	1—5	2—6	1—5	1—4	2—5	4—6	3—5	3—5	2—4	4—5	4—5	4—5
	0	—·002	·108	·000	—·002	·038	·181	·057	·056	·011	·066	·065	·063
	0	—·015	·163	—·017	—·016	·056	·265	·096	·089	·010	·110	·104	·098
	0	—·036	·124	—·044	—·037	·040	·194	·082	·075	—·004	·094	·088	·082
7	1—7	2—7	2—7	3—7	2—6	4—7	2—5	2—5	4—6	4—6	3—5	5—6	2—4
	0	·124	·123	·185	·057	·218	·025	·024	·091	·092	·034	·100	·006
	0	·200	·179	·278	·089	·309	·039	·033	·152	·147	·053	·161	—·000
	0	·156	·134	·203	·074	·214	·027	·020	·127	·121	·042	·133	—·015
8	1—7	2—8	1—6	2—7	1—5	2—6	2—6	4—7	3—6	5—7	5—7	4—6	6—7
	·002	·142	·007	·073	·006	·038	·037	·116	·051	·126	·128	·058	·136
	—·003	·220	·004	·117	·001	·060	·055	·188	·053	·204	·199	·096	·211
	—·018	·168	—·010	·098	—·014	·048	·041	·152	·071	·164	·158	·083	·166
9	1—8	1—7	2—8	1—6	3—8	3—8	2—6	3—7	5—8	4—7	6—8	3—6	5—7
	·002	·006	·086	·008	·119	·123	·028	·067	·150	·076	·159	·035	·083
	—·002	·006	·144	·007	·195	·188	·046	·109	·238	·129	·249	·055	·137
	—·017	—·005	·122	—·006	·158	·149	·037	·094	·184	·112	·190	·044	·117
10	1—9	1—8	2—9	2—8	3—9	2—7	3—8	3—8	3—7	3—7	6—9	4—7	3—6
	·002	·007	·099	·060	·140	·037	·085	·035	·048	·049	·192	·055	·027
	—·002	·007	·160	·102	·218	·062	·135	·130	·082	·078	·279	·094	·041
	—·017	—·003	·132	·090	·170	·054	·115	·103	·073	·069	·202	·083	·034
11	1—10	1—9	2—10	2—9	2—8	3—9	2—7	4—9	3—8	2—6	4—8	4—8	8—10
	·003	·008	·112	·021	·045	·096	·030	·110	·062	·020	·070	·071	·225
	—·002	·008	·175	·118	·079	·159	·050	·182	·100	·028	·117	·113	·324
	—·017	—·002	·142	·102	·070	·135	·043	·151	·085	·022	·102	·097	·221
12	1—10	1—9	1—8	3—11	2—9	2—8	3—9	4—10	6—11	4—9	7—11	5—9	4—8
	·005	·010	·012	·172	·055	·037	·072	·131	·232	·084	·248	·090	·054
	·007	·015	·019	·264	·092	·065	·124	·204	·330	·140	·341	·151	·095
	·001	·007	·012	·197	·080	·057	·109	·163	·224	·120	·226	·130	·080
13	2—13	2—12	2—11	2—10	3—11	2—9	2—8	2—8	4—10	2—7	4—9	4—9	7—11
	·211	·130	·088	·062	·123	·045	·031	·033	·097	·024	·065	·067	·178
	·319	·215	·150	·108	·200	·075	·055	·053	·161	·036	·112	·111	·265
	·221	·173	·128	·096	·163	·066	·049	·045	·136	·031	·100	·096	·197
14	2—14	1—10	2—12	2—11	2—10	3—11	4—12	4—11	4—11	4—10	5—11	3—8	5—10
	·225	·012	·099	·070	·051	·095	·163	·108	·114	·076	·123	·033	·084
	·331	·022	·162	·118	·088	·159	·246	·180	·180	·131	·194	·058	·140
	·224	·017	·137	·104	·079	·135	·187	·151	·147	·115	·157	·052	·121
15	1—12	1—11	2—13	2—12	4—14	2—10	3—11	5—13	2—8	5—12	3—9	5—11	5—11
	·007	·013	·108	·079	·256	·044	·077	·188	·026	·136	·042	·096	·100
	·012	·023	·172	·130	·351	·076	·131	·283	·045	·214	·066	·160	·159
	·008	·018	·142	·111	·230	·068	·114	·206	·041	·170	·063	·135	·132

TABLE 7.4-III

Same data as in table 7.4-I: $\alpha = {\cdot}05$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3	1—3	1—3	1—3	2—3	2—3	2—3	3—3	3—3	3—3	4—3	4—3	4—3	5—3
	0	0	0	·043	·044	·044	·049	·050	·050	·054	·055	·055	·058
	0	0	0	·076	·072	·069	·085	·082	·079	·094	·092	·089	·101
	0	0	0	·062	·058	·054	·070	·066	·063	·078	·074	·071	·082
4	1—4	1—4	1—3	1—3	3—4	3—4	4—4	2—3	2—3	1—2	1—2	3—3	3—3
	0	0	—·005	—·004	·079	·081	·092	·006	·003	—·010	—·012	·010	·009
	0	0	—·025	—·025	·140	·132	·163	—·002	—·009	—·037	—·041	·010	·004
	0	0	—·051	—·053	·114	·104	·133	—·022	—·032	—·067	—·073	—·006	—·014
5	1—5	2—5	1—4	3—5	2—4	4—5	2—4	3—4	3—4	6—5	4—4	4—4	4—4
	0	·081	—·003	·112	·026	·133	·021	·036	·035	·154	·041	·041	·039
	0	·147	—·021	·199	·044	·231	·025	·067	·061	·257	·078	·072	·067
	0	·122	—·046	·160	·035	·184	·006	·061	·053	·197	·071	·064	·058
6	1—5	2—6	1—4	2—5	4—6	4—6	2—4	2—4	4—5	7—6	5—5	5—5	3—4
	·001	·107	·001	·042	·164	·177	·015	·014	·065	·191	·068	·071	·017
	—·004	·181	—·004	·076	·279	·275	·024	·020	·119	·312	·129	·127	·025
	—·020	·144	—·017	·068	·208	·203	·014	·008	·106	·220	·116	·112	·014
7	1—6	1—5	2—6	1—4	1—4	2—5	2—5	4—6	2—4	5—6	5—6	4—5	4—5
	·002	·003	·055	·003	·003	·027	·025	·089	·010	·095	·099	·039	·039
	—·004	·002	·104	—·001	—·002	·045	·039	·157	·013	·169	·166	·071	·068
	—·019	—·009	·095	—·013	—·016	·036	·027	·134	·004	·144	·139	·064	·059
8	1—7	1—6	1—5	1—5	3—7	1—4	3—6	3—6	5—7	4—6	3—5	5—6	5—6
	·002	·005	·004	·005	·097	·005	·049	·050	·120	·056	·024	·059	·061
	—·003	·004	·004	·003	·170	·004	·092	·088	·206	·103	·041	·113	·110
	—·018	—·006	—·006	—·010	·144	—·008	·085	·078	·169	·094	·035	·104	·099
9	1—7	1—6	2—8	3—8	3—8	3—7	3—7	2—5	3—6	6—8	5—7	4—6	6—4
	·004	·006	·086	·108	·119	·062	·066	·015	·035	·149	·078	·040	·080
	·005	·008	·144	·199	·195	·118	·114	·027	·065	·249	·142	·073	·149
	—·002	·001	·122	·167	·158	·108	·101	·020	·059	·193	·126	·068	·132
10	1—8	2—9	2—8	3—9	2—7	3—8	5—9	3—7	6—9	4—7	3—6	6—8	5—7
	·004	·092	·055	·131	·036	·079	·152	·047	·167	·051	·027	·099	·056
	·006	·169	·107	·224	·066	·139	·265	·086	·278	·098	·049	·178	·104
	—·000	·146	·100	·179	·060	·122	·203	·078	·207	·091	·042	·152	·096
11	1—9	2—10	2—9	2—8	4—10	2—7	4—9	2—6	4—8	7—10	5—8	4—7	9—10
	·005	·105	·067	·042	·159	·029	·102	·019	·064	·193	·066	·040	·197
	·007	·184	·122	·081	·272	·053	·183	·034	·120	·311	·128	·076	·320
	·000	·155	·111	·076	·206	·048	·156	·030	·110	·221	·118	·071	·225
12	1—9	3—12	1—7	1—6	2—8	3—9	2—7	4—9	2—6	4—8	4—8	7—10	4—7
	·005	·268	·009	·009	·035	·067	·024	·074	·016	·048	·051	·136	·031
	·010	·395	·018	·019	·066	·125	·045	·141	·031	·095	·094	·236	·062
	·006	·243	·015	·016	·062	·114	·040	·128	·027	·091	·087	·188	·059
13	1—10	3—13	2—10	2—9	2—9	4—11	3—9	5—11	4—9	5—10	3—7	5—9	8—11
	·006	·291	·056	·039	·043	·128	·053	·143	·057	·097	·024	·064	·156
	·011	·408	·109	·078	·078	·224	·101	·241	·113	·172	·048	·121	·264
	·007	·244	·103	·075	·071	·182	·094	·190	·106	·148	·045	·111	·202
14	1—11	2—12	2—11	2—10	3—11	3—10	4—11	3—9	5—11	3—8	5—10	5—10	5—9
	·006	·087	·064	·047	·086	·059	·098	·044	·106	·031	·075	·080	·050
	·011	·164	·120	·090	·158	·116	·179	·084	·191	·060	·140	·141	·100
	·007	·145	·111	·085	·140	·108	·154	·079	·162	·057	·126	·124	·095
15	1—11	2—13	2—12	3—13	3—12	3—11	3—10	4—11	2—7	6—12	3—8	6—11	5—10
	·007	·097	·073	·140	·099	·071	·050	·078	·017	·118	·026	·086	·064
	·014	·176	·132	·232	·174	·131	·097	·147	·036	·217	·053	·164	·117
	·013	·152	·119	·184	·149	·118	·092	·132	·034	·1	·050	·145	·106

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Summary

In this thesis the construction of simply applicable distribution-free confidence intervals for location difference of two populations with the same shape for their distributions is considered. The simplest confidence bounds are based on the difference of the ith order statistic of the first sample of size m and the jth order statistic of the second sample of size n. Generalizations of these confidence bounds are also considered. With a certain selection procedure "optimal" confidence bounds are selected for sample sizes $m,n = 3, 4, \ldots, 15$. This selection procedure is based on the power function of the tests to which these confidence bounds correspond. Normal and Uniform shift alternatives, as well as Lehmann alternatives have been taken into consideration. The selected tests are compared with Student's two-sample test in the case of Normal shift alternatives and with Wilcoxon's two-sample test in the case of Lehmann alternatives. The asymptotic relative efficiency of an asymptotic form of the test relative to Student's and Wilcoxon's test is determined in various cases. An extensive analytical and numerical study is made of the behaviour of moments and densities of $F^k(x)$ for various classes of distribution functions $F(x)$.

Curriculum vitae

De auteur van dit proefschrift werd geboren te Utrecht op 16 februari 1937. Hij doorliep de H.B.S. "Henegouwerplein" te Rotterdam en behaalde in 1954 het eindexamen van de b-afdeling. Daarna studeerde hij Wis- en Natuurkunde aan de Universiteit van Leiden waar hij in 1957 het kandidaatsexamen en in 1960 het doktoraalexamen aflegde.

Van 1958 tot 1960 was hij verbonden als assistent statistiek bij het Instituut voor Theoretische Biologie van de Universiteit van Leiden en van 1960 tot 1965 als medewerker van de afdeling Mathematische Statistiek van het Mathematisch Centrum te Amsterdam.

Sinds 1965 is hij werkzaam bij de N.V. Philips' Gloeilampenfabrieken te Eindhoven, tot 1967 op het Natuurkundig Laboratorium en vanaf 1967 bij de hoofdgroep Informatie Systemen en Automatie, afdeling Research, als leider van de groep Statistiek en Waarschijnlijkheidsrekening.

STELLINGEN

bij het proefschrift van P. van der Laan

26 juni 1970
T.H. Eindhoven

I

De sequente „Configural Rank" toets voor twee beslissingsmogelijkheden (éénzijdige toetsing) van Wilcoxon, Rhodes en Bradley kan met behulp van een methode van Sobel en Wald uitgebreid worden tot een sequente toets met drie beslissingsmogelijkheden (tweezijdige toetsing). Deze uitbreiding tot een tweezijdige toets is algemener dan de uitbreiding aangegeven door Wilcoxon en Bradley.

P. van der Laan (1966), A sequential distribution-free two-sample grouped rank test with three possible decisions, Statistica Neerlandica **20**, 31-41.

M. Sobel and A. Wald (1949), A sequential decision procedure for choosing one of three hypotheses concerning the unknown mean of a normal distribution, Ann. Math. Stat. **20**, 502-522.

F. Wilcoxon, R. J. Rhodes and R. A. Bradley (1963), Two sequential two-sample grouped rank tests with applications to screening experiments. Biometrics **19**, 58-84.

F. Wilcoxon and R. A. Bradley (1964), A note on the paper: Two sequential two-sample grouped rank tests with applications to screening experiments, Biometrics **20**, 892-895.

II

De numerieke methode ontwikkeld door Adriaanse en Van der Laan om voor een speciaal Markov proces onder- en bovengrenzen voor de verdelingsfuncties op de tijdstippen n_0, $n_0 + 1, \ldots$ ($n_0 \geqslant 1$) te bepalen, kan uitgebreid worden tot een algemene klasse van Markov processen met discrete tijdparameter en (in het algemeen) afhankelijke aangroeiingen:

$$\{\underline{x}_n;\ n = 0, 1, 2, \ldots\}$$

met

$$-\infty < a \leqslant \underline{x}_0 \leqslant b < \infty$$

en

$$\Pr\,[\underline{x}_{n+1} = h_j(x) \mid \underline{x}_n = x] = p_j > 0 \qquad (j = 1, 2, \ldots, k;\ \sum_{j=1}^{k} p_j = 1),$$

waarbij de functies $h_j(.)$ voldoen aan de volgende voorwaarden:

(i) $h_j(.)$ strikt stijgend

(ii) $h_j(b) \leqslant b$ voor alle j

(iii) $h_j(a) \geqslant a$ voor alle j

(iv) $\forall\, x\ (a < x < b)\ \exists_{j_1, j_2}$ waarvoor geldt: $h_{j_1}(x) > x$ en $h_{j_2}(x) < x$.

R. P. Adriaanse en P. van der Laan (1968), Statistische eigenschappen van een vlambeveiliging met behulp van een gasontladingsbuis welke gevoelig is voor ultraviolet licht, Statistica Neerlandica **22**, 159-172

R. P. Adriaanse en P. van der Laan (1970), Some remarks on a general class of Markov processes with discrete time parameter and dependent increments, Technometrics, ter perse.

III

Voor positieve gehele getallen i_1, i_2, j_1, j_2, m, n, waarbij $i_1 < i_2 \leqslant m$ en $j_1 \leqslant j_2 \leqslant n$, geldt de volgende identiteit:

$$\sum_{r_1=j_1}^{n} \sum_{r_2=j_2}^{n} \binom{i_1+r_1-1}{r_1} \binom{i_2-i_1+r_2-r_1-1}{r_2-r_1} \binom{m+n-i_2-r_2}{n-r_2}$$

$$= \sum_{r_1=j_1}^{n} \sum_{r_2=j_2}^{n} \binom{i_1+j_1-1}{r_1} \binom{i_2-i_1+j_2-j_1}{r_2-r_1} \binom{m+n-i_2-j_2+1}{n-r_2}.$$

IV

Voor het berekenen van de onderscheidingsvermogens van verdelingsvrije rangnummertoetsen, als die van Wilcoxon, Van der Waerden, enz., tegen homogene en exponentiële verschuivingsalternatieven zijn de volgende beweringen van belang.

Gegeven zijn de onafhankelijke steekproeven $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m$ en $\underline{y}_1, \underline{y}_2, \ldots, \underline{y}_n$ uit populaties met continue verdelingsfuncties $F(x)$ respectievelijk

$$G(y) = F(y-\mu) \quad (\mu \geqslant 0)$$

en de geordende gecombineerde steekproef $\underline{z}_1 < \underline{z}_2 < \ldots < \underline{z}_{m+n}$. Definieer de vector $\underline{t} = (\underline{t}_1, \underline{t}_2, \ldots, \underline{t}_{m+n})$ waarbij

$$\underline{t}_i = \begin{cases} 0 \text{ indien } z_i \text{ tot de } x\text{-steekproef behoort} \\ 1 \text{ indien } z_i \text{ tot de } y\text{-steekproef behoort.} \end{cases}$$

Indien $F(x)$ de exponentiële verdelingsfunctie is dan geldt dat vectoren $\underline{t}$, waarbij $t_1 = t_2 = \ldots = t_k = 0$, $t_{k+1} = 1$ $(n > 1,\ k = 0, 1, \ldots, m-1)$ even waarschijnlijk zijn.

Indien $F(x)$ de homogene verdelingsfunctie is dan geldt dat vectoren $\underline{t}$, waarbij $t_1 = t_2 = \ldots = t_k = 0$, $t_{k+1} = 1$, $t_{m+n-l} = 0$, $t_{m+n-l+1} = t_{m+n-l+2} = \ldots = t_{m+n} = 1$ $(m,n > 1;\ k = 0, 1, \ldots, m-2;\ l = 0, 1, \ldots, n-2)$, even waarschijnlijk zijn. Dit geldt ook voor de vectoren $\underline{t}^{(1)}$ en $\underline{t}^{(2)}$, $\underline{t}^{(i)} = (\underline{t}^{(i)}{}_1, \underline{t}^{(i)}{}_2, \ldots, \underline{t}^{(i)}{}_{m+n})$ $(i = 1, 2)$, waarbij $t^{(i)}{}_1 = t^{(i)}{}_2 = \ldots = t^{(i)}{}_{r(i)} = 0$, $t^{(i)}{}_{r(i)+1} = 1$, $t^{(i)}{}_{m+n-s(i)} = 0$, $t^{(i)}{}_{m+n-s(i)+1} = t^{(i)}{}_{m+n-s(i)+2} = \ldots = t^{(i)}{}_{m+n} = 1$, $r(1) = s(2) = k$ en $r(2) = s(1) = l$ $(k \neq l;\ k,l = 0, 1, \ldots, \min(m,n) - 1)$.

P. van der Laan (1964), Exact power of some rank tests. Publ. de l'Inst. de Stat. de l'Univ. de Paris **18**, 211-233.

V

De stelling van Bowker dat de toets van Mathisen voor populaties met continue verdelingsfuncties bruikbaar (asymptotisch onderscheidend) is tegen verschuivingsalternatieven, is in haar algemeenheid niet juist.

A. H. Bowker (1944), Note on consistency of a proposed test for the problem of two samples, Ann. Math. Stat. **15**, 98-101.

VI

Men kan verwachten dat voor de gebruikelijke onbetrouwbaarheden het relatieve verlies aan onderscheidingsvermogen tegen eenzijdige normale verschuivingsalternatieven van de „Normal scores” toetsen ten opzichte van de toets voor twee steekproeven van Student voor gelijke steekproefomvang ter grootte zes groter is dan voor gelijke steekproefomvang ter grootte tien.

P. van der Laan and J. Oosterhoff (1965), Monte-Carlo estimation of the powers of the distribution-free two-sample tests of Wilcoxon, Van der Waerden and Terry and comparison of these powers, Statistica Neerlandica **19**, 265-275.
P. van der Laan and J. Oosterhoff (1967), Experimental determination of the power functions of the two-sample rank tests of Wilcoxon, Van der Waerden and Terry by Monte Carlo techniques - I. Normal parent distributions, Statistica Neerlandica **21**, 55-68.

VII

Het kan misleidend zijn om benaderingen van verdelingsdichtheden van toetsingsgrootheden te geven in de vorm van figuren waarbij alleen het „middengebied” van de verdelingen gegeven wordt.

VIII

Sterke eentoppigheid (Engels: „strong unimodality”) van een verdelingsdichtheid behorende bij een verdelingsfunctie $F(x)$ is niet voldoende als voorwaarde voor het afnemen van de variantie behorende bij $F^k(x)$ bij toenemende $k(> 0)$.

IX

Gegeven zijn twee onafhankelijke steekproeven $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m$ en $\underline{y}_1, \underline{y}_2, \ldots, \underline{y}_n$ uit populaties met, op verschuiving na, gelijke homogene verdelingen. Voor het toetsen van de nulhypothese dat de verdelingen identiek zijn tegen de eenzijdige alternatieve hypothese dat de verdelingen ten opzichte van elkaar verschoven zijn, zal de meest onderscheidende V_1 toets voor voldoende grote m en n van de vorm V_{1i}, V_{ni}, V_{j1} of V_{jm} zijn.

X

Voor een betere verwezenlijking van de doelstelling van de Vereniging Voor Statistiek verdient het aanbeveling de banden tussen deze Vereniging en het Wiskundig Genootschap nauwer aan te halen.