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Svetla Iordanova Nikova

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PROEFSCHRIFT

ter verkrijging van de graad van doctor aan de Technische Universiteit Eindhoven, op gezag van de Rector Magnificus, prof.dr. M. Rem, voor een commissie aangewezen door het College voor Promoties in het openbaar te verdedigen op

maandag 21 september 1998 om 16.00 uur

door

Svetla Iordanova Nikova

geboren te Veliko Tarnovo, Bulgarije

Dit proefschrift is goedgekeurd door de promotoren:

prof.dr.ir. H.C.A. van Tilborg
en
prof.dr. S.M. Dodunekov

Co-promotor: dr. P.G. Boyvalenkov

This thesis was published by University Press Facilities
Eindhoven University of Technology

To my parents.
To Ioana, Simona and Venci.

Acknowledgments

Here I would like to thank all the people, who helped me during the preparation of the thesis. First of all I would like to express my gratitude to my supervisors Prof. dr. ir. H.C.A. van Tilborg and Prof. dr. S. Dodunekov, who gave me opportunity to carry out my doctorate research in the group of Discrete Mathematics at Eindhoven University of Technology and in Veliko Tarnovo University, Bulgaria. I would like to thank them for their advises, encouragement and patience. Another person who has played an important role in my research is my co-promotor Dr. Peter Boyvalenkov from Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, who direct my attention to these topics. I am indebted to him for his many advises and many stimulating discussions, and for his rapid critical reading of my manuscripts. Furthermore, my thanks go to my co-authors P.Boyvalenkov, D.Danev and V. Nikov for sharing their research interests with me. Special thanks go to Prof.dr. J.J. Seidel for the interesting discussions, for the careful reading of the thesis and helpful comments and advises. I am also grateful to Prof.dr. Simonis, Prof.dr A. Blokhuis and, of course, to my promotores for the reading of the thesis. Their suggestions and constructive comments proved most valuable. I thank all the members of the Discrete Mathematics group for providing a pleasant and stimulating working environment during my last year as Ph. D. student. Last but not at least I would like to thank my family and especially to my husband V. Nikov for giving me moral support and love.

July 1998, Eindhoven

Svetla Nikova

Preface

This thesis presents author's research in the period 1994 to 1998. It was done in Veliko Tarnovo University, Bulgaria (June 1994 - September 1997) and in Eindhoven University of Technology (October 1997 - September 1998) as Ph.D. student. Most of the material in the thesis is based on the papers published during this period. In the order of appearance as preprints they are:

1. P.G.Boyvalenkov, S.Nikova, *New lower bounds for some spherical designs*, **Lecture Notes in Computer Science** 781, Proceedings, ed. G. Cohen, S. Litsyn, A. Lobstein, G. Zémor, Springer-Verlag, 207-216, 1994.

2. P.G.Boyvalenkov, S.Nikova, *Improvements of the lower bounds for the size of some spherical designs*, **Mathematica Balkanica**, vol. 11, to appear.

3. S.Nikova, *On Bounds for the Size of Designs in Complex Projective Spaces*, Proc. International Workshop on Optimal Codes and Related Topics, Sozopol, May 1995, 121-126.

4. P.G.Boyvalenkov, S.Nikova, *On Lower Bounds on the Size of Designs in Compact Symmetric Spaces of Rank 1*, **Archiv der Mathematik** 68, 1997, 81-88.

5. P.G.Boyvalenkov, S.Nikova, *Some Characterizations of Spherical Designs with small Cardinalities*, Proc. Fifth International Workshop on Algebraic and Combinatorial Coding Theory, Sozopol, June 1996, 77-80.

6. P.G.Boyvalenkov, S.Nikova, V.Nikov, *Nonexistence Results for Spherical 3-Designs of Small Cardinalities*, International Symposium on Information Theory IEEE, June, 1997

7. P.G.Boyvalenkov,D.Danev, S.Nikova, *Nonexistence of Certain Spherical Designs of Odd Strengths and Cardinalities*, **Discrete and Computational Geometry** to appear.

8. S.Nikova, V.Nikov, *Necessary and sufficient conditions for improving the Delsarte bound for τ-designs*, Sixth International Workshop on Algebraic and Combinatorial Coding Theory, Pskov, Russia, September 6-12, 1998.

9. S. Nikova, *Extremal polynomials of degree $\tau + 2$ and $\tau + 3$*, **preprint**.

Papers [1] and [2] discuss the improvements of the lower bounds of spherical τ - designs. In [3] and [4] some new lower bounds for designs in projective spaces are given. Restrictions for the inner products and necessary conditions for existence of spherical designs with odd strengths and cardinalities are presented in [5], [6] and [7]. In the last two works [8] and [9] necessary and sufficient conditions for optimality of the Delsarte bound and analytical expression of the extremal polynomials are given. The introduction further explain this topics.

Contents

Introduction

Polynomial metric spaces are finite metric spaces represented by P- and Q- polynomial association schemes [33] as well as infinite metric spaces, which are completely classified by Wang [75] as the real sphere, a real, complex or quaternions projective space and the Cayley projective plane. Hamming, Johnson and Grassmann spaces are the most important examples of finite polynomial metric spaces.

Every polynomial metric space $\mathcal{M}$ is characterized by its metric $d(x, y)$, and normalized measure $\mu_{\mathcal{M}}(.)$.

A basic property of a polynomial metric space $\mathcal{M}$ is the existence of a decomposition of the Hilbert space $\mathcal{L}_2(\mathcal{M}, \mu)$ of complex-valued quadratic-integrable functions with the usual inner product, into a direct sum of mutually orthogonal subspaces V_i of dimension r_i. Besides, there exist real polynomials $Q_i(t)$, $i = 0, 1, \ldots$, ($Q_i(t)$ of degree i), called **zonal spherical functions**, such that for all $x, y \in \mathcal{M}$

$$Q_i(\sigma_{\mathcal{M}}(d(x,y))) = \frac{1}{r_i}\sum_{j=1}^{r_i} v_{ij}(x)\overline{v_{ij}(y)},$$

where $\{v_{ij}(x) : 1 \leq j \leq r_i\}$ is an orthonormal basis of V_i and $\sigma_{\mathcal{M}}(d)$ is a continuous, strictly decreasing function (called *substitution*) such that

$$\sigma_{\mathcal{M}}(0) = 1, \quad \sigma_{\mathcal{M}}(D) = -1.$$

Let G be the group of isometries of $\mathcal{M}$ with $d(gx, gy) = d(x, y)$ for any $x, y \in \mathcal{M}$ and $g \in G$. A connected compact metric space $\mathcal{M}$ is called *two-point homogeneous* (with respect to G) if G acts distance-transitive on M, i.e. $d(x_1, y_1) = d(x_2, y_2)$ implies the existence of an isometry $g \in G$ with $gx_1 = x_2$ and $gy_1 = y_2$.

The only infinite spaces which are two-point homogeneous with respect to their full isometry group are already mentioned above: the real sphere, a real, complex or quaternions projective space and the Cayley projective plane.

Each infinite polynomial metric space is connected with a system of orthogonal polynomials $\{Q_i(t)\}_{i=0}^{\infty}$ and its adjacent system of orthogonal polynomials $\{Q_i^{a,b}(t)\}_{i=0}^{\infty}$, which we will define in Chapter 1.

Definition 1. A finite nonempty subset $C \subset \mathcal{M}$ is called a **τ-design** if

$$\sum_{x \in C} v(x) = 0$$

for all $v(x) \in V_1 \oplus V_2 \oplus \cdots \oplus V_\tau$, where $V_1, \ldots, V_\tau$ are ordered subspaces. The maximal integer τ for which C is a τ-design is called the **strength** of the design C and is denoted by $\tau(C)$.

Let $\mathcal{M}$ be a polynomial metric space and let $\tau \geq 1$ be a fixed integer. We consider the quantity

$$B(\mathcal{M},\tau) = \min\{|C| : C \subset \mathcal{M}, \tau(C) = \tau\}.$$

Bounds for $B(\mathcal{M},\tau)$ in different (finite and infinite) PMS have been obtained by many authors [16, 17, 32, 33, 36, 38, 45, 22, 23, 24]. Classical lower bounds for $B(\mathcal{M},\tau)$ was obtained in infinite polynomial metric spaces by Delsarte, Goethals and Seidel (for the unit sphere $\mathbf{S}^{n-1}$) and Dunkl (for the projective spaces $\mathbb{F}P^{n-1}$). It can be presented as follows [53]. For any polynomial metric spaces and for any τ

$$B(\mathcal{M},\tau) \geq R(\mathcal{M},\tau) = \begin{cases} \displaystyle\sum_{j=0}^{e} r_j & \text{for } \tau = 2e, \\ \displaystyle(1 - \frac{Q_e^{1,0}(-1)}{Q_{e+1}(-1)})\sum_{j=0}^{e} r_j & \text{for } \tau = 2e+1. \end{cases}$$

Spherical τ-designs were introduced by Delsarte, Goethals and Seidel in 1977. A *spherical τ-design* in $\mathbb{R}^n$ is a finite set $C \subset \mathbf{S}^{n-1}$ with the property that for every polynomial f of degree at most τ, the average value of f on C equals the average value of f on $\mathbf{S}^{n-1}$. A specialization of $R(\mathcal{M},\tau)$ for $\mathcal{M} = \mathbf{S}^{n-1}$ is as follows:

$$B(n,\tau) \geq R(n,\tau) = \begin{cases} \binom{n+e-1}{n-1} + \binom{n+e-2}{n-1} & \text{if } \tau = 2e; \\ 2\binom{n+e-1}{n-1} & \text{if } \tau = 2e+1. \end{cases}$$

The so called Delsarte bounds were obtained by using suitable polynomials of degree τ in the following theorem.

Theorem 2. (The Linear Programming Bound for designs [32, 36, 49]) *Let $\mathcal{M}$ be a polynomial metric space, let $\tau \geq 1$ be integer and let $f(t)$ be a real nonzero polynomial such that*

(B1) $f(t) \geq 0$, *for* $-1 \leq t \leq 1$,

(B2) *the coefficients in the zonal spherical function expansion* $f(t) = \sum_{i=0}^{k} f_i Q_i(t)$ *satisfy* $f_0 > 0$, $f_i \leq 0$ *for* $i = \tau+1, \ldots, k$.

Then, $B(\mathcal{M},\tau) \geq f(1)/f_0$.

In this thesis we propose a method for improving the Delsarte bound in infinite polynomial metric spaces by means of linear programming and other arguments. We investigate some properties of polynomials, used in the linear programming bound of Theorem 2. This allows us to search effectively for suitable polynomials in order to improve the Delsarte bound.

Another approach is based on a deeper investigation of the structure of feasible designs of relatively small cardinalities. We apply such an argument for spherical designs. This gives many non-existence results including an asymptotic improvement of the Delsarte bound for τ odd and for odd cardinalities.

Overview

In Chapter 1, we define polynomial metric spaces. We give the most important examples of finite polynomial metric space and describe the infinite ones. Some well known properties of systems of orthogonal polynomials associated with the polynomial metric spaces are discussed. In this chapter, we also give definitions for codes and designs in such spaces and discuss the linear programming bound for codes and designs. In the last section, the universal Delsarte bound for designs is presented.

In Chapter 2, we propose a method for improving the Delsarte bound for some spherical τ-designs using linear programming techniques and extremal polynomials of degree $\tau+3$. We investigate some properties of extremal polynomials, namely the number of double zeros and the number of zero coefficients. In Section 2.5, we present new lower bounds for $6 \leq \tau \leq 11$ in some dimensions.

In Chapter 3, we find new lower bounds for some τ-designs in infinite projective spaces. Our approach is similar to the method we have used for spherical designs in Chapter 2. In Section 3.4, we give some examples of new bounds and we present some tables to compare our results with the Delsarte bounds.

In Chapter 4 we give necessary and sufficient conditions for improving the Delsarte bound for τ-designs. In Section 4.3 we define test functions $G_\tau(\mathcal{M}, Q_j)$ with the property that $G_\tau(\mathcal{M}, Q_j) < 0$ for some $j > \tau$ if and only if the Delsarte bound $B(\mathcal{M}, \tau) \geq R_(\mathcal{M}, \tau)$ can be improved by linear programming. Then we investigate when the Delsarte bound is optimal. If it is not optimal in Section 4.4 we give improving polynomials of degree $\tau + 2$ in non-antipodal PMS and of degree $\tau + 3$ in antipodal PMS.

In Chapter 5, we obtain some necessary conditions for the existence of spherical τ-designs of odd strength and cardinality. These conditions imply nonexistence results in many cases. In Section 5.2, we derive a general nonexistence rule. It gives a bound which is asymptotically better than the corresponding estimation based on the Delsarte-Goethals-Seidel bound. It turns out that our approach works well in small dimensions too. In Section 5.3 and Section 5.4, we consider in detail the strengths $\tau = 3$ and $\tau = 5$ respectively. We rule out the first open cases by showing the nonexistence of 3-designs with 7 points and 5-designs with 13 points. When the nonexistence argument does not work, we obtain (Section 5.5) bounds on the maximal inner product of a τ-design of a fixed cardinality.

Chapter 1

Codes and Designs in Polynomial Metric Spaces

1.1 Introduction

In this chapter we introduce the notion of polynomial metric space. Some general definitions are given in Section 1.2. In Section 1.3 some general properties of orthogonal polynomials associated with a polynomial metric space are described. In Section 1.4 we give some definitions for codes and designs in such spaces. In the last two sections we discuss the linear programming bounds for codes and designs and the universal Delsarte bound for designs.

1.2 Polynomial Metric Spaces

In this section we define polynomial metric space (PMS). Finite PMS are nothing but $P-$ and $Q-$ polynomial association schemes [13, 74]. The infinite ones [35, 36, 49, 53, 75] are compact, connected, two-point homogeneous spaces and they are completely classified by Wang [75] to be the Euclidean unit spheres, the real, the complex and the quaternionic projective spaces and the Cayley projective plane. Let $\mathcal{M}$ be a compact connected, two-point homogeneous metric space with a (finite) diameter

$$D = D(\mathcal{M}) = \max_{x,y \in \mathcal{M}} d(x,y).$$

This means that an *isometry* group G acts transitively on $\mathcal{M}$ (i.e. for any $x, y \in \mathcal{M}$ $\exists g \in G$ such that $gx = y$). Therefore on $\mathcal{M}$ there exists unique normalized invariant measure, the Haar measure, μ ($\mu(gM) = \mu(M)$ for any measurable $M \subset \mathcal{M}$ and any $g \in G$; $\mu(\mathcal{M}) = 1$).

We shall assume that $\sigma_{\mathcal{M}}(d)$ is a continuous, strictly decreasing function (called *substitution*) such that

$$\sigma_{\mathcal{M}}(0) = 1, \quad \sigma_{\mathcal{M}}(D) = -1. \tag{1.1}$$

Definition 1.1 *Let $\mathcal{L}_2(\mathcal{M},\mu)$ denote the Hilbert space of complex-valued square-integrable functions with the usual inner product*

$$\langle u,v\rangle = \int_{\mathcal{M}} u(x)\overline{v(x)}d\mu(x).$$

Suppose that $\mathcal{L}_2(\mathcal{M},\mu)$ decomposes into a countable (when $\mathcal{M}$ is infinite) or finite (with $D+1$ members when $\mathcal{M}$ is finite) direct sum of mutually orthogonal finite-dimensional subspaces

$$\mathcal{L}_2(\mathcal{M},\mu) = V_0 \oplus V_1 \oplus \cdots.$$

Then $\mathcal{M}$ is called **polynomial metric space**, *if there exist*

a) an ordering of the spaces $V_0, V_1, \ldots$ (V_0 is the space of constant functions), where $r_i = \dim(V_i)$ and $\{v_{ij}(x) : 1 \le j \le r_i\}$ is an orthonormal basis of V_i;

b) real polynomials $Q_i(t)$, $i = 0,1,\ldots$, ($Q_i(t)$ of degree i), called **zonal spherical functions** *(ZSF),*

such that for all $x, y \in \mathcal{M}$

$$Q_i(\sigma_{\mathcal{M}}(d(x,y))) = \frac{1}{r_i}\sum_{j=1}^{r_i} v_{ij}(x)\overline{v_{ij}(y)}. \tag{1.2}$$

The function $\sigma_{\mathcal{M}}(d)$ will be referred to as a *standard substitution* for $\mathcal{M}$. The inverse of $\sigma_{\mathcal{M}}(d)$ will be denoted by $\sigma_{\mathcal{M}}^{-1}$, i.e. $\sigma_{\mathcal{M}}^{-1}(t) = d$ if and only if $t = \sigma_{\mathcal{M}}(d)$.

For $x, y \in \mathbf{S}^{n-1}$, the real number $\sigma_{\mathcal{M}}(d(x,y)) \in [-1,1]$ is called *inner product* of x and y.

In the interval $[-1,1]$ we consider the function defined by

$$\nu(t) = 1 - \mu(\sigma^{-1}(t)),$$

which increases with t from $\nu(-1) = 0$ up to $\nu(1) = \mu(0)$. It generates the Lebesgue-Stiltjies measure ν on the interval $[-1,1]$, which is normalized ($\nu([-1,1]) = 1$).

Relations (1.2) and (1.1) imply the orthogonality relations

$$r_i \int_{-1}^{1} Q_i(t)Q_j(t)d\nu(t) = \delta_{ij}, \quad i,j = 0,1,\ldots, \tag{1.3}$$

where δ_{ij} is the Kronecker symbol and the integral is taken in the Lebesgue-Stieltjes sense. These equalities show that the polynomials $\{Q_i(t)\}_{i=0}^{N}$, (where $N = D+1$, when $\mathcal{M}$ is finite and $N = \infty$ otherwise) are orthogonal in the interval $[-1,1]$ with weight $w(t)$, such that $w(t)dt = d\nu(t)$ (see Theorem 1.5 below). It also follows by (1.1) and (1.2) that $Q_i(1) = 1$.

Definition 1.2 *Any finite nonempty subset of $\mathcal{M}$ is called a* **code**.

Definition 1.3 *A code $C \subset \mathcal{M}$ is called a* τ-**design** *if*

$$\sum_{x \in C} v_{ij}(x) = 0 \tag{1.4}$$

for all $i = 1, 2, \ldots, \tau$ and all $j = 1, 2, \ldots, r_i$. The maximal integer τ for which C is a τ-design is called the **strength** *of the design C and is denoted by $\tau(C)$.*

There is no complete classification of finite PMS. We mention the most important examples [33]:

- The Hamming space $H(n, r)$ consists of all n-tuples with components from an alphabet of cardinality r. The metric is the Hamming distance between two points $x, y \in H(n, r)$ which equals to the number of positions where they differ. The zonal spherical functions are the r-ary Krawtchouk polynomials. The codes in the Hamming spaces are extensively studied in coding theory and information theory. The designs in $H(n, r)$ are known as orthogonal arrays and are applied in statistics.

- The Johnson space $J(n, w)$ consists of all w-subsets of a n-set. The distance between $x, y \in J(n, w)$ is defined as $d(x, y) = w - |x \cap y|$. The zonal spherical functions now are the Hahn polynomials. The designs in the Johnson space are nothing but the classical $t - (v, k, \lambda)$ designs.

- The Grassmann space $J(n, w, q)$ is the set of all w-dimensional subspaces of the vector space F_q^n over the finite field of q elements F_q. The distance is defined by $d(x, y) = w - \dim|x \cap y|$.

The above spaces are extensively studied from the points of view of combinatorics, coding theory and information theory. In the present work, we are mainly interested in infinite PMS.

As we mentioned before, infinite polynomial metric spaces are completely classified [44, 75]. The classical example is given by the Euclidean spheres S^{n-1}, with the usual metric and inner product. The measure $\mu(.)$ in this case is the normalized Lebesgue measure (i.e. $\mu(S^{n-1}) = 1$). The standard substitution is $\sigma(d) = 1 - d^2/2$ and maps the distance into the inner product. It turns out that this very familiar situation creates ample room for investigations.

A real polynomial in n variables is called harmonic if it belongs to the kernel of the Laplace operator

$$\Delta = \frac{\partial^2}{\partial {x_1}^2} + \cdots + \frac{\partial^2}{\partial {x_n}^2}.$$

A polynomial in n variables is called homogeneous of degree d if all monomials occurring in it have total degree d (the total degree of the monomial $x_1^{s_1} x_2^{s_2} \ldots x_n^{s_n}$ is $s_1 + s_2 + \cdots + s_n$). The subspaces V_i, $i = 0, 1, \ldots$, consist of all homogeneous harmonic polynomials of n

variables of total degree i. The dimension of V_i is

$$r_i = \binom{n+i-1}{n-1} - \binom{n+i-3}{n-1}.$$

The weight $w(t)$ equals $(1-t^2)^{(n-3)/2}$. The zonal spherical functions are the *Gegenbauer polynomials* $\{Q_i^{(n)}(t)\}_{i=0}^{\infty}$, normalized by $Q_i^{(n)}(1) = 1$. They can be defined by the recurrence relation

$$(i+n-2)Q_{i+1}^{(n)}(t) = (2i+n-2)tQ_i^{(n)}(t) - iQ_{i-1}^{(n)}(t), \tag{1.5}$$

for $i \geq 1$, where $Q_0^{(n)}(t) = 1$ and $Q_1^{(n)}(t) = t$.

The other examples of infinite PMS are the projective spaces $\mathbb{F}P^{n-1}$, where $\mathbb{F}$ is the field $\mathbb{R}$ of real numbers, the field $\mathbb{C}$ of complex numbers, the (non-commutative) algebra $\mathbb{H}$ of quaternions, or the (non-associative) algebra $\mathbb{O}$ of Cayley numbers ($\mathbb{O}P^{n-1}$ exists only for $n = 2, 3$). Together with the Euclidean sphere they are all **compact symmetric spaces of rank 1** (cf. [44, 75]). We give the following model for the projective spaces $\mathbb{F}P^{n-1}$. Denote by $\mathbb{F}^n$ the set of vectors $u = (u_1, u_2, \ldots, u_n)$ over the field $\mathbb{F}$. For an element $u \in \mathbb{F}^n$ we define its conjugate $\overline{u} = (\overline{u}_1, \ldots, \overline{u}_n)$ and a norm $|u| = \sqrt{u\overline{u}}$. The inner product of vectors $u = (u_1, u_2, \ldots, u_n), v = (v_1, v_2, \ldots, v_n) \in \mathbb{F}^n$ is defined by $(u, v) = u_1\overline{v}_1 + \cdots + u_n\overline{v}_n$. For the case that $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ the points of the projective space $\mathbb{F}P^{n-1}$ are defined as the lines through the origin

$$U = \{\lambda u | \lambda \in \mathbb{F}\backslash\{0\}\} \quad \text{for} \quad u \in \mathbb{F}^n.$$

The function

$$\rho(U, V) = \frac{|(u, v)|}{|u|.|v|} = \cos \angle(U, V)$$

does not depend on the particular choice of vectors $u \in U$ and $v \in V$, $U, V \in \mathbb{F}P^n$, and therefore we can define a metric on $\mathbb{F}P^n$ by:

$$d(U, V) = \sqrt{2(1-\rho(U, V))}.$$

The spaces $\mathbb{R}P^{n-1}$, $\mathbb{C}P^{n-1}$, $\mathbb{H}P^{n-1} (n \geq 2)$ and $\mathbb{O}P^2$ are polynomial with standard substitution $t(d) = 2(1-d^2/2)^2 - 1$ (see [44, 69, 75]).

To describe the zonal spherical functions simultaneously, we denote by 2m the dimension of $\mathbb{F}$ over $\mathbb{R}$, i.e. $m = 1/2$, 1, 2 or 4 in the different cases. Then the ZSF of $\mathbb{F}P^{n-1}$ are the Jacobi polynomials $\{P_i^{(\alpha,\beta)}(t)\}_{i=0}^{\infty}$ (normalized by $P_k^{(\alpha,\beta)}(1) = 1$), where

$$(\alpha, \beta) = (mn - m - 1, m - 1). \tag{1.6}$$

An explicit formula for the normalized Jacobi polynomials is the following [1, Chapter 22]

$$P_i^{(\alpha,\beta)}(t) = \frac{2^{-i}}{\binom{i+\alpha}{i}} \sum_{j=0}^{i} \binom{i+\alpha}{j} \binom{i+\beta}{i-j} (t+1)^j (t-1)^{i-j} \tag{1.7}$$

Definition 1.4 *A polynomial metric space $\mathcal{M}$ is called* **antipodal** *if for every point $x \in \mathcal{M}$ there exists a point $\overline{x} \in \mathcal{M}$ such that for any point $y \in \mathcal{M}$ we have*

$$\sigma(d(x,y)) + \sigma(d(\overline{x},y)) = 0. \tag{1.8}$$

It follows from (1.8) that for every point x of an antipodal space $\mathcal{M}$ the point $\overline{x} \in \mathcal{M}$ is the unique point with $d(x,\overline{x}) = D$. Indeed, by $y = \overline{x}$ in (1.8) we see that $\sigma(d(x,\overline{x})) = -1$, i.e. $d(x,\overline{x}) = D$. If $d(x,\overline{x}') = D$ for some $\overline{x}' \in \mathcal{M}$ then by $y = \overline{x}'$ in (1.8) we have $\sigma(d(\overline{x},\overline{x}')) = 1$, which means that $\overline{x} = \overline{x}'$.

The most important examples of finite antipodal PMS are the binary Hamming space $H(n,2)$ and Johnson space $J(2w,w)$. Among the infinite PMS only the Euclidean spheres S^{n-1} are antipodal. The advantages of this fact will be seen in Chapter 5.

1.3 General Properties of Orthogonal Polynomials Systems and Their Adjacent Systems

In the sequel, we assume that M is an infinite PMS. The corresponding ZSF constitute an infinite system of orthogonal polynomials. In this section we collect some well known facts about such systems.

Let the function $\nu(t)$ be differentiable on the interval $[-1,1]$ and let its corresponding weight function $w(t) = \nu'(t)$ be continuous on $[-1,1]$ and positive inside $[-1,1]$. Suppose in addition that the Lebesgue-Stieltjes measure ν on $[-1,1]$ corresponding to the function $\nu(t)$ is normalized, i.e.

$$1 = \int_{-1}^{1} d\nu(t) = \int_{-1}^{1} w(t)dt$$

The inner product is defined as usually

$$(u(t), v(t)) = \int_{-1}^{1} u(t)v(t)w(t)dt. \tag{1.9}$$

The ZSF are orthogonal with respect to the inner product (1.9).

Theorem 1.5 *[73] There exist a unique sequence of polynomials $\{Q_i(t)\}_{i=0}^{\infty}$, $Q_i(t)$ of degree i, and a corresponding unique sequence of positive constants $\{r_i\}_{i=0}^{\infty}$, such that for any $i, j, (i,j \geq 0)$*

$$r_i \int_{-1}^{1} Q_i(t)Q_j(t)d\nu(t) = \delta_{ij}, \quad Q_i(1) = 1.$$

Note that $Q_0(t) \equiv 1$ and $r_0 = 1$ due to the normalization of measure ν.

Any real polynomial $f(t)$ of degree k can be uniquely written in the form

$$f(t) = \sum_{i=0}^{k} f_i Q_i(t) \tag{1.10}$$

(actually this is its Fourier expansion). The coefficients f_i, $i = 0, 1, \ldots, k$, can be found by the formula

$$f_i = r_i.(Q_i(t), f(t)) = r_i \int_{-1}^{1} Q_i(t) f(t) d\nu(t). \tag{1.11}$$

Remark 1.6 *The ZSF coefficients f_i, $i = 0, 1, \ldots, k$ can be also computed by a triangular system of linear equations which is obtained by comparing the coefficients of the equal degrees of t in both sides of (1.10). It is clear that this way is more convenient for large i, while (1.11) must be used for small i.*

It is convenient to introduce the notation

$$b_i = \int_{-1}^{1} t^i d\nu(t). \tag{1.12}$$

By the normalization, we have $b_0 = 1$. Now, the coefficient f_0, which is very important for our investigations, can be expressed as follows

$$f_0 = \int_{-1}^{1} f(t) d\nu(t) = \sum_{i=0}^{k} a_i b_i, \tag{1.13}$$

where $f(t) = a_0 + a_1 t \cdots + a_k t^k = \sum_{i=0}^{k} a_i t^i$.

Example 1.7 *For $\mathcal{M} = \mathbf{S}^{n-1}$, a straightforward calculation of the corresponding integrals (cf. [73, p. 82], [61, Lemma 2.1], [31]; see also Lemma 2.4)*

$$\int_{-1}^{1} t^i (1 - t^2)^{\frac{n-3}{2}} \, dt$$

yields that the numbers $b_i, i \geq 1$, are given by

$$b_i = \begin{cases} 0 & \text{if } i \text{ is odd,} \\ \dfrac{(2j-1)!!}{n(n+2)\ldots(n+2j-2)} & \text{if } i{=}2j \text{ is even.} \end{cases}$$

The next theorem gives another well known property of the orthogonal polynomials.

Theorem 1.8 *Any polynomial $Q_i(t)$, $i \geq 1$, has i different simple roots inside the interval $[-1, 1]$.*

Denote by $t_{i,j}$ ($j = 1, \dots, i$) the roots of $Q_i(t)$, $i \geq 1$, written in increasing order and let $t_i = t_{i,i}$. Note that by Theorem 1.8 and the normalization condition $Q_i(1) = 1$, the leading coefficient of the polynomial $Q_k(t)$ is positive and $\mathrm{sign} Q_i(-1) = (-1)^i$ for $i \geq 1$.

We write

$$Q_i(t) = \sum_{j=0}^{i} a_{ij} t^j$$

and put $m_i = a_{i,i}/a_{i+1,i+1}$. Then the following recurrence relation holds

$$m_i Q_{i+1}(t) = (t + m_i + \frac{m_{i-1} r_{i-1}}{r_i} - 1) Q_i(t) - \frac{m_{i-1} r_{i-1}}{r_i} Q_{i-1}(t) \tag{1.14}$$

for $i \geq 0$, where $r_{-1} = m_{-1} = 0$ and $Q_{-1}(t) \equiv 0$.

For each a and b in $\mathbb{N}$, one can associate with the system $\{Q_i(t)\}_{i=0}^{\infty}$ another system denoted by $\{Q_i^{a,b}(t)\}_{i=0}^{\infty}$. These systems are called the *adjacent systems* of $\{Q_i(t)\}_{i=0}^{\infty}$. They are again systems of orthogonal polynomials with the new measure $\nu^{a,b}(t)$ defined by

$$(\nu^{a,b}(t))' = c^{a,b}(1-t)^a(1+t)^b w(t).$$

The constant $c^{a,b}$ here is chosen in such a way that the Lebesgue-Stieltjes measure of the corresponding function $\nu^{a,b}(t)$ is normalized, i.e.

$$\int_{-1}^{1} d\nu^{a,b}(t) = c^{a,b} \int_{-1}^{1} (1-t)^a(1+t)^b d\nu(t) = 1.$$

Theorem 1.5 applies in this case as well there is a corresponding unique system of orthogonal polynomials $\{Q_i^{a,b}(t)\}_{i=0}^{\infty}$.

Definition 1.9 *The unique system $\{Q_i^{a,b}(t)\}_{i=0}^{\infty}$ of orthogonal polynomials is called* **adjacent** *to the original system $\{Q_i(t)\}_{i=0}^{\infty}$.*

Note that the orthogonality and normalization conditions for the system $\{Q_i^{a,b}(t)\}_{i=0}^{\infty}$ may be rewritten in the following form

$$r_i^{a,b} \int_{-1}^{1} Q_i^{a,b}(t) Q_j^{a,b}(t) d\nu^{a,b}(t) = \delta_{ij}, \tag{1.15}$$

for $i, j \geq 0$, where $Q_i^{a,b}(1) = 1$, $Q_0^{a,b}(t) \equiv 1$, $r_0^{a,b} = 1$.

Example 1.10 *For $\mathcal{M} = S^{n-1}$, the polynomials $Q_i^{1,0}(t)$ are the Jacobi polynomials $P_i^{(\frac{n-1}{2}, \frac{n-3}{2})}(t)$ and the polynomials $Q_i^{1,1}(t)$ are the Gegenbauer polynomials $Q_i^{(n+2)}(t)$. Analogously, for the infinite projective spaces we have $Q_i^{a,b}(t) = P_i^{\alpha+a, \beta+b}(t)$, where α and β are given by (1.6).*

Adjacent systems can be defined for all positive integers a and b. However, we need only the cases $a, b \in \{0, 1\}$. Since $Q_i^{0,0}(t) = Q_i(t)$, we shall omit the upper index when $a = b = 0$.

1.4 Codes and Designs in Polynomial Metric Spaces

For a code C in a PMS $\mathcal{M}$ we consider its *minimum distance*

$$d(C) = \min\{d(x,y) | x, y \in C, x \neq y\}$$

and its *maximal inner product*

$$s(C) = \max\{\sigma(d(x,y)) | x, y \in C, x \neq y\}.$$

The number $s(C)$ is called also a *maximal cosine* of C (this is exactly the case for the Euclidean sphere). It is clear that the minimum distance and the maximal inner product are related by the equality $s(C) = \sigma(d(C))$.

A code $C \subset \mathcal{M}$ with cardinality $M = |C|$ and a maximal inner product $s = s(C)$ is referred to as an *$(\mathcal{M}, M, s)$ code.*

Let $\mathcal{M}$ be a PMS and let $s \in [-1, 1)$ be a real number. The problem of finding bounds on the quantity

$$A(\mathcal{M}, s) = \max\{|C| : C \subset \mathcal{M}, s(C) = s\},$$

in different (finite and infinite) PMS has been investigated by many authors.

Relatively few exact values of $A(\mathcal{M}, s)$ are known. In general, different methods are employed to find lower bounds (usually by constructions) or upper bounds (usually by linear programming techniques) for $A(\mathcal{M}, s)$ (cf. [15, 20, 18, 29, 32, 33, 49, 39, 52, 53, 58, 57] and the references therein).

Let $\mathcal{M}$ be a PMS and let $\tau \geq 1$ be a fixed integer. We consider the quantity

$$B(\mathcal{M}, \tau) = \min\{|C| : C \subset \mathcal{M}, \tau(C) = \tau\}. \tag{1.16}$$

Bounds for $B(\mathcal{M}, \tau)$ in different (finite and infinite) PMS are obtained in [16, 17, 32, 33, 36, 38, 45, 22, 23, 24] (see also references therein).

The next theorem is useful in the investigation of the cardinalities of codes and designs in polynomial metric spaces. It was proved in different settings and terminology in [36, 49, 52]. Here, we write it in the form which was given in [52].

Theorem 1.11 *[52]. For any code $C \subset \mathcal{M}$ and any real polynomial $f(t) = \sum_{i=0}^{k} f_i Q_i(t)$ we have*

$$|C|f(1) + \sum_{x,y \in C, x \neq y} f(\sigma(d(x,y))) = |C|^2 f_0 + \sum_{i=1}^{k} \frac{f_i}{r_i} \sum_{j=1}^{r_i} |\sum_{x \in C} v_{ij}(x)|^2. \tag{1.17}$$

Proof. We calculate $\sum_{x,y \in C} f(\sigma(d(x,y)))$ in two ways. First, we have

$$\begin{aligned}
\sum_{x,y \in C} f(\sigma(d(x,y))) &= \sum_{x \in C} f(\sigma(d(x,x)) + \sum_{x,y \in C, x \neq y} f(\sigma(d(x,y))) \\
&= |C|f(1) + \sum_{x,y \in C, x \neq y} f(\sigma(d(x,y))).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sum_{x,y\in C} f(\sigma(d(x,y))) &= \sum_{x,y\in C}\sum_{i=0}^{k} f_i Q_i(\sigma(d(x,y))) \\
&= \sum_{x,y\in C} f_0 Q_0(\sigma(d(x,y))) + \sum_{i=1}^{k} f_i \sum_{x,y\in C} Q_i(\sigma(d(x,y))) \\
&= \sum_{x,y\in C} f_0 + \sum_{i=1}^{k} f_i \sum_{x,y\in C} \frac{1}{r_i}\sum_{j=1}^{r_i} v_{ij}(x)\overline{v_{ij}(y)} \\
&= |C|^2 f_0 + \sum_{i=1}^{k}\frac{f_i}{r_i}\sum_{j=1}^{r_i}\sum_{x,y\in C} v_{ij}(x)\overline{v_{ij}(y)} \\
&= |C|^2 f_0 + \sum_{i=1}^{k}\frac{f_i}{r_i}\sum_{j=1}^{r_i}\Big(\sum_{x\in C} v_{ij}(x)\Big)\overline{\Big(\sum_{y\in C} v_{ij}(y)\Big)} \\
&= |C|^2 f_0 + \sum_{i=1}^{k}\frac{f_i}{r_i}\sum_{j=1}^{r_i}\Big|\sum_{x\in C} v_{ij}(x)\Big|^2.
\end{aligned}$$

□

We are now in a position to obtain the so-called linear programming bounds for codes and designs in PMS as an immediate consequence of Theorem 1.11.

Theorem 1.12 *(The Linear Programming Bound for codes [32, 36, 49]) Let $\mathcal{M}$ be a PMS, let $s \in [-1,1)$ and let $f(t)$ be a real nonzero polynomial such that*

(A1) *$f(t) \le 0$, for $-1 \le t \le s$,*

(A2) *the coefficients in the ZSF expansion $f(t) = \sum_{i=0}^{k} f_i Q_i(t)$ satisfy $f_0 > 0$, $f_i \ge 0$ for $i = 1, \dots, k$.*

Then, $A(\mathcal{M}, s) \le f(1)/f_0$.

Proof. Consider an arbitrary $(\mathcal{M}, M, s)$ code C and let $f(t)$ be any real nonzero polynomial satisfying **(A1)** and **(A2)**. We now apply (1.17). Because of condition **(A1)**, the left-hand side of (1.17) does not exceed $f(1)M$ and, because of **(A2)**, the right-hand side is greater than or equal to $f_0 M^2$. Therefore $M \le f(1)/f_0$.

□

Theorem 1.13 *(The Linear Programming Bound for designs [32, 36, 49]) Let $\mathcal{M}$ be a PMS, let $\tau \ge 1$ be integer and let $f(t)$ be a real nonzero polynomial such that*

(B1) $f(t) \geq 0$, *for* $-1 \leq t \leq 1$,

(B2) *the coefficients in the ZSF expansion* $f(t) = \sum_{i=0}^{k} f_i Q_i(t)$ *satisfy* $f_0 > 0$, $f_i \leq 0$ *for* $i = \tau + 1, \ldots, k$.

Then, $B(\mathcal{M}, \tau) \geq f(1)/f_0$.

Proof. We apply (1.17) with the polynomial $f(t)$ and an arbitrary τ-design $C \subset \mathcal{M}$. Because of condition **(B1)** the left-hand side is greater than or equal to $f(1)|C|$ and, because of **(B2)**, the right-hand side does not exceed $f_0|C|^2$. Therefore $|C| \geq f(1)/f_0$.

□

Remark 1.14 *For a finite PMS, the conditions* **(A1)** *and* **(B1)** *re stronger than is really required. Indeed, in the finite PMS, all possible inner products form a discrete set. Therefore, our polynomials have to be non-positive (resp. nonnegative) in the intersection of this set with the interval* $[-1, s]$ *(resp.* $[-1, 1]$*, which is in fact the whole set of inner products).*

Remark 1.15 *For an infinite PMS, the condition* $f_0 > 0$ *is a trivial consequence of the requirement* $f(t) \not\equiv 0$ *and* **(B1)**. *In particular, if* $\deg(f) \leq \tau$, *then condition* **(B2)** *is automatically satisfied.*

Theorem 1.11 implies a second characterization of designs in PMS which will be crucial for our investigations in Chapter 5.

Theorem 1.16 *Let* $\mathcal{M}$ *be a PMS, let* $\tau \geq 1$ *be integer and let* $C \subset \mathcal{M}$ *be a* τ*-design. Then, for every point* $y \in \mathcal{M}$ *and every real polynomial* $f(t)$ *of degree at most* τ *we have*

$$\sum_{x \in C} f(\sigma(d(x, y))) = f_0|C|. \tag{1.18}$$

Conversely, if (1.18) is satisfied for every point $y \in C$ *and every real polynomial of degree at most* τ*, then* C *is a* τ*-design.*

Proof. Let $C \subset \mathcal{M}$ be a τ-design. Any polynomial $f(t)$ can be written as $f(t) = \sum_{i=0}^{k} f_i Q_i(t)$. Now using (1.2) and (1.4) for $i \geq 1$ we obtain

$$\sum_{x \in C} Q_i(\sigma_{\mathcal{M}}(d(x, y))) = \frac{1}{r_i} \sum_{x \in C} \sum_{j=1}^{r_i} v_{ij}(x)\overline{v_{ij}(y)} = \frac{1}{r_i} \sum_{j=1}^{r_i} \overline{v_{ij}(y)} \sum_{x \in C} v_{ij}(x) = 0.$$

To prove the converse assertion, we use (1.17) for C and any polynomial $f(t)$ of degree τ with $f_i > 0$ for every $i = 0, 1, \ldots, \tau$ (for example, $f(t) = \sum_{i=0}^{\tau} Q_i(t)$ is such a polynomial). The sum

$$\sum_{x, y \in C, x \neq y} f(\sigma(d(x, y)))$$

decomposes into $|C|$ sums of the form (1.18), each of them equal to $f_0|C|-f(1)$. Therefore (1.17) becomes

$$\sum_{i=1}^{k}\frac{f_i}{r_i}\sum_{j=1}^{r_i}|\sum_{x\in C}v_{ij}(x)|^2=0$$

which implies $\sum v_{i,j}(x)=0$ for all $j=1,\ldots,r_i, i=1,\ldots,\tau$.

□

Definition 1.17 *We denote by $B_{\mathcal{M},\tau}$ the set of real polynomials which satisfy the conditions* **(B1)** *and* **(B2)** *of Theorem 1.13.*

Lemma 1.18 *If $f,g\in B_{\mathcal{M},\tau}$ and $\alpha,\beta>0$ then $\alpha f+\beta g\in B_{\mathcal{M},\tau}$.*

Proof. If $f(t)\geq 0$ and $g(t)\geq 0$ for $t\in[-1,1]$ then we obviously have $\alpha f(t)+\beta g(t)\geq 0$ in $[-1,1]$, i.e. **(B1)** is satisfied. The coefficients in the ZSF expansion of $\alpha f(t)+\beta g(t)$ are given by the expression $\alpha f_i+\beta g_i$, where f_i's and g_i's are the ZSF coefficients of f and g respectively. Therefore, condition **(B2)** is also satisfied.

□

We consider the functional $\Omega(f)=f(1)/f_0$, which is well defined for every real polynomial $f(t)$ such that $f_0\neq 0$. Obviously, $\Omega(af)=\Omega(f)$ for any real $a\neq 0$.

The following notion of extremality is very important for our purposes.

Definition 1.19 *A polynomial $f(t)\in B_{\mathcal{M},\tau}$ is called $B_{\mathcal{M},\tau}$-extremal if*

$$\Omega(f)=\max\{\Omega(g):\ g(t)\in B_{\mathcal{M},\tau},\ \deg(g)\leq\deg(f)\}.$$

The next lemma, although trivial again, plays an important role in the investigations of extremality properties of polynomials used for obtaining linear programming bounds for codes and designs.

Lemma 1.20 *Let $f(t)$ and $g(t)\in B_{\mathcal{M},\tau}$ and $\Omega(f)<\Omega(g)$. Then $\Omega(\alpha f+\beta g)\in(\Omega(f),\Omega(g))$ for arbitrary positive reals α and β, provided all functions are well defined.*

Proof. We obviously have

$$\Omega(\alpha f+\beta g)=\frac{\alpha f(1)+\beta g(1)}{\alpha f_0+\beta g_0}.$$

Then it is easy to check that the inequalities $\Omega(f)<\Omega(\alpha f+\beta g)$ and $\Omega(\alpha f+\beta g)<\Omega(g)$ are equivalent to the assumption $\Omega(f)<\Omega(g)$.

□

To conclude this section we write the linear programming bound for antipodal designs in antipodal spaces. A code C in an antipodal PMS $\mathcal{M}$ is called antipodal if $C = \bar{C}$. This shows that equality (1.18) is an identity for odd polynomial functions. Therefore, the strength $\tau(C)$ is an odd number, and we need not to pay attention to the ZSF coefficients with odd indices. This is expressed in the following assertion.

Theorem 1.21 *Let $\mathcal{M}$ be an antipodal PMS, let $\tau = 2e+1 \geq 1$ be odd integer and let $f(t)$ be a real nonzero polynomial such that*

(B1) *$f(t) \geq 0$, for $-1 \leq t \leq 1$,*

(B2′) *the coefficients in the ZSF expansion $f(t) = \sum_{i=0}^{k} f_i Q_i(t)$ satisfy $f_0 > 0$, $f_i \leq 0$ for all even $i \geq 2e+2$.*

Then, any antipodal τ-design $C \subset \mathcal{M}$ satisfies

$$|C| \geq \frac{f(1)}{f_0}.$$

1.5 The Delsarte Bound for Designs in PMS

The classical lower bounds for $B(\mathcal{M},\tau)$ were obtained in the finite PMS for $\tau = 2k$ by Delsarte [32, 33] and for $\tau = 2k+1$ by Dunkl [38]. In the case of infinite PMS they were proved by Delsarte, Goethals and Seidel in [36] for the Euclidean sphere and by Dunkl [38] (see also Bannai-Hoggar [11]) for the projective spaces. Each of these bounds is commonly called the **Delsarte bound**. The following presentation of the Delsarte bound is due to Levenshtein [53].

Theorem 1.22 *For any PMS and for any τ*

$$B(\mathcal{M},\tau) \geq R(\mathcal{M},\tau) = \begin{cases} \sum_{j=0}^{e} r_j & \text{for } \tau = 2e, \\ \left(1 - \frac{Q_e^{1,0}(-1)}{Q_{e+1}(-1)}\right) \sum_{j=0}^{e} r_j & \text{for } \tau = 2e+1. \end{cases} \tag{1.19}$$

Proof. We apply Theorem 1.13 with the polynomial $(Q_e^{1,0}(t))^2$ for $\tau = 2e$, and with the polynomial $(t+1)(Q_e^{1,1}(t))^2$ for $\tau = 2e+1$. Obviously, these two polynomials are nonnegative in the interval $[-1,1]$, i.e. **(B1)** is satisfied. Since their degrees are exactly τ, condition **(B2)** is also satisfied. Therefore, both polynomials belong to $B_{\mathcal{M},\tau}$. Without repeating the calculations of Levenshtein we quote

$$\Omega((Q_e^{1,0}(t))^2) = \sum_{j=0}^{e} r_j$$

and

$$\Omega((t+1)(Q_e^{1,1}(t))^2) = (1 - \frac{Q_e^{1,0}(-1)}{Q_{e+1}(-1)}) \sum_{j=0}^{e} r_j.$$

□

The specializations of the Delsarte bounds for the Euclidean spheres and the infinite projective spaces are given in the next two chapters respectively.

A classical result by Schoenberg and Szegö [69] shows that the polynomials $(Q_e^{1,0}(t))^2$ and $(Q_e^{1,1}(t))^2(t+1)$ are $B_{\mathcal{M},\tau}$-extremal for the corresponding values of τ. Therefore, improvements of the Delsarte bound by means of the pure linear programming approach could possibly already be obtained by using polynomials of degree at least $\tau+1$. We shall find such polynomials in the next two chapters, for the Euclidean spheres and the infinite projective spaces, respectively.

Definition 1.23 *Designs which attain the bound (1.19) are called tight.*

The most general necessary conditions for the existence of tight designs follow from the proofs of Theorem 1.13 and Theorem 1.22.

Theorem 1.24 *Let $C \subset \mathcal{M}$ be a tight τ-design. If $(x,y) = \alpha$ for some $x, y \in C, x \neq y$, then $Q_e^{1,0}(\alpha) = 0$ for $\tau = 2e$ (resp. $(\alpha+1)Q_e^{1,1}(\alpha) = 0$ for $\tau = 2e+1$).*

Conversely, if $Q_e^{1,0}(\alpha) = 0$ for $\tau = 2e$ (resp. $(\alpha+1)Q_e^{1,1}(\alpha) = 0$ for $\tau = 2e+1$), then there exist $x, y \in C$, such that $(x,y) = \alpha$.

In particular, it follows that any tight $(2k+1)$-design C in an antipodal PMS is antipodal, i.e. $C = \bar{C}$.

The following assertion is known as a Lloyd-type theorem since its analog in the Hamming space was first proved by Lloyd during investigations of perfect codes. Lloyd's Theorem for tight designs in PMS was proved by Delsarte [32, 33] for finite PMS, by Bannai-Damerell [9, 10] for the Euclidean sphere and by Bannai-Hoggar [11, 12] for the infinite projective spaces (see also [60]).

Theorem 1.25 *Let $C \subset \mathcal{M}$ be a tight τ-design and let $(x,y) = \alpha$ for some $x, y \in C$. Then α is a rational number.*

From Theorem 1.25 and the converse assertion of Theorem 1.24 it follows that if C is a tight τ-design, then all roots of the polynomials $Q_e^{1,0}(t)$ for $\tau = 2e$ (resp. $Q_e^{1,1}(t)$ for $\tau = 2e+1$) are rational numbers. This turns out to be a very strong restriction. It was used by Delsarte [32, 33], Bannai-Damerell [9, 10] and by Bannai-Hoggar [11, 12] to prove nonexistence of tight designs in many cases.

Other nonexistence results can be obtained by computing the distance distribution of tight designs [21].

Chapter 2

Linear Programming Bound for Spherical Designs

2.1 Introduction

In this chapter we give a method for finding new lower bounds for some spherical τ-designs using pure linear programming. This method is similar to the method for obtaining upper bounds for spherical codes proposed by Boyvalenkov [20] (see also [18]). Here, we use $B_{\mathbf{S}^{n-1},\tau}$-extremal polynomials of degree $\tau+3$, in combination with Theorem 1.13.

In Section 2.5 we give improvements of the Delsarte bound for $\tau = 6$ ($4 \leq n \leq 10$, Table 2.1), for $\tau = 7$ ($5 \leq n \leq 7$, Table 2.2), for $\tau = 8$ ($4 \leq n \leq 17$, Table 2.3), for $\tau = 9$ ($4 \leq n \leq 14$, Table 2.4), for $\tau = 10$ ($4 \leq n \leq 26$, Table 2.5) and for $\tau = 11$ ($4 \leq n \leq 23$, Table 2.6). The chapter is based on [22] and [23].

2.2 Spherical Harmonics and Spherical τ-designs

In the beginning of this section we will give some definitions and properties of spherical harmonics and spherical τ-designs following [36], [29, Chapter 3.2] and [71].

Let us denote by $Pol_m(\mathbb{R}^n)$ the linear space of real polynomials in n variables of degree at most m. Then $Hom_m(\mathbb{R}^n)$ and $Harm_m(\mathbb{R}^n)$ are the subspaces of the homogeneous, and of the homogeneous harmonic polynomials of degree m, respectively (see the definitions in Section 1.2). If we consider the linear space $F(M)$ consisting of real-valued functions defined over a set M, and a given subset N of M, we shall denote by $F(N)$ the homomorphic image of $F(M)$ obtained by restricting all functions in $F(M)$ to the domain N. In particular, we shall need the spaces $Hom_m(\mathbf{S^{n-1}})$ and $Pol_m(\mathbf{S^{n-1}})$ which are the restrictions of $Hom_m(\mathbb{R}^n)$ and $Pol_m(\mathbb{R}^n)$ to the unit sphere.

The inner product which we use is the usual one:

$$\langle f, g \rangle = \int_{S^{n-1}} f(x)g(x)d\mu(x),$$

for $f, g \in Pol_m(\mathbb{R}^n)$. We have the following well known direct sum decomposition of $Pol_m(\mathbb{R}^n)$

$$Pol_m(\mathbb{R}^n) = \sum_{i=0}^{m} \oplus Hom_i(\mathbb{R}^n).$$

Now the following decomposition theorem holds.

Theorem 2.1 *[37] For any integer m, one has the direct sum decomposition*

$$Pol_m(\mathbf{S}^{n-1}) = Hom_m(\mathbf{S}^{n-1}) \oplus Hom_{m-1}(\mathbf{S}^{n-1}).$$

Proof. On the sphere we have $(x, x) = 1$ and therefore the inclusions

$$Hom_{m-2i}(\mathbf{S}^{n-1}) \cong (x, x)^i Hom_{m-2i}(\mathbf{S}^{n-1}) \subset Hom_m(\mathbf{S}^{n-1})$$

$$Hom_{m-2i-1}(\mathbf{S}^{n-1}) \cong (x, x)^i Hom_{m-2i-1}(\mathbf{S}^{n-1}) \subset Hom_{m-1}(\mathbf{S}^{n-1}).$$

The orthogonality holds since the integral of an odd function over $\mathbf{S}^{\mathbf{n-1}}$ vanishes.

□

The dimensions of these spaces are as follows:

$$\dim\, Hom_m(\mathbf{S}^{n-1}) = \dim\, Hom_m(\mathbb{R}^n) = \binom{n+m-1}{n-1},$$

since $f(tx) = t^m f(x)$,

$$\dim\, Pol_m(\mathbf{S}^{n-1}) = \binom{n+m-1}{n-1} + \binom{n+m-2}{n-1}$$

by Theorem 2.1, and

$$\dim\, Harm_m(\mathbb{R}^n) = \binom{n+m-1}{n-1} - \binom{n+m-3}{n-1}.$$

The last equation holds because the space $Harm_m(\mathbb{R}^n)$ is the kernel and the space $Hom_{m-2}(\mathbf{S}^{n-1})$ is the image of the Laplace operator, when applied to $Hom_m(\mathbf{S}^{n-1})$. Hence, we obtain

$$Hom_m(\mathbf{S}^{n-1}) \cong Harm_m(\mathbb{R}^n) \oplus Hom_{m-2}(\mathbf{S}^{n-1}).$$

The elements of $Harm_m(\mathbb{R}^n)$ are called the *spherical harmonics* of degree m for the sphere S^{n-1}.

Let $H := Harm_m(\mathbb{R}^n)$ be a linear space provided with a non-degenerate inner product $\langle , \rangle$. For any linear functional $L(h)$ defined on H there exist an element $\hat{L} \in H$ such that

$$L(h) = \langle \hat{L}, h \rangle,$$

for $h \in H$. If we fix $\xi \in S^{n-1}$ and define a linear functional on H by $h \to h(\xi)$, for any $h \in H$, then there exist a unique $\hat{\xi} \in H$ such that $\langle \hat{\xi}, h \rangle = h(\xi)$. This polynomial $\hat{\xi}$, also written as $Q_m(\xi, \cdot)$, is called *the* m^{th} *zonal spherical harmonic* with pole ξ. The polynomial $Q_m((\xi, \cdot))$ is constant on the parallels which are perpendicular to ξ, so that $Q_m((\xi, \eta))$ depends on the value of the inner product (ξ, η) only (so $Q_m((\xi, \eta)) = Q_m(t)$, where $t = (\xi, \eta)$).

It is well known [44, 71] that $Q_m(t)$ is a Gegenbauer polynomial. The Gegenbauer polynomials constitute a family of polynomials in one variable t which is orthogonal with respect to the weight function

$$w(t) = c_n(1 - t^2)^{(n-3)/2},$$

where

$$c_n = \left(\int_{-1}^{1} (1 - t^2)^{\frac{n-3}{2}} dt \right)^{-1} = \frac{\Gamma(n-1)}{2^{n-2}(\Gamma(\frac{n-1}{2}))^2}, \tag{2.1}$$

where $\Gamma(x)$ is the Gamma function.

A three-terms recurrence relation for the Gegenbauer polynomials was given in (1.5) (see also (1.14)). The next lemma is well known [1, Chapter 22].

Lemma 2.2 *The polynomials* $Q_{2i-1}^{(n)}(t)$ *(respectively* $Q_{2i}^{(n)}(t)$*) are odd (respectively even) functions and their nonzero coefficients alternate in sign.*

Proof. By induction, using the recurrence relation.

□

We will denote

$$Q_i^{(n)}(t) = a_{i,i}^{(n)} t^i + a_{i,i-2}^{(n)} t^{i-2} + \cdots,$$

where the last term is a_{i0} for i even, and $a_{i1}t$ for i odd.

Spherical designs were introduced by Delsarte, Goethals and Seidel in 1977 [36]. Just as the classical t-designs are a special class of constant weight codes, so the spherical τ-designs are a special class of spherical codes. The original motivation for studying these objects came from the numerical evaluation of multi-dimensional integrals. The integral of a polynomial function over the sphere may be approximated by its average value at the code points; if the code is spherical τ-design the approximation is exact for all polynomials of degree at most τ.

Definition 2.3 *A finite subset C is a spherical τ-design on $\mathbf{S}^{n-1}$ if and only if the equality*

$$\int_{\mathbf{S}^{n-1}} f(x)d\mu(x) = \frac{1}{|C|}\sum_{x\in C} f(x)$$

holds for all polynomials $f(x)$ of degree at most τ.

The equivalence between definition 1.3 and 2.3 follows by the fact that $\int_{\mathbf{S}^{n-1}} f(x)d\mu(x) = 0$ for harmonic polynomials. We recall that the measure μ is the normalized Lebesgue measure, i.e. $\mu(\mathbf{S}^{n-1}) = 1$.

It follows by Definition 2.3 that spherical designs can be considered as a set of nodes for the Tchebichev-type quadrature formulas (i.e. formulas with equal weights and distinct nodes) with algebraic precision τ. This explains the attention which was paid to the spherical designs from a point of view of the numerical analysis [2, 3, 40, 62].

The *strength* $\tau(C)$ is the maximum value of τ for which C is a spherical τ-design. A spherical 1-design is any subset C of $\mathbf{S}^{n-1}$ which has its center of mass in the center of the sphere, so in 0. A spherical 2-design is what Schläffli calls a eutectic star which essentially is the projection into $\mathbb{R}^n$ of $|C|$ mutually orthogonal vectors [30].

2.3 Bounds for Spherical Designs

We abbreviate $B(\mathbf{S}^{n-1},\tau)$ and $B_{\mathbf{S}^{n-1},\tau}$ to $B(n,\tau)$ and $B_{n,\tau}$, respectively (see Definition 1.17).

There is no upper bound on the number of points of a spherical τ-design, since the union of two disjoint τ-designs with r and s points respectively again is a τ-design (with $r+s$ points). Upper bounds on $B(n,\tau)$ are normally obtained by explicit constructions [2, 42] (see the end of this section).

The Delsarte bound for spherical designs specializes to the following bound for $B(n,\tau)$. This bound was proved by Delsarte, Goethals and Seidel [36, Theorems 5.11, 5.12] who in the same paper introduced the notion of spherical designs.

$$B(n,\tau) \geq R(n,\tau) = \begin{cases} \binom{n+e-1}{n-1} + \binom{n+e-2}{n-1} & \text{if } \tau = 2e; \\ 2\binom{n+e-1}{n-1} & \text{if } \tau = 2e+1. \end{cases} \tag{2.2}$$

A spherical design is called *tight* if it attains the above bound. It is clear that a tight τ-design can not be a $(\tau+1)$-design.

We shall now present all known examples of tight τ-designs. For $n=2$ and any τ, a tight τ-design is nothing but a regular $(\tau+1)$-gon. Any pair of antipodal points on $\mathbf{S}^{n-1}$ is a tight 1-design. The $n+1$ vertices of a regular simplex in $\mathbb{R}^n$ provide a tight 2-design. The $2n$ vertices of the cross polytope form a tight 3-design on $\mathbf{S}^{n-1}$.

For $\tau \geq 4$ and $n \geq 3$, exactly eight examples of tight τ-designs are known. All of them are unique up to isometry [29, Chapter 14].

Tight spherical 4-designs are constructed by Delsarte, Goethals and Seidel for $n = 6$ and $n = 22$. Tight spherical 5-designs are known to exist for $n = 3, 7, 23$. In general, if a tight spherical 4-design on $\mathbf{S}^{n-1}$ exists then $n = m^2 - 3$, and if a tight 5-design on S^{n-1} exists then $n = m^2 - 2$. Moreover, a tight spherical 4-design on $\mathbf{S}^{m^2-4}$ exists if and only if a tight spherical 5-design on $\mathbf{S}^{m^2-3}$ exists (see [21, 50]). In both cases the number m must be odd [21, 50]. Examples are known for $m = 3$ and $m = 5$ only. Tight 7-designs can only exist in dimension $n = 3m^2 - 4$. Examples are known for $m = 2$ and $m = 3$ only. The only possible tight 11-design is realized by the minimal norm vectors in the famous Leech lattice [29]. All these designs are unique up to isometry [29, Chapter 14].

On the other hand, Bannai and Damerell [9, 10] proved that for $n \geq 3$ tight spherical τ-designs on $\mathbf{S}^{n-1}$ do not exist if $\tau = 2e$ and $e \geq 3$ (the case $e = 3$ was already considered in [36]) or $\tau = 2e + 1$ and $e \geq 4$ except for the case $\tau = 11$, $n = 24$ (this is the Leech lattice).

Seymour and Zaslavsky [70] proved the existence of spherical τ-designs on $\mathbf{S}^{n-1}$ for all values of n and τ, provided $|C|$ is sufficiently large. Since spherical designs can be used for numerical integration, it is of interest to give explicit constructions.

Mimura [59] gave a construction for $\tau = 2$ in all dimensions and all cardinalities $|C| \geq n_2$ for some positive integer n_2. General constructions of τ-designs were given by Bajnok [2, 3, 4, 5]. Hardin and Sloane [42] constructed 4-designs for the following values of $|C|$ and $n \geq 3$: $|C| = 12, 14, \geq 16$ for $n = 3$; $|C| \geq 20$ for $k = 4$; $|C| \geq 29$ for $n = 5$; $|C| = 27, 36, \geq 39$ for $n = 6$; $|C| \geq 53$ for $n = 7$; and $|C| \geq 69$ for $n = 8$.

In three dimensions, Reznick [67] and Hardin-Sloane [43] showed that 5-designs exist for $|C| = 12, 16, 18, 20, \geq 22$. In [43] it is shown that 6-designs exist for $|C| = 24, 26, \geq 28$; 7-designs for $|C| = 24, 30, 32, 34, \geq 36$; 8-designs for $|C| = 36, 40, 42, \geq 44$; 9-designs for $|C| = 48, 50, 52, \geq 54$; 10-designs for $|C| = 60, 62, \geq 64$; 11-designs for $|C| = 70, 72, \geq 74$ and 12-designs for $|C| = 84, \geq 86$.

Some of the known polytopes in dimensions $n \geq 4$ are known to be spherical designs of large strengths [40]. For example, the 600-cell in $\mathbb{R}^4$ (with 120 vertices) is spherical 11-design (see also the end of this chapter).

The above can be summarized by the following:

$$\begin{gathered}
B(2,\tau) = \tau + 1, \\
B(n,1) = 2, B(n,2) = n+1, B(n,3) = 2n, \\
B(3,4) \leq 12, B(4,4) \leq 20, B(5,4) \leq 29, B(6,4) = 27, \\
B(7,4) \leq 53, B(8,4) \leq 69, B(22,4) = 275, \\
B(3,5) = 12, B(7,5) = 56, B(23,5) = 552, \\
B(3,6) \leq 24, B(3,7) \leq 24, B(8,7) = 240, B(23,7) = 4600, \\
B(3,8) \leq 36, B(3,9) \leq 48, B(3,10) \leq 60, \\
B(3,11) \leq 70, B(4,11) \leq 120, B(24,11) = 196960.
\end{gathered}$$

2.4 A Method for Obtaining New Lower Bounds on $B(n,\tau)$

In this section, we improve the lower bounds in (2.2) in some cases. Our method proposes suitable polynomials for applying in Theorem 1.13.

Delsarte, Goethals, and Seidel [36, Theorems 5.11, 5.12] obtain bound (2.2) by using Theorem 1.13 with $B_{n,\tau}$-extremal polynomials (see Definition 1.17) of degree τ. Our polynomials have degree $\tau+3$. It can be proved (see [20, Theorem 5.3]) that they are also $B_{n,\tau}$-extremal.

By formula (1.13), in the context of this chapter, the coefficient f_0 in the Gegenbauer expansion

$$f(t)=\sum_{i=0}^{k} f_i Q_i^{(n)}(t)$$

can be computed by

$$f_0=c_n\int_{-1}^{1} f(t)(1-t^2)^{\frac{n-3}{2}}dt>0, \tag{2.3}$$

where c_n is given by (2.1).

The calculation of the numbers b_i, $i\geq 0$, (see (1.12) and (1.13)) in the next lemma gives a useful expression of f_0 in the coefficients of $f(t)$.

Lemma 2.4 *Let*

$$f(t)=\sum_{i=0}^{k} a_i t^i=\sum_{i=0}^{k} f_i Q_i^{(n)}(t)$$

be a real polynomial. Then

$$\begin{aligned} f_0 &= a_0+\frac{a_2}{n}+\frac{3a_4}{n(n+2)}+\cdots && (2.4)\\ &= a_0+\sum_{i=1}^{[k/2]}\frac{(2i-1)!!a_{2i}}{n(n+2)\cdots(n+2i-2)}. && (2.5)\end{aligned}$$

Proof. We have to compute the numbers b_i in (1.13). Obviously

$$b_{2i+1}=c_n\int_{-1}^{1} t^{2i+1}(1-t^2)^{\frac{n-3}{2}}dt=0.$$

For the even case we have [61]

$$\begin{aligned} b_{2i} &= c_n\int_{-1}^{1} t^{2i}(1-t^2)^{\frac{n-3}{2}}dt\\ &= c_n\int_{0}^{1} u^{\frac{2i-1}{2}}(1-u)^{\frac{n-3}{2}}du\end{aligned}$$

$$= \frac{\Gamma(n-1)\Gamma(i+\frac{1}{2})}{2^{n-2}\Gamma(\frac{n-1}{2})\Gamma(i+\frac{n}{2})}$$
$$= \frac{(2i-1)!!}{n(n+2)\cdots(n+2i-2)}.$$

□

In particular, it follows from Lemma 2.4 that if the coefficients of $f(t)$ are rational functions of several variables, then $f(1)/f_0$ is also a rational function of the same variables and the dimension n. This allows us to use the standard way (by calculating partial derivatives) for the investigation of the function $f(1)/f_0$.

To investigate $B_{n,\tau}$-extremal polynomials, we study two of their properties, which turn out to be very important for our purposes. First, we obtain a lower bound on the number of double zeros of extremal polynomials and, secondly, we find sufficient conditions for $f_i = 0$ for some $i > \tau$.

The next theorem gives lower bounds on the number of double zeros of the extremal polynomials. By double zero we mean a zero of multiplicity two. So we shall count a zero of multiplicity four as two double zeros, etc.

Theorem 2.5 *Let $f(t)$ be a $B_{n,\tau}$-extremal polynomial ($n \geq 3$, $\tau \geq 4$) of degree $k \geq \tau+3$.*

(i) *If τ is odd or if τ is even and -1 is a zero of $f(t)$ of an even multiplicity, then $f(t)$ has at least $[\tau/2]+1$ double zeros in $[-1,1]$.*

(ii) *If τ is even and -1 is a zero of $f(t)$ of an odd multiplicity, then $f(t)$ has at least $\tau/2$ double zeros in $[-1,1]$.*

Proof. By (B1) in Theorem 1.13 any $B_{n,\tau}$-extremal polynomial can be written as $f(t) = A^2(t)G(t)$, where $2\deg(A) \leq \tau$, $G(t) \geq 0$ for $-1 \leq t \leq 1$, $G(1) > 0$, $A(t)$ has $\deg(A)$ zeros in $[-1,1]$. Note that this includes the case $A(t) = const$ as well.

Assume that $G(t)$ has no double zeros in the interval $[-1,1]$. Then, it is clear that $G(t) = 0$ is possible only for $t = -1$. We shall consider two cases.

Case 1. $G(-1) > 0$.

We have $G(t) > 0$ for $t \in [-1,1]$. Then there exists $\varepsilon > 0$ such that $G(t) \geq \varepsilon > 0$ for $t \in [-1,1]$. Let us consider the polynomial

$$\begin{aligned} P_\varepsilon(t) &= f(t) - \varepsilon A^2(t) \\ &= A^2(t)(G(t)-\varepsilon). \end{aligned}$$

Then, $P_\varepsilon(t) \geq 0$ for $-1 \leq t \leq 1$ so condition (B1) is satisfied by $P_\varepsilon(t)$. This also implies $f_0(P_\varepsilon) > 0$ (see Remark 1.15).

Write $f(t) = \sum_{i=0}^{k} f_i Q_i^{(n)}(t)$ and

$$\begin{aligned} P_\varepsilon(t) &= \sum_{i=0}^{k} f_i(P_\varepsilon) Q_i^{(n)}(t) \\ &= \sum_{i=0}^{k} f_i Q_i^{(n)}(t) - \varepsilon \sum_{i=0}^{2deg(A)} f_i(A^2) Q_i^{(n)}(t). \end{aligned}$$

It follows from the last representation of $P_\varepsilon(t)$ that $f_i(P_\varepsilon) = f_i$ for $i \geq 2\deg(A)+1$. In particular, $f_i(P_\varepsilon) = f_i \leq 0$ for $i \geq \tau + 1 \geq 2\deg(A)+1$. Thus $P_\varepsilon(t) \in B_{n,\tau}$.

Since $A^2(t) \in B_{n,\tau}$ (see Remark 1.15) we have $\Omega(A^2) < \Omega(f)$. But one can easily check that this inequality is equivalent to

$$\Omega(P_\varepsilon) = \frac{P_\varepsilon(1)}{f_0(P_\varepsilon)} = \frac{f(1) - \varepsilon A^2(1)}{f_0 - \varepsilon f_0(A^2)} > \frac{f(1)}{f_0} = \Omega(f) ,$$

a contradiction. This proves the theorem in the case when -1 is an even zero of $f(t)$.

Case 2. $G(-1) = 0$.

Now, we have $f(t) = A^2(t)(t+1)G_1(t)$, where $G_1(t) > 0$ for $-1 \leq t \leq 1$ (otherwise $G_1(-1) = 0$ and -1 would be a double zero of $G(t)$). As above, there exists $\varepsilon > 0$ such that $G_1(t) > \varepsilon > 0$ for $t \in [-1,1]$.

If τ is odd and $B(t) = A^2(t)(t+1)$, then we have $\deg(B) = 2\deg(A)+1 \leq \tau$. Therefore $B(t) \in B_{n,\tau}$. Continuing in the same way as in *Case 1* one obtains a contradiction.

If τ is even and $\deg(A) \leq \tau/2 - 1$ we get a contradiction by a similar argument.

□

The next lemma gives further information in the case $\tau = 2e+1$.

Lemma 2.6 *If $\tau = 2e+1$ and $f(t)$ is a $B_{n,\tau}$-extremal polynomial of degree $2e+4$, then* $f(-1) = 0$.

Proof. By Theorem 2.5 (i) the polynomial $f(t)$ has at least $e+1$ double zeros. In fact, their number must be exactly $e+1$, otherwise $f(t)$ would be a square and its leading coefficient in the Gegenbauer expansion would be positive, contradicting (**B2**) in Theorem 1.13. Therefore, $f(t) = A^2(t)G(t)$ where $G(t)$ is a second degree polynomial.

Obviously, $G(t)$ could vanish in $[-1,1]$ only for $t = -1$. Assume that $G(-1) > 0$ and let $\varepsilon > 0$ be such that $G(t) \geq \varepsilon$ for $t \in [-1,1]$. Then we consider the polynomial $P_\varepsilon(t) = A^2(t)(G(t) - \varepsilon)$. Since $P_\varepsilon(t) \geq 0$ for $-1 \leq t \leq 1$, condition (**B1**) is satisfied by $P_\varepsilon(t)$. Moreover, we have $f_0(P_\varepsilon) > 0$, $f_i(P_\varepsilon) = f_i \leq 0$ for $i = 2e+3, 2e+4$ and $f_{2e+2}(P_\varepsilon) = f_{2e+2} - \varepsilon f_{2e+2}(A^2) < 0$, so (**B2**) is also satisfied by P_ε. However, we have

$$\Omega(P_\varepsilon) = \frac{P_\varepsilon(1)}{f_0(P_\varepsilon)} = \frac{f(1) - \varepsilon A^2(1)}{f_0 - \varepsilon f_0(A^2)} > \frac{f(1)}{f_0} = \Omega(f),$$

which contradicts the extremality of $f(t)$. Therefore $G(-1) = f(-1) = 0$.

□

We restrict ourselves to extremal polynomials of degree $\tau+3$. These polynomials can have the following form

$$f(t) = \begin{cases} A^2(t)[q(t+1)+1-t] & \text{if } \tau = 2e \\ A^2(t)[q(t+1)+1-t](t+1) & \text{if } \tau = 2e+1 \end{cases} \tag{2.6}$$

where $\deg(A) = e+1$ and $0 < q < 1$. The polynomial

$$A(t) = t^{e+1} + a_1 t^e + \cdots + a_{e-1}t + a_e$$

has $e+1$ zeros in $[-1,1]$. Indeed, it follows from Theorem 2.5 that for $\tau = 2e$ we have to consider two cases. Namely, we may have $\deg(A) = e+1$ or $\deg(A) = e$ but $A(-1) = 0$. Our numerical experiments in the second case did not give any good results. Thus, we shall be looking for polynomials of degree $2e+3$ with $e+1$ double zeros.

Equation (2.6) shows that (**B1**) is already satisfied for our polynomials. It also implies $f_0 > 0$ (see Remark 1.15).

In order to reach condition (**B2**), we have to require $f_i \leq 0$ for $i = \tau+1, \tau+2, \tau+3$. The inequality $f_{\tau+3} < 0$ is equivalent to $q < 1$. The next theorem shows that we can take $f_{\tau+1} = 0$ without loss of generality.

Theorem 2.7 *Let $f(t)$ be a $B_{n,\tau}$-extremal polynomial ($n \geq 3$, $\tau \geq 4$) of degree $k \geq \tau+1$. If $\Omega(f) < R(n,\tau+1)$ then $f_{\tau+1} = 0$ in the Gegenbauer expansion $f(t) = \sum_{i=0}^{k} f_i Q_i^{(n)}(t)$.*

Proof. Let us suppose that $f_{\tau+1} < 0$ under the assumptions of the Theorem. In [36], Delsarte, Goethals, and Seidel introduced the polynomial $C_{n,\tau}(t) = (Q_e^{1,0}(t))^2$ for $\tau = 2e$ and $C_{n,\tau}(t) = (t+1)(Q_e^{1,1}(t))^2$ for $\tau = 2e+1$. They used this polynomial to obtain the bound

$$B(n,\tau+1) \geq R(n,\tau+1).$$

Note that $C_{n,\tau+1}(t)$ has degree $\tau+1$ and that $c_{\tau+1} > 0$ (see Remark 1.6) in the Gegenbauer expansion $C_{n,\tau+1}(t) = \sum_{i=0}^{\tau+1} c_i Q_i^{(n)}(t)$.

Since $f_{\tau+1} < 0$, there exist positive numbers ξ and η, such that $\xi f_{\tau+1} + \eta c_{\tau+1} \leq 0$. We consider the polynomial $H(t) = \xi f(t) + \eta C_{n,\tau+1}(t)$. It is easy to see (as in Lemma 1.18), that $H(t)$ belongs to the set $B_{n,\tau}$ and $\deg(H) \leq \deg(f) = k$. By Lemma 1.20, the number $\Omega(H) = H(1)/f_0(H)$ lies between the numbers $\Omega(f)$ and $R(n,\tau+1) = \Omega(C_{n,\tau+1})$, i.e. we have

$$\Omega(f) < \Omega(H) < R(n,\tau+1),$$

a contradiction with the extremality of the polynomial $f(t)$.

□

The number $R(n, \tau+1)$ is relatively large with respect to $R(n,\tau)+1$. Therefore, without loss of generality, one can search for improvements of (2.2), assuming $f_{\tau+1} = 0$ (otherwise we would have $B(n,\tau) \geq R(n,\tau+1)$ which is rather good bound). In particular, if extremal polynomials of degree $\tau+1$ do exist they would give quite nice bounds.

One can use some polynomials of degree $\tau+3$ to obtain assertions that are similar to Theorem 2.7 and concern the coefficient $f_{\tau+2}$. Such polynomials have $f_{\tau+1} = 0$ and $f_{\tau+2} > 0$. As before, the number $\Omega(f)$ is large with respect to $R(n,\tau)$. We may conclude in the same way that, without loss of generality, $f_{\tau+2} = 0$ must hold for any $B_{n,\tau}$-extremal polynomial of degree at least $\tau+3$.

We use the equalities $f_{\tau+1} = f_{\tau+2} = 0$ as equations with respect to the first two unknown coefficients a_1 and a_2 of $A(t)$. As we shall see in Section 2.5, it is easy to express them as functions of q and n. What remains to be done is to find q and the remaining coefficients of $A(t)$ in order to maximize the rational function

$$F(q, a_3, \ldots, a_e) = \Omega(f) = \frac{2qA^2(1)}{f_0}.$$

Equating to zero the partial derivatives

$$F'_{a_i} = \frac{2qA(1).2f_0 - 2qA^2(1)(f_0)'_{a_i}}{f_0^2}$$

for $i = 3, \ldots, e$, we obtain $2f_0 = (f_0)'_{a_i} A(1)$. This gives us the following system of linear equations with respect to the unknowns $a_3, \ldots, a_e$

$$\begin{aligned}
(f_0)'_{a_3} - (f_0)'_{a_4} &= 0 \\
(f_0)'_{a_3} - (f_0)'_{a_5} &= 0 \\
&\cdots \\
(f_0)'_{a_3} - (f_0)'_{a_e} &= 0 \\
2f_0 - A(1)(f_0)'_{a_3} &= 0.
\end{aligned} \tag{2.7}$$

In fact, the last equation is not linear as it stands. However, it becomes linear by replacing the parameters $a_4, \ldots, a_e$ by the corresponding functions of a_3 by means of the first $e-3$ equations.

Therefore, one can resolve the system (2.7) with respect to the parameters $a_3, a_4, \ldots, a_e$. Of course, they are rational functions of still unknown parameter q and the dimension n. It does not seem possible to apply further analytical methods in order to find the optimal values of $q \in (0,1)$. We use a computer and a simple numerical method to find good approximations of the extremal polynomials. Of course, only the integer part of the final result is important.

The lower bounds we have obtained are better than (2.2) in the cases, that extremal polynomials of degree $\tau+3$ exist, which satisfy the requirements $f_{\tau+1} = f_{\tau+2} = 0$ and which have $[\tau/2]+1$ double zeros.

2.5 Examples of New Lower Bounds

Case 1. $\tau = 4$.

We shall improve bound (2.2) in dimensions 3, 4 and 5 by one. It is well known (see [9]) that (2.2) can not be attained in these dimensions. Thus we shall not obtain new bounds. However, it is easier to give a detailed explanation of our approach in this small case.

We must consider polynomials of degree 7 having the following form

$$\begin{aligned} f(t) &= (t^3 + at^2 + bt + c)^2[q(t+1) + 1 - t] \\ &= \sum_{i=0}^{7} f_i Q_i^{(n)}(t), \end{aligned}$$

where $f_5 = f_6 = 0$ and $0 < q < 1$ (the last implies $f_7 < 0$). By $f_5 = f_6 = 0$ one can express (see Remark 1.6)

$$\begin{aligned} a &= \frac{q+1}{2(1-q)}, \\ b &= \frac{3a^2}{2} - \frac{21}{2(n+10)} \\ &= \frac{3(q+1)^2}{4(1-q)^2} - \frac{21}{2(n+10)}. \end{aligned}$$

Using (2.3) we obtain

$$\begin{aligned} f_0 &= f_0(c,q,n) \\ &= c^2(q+1) + \frac{1}{n}[2bc(q-1) + (b^2 + 2ac)(1+q)] \\ &\quad + \frac{3}{n(n+2)}[(2ab + 2c)(q-1) + (a^2 + 2b)(1+q)] \\ &= \alpha c^2 + \beta c + \gamma, \end{aligned}$$

where

$$\begin{aligned} \alpha &= q+1, \\ \beta &= \frac{2}{n}\left[a(q+1) + (q-1)(b + \frac{3}{n+2})\right], \\ \gamma &= \frac{1}{n}\left\{b^2(q+1) + \frac{3}{n+2}\left[2ab(q-1) + (a^2+2b)(q+1)\right]\right\}, \end{aligned}$$

In order to determine c as a function of q and n, we have to consider the function

$$F(c,q,n) = \frac{f(1)}{f_0(c,q,n)}.$$

By the equality $F_c' = 0$ we obtain the following equation

$$2f_0 - (f_0)_c'(1 + a + b + c) = 0. \tag{2.8}$$

From (2.8) one can express

$$c = \frac{2\gamma - \beta(1 + a + b)}{2\alpha(1 + a + b) - \beta}.$$

Finally, for fixed n, we have to find $q \in (0, 1)$ maximizing $f(1)/f_0$. The best polynomials we have obtained in dimensions 3, 4 and 5 give bounds 10, 15, and 21 respectively while (2.2) gives 9, 14, and 21. The smallest values for which 4-designs in these dimensions have been found by Hardin and Sloane [42] are 12, 20, and 29 respectively. Therefore, $10 \le B(3, 4) \le 12, 15 \le B(4, 4) \le 20$ and $21 \le B(5, 4) \le 29$.

Case 2. $\tau = 5$.

For $\tau = 5$ we found $B_{n,5}$-extremal polynomials of degree 7. Their form led us to the polynomial

$$f(t) = (t^2 - \frac{1}{n+2})^2(t + 1)^2(2 - t),$$

which has $f_7 < 0$, $f_6 = 0$ and $f(1)/f_0 = R(n, 5) = n(n + 1)$, i.e. we rediscover the Delsarte, Goethals, Seidel bound $B(n, 5) \ge R(n, 5)$.

Case 3. $\tau = 6$.

We consider polynomials of degree 9 of the following form

$$\begin{aligned} f(t) &= (t^4 + at^3 + bt^2 + ct + d)^2[q(t + 1) + 1 - t] \\ &= \sum_{i=0}^{9} f_i Q_i^{(n)}(t), \end{aligned}$$

where $f_7 = f_8 = 0$ and $0 < q < 1$.

Similarly to Case 1, we express a, b, c and d as functions of q and n. The new bounds we have obtained are given in Table 2.1. Delsarte, Goethals and Seidel proved in [36, Theorem 7.7], that (2.2) can not be attained for $\tau = 6$ and $n \ge 3$ (see also [10, Theorem 1]). Therefore, only improvements by more than one of the bound (2.2) are really of interest. We obtain such improvements in dimensions $4 \le n \le 10$.

Case 4. $\tau = 7$.

In this case we work with polynomials of degree 10 of the form

$$\begin{aligned} f(t) &= (t^4 + at^3 + bt^2 + ct + d)^2[q(t + 1) + 1 - t](t + 1) \\ &= \sum_{i=0}^{10} f_i Q_i^{(n)}(t), \end{aligned}$$

where $f_8 = f_9 = 0$, $0 < q < 1$. One can express the coefficients a, b, c and d as functions of q and n. Maximizing, we obtain new bounds in dimensions 5, 6 and 7. The results are given in Table 2.2.

n	$R(n,\tau)+1$	New bounds
4	31	32
5	51	54
6	78	84
7	113	121
8	157	167
9	211	221
10	276	283

Table 2.1: New lower bounds for the cardinality of spherical 6-designs on $\mathbf{S}^{n-1}$, $4 \le n \le 10$.

n	$R(n,\tau)+1$	New bounds
5	71	73
6	113	116
7	169	172

Table 2.2: New lower bounds for the cardinality of spherical 7-designs on $\mathbf{S}^{n-1}$, $5 \le n \le 7$.

Case 5. $\tau = 8$.

We use polynomials of the form

$$\begin{aligned} f(t) &= (t^5 + at^4 + bt^3 + ct^2 + dt + e)^2[q(t+1) + 1 - t] \\ &= \sum_{i=0}^{11} f_i Q_i^{(n)}(t) \end{aligned}$$

($f_9 = f_{10} = 0$ and $0 < q < 1$) to obtain new lower bounds improving (2.2) by more than 1 in dimensions $4 \le n \le 7$. The results are given in Table 2.3.

Case 6. $\tau = 9$.

We consider polynomials of degree 12 having the following form:

$$\begin{aligned} f(t) &= (t^5 + at^4 + bt^3 + ct^2 + dt + e)^2[q(t+1) + 1 - t](t+1) \\ &= \sum_{i=0}^{12} f_i Q_i^{(n)}(t), \end{aligned}$$

where $f_{10} = f_{11} = 0$ and $0 < q < 1$. The new lower bounds are presented in Table 2.4.

n	$R(n,\tau)+1$	New bounds
4	56	59
5	106	115
6	183	203
7	295	332
8	451	511
9	661	750
10	936	1060
11	1288	1450
12	1730	1930
13	2276	2507
14	2941	3191
15	3741	3989
16	4693	4908
17	5815	5951

Table 2.3: New lower bounds for the cardinality of spherical 8-designs on $\mathbf{S}^{n-1}$, $4 \le n \le 17$.

n	$R(n,\tau)+1$	New bounds
4	71	73
5	141	149
6	253	272
7	421	458
8	661	724
9	991	1087
10	1431	1565
11	2003	2173
12	2731	2924
13	3641	3828
14	4761	4892

Table 2.4: New lower bounds on the size of the spherical 9-designs on $\mathbf{S}^{n-1}$, $4 \le n \le 14$.

Case 7. $\tau = 10$.

Now using polynomials of degree 13 of the form

$$\begin{aligned} f(t) &= (t^6 + at^5 + bt^4 + ct^3 + dt^2 + et + f)^2[q(t+1) + 1 - t] \\ &= \sum_{i=0}^{13} f_i Q_i^{(n)}(t), \end{aligned}$$

where $f_{11} = f_{12} = 0$ and $0 < q < 1$, we express a and b by the equations $f_{11} = f_{12} = 0$. Further, we follow the above described method and obtain improvements for the classical bound for dimensions $4 \le n \le 26$. They are listed in Table 2.5 below. In some cases the improvements are almost 20 percent.

n	$R(n,\tau)+1$	New bounds
4	92	97
5	197	215
6	379	424
7	673	770
8	1123	1305
9	1783	2097
10	2718	3222
11	4005	4771
12	5734	6845
13	8009	9556
14	10949	13028
15	14689	17394
16	19381	22798
17	25195	29392
18	32320	37332
19	40965	46784
20	51360	57916
21	63757	70900
22	78431	85908
23	95681	103113
24	115831	122689
25	139231	144805
26	166258	169626

Table 2.5: New lower bounds on the size of the spherical 10-designs on $\mathbf{S}^{n-1}$, $4 \le n \le 26$.

Case 8. $\tau = 11$.

In this case we use polynomials of degree 14 of the form

$$\begin{aligned} f(t) &= (t^6+at^5+bt^4+ct^3+dt^2+et+f)^2[q(t+1)+1-t](t+1) \\ &= \sum_{i=0}^{14} f_i Q_i^{(n)}(t) \end{aligned}$$

where $f_{12}=f_{13}=0$ and $0<q<1$. The new bounds in dimensions $4 \le n \le 23$ are presented in Table 2.6.

We note that the regular polytope (3,3,5) in $\mathbb{R}^4$ is an 11-design with 120 points [36]. So, in this case our new bound 117 (instead of the Delsarte's bound 112) seems reasonably tight.

For $n = 24$ we obtain the bound (2.2) again. As we already mentioned this bound is attained by the 11-design formed by the minimum norm vectors in the Leech lattice [36].

n	$R(n,\tau)+1$	New bounds
4	112	117
5	252	270
6	504	552
7	924	1035
8	1584	1808
9	2574	2985
10	4004	4701
11	6006	7117
12	8736	10413
13	12376	14790
14	17136	20464
15	23256	27664
16	31008	36623
17	40698	47574
18	52668	60744
19	67298	76344
20	85008	94566
21	106260	115577
22	131560	139514
23	161460	166483

Table 2.6: New lower bounds on the cardinality of the spherical 11-designs on $\mathbf{S}^{n-1}$, $4 \le n \le 23$.

Our result by a polynomial of degree 14 implies the following characterization of the Leech lattice which seems to be new.

Definition 2.8 *A spherical code $C \subset \mathbf{S}^{n-1}$ is said to have index k if*

$$\int_{\mathbf{S}^{n-1}} f(x)d\mu(x) = \frac{1}{|C|}\sum_{x\in C} f(x)$$

holds for all homogeneous polynomials $f(x) = f(x_1, \ldots, x_n)$ of degree k.

In this context, a spherical τ-design is nothing but a code with indices $1, 2, \ldots, \tau$. The examination of the possible indices of a spherical code is interesting from a point of view of the numerical analysis.

Theorem 2.9 *The unique tight spherical 11-design in $\mathbf{S}^{23}$ has index 14.*

Proof. By our polynomial of degree 14 in (1.17) we obtain

$$\sum_{x\in C} v_{14,j}(x) = 0$$

for all $j = 1, 2, \ldots, r_{14}$. This is equivalent to the required property (see the remark following Definition 2.3).

□

Chapter 3

Linear Programming Bounds for Designs in Projective Spaces

3.1 Introduction

In this chapter we consider τ-designs in infinite projective spaces and improve the Delsarte bound in some cases. In Section 3.3 we explain our approach which is similar to the method we have used for spherical designs in Chapter 2. In Section 3.4 we give some examples of new bounds and compare our results with the classical bounds in tables. This chapter is based on [24].

3.2 Designs in Infinite Projective Spaces

We consider τ-designs in the projective space $\mathbb{F}P^{n-1}$ consisting of the lines through the origin in $\mathbb{F}^n$. Here $\mathbb{F}$ denotes the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, or the Cayley octonions $\mathbb{O}$. Together with the Euclidean spheres $\mathbf{S}^{n-1}$, they constitute all connected compact symmetric spaces of rank 1. A model of these spaces is described in Chapter 1.

As we mentioned in Section 1.2, the spaces $\mathbb{R}P^{n-1}$, $\mathbb{C}P^{n-1}$, $\mathbb{H}P^{n-1}$ $(n \geq 2)$ and $\mathbb{O}P^{n-1}$ $(n = 2, 3)$ are polynomial with standard substitution

$$\sigma(d) = 2(1 - \frac{d^2}{2})^2 - 1.$$

For $i = 0, 1, \ldots$ we have

$$r_i = \frac{2i + \alpha + \beta + 1}{\alpha + \beta + 1} \frac{\binom{i+\alpha+\beta}{i}\binom{i+\alpha}{i}}{\binom{i+\beta}{i}}$$

where r_i is the dimension of the subspace V_i and $(\alpha, \beta) = (mn - m - 1, m - 1)$.

In this context, the zonal spherical functions are the normalized Jacobi polynomials [45]. They can be written explicitly by (1.7).

Any real polynomial

$$f(t) = \sum_{i=0}^{k} a_i t^i$$

is associated with its Jacobi expansion

$$f(t) = \sum_{i=0}^{k} f_i P_i^{(\alpha,\beta)}(t)$$

for well-defined Jacobi coefficients f_i. The coefficient f_0 is the most interesting for our investigation. One can compute it by the following formula

$$f_0 \int_{-1}^{1} (1-t)^\alpha (1+t)^\beta \, dt = \int_{-1}^{1} f(t)(1-t)^\alpha (1+t)^\beta dt. \tag{3.1}$$

Following the notation of (1.13) we have

$$f_0 = \sum_{i=0}^{k} a_i b_i, \tag{3.2}$$

where (see 1.12)

$$b_i \int_{-1}^{1} (1-t)^\alpha (1+t)^\beta \, dt = \int_{-1}^{1} t^i (1-t)^\alpha (1+t)^\beta dt.$$

For another way of calculating the Jacobi coefficients see Remark 1.6. It will be used in the proof of Theorem 3.4 and Corrolary 3.5.

Seymour and Zaslavsky [70] have shown the existence of τ-designs in $\mathbb{F}P^{n-1}$ for any τ, $\mathbb{F}$, and n provided $|C|$ is sufficiently large. So, we are interested in the lower bounds for the minimum possible size of τ-designs. We write $B(m,n,\tau)$, $B_{m,n,\tau}$ and $R(m,n,\tau)$ instead of $B(\mathbb{F}P^{n-1},\tau)$, $B_{\mathbb{F}P^{n-1},\tau}$ and $R(\mathbb{F}P^{n-1},n,\tau)$, respectively.

Lower bounds on $B(m,n,\tau)$ were obtained by linear programming techniques (cf. Dunkl [38], Hoggar [45]). The explicit form of the Delsarte bounds is [11, 36, 38]

$$B(m,n,\tau) \geq R(m,n,\tau) = \begin{cases} \dfrac{(mn)_e \cdot (mn-m+1)_e}{(m)_e \cdot e!} & \text{for } \tau = 2e, \\[2ex] \dfrac{(mn)_{e+1} \cdot (mn-m+1)_e}{(m)_{e+1} \cdot e!} & \text{for } \tau = 2e+1, \end{cases} \tag{3.3}$$

where $(p)_a = p(p+1)\cdots(p+a-1)$ for $a \in \mathbb{N}$, and $(p)_0 = 1$.

The theory of tight designs in infinite projective spaces has been developed by Bannai and Hoggar [11, 12, 45, 47, 46]. They showed that tight τ-designs in projective spaces do not exist for $\tau = 4$ and $\tau \geq 6$ in dimensions $n \geq 3$. In particular, in these cases the bound (3.3) can be increased by one.

In the next section we shall obtain further improvements by substituting suitable polynomials of degree $\tau + 2$ in Theorem 1.13.

3.3 A Method for Obtaining New Lower Bounds on Cardinality of Designs in $\mathbb{F}P^{n-1}$

To obtain the classical bound (3.3) one can use the polynomials

$$f_{m,n,\tau}(t) = \begin{cases} (Q_e^{1,0}(t))^2 = (P_e^{\alpha+1,\beta}(t))^2 & \text{for } \tau = 2e, \\ (Q_e^{1,1}(t))^2(t+1) = (P_e^{\alpha+1,\beta+1}(t))^2(t+1) & \text{for } \tau = 2e+1, \end{cases}$$

of degree τ, as in the spherical case (cf. [11, 12]).

In general, it is not known which polynomials are the optimal choices for Theorem 1.13. However, it is known [38, 69] that the polynomials $f_{m,n,\tau}(t)$ are $B_{m,n,\tau}$-extremal. In this chapter, we propose a method for improving bound (3.3) in some cases by using good polynomials of degree $\tau + 2$. This approach is similar to our approach in the previous chapter where we wanted to obtain new bounds for spherical designs.

While searching for good polynomials $f(t)$ of degree $k \geq \tau+2$, we require $f_{\tau+1} = 0$ in the Jacobi expansion of $f(t)$. A more general setting for this situation is given by the following assertion which suggests how to find indices of zero coefficients in Jacobi expansions of $B_{m,n,\tau}$-extremal polynomials.

Theorem 3.1 *Let $j > \tau$ be an integer and let $f_j(t)$ be a real polynomial such that*

(i) *$f_j(t) \geq 0$ for $-1 \leq t \leq 1$,*

(ii) *the coefficients in the Jacobi expansion $f_j(t) = \sum_{i=0}^{k} f_{i,j} P_i^{(\alpha,\beta)}(t)$ satisfy $f_{i,j} \leq 0$ for $i \neq j$ and $\tau + 1 \leq i \leq k$, and $f_{j,j} > 0$.*

Then, any $B_{m,n,\tau}$-extremal polynomial $f(t)$ of degree $k_1 \geq k$ such that $f(1)/f_0 < f_j(1)/f_{0,j}$ satisfies $f_j = 0$ in its Jacobi expansion.

Proof. Suppose that $f_j < 0$ under the assumptions of the theorem. It is easy to see that there exist linear combinations

$$g_{\xi,\eta}(t) := \xi f(t) + \eta f_j(t) = \sum_{i=0}^{k_1} g_i P_i^{(\alpha,\beta)}(t)$$

with ξ and η positive, such that $g_j \leq 0$. As in Lemma 1.18 we check that such polynomials belong to the set $B_{m,n,\tau}$. However, by Lemma 1.20, the number $g_{\xi\eta}(1)/g_0$ lies in the interval $(f(1)/f_0, f_j(1)/f_{0,j})$, a contradiction, because $f(t)$ is $B_{m,n,\tau}$-extremal.

□

The next assertion is a projective analog of Lemma 2.7.

Corollary 3.2 *Let $f(t)$ be a $B_{m,n,\tau}$-extremal polynomial of degree $k \geq \tau+2$. If $f(1)/f_0 < R(m,n,\tau+1)$ then $f_{\tau+1} = 0$ in the Jacobi expansion $f(t) = \sum_{i=0}^{k} f_i P_i^{(\alpha,\beta)}(t)$.*

Proof. Apply Theorem 3.1 to the polynomial $f_{m,n,\tau+1}(t)$ (with $j = \tau+1 = \deg(f_{m,n,\tau+1})$).

□

Corollary 3.3 *No extremal polynomials of degree* $k = \tau + 1$ *will satisfy* $f(1)/f_0 < R(m,n,\tau+1)$.

□

We conclude that when, looking for improvements of (3.3) which are not greater than $R(m,n,\tau+1)$, one can assume $f_{\tau+1} = 0$ without loss of generality.

Another important problem, which concerns the form of extremal polynomials, deals with their number of double zeros. Polynomials $f_{m,n,\tau}(t)$ have $[\tau/2]$ double zeros. The following theorem shows that the higher degree extremal polynomials must have one double zero more for τ odd. It is the projective analog of Theorem 2.5.

Theorem 3.4 *Let* $f(t)$ *be a* $B_{m,n,\tau}$*-extremal polynomial of degree* $k \geq \tau + 2$.

a) *If* τ *is odd then* $f(t)$ *has at least* $(\tau+1)/2$ *double zeros in the interval* $[-1,1]$.

b) *If* τ *is even then* $f(t)$ *has at least* $\tau/2$ *double zeros in the interval* $[-1,1]$.

Proof. We can write $f(t) = A^2(t)G(t)$, where $G(t) \geq 0$ for $-1 \leq t \leq 1$, $A(t)$ has $\deg(A)$ zeros in $[-1,1]$, and $G(t)$ has no double zeros in $[-1,1]$. Thus $G(t) = 0$ is possible only for $t = -1$. We consider two cases.

Case 1. $G(-1) > 0$.

Suppose that $2\deg(A) \leq \tau - 1$ contradicting the assertion. Then, there exists $\varepsilon > 0$ such that $G(t) \geq \varepsilon > 0$ for all $t \in [-1,1]$. We consider the nonzero polynomial

$$R_\varepsilon(t) = f(t) - \varepsilon A^2(t) = A^2(t)(G(t) - \varepsilon).$$

Then $R_\varepsilon(t) \geq 0$ for $-1 \leq t \leq 1$ so condition (B1) is satisfied by $R_\varepsilon(t)$). This also implies $f_0(R_\varepsilon) > 0$ (see Remark 1.15).

Write $f(t) = \sum_{i=0}^{k} f_i P_i^{(\alpha,\beta)}(t)$ and

$$\begin{aligned} R_\varepsilon(t) &= \sum_{i=0}^{k} f_i(R_\varepsilon) P_i^{(\alpha,\beta)}(t) \\ &= \sum_{i=0}^{k} f_i P_i^{(\alpha,\beta)}(t) - \varepsilon \sum_{i=0}^{2deg(A)} f_i(A^2) P_i^{(\alpha,\beta)}(t). \end{aligned}$$

It follows that we have $f_i(R_\varepsilon) = f_i$ for $i \geq 2\deg(A)+1$. In particular, $f_i(R_\varepsilon) = f_i \leq 0$ for $i \geq \tau+1$ (since $\tau \geq 2\deg(A)+1$). Therefore $R_\varepsilon(t) \in B_{m,n,\tau}$.

Since $A^2(t) \in B_{m,n,\tau}$ and $\deg(A^2) \leq \tau - 1$ we have $A^2(1)/f_0(A^2) < R(m,n,\tau) < f(1)/f_0$. But one can easily check that this inequality is equivalent to

$$\frac{R_\varepsilon(1)}{f_0(R_\varepsilon)} = \frac{f(1) - \varepsilon A^2(1)}{f_0 - \varepsilon f_0(A^2)} > \frac{f(1)}{f_0},$$

a contradiction.

Case 2. $G(-1) = 0$.

We set $f(t) = A^2(t)(t+1)G_1(t)$, where $G_1(t) > 0$ for $-1 \leq t \leq 1$. If τ is odd and $2\deg(A) \leq \tau - 1$, then we apply the same argument as in Case 1 with $A^2(t)(t+1) \in B_{m,n,\tau}$ instead of $A^2(t)$. If τ is even, then the weaker assumption $2\deg(A) \leq \tau - 2$ ensures a contradiction by our argument.

□

For even τ, Theorem 3.4 gives no more double zeros. However, in this case a useful consequence for degree $\tau + 2$ follows. It is the projective analog of Lemma 2.6.

Corollary 3.5 *If τ is even and $f(t)$ is a $B_{m,n,\tau}$-extremal polynomial of degree $\tau+2$, then $f(-1) = 0$.*

Proof. By Theorem 3.4 the polynomial $f(t)$ has at least $\tau/2$ double zeros. In fact, their number must be exactly $\tau/2$, otherwise $f(t)$ would be a square and its leading Jacobi coefficient would be positive (see Remark 1.6), contradicting (B2) in Theorem 1.13. Therefore $f(t) = A^2(t)G(t)$ where $G(t)$ is a second degree polynomial. Obviously, $G(t)$ could vanish in $[-1,1]$ only for $t = -1$. However, the assumption $G(-1) > 0$ leads to a contradiction as in the proof of Theorem 3.4. Therefore $G(-1) = 0$.

□

We now search for good polynomials of degree $\tau + 2$. Theorem 3.4 and Corollary 3.5 determine the form of the extremal polynomials of degree $\tau + 2$, i.e. we must take

$$\begin{aligned} f(t) &= A^2(t)G(t) \\ &= (t^p + a_1 t^{p-1} + \cdots + a_{p-1}t + a_p)^2 G(t) \end{aligned}$$

where $\deg(A) = p = [(\tau+1)/2]$, and

$$G(t) = \begin{cases} q(t+1) + 1 - t & \text{if } \tau \text{ is odd,} \\ [q(t+1) + 1 - t](t+1) & \text{if } \tau \text{ is even.} \end{cases}$$

Here $0 < q < 1$ ensures $f(t) \geq 0$ for $-1 \leq t \leq 1$ and $f_{\tau+2} < 0$ simultaneously. From the condition $f_{\tau+1} = 0$ (see Corollary 3.2) it follows that we can express coefficient $a_1 = a_1(q,n)$ of $A(t)$ as a function of the parameter q and the dimension n.

We can now consider $F = f(1)/f_0$ as a function of the unknown coefficients $a_2, \ldots, a_p$, q, and n. Just as in (2.7) one can use the equations obtained by equating the partial derivatives of F to zero, to express the coefficients $a_2, \ldots, a_p$ as functions of q and n. The denominator f_0 is given by the formula (3.2) as function of $a_1, a_2, \ldots, a_p, q$, and n.

It does not seem possible to use further analytical methods. So we have searched for $q \in (0,1)$ in order to maximize the ratio $f(1)/f_0$ by means of a computer using simple numerical method. It turns out that usually, new bounds can be found in some range $n_1(t) \leq n \leq n_2(t)$.

3.4 Some Examples of New Bounds

3.4.1 Bounds in Complex Projective Space

In complex projective space we can improve the classical bound (3.3) for $\tau = 5$, 6 and 7. The parameters α and β for $\mathbb{C}\ P^{n-1}$ are respectively $\alpha = n-2$ and $\beta = 0$ by (1.6).

Case 1. $\tau = 5$.

As follows from the previous section, we have to work with polynomials of degree 7 having three double zeros:

$$\begin{aligned} f(t) &= (t^3 + at^2 + bt + c)^2[q(t+1) + 1 - t] \\ &= \sum_{i=0}^{7} f_i P_i^{(n-2,0)}(t). \end{aligned}$$

Using the equation $f_6 = 0$ we express (see Remark 1.6)

$$a = \frac{1}{2}\left(\frac{a_{7,6}}{a_{7,7}} + \frac{1+q}{1-q}\right),$$

where $P_7^{(n-2,0)}(t) = a_{7,7}t^7 + a_{7,6}t^6 + \cdots + a_{7,0}$.

Next, we examine the function

$$F(a,b,q,n) = \frac{f(1)}{f_0} = \frac{2q(1+a+b+c)^2}{f_0(a,b,q,n)}$$

in the usual way. The equalities

$$F'_b(b,c,q,n) = F'_c(b,c,q,n) = 0$$

give (after some simplifications) a system of two equations which are linear with respect to b and c. Therefore, we can express these parameters as functions of q and n.

Finally, for fixed dimension n, we search for $q \in (0,1)$ maximizing the function F by means of a computer. The new bounds we have obtained are in dimensions $3 \leq n \leq 11$. They are presented in Table 3.1.

Case 2. $\tau = 6$.

In this case we use polynomials of degree 8 of the form

$$\begin{aligned} f(t) &= (t^3 + at^2 + bt + c)^2[q(t+1) + 1 - t](t+1) \\ &= \sum_{i=0}^{8} f_i P_i^{(n-2,0)}(t). \end{aligned}$$

where $f_7 = 0$ and $0 < q < 1$. By the equation $f_7 = 0$ we obtain

$$a = \frac{1}{2}\left(\frac{a_{8,7}}{a_{8,8}} + \frac{q}{1-q}\right),$$

where $P_8^{(n-2,0)}(t) = a_{8,8)}t^8 + a_{8,7}t^7 + \cdots + a_{8,0}$.

Next, we use the partial derivatives of $F(q, b, c)$ to express the parameters b and c. Finally, we fix n and search for $q \in (0,1)$ in order to maximize the ratio $f(1)/f_0$. We obtain improvements of the Delsarte bound in dimensions $3 \leq n \leq 16$. The results are listed in Table 3.2.

Case 3. $\tau = 7$.

Using polynomials of degree 9, with $f_8 = 0$ and the same arguments, as above, we obtain new lower bounds in dimensions $3 \leq n \leq 23$. They are presented in Table 3.3.

n	$R(1, n, 5)$	New bounds
3	60	63
4	200	218
5	525	591
6	1176	1350
7	2352	2720
8	4320	4966
9	7425	8380
10	12100	13252
11	18876	19848

Table 3.1: New lower bounds on $B(1, n, 5), 3 \leq n \leq 11$.

n	R(1,n,6)	New bounds
3	100	102
4	400	427
5	1225	1365
6	3136	3619
7	7056	8362
8	14400	17363
9	27225	33101
10	48400	58818
11	81796	98515
12	132496	156855
13	207025	238990
14	313600	350306
15	462400	496134
16	665856	681473

Table 3.2: New lower bounds on $B(1, n, 6)$,$3 \leq n \leq 16$

n	R(1,n,7)	New bounds
3	150	158
4	700	778
5	2450	2885
6	7056	8749
7	17640	22858
8	39600	53193
9	81675	112845
10	157300	221884
11	286286	409391
12	496860	715479
13	828100	1193093
14	1332800	1909363
15	2080800	2946241
16	3162816	4400228
17	4694805	6381021
18	6822900	9009106
19	9728950	12412451
20	13636700	16722740
21	18818646	22071790
22	25603600	28588951
23	34385000	36400275

Table 3.3: New lower bounds on $B(1, n, 7)$, $3 \leq n \leq 23$

3.4.2 Bounds in Quaternionic Projective Space

We have applied our method for $\tau = 5$, 6 and 7 to the quaternionic projective space. New lower bounds were found in dimensions $3 \le n \le 5$ for $\tau = 5$, $4 \le n \le 6$ for $\tau = 6$, and $3 \le n \le 11$ for $\tau = 7$. The corresponding results are given in Tables 3.4 and 3.5.

n	$R(2, n, 5)$	New bound
3	210	221
4	840	903
5	2475	2613

Table 3.4: New lower bounds on $B(2, n, 5)$, $3 \le n \le 5$.

n	$R(2, n, 6)$	New bound
4	2520	2562
5	9075	9359
6	26026	26474

Table 3.5: New lower bounds on $B(2, n, 6)$, $4 \le n \le 6$.

n	$R(2, n, 7)$	New bound
3	882	962
4	5544	6571
5	23595	29671
6	78078	101403
7	216580	283258
8	527136	678399
9	1159893	1438197
10	2355430	2760958
11	4480630	4883694

Table 3.6: New lower bounds on $B(2, n, 7)$, $3 \le n \le 11$.

3.4.3 Bounds in the Cayley plane

Projective spaces over the non-associative algebra of the Cayley octonions exist only for $n = 2$ and $n = 3$ (the so-called Cayley line and plane). An appropriate model for $\mathbb{O}P^{n-1}$ can be found in [45, Section 1]. In this case we obtained only one new bound, $D(7, 3, \mathbb{O}) \ge 7060$ instead of $D(7, 3, \mathbb{O}) \ge 6435$ by (3).

3.4.4 Bounds in Real Projective Space

In this case, bound (3.3) can be written as

$$B(\frac{1}{2}, n, \tau) \geq R(\frac{1}{2}, n, \tau) = \binom{n+\tau-1}{n-1}.$$

In the real projective space we use a modification of the method we have applied in the Chapter 2 for estimating the size of spherical designs. Indeed, there is a one-to-one correspondence between the τ-designs in $\mathbb{R}\,P^{n-1}$ and the antipodal spherical $(2\tau+1)$-designs on $\mathbf{S}^{n-1}$ [53, Theorem 9.2]. Therefore we can search for linear programming bounds for spherical designs as in Chapter 2 paying attention to the antipodality. This is expressed by the following theorem, which is, in some sense, a specialization of Theorem 1.21.

Theorem 3.6 *Let $f(t)$ be a nonzero real polynomial such that*

(B1) $f(t) \geq 0$ *for* $-1 \leq t \leq 1$,

(B2′) *the coefficients in the Gegenbauer expansion* $f(t) = \sum_{i=0}^{k} f_i Q_i^{(n)}(t)$ *satisfy* $f_{2j} \leq 0$ *for* $2\tau+1 < 2j \leq k = \deg(f)$.

Then, $B(1/2, n, \tau) \geq f(1)/2f_0$.

Proof. If $C \subset \mathbb{R}\,P^{n-1}$ is a τ-design, then its realization C' on $\mathbf{S}^{n-1}$ is an antipodal spherical $(2\tau+1)$-design. We have $|C'| = 2|C|$ and $|C'| \geq f(1)/f_0$ by the linear programming bound (Theorem 1.21) for antipodal spherical designs.

□

We now can apply the method from the previous chapter for obtaining linear programming bounds for spherical $(2\tau+1)$-designs with polynomials of degree $2\tau+4$ without consideration of the coefficient $f_{2\tau+3}$ (it can be arbitrarily chosen by Theorem 1.21).

We considered the cases $\tau = 3$, 4, and 5. The new bounds we have obtained are in dimensions $4 \leq n \leq 7$ for $\tau = 3$, $4 \leq n \leq 11$ for $\tau = 4$, and $3 \leq n \leq 23$ for $\tau = 5$. Examples are presented in Tables 3.6, 3.7 and 3.8.

Reznick [67, Section 4] has shown the nonexistence of antipodal spherical 5-designs with $n(n+1)+2$ points. This implies that $B(1/2, n, 2) \geq R(1/2, n, 2) + 2 = n(n+1)/2 + 2$ for $n \geq 3$. We are not aware of other improvements of (3.3) by more than one.

n	R(1/2,n,3)	New bound
5	35	37
6	56	58
7	84	86

Table 3.7: New lower bounds on $B(1/2, n, 3)$,$5 \leq n \leq 7$.

n	R(1/2,n,4)	New bound
4	35	37
5	70	75
6	126	137
7	210	231
8	330	365
9	445	549
10	715	789
11	1001	1094
12	1365	1470
13	1820	1922
14	2380	2451

Table 3.8: New lower bounds on $B(1/2, n, 4)$, $4 \leq n \leq 14$.

n	R(1/2,n,5)	New bound
4	56	59
5	126	136
6	252	279
7	462	524
8	792	918
9	1287	1517
10	2002	2393
11	3003	3624
12	4368	5302
13	6188	7528
14	8568	10409
15	11628	14056
16	15504	18583
17	20349	24104
18	26334	30724
19	33649	38545
20	42504	47655
21	53130	58130
22	65780	70031
23	80730	83404

Table 3.9: New lower bounds on $B(1/2, n, 5)$, $3 \leq n \leq 23$.

Chapter 4

Necessary and sufficient conditions for optimality of the Delsarte bound

4.1 Introduction

In this chapter are summarized the last investigations which were done during the last months of the research period. The results are rather new and this work is still in process. We give necessary and sufficient conditions for improving the Delsarte bound for τ-designs. We define test functions $G_\tau(\mathcal{M}, Q_j)$ with the property that $G_\tau(\mathcal{M}, Q_j) < 0$ for some $j > \tau$ if and only if the Delsarte bound $B(\mathcal{M}, \tau) \geq R(\mathcal{M}, \tau)$ can be improved by linear programming. Then we investigate when the Delsarte bound is optimal. If it is not optimal we obtain some polynomials of degree $\tau + 2$ in non-antipodal PMS and of degree $\tau + 3$ in antipodal PMS, which improve the Delsarte bound. This chapter is based on [64] and [65].

4.2 Preliminaries

First we will recall some definitions and notations. Let $\mathcal{M}$ be a polynomial metric space with metric $d(x, y)$ and standard substitution $\sigma(d(x, y))$. Any finite nonempty subset C of $\mathcal{M}$ is called a code. A code for which $\sigma(d(x, y)) \leq \sigma(d)$, where d is the minimum distance of C we call an $(\mathcal{M}, |C|, \sigma)$-**code**.

For each a and $b \in I\!\!N$, one can associate the ZSF with their *adjacent systems* of orthogonal polynomials $\{Q_i^{a,b}(t)\}_{i=0}^{\infty}$. These polynomials are orthogonal with respect to the measure $\nu^{a,b}(t)$ defined by $d\nu^{a,b}(t) = c^{a,b}(1-t)^a(1+t)^b \, d\nu(t)$ ($c^{a,b}$ is a constant).

The most important properties of these systems of orthogonal polynomials and the corresponding adjacent systems were given in Section 1.3. In this section we present without proofs some additional properties of the systems $Q_i^{a,b}(t)$ [1, 53, 56, 73].

Denote by $t_{k,i}^{a,b}$, $i = 1, \ldots, k$, the roots of polynomial $Q_k^{a,b}(t)$, $k \geq 0$, ordered in increasing order and by $t_k^{a,b}$ the greatest zero $t_{k,k}^{a,b}$ of the polynomial $Q_k^{a,b}(t)$. Note that by Lemma 1.8 and the normalization $Q_k^{a,b}(1) = 1$ the leading coefficient $a_{k,k}^{a,b}$ of polynomials $Q_k^{a,b}(t)$ is positive and $sgnQ_k^{a,b}(-1) = (-1)^k$ for $k \geq 0$. Note also that $Q_0^{a,b}(t) \equiv 1$ and $r_0^{a,b} = 1$. We introduce the notation

$$Q_k^{a,b}(t) = \sum_{i=0}^{k} a_{k,i}^{a,b} t^i.$$

When $a = b = 0$ we omit the upper indices.

Lemma 4.1 *(Christoffel-Darboux formulae) For any integer k, and reals x and y we have*

$$\sum_{i=0}^{k} r_i Q_i(x) Q_i(y) = \begin{cases} r_k m_k \dfrac{Q_{k+1}(x)Q_k(y) - Q_k(x)Q_{k+1}(y)}{x-y} & \text{if } x \neq y, \\ r_k m_k (Q'_{k+1}(x)Q_k(x) - Q'_k(x)Q_{k+1}(x)) & \text{if } x = y. \end{cases}$$

Corollary 4.2 *(monotonicity) For any k the ratio $Q_{k+1}(t)/Q_k(t)$ increases with t in every interval which does not contain zero(s) of the denominator.*

Corollary 4.3 *(separation of roots) For any k and j, if $1 \leq j \leq k$ $t_{k+1,j} < t_{k,j} < t_{k+1,j+1}$.*

Corollary 4.4 *Let $t_{k-1} < t < t_k$, with $k \geq 1$. Then $Q_k(t) < 0$ and $Q_i(t) > 0$ for any i, $i = 0, 1, \ldots, k-1$. If $t \geq t_k$, then $Q_k(t) \geq 0$.*

Let us consider the symmetric function

$$T_k(x,y) = \sum_{i=0}^{k} r_i Q_i(x) Q_i(y). \tag{4.1}$$

Introduce in addition the function

$$R_k(x,y,z) = T_{k-1}(x,y)Q_k(z) - T_{k-1}(y,z)Q_k(x)$$

for $k \geq 1$.

The next theorem gives a connection between the parameters of the adjacent system of orthogonal polynomials and the original system when $a, b \in \{0,1\}$ (which are in fact all cases we need to consider).

Theorem 4.5 *[53] For any nonnegative integer k*

$$\begin{aligned} Q_k^{0,1}(t) &= \frac{T_k(t,-1)}{T_k(1,-1)}, & r_k^{0,1} &= \frac{(T_k(1,-1))^2}{-c^{0,1} r_k m_k Q_k(-1) Q_{k+1}(-1)}; \\ Q_k^{1,0}(t) &= \frac{T_k(t,1)}{T_k(1,1)}, & r_k^{1,0} &= \frac{(T_k(1,1))^2}{c^{1,0} r_k m_k Q_k(1) Q_{k+1}(1)}; \\ Q_k^{1,1}(t) &= \frac{R_{k+1}(-1,t,1)}{R_{k+1}(-1,1,1)}, & r_k^{1,1} &= \frac{r_{k+1}(R_{k+1}(-1,1,1))^2}{-4c^{1,1} T_k(1,-1) T_{k+1}(1,-1)} \end{aligned}$$

or more general,

$$T_k^{a,b}(x,y) = \sum_{i=0}^{k} r_i^{a,b} Q_i^{a,b}(x) Q_i^{a,b}(y)$$

$$Q_k^{a,b+1}(t) = \frac{T_k^{a,b}(t,-1)}{T_k^{a,b}(1,-1)}, \quad Q_k^{a+1,b}(t) = \frac{T_k^{a,b}(t,1)}{T_k^{a,b}(1,1)}$$

The next lemmas concern the separation of adjacent polynomials roots [53].

Lemma 4.6 *For any j and k, $1 \le j \le k+1$*

$$t_{k,j-1} < t_{k,j-1}^{0,1} < t_{k-1,j-1}^{1,1} < t_{k,j}^{1,0} < t_{k,j},$$

in all cases when the corresponding entries are defined.

Lemma 4.7 *For any j and k, $1 \le j \le k-1$*

$$t_{k,j} < t_{k-1,j}^{1,0} < t_{k-1,j}^{1,1} < t_{k-1,j}^{0,1} < t_{k,j+1}.$$

Lemma 4.8 *For any integer $k \ge 1$*

$$t_{k-1}^{1,1} < t_k^{1,0} < t_k^{1,1}, (t_0^{1,1} = -\infty).$$

Lemma 4.9 *The functions*

$$\frac{Q_k(t)}{Q_k^{1,0}(t)}, \quad \frac{Q_k^{0,1}(t)}{Q_k(t)}, \quad \frac{Q_k^{0,1}(t)}{Q_k^{1,0}(t)}$$

increase with t in every interval which does not contain zero(s) of the denominator.

Lemma 4.10 *The functions*

$$\frac{Q_{k-1}^{0,1}(t)}{Q_k(t)}, \quad \frac{Q_{k-1}^{1,1}(t)}{Q_k(t)}, \quad \frac{Q_{k-1}^{1,0}(t)}{Q_k(t)}$$

decrease with t in every interval which does not contain zero(s) of the denominator.

The universal upper (resp. lower) bounds $L_{2k-1+\varepsilon}(\sigma)$ (resp. $R(\mathcal{M},\tau)$) for the cardinality of an $(\mathcal{M},|C|,\sigma)$-code (resp. a τ-design) can be presented in the following form [53, 36]:

$$|C| \le L_{2k-1+\varepsilon}(\sigma) = \left(1 - \frac{Q_{k-1+\varepsilon}^{1,0}(\sigma)}{Q_k^{0,\varepsilon}(\sigma)}\right) \sum_{i=0}^{k-1+\varepsilon} r_i, \tag{4.2}$$

where $\varepsilon = 0$ if $t_{k-1}^{1,1} \leq \sigma < t_k^{1,0}$ and $\varepsilon = 1$ if $t_k^{1,0} \leq \sigma < t_k^{1,1}$, resp.

$$|C| \geq R(\mathcal{M}, \tau) = 2^\theta c^{0,\theta} \sum_{i=0}^{k} r_i^{0,\theta}, \tag{4.3}$$

where $\theta \in \{0,1\}$ and $\tau = 2k + \theta$. Bound (4.2) can be obtained by using the polynomial

$$f^{(\sigma)}(t) = (t-\sigma)(t+1)^\varepsilon (T_{k-1}^{1,\varepsilon}(t,\sigma))^2,$$

and bound (4.3) can be obtained by using the polynomial

$$f^{(\tau)}(t) = (t+1)^\theta ((Q_k^{1,\theta}(t))^2$$

in Theorem 1.12 (resp. Theorem 1.13).

4.3 Test functions

First we will give a modification of the Gauss-Jacobi formula due to Levenshtein [56] .

Theorem 4.11 *For any σ, $-1 \leq \sigma < 1$, and $\epsilon \in \{0,1\}$, the polynomial $(t-\sigma)(t+1)^\epsilon T_{k-1}^{1,\epsilon}(t,\sigma)$ has $k+\epsilon$ simple roots $\beta_0 < \beta_1 < \cdots < \beta_{k+\epsilon-1}$, where $\beta_{k+\epsilon-1} = \sigma$ and $\beta_0 \geq -1$. Moreover $\beta_0 = -1$ if and only if $\epsilon = 1$ or $\epsilon = 0$ and $\sigma = t_{k-1}^{1,1}$. Further, for any polynomial $f(t)$ of degree $2k-1+\epsilon$*

$$f_0 = \frac{f(1)}{L_{2k-1+\epsilon}(\sigma)} + \sum_{j=1}^{k+\epsilon-1} \rho_j^{(\sigma)} f(\beta_j),$$

where coefficients $\rho_j^{(\sigma)}$ are positive for $j \geq 1$ and $\rho_0^{(\sigma)} \geq 0$ with equality if and only if $\sigma = t_k^{1,0}$.

Using Theorem 4.11 Boyvalenkov, Danev and Bumova [18] (see also [19]) obtained necessary and sufficient conditions for the optimality of $f^{(\sigma)}(t)$ over $A_{\mathcal{M},\sigma}$ [1] without restriction on their degree. To describe this result they introduced the following linear functional

$$G_\sigma(\mathcal{M}, f) = \frac{f(1)}{L_{2k-1+\epsilon}(\sigma)} + \sum_{i=0}^{k+\epsilon-1} \rho_i^{(\sigma)} f(\beta_i) \tag{4.4}$$

The next theorem is similar to Theorem 4.11 for τ-designs.

Theorem 4.12 *Let the zeros of $Q_k^{1,\theta}(t)$ be denoted by α_i, $1 \leq k$ and let $\alpha_{k+\theta} = -1$ for $\theta = 1$. Then for any polynomial $f(t)$ of degree at most $\tau = 2k + \theta$ the following equality holds*

$$f_0 = \frac{f(1)}{R(\mathcal{M},\tau)} + \sum_{i=0}^{k+\theta} \rho_i^{(\tau)} f(\alpha_i), \tag{4.5}$$

where $\rho_i^{(\tau)}$ are positive numbers s.t. $\sum_{j=0}^{k+\theta} \rho_j^{(\tau)} = 1$.

[1] $A_{\mathcal{M},\sigma}$ is the set of real polynomials which satisfy the conditions **(A1)** and **(A2)** of the Theorem 1.12

Proof. Let $g(t) = (t-1)(t+1)^\theta Q_k^{1,\theta}(t)$ with $k+1+\theta$ simple roots:

$$-1 < \alpha_1 < \alpha_2 < \ldots < \alpha_k < 1, \ \alpha_0 = 1, \ \text{ if } \ \theta = 1 \ \alpha_{k+\theta} = -1.$$

For $g(t)$ we consider the Lagrange polynomials $l_j(g;t)$, $j = 0, 1, \ldots, k+\theta$. They are polynomials of degree $k+\theta$ with the property that have a value 0 for $t = \alpha_i$, $\forall i$ except for $t = \alpha_j$ when $l_j(g;\alpha_j) = 1$, i.e. $l_j(g;\alpha_i) = \delta_{i,j}$ ($\delta_{i,j}$ is the Kronecker symbol). We have

$$\begin{aligned} l_{k+\theta}(g;t) &= \frac{(t-1)Q_k^{1,\theta}(t)}{-2Q_k^{1,\theta}(-1)} \quad for \quad \theta = 1; \\ l_j(g;t) &= \frac{(t+1)^\theta(t-1)T_{k-1}^{1,\theta}(t;\alpha_j)}{(\alpha_j+1)^\theta(\alpha_j-1)T_{k-1}^{1,\theta}(\alpha_j,\alpha_j)} \quad for \quad j = 1, \ldots, k; \\ l_0(g;t) &= \frac{(t+1)^\theta Q_k^{1,\theta}(t)}{2^\theta Q_k^{1,\theta}(1)} \quad for \quad j = 0; \end{aligned}$$

For any polynomial $f(t)$ of degree at most τ the polynomial

$$f(t) - \sum_{j=0}^{k+\theta} f(\alpha_j) l_j(g;t) \tag{4.6}$$

equals zero at all (simple) roots of $g(t)$ and hence it can be represented as

$$f(t) - \sum_{j=0}^{k+\theta} f(\alpha_j) l_j(g;t) = g(t)h(t), \tag{4.7}$$

where $h(t)$ is a polynomial of degree at most $k-1$. Using the orthogonality relation for the system of orthogonal polynomials $\{Q_k^{1,\theta}(t)\}$ we can calculate

$$\int_{-1}^{1} g(t)h(t)\, d\nu(t) = -\frac{1}{c^{1,\theta}} \int_{-1}^{1} h(t) Q_k^{1,\theta}(t)\, d\nu^{1,\theta}(t) = 0$$

This gives:

$$f_0 = \int_{-1}^{1} f(t)\, d\nu(t) = \sum_{j=0}^{k+\theta} f(\alpha_j). \int_{-1}^{1} l_j(g;t)\, d\nu(t) = \sum_{j=0}^{k+\theta} \rho_j.f(\alpha_j),$$

It remains to prove the two properties of ρ_j. That they are positive follows from the following three arguments:

$$\begin{aligned} \rho_{k+\theta} &= \int_{-1}^{1} l_{k+\theta}(g;t)\, d\nu(t) = \int_{-1}^{1} \frac{(t-1)Q_k^{1,\theta}(t)}{-2Q_k^{1,\theta}(-1)}\, d\nu(t) \\ &= \frac{1}{2c^{1,0}Q_k^{1,1}(-1)} \int_{-1}^{1} \frac{T_k^{1,0}(t,-1)}{T_k^{1,0}(1,-1)}\, d\nu^{1,0}(t) = \frac{1}{2c^{1,0}T_k^{1,0}(-1,-1)} > 0 \end{aligned}$$

Analogously, we have

$$\rho_j = \int_{-1}^{1} l_j(g;t)\ d\nu(t) = \frac{1}{c^{1,\theta}(\alpha_j+1)^\theta(1-\alpha_j)T_{k-1}^{1,\theta}(\alpha_j,\alpha_j)} > 0$$

and

$$\rho_0 = \int_{-1}^{1} l_0(g;t)\ d\nu(t) = \frac{1}{2^\theta c^{0,\theta} T_k^{0,\theta}(1,1)}.$$

Since

$$2^\theta c^{0,\theta} T_k^{0,\theta}(1,1) = 2^\theta c^{0,\theta} \sum_{i=0}^{k} r_i^{0,\theta} = R(\mathcal{M},\tau).$$

It follows that

$$\rho_0 = \frac{1}{R(\mathcal{M},\tau)} > 0.$$

Finally, from $\rho_j = \int_{-1}^{1} l_j(g;t)\ d\nu(t)$ we conclude that

$$\sum_{j=0}^{k+\theta} \rho_j = \int_{-1}^{1} \sum_{j=0}^{k+\theta} l_j(g;t)\ d\nu(t) = \int_{-1}^{1} 1\ d\nu(t) = 1$$

(we use that $\sum_{i=0}^{k+\theta} l_i(g;\alpha_j) = 1$ for every α_j whence $\sum_{i=0}^{k+1} l_i(g;t) \equiv 1$).

□

Theorem 4.13 *[56, 69] For any $\tau = 2k+\theta$, $\theta \in \{0,1\}$,*

$$R(\mathcal{M},\tau) = \max \Omega(f), \tag{4.8}$$

where the maximum is taken over the class of polynomials $f(t) \in B_{\mathcal{M},\tau}$ of degree at most τ. The maximum in (4.8) is realized if and only if $f(t)$ is proportional to $f^\tau(t)$.

Proof. Theorem 4.12 states that

$$f(1) = R(\mathcal{M},\tau) f_0 - R(\mathcal{M},\tau) \sum_{j=1}^{k+\theta} \rho_j f(\alpha_j)$$

Hence

$$\Omega(f) = \frac{f(1)}{f_0} = R(\mathcal{M},\tau) - \frac{R(\mathcal{M},\tau)}{f_0} \sum_{j=1}^{k+\theta} \rho_j f(\alpha_j) \le R(\mathcal{M},\tau),$$

where equality holds if and only if $f(\alpha_j) = 0$ for $j = 1,\ldots,k+\theta$. Therefore $f(t)$ is divisible by $(t+1)^\theta Q_k^{1,\theta}(t)$. $f(t)$ must be positive in $[-1,1]$ hence $f(t) = const. f^\tau(t)$.

□

We present a second proof which illustrates in a good way our approach in the next section.

Alternative Proof. It follows from Lukács Theorem [73, p.4] that every nonnegative polynomial of degree τ in $[-1,1]$ can be represented in the following way:

$$f(t) = \begin{cases} (A_k(t))^2 + (B_{k-1}(t))^2(1-t^2) & \text{if } \tau = 2k \\ (A_k(t))^2(t+1) + (1-t)(B_k(t))^2 & \text{if } \tau = 2k+1, \end{cases} \tag{4.9}$$

where $\deg(A_k) = deg(B_k) = k$.

We now need to distinguish between the case that τ is even or odd.

Case 1. $\tau = 2k$. Putting

$$A_0 = \int_{-1}^{1} (A_k(t))^2 \, d\nu(t)$$

$$B_0 = \int_{-1}^{1} (B_{k-1}(t))^2(1-t^2) \, d\nu(t).$$

we obtain

$$\Omega(f) = \frac{(A_k(1))^2}{A_0 + B_0} \leq \frac{(A_k(1))^2}{A_0} = \Omega((A_k(t))^2).$$

Case 2. $\tau = 2k+1$. Putting

$$A_0 = \int_{-1}^{1} (A_k(t))^2(1+t) \, d\nu(t)$$

$$B_0 = \int_{-1}^{1} (B_k(t))^2(1-t) \, d\nu(t).$$

we get

$$\Omega(f) = \frac{2(A_k(1))^2}{A_0 + B_0} \leq \frac{2(A_k(1))^2}{A_0} = \Omega((A_k(t))^2(t+1)).$$

Hence we have to consider polynomials of the following form:

$$f(t) = \begin{cases} (A_k(t))^2 & \text{if } \tau = 2k \\ (A_k(t))^2(t+1) & \text{if } \tau = 2k+1 \end{cases} \tag{4.10}$$

i.e. $f(t) = (t+1)^\theta (A_k(t))^2$.

Let $A_k(t) = \sum_{i=0}^{k} f_i^{0,\theta} Q_i^{0,\theta}(t)$ (the expansion of $A_k(t)$ in terms of the adjacent system of polynomials). Then $A_k(1) = \sum_{i=0}^{k} f_i^{0,\theta}$ and

$$\begin{aligned} f_0 &= \int_{-1}^{1} (A_k(t))^2(1+t)^\theta \, d\nu(t) = \frac{1}{c^{0,\theta}} \int_{-1}^{1} (A_k(t))^2 \, d\nu^{0,\theta}(t) \\ &= \frac{1}{c^{0,\theta}} \int_{-1}^{1} (\sum_{i=0}^{k} f_i^{0,\theta} Q_i^{0,\theta}(t))^2 \, d\nu^{0,\theta}(t) \end{aligned}$$

$$= \frac{1}{c^{0,\theta}} \int_{-1}^{1} \sum_{i,j=0}^{k} f_i^{0,\theta} f_j^{0,\theta} Q_i^{0,\theta}(t) Q_j^{0,\theta}(t) \; d\nu^{0,\theta}(t)$$

$$= \frac{1}{c^{0,\theta}} \int_{-1}^{1} \sum_{i=0}^{k} (f_i^{0,\theta})^2 (Q_i^{0,\theta}(t))^2 \; d\nu^{0,\theta}(t)$$

$$= \frac{1}{c^{0,\theta}} \sum_{i=0}^{k} \frac{(f_i^{0,\theta})^2}{r_i^{0,\theta}}. \tag{4.11}$$

Hence

$$\Omega(f) = 2^\theta c^{0,\theta} \frac{(\sum_{i=0}^{k} f_i^{0,\theta})^2}{\sum_{i=0}^{k} \frac{(f_i^{0,\theta})^2}{r_i^{0,\theta}}}.$$

Now we use the Cauchy inequality

$$(\sum_{i=0}^{k} f_i)^2 \leq (\sum_{i=0}^{k} \frac{(f_i)^2}{r_i})(\sum_{i=0}^{k} r_i),$$

where equality holds if and only if f_i and r_i are proportional. Applying this to $\Omega(f)$ we obtain

$$\Omega(f) \leq 2^\theta c^{0,\theta} \sum_{i=0}^{k} r_i^{0,\theta} = R(\mathcal{M}, \tau).$$

Equality holds if and only if

$$A_k(t) = const. \sum_{i=0}^{k} r_i^{0,\theta} Q_i^{0,\theta}(t) = const. Q_k^{1,\theta}(t).$$

i.e. $f(t)$ is proportional to $(t+1)^\theta (Q_k^{1,\theta}(t))^2$.

□

Analogously to the definition (4.4) of $G_\sigma(\mathcal{M}, f)$ we consider the following linear functional

$$G_\tau(\mathcal{M}, f) = \frac{f(1)}{R(\mathcal{M}, \tau)} + \sum_{i=1}^{k+\theta} \rho_i^{(\tau)} f(\alpha_i) \tag{4.12}$$

where $\alpha_i, \rho_i^{(\tau)}$ are defined as in Theorem 4.12.

This linear functional maps the set of real polynomials to the set of real numbers. The reader should be careful with the notations $G_\sigma(\mathcal{M}, f)$ and $G_\tau(\mathcal{M}, f)$. The subscripts σ $(-1 \leq \sigma < 1)$ and τ ($\tau \geq 2$ - integer) refer to the case of a $(\mathcal{M}, |C|, \sigma)$-code and a τ-design, respectively. As we may expect the duality between the optimal choice of polynomials and resulting bound for codes and this polynomials with the resulting bound for designs can be extended to duality between the corresponding test functions for codes and designs. This duality follows from the fact that $G_\tau(\mathcal{M}, f)$ can be obtained from $G_\sigma(\mathcal{M}, f)$ by taking $\sigma = t_k^{1,\theta}$. By Theorem 4.12, we have $-1 \leq G_\tau(\mathcal{M}, f) \leq 1$ and $G_\tau(\mathcal{M}, f) = f_0$ for any polynomial $f(t)$ of degree at most τ. Also $G_\tau(\mathcal{M}, f) = f(1)/R(\mathcal{M}, \tau)$ if $f(t)$ vanishes at the zeros of $f^{(\tau)}(t)$.

Lemma 4.14 *Let* $f(t) = g(t)q(t) + r(t)$, *where* $g(t) = (t-1)(t+1)^\theta Q_k^{1,\theta}(t)$. *Then* $G_\tau(\mathcal{M}, f) = \int_{-1}^1 r(t)d\nu(t)$.

Proof. Using the definition of $G_\tau(\mathcal{M}, f)$ we have

$$G_\tau(\mathcal{M}, f) = \sum_{i=0}^{k+\theta} \rho_i^{(\tau)} f(\alpha_i) = \int_{-1}^{1} \sum_{j=0}^{k+\theta} l_j(g;t) f(\alpha_j) d\nu(t) = \int_{-1}^{1} r(t) d\nu(t).$$

The polynomial $r(t)$ is of degree $k+\theta$ and $r(\alpha_i) = f(\alpha_i)$ for $i = 0, \ldots, k+\theta$. This uniquely determines $r(t)$. It must be the remainder of the division of $f(t)$ by $g(t)$.

□

Now we will prove necessary and sufficient conditions for an improvement of the Delsarte bound. Later on we will investigate some properties of the test functions, which turns out to be very useful.

Theorem 4.15 *The bound* $R(\mathcal{M}, \tau)$ *can be improved by a polynomial* $f(t) \in B_{\mathcal{M},\tau}$ *of degree at least* $\tau + 1$, *if and only if* $G_\tau(\mathcal{M}, Q_j) < 0$ *for some* $j \geq \tau + 1$. *Moreover, if* $G_\tau(\mathcal{M}, Q_j) < 0$ *for some* $j \geq \tau + 1$, *then* $R(\mathcal{M}, \tau)$ *can be improved by a polynomial in* $B_{\mathcal{M},\tau}$ *of degree* j.

Proof. Suppose that $G_\tau(\mathcal{M}, Q_j) \geq 0$ for all $j \geq \tau+1$. Consider a polynomial $f(t) \in B_{\mathcal{M},\tau}$ of degree $m \geq \tau + 1$. We write

$$f(t) = \tilde{g}(t) + \sum_{i=\tau+1}^{m} f_i Q_i(t) = \tilde{g}(t) + F(t), \qquad (4.13)$$

where $\deg(\tilde{g}) \leq \tau$. Then Theorem 4.12 applied to $\tilde{g}$ and relations (4.12), (4.13) imply

$$\begin{aligned} f_0 = \tilde{g}_0 = G_\tau(\mathcal{M}, \tilde{g}) &= G_\tau(\mathcal{M}, f) - G_\tau(\mathcal{M}, F) \\ \geq \frac{f(1)}{R(\mathcal{M}, \tau)} - G_\tau(\mathcal{M}, F) &\geq \frac{f(1)}{R(\mathcal{M}, \tau)} \end{aligned}$$

Therefore $R(\mathcal{M}, \tau) \geq \frac{f(1)}{f_0}$ i.e. $f(t)$ does not improve the bound in (4.3).

Conversely, let $G_\tau(\mathcal{M}, Q_j) < 0$ for some fixed $j \geq \tau + 1$ and $-Q_j(t) = f^{(\tau)}(t)a(t) + b(t)$. Consider $f(t) = f^{(\tau)}(t)(a(t) + c) = -Q_j(t) + cf^{(\tau)}(t) - b(t)$, where $c = -\min\{a(t) : t \in [-1, 1]\}$. This choice of c ensures that $f_{\tau+1} = \ldots = f_{j-1} = 0$, $f_j = -1$ and $f(t) \geq 0$ in $[-1, 1]$. On the other hand we have $f_0 = G_\tau(\mathcal{M}, f) - G_\tau(-Q_j) < \frac{f(1)}{R(M,\tau)}$, i.e we have improved the Delsarte bound.

□

Let us recall that among the infinite PMS only the Euclidean spheres $\mathbf{S}^{n-1}$ are antipodal.

Theorem 4.16 *Let $\mathcal{M}$ be antipodal. If τ and j are odd, then $G_\tau(\mathcal{M}, Q_j) = 0$.*

Proof. We use the notations introduced in Theorem 4.12. The zeros of $Q_k^{(n+2)}(t) = Q_k^{1,1}(t)$ are symmetric with respect to the origin, i.e. $\alpha_j = -\alpha_{k+1-j}$, for $j = 1, \ldots, \lfloor k/2 \rfloor$ and for k odd $\alpha_{\frac{k+1}{2}} = 0$ (i.e. 0 is a root of $Q_k^{(n+2)}(t)$).

On the other hand by $Q_e^{(n+2)}(-t) = (-1)^e Q_e^{(n+2)}(t)$ and $T_{k-1}^{1,1}(-\alpha_j, -\alpha_j) = \sum_{e=0}^{k-1} r_e^{1,1}(Q_e^{(n+2)}(-\alpha_j))^2 = \sum_{e=0}^{k-1} r_e^{1,1}(Q_e^{(n+2)}(\alpha_j))^2 = T_{k-1}^{1,1}(\alpha_j, \alpha_j)$ we have

$$\begin{aligned}
\rho_j^{(\tau)} &= \frac{1}{c^{1,1}(1-\alpha_j^2)T_{k-1}^{1,1}(\alpha_j, \alpha_j)} \\
&= \frac{1}{c^{1,1}(1-\alpha_{k+1-j}^2)T_{k-1}^{1,1}(-\alpha_{k+1-j}, -\alpha_{k+1-j})} \\
&= \frac{1}{c^{1,1}(1-\alpha_{k+1-j}^2)T_{k-1}^{1,1}(\alpha_{k+1-j}, \alpha_{k+1-j})} = \rho_{k+1-j}^{(\tau)}.
\end{aligned}$$

Therefore

$$\sum_{j=1}^{k} \rho_j^{(\tau)} Q_e^{(n)}(\alpha_j) = 0.$$

For $\rho_0^{(\tau)}$ and $\rho_{k+1}^{(\tau)}$ we have

$$\rho_0^{(\tau)} = \frac{1}{R(\mathcal{M}, \tau)} = \rho_{k+1}^{(\tau)} \tag{4.14}$$

and $Q_e(-1) = (-1)^e$.

□

We recall that the ZSF for infinite PMS are Jacobi polynomials $P_i^{\alpha,\beta}(t)$. They are defined by the recurrence formula

$$(t + m_i + c_i - 1)Q_i(t) = m_i Q_{i+1}(t) + c_i Q_{i-1}(t), \tag{4.15}$$

for $i \geq 0$, where $r_{-1} = m_{-1} = 0$, $m_i = \frac{a_{i,i}}{a_{i+1,i+1}}$, $c_i = \frac{r_{i-1}m_{i-1}}{r_i}$ and $Q_{-1}(t) \equiv 0$, $Q_0(t) \equiv 1$. For antipodal spaces we have $\alpha = \beta = \frac{n-3}{2}$.

The parameters m_i and c_i can be written as follows:

$$\begin{aligned}
m_i &= \frac{2(i+\alpha+1)(i+\alpha+\beta+1)}{(2i+\alpha+\beta+1)(2i+\alpha+\beta+2)}, \\
c_i &= \frac{2i(i+\beta)}{(2i+\alpha+\beta)(2i+\alpha+\beta+1)}
\end{aligned} \tag{4.16}$$

Let us introduce

$$n_i^{a,b} = \frac{a_{i,i-1}^{a,b}}{a_{i,i}^{a,b}}, \quad \tilde{n}_i^{a,b} = \frac{a_{i,i-2}^{a,b}}{a_{i,i}^{a,b}}. \tag{4.17}$$

Lemma 4.17 *With the notations introduced above we have*

a)

$$G_\tau(\mathcal{M}, Q_{\tau+1}) = \frac{1}{c^{1,\theta} r_k^{1,\theta}} \frac{a_{\tau+1,\tau+1}}{(a_{k,k}^{1,\theta})^2}$$

b)

$$G_\tau(\mathcal{M}, Q_{\tau+2}) = \frac{1}{c^{1,\theta} r_k^{1,\theta}} \frac{a_{\tau+2,\tau+2}}{(a_{k,k}^{1,\theta})^2} [n_{\tau+2} + 1 - \theta - n_k^{1,\theta} - n_{k+1}^{1,\theta}]$$

c)

$$G_\tau(\mathcal{M}, Q_{\tau+3}) = \begin{cases} \frac{1}{c^{1,\theta} r_k^{1,\theta}} \frac{a_{\tau+3,\tau+3}}{(a_{k,k}^{1,\theta})^2} [\tilde{n}_{\tau+3} - \tilde{n}_k^{1,0} - n_k^{1,0} + (n_k^{1,0})^2 + 1 & \\ -\tilde{n}_{k+2}^{1,0} - n_{k+1}^{1,0} + n_k^{1,0} n_{k+1}^{1,0} - n_{k+1}^{1,0} n_{k+2}^{1,0}] & \textit{for } \tau = 2k \\ \frac{1}{c^{1,\theta} r_k^{1,\theta}} \frac{a_{\tau+1,\tau+1}}{(a_{k,k}^{1,\theta})^2} [\tilde{n}_{\tau+3} + 1 - \tilde{n}_k^{1,1} - \tilde{n}_{k+2}^{1,1}] & \textit{for } \tau = 2k+1 \end{cases}$$

Proof. Let e be positive integer. By Lemma 4.14 we have

$$G_\tau(\mathcal{M}, Q_{\tau+e}) = \int_{-1}^{1} r(t) d\nu(t), \quad Q_{\tau+e}(t) = g(t)q(t) + r(t),$$

where $g(t) = (t-1)(t+1)^\theta Q_k^{1,\theta}(t)$. On the other hand

$$\begin{aligned} \int_{-1}^{1} Q_{\tau+e}(t) d\nu(t) &= \int_{-1}^{1} g(t)q(t) d\nu(t) + \int_{-1}^{1} r(t) d\nu(t) \\ 0 &= \frac{-1}{c^{1,\theta}} \int_{-1}^{1} Q_k^{1,\theta}(t) q(t) d\nu^{1,\theta} + \int_{-1}^{1} r(t) d\nu(t) \end{aligned}$$

Hence,

$$G_\tau(\mathcal{M}, Q_{\tau+e}) = \frac{1}{c^{1,\theta}} \int_{-1}^{1} Q_k^{1,\theta}(t) q(t) d\nu^{1,\theta} = \frac{f_k^{1,\theta}(q)}{c^{1,\theta} r_k^{1,\theta}},$$

We will omit the technical details further and we will prove only **b)** when τ is even.

$$\begin{aligned} Q_{\tau+2}(t) &= a_{2k+2,2k+2} t^{2k+2,2k+2} + a_{2k+2,2k+1} t^{2k+1} + \ldots \\ g(t) &= a_{k,k}^{1,0} t^{k+1} + (a_{k,k-1}^{1,0} - a_{k,k}^{1,0}) t^k + \ldots, \end{aligned}$$

Hence

$$q(t) = \frac{a_{2k+2,2k+2}}{a^{1,0}_{k,k}} t^{k+1} + \frac{a_{2k+2,2k+1} - \frac{a_{2k+2,2k+2}(a^{1,0}_{k,k-1} - a^{1,0}_{k,k})}{a^{1,0}_{k,k}}}{a^{1,0}_{k,k}} t^k + \dots$$

and

$$f_k^{1,0}(q(t)) = \frac{1}{a^{1,0}_{k,k}} \left[a_{2k+2,2k+1} - \frac{a_{2k+2,2k+2}(a^{1,0}_{k,k-1} - a^{1,0}_{k,k})}{a^{1,0}_{k,k}} - \frac{a_{2k+2,2k+2}}{a^{1,0}_{k+1,k+1}} a^{1,0}_{k+1,k} \right].$$

□

The following relations hold (by Theorem 4.5 and (4.15)):

$$n_k^{1,0} = n_k + c_k, \quad n_{k+1} = n_k + (c_k + m_k - 1)$$

$$\tilde{n}_{k+1} = \tilde{m}_k + n_k(c_k + m_k - 1) + c_k m_{k-1}, \quad \tilde{n}^{1,0}_{k+1} = \tilde{n}_k + \frac{r_{k-2}}{r_k} m_{k-1} m_{k-2} + c_k \tilde{n}_{k-1}. \quad (4.18)$$

For antipodal PMS

$$n_k = n_k^{1,1} = 0, \quad n_k^{1,0} = c_k = \frac{k}{2k+n-2}, \quad m_k = \frac{k+n-2}{2k+n-2}, \quad (4.19)$$

$$\tilde{n}_{k+1} = \frac{k(k+1)}{2(2k+n-2)}, \quad \tilde{n}^{1,1}_{k+1} = \frac{k(k+1)}{2(2k+n)}$$

Corollary 4.18 *Let $\mathcal{M}$ be antipodal PMS. Then*

$$G_\tau(\mathcal{M}, Q_{\tau+2}) \begin{cases} > 0, & \text{for } \tau = 2k \\ = 0, & \text{for } \tau = 2k+1. \end{cases}$$

Now we investigate the asymptotical behavior of the test functions for designs.

Theorem 4.19 *Let $\mathcal{M}$ be PMS. Then the following limits hold*

$$\lim_{j\to\infty} G_\tau(\mathcal{M}, Q_j) = \begin{cases} \frac{1}{R(\mathcal{M},\tau)} & \text{for } \tau \text{ even} \\ \frac{1}{R(\mathcal{M},\tau)} + (-1)^j \rho^{(\tau)}_{k+1} & \text{for } \tau \text{ odd,} \end{cases}$$

$$(\rho^{(\tau)}_{k+1} = \frac{1}{R(\mathcal{M},\tau)} \text{ for } \mathcal{M}\text{- antipodal})$$

For τ odd we take the limit for j odd/even separately.

All limits are nonnegative numbers.

Proof. Note that for e large enough we have $Q_e(t) \sim 0$, $t \in (-1,1)$. The assertion now follows from (4.14) and $Q_e(1) = 1$.

□

As we can see for $\mathcal{M}$ and τ fixed, there exists a constant $j_0 = j_0(\mathcal{M}, \tau)$ such that $G_\tau(\mathcal{M}, Q_j) \geq 0$ for all $j \geq j_0$. That means that, for fixed $\mathcal{M}$ and τ, we can not expect to obtain better bounds when we use polynomials of very high degree.

As we mentioned before, the test functions for codes were introduced and investigated by Boyvalenkov, Danev and Bumova in [18] (for $\mathcal{M} = \mathbf{S}^{n-1}$) and by Boyvalenkov and Danev [19] (in the general case). In particular, Theorem 4.9 from [18] implies the following result about the possibility for improving the Delsarte-Goethals-Seidel bound for spherical $(2k)$-designs.

Corollary 4.20 *If $\tau = 2k$ and $2 \leq n \leq k^2 + 1$, then $G_\tau(\mathbf{S}^{n-1}, Q_{\tau+3}) < 0$.*

In obtaining the complete understanding where it is possible to improve the Delsarte bound we have to examine the sign of $G_\tau(\mathbb{F}P^{n-1}, Q_{\tau+2})$ and $G_\tau(\mathbf{S}^{\mathbf{n}-1}, Q_{\tau+3})$. Using (4.18), (4.19) and Lemma 4.17 we got the following results.

Corollary 4.21 *If $\tau = 2k+1$ and $2 \leq n \leq k^2 - 2$, then $G_\tau(\mathbf{S}^{n-1}, Q_{\tau+3}) < 0$.*

Corollary 4.22 *Let $\mathcal{M} = \mathbb{F}P^{n-1}$. Then $G_\tau(\mathbb{F}P^{n-1}, Q_{\tau+2}) < 0$ in the following cases:*

a) $\mathbb{F} = \mathbb{R}$ *and*
$2 \leq n \leq \tau^2 - 2$.

b) $\mathbb{F} = \mathbb{C}$ *and*
$2 \leq n \leq 2k^2 + 2k - 1$, *if* $\tau = 2k+1$
$3 \leq n \leq \lfloor k^2 + 1/2 + \sqrt{4k^4 - 12k^2 - 8k + 1}/2 \rfloor$, *if* $\tau = 2k$ *and* $k \geq 3$
$n = 4, 5$, *if* $\tau = 4$.

c) $\mathbb{F} = \mathbb{H}$ *and*
$2 \leq n \leq k^2 + k - 1$, *if* $\tau = 2k+1$
$3 \leq n \leq \lfloor k^2/2 + 1/2 + \sqrt{k^4 - 6k^2 - 6k}/2 \rfloor$, *if* $\tau = 2k$ *and* $k \geq 4$
$n = 4, 5, 6$, *if* $\tau = 6$.
For $\tau = 4$ the test function is positive.

d) $\mathbb{F} = \mathbb{O}$, $n = 3$ *and*
$k \geq 3$ *if* $\tau = 2k+1$
$k \geq 5$ *if* $\tau = 2k$.

4.4 Extremal polynomials of degree $\tau+2$ and $\tau+3$

In the previous section we obtained necessary and sufficient conditions for improving the Delsarte bound by using linear programming. The investigations of the test functions for designs show that the smallest possible degree of an improving polynomial in non-antipodal PMS is $\tau+2$ and for antipodal and is $\tau+3$ (see Lemma 4.17 and Corollary 4.18).

Theorem 4.23 *Let $\mathcal{M}$ be non-antipodal PMS. Then, up to multiplication with a positive constant, any $B_{\mathcal{M},\tau}$-extremal polynomial of degree $\tau+2$ has the form*

$$f^{\star}(t)=\begin{cases}[T_{k-1}(t,1)+f_k(A)Q_k(t)+f_{k+1}(A)Q_{k+1}(t)]^2 \\ +(1-t^2)[f_{k-1}(B)Q^{1,1}_{k-1}(t)+f_k(B)Q^{1,1}_k(t)]^2 & \text{if } \tau=2k, \\ \\ (1+t)[2c^{0,1}T^{0,1}_{k-1}(t,1)+f_k(A)Q^{0,1}_k(t)+f_{k+1}(A)Q^{0,1}_{k+1}(t)]^2 \\ +(1-t)[f_k(B)Q^{1,0}_k(t)+f_{k+1}(B)Q^{1,0}_{k+1}(t)]^2 & \text{if } \tau=2k+1,\end{cases} \tag{4.20}$$

where the coefficients f_k are defined below by (4.23), (4.24), (4.36).

Proof. By Lukács Theorem [73, p.4] $f(t)$ must have the following form:

$$f(t)=\begin{cases}(A_{k+1}(t))^2+(1-t^2)(B_k(t))^2 & \text{when } deg(f)=2k+2, \\ (1+t)(A_{k+1}(t))^2+(1-t)(B_{k+1}(t))^2 & \text{when } deg(f)=2k+3,\end{cases} \tag{4.21}$$

where the indices show the degrees of the corresponding polynomials.

For our purposes we need to introduce some notation. We will first distinguish between the case that τ is even or τ is odd and after that we can introduce a unified notation which generalizes both of them.

Case1. $\tau=2k$

Now let us express $A_{k+1}(t)$ and $B_{k+1}(t)$ in terms of suitable adjacent systems of orthogonal polynomials. This will be helpful to surmount the main difficulty, namely to calculate f_0.

$$A_{k+1}(t)=\sum_{i=0}^{k+1} f_i(A)Q_i(t)$$

$$B_k(t)=\sum_{i=0}^{k} f_i(B)Q^{1,1}_i(t).$$

Now using (4.11) we obtain

$$\begin{aligned}\Omega(f) &= \frac{f(1)}{f_0} \\ &= \frac{(\sum_{i=0}^{k+1} f_i(A))^2}{\sum_{i=0}^{k+1} \frac{(f_i(A))^2}{r_i} + \frac{1}{c^{1,1}} \sum_{i=0}^{k} \frac{(f_i(B))^2}{r_i^{1,1}}}\end{aligned}$$

Let

$$f(t) = \sum_{i=0}^{2k+2} f_i Q_i(t). \tag{4.22}$$

be the ZSF expansion of $f(t)$.

Comparing the coefficients of t^{2k+2} and t^{2k+1} in the representations (4.22) and (4.21) and using the conditions $f_{\tau+1} = 0$ and $f_{\tau+2} < 0$ we obtain the following restrictions

$$(f_{k+1}(A))^2 < (f_k(B))^2 . \left(\frac{a_{k,k}^{1,1}}{a_{k+1,k+1}} \right)^2$$

$$\begin{aligned} & (f_{k+1}(A))^2 [2a_{k+1,k+1}a_{k+1,k} - (a_{k+1,k+1})^2 \frac{a_{2k+2,2k+1}}{a_{2k+2,2k+2}}] \\ + \;& 2f_{k+1}(A) f_k(A) a_{k+1,k+1} a_{k,k} \\ + \;& (f_k(B))^2 [(a_{k,k}^{1,1})^2 \frac{a_{2k+2,2k+1}}{a_{2k+2,2k+2}} - 2a_{k,k}^{1,1} a_{k,k-1}^{1,1}] \\ - \;& 2f_k(B) f_{k-1}(B) a_{k,k}^{1,1} a_{k-1,k-1}^{1,1} = 0 \end{aligned}$$

In this case we introduce the following notations:

$$\begin{aligned} & x_i = f_i(A), && i = 0, \ldots, k+1, \\ & y_{i+1} = f_i(B), && i = 0, \ldots, k, \\ & \alpha_i = \tfrac{1}{r_i}, && i = 0, \ldots, k+1, \\ & \beta_{i+1} = \tfrac{1}{c^{1,1} r_i^{1,1}}, && i = 0, \ldots, k, \\ & A_{11} = 2a_{k+1,k+1}a_{k+1,k} - (a_{k+1,k+1})^2 \tfrac{a_{2k+2,2k+1}}{a_{2k+2,2k+2}}, \\ & A_{12} = 2a_{k+1,k+1}a_{k,k}, \\ & B_{11} = (a_{k,k}^{1,1})^2 \tfrac{a_{2k+2,2k+1}}{a_{2k+2,2k+2}} - 2a_{k,k}^{1,1}a_{k,k-1}^{1,1}, \\ & B_{12} = -2a_{k,k}^{1,1}a_{k-1,k-1}^{1,1}, \\ & C = (\tfrac{a_{k,k}^{1,1}}{a_{k+1,k+1}})^2. \end{aligned} \tag{4.23}$$

Case2. $\tau = 2k+1$

As in the previous case we take the expansion of $A_{k+1}(t)$ and $B_{k+1}(t)$ in terms of $Q_i^{0,1}(t)$ and $Q_i^{1,0}(t)$, respectively.

$$A_{k+1}(t) = \sum_{i=0}^{k+1} f_i(A) Q_i^{0,1}(t)$$

$$B_{k+1}(t) = \sum_{i=0}^{k+1} f_i(B) Q_i^{1,0}(t).$$

Then by (4.11) we have

$$\begin{aligned}\Omega(f) &= \frac{f(1)}{f_0} \\ &= \frac{2(\sum_{i=0}^{k+1} f_i(A))^2}{\frac{1}{c^{0,1}}\sum_{i=0}^{k+1}\frac{(f_i(A))^2}{r_i^{0,1}} + \frac{1}{c^{1,0}}\sum_{i=0}^{k+1}\frac{(f_i(B))^2}{r_i^{1,0}}}\end{aligned}$$

The corresponding ZSF expansion is $f(t) = \sum_{i=0}^{2k+3} f_i(f) Q_i(t)$.
As in the previous case we have the conditions

$$(f_{k+1}(A))^2 < (f_k(B))^2 (\frac{a_{k+1,k+1}^{1,0}}{a_{k+1,k+1}^{0,1}})^2$$

$$\begin{aligned} & (f_{k+1}(A))^2 [2a_{k+1,k+1}^{0,1} a_{k+1,k}^{0,1} + (a_{k+1,k+1}^{0,1})^2 (1 - \frac{a_{2k+3,2k+2}}{a_{2k+3,2k+3}})] \\ + \ & 2 f_{k+1}(A) f_k(A) a_{k+1,k+1}^{0,1} a_{k,k}^{0,1} \\ + \ & (f_{k+1}(B))^2 [(a_{k+1,k+1}^{1,0})^2 (1 + \frac{a_{2k+3,2k+2}}{a_{2k+3,2k+3}}) - 2a_{k+1,k+1}^{1,0} a_{k+1,k}^{1,0}] \\ - \ & 2 f_{k+1}(B) f_k(B) a_{k+1,k+1}^{1,0} a_{k,k}^{1,0} = 0 \end{aligned}$$

Here we introduce similar notations as in the previous case.

$$\begin{aligned} & x_i = f_i(A) && i = 0, \ldots, k+1 \\ & y_i = f_i(B) && i = 0, \ldots, k+1 \\ & \alpha_i = \frac{1}{2c^{0,1} r_i^{0,1}} && i = 0, \ldots, k+1 \\ & \beta_i = \frac{1}{2c^{1,0} r_i^{1,0}} && i = 0, \ldots, k+1 \\ & A_{11} = (a_{k+1,k+1}^{0,1})^2 (1 - \tfrac{a_{2k+3,2k+2}}{a_{2k+3,2k+3}}) + 2a_{k+1,k+1}^{0,1} a_{k+1,k}^{0,1} \\ & A_{12} = 2a_{k+1,k+1}^{0,1} a_{k,k}^{0,1} \\ & B_{11} = (a_{k+1,k+1}^{1,0})^2 (1 + \tfrac{a_{2k+3,2k+2}}{a_{2k+3,2k+3}}) - 2a_{k+1,k+1}^{1,0} a_{k+1,k}^{1,0} \\ & B_{12} = -2a_{k+1,k+1}^{1,0} a_{k,k}^{1,0} \\ & C = (\tfrac{a_{k+1,k+1}^{1,0}}{a_{k+1,k+1}^{0,1}})^2 \end{aligned} \tag{4.24}$$

Now we are ready to prove the assertion of the theorem. We unify the notations of the two cases above by putting $\tau = 2k+\theta$, $\theta \in \{0,1\}$. Later on we will abbreviate $\underline{x} = (x_0, \dots, x_{k+1})$ and $\underline{y} = (y_\theta, \dots, y_{k+1})$. We have to solve the following optimization problem.

Maximize the function

$$F(\underline{x},\underline{y}) = \frac{(\sum_{i=0}^{k+1} x_i)^2}{\sum_{i=0}^{k+1} \alpha_i x_i^2 + \sum_{i=\theta}^{k+1} \beta_i y_i^2} \tag{4.25}$$

under the conditions:

$$\left| \begin{array}{l} x_{k+1}^2 < C y_{k+1}^2 \\ G(\underline{x},\underline{y}) = A_{11} x_{k+1}^2 + A_{12} x_{k+1} x_k + B_{11} y_{k+1}^2 + B_{12} y_{k+1} y_k = 0 \end{array} \right. \tag{4.26}$$

The coefficients α_i, β_i are positive. Hence $F(\underline{x},\underline{y}) \geq 0$. We solve this optimization problem by means of the Lagrange multiplier method as follows. We consider the function

$$\tilde{F}(\underline{x},\underline{y},\lambda) = F(\underline{x},\underline{y}) - \lambda G(\underline{x},\underline{y})$$

and maximize it for $\underline{x}$, $\underline{y}$. A necessary condition for this is the vanishing of the first derivatives of $\tilde{F}$. We denote

$$\mu = \frac{\sum_0^{k+1} x_i}{\sum_{i=0}^{k+1} \alpha_i x_i^2 + \sum_{i=\theta}^{k+1} \beta_i y_i^2}.$$

We have to consider only the case $\mu \neq 0$, because otherwise we obtain $\min F(\underline{x},\underline{y}) = 0$. After simplification of the derivatives we obtain

$$\begin{array}{rl} x_i = \dfrac{1}{\mu \alpha_i} & \text{for} \quad i = 0, \dots, k-1 \\ y_i = 0 & \text{for} \quad i = 0, \dots, k-1 \end{array} \tag{4.27}$$

$$\left| \begin{array}{rcrcrcl} (2\mu^2 \alpha_k) & x_k & + & (\lambda A_{12}) & x_{k+1} & = & 2\mu \\ (\lambda A_{12}) & x_k & + & (2\mu^2 \alpha_{k+1} + 2\lambda A_{11}) & x_{k+1} & = & 2\mu \end{array} \right. \tag{4.28}$$

$$\left| \begin{array}{rcrcrcl} (2\mu^2 \beta_k) & y_k & + & (\lambda B_{12}) & y_{k+1} & = & 0 \\ (\lambda B_{12}) & y_k & + & (2\mu^2 \beta_{k+1} + 2\lambda B_{11}) & y_{k+1} & = & 0 \end{array} \right. \tag{4.29}$$

$$A_{11} x_{k+1}^2 + A_{12} x_{k+1} x_k + B_{11} y_{k+1}^2 + B_{12} y_{k+1} y_k = 0 \tag{4.30}$$

The system (4.29) must have nonzero solutions because of the inequality in (4.26). Therefore

$$\left| \begin{array}{cc} 2\mu^2 \beta_k & \lambda B_{12} \\ \lambda B_{12} & 2\mu^2 \beta_{k+1} + 2\lambda B_{11} \end{array} \right| = 0, \tag{4.31}$$

whence

$$\lambda = \mu^2 \tilde{\lambda},$$

where

$$\tilde{\lambda} = 2\frac{\beta_k B_{11} \pm \sqrt{(B_{11}^2 \beta_k + \beta_{k+1} B_{12}^2)\beta_k}}{B_{12}^2}. \tag{4.32}$$

Solving (4.29) we obtain

$$y_k = \frac{-\lambda B_{12}}{2\mu^2 \beta_k} y_{k+1} = \frac{-\tilde{\lambda} B_{12}}{2\beta_k} y_{k+1} \tag{4.33}$$

By (4.28) we can express x_k and x_{k+1}

$$\begin{aligned} x_k &= \frac{1}{\mu} \frac{4\alpha_{k+1} + 4\tilde{\lambda} A_{11} - 2\tilde{\lambda} A_{12}}{4\alpha_k \alpha_{k+1} + 4\tilde{\lambda}\alpha_k A_{11} - \tilde{\lambda}^2 A_{12}^2} \\ x_{k+1} &= \frac{1}{\mu} \frac{4\alpha_k - 2\tilde{\lambda} A_{12}}{4\alpha_k \alpha_{k+1} + 4\tilde{\lambda}\alpha_k A_{11} - \tilde{\lambda}^2 A_{12}^2} \end{aligned} \tag{4.34}$$

Now from (4.30) and (4.33) we find

$$y_{k+1} = \pm \sqrt{\frac{-2\beta_k}{2\beta_k B_{11} - \tilde{\lambda} B_{12}^2} (A_{11} x_{k+1}^2 + A_{12} x_{k+1} x_k)} \tag{4.35}$$

Replacing our solution in $F(\underline{x}, \underline{y})$ and using the homogeneity of $F(\underline{x}, \underline{y})$ we can cancel μ in the numerator and denominator. Hence our solutions actually do not depend on μ. We can chose $\mu = 1$ and $\tilde{\lambda}$ to be positive.

Finally, we arrive at the following solution

$$\begin{aligned} x_i &= \frac{1}{\alpha_i}, \quad \text{for} \quad i = 0, \ldots, k-1 \\ y_i &= 0, \quad \text{for} \quad i = 0, \ldots, k-1 \\ x_k &= \frac{4\alpha_{k+1} + 4\tilde{\lambda} A_{11} - 2\tilde{\lambda} A_{12}}{4\alpha_k \alpha_{k+1} + 4\tilde{\lambda}\alpha_k A_{11} - \tilde{\lambda}^2 A_{12}^2} \\ x_{k+1} &= \frac{4\alpha_k - 2\tilde{\lambda} A_{12}}{4\alpha_k \alpha_{k+1} + 4\tilde{\lambda}\alpha_k A_{11} - \tilde{\lambda}^2 A_{12}^2} \\ y_{k+1} &= \pm \sqrt{\frac{-2\beta_k}{2\beta_k B_{11} - \tilde{\lambda} B_{12}^2} [A_{11} x_{k+1}^2 + A_{12} x_{k+1} x_k]}, \\ y_k &= \frac{-\tilde{\lambda} B_{12}}{2\beta_k} y_{k+1} \end{aligned} \tag{4.36}$$

where

$$\tilde{\lambda} = 2\frac{\beta_k B_{11} + \sqrt{\beta_k[\beta_k B_{11}^2 + \beta_{k+1} B_{12}^2]}}{B_{12}^2}$$

□

Note that the inequality in (4.26) exclude the possibility the polynomial $f^{(\tau)}(t)$ to be a solution of our optimization problem.

Corollary 4.24 *Let $\mathcal{M}$ be a non-antipodal PMS and let τ be an integer. Then*

$$B(\mathcal{M},\tau) \geq S(\mathcal{M},\tau) = R(\mathcal{M},\tau-2) + x_k + x_{k+1} = \Omega(f^*).$$

As we mentioned before, for antipodal PMS the corresponding $B_{\mathcal{M},\tau}$-extremal polynomial is of degree $\tau+3$. We can prove in a similar way analogous theorem for the form of this polynomials.

Theorem 4.25 *Let $\mathcal{M}$ be antipodal PMS. Then, up to multiplication with a positive constant, any $B_{\mathcal{M},\tau}$-extremal polynomial of degree $\tau+3$ has the form*

$$f^{**}(t) = \begin{cases} (1+t)\{2c^{0,1}T_{k-2}^{0,1}(t,1) + f_{k-1}(A)Q_{k-1}^{0,1}(t) + f_k(A)Q_k^{0,1}(t) \\ +f_{k+1}(A)Q_{k+1}^{0,1}(t)\}^2 + (1-t)\{f_{k-1}(B)Q_{k-1}^{1,0}(t) \\ +f_k(B)Q_k^{1,0}(t) + f_{k+1}(B)Q_{k+1}^{1,0}(t)\}^2 & \text{if } \tau = 2k, \\ \\ \{T_{k-1}(t,1) + f_k(A)Q_k(t) + f_{k+1}(A)Q_{k+1}(t) \\ +f_{k+2}(A)Q_{k+2}(t)\}^2 + (1-t^2)\{f_{k-1}(B)Q_{k-1}^{1,1}(t) \\ +f_k(B)Q_k^{1,1}(t) + f_{k+1}(B)Q_{k+1}^{1,1}(t)\}^2 & \text{if } \tau = 2k+1. \end{cases} \tag{4.37}$$

where the coefficients f_k are defined in the proof.

The proof is similar to the proof of the previous theorem and we present it in the Appendix.

Corollary 4.26 *Let $\mathcal{M}$ be an antipodal PMS and let τ be an integer. Then*

$$B(\mathcal{M},\tau) \geq S(\mathcal{M},\tau) = R(\mathcal{M},\tau-3) + x_k + x_{k+1} + x_{k+2} = \Omega(f^{**}),$$

where x_k, x_{k+1}, x_{k+2} are defined in the proof of the Theorem 4.25.

Summary of Chapter 4: The investigations in this chapter are a natural continuation of the results from the previous two chapters. First of all we proved necessary and sufficient conditions for optimality of the Delsarte bound (see Theorem 4.15). In Corollary 4.22 we present for every $\mathcal{M}$ and τ precise intervals for the dimension n, where the test function is negative. As a consequence of the above we obtain that the smallest possible degree of the $B_{\mathcal{M},\tau}$-extremal polynomials is $\tau+2$ for non-antipodal spaces and $\tau+3$ for antipodal PMS. In Theorems 4.23 and 4.25 we proved the analytical form for these polynomials.

This allows us to give analytical expression for new lower bound $S(\mathcal{M}, \tau)$. Unfortunately these expressions are too complicated to be compared with the Delsarte bound in general.

We compared by computer the results in Chapter 3 and Chapter 2 and the bounds from Corollary 4.24 and 4.26. We also compared the polynomials, which we use for obtaining new bounds in Chapter 2 and Chapter 3, and the polynomials described in (4.20) and (4.37). This investigation showed the coincidence between the corresponding polynomials and coincidence between the corresponding bounds. Our next goal will be to simplify the expressions of $S(\mathcal{M}, \tau)$.

Chapter 5

Non-existence of Certain Spherical Designs

5.1 Introduction

In this chapter we obtain some necessary conditions for the existence of spherical τ-designs of odd strengths and odd cardinalities. These conditions imply non-existence results in many cases. In Section 5.2 we derive a general non-existence rule. It gives a bound which is asymptotically better than the corresponding estimation based on the Delsarte-Goethals-Seidel bound.

It turns out that our approach works in small dimensions as well. In Section 5.3 and Section 5.4 we consider in detail the strengths $\tau = 3$ and $\tau = 5$ respectively. We rule out the first open cases by showing the non-existence of 3-designs with 7 points and 5-designs with 13 points. For large odd cardinalities, when the non-existence argument does not work, we obtain in Section 5.5 bounds on the maximal inner product of a τ-design of a fixed cardinality. This chapter is based on [27].

5.2 Necessary Conditions for the Existence of Spherical Designs

In Chapter 1 and Chapter 2, we gave some definitions for the spherical designs. We shall now proceed with a more detailed investigation of the structure of spherical τ-designs with odd τ. It is natural to study the distribution of the inner products of the points of a τ-design.

Let $C \subset \mathbf{S}^{n-1}$ be a τ-design and $y \in C$. We shall study the multi-set

$$I(y) = \{(x, y) : x \in C, x \neq y\},$$

i.e. we also count the multiplicities. Without loss of generality we may assume that

$$I(y) = \{t_1, t_2, \ldots, t_{|C|-1}\},$$

where

$$-1 \le t_1 \le t_2 \le \cdots \le t_{|C|-1} < 1.$$

Then, Equality (1.18) becomes

$$\sum_{i=1}^{|C|-1} f(t_i) = |C| f_0 - f(1) \tag{5.1}$$

and we use it in this form.

Our approach will be the following. We shall use some polynomials in (5.1) to obtain bounds on the smallest inner product t_1. Then, for odd τ and $|C|$, we shall conclude that the same bound is satisfied for the second smallest inner product t_2 for at least one point $y \in C$. The last information already implies a strong necessary condition for the existence of spherical designs of odd strengths and odd cardinalities.

The characterization (1.18) given by Theorem 1.16 was used in [39] for obtaining bounds on the minimum distance and the covering radius of τ-designs in polynomial metric spaces. In [25], Equation (5.1) was already used for obtaining results for the distribution of the inner products of spherical τ-designs on the interval $[-1, 1]$.

Let $\tau = 2e + 1$ be odd, let $C \subset \mathbf{S}^{n-1}$ be a τ-design of cardinality $|C| = R(n, \tau) + k$, and let $y \in C$. We first derive an upper bound on the least inner product $t_1 \in I(y)$. We set

$$g(t) = [Q_e^{(n+2)}(t)]^2$$

and

$$\delta = -\frac{R(n, \tau)}{R(n, \tau) + 2k} < 0.$$

Theorem 5.1 *With the above definitions, $t_1 \le \delta$.*

Proof. As we already mentioned (cf. Section 2.5), bound (2.2) for $\tau = 2e + 1$ was obtained [36, Theorem 5.12] by substituting the polynomial $(t + 1)g(t)$ in Theorem 1.13. Since $g(t)$ is an even function, (2.3) shows that the first coefficients in the Gegenbauer expansions of $(t + 1)g(t)$ and $g(t)$ coincide. We denote this common coefficient by g_0 (in fact, $2g(1)/g_0 = 2/g_0 = R(n, \tau)$).

We set

$$f(t) = (t - t_1)g(t) = (t + 1)g(t) - (t_1 + 1)g(t)$$

in (5.1). Then the left-hand side of (5.1) is nonnegative, and the right-hand side is

$$f_0|C| - f(1) = -t_1 g_0(R(n, \tau) + k) - 1 + t_1$$

which implies $t_1 \le -R(n, \tau)/(R(n, \tau) + 2k) = \delta$.

□

Remark 5.2 *In fact, an upper bound for the smallest inner product was proved in [25, Corollary 2.3], but unstressed there. However, the idea for using such bounds for obtaining nonexistence results for odd cardinalities came later [26, 27].*

We recall (see (1.12) and Lemma 2.4) that the first coefficients in the Gegenbauer expansion of t^i are denoted by b_i. In fact, in Lemma 2.3 we computed the exact values of the b_i's which are relevant for Theorem 5.4 below.

Definition 5.3 *For $\varepsilon \in [-1, 0)$, we say that a point x is ε-near antipodal to y, if $(x, y) \leq \varepsilon$.*

For $\varepsilon = -1$, Definition 5.3 gives, of course, the usual antipodality.

Theorem 5.4 *Let $n \geq 3$, $\tau = 2e + 1 \geq 3$, and $k \geq 1$ be such that*

$$b_{2e}[R(n, \tau) + k] - 1 < 2\delta^{2e}. \tag{5.2}$$

Then, each point of C has a unique δ-near antipode from C. In particular, k must be even.

Proof. If $t_2 \leq \delta$ for some $y \in C$, then substituting $f(t) = t^{2e}$ in (5.1) yields

$$b_{2e}|C| - 1 = \sum_{i=1}^{|C|-1} t_i^{2e} \geq t_1^{2e} + t_2^{2e} \geq 2t_2^{2e} \geq 2\delta^{2e}, \tag{5.3}$$

which contradicts (5.2). Therefore

$$t_1 \leq \delta < t_2,$$

for all $y \in C$. We conclude that each point $y \in C$ has a unique δ-near antipode $x \in C$. Therefore, the points of C can be divided into disjoint pairs, so $|C|$ is even. Since $R(n, \tau)$ is even for τ odd, the number $k = |C| - R(n, \tau)$ must be even.

□

Theorem 5.4 gives the following non-existence rule.

Corollary 5.5 *If $n \geq 3$ and the odd numbers $\tau = 2e + 1 \geq 3$ and $k \geq 1$ are such that (5.2) is satisfied, then there exist no spherical τ-designs on $\mathbf{S}^{n-1}$ with $R(n, \tau) + k$ points.*

□

Substituting suitable polynomials in (5.1) and using the estimation from Theorem 5.1 one can obtain better non-existence results for $(2e+1)$-designs with odd cardinalities. In the next two sections the cases $e=1$ and $e=2$ will be considered in detail. Before we do so we shall derive another universal non-existence rule. We do this by improving the argument of Theorem 5.4 in general.

We recall that (see Theorem 1.8 and the subsequent paragraph) the smallest zero of the Gegenbauer polynomial $Q_e^{(n+2)}(t) = Q_e^{1,1}(t)$ is denoted by $t_{e,1}^{(n+2)}$. Obviously, $t_{e,1}^{(n+2)}$ is the least zero of $g(t)$ as well.

Theorem 5.6 *If $n \geq 3$ and the odd numbers $\tau = 2e+1 \geq 3$ and $k \geq 1$ satisfy the conditions*

$$\delta < t_{e,1}^{(n+2)} \tag{5.4}$$

and

$$-2\delta g(\delta) > 1 \tag{5.5}$$

then no τ-designs exist with $R(n,\tau)+k$ points on $\mathbf{S}^{n-1}$.

Proof. Assume that $t_2 \leq \delta$ for some $y \in C$. Since $g(1) = (Q_e^{(n+2)}(1))^2 = 1$ and $g_0 = 2g(1)/R(n,\tau) = 2/R(n,\tau)$, we have by the definition of δ and (5.1) for y and $g(t)$

$$\begin{aligned} -\frac{1}{\delta} &= \frac{R(n,\tau)+2k}{R(n,\tau)} = \frac{2|C|-R(n,\tau)}{R(n,\tau)} \\ &= g_0|C| - g(1) = \sum_{i=1}^{|C|-1} g(t_i). \end{aligned}$$

The non-negative even function $g(t)$ decreases in the interval $(-\infty, t_{e,1}^{(n+2)}]$. Therefore, the last sum can be bounded from bellow by

$$\sum_{i=1}^{|C|-1} g(t_i) \geq g(t_1) + g(t_2) \geq 2g(t_2) \geq 2g(\delta)$$

(for the last estimation, we use (5.4)), which contradicts (5.5). Hence $t_2 > \delta$ for all $y \in C$ and we can repeat the non-existence argument in the proof of Theorem 5.4.

□

Lemma 5.7 *Let k and τ be fixed. Then $t_{e,1}^{(n+2)}$ tends to zero for $n \to \infty$.*

Proof. By the recurrence relation (1.5) for the Gegenbauer polynomials, one has

$$Q_e^{(n)}(t) \stackrel{n\to\infty}{\longrightarrow} tQ_{e-1}^{(n)}(t),$$

whence $Q_e^{(n)}(t) \stackrel{n\to\infty}{\longrightarrow} t^e$ and our claim follows.

□

We now discuss the non-existence results ensured by Theorem 5.6 for τ fixed and $n \to \infty$.

Theorem 5.8 *For fixed $\tau = 2e + 1 \geq 3$ and every positive $p < p_0 = (2^{1/\tau} - 1)/e!$ there exists a constant $n_0 = n_0(p)$ such that for $n \geq n_0$ τ-designs with cardinality $R(n,\tau) + k$ do not exist on $\mathbf{S}^{n-1}$ for all odd positive $k \leq pn^e$.*

Proof. Since $k \leq 2pn^e$, we have

$$\delta \leq -\frac{R(n,\tau)}{R(n,\tau) + 2pn^e}.$$

However,

$$\frac{R(n,\tau)}{R(n,\tau) + 2pn^e} \stackrel{n\to\infty}{\longrightarrow} \frac{1}{1 + pe!} > 0$$

because $R(n,\tau) = 2(n+e-1)\cdots(n+1)n/e! \approx 2n^e/e!$ as $n \to \infty$. Therefore, we have

$$\delta \leq -\frac{1}{1 + pe!} < 0$$

for all large enough n. This and Lemma 5.7 imply that condition (5.4) is satisfied for large enough n.

It remains to check (5.5). It follows from (5.4) that $g(t)$ is decreasing in $(-\infty, \delta]$ (in fact, this is the sense of this requirement). Thus, we obtain

$$\begin{aligned} -2\delta g(\delta) &\geq \frac{2R(n,\tau)}{R(n,\tau) + 2pn^e} \cdot g\left(\frac{R(n,\tau)}{R(n,\tau) + 2pn^e}\right) \\ &= \frac{2R(n,\tau)}{R(n,\tau) + 2pn^e} \cdot \left[Q_e^{(n+2)}\left(\frac{R(n,\tau)}{R(n,\tau) + 2pn^e}\right)\right]^2. \end{aligned}$$

As in the proof of Lemma 5.7 we see that the right-hand side of the last expression tends to

$$\frac{2}{(1+pe!)} \cdot \frac{1}{(1+pe!)^{2e}} = \frac{2}{(1+pe!)^\tau}$$

when $n \to \infty$. Therefore,

$$-\delta g(\delta) \geq \frac{2}{(1+pe!)^\tau} > \frac{2}{(1+p_0 e!)^\tau} = 1 \tag{5.6}$$

for all large enough n. This completes the proof.

□

Corollary 5.9 *For fixed $\tau = 2e + 1 \geq 3$ and $n \to \infty$, we have*

$$B_{odd}(n,\tau) \geq \frac{1 + 2^{1/\tau}}{e!} n^e.$$

The above approach can be further refined and improved. We show this in the next two sections.

5.3 Non-existence of Certain 3-designs

Bajnok [5] has constructed spherical 3-designs of all possible even cardinalities. So only odd cardinalities are interesting for us.

For $\tau = 3$, we have $R(n,3) = 2n$, $g(t) = t^2$, $\delta = -n/(n+k)$, and $t_{e,1}^{(n+2)} = 0$. Let $C \subset \mathbf{S}^{n-1}$ be a 3-design with $|C| = R(n,3) + k = 2n + k$ points and $y \in C$. We set $C = \{y, x_1, x_2, \ldots, x_{2n+k-1}\}$ and $(x_i, y) = t_i$ for $1 \leq i \leq 2n+k-1$, where $-1 \leq t_1 \leq t_2 \leq \cdots \leq t_{2n+k-1}$. By Corollary 5.5 we obtain the following:

Theorem 5.10 *For odd $k < (2^{1/3} - 1)n \approx 0.26n$ there exists no spherical 3-design on $\mathbf{S}^{n-1}$ with $2n + k$ points.*

Proof. For $\tau = 2e + 1 = 3$, (5.2) is equivalent to $k < (2^{1/3} - 1)n$.

□

Example 5.11 *By Theorem 5.10, we obtain that there exist no 3-designs with $2n+1$ points ($k = 1$) in dimensions $n \geq 4$, no 3-designs with $2n+3$ points ($k = 3$) in dimensions $n \geq 12$, etc.*
Notice that Theorem 5.6 gives the same result because $Q_1^{(n+2)}(t) = t$ (and $t_{e,1}^{(n+2)} = 0 > \delta = -n/(n+k)$).

To obtain further non-existence results for spherical 3-designs we need better estimations. As a simple consequence, we shall prove the non-existence of spherical 3-designs on $\mathbf{S}^2$ with 7 points which solves the first open case.

Lemma 5.12 *For every real a,*

$$t_1 \leq F(a) \leq t_{2n+k-1},$$

where

$$F(a) = -\frac{na^2 + 2(n+k)a + n}{n(2n+k-1)a^2 + 2na + n + k} = \frac{h_1(a)}{h_2(a)}.$$

Proof. Set $f(t) = (t-a)^2(t-t_i)$, where $i = 1$ or $2n+k-1$, in (5.1). The left-hand side of (5.1) is non-negative for $i = 1$ and non-positive for $i = 2n+k-1$. By (2.4), we compute

$$f_0 = a^2 t_i - \frac{1}{n}(2a + t_i).$$

Solving the inequalities $f_0|C| - f(1) \geq 0$ for $i = 1$ and $f_0|C| - f(1) \leq 0$ for $i = 2n+k-1$, leads to the claimed estimations.

□

The argument in Theorem 5.1 corresponds to $a = 0$ in Lemma 5.12. Now we investigate the function $F(a)$ to obtain better estimations for t_1. The equation $F'(a) = 0$ is equivalent to

$$h_1'(a)h_2(a) - h_1(a)h_2'(a) = 0,$$

which gives the quadratic equation

$$n(n+k-1)a^2 + n(n-1)a - k = 0. \tag{5.7}$$

Let a_1 and a_2 be the positive and the negative root of (5.7) respectively, i.e.

$$a_{1,2} = \frac{-n(n-1) \pm \sqrt{n^2(n-1)^2 + 4nk(n+k-1)}}{2n(n+k-1)}.$$

Lemma 5.13 *With the above definitions*

$$t_1 \leq F(a_1) = a_2 < 0$$

and

$$t_{2n+k-1} \geq F(a_2) = a_1 > 0.$$

Proof. The function $F(a)$ has its maximum for $a = a_2$ and its minimum for $a = a_1$. Since $F'(a_1) = 0$, we have

$$F(a_1) = \frac{h_1'(a_1)}{h_2'(a_1)} = \frac{-na_1 - (n+k)}{n(2n+k-1)a_1 + n}.$$

To check the identity $F(a_1) = a_2$, we apply the Viète formulae in its equivalent equality

$$n(a_1 + a_2) + n(2n+k-1)a_1a_2 = -(n+k).$$

Analogously, one obtains $F(a_2) = a_1$.

□

We now obtain a necessary condition for the existence of 3-designs which in fact refines (5.2) and Theorem 5.6.

Theorem 5.14 *If k is odd, then*

$$a_2 \geq -\frac{2n^3 + (5k-7)n^2 + (4k^2 - 15k + 5)n + k(k-1)(k-5)}{2n[2n^2 + (k-5)n + 3(1-k)]}. \tag{5.8}$$

Proof. If $t_2 > a_2$ for all $y \in C$ we can apply the non-existence argument from Corollary 5.5. So, we can assume that $t_2 \leq a_2$ for some $y \in C$.

For this point y, we set $f(t) = (t-a)^2$ in (5.1) assuming $a \geq a_2$. We have (computing f_0 by (2.4))

$$\begin{aligned}(2n+k-1)a^2 + 2a + \frac{n+k}{n} &= f_0|C| - f(1) = \sum_{i=1}^{2n+k-1} (t_i - a)^2 \\ &\geq (t_1 - a)^2 + (t_2 - a)^2 \geq 2(t_2 - a)^2 \geq 2(a_2 - a)^2\end{aligned}$$

(for the last inequality, one has to use $t_2 \leq a_2 \leq a$). This gives the quadratic inequality

$$(2n+k-3)a^2 + 2(1+2a_2)a + \frac{n+k}{n} - 2a_2^2 \geq 0.$$

The quadratic function in a on the left-hand side has its minimum at the point

$$a_0 = -\frac{1+2a_2}{2n+k-3},$$

which is grater than or equal to a_2. The value of this minimum equals

$$-\frac{2}{2n+k-3}\left((2n+k-1)a_2^2 + 2a_2 + \frac{1}{2} - \frac{(n+k)(2n+k-3)}{2n}\right).$$

Since the minimum must be non-negative, we obtain

$$(2n+k-1)a_2^2 + 2a_2 + \frac{1}{2} \leq \frac{(n+k)(2n+k-3)}{2n}.$$

Since a_2 is a root of (5.7), we express

$$a_2^2 = \frac{k - n(n-1)a_2}{n(n+k-1)}$$

from (5.7) to obtain a linear inequality with respect to a_2 which is equivalent to (5.8).

□

Theorem 5.15 *There exists no spherical 3-design with* 7 *points on* $\mathbf{S}^2$.

Proof. In this case (5.8) is violated since $a_2 = -(1+\sqrt{2})/3 \approx -0.804$ while the right-hand side of (5.8) is equal to $-1/2$.

□

Example 5.11 and Theorem 5.15 complete the case $k = 1$, i.e. we have shown the non-existence of spherical 3-designs on $\mathbf{S}^{n-1}$ with $2n+1$ points (this is the first possible cardinality of a non-tight 3-design) in all dimensions $n \geq 3$. The precise investigation of condition (5.8) implies the following result, which slightly improves Theorem 5.8 for $\tau = 3$.

Corollary 5.16 *No spherical 3-design on* $\mathbf{S}^{n-1}$ *exists with* $R(n,3)+k=2n+k$ *points for* $n \geq 3$*, where* k *is any positive odd integer* $< n(2^{1/3}-1)+p$*, and where* $p = 2(14 - 5.2^{1/3} - 4.2^{2/3})/9 \approx 0.30018$*. In other words,*

$$B_{odd}(n,3) \geq (1+2^{1/3})n + p \quad (5.9)$$

$$\approx (1+2^{1/3})n + 0.30018. \quad (5.10)$$

Proof. We are interested in the pairs (n,k) for which $n \geq 3$ and $k \geq 1$. In what follows we shall take only such pairs under consideration. After a routine calculation (which we made using Maple V), inequality (5.8) takes the form

$$\begin{aligned} 0 \leq\ & h_3(n,k) = k^6 + 4(2n-3)k^5 + 2(13n^2 - 39n + 23)k^4 \\ & +2(21n^3 - 93n^2 + 112n - 30)k^3 \\ & +(31n^4 - 186n^3 + 349n^2 - 210n + 25)k^2 \\ & +2n(2n^4 - 25n^3 + 77n^2 - 82n + 28)k \\ & -n^2(4n^4 + 20n^3 - 33n^2 + 22n - 5). \end{aligned}$$

The constant p was "conjectured" by setting $k = (2^{1/3}-1)n+p$, forgetting the small (with respect to the degrees of n) terms and resolving a linear (with respect to p) equation. In this case we get a polynomial in the variable n of degree five with a leading coefficient which is equal to

$$6(1+2^{1/3})^2 p + 4(4 - 5.2^{1/3}).$$

Since this coefficient must be non-positive, we see that the largest p which can be used is exactly $2(14 - 5.2^{1/3} - 4.2^{2/3})/9$.

We have

$$h_3(n, n(2^{1/3}-1)+p) < -15n^4 + 49n^3 - 32n^2 + 5n + 1 < 0,$$

whenever $n \geq 3$. The standard investigation of $h_3(n,k)$ shows that it is an increasing function of the variable k in the interval $[1, +\infty)$. Thus, for every positive integer k such that $k < n(2^{1/3}-1)+p$, we have $h_3(n,k) < h_3(n, n(2^{1/3}-1)+p) < 0$. In this case, condition (5.8) is violated, which completes the proof.

□

Remark 5.17 *Bajnok [5] has constructed 3-designs on* $\mathbf{S}^{n-1}$ *for all odd cardinalities greater than or equal to* $R(n,3)+n/2 = 5n/2$ *for* $n \geq 6$*, for cardinality 11 when* $n = 3,4$*, and for cardinality 15 when* $n = 5$*, and for all possible even cardinalities (see Table 4.1). Thus, Corollary 5.16 shows that all possible cardinalities of 3-designs on* $\mathbf{S}^{n-1}$ *are already known for* $n = 4,6$ *and only one open case remains in each of the dimensions* $n = 3,5$*, and* $7 \leq n \leq 14$*. The situation is explained also in Table 5.3 below. Our non-existence results are given in the second column, where the asterisk* ($\star$) *show non-existence and the last entry gives the bound from Theorem 5.10.*

n	Non-existence?	Constructions(Bajnok [5])
3	7^*,9	≥ 11
4	9^*	≥ 11
5	11^*,13	≥ 15
6	13^*	≥ 15
≥ 7	$\geq 2.26n$	$\geq 5n/2 = 2.5n$

Table 5.1: Spherical 3-designs with odd cardinalities.

5.4 Non-existence of Certain 5-designs

Let $C \subset \mathbf{S}^{n-1}$ be a 5-design with $|C| = R(n,5) + k = n^2 + n + k$ points and $y \in C$. We set $C = \{y, x_1, x_2, \ldots, x_{n^2+n+k-1}\}$ and $(x_i, y) = t_i$ for $i = 1, 2, \ldots, n^2 + n + k - 1$, where $-1 \leq t_1 \leq t_2 \leq \cdots \leq t_{n^2+n+k-1}$. Theorem 5.1 gives

$$t_1 < \delta = -\frac{n(n+1)}{(n^2+n+2k)}. \tag{5.11}$$

Inequality (5.2) is equivalent to

$$G(n,k) = \frac{n^2+n+2k}{n(n+1)} \cdot \sqrt[4]{\frac{2n^2+n+3k}{2n(n+2)}} < 1.$$

Theorem 5.18 *Let $n \geq 3$ and let k be an odd integer such that*

$$G(n,k) < 1.$$

Then there exists no spherical 5-design on $\mathbf{S}^{n-1}$ with $n^2 + n + k$ points.

Remark 5.19 *For $k = 1$, a simple analysis of the function $G(n,1)$ implies the non-existence of spherical 5-designs with $n^2 + n + 1$ points in all dimensions $n \geq 7$. Similarly, for $k = 3$, one obtains the non-existence of spherical 5-designs with $n^2 + n + 3$ points in all dimensions $n \geq 20$. In three dimensions, Theorem 5.18 provides no information.*

We now discuss the non-existence results guaranteed by Theorem 5.6 when $\tau = 5$. It already gives better results (not as in the case $\tau = 3$). We have

$$g(t) = \frac{(n+2)t^2 - 1}{n+1} \tag{5.12}$$

Theorem 5.20 *For $n \geq 3$ and any odd positive $k < n(n+1)(\sqrt{n+2} - 1)/2$, there exists no spherical 5-design on $\mathbf{S}^{n-1}$ if*

$$(n+1)(n^2+n+2k)^5 < 2n[n^2(n+1)^3 - 4k(n^2+n+k)]^2. \tag{5.13}$$

Proof. Since

$$P_2^{(n+2)}(t) = ((n+2)t^2 - 1)/(n+1)$$

(i.e. $t_{e,1}^{(n+2)} = -1/\sqrt{n+2}$), condition (5.4) is equivalent to inequality

$$k < \frac{n(n+1)(\sqrt{n+2}-1)}{2}.$$

A little algebra (use (5.11) and (5.12)) shows that (5.5) is equivalent to (5.13).

□

Remark 5.21 *For $k = 1$, inequality (5.13) implies the non-existence of 5-designs with $n^2 + n + 1$ points in all dimensions $n \geq 4$ (in fact, after Remark 5.19, we need to check (5.13) for $n = 4, 5$ and 6 only).*

Analogously, for $k = 3$, one obtains the non-existence of spherical 5-designs with n^2+n+3 points in all dimensions $n \geq 7$.

A careful analysis of condition (5.13) leads to the following asymptotic result.

Corollary 5.22 *For $n \geq 3$, there exists no spherical 5-design on $\mathbf{S}^{n-1}$ with $n^2 + n + k$ points for all odd positive $k < p_0 n^2 + p_1 n$, where $p_0 = (2^{1/5} - 1)/2 \approx 0.074349$ and $p_1 = (-5 + 7.2^{1/5} - 2.2^{3/5})/10 \approx 0.00095$. In other words,*

$$B_{odd}(n, 5) \geq \frac{1+2^{1/5}}{2} n^2 + (1 + p_1)n \tag{5.14}$$

$$\approx \frac{1+2^{1/5}}{2} n^2 + 1.00095n. \tag{5.15}$$

Proof. Using Maple V again (as in Corollary 5.16), we found that inequality (5.13) is equivalent to

$$\begin{aligned} 0 &< h_5(n,k) = -32(n+1)k^5 - 16(5n^2 + 10n + 3)nk^4 \\ &- 16n^2(n+1)(5n^2 + 10n + 1)k^3 \\ &- 8n^3(n+1)^2(5n^2 + 12n + 3)k^2 - 2n^4(n+1)^4(5n+13)k + n^5(n+1)^6. \end{aligned}$$

It is obvious that for every fixed positive number n, the function $h_5(n,k)$ in the variable k is decreasing on $(0, +\infty)$. Constant p_1 was found in the same way as in the case $\tau = 3$ (cf. Corollary 5.16). We now substitute k with $p_0 n^2 + p_1 n$ in $h_5(n,k)$ and obtain

$$\begin{aligned} h_5(n,k) &> h_5(n, p_0 n^2 + p_1 n) \\ &> n^9 + 4n^8 + 6n^7 + 3n^6 > 0 \end{aligned}$$

for $k \in [1, p_0 n^2 + p_1 n)$ and for every positive n. Thus (5.13) is satisfied for every positive $k < p_0 n^2 + p_1 n$.

To complete the proof we check the other necessary condition in Theorem 5.20 by seeing that $k < p_0 n^2 + p_1 n < n(n+1)(\sqrt{n+2} - 1)/2$ for every $n \geq 3$.

□

We have to mention that neither of the constants p_0 nor p_1 can be made larger by our method. Indeed, if we try to increase some of them, a negative coefficient will appear in the front of the highest power of n in $h_5(n, p_0n^2 + p_1n)$. In this case $h_5(n, p_0n^2 + p_1n)$ will be negative for large enough n.

Using Maple in a slightly more complicated way, one can prove that

$$B_{odd}(n,5) \geq \frac{1+2^{1/5}}{2}n^2 + (1+p_1)n + p_2 \tag{5.16}$$

$$\approx \frac{1+2^{1/5}}{2}n^2 + 1.00095n + 0.0428 \tag{5.17}$$

instead of $B_{odd}(n,5) \geq n^2+n+1$ by (2.2). In this case, the function $h_5(n, p_0n^2+p_1n+p_2)$ is a polynomial which vanishes at $n = 3$ and is positive for all $n > 3$.

To refine our approach we follow the argument from the previous section. We have to consider in (5.1) the polynomials

$$f_1(t) = (t^2 + at + b)^2(t - t_1)$$

and

$$f_2(t) = (t^2 + at + b)^2(t - t_{n^2+n+k-1})$$

for the best choices of the parameters a and b. The following lemma is analogous to Lemma 5.12 and can be proved in the same way.

Lemma 5.23 *For every real a and b*

$$t_1 \leq F(a,b) \leq t_{n^2+n+k-1},$$

where

$$F(a,b) = -\frac{n(1+a+b)^2 - 2a(n^2+n+k)(b+\frac{3}{n+2})}{(n^2+n+k)(a^2+nb^2+2b+\frac{3}{n+2}) - n(1+a+b)^2}.$$

□

We now describe a simple algorithm for proving further non-existence results for 5-designs. Given $n \geq 3$ and odd $k \geq 1$, we first use Lemma 5.23 to obtain some bounds $t_1 \leq b_1$ and $t_{n^2+n+k-1} \geq b_2$. Then, we search for polynomials

$$f(t) = (t-c)^2(t-d)^2,$$

where $b_1 \leq c \leq d \leq b_2$. The non-existence argument will hold if

$$f_0|C| - f(1) < 2f(c) + f(d).$$

Of course, it is enough to find just one pair (c,d) for which the last inequality holds. The above technique works well enough to rule out the first open case.

Theorem 5.24 *There exists no spherical 5-design with* 13 *points on* $\mathbf{S}^2$.

Proof. We obtain the bounds $t_1 \leq -0.898 = b_1$ and $t_{12} \geq 0.489 = b_2$ by using the pairs $(a,b) = (-0.148, -0.167)$ and $(1.24, 0.307)$ respectively. Then the pair $(c,d) = (-0.341, b_2)$ works.

□

The above technique works in other cases where (5.13) does not give non-existence results. For example, one can prove the non-existence of 5-designs with $45 = R(6,5) + 3$ points on $\mathbf{S}^5$ (in some sense, this is the next case after Theorem 5.24).

5.5 Bounds on the Maximal Inner Product of Designs with odd Strengths and Cardinalities

The non-existence argument from Theorem 5.4 does not work for very large odd cardinalities. In this case we are able to obtain a lower bound on the maximal inner product $s(C)$ (equivalently, an upper bound on the minimum distance $d(C) = \sqrt{2(1-s(C))}$) of all $(2e+1)$-designs $C \subset \mathbf{S}^{n-1}$ with fixed odd cardinality $R(n, 2e+1) + k$.

Theorem 5.25 *Let* $\tau = 2e+1 \geq 3$, $k \geq 3$ *be odd,* $n \geq 3$, *and* $C \subset \mathbf{S}^{n-1}$ *be a* τ*-design with* $R(n,\tau) + k$ *points. Then*

$$s(C) \geq 2\delta^2 - 1$$

and

$$d(C) \leq 2\sqrt{1-\delta^2}.$$

Proof. There exists a point $y \in C$ such that $t_2 \leq \delta$. Let the acute angle φ be such that $\cos\varphi = -\delta$. Then the angle between the vectors x_1 and x_2 does not exceed 2φ. Thus, we have

$$s(C) \geq (x_1, x_2) \geq \cos 2\varphi = 2\cos^2\varphi - 1 = 2\delta^2 - 1.$$

The bound for $d(C)$ is obtained by $d(C) = \sqrt{2(1-s(C))}$.

□

An universal bound on the maximal possible cardinality $A(n,s) = A(\mathbf{S}^{n-1}, s)$ of an $(\mathbf{S}^{n-1}, M, s)$ code is the so-called Levenshtein bound [51, 52, 53].

$$A(n,s) \leq \begin{cases} L_{2e-1}(n,s) & \text{for } t_{e-1}^{1,1} \leq s \leq t_e^{1,0}, \\ L_{2e}(n,s) & \text{for } t_e^{1,0} \leq s \leq t_e^{1,1}, \end{cases} \tag{5.18}$$

where

$$L_{2e-1}(n,s) = \binom{e+n-3}{e-1}\left[\frac{2e+n-3}{n-1} - \frac{P_{e-1}^{(n)}(s) - P_e^{(n)}(s)}{(1-s)P_e^{(n)}(s)}\right],$$

and

$$L_{2e}(n,s) = \binom{e+n-2}{e}\left[\frac{2e+n-1}{n-1} - \frac{(1+s)(P_e^{(n)}(s) - P_{e+1}^{(n)}(s))}{(1-s)(P_e^{(n)}(s) + P_{e+1}^{(n)}(s))}\right].$$

The numbers $t_e^{1,1}$ and $t_e^{1,0}$ are the greatest zeros of the Jacobi polynomials $P_e^{(\frac{n-1}{2},\frac{n-1}{2})}(t) = Q_e^{(n+2)}(t)$ and $P_e^{(\frac{n-1}{2},\frac{n-3}{2})}(t)$, respectively. The Levenshtein bound has been obtained by the linear programming approach using Theorem 1.12.

We proceed with a short explanation of the logic of the bounds (5.18). The real numbers $\{t_e^{1,1}\}_{e=0}^{\infty}$ (set $t_0^{1,1} = -1$) and $\{t_e^{1,0}\}_{e=1}^{\infty}$ divide the interval $[-1,1)$ into consecutive closed non-overlapping intervals $\{I_m\}_{m=1}^{\infty}$. For each positive integer m and all $s \in I_m$ one has $A(n,s) \leq L_m(n,s)$.

In the common boundary points of I_m and I_{m+1} we have

$$L_{2e-1}(n,t_e^{1,0}) = L_{2e}(n,t_e^{1,0}) = \binom{n+e-1}{n-1} + \binom{n+e-2}{n-1} = R(n,2e),$$

and

$$L_{2e}(n,t_e^{1,1}) = L_{2e+1}(n,t_e^{1,1}) = 2\binom{n+e-1}{n-1} = R(n,2e+1).$$

Using these relations between the Levenshtein bounds and the Delsarte-Goethals-Seidel bound, one obtains bounds on the maximal possible inner product of a spherical τ-design of a fixed cardinality. What we give below is a reformulation of (a part of) Theorem 1 in [39].

Theorem 5.26 *[39] For any spherical* $(2e+1)$*-design* $C \subset \mathbf{S}^{n-1}$,

$$s(C) \geq t_e^{(n+2)} \ (resp.\ d(C) \leq \sqrt{2(1-t_e^{(n+2)})}), \tag{5.19}$$

We recall (see Theorem 1.8 and the subsequent paragraph) that $t_e^{(n+2)} = t_e^{1,1}$ is the greatest zero of the Gegenbauer polynomial $Q_e^{(n+2)}(t)$.

To describe the asymptotic form of the bound (5.19), we need the Hermite polynomials and their greatest zeros. The Hermite polynomials can be defined [1, Chapter 22] by

$$H_0(t) = 1,\ H_1(t) = 2t,\ H_{i+1}(t) = 2tH_i(t) - 2iH_{i-1}(t),\ i \geq 1.$$

Let h_e is the greatest zero of the polynomial $H_e(t)$. Then $h_1 = 0$, $h_2 = 1/\sqrt{2}$, $h_3 = \sqrt{3/2}$ and $h_e = \sqrt{2e} + O(e^{-1/6})$ as $e \to \infty$.

Corollary 5.27 *[39, Theorem 4]*

$$s(C) \geq \sqrt{\frac{2}{n}}h_e + O(n^{-3/2}) \ as\ n \to \infty. \tag{5.20}$$

□

For τ fixed, the right-hand side of (5.20) tends to zero as n tends to ∞. For odd $|C|$, we obtain a positive lower bound on $s(C)$ that does not depend (explicitly) on n.

Theorem 5.28 *Let $\tau = 2e+1$, $k = \gamma n^e$ be odd, and*

$$(2^{1/\tau} - 1)/e! < \gamma < (\sqrt{2} - 1)/e!.$$

Then for any spherical τ-design $C \subset \mathbf{S}^{n-1}$ with odd cardinality $|C| = R(n,\tau) + k$ and as $n \to \infty$,

$$s(C) \geq h(\gamma) = \frac{1 - 2\gamma e! - \gamma^2 (e!)^2}{(1 + \gamma e!)^2} = \frac{2}{(1+\gamma e!)^2} - 1.$$

Proof. For large enough n, we have $R(n,\tau) \approx 2n^e/e!$ and Theorem 5.25 implies the assertion.

□

The function $h(\gamma)$ is strictly decreasing for $\gamma > 0$. Since $h((\sqrt{2}-1)/e!) = 0$, we have $h(\gamma) > 0$ for all $(2^{1/\tau} - 1)/e! < \gamma < (\sqrt{2}-1)/e!$. Therefore, Theorem 5.28 gives better results than (5.19) for all large enough n.

Finally, we show some improvements of the bound (5.19) by Theorem 5.25. In other words, our technique works well in small cases also.

Example 5.29 *For $\tau = k = 3$ and $n = 8, 9, 10$, bound (5.19) gives $s(C) \geq 0$, while Theorem 5.25 implies $s(C) \geq 7/121, 1/8, 31/169$ respectively. For $\tau = 5$, $k = 3$, and $n = 7, 8$, (5.13) gives $s(C) \geq 1/3, 1/\sqrt{10}$ while our Theorem 5.25 gives $s(C) \geq 607/961, 119/169$ respectively.*

Appendix

Tables with values of parameters of the $B_{n,7}$-extremal polynomials

n	a	b	c	d	q
4	0.6774	-0.3116	-0.1926	-0.0106	0.1506
5	0.6711	-0.2718	-0.1748	-0.0088	0.1461
6	0.6648	-0.2371	-0.1586	-0.0079	0.1415
7	0.6585	-0.2067	-0.1442	-0.0073	0.1368
8	0.6524	-0.1796	-0.1317	-0.0070	0.1323
9	0.6467	-0.1553	-0.1207	-0.0069	0.1279
10	0.6413	-0.1332	-0.1110	-0.0068	0.1238

Table 1. Values of the parameters a, b, c, d, q of the $B_{n,6}$-extremal polynomial

n	a	b	c	d	q
5	0.1731	-0.3533	-0.057	0.0029	0.1476
6	0.1659	-0.3155	-0.0498	0.0019	0.1423
7	0.1590	-0.2813	-0.0433	0.0010	0.1372

Table 2. Values of the parameters a, b, c, d, q of the $B_{n,7}$-extremal polynomial.

n	a	b	c	d	e	q
4	0.6833	-0.5497	-0.3436	0.0187	0.0146	0.1549
5	0.6802	-0.5016	-0.3177	0.0157	0.0130	0.1527
6	0.6768	-0.4587	-0.2937	0.0126	0.0113	0.1503
7	0.6730	-0.4205	-0.2720	0.0098	0.0098	0.1475
8	0.6689	-0.3865	-0.2525	0.0073	0.0085	0.1445
9	0.6646	-0.3560	-0.2350	0.0052	0.0074	0.1413
10	0.6601	-0.3285	-0.2194	0.0034	0.0065	0.1380
11	0.6556	-0.3035	-0.2053	0.0019	0.0057	0.1346
12	0.6511	-0.2807	-0.1926	0.0006	0.0050	0.1313
13	0.6467	-0.2597	-0.1812	-0.0006	0.0045	0.1280
14	0.6425	-0.2401	-0.1709	-0.0017	0.0040	0.1247
15	0.6385	-0.2219	-0.1615	-0.0025	0.0036	0.1216
16	0.6346	-0.2048	-0.1529	-0.0034	0.0032	0.1186
17	0.6309	-0.1886	-0.1451	-0.0041	0.0029	0.1158

Table3. Values of the parameters a, b, c, d, e, q of the $B_{n,8}$-extremal polynomial.

n	a	b	c	d	e	q
4	0.1852	-0.6384	-0.1110	0.0519	0.0069	0.1582
5	0.1807	-0.5903	-0.0986	0.0452	0.0055	0.1530
6	0.1757	-0.5470	-0.0879	0.0390	0.0044	0.1496
7	0.1707	-0.5078	-0.0788	0.0336	0.0036	0.1458
8	0.1654	-0.4722	-0.0709	0.0288	0.0030	0.1419
9	0.1600	-0.4396	-0.0640	0.0245	0.0025	0.1379
10	0.1546	-0.4096	-0.0580	0.0208	0.0021	0.1339
11	0.1493	-0.3817	-0.0527	0.0175	0.0018	0.1299
12	0.1443	-0.3557	-0.0481	0.0146	0.0015	0.1261
13	0.1396	-0.3312	-0.0441	0.0120	0.0013	0.1225
14	0.1352	-0.3080	-0.0406	0.0097	0.0011	0.1191

Table 4 Values of the parameters a, b, c, d, e, q of the $B_{n,9}$-extremal polynomial.

n	a	b	c	d	e	f	q
4	0.6872	-0.7916	-0.5039	0.1000	0.0601	0.0016	0.1577
5	0.6867	-0.7371	-0.4727	0.0874	0.0535	0.0010	0.1573
6	0.6860	-0.6871	-0.4435	0.0757	0.0473	0.0007	0.1568
7	0.6846	-0.6417	-0.4164	0.0654	0.0418	0.0005	0.1559
8	0.6827	-0.6008	-0.3915	0.0566	0.0369	0.0004	0.1545
9	0.6803	-0.5639	-0.3686	0.0491	0.0327	0.0004	0.1527
10	0.6774	-0.5304	-0.3477	0.0426	0.0291	0.0003	0.1507
11	0.6743	-0.4999	-0.3285	0.0371	0.0260	0.0003	0.1484
12	0.6709	-0.4720	-0.3109	-0.0323	0.0234	0.0003	0.1459
13	0.6673	-0.4463	-0.2948	0.0282	0.0210	0.0003	0.1433
14	0.6637	-0.4227	-0.2800	0.0245	0.0190	0.0003	0.1406
15	0.6600	-0.4007	-0.2664	0.0213	0.0172	0.0003	0.1379
16	0.6563	-0.3803	-0.2538	0.0185	0.0157	0.0002	0.1351
17	0.6526	-0.3612	-0.2422	0.0160	0.0143	0.0002	0.1324
18	0.6490	-0.3433	-0.2315	0.0137	0.0131	0.0002	0.1296
19	0.6454	-0.3264	-0.2215	0.0117	0.0121	0.0002	0.1269
20	0.6419	-0.3105	-0.2122	0.0098	0.0111	0.0002	0.1243
21	0.6386	-0.2953	-0.2036	0.0082	0.0102	0.0002	0.1217
22	0.6353	-0.2809	-0.1956	0.0066	0.0095	0.0002	0.1192
23	0.6322	-0.2672	-0.1881	0.0052	0.0088	0.0002	0.1168
24	0.6292	-0.2540	-0.1811	0.0039	0.0082	0.0002	0.1144
25	0.6263	-0.2413	-0.1745	0.0027	0.0076	0.0002	0.1122
26	0.6236	-0.2292	-0.1684	0.0016	0.0071	0.0002	0.1100

Table 5. Values of the parameters a, b, c, d, e, f, q of the $B_{n,10}$-extremal polynomial.

n	a	b	c	d	e	f	q
4	0.1885	-0.8831	-0.1571	0.1549	0.0236	-0.0019	0.1586
5	0.1865	-0.8304	-0.1434	0.1382	0.0196	-0.0019	0.1572
6	0.1840	-0.7819	-0.1314	0.1231	0.0164	0.0017	0.1554
7	0.1811	-0.7374	-0.1207	0.1096	0.0139	-0.0015	0.1533
8	0.1778	-0.6967	-0.1111	0.0977	0.0119	0.0013	0.1509
9	0.1741	-0.6593	-0.1024	0.0872	0.0102	-0.0011	0.1483
10	0.1701	-0.6248	-0.0945	0.0780	0.0089	-0.0009	0.1453
11	0.1658	-0.5929	-0.0873	0.0700	0.0077	-0.0008	0.1422
12	0.1614	-0.5634	-0.0807	0.0628	0.0067	-0.0007	0.1390
13	0.1570	-0.5358	-0.0747	0.0564	0.0059	-0.0006	0.1357
14	0.1525	-0.5101	-0.0693	0.0507	0.0052	-0.0005	0.1323
15	0.1480	-0.4858	-0.0643	0.0456	0.0046	-0.0004	0.1289
16	0.1436	-0.4630	-0.0598	0.0410	0.0041	-0.0003	0.1256
17	0.1393	-0.4414	-0.0558	0.0369	0.0036	-0.0003	0.1223
18	0.1352	-0.4208	-0.0520	0.0331	0.0032	-0.0002	0.1191
19	0.1312	-0.4011	-0.0486	0.0296	0.0029	-0.0002	0.1160
20	0.1275	-0.3822	-0.0455	0.0264	0.0026	-0.0001	0.1131
21	0.1241	-0.3640	-0.0428	0.0234	0.0024	-0.0001	0.1104
22	0.1209	-0.3463	-0.0403	0.0206	0.0022	-0.0001	0.1078
23	0.1180	-0.3292	-0.0381	0.0181	0.0020	-0.0000	0.1055

Table 6. Values of the parameters a, b, c, d, e, f, q of the $B_{n,11}$-extremal polynomial.

n	a	b	c	q
3	1.2061	0.0898	-0.1375	0.3210
4	1.3891	0.3334	-0.0767	0.3111
5	1.5477	0.5477	-0.0170	0.3007
6	1.6853	0.7392	0.0412	0.2895
7	1.8056	0.9121	0.0974	0.2777
8	1.9120	1.0699	0.1514	0.2658
9	2.0076	1.2153	0.2035	0.2543
10	2.0946	1.3508	0.2537	0.2435
11	2.1749	1.4784	0.3024	0.2339

Table 7. Values of the parameters a, b, c, q of the $B_{1,n,5}$-extremal polynomial.

n	a	b	c	q
3	0.7168	-0.2238	-0.1024	0.3250
4	0.9218	-0.0391	-0.0896	0.3231
5	1.1006	0.1337	-0.0667	0.3193
6	1.2559	0.2956	-0.0375	0.3131
7	1.3914	0.4467	-0.0047	0.3051
8	1.5105	0.5875	0.0299	0.2956
9	1.6162	0.7186	0.0653	0.2851
10	1.7108	0.8409	0.1008	0.2740
11	1.7963	0.9555	0.1360	0.2627
12	1.8743	1.0634	0.1707	0.2514
13	1.9461	1.1655	0.2049	0.2404
14	2.0129	1.2628	0.2385	0.2299
15	2.0756	1.3561	0.2717	0.2203
16	2.1350	1.4462	0.3046	0.2116

Table 8. Values of the parameters a, b, c, q of the $B_{1,n,6}$-extremal polynomial.

n	a	b	c	d	q
3	0.9534	-0.2850	-0.2995	-0.0069	0.4552
4	1.2787	0.0787	-0.2667	-0.0361	0.4008
5	1.5375	0.4269	-0.1773	-0.0480	0.3707
6	1.7463	0.7450	-0.0605	-0.0469	0.3520
7	1.9193	1.0327	0.0678	-0.0371	0.3382
8	2.0662	1.2937	0.1996	-0.0217	0.3265
9	2.1935	1.5317	0.3309	-0.0028	0.3158
10	2.3058	1.7504	0.4597	0.0185	0.3053
11	2.4059	1.9522	0.5850	-0.0413	0.2948
12	2.4962	2.1396	0.7064	0.0649	0.2843
13	2.5783	2.3144	0.8237	0.0851	0.2738
14	2.6535	2.4782	0.9370	0.1135	0.2633
15	2.7229	2.6324	1.0466	0.1380	0.2530
16	2.7875	2.7783	1.1526	0.1625	0.2429
17	2.8479	2.9171	1.2556	0.1869	0.2331
18	2.9048	3.0497	1.3558	0.2112	0.2237
19	2.9588	3.1772	1.4536	0.2355	0.2150
20	3.0103	3.3004	1.5496	0.2597	0.2069
21	3.0598	3.4201	1.6441	0.2840	0.1997
22	3.1077	3.5371	1.7376	0.3083	0.1933
23	3.1543	3.6519	1.8304	0.3328	0.1880

Table 9. Values of the parameters a, b, c, d, q of the $B_{1,n,7}$-extremal polynomial.

n	a	b	c	q
3	1.3249	0.2833	-0.0667	0.3036
4	1.5928	0.6298	0.0209	0.2820
5	1.8133	0.9308	0.1108	0.2640

Table 10. Values of the parameters a, b, c, d, q of the $B_{2,n,5}$-extremal polynomial.

n	a	b	c	q
4	1.1423	0.2231	-0.0353	0.2933
5	1.3835	0.4716	0.0134	0.2772
6	1.5805	0.6982	0.0690	0.5691

Table 11 Values of the parameters a, b, c, q of the $B_{2,n,6}$-extremal polynomial.

n	a	b	c	d	q
3	1.3597	0.2034	-0.2196	-0.0361	0.3106
4	1.6761	0.6812	-0.0528	-0.0376	0.2988
5	1.9406	1.1169	0.1426	-0.0207	0.2868
6	2.1614	1.5113	0.3492	0.0075	0.2731
7	2.3482	1.8678	0.5571	0.0424	0.2582
8	2.5090	2.1920	0.7614	0.0815	0.2433
9	2.6502	2.4895	0.9605	0.1230	0.2291
10	2.7768	2.7662	1.1546	0.1662	0.2168
11	2.8925	3.0270	1.3447	0.2106	0.2069

Table 12. Values of the parameters a, b, c, d, q of the $B_{2,n,7}$-extremal polynomial.

n	a	b	c	d	q
3	1.5469	0.5414	-0.0596	-0.0259	0.2878

Table 13. Values of the parameters a, b, c, d, q of the $B_{4,3,7}$-extremal polynomial.

n	a	b	c	d	q
5	0.0000	-0.3520	0.0000	0.0027	0.1799
6	0.0000	-0.3145	0.0000	0.0018	0.1724
7	0.0000	-0.2808	0.0000	0.0009	0.1649

Table 14. Values of the parameters a, b, c, d, q of the $B_{1/2,n,3}$-extremal polynomial.

n	a	b	c	d	e	q
4	0.0000	-0.6356	0.0000	0.0509	0.0000	0.1931
5	0.0000	-0.5872	0.0000	0.0442	0.0000	0.1889
6	0.0000	-0.5437	0.0000	0.0381	0.0000	0.1840
7	0.0000	-0.5047	0.0000	0.0327	0.0000	0.1787
8	0.0000	-0.4693	0.0000	0.0280	0.0000	0.1731
9	0.0000	-0.4371	0.0000	0.0239	0.0000	0.1673
10	0.0000	-0.4074	0.0000	0.0203	0.0000	0.1615
11	0.0000	-0.3800	0.0000	0.0172	0.0000	0.1558
12	0.0000	-0.3544	0.0000	0.0144	0.0000	0.1503
13	0.0000	-0.3303	0.0000	0.0119	0.0000	0.1451
14	0.0000	-0.3075	0.0000	0.0096	0.0000	0.1402

Table 15. Values of the parameters a, b, c, d, e, q of the $B_{1/2,n,4}$-extremal polynomial.

n	a	b	c	d	e	f	q
4	0.0000	-0.8795	0.0000	0.1527	0.0000	-0.0017	0.1971
5	0.0000	-0.8260	0.0000	0.1358	0.0000	-0.0017	0.1955
6	0.0000	-0.7770	0.0000	0.1207	0.0000	-0.0016	0.1933
7	0.0000	-0.7324	0.0000	0.1073	0.0000	-0.0014	0.1905
8	0.0000	-0.6917	0.0000	0.0956	0.0000	0.0012	0.1871
9	0.0000	-0.6544	0.0000	0.0853	0.0000	-0.0010	0.1833
10	0.0000	-0.6202	0.0000	0.0763	0.0000	-0.0009	0.1790
11	0.0000	-0.5886	0.0000	0.0684	0.0000	-0.0007	0.1745
12	0.0000	-0.5593	-0.0000	0.0614	0.0000	-0.0006	0.1698
13	0.0000	-0.5321	0.0000	0.0552	0.0000	-0.0005	0.1650
14	0.0000	-0.5066	0.0000	0.0497	0.0000	-0.0004	0.1602
15	0.0000	-0.4828	0.0000	0.0448	0.0000	-0.0004	0.1553
16	0.0000	-0.4603	0.0000	0.0403	0.0000	-0.0003	0.1505
17	0.0000	-0.4390	0.0000	0.0363	0.0000	-0.0002	0.1459
18	0.0000	-0.4187	0.0000	0.0326	0.0000	-0.0002	0.1413
19	0.0000	-0.3994	0.0000	0.0292	0.0000	-0.0002	0.1370
20	0.0000	-0.3808	0.0000	0.0261	0.0000	-0.0001	0.1329
21	0.0000	-0.3630	-0.0000	0.0232	0.0000	-0.0001	0.1290
22	0.0000	-0.3457	0.0000	0.0205	0.0000	-0.0001	0.1255
23	0.0000	-0.3282	0.0000	0.0180	0.0000	0.0000	0.1222

Table 16. Values of the parameters a, b, c, d, e, f, q of the $B_{1/2,n,5}$-extremal polynomial.

Proof of the Theorem 4.25

Using Lukács Theorem [73, p.4] $f(t)$ must have the following form:

$$f(t) = \begin{cases} (1+t)(A_{k+1}(t))^2 + (1-t)(B_{k+1}(t))^2 & \text{when } deg(f) = 2k+3, \\ (A_{k+2}(t))^2 + (1-t^2)(B_{k+1}(t))^2 & \text{when } deg(f) = 2k+4, \end{cases}$$

where the indices coincide with the degrees of the corresponding polynomials.

Case1. $\tau = 2k$

Now let

$$A_{k+1}(t) = \sum_{i=0}^{k+1} f_i(A) Q_i^{0,1}(t), \quad B_{k+1}(t) = \sum_{i=0}^{k+1} f_i(B) Q_i^{1,0}(t).$$

Then

$$\Omega(f) = \frac{f(1)}{f_0} = \frac{2(\sum_{i=0}^{k+1} f_i(A))^2}{\frac{1}{c^{0,1}} \sum_{i=0}^{k+1} \frac{(f_i(A))^2}{r_i^{0,1}} + \frac{1}{c^{1,0}} \sum_{i=0}^{k+1} \frac{(f_i(B))^2}{r_i^{1,0}}}$$

The ZSF expansion is $f(t) = \sum_{i=0}^{2k+3} f_i(f) Q_i(t)$. We will denote by

$$\begin{aligned} x_{i+1} &= f_i(A) & i &= 0, \ldots, k+1 \\ y_{i+1} &= f_i(B) & i &= 0, \ldots, k+1 \\ \alpha_{i+1} &= \tfrac{1}{2c^{0,1} r_i^{0,1}} & i &= 0, \ldots, k+1 \\ \beta_{i+1} &= \tfrac{1}{2c^{1,0} r_i^{1,0}} & i &= 0, \ldots, k+1 \end{aligned}$$

$$\begin{aligned}
A_{11} &= (a_{k+1,k-1}^{0,1})^2 + 2a_{k+1,k+1}^{0,1} a_{k+1,k-1}^{0,1} + 2a_{k+1,k+1}^{0,1} a_{k+1,k}^{0,1} - (a_{k+1,k+1}^{0,1})^2 \tfrac{a_{2k+3,2k+1}}{a_{2k+3,2k+3}} \\
A_{12} &= 2a_{k,k-1}^{0,1} a_{k+1,k+1}^{0,1} + 2a_{k,k}^{0,1} a_{k+1,k}^{01} + 2a_{k,k}^{0,1} a_{k+1,k+1}^{0,1} \\
A_{13} &= 2a_{k-1,k-1}^{0,1} a_{k+1,k+1}^{0,1} \\
A_{22} &= (a_{k,k}^{0,1})^2 \\
B_{11} &= -(a_{k+1,k}^{1,0})^2 - 2a_{k+1,k+1}^{1,0} a_{k+1,k-1}^{1,0} + 2a_{k+1,k+1}^{10} a_{k+1,k}^{1,0} + (a_{k+1,k+1}^{10})^2 z \\
B_{12} &= 2a_{k,k}^{1,0} a_{k+1,k+1}^{1,0} - 2a_{k,k-1}^{1,0} a_{k+1,k+1}^{1,0} - 2a_{k,k}^{1,0} a_{k+1,k}^{1,0} \\
B_{13} &= -2a_{k-1,k-1}^{1,0} a_{k+1,k+1}^{1,0} \\
B_{22} &= -(a_{k,k}^{1,0})^2 \\
C_{11} &= (a_{k+1,k+1}^{0,1})^2 + 2a_{k+1,k+1}^{0,1} a_{k+1,k}^{0,1} \\
C_{12} &= 2a_{k,k}^{0,1} a_{k+1,k+1}^{0,1} \\
D_{11} &= -2a_{k+1,k+1}^{1,0} a_{k+1,k}^{1,0} + (a_{k+1,k+1}^{1,0})^2 \\
D_{12} &= -2a_{k,k}^{1,0} a_{k+1,k+1}^{1,0} \\
E &= \left(\tfrac{a_{k+1,k+1}^{1,0}}{a_{k+1,k+1}^{0,1}}\right)^2 \\
\tilde{f} &= f_{\tau+2} a_{\tau+2,\tau+2}
\end{aligned}$$

Case2. $\tau = 2k+1$

Now

$$A_{k+2}(t) = \sum_{i=0}^{k+2} f_i(A) Q_i(t), \quad B_{k+1}(t) = \sum_{i=0}^{k+1} f_i(B) Q_i^{1,1}(t).$$

Then

$$\Omega(f) = \frac{f(1)}{f_0} = \frac{(\sum_{i=0}^{k+2} f_i(A))^2}{\sum_{i=0}^{k+2} \frac{(f_i(A))^2}{r_i} + \frac{1}{c^{1,1}} \sum_{i=0}^{k+1} \frac{(f_i(B))^2}{r_i^{1,1}}}$$

In terms of ZSF we have the expansion $f(t) = \sum_{i=0}^{2k+4} f_i Q_i(t)$. Let us denote by

$$\begin{array}{ll}
x_i = f_i(A) & i = 0, \dots, k+2 \\
y_{i+1} = f_i(B) & i = 0, \dots, k+1 \\
\alpha_i = \frac{1}{r_i} & i = 0, \dots, k+2 \\
\beta_{i+1} = \frac{1}{c^{1,1} r_i^{1,1}} & i = 0, \dots, k+1 \\
A_{11} = (a_{k+2,k+1})^2 + 2a_{k+2,k+2}a_{k+2,k} - (a_{k+2,k+2})^2 \frac{a_{k+4,k+2}}{a_{k+4,k+4}} & \\
A_{12} = 2a_{k+1,k}a_{k+2,k+2} + 2a_{k+1,k+1}a_{k+2,k+1} & \\
A_{13} = a_{k,k}a_{k+2,k+2} & \\
A_{22} = (a_{k+1,k+1})^2 & \\
B_{11} = (a^{1,1}_{k+1,k+1})^2 - (a^{1,1}_{k+1,k})^2 - 2a^{1,1}_{k+1,k+1}a^{1,1}_{k+1,k-1} + (a^{1,1}_{k+1,k+1})^2 \frac{a_{k+4,k+2}}{a_{k+4,k+4}} & \\
B_{12} = -2a^{1,1}_{k,k}a^{1,1}_{k+1,k} - 2a^{1,1}_{k,k-1}a^{1,1}_{k+1,k+1} & \\
B_{13} = -2a^{1,1}_{k-1,k-1}a^{1,1}_{k+1,k+1} & \\
B_{22} = -(a^{1,1}_{k,k})^2 & \\
C_{11} = 2a_{k+2,k+2}a_{k+2,k+1} & \\
C_{12} = 2a_{k+1,k+1}a_{k+2,k+2} & \\
D_{11} = -2a^{1,1}_{k+1,k+1}a^{1,1}_{k+1,k} & \\
D_{12} = -2a^{1,1}_{k,k}a^{1,1}_{k+1,k+1} & \\
E = (\frac{a^{1,1}_{k+1,k+1}}{a_{k+2,k+2}})^2 & \\
\tilde{f} = f_{\tau+2}a_{\tau+2,\tau+2} &
\end{array}$$

In general we can unified the notations for $\tau = 2k+\theta$ and will abbreviate $\underline{x} = (x_{1-\theta}, \dots, x_{k+2})$ and $\underline{y} = (y_1, \dots, y_{k+2})$. Now we have to solve the following optimization problem :

Maximize the function

$$F(\underline{x}, \underline{y}) = \frac{(\sum_{i=1-\theta}^{k+2} x_i)^2}{\sum_{i=1-\theta}^{k+2} \alpha_i x_i^2 + \sum_{i=1}^{k+2} \beta_i y_i^2}$$

under the conditions: [1].

$$\left| \begin{array}{l}
x_{k+2}^2 < E y_{k+2}^2 \\
G_1(\underline{x}, \underline{y}) = A_{11}x_{k+2}^2 + A_{12}x_{k+2}x_{k+1} + A_{13}x_{k+2}x_k + A_{22}x_{k+1}^2 \\
+B_{11}y_{k+2}^2 + B_{12}y_{k+2}y_{k+1} + B_{13}y_{k+2}y_k + B_{22}y_{k+1}^2 = 0 \\
G_2(\underline{x}, \underline{y}) = C_{11}x_{k+2}^2 + C_{12}x_{k+2}x_{k+1} + D_{11}y_{k+2}^2 + D_{12}y_{k+2}y_{k+1} - \tilde{f} = 0
\end{array} \right.$$

The coefficients α_i, β_i are positive. Hence $F(\underline{x}, \underline{y}) \geq 0$. We solve the problem by means of the Lagrange multiplier method as follows. We consider the function

[1] The conditions were obtained equating the coefficients in front of $t^{\tau+1}$, $t^{\tau+2}$, $t^{\tau+3}$ in ZSF expansion of $f(t)$ and using the antipodality of the PMS (see Lemma 2.2)

$$\tilde{F}(\underline{x}, \underline{y}, \lambda_1, \lambda_2) = F(\underline{x}, \underline{y}) - \lambda_1 G_1(\underline{x}, \underline{y}) - \lambda_2 G_2(\underline{x}, \underline{y})$$

and maximize it for $\underline{x}$, $\underline{y}$. A necessary condition is the first derivatives of $\tilde{F}$ to be zero. We denote

$$\mu = \frac{\sum_{i=1-\theta}^{k+2} x_i}{\sum_{i=1-\theta}^{k+2} \alpha_i x_i^2 + \sum_{i=1}^{k+2} \beta_i y_i^2}$$

and consider only the case $\mu \neq 0$, because otherwise we obtain $\min F(\underline{x}, \underline{y}) = 0$. After the simplification of the derivatives we have

$$\begin{aligned} x_i &= \frac{1}{\mu \alpha_i} \quad \text{for} \quad i = 1-\theta, \ldots, k-1 \\ y_i &= 0, \quad \text{for} \quad i = 1, \ldots, k-1 \end{aligned}$$

$$\left| \begin{array}{llllllll} \lambda_1 A_{13} & x_{k+2} & + & & & + & 2\mu^2\alpha_k \; x_k & = 2\mu \\ (\lambda_1 A_{12} + \lambda_2 C_{12}) & x_{k+2} & + & (2\lambda_1 A_{22} + 2\mu^2 \alpha_{k+1}) & x_{k+1} & & & = 2\mu \\ (2\lambda_2 C_{11} + 2\lambda_1 A_{11} + 2\mu^2 \alpha_{k+2}) & x_{k+2} & + & (\lambda_1 A_{12} + \lambda_2 C_{12}) & x_{k+1} & + & \lambda_1 A_{13} \; x_k & = 2\mu \end{array} \right.$$

$$\left| \begin{array}{llllllll} \lambda_1 B_{13} & y_{k+2} & + & & & + & 2\mu^2\beta_k \; y_k & = 0 \\ (\lambda_1 B_{12} + \lambda_2 D_{12}) & y_{k+2} & + & (2\lambda_1 B_{22} + 2\mu^2 \beta_{k+1}) & y_{k+1} & & & = 0 \\ (2\lambda_2 D_{11} + 2\lambda_1 B_{11} + 2\mu^2 \beta_{k+2}) & y_{k+2} & + & (\lambda_1 B_{12} + \lambda_2 D_{12}) & y_{k+1} & + & \lambda_1 B_{13} \; y_k & = 0 \end{array} \right.$$

$$\left| \begin{array}{rcl} G_1(\underline{x}, \underline{y}) & = & A_{11} x_{k+2}^2 + A_{12} x_{k+2} x_{k+1} + A_{13} x_{k+2} x_k + A_{22} x_{k+1}^2 \\ & + & B_{11} y_{k+2}^2 + B_{12} y_{k+2} y_{k+1} + B_{13} y_{k+2} y_k + B_{22} y_{k+1}^2 = 0 \\ G_2(\underline{x}, \underline{y}) & = & C_{11} x_{k+2}^2 + C_{12} x_{k+2} x_{k+1} + D_{11} y_{k+2}^2 + D_{12} y_{k+2} y_{k+1} - \tilde{f} = 0 \end{array} \right.$$

The second linear system must have nonzero solution, hence the determinant Δ

$$\Delta := \begin{vmatrix} \lambda_1 B_{13} & 0 & 2\mu^2 \alpha_k \\ \lambda_1 B_{12} + \lambda_2 D_{12} & 2\lambda_1 B_{22} + 2\mu^2 \alpha_{k+1} & 0 \\ 2\lambda_2 D_{11} + 2\lambda_1 B_{11} + 2\mu^2 \beta_{k+2} & \lambda_1 B_{12} + \lambda_2 D_{12} & \lambda_1 B_{13} \end{vmatrix} = 0.$$

Now solving the same system we obtain

$$\begin{aligned} y_k &= -\frac{\lambda_1 B_{13}}{2\mu^2 \beta_k} y_{k+2} \\ y_{k+1} &= -\frac{\lambda_1 B_{12} + \lambda_2 D_{12}}{2\lambda_1 B_{22} + 2\mu^2 \beta_{k+1}} y_{k+2}. \end{aligned}$$

From the first linear system we can express x_k and x_{k+1} and x_{k+2}. Substituting x_k, x_{k+1} x_{k+2}, y_k and y_{k+1} in $G_1(\underline{x}, \underline{y}) = 0$ we obtain y_{k+2}. Now we do the same with x_k, x_{k+1}, x_{k+2}, y_k, y_{k+1} and y_{k+2} in $G_2(\underline{x}, \underline{y}) = 0$. Then we solve the obtained equation together with the equation $\Delta = 0$ as a nonlinear system with respect to λ_1 and λ_2.

Using the same arguments as in Theorem 4.23 we can chose $\mu = 1$.

□

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Curriculum Vitae

Svetla Nikova was born on January 22, 1969 in Veliko Tarnovo, Bulgaria. From October 1987 till January 1993 she studied Mathematics in Sofia University "St. Kliment Ochridski", where she received her Master of Science degree in Algebra. From June 1994 till June 1997 she was a Ph.D. Student in Veliko Tarnovo University and from October 1997 till October 1998 a Ph.D. Student in Eindhoven University of Technology at the Discrete Mathematics group.

STATEMENTS

accompanying the dissertation

Bounds for Designs in Infinite Polynomial Metric Spaces

Svetla Iordanova Nikova

1.For any polynomial metric space $\mathcal{M}$ and for any τ-design C

$$|C| \geq R(\mathcal{M}, \tau) = 2^{\theta} c^{0,\theta} \sum_{i=0}^{e} r_i^{0,\theta}, \qquad (1)$$

where $\tau = 2e + \theta$, $\theta \in \{0, 1\}$. For definitions of $c^{0,\theta}$ and $r_i^{0,\theta}$ see Chapter 1 of this thesis.

[1] P.Delsarte, An Algebraic Approach to Association Schemes in Coding Theory, Philips Research Reports Suppl., 10, 1973.
[2] P.Delsarte, J.-M.Goethals, J.J.Seidel, Spherical codes and designs, *Geom. Dedicata* 6, 1977, 363-388.
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[4] V.I.Levenshtein, Designs as maximum codes in polynomial metric spaces, *Acta Applicandae Math.* 25, 1992, 1-82.

2.For any $\tau = 2e + \theta$, $\theta \in \{0, 1\}$,

$$R(\mathcal{M}, \tau) = \max \Omega(f), \qquad (2)$$

where the maximum is taken over the class of polynomials $f(t) \in B_{\mathcal{M},\tau}$ of degree at most τ. The maximum in (2) is realized if and only if $f(t)$ is proportional to $f^{(\tau)}(t) = (t+1)^{\theta}((Q_e^{1,\theta}(t))^2$.
For definitions of $\Omega(f)$ and $B_{\mathcal{M},\tau}$ see p.15 of this thesis.

[1] I.Schoenberg, G.Szegö, An extremum problem for polynomials, *Comp. Math.* 14, 1960, 260-268.
[2] V.I.Levenshtein, Universal bounds for codes and designs, Chapter in Handbook of Coding Theory, V.Pless, W.C.Huffman, and R.A.Brualdi, Eds. Amsterdam: Elsevier, to appear.

3. The bound $R(\mathcal{M}, \tau)$ can be improved by a polynomial $f(t) \in B_{\mathcal{M},\tau}$ of degree at least $\tau + 1$, if and only if $G_{\tau}(\mathcal{M}, Q_j) < 0$ for some $j \geq \tau + 1$. Moreover, if $G_{\tau}(\mathcal{M}, Q_j) < 0$ for some $j \geq \tau + 1$, then $R(\mathcal{M}, \tau)$ can be improved by a polynomial in $B_{\mathcal{M},\tau}$ of degree j.

The linear functional $G_\tau(\mathcal{M}, Q_j)$ is defined on p.56 of this thesis.
Chapter 4 of this Ph.D. Thesis.

4. If C is a $2e$-design on the unit sphere in $\mathbb{R}^n$, which contains an antipodal pair then

$$|C| \geq 2\binom{n+e-1}{n-1}.$$

[1] C.D. Godsil, Algebraic Combinatorics, Chapman and Hall, London, Chapter 16, p.346.

5.Let $C \subset \mathbf{S}^{n-1}$ be a τ-design and $y \in C$. We consider the multi-set

$$I(y) = \{(x, y) : x \in C, x \neq y\} = \{t_1, t_2, \dots, t_{|C|-1}\},$$

Let $\tau = 2e+1$ and $|C| = R(n, \tau) + k$. Then the smallest inner product t_1 satisfy

$$t_1 \leq -\frac{R(n,\tau)}{R(n,\tau)+2k}.$$

Chapter 5 of this Ph.D. Thesis.

6. For fixed $\tau = 2e + 1 \geq 3$ and $n \to \infty$, we have

$$B_{odd}(n, \tau) \geq \frac{1+2^{1/\tau}}{e!} n^e.$$

Chapter 5 of this Ph.D. Thesis.

7. The number of people who can understand simple mathematical ideas is not smaller than the number of those who are commonly called musical.

8. The best way to learn mathematics is to *do* mathematics.