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Moore-Smith theory for Uniform Spaces through Asymptotic Equivalence

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Abstract

Moore-Smith theory tells us how to generate a topology by defining convergence of nets rather than using a definition of open set. In this report, we extend this theory to uniform spaces, and show how a uniform space can be generated by defining asymptotic equivalence of nets rather than using a definition of entourage.

Keywords: Moore-Smith theory, topology, uniform spaces, asymptotic equivalence

1 Introduction

That limits can be used to characterize a topological space is very well known, and every standard textbook on topology discusses the theorem that a function f is continuous if and only if for every converging net $n = \{n_d \mid d \in D\}$ the net $f \cdot n = \{f(n_d) \mid d \in D\}$ is also converging, and for every limit point n_ω of n , $f(n_\omega)$ is a limit point of $f \cdot n$.

The message of Moore-Smith theory (see e.g. [1]), is that one can actually define any topological space using a suitable notion of limit on nets, known as a convergence class, rather than starting from the usual notion of open set.

In this report, we show a variant on that theme, namely that one can define any uniform space using a suitable notion of asymptotic equivalence of nets, rather than starting from the usual notion of entourage.

That asymptotic equivalence can be used to characterize a uniform space, i.e. that a function f is uniformly continuous if and only if for every asymptotically equivalent pair of nets $n \approx m$ in the domain of f we find $f \cdot n \approx f \cdot m$, is a result due to Fuller [2]. However, this result did not get included in the standard textbooks. In fact, most of the textbooks do not get to discussing the notion of asymptotic equivalence in uniform spaces at all (exercise 1 of section 23 in the book of Čech [3] forming a notable exception).

In the remainder of this report, we start by giving the necessary formal definitions, and consecutively prove four theorems that together show that every uniform space can be generated by a notion of asymptotic equivalence on nets. Since the definitions that are used in this report very closely follow the the

definitions in chapter 2 of [1], we refer to this standard textbook for the intuitions behind, and explanations of, these definitions. Our only deviation from [1] in this respect, is the notation we use for a product of a directed family of nets, which is known in [1] as the 'iterative limit'.

2 Preliminary definitions

Definition 1 (Directed Set) A directed set $\langle D, \leq \rangle$, is a set D with a binary relation $\leq \subseteq D \times D$ on it that is:

- *Transitive:* $\forall d, e, f \in D \ d \leq e \wedge e \leq f \Rightarrow d \leq f$;
- *Reflexive:* $\forall d \in D \ d \leq d$;
- *Directed:* $\forall d, e \in D \exists f \in D \ d \leq f \wedge e \leq f$.

A cofinality between directed sets $\langle D, \leq \rangle$ and $\langle E, \preceq \rangle$ is a function $h : D \rightarrow E$ such that

- *Cofinality:* $\forall e \in E \exists d \in D \forall d' \in D \ d \leq d' \Rightarrow e \preceq h(d')$.

Definition 2 (Net) A net in \mathbb{X} is a function $f : D \rightarrow \mathbb{X}$ from a directed set D to a set \mathbb{X} , where the relation \leq is usually left implicit. A net $f : D \rightarrow \mathbb{X}$ is a subnet of a net $g : E \rightarrow \mathbb{X}$ if there exists a cofinality $h : D \rightarrow E$ such that $f(d) = g(h(d))$ for all $d \in D$. Given a family $\{f_e : D_e \rightarrow \mathbb{X} \mid e \in E\}$ of nets in \mathbb{X} , the product $\prod_{e \in E} f_e$ of this family is the net $\pi : (E \times \prod_{e \in E} D_e) \rightarrow \mathbb{X}$ such that $\pi(e, p) = f_e(p(e))$, where the product set $E \times \prod_{e \in E} D_e$ is ordered, regardless of a possible ordering on E , by $(e, p) \leq (e', p')$ iff for all $e \in E$ we have $p(e) \leq p'(e)$.

Often, we'll say that a property P holds *eventually* for a net $f : D \rightarrow \mathbb{X}$. By this, we formally mean that there exists a $d \in D$ such that for every $e \geq d$ the property P holds for $f(e)$. Clearly, eventuality of properties is preserved when taking subnets.

Definition 3 (Uniform Structure) A uniform structure on a set \mathbb{X} , is a set $\mathcal{U} \subseteq \mathcal{P}(\mathbb{X} \times \mathbb{X})$ of binary relations on \mathbb{X} (called entourages) such that

- *Identity:* for every $U \in \mathcal{U}$ and every $x \in \mathbb{X}$, $(x, x) \in U$;
- *Filtering:* if $U \in \mathcal{U}$ and $U \subseteq V \subseteq \mathbb{X} \times \mathbb{X}$ then $V \in \mathcal{U}$;
- *Finite intersection:* if $U, V \in \mathcal{U}$ then $U \cap V \in \mathcal{U}$;
- *Transitivity:* for every $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $V \cdot V \subseteq U$;
- *Symmetry:* if $U \in \mathcal{U}$ then $U^{-1} \in \mathcal{U}$.

where we write $U \cdot V = \{(u, w) \mid \exists v \ (u, v) \in U \wedge (v, w) \in V\}$ and $U^{-1} = \{(v, u) \mid (u, v) \in U\}$.

Definition 4 (Asymptotic Equivalence) Given a set \mathbb{X} , a relation \approx between nets in \mathbb{X} is an asymptotic equivalence if for all nets f, g, h in \mathbb{X} we find:

- Domain: if $f \approx g$ then $\text{dom}(f) = \text{dom}(g)$;
- Reflexivity: $f \approx f$;
- Transitivity: $f \approx g$ and $g \approx h$ implies $f \approx h$;
- Symmetry: $f \approx g$ implies $g \approx f$;
- Subnet closure 1: if $f \approx g$, then for every cofinality $h : D \rightarrow \text{dom}(f)$ we find $f \cdot h \approx g \cdot h$;
- Subnet closure 2: if $f \not\approx g$ and $\text{dom}(f) = \text{dom}(g)$, then there exists a cofinality $h : D \rightarrow \text{dom}(f)$ such that for every cofinality $k : E \rightarrow D$ we have $f \cdot h \cdot k \not\approx g \cdot h \cdot k$;
- Product closure: given two families $\{f_d \mid d \in D\}$ and $\{g_d \mid d \in D\}$ of nets in \mathbb{X} such that $f_d \approx g_d$ for all $d \in D$, we find $\prod_{d \in D} f_d \approx \prod_{d \in D} g_d$.

where $(f \cdot h)(x) = f(h(x))$ for all $x \in \text{dom}(h)$.

3 Results

Theorem 1 Given a uniform structure \mathcal{U} on a set \mathbb{X} , let \sim be the relation between nets in \mathbb{X} defined by $f \sim g$ iff $\text{dom}(f) = \text{dom}(g)$ and for every $U \in \mathcal{U}$ there exists a $d \in \text{dom}(f)$ such that for every $e \geq d$ we find $(f(e), g(e)) \in U$. This relation \sim is an asymptotic equivalence (and we call it the natural asymptotic equivalence for \mathcal{U} from now on).

Proof

- Domain: by definition, if $f \sim g$ then $\text{dom}(f) = \text{dom}(g)$;
- Reflexivity: since every $U \in \mathcal{U}$ contains the identity, we find $f \sim f$;
- Transitivity: Assume $f \sim g$ and $g \sim h$. Since every $U \in \mathcal{U}$ contains a $V \in \mathcal{U}$ with $V \cdot V \subseteq U$, by transitivity of the uniform structure, we may conclude from $f \sim g$ and $g \sim h$ that there exist $d, d' \in \text{dom}(f)$ such that $(f(e), g(e)) \in V$ and $(g(e), h(e)) \in V$ whenever $e \geq d$ and $e \geq d'$. As $\text{dom}(f)$ is directed we can take a $d'' \geq d$ and $d'' \geq d'$ and conclude that $(f(e), h(e)) \in V \cdot V \subseteq U$ whenever $e \geq d''$. From this we conclude $f \sim h$;
- Symmetry: Assume $f \sim g$, and take $U \in \mathcal{U}$. Recall that also $U^{-1} \in \mathcal{U}$ by definition of uniform structure. Hence, for sufficiently large $e \in \text{dom}(f)$ we have $(f(e), g(e)) \in U^{-1}$ and thus $(g(e), f(e)) \in U$, from which we conclude $g \sim f$;

- Subnet closure 1: Assume $f \sim g$ and take a cofinality $h : D \rightarrow \text{dom}(f)$. For every $U \in \mathcal{U}$ there exists a $d \in \text{dom}(f)$ such that for every $e \geq d$ we have $(f(e), g(e)) \in U$. By definition of cofinality, there exists a $d' \in D$ with $h(e') \geq d$ for every $e' \geq d'$. Hence, for every $e' \in D$ with $e' \geq d'$ we have $(f(h(e')), g(h(e'))) \in U$, from which we conclude $f \cdot h \sim g \cdot h$;
- Subnet closure 2: if $f \not\sim g$ and $\text{dom}(f) = \text{dom}(g)$, then there exists a $U \in \mathcal{U}$ such that for every $d \in \text{dom}(f)$ there exists a $e \geq d$ with $(f(e), g(e)) \notin U$. From this we conclude that the identity function $i : D \rightarrow \text{dom}(f)$ from $D = \{e \in \text{dom}(f) \mid (f(e), g(e)) \notin U\}$ to $\text{dom}(f)$ is a cofinality. Obviously, for any cofinality $h : E \rightarrow D$ we find $f \cdot i \cdot h \not\sim g \cdot i \cdot h$, since $(f(d), g(d)) \notin U$ for any $d \in D$.
- Product closure: given two families $\{f_d \mid d \in D\}$ and $\{g_d \mid d \in D\}$ of nets in \mathbb{X} such that $f_d \sim g_d$ for all $d \in D$, take any $U \in \mathcal{U}$, and for each $d \in D$ pick an element $p(d) \in E_d$ such that $(f_d(e), g_d(e)) \in U$ for all $e \geq p(d)$. Then by definition of the product ordering also $(\prod_{d \in D} f_d(a, p'), \prod_{d \in D} g_d(a, p')) \in U$ for all $p' \geq p$ (regardless of $a \in D$), from which we may conclude $\prod_{d \in D} f_d(a, p') \sim \prod_{d \in D} g_d(a, p')$.

Theorem 2 *Given an asymptotic equivalence \sim between nets in \mathbb{X} , let \mathcal{U} be the set of binary relations on \mathbb{X} such that $U \in \mathcal{U}$ if for every pair of nets $f, g : D \rightarrow \mathbb{X}$ with $f \sim g$ it holds that (f, g) is eventually in U . This set of relations forms a uniform structure on \mathbb{X} (and we call it the natural uniform structure for \sim from now on).*

Proof We start out by proving that, whenever f and g have the same domain but $f \not\sim g$, there is a U such that (f, g) is not eventually in U (i.e. infinitely often it is not in U). For this, assume $f \not\sim g$. Then by subnet closure of \sim , there exists a cofinality $h : H \rightarrow \text{dom}(f)$ such that for every cofinality $k : K \rightarrow H$ we find $f \cdot h \cdot k \not\sim g \cdot h \cdot k$. Given this h , we define the families $B_m = \{n \in H \mid n \geq m\}$ and $A_m = \{(f \cdot h)(n), (g \cdot h)(n) \mid n \in B_m\}$ and make the following case distinction:

- The easy case, is when for some m it holds that every pair (u, v) with $u \sim v$ is eventually in the complement of A_m . If this is the case then we take U to be that complement, thus by construction in \mathcal{U} . Furthermore, as $(f \cdot h, g \cdot h)$ is eventually in A_m , (f, g) is not eventually in U .
- The difficult case, is when for every A_m there is a pair (u, v) with $u_m \sim v_m$ and (u, v) eventually in A_m . Using the subnet and product closures, we find that this case leads to a contradiction, because using these pairs, we can construct a net $w_m : \text{dom}(u_m) \rightarrow H$ such that $f \cdot h \cdot w_m = u_m \sim v_m = g \cdot h \cdot w_m$. Taking the product over the family $\{w_m \mid m \in H\}$ gives us $f \cdot h \cdot \prod_{m \in H} w_m = \prod_{m \in H} (f \cdot h \cdot w_m) \sim \prod_{m \in H} (g \cdot h \cdot w_m) = g \cdot h \cdot \prod_{m \in H} w_m$. Finally, we obtain the contradiction by observing that $\prod_{m \in H} w_m$ is a cofinality, because by construction $d \geq e$ implies $(\prod_{m \in H} w_m)(d) = w_d(p(d)) \in B_d \subseteq B_e$ and so $(\prod_{m \in H} w_m)(d, p) \geq e$.

Now, we are ready to show that \mathcal{U} has all the properties of a uniform structure.

- Identity: for any $U \in \mathcal{U}$ and any $x \in \mathbb{X}$, we take any constant net $f_x : D \rightarrow \mathbb{X}$ with $f(d) = x$ for all $d \in D$ (for simplicity, one may take the singleton directed set for D , but any directed set will do). By reflexivity of \sim we know $f_x \sim f_x$, and by construction there exists a $d \in D$ with $(x, x) = (f_x(d), f_x(d)) \in U$;
- Filtering: take any $U \in \mathcal{U}$ and any V such that $U \subseteq V \subseteq \mathbb{X} \times \mathbb{X}$. Trivially, if any asymptotically equivalent pair of filters is eventually related by U then it is eventually related by V , hence $V \in \mathcal{U}$;
- Finite intersection: trivially, if every related pair of sequences is eventually in U and V , then it is eventually in $U \cap V$;
- Transitivity: We prove by contradiction that for every $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $V \cdot V \subseteq U$. Assume that we have a $U \in \mathcal{U}$ and that there is no $V \in \mathcal{U}$ with $V \cdot V \subseteq U$. Then we can find, for every $V \in \mathcal{U}$ three points x_V, y_V and z_V such that $(x_V, y_V) \in V$ and $(y_V, z_V) \in V$ but $(x_V, z_V) \notin U$. Observe, that the set \mathcal{U} is directed under the ordering relation \subseteq , because we already concluded it to be closed under intersection. This means that the families x_V, y_V and z_V we just constructed can be interpreted as nets over \mathcal{U} . By construction, we find for every $V, W \in \mathcal{U}$ with $W \subseteq V$ that $(x_W, y_W) \in V$ and $(y_W, z_W) \in V$, hence (x, y) and (y, z) are eventually in V . We started this proof by showing that, modus tollens, this means that $x \sim y$ and $y \sim z$, and by transitivity of \sim we then have $x \sim z$. But by assumption the pairs $(x_W, z_W) \notin U$ for any value of W , hence $U \notin \mathcal{U}$. A contradiction.
- Symmetry: if any equivalent pair of nets (f, g) is eventually related in $U \in \mathcal{U}$, then by symmetry of \sim also (g, f) is eventually related in U . Hence U^{-1} eventually relates every pair, and we conclude that $U^{-1} \in \mathcal{U}$.

Theorem 3 *Given a uniform structure \mathcal{U} , the natural uniform structure for the natural asymptotic equivalence for \mathcal{U} coincides with \mathcal{U} .*

Proof Let \sim be the natural asymptotic equivalence for \mathcal{U} , then by definition any $U \in \mathcal{U}$ eventually contains every pair $x_d \sim y_d$ of equivalent nets, hence it is an entourage in the natural uniform structure for \sim . Reversely, assume that $U \notin \mathcal{U}$, then by definition of uniformity we find for any $V \in \mathcal{U}$ that $V \not\subseteq U$. Hence, for any $V \in \mathcal{U}$ there is a pair $(x_V, y_V) \in V$ such that $(x_V, y_V) \notin U$. As the set \mathcal{U} is directed by \supseteq (because \mathcal{U} is closed under \cap) these pairs give us two nets x_V and y_V that, by construction, are asymptotically equivalent ($x_V \sim y_V$), but are never in U . Hence, U is not an entourage in the natural uniform structure arising from \sim , and thus the elements of \mathcal{U} are exactly the elements of the natural uniform structure arising from the natural asymptotic equivalence for \mathcal{U} .

Theorem 4 *Given an asymptotic equivalence \sim , the natural asymptotic equivalence of the natural uniformity for \sim coincides with \sim .*

Proof Let \mathcal{U} be the natural uniformity for \sim . By definition, whenever we have equivalent nets $f \sim g$, they are eventually related by any $U \in \mathcal{U}$, hence the nets are related by the natural asymptotic equivalence for \mathcal{U} . Reversely, assume that f and g have the same domain but $f \not\sim g$, then (by similar reasoning as was used in beginning of the proof of Theorem 2, we can find a $U \in \mathcal{U}$ such that (f, g) are not eventually in U .

Remark 1 (Quasi-Uniform Spaces) *If we drop the symmetry property from the definitions of uniformity and asymptotic equivalence, we get the usual definitions for quasi-uniformity [1] and asymptotic pre-order. The same theorems still apply, since symmetry of the one is only used in the proof of symmetry of the other. As every topological space is quasi-uniformizable, this means that every topological space can therefore also be characterized (but not uniquely) by a pre-ordering on nets over that space.*

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