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PROEFSCHRIFT

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C O N T E N T S

G E N E R A L   I N T R O D U C T I O N

The main subject of this thesis is the description of the motion of a charged particle in the combination of a high-frequency electromagnetic wave of large-amplitude, and of a (primary) magnetic field that may weakly depend on space and time.

In the study of wave-plasma interactions the calculation of single-particle orbits plays an important role. These orbits are of great use, e.g. in the search for nonlinear, selfconsistent solutions of the collisionless Boltzmann equation. It is impossible to obtain the exact analytic solution for the trajectory of an individual particle moving in arbitrary fields. However, when the oscillating field is due to a circularly polarized wave propagating in the direction of a uniform magnetostatic field, such a solution can be found[1-9].

The behaviour of the particle in the combined fields mentioned above exhibits resonance effects when the Doppler-shifted frequency of the wave is near the local, relativistic cyclotron frequency.

In the context of thermonuclear research this cyclotron resonance mechanism is used for the heating and confinement of laboratory plasmas. On the other hand, both electron and ion cyclotron resonances seem to play an important role in the physics of the magnetosphere, e.g. the loss of particles from the radiation belts, and the emission of VLF and ELF radiation. Further, recent literature shows the strong interest in large-amplitude electron whistlers.

In Chapter I we shall formulate the relativistic and nonlinear equations of motion for a charged particle when the primary field is axially symmetric, while the additional high-frequency wave propagates in the direction of the axis of symmetry. The WKB approximation for the transverse motion will be derived and discussed shortly. This approximation holds when the primary field depends only weakly on space and time, that is when the relative change of the cyclotron frequency (as seen by the particle) during one gyration period is small.

In several situations explicit expressions can be obtained for one to four of the constants of motion. It is shown that these constants hold under less restrictive conditions than stated in the literature. In the case of a circularly polarized wave propagating along a uniform magnetostatic field four constants of motion can be obtained.

In this first Chapter we shall introduce a generalized velocity (generalized only with respect to the wave field), and express the position and the velocity in frames that rotate around the axis with the Doppler-shifted frequency. Represented in these variables, the constants of motion assume a rather simple form, and give a direct insight into the character of the motion.

In Chapter II we shall discuss the possible modes of oscillation. Many features of the motion shall be described without recourse to the complete solution given there in terms of elliptic functions. All final expressions shall be represented in a form, normalized such that they contain two dominating parameters. The latter depend on the initial conditions, on the properties of the wave and of the medium.

Chapter III concerns the problem of the motion of a charged particle when the primary field varies only slowly. This special problem has led to a general study of problems mainly described by the WKB approximation to the solution of some Helmholtz equation. Successive higher-order corrections to this approximation will be constructed; these corrections prove to be correct up to increasing powers of a proper smallness parameter, and thus lead to an asymptotic solution of the equation. Moreover, the related problem of the adiabatic invariant associated with the Helmholtz equation is also considered. The exact invariant can again be approximated, in an asymptotic sense, by a corresponding sequence of functions.

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C H A P T E R I

EQUATIONS AND RESULTING CONSTANTS OF MOTION FOR AN ELECTRON NEAR CYCLOTRON RESONANCE IN EXTERNAL FIELDS

I.1 Introduction

In this Chapter we shall derive some basic properties of the motion of an electron in an axially symmetric (primary) electromagnetic field, and an additional high-frequency field propagating in the axial direction.

In section I.2 we shall state the basic equations that govern the motion, assuming that the characteristic length in the radial direction of the inhomogeneous primary magnetic field is large compared to the gyration radius. This means that we assume that the change of the cyclotron frequency, as seen by the particle, is mainly due to the time-dependence and to the axial variation of the magnetic part of the primary field. However, as we shall explicitly mention later on, some results even hold when this restriction is released.

In section I.3 we shall consider the case in which the primary magnetic field is only slightly inhomogeneous. Supposing that the relative change of the cyclotron frequency during one gyration period is small, we shall derive the WKB approximation for the transverse motion.

In section I.4 the basic set of equations and the WKB approximation for the transverse motion shall be written in normalized dimensionless form. Moreover, we shall express the transverse position and momentum in frames that rotate with the local Doppler-shifted frequency of the wave. The expressions resulting in these particular rotating systems turn out to be very useful because, near cyclotron resonance, the particle orbits then prove to be relatively simple.

Finally, in section I.5 we shall discuss some cases in which one or more constants of motion can be derived directly from the basic set of equation.

I.2 Equations of motion

We consider the relativistic motion of an electron in the external field consisting of the combination of an axially symmetric (primary) field, and the field of a high-frequency electromagnetic wave.

The primary electromagnetic field $\underline{E}_o$, $\underline{B}_o$ is assumed to be axially symmetric, with a vanishing azimuthal component of $\underline{B}_o$. Introducing cylindrical coordinates ($|r| = |x+iy|, \phi, z$) the associated vector potential, to be taken in the azimuthal direction, is given by:

$$A_{o\phi} = \frac{1}{2}\,|r| B_o(z,t) \quad , \tag{1}$$

the z-axis being the axis of symmetry, and B_o being a given scalar function. The Cartesian components of the electric field $\underline{E}_o$ and the magnetic field $\underline{B}_o$ read (MKS units)

$$E_{ox} = \frac{1}{2}\,y\,\frac{\partial B_o}{\partial t} \quad , \quad E_{oy} = -\frac{1}{2}\,x\,\frac{\partial B_o}{\partial t} \quad , \quad E_{oz} = 0 \quad ,$$

$$B_{ox} = -\frac{1}{2}\,x\,\frac{\partial B_o}{\partial z} \quad , \quad B_{oy} = -\frac{1}{2}\,y\,\frac{\partial B_o}{\partial z} \quad , \quad B_{oz} = B_o \quad . \tag{2}$$

If B_o also depends on $|r|$, then (1) represents a general, axially symmetric field. The expression in (2) for B_{oz} then should be replaced by $B_{oz} = B_o + \frac{1}{2}\,|r|\,\partial B_o/\partial |r|$.
The field (2) can represent different configurations. It may fix, e.g., the approximation to first order in $|r|/L$ of an axially symmetric field with characteristic scale length L, while it constitutes the field of a cusped configuartion, if $\partial B_o/\partial z$ is independent of z. Equation (2) can also be used in the description of magnetic pumping.

The high-frequency field is assumed to originate from a wave propagating along the axial direction, fixed by its associated vector potential $\underline{A}(z,t) = (A_x, A_y, 0)$; in addition we admit axial electric fields with scalar potential $V(z,t)$. Hence, in addition to the primary field we introduce the fields:

$$\underline{E}(z,t) = \left(-\frac{\partial A_x}{\partial t}, -\frac{\partial A_y}{\partial t}, -\frac{\partial V}{\partial z}\right) \quad , \quad \underline{B}(z,t) = \left(-\frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z}, 0\right) \quad . \tag{3}$$

We emphasize that both the primary and the h.f. field need not be current free.

Considering a wave packet propagating along the axial direction we assume for the transverse components of the vector potential the form

$$A_x(z,t) + iA_y(z,t) = A(z,t)\ \exp\ i\left[\omega(t-t_o) - k(z-z_o) + \eta\right] \quad , \tag{4}$$

where t_o, z_o, the frequency ω, and the wave number k are to be real constants; η is a phase constant. The amplitude $A(z,t)$ may be any complex function of its arguments. If $A(z,t)$ is constant, (4) represents a right-circularly polarized wave packet, whose wave vector rotates in the same direction as the electron does in its cyclotron orbit around the primary magnetic field B_{oz}. When $A(z,t)$ depends on z and t only in the combination $\omega t - kz$, we deal with a packet of plane waves with equal phase velocities, but yet unspecified polarization.

It is convenient to use, instead of the ordinary time t, the particle's proper time τ defined by the equation

$$\frac{dt}{d\tau} = \gamma \quad , \tag{5}$$

where γ is the Lorentz factor

$$\gamma = \left[1 + \frac{1}{c^2}\left\{\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2\right\}\right]^{\frac{1}{2}} \quad . \tag{6}$$

The proper time τ is an increasing function of the ordinary time t.

Rotating coordinates $r = x + iy$ and $r^* = x - iy$ instead of Cartesian coordinates will be used for the moving electron, and also for the corresponding velocities as well as for the vector potential. We thus define the quantities

$$r(\tau) \equiv x(\tau) + iy(\tau) \quad ; \quad A_r(z,t) \equiv A_x(z,t) + iA_y(z,t) \quad ;$$

$$u_r(\tau) \equiv u_x(\tau) + iu_y(\tau) = \frac{dr}{d\tau} \quad ; \quad u_z(\tau) = \frac{dz}{d\tau} \quad . \tag{7}$$

The momentum equations that describe the relativistic motion of a test electron in the combined primary and high-frequency fields can be put in the form:

$$\frac{du_r}{d\tau} - i\omega_c u_r - \frac{1}{2}\, ir\, \frac{d\omega_c}{d\tau} = \frac{e}{m}\frac{dA_r}{d\tau} \quad , \tag{8}$$

$$\frac{du_z}{d\tau} = -\frac{e\gamma}{m} E_z - \frac{e}{m}\,\mathrm{Re}\left(u_r^* \frac{\partial A_r}{\partial z}\right) - \frac{1}{2}\frac{\partial \omega_c}{\partial z}\,\mathrm{Re}\left(iru_r^*\right) \quad , \tag{9}$$

where $\omega_c = eB_o(z,t)/m$ is the electron cyclotron frequency related to the particle's rest mass, while an asterisk denotes a conjugate complex quantity. Equation (8) for $A_r \equiv 0$ has been the subject of several investigations about the constancy of the magnetic moment[1,2,3].

The equation for the total kinetic energy $mc^2\gamma$ of the particle can be deduced from (8) and (9). In fact, by adding (8) multiplied by u_r^*, the complex conjugate of this relation, and (9) multiplied by $2u_z$, we get:

$$c^2 \frac{d\gamma}{d\tau} = -\frac{e}{m} E_z u_z + \frac{e}{m} \mathrm{Re}\left[u_r^* \frac{\partial A_r}{\partial t}\right] + \frac{1}{2}\frac{\partial \omega_c}{\partial t} \mathrm{Re}\left[iru_r^*\right] \quad . \tag{10}$$

The equations (5) - (10) constitute a complete set enabling to determine the τ-dependence of the unknowns x, y, z, t, u_x, u_y, u_z, γ.

For later use we also express the azimuthal component of the canonical momentum in terms of the rotating coordinates. This momentum p_ϕ is given by each of the following expressions

$$\frac{p_\phi}{m} = |r|^2 \frac{d\phi}{d\tau} - |r| \left[\frac{eA_{o\phi}}{m} + \frac{eA_\phi}{m}\right] = -\frac{1}{2}\omega_c |r|^2 - \mathrm{Re}\left\{ir^*(u_r - \frac{e}{m} A_r)\right\} , \tag{11}$$

A_ϕ being the azimuthal component of the high-frequency vector potential.

With the aid of (8), we obtain from (11) the time derivative of p_ϕ,

$$\frac{d}{d\tau}\frac{p_\phi}{m} = \mathrm{Re}\left[iu_r^* \frac{eA_r}{m}\right] \quad . \tag{12}$$

The system is independent of the azimuthal coordinate ϕ if $A_r \equiv 0$. The azimuthal momentum p_ϕ is then conserved during the motion.

The equations (9) - (12) still hold when ω_c and A_r would even depend on the radial coordinate $|r|$. Hence, all conclusions only involving (9) - (12) are also justified when the primary field and the transverse components of the high-frequency field are yet arbitrary functions of $|r|$.

The particle travelling in the combined fields (2) and (4) experiences a resonance when the Doppler-shifted frequency of the wave in proper time, that is $\omega\gamma - ku_z$, happens to be near the local electron cyclotron frequency ω_c (in ordinary time this occurs when $\omega - k\,dz/dt$ is near the local relativistic cyclotron frequency ω_c/γ). The total variation of the cyclotron frequency ω_c, as seen by the particle during its motion, is a consequence of the space and time-dependence of the axial component B_{oz} of the primary magnetic field. The axial velocity and the total kinetic energy, and therefore, the Doppler-shifted frequency $\omega\gamma - ku_z$, are not only modified by the high-frequency field, but also by the transverse components B_{ox} and B_{oy} (mirror effect), these being related to the spatial inhomogeneity of

ω_c, and by the primary electric field which is related to the time dependence of ω_c.

I.3 The adiabatic approximation

We shall restrict ourselves to the case in which the primary field is weakly inhomogeneous. In this section we want to clarify what we mean by the word "weakly".

A substitution of $u_r = dr/d\tau$ into (8) leads to the following equation for $r(\tau)$.

$$\frac{d^2 r(\tau)}{d\tau^2} - i\omega_c \frac{dr}{d\tau} - \frac{i}{2} r \frac{d\omega_c}{d\tau} = \frac{e}{m}\frac{dA_r}{d\tau} \quad . \tag{13}$$

This equation for the transverse motion is coupled to the axial motion through the z-dependence, and to the kinetic energy through the t-dependence of both the cyclotron frequency ω_c and the high-frequency field A_r.

In this section we shall derive an approximate formal solution of this equation under the condition that the relative change of the cyclotron frequency as seen by the particle is small over one gyration period. This implies that we only look for situations in which ω_c never vanishes along the particle's trajectory.

First we want to put equation (13) for $r(\tau)$ into the reduced form. By applying the substitution

$$r(\tau) = \xi(\tau) \exp i \int^{\tau} \frac{\omega_c}{2} d\tau' \tag{14}$$

to (8) we obtain the following equation for $\xi(\tau)$,

$$\frac{d^2\xi}{d\tau^2} + \frac{1}{4}\omega_c^2(\tau)\xi = \frac{e}{m}\frac{dA_r}{d\tau} \exp\left\{- i \int^{\tau} \frac{\omega_c}{2} d\tau'\right\} \quad . \tag{15}$$

This equation corresponds to a driven harmonic oscillator with a time dependent frequency.

We have mentioned that the time-scale of ω_c should be long compared to that of the gyration. To express this fact we introduce the variable q defined by the equation $dq/d\tau = \varepsilon$, in which ε is assumed to be a small positive constant, while we consider ω_c as a function of q instead of τ. Equation (15) can then be represented as follows

$$\frac{d^2\xi(\tau)}{d\tau^2} + f^2(q)\xi(\tau) = k(\tau) \quad ; \quad \frac{dq}{d\tau} = \varepsilon \quad , \tag{16a}$$

with

$$f(q) = \frac{1}{2}\,\omega_c(q) \quad , \quad k(\tau) = \left[\frac{d}{d\tau}\,\frac{eA_r}{m}\right] \exp\left\{-\,i\int^{\tau}\frac{\omega_c}{2}\,d\tau'\right\} \quad . \qquad (16b)$$

In the limit $\varepsilon = 0$, q and hence $f(q)$, become independent of τ.

We next apply the following Liouville transformation to (16):

$$\bar{\xi} = \xi\, F^{\frac{1}{2}}(q,\varepsilon) \quad , \quad \bar{\tau} = \int^{\tau} F\{q(\tau'),\varepsilon\}d\tau' \quad , \quad q = \int^{q} f(q',\varepsilon)dq' \quad , \qquad (17a)$$

where F is to be considered as a function of q and the parameter of smallness ε. Choosing the function F such that it constitutes a solution of the following nonlinear equation and additional condition

$$F^2(q,\varepsilon) = f^2(q) + \varepsilon^2 F^{\frac{1}{2}}\,\frac{d^2}{dq^2}\,F^{-\frac{1}{2}} \quad , \quad F(q,0) = f(q) \quad , \qquad (17b)$$

we obtain from (16a) an equation for $\bar{\xi}$ which has again the reduced form

$$\frac{d^2\xi(\bar{\tau})}{d\bar{\tau}^2} + \bar{\xi}(\bar{\tau}) = \bar{k}(\bar{\tau},\bar{q},\varepsilon) \quad ; \quad \frac{d\bar{q}}{d\bar{\tau}} = \varepsilon \quad , \qquad (18a)$$

with

$$\bar{k}(\bar{\tau},\bar{q},\varepsilon) = F^{-3/2}(\bar{q},\varepsilon)\; k(\tau) \quad . \qquad (18b)$$

As can be verified by direct substitution, the general solution of (18a) is given by

$$\bar{\xi} = \left[c_1 - \frac{i}{2}\int_{\bar{\tau}_o}^{\bar{\tau}} \bar{k}(\bar{\tau}')e^{-i\bar{\tau}'}\,d\bar{\tau}'\right] e^{i\bar{\tau}} + \left[c_2 + \frac{i}{2}\int_{\bar{\tau}_o}^{\bar{\tau}} \bar{k}(\bar{\tau}')e^{+i\bar{\tau}'}\,d\bar{\tau}'\right] e^{-i\bar{\tau}} \quad ,$$

$$\bar{q} = \bar{q}_o + \varepsilon(\bar{\tau}-\bar{\tau}_o) \quad , \qquad (19)$$

where c_1 and c_2 are constants, fixed by the initial conditions at $\bar{\tau} = \bar{\tau}_o$.

Passing with the aid of (17a) and (14), from the variables $\bar{\xi}$, $\bar{\tau}$, $\bar{q}$ to r, τ and q, and making use of (18b) and (16b) in order to express $\bar{k}$ in terms of A_r, the representation (19) transforms into

$$r(\tau) = \left[c_1 - \frac{i}{2}\int_{\tau_o}^{\tau} d\tau' \, \frac{\exp\left\{-i\int_{\tau_o}^{\tau'}\left(F + \frac{\omega_c}{2}\right)d\tau''\right\}}{F^{\frac{1}{2}}} \, \frac{d}{d\tau'}\frac{eA_r}{m}\right] \frac{\exp\left\{i\int_{\tau_o}^{\tau}\left(F + \frac{\omega_c}{2}\right)d\tau'\right\}}{F^{\frac{1}{2}}} +$$

$$+ \left[c_2 + \frac{i}{2}\int_{\tau_o}^{\tau} d\tau' \, \frac{\exp\left\{i\int_{\tau_o}^{\tau'}\left(F - \frac{\omega_c}{2}\right)d\tau''\right\}}{F^{\frac{1}{2}}} \, \frac{d}{d\tau'}\frac{eA_r}{m}\right] \frac{\exp\left\{-i\int_{\tau_o}^{\tau}\left(F - \frac{\omega_c}{2}\right)d\tau'\right\}}{F^{\frac{1}{2}}} \quad ,$$

$$q = q_o + \varepsilon(\tau-\tau_o) \quad , \qquad (20)$$

the constants $c_{1,2}$ now being fixed by the initial conditions at $\tau = \tau_o$.

The expression (20) for $r(\tau)$ is the general solution of (13), if F is the proper solution of (17b), reducing to f for $\varepsilon \to 0$. Hence, we have transformed the problem of solving (13) into the problem of solving (17b).

However, in general we cannot find explicit expressions for the function F since we cannot solve (17b) for arbitrary functions $f(q)$. If the latter is such that two independent solutions of the homogeneous part of (16a) are known, we can also find a solution of (17b). In that case the function F can be expressed in terms of these two independent solutions[4].

On the other hand, we could solve (17b) by iteration which leads to a series in powers of the smallness parameter. However, the series thus obtained will in general not converge, but only be asymptotic. With the aid of the related method described in Chapter III, an approximate solution of (16a) and thus also of (13), can be obtained that is correct to any order in the smallness parameter. By this we mean that the deviation of this approximate solution from the exact one, decreases in the limit $\varepsilon \to 0$ more rapidly than any power of ε.

However, for the present purpose we shall neglect in $r(\tau)$ all terms of second and higher order in ε. It then follows from (17b) and (16b), that $F = \frac{1}{2}\omega_c + O(\varepsilon^2)$.

We deduce from (20) that the particle experiences a resonance when its Doppler-shifted frequency is near $\frac{1}{2}\omega_c \pm F$. By neglecting terms of second and higher order in ε, we not only have to assume that the time-scale of ω_c is long compared to the gyration period, but also that the influence of these terms on the resonant behaviour is negligible.

Partial integrating the terms occurring under the integral signs in (20), while replacing F by $\omega_c/2$, we obtain the following approximate solution of (13),

$$\omega_c^{\frac{1}{2}} r(\tau) = \left[C_1 + \int_{\tau_o}^{\tau} d\tau' \frac{eA_r}{m} \omega_c^{\frac{1}{2}} \left(1 - \frac{i\dot{\omega}_c}{2\omega_c^2}\right) \exp\left\{-i \int_{\tau_o}^{\tau'} \omega_c d\tau''\right\}\right] \exp\left\{i\int_{\tau_o}^{\tau} \omega_c d\tau'\right\} +$$

$$+ C_2 + \int_{\tau_o}^{\tau} d\tau' \frac{eA_r}{m} \frac{i\dot{\omega}_c}{2\omega_c^{3/2}} \quad , \tag{21}$$

where we have replaced $d\omega_c/d\tau$ by $\dot{\omega}_c$ for short; $C_{1,2}$ being constants. This expression is the WKB solution of (13). It is correct up to first order in ε.

For our purpose we need an alternative form of (21). Differentiating this equation, and combining the result with (21) itself, we find the following two relations (remembering that $u_r = dr/d\tau$):

$$\frac{1}{\omega_c^{\frac{1}{2}}} \left(u_r - \frac{e}{m} A_r + \frac{\dot{\omega}_c}{2\omega_c} r\right) = iC_1 \exp\left\{ i \int_{\tau_o}^{\tau} \omega_c d\tau'\right\} +$$

$$+ i \int_{\tau_o}^{\tau} d\tau' \frac{eA_r}{m} \omega_c^{\frac{1}{2}} \left(1 - \frac{i\dot{\omega}_c}{2\omega_c^2}\right) \exp\left\{ i \int_{\tau'}^{\tau} \omega_c d\tau''\right\} , \tag{22a}$$

$$\omega_c^{\frac{1}{2}} \left(r - \frac{u_r - \frac{e}{m} A_r + \frac{\dot{\omega}_c}{2\omega_c} r}{i\omega_c}\right) = C_2 + \int_{\tau_o}^{\tau} d\tau' \frac{i\dot{\omega}_c}{2\omega_c^{3/2}} \frac{eA_r}{m} \quad , \tag{22b}$$

with $C_{1,2}$ being fixed by the initial conditions, viz.

$$iC_1 = \frac{1}{\omega_{co}^{\frac{1}{2}}} \left(u_{ro} - \frac{e}{m} A_{ro} + \frac{\dot{\omega}_{co}}{2\omega_{co}} r_o\right) \quad ; \quad C_2 = \omega_{co}^{\frac{1}{2}} r_o - C_1 \quad , \tag{22c}$$

here and henceforth a subscript zero denotes the value of the relevant quantity at the initial time τ_o.

The quantities on the left-hand sides of (22) are related to the azimuthal component of the canonical momentum. In fact, it follows from (11) that p_ϕ can also be represented by

$$\frac{p_\phi}{m} = \frac{\left|u_r - \frac{e}{m} A_r + \frac{\dot{\omega}_c}{2\omega_c} r\right|^2}{2\omega_c} - \frac{1}{2}\omega_c \left|r - \frac{u_r - \frac{e}{m} A_r + \frac{\dot{\omega}_c}{2\omega_c} r}{i\omega_c}\right|^2 \quad . \tag{23}$$

The terms occurring in (22) and (23) can be interpreted as follows. The quantity $-r\dot{\omega}_c/2\omega_c$ represents the first-order radial drift of the particle due to the slowly varying primary electromagnetic field defined by (2). In view of this, the quantity r_g defined by

$$r_g \equiv r - \frac{u_r - \frac{e}{m} A_r + \frac{\dot{\omega}_c}{2\omega_c} r}{i\omega_c} ,$$

can be considered as the position of a guiding centre. The second term on the right-hand side of the above expression then represents the particle's position r_L with respect to this centre, so that

$$r_L = r - r_g = \frac{u_r - \frac{e}{m} A_r + \frac{\dot{\omega}_c}{2\omega_c} r}{i\omega_c} .$$

Although the vector that corresponds to r_L is not at a right angle to the particle's trajectory, its modulus may be interpreted as a generalized gyration radius. The qualification "generalized" stresses the fact that this radius depends on $u_r - \frac{e}{m} A_r$ instead of u_r.

The first term on the right-hand side of (23) proves to be proportional to the magnetic flux through a circle with radius $|r_L|$ and centered at the guiding centre. In the absence of a high-frequency wave ($A_r \equiv 0$) it is proportional to the relativistic magnetic moment. The second term on the right-hand side of (23) is proportional to the magnetic flux through a circle of radius $|r_g|$ and centered at the axis.

Substituting (22b) into (23), and taking the average over the initial phase of the guiding centre position, i.e. averaging over the phase of C_2, while neglecting products of small quantities, we find the following expression for the average value $\langle p_\phi \rangle$:

$$\frac{\langle p_\phi \rangle}{m} = \frac{\left| u_r - \frac{e}{m} A_r + \frac{\dot{\omega}_c}{2\omega_c} r \right|^2}{2\omega_c} - \frac{1}{2} |C_2|^2 . \tag{24}$$

Hence, apart from an additional constant $\langle p_\phi \rangle$ equals the angular momentum of the particle with respect to its guiding centre.

Combining (22a) with (23) and (24), we arrive at relations which express p_ϕ and $\langle p_\phi \rangle$, respectively, in terms of the initial conditions, of the cyclotron frequency ω_c, and of the wave vector A_r. Since ω_c and A_r are not given as functions of the proper time τ, but as functions of the ordinary time t and the axial position z, the equations (22) - (24) are coupled to (9) and (10); these latter equations determine the behaviour in proper time of the axial velocity and of the kinetic energy, respectively.

I.4 Normalization of the equations of motion

For convenience we change to normalized dimensionless variables, basing the time-scale on the wave frequency ω, and the length-scale on the free-space wavelength $\frac{\omega}{c}$; thus we introduce

$$s = \omega\,\tau \; ; \quad X,Y,Z = \frac{\omega}{c}\,(x,y,z) \quad . \tag{25}$$

We shall further use the normalized quantities:

$$P_z(s) = \frac{dZ}{ds} = \frac{u_z}{c} \quad ; \quad P_\phi(s) = \frac{\omega p_\phi(\tau)}{mc^2} \quad ;$$

$$\Omega(Z,\omega t) = \frac{\omega_c(z,t)}{\omega} \quad ; \quad g(Z,\omega t) = \frac{eA(z,t)}{mc} \quad ; \quad N = \frac{kc}{\omega} \quad ;$$

$$F_z(Z,\omega t) = \frac{eE_z(z,t)}{m\omega c} \quad ; \quad \chi(Z,\omega t) = \frac{eV(z,t)}{mc^2} \quad . \tag{26}$$

In addition the position and velocity in the planes perpendicular to the axis will be fixed with the aid of the quantities R(s) and P(s), to be defined by

$$R(s) \equiv \frac{\omega}{c}\, r \exp\left\{-i \int_{s_o}^{s} (\gamma - NP_z)\,ds' - i\eta\right\} \quad , \tag{27}$$

$$P(s) \equiv \left(\frac{u_r}{c} - \frac{eA_r}{mc}\right) \exp\left\{-i \int_{s_o}^{s} (\gamma - NP_z)\,ds' - i\eta\right\} \quad . \tag{28}$$

These expressions represent the normalized position and the normalized form of the generalized velocity (generalized only with respect to A_r), respectively, in frames that rotate with the local Doppler-shifted frequency (amounting to $\gamma - NP_z$ in proper time) around the z-axis. In general, this Doppler-shifted frequency is not a constant. In the special case of a constant wave amplitude A, the wave vector g will also be constant in the rotating system. The transformation to these particular rotating systems turns out to be very useful because, near cyclotron resonance, the particle orbits in these frames prove to be relatively simple (see Chapter II).

In terms of these variables and quantities the equations (9) and (10), which govern the axial motion and the time dependence of the kinetic energy, respectively, become

$$\frac{dP_z}{ds} = -\gamma F_z + N\mathrm{Re}\left(igP^*\right) - \mathrm{Re}\left\{\left(P^* + g^*\right)\frac{\partial g}{\partial Z}\right\} - \frac{\partial \Omega}{\partial Z}\,\mathrm{Re}\left\{ i\,\frac{1}{2}\,R\left(P^* + g^*\right)\right\}, \tag{29}$$

$$\frac{d\gamma}{ds} = - P_z F_z + \mathrm{Re}\left(igP^*\right) + \mathrm{Re}\left\{\left(P^* + g^*\right)\frac{\partial g}{\partial \omega t}\right\} + \frac{\partial \Omega}{\partial \omega t}\,\mathrm{Re}\left\{ i\,\frac{1}{2}\,R\left(P^* + g^*\right)\right\}. \tag{30}$$

An additional relation for the three quantities P_z, P, and γ results from (6), reading in the normalized quantities

$$\gamma = \left[1 + |P + g|^2 + P_z^2\right]^{\frac{1}{2}} \quad . \tag{31}$$

We shall also express the equation (12) for the angular momentum in the normalized quantities, giving

$$\frac{dP_\phi}{ds} = \mathrm{Re}\left(igP^*\right) \quad , \tag{32}$$

where, in view of (23) the normalized angular momentum is given by

$$P_\phi = \frac{\omega}{mc^2}\,p_\phi = \frac{\left|P + \frac{\dot\Omega}{2\Omega}\,R\right|^2}{2\Omega} - \frac{\Omega}{2}\left|R + i\,\frac{P + \frac{\dot\Omega}{2\Omega}\,R}{\Omega}\right|^2 \quad . \tag{33}$$

Since (9) - (12) also hold when ω_c and A_r are arbitrary functions of the radial coordinate $|r|$, the equations (29) - (33) are still justified when Ω and g depend on $|R|$.

We do not want to transform equation (8) for the transverse motion in terms of the new variables and quantities since we shall use the adiabatic approximation, whenever we might need it. The relations (22a) and (22b) become in the new variables:

$$\frac{1}{\Omega^{\frac{1}{2}}(s)}\left[P(s) + \frac{\dot\Omega(s)}{2\Omega(s)}\,R(s)\right] = \frac{1}{\Omega_o^{\frac{1}{2}}}\left[P_o + \frac{\dot\Omega_o}{2\Omega_o}\,R_o\right]\exp\left\{-i\int_{s_o}^{s}\psi ds'\right\} +$$

$$+ i\int_{s_o}^{s} ds' g\Omega^{\frac{1}{2}}\left(1 - \frac{i\dot\Omega}{2\Omega^2}\right)\exp\left\{-i\int_{s'}^{s}\psi ds''\right\} \quad , \tag{34}$$

$$\Omega^{\frac{1}{2}}(s)\left[R(s) + i\,\frac{P(s) + \frac{\dot\Omega(s)}{2\Omega(s)}\,R(s)}{\Omega(s)}\right] = \frac{\omega^{\frac{1}{2}}}{c}\,C_2\exp\left\{-i\int_{s_o}^{s}(\gamma - NP_z)ds' - in\right\} +$$

$$+ \int_{s_o}^{s} ds' g\,\frac{i\dot\Omega}{2\Omega^{3/2}}\exp\left\{-i\int_{s'}^{s}(\gamma - NP_z)ds'' - in\right\} \quad , \tag{35}$$

where C_2 is given by (22c), and $d\Omega/ds$ has been replaced by $\dot\Omega$ for short; ψ is the resonance function defined by

$$\psi(s) \equiv \gamma - NP_z - \Omega \quad . \tag{36}$$

This function governs the resonant behaviour of the particle; it vanishes if the local normalized Doppler-shifted frequency γ-NP_z equals the local normalized cyclotron frequency Ω.

Substituting (35) into (33), and next averaging over the initial phase of the guiding centre position, we find the following normalized form of (24)

$$\langle P_\phi \rangle = \frac{|P + \frac{\dot{\Omega}}{2\Omega} R|^2}{2\Omega} - \frac{\omega}{2C^2} |C_2|^2 \quad . \tag{37}$$

Under cyclotron resonance conditions ($\psi \sim 0$), when the normalized Doppler-shifted frequency γ-NP_z is near the normalized local cyclotron frequency Ω, the right-hand side of (34) only contains slowly varying quantities (slowly with respect to the gyration period). Equation (35) fixes the position of the guiding centre. On the right-hand side of this equation fast oscillating terms occur, which only slightly influence the resonant behaviour of the particle. Therefore, near cyclotron resonance, the right-hand side of (35) may be neglected.

I.5 Constants of motion

I.5.1 Introduction

Only in a few cases it is possible to find analytic solutions of the equations of motion. Constants of motion play an important role in obtaining such solutions. We shall show that under some restrictive conditions one can obtain one to four integrals of motion from the equations in the preceding section. In the homogeneous situation $(\partial\Omega/\partial z = \partial\Omega/\partial t = 0)$, with a single (constant wave amplitude g) right-circularly polarized wave propagating along the magnetostatic field, we easily find four of them. In this situation it is even possible to find analytically exact solutions of the equations of motion, but in most other cases one has to use approximate methods or numerical calculations. The constants of motion are useful to reduce the order of the equations and often give an insight into the character of the motion without recourse to the exact solutions. Moreover, they are of great use for constructing solutions of the Vlasov equation in the search for consistent solutions[5,9,10].

We have already found the relation (22b) or, expressed in normalized coordinates, the equivalent relation (35). The real and imaginary parts of this expression constitute two adiabatic constants relating the particle's position to the generalized velocity. Neglecting the off-axis position of the guiding centre (i.e. omitting the right-hand side of (35)) we obtain:

$$R(s) = \frac{P}{i\Omega}\left(1 - \frac{i\dot{\Omega}}{2\Omega^2}\right) \quad . \tag{38}$$

If the primary field is homogeneous ($\dot{\Omega} = 0$), then we may choose, without loss of generality, the origin of our coordinate system such that

$$\frac{\omega^{\frac{1}{2}}}{c} C_2 \equiv \Omega_o^{\frac{1}{2}}\left(R_o + \frac{iP_o}{\Omega_o}\right) = 0 \quad .$$

It then follows from (35) that

$$R(s) = \frac{P(s)}{i\Omega} \tag{39}$$

holds exactly.

We conclude from this relation that the particle orbits in the R-plane and in the P-plane then are similar; apart from the scaling factor Ω for the corresponding rectangular Cartesian coordinates their positions in the complex plane transform into each other by rotation over an angle $\frac{\pi}{2}$.

Under some restrictions we obtain from (29) and (30) one or two integrals of motion. In view of (32) these equations can be put in the form

$$\frac{d}{ds}\left(P_z - NP_\phi\right) = -\gamma F_z - \text{Re}\left\{\left(P^* + g^*\right)\frac{\partial g}{\partial Z}\right\} - \frac{\partial\Omega}{\partial Z}\text{Re}\left\{i\,\frac{1}{2}\,R\left(P^* + g^*\right)\right\} , \tag{40}$$

$$\frac{d}{ds}\left(\gamma - P_\phi\right) = -P_zF_z + \text{Re}\left\{\left(P^* + g^*\right)\frac{\partial g}{\partial\omega t}\right\} + \frac{\partial\Omega}{\partial\omega t}\text{Re}\left\{i\,\frac{1}{2}\,R\left(P^* + g^*\right)\right\} . \tag{41}$$

In the next subsections we shall discuss, successively, the different cases in which we can deduce one or two constants of motion from the above equations. In all these cases analogous results can be obtained for a left-circularly polarized wave packet and for ions instead of electrons as test particles.

I.5.2 Space-dependent primary field

We shall consider the situation in which the primary field and the amplitude A of the high-frequency field are time-independent ($\partial\Omega/\partial t = 0$, $\partial g/\partial t = 0$); but otherwise Ω may be an arbitrary function of the space coordinates $|r|$ and z. The wave field given by (4) then has the form

$$A_r(z,t) = h(z)e^{i\omega t} ,$$

where h(z) may be an arbitrary complex function of its argument. This expression can represent the field of a circularly polarized wave propagating through an inhomogeneous medium, that of a damped wave, or also that of standing waves. In addition we assume that the axial electric field can be derived from a time-independent potential. Noting that then $F_z = - d\chi/dZ$, we can integrate (41) to obtain

$$\gamma - \chi(Z) - P_\phi = \text{constant} \quad . \tag{42}$$

The increase of the normalized total energy $\gamma-\chi$ thus equals the increase of the normalized angular momentum P_ϕ. Expressed in unnormalized quantities, we could say that the circularly polarized waves carry angular momentum, and that the interaction between these waves and the particle is such that the increase of the total particle energy divided by the frequency, i.e. $(mc^2\gamma-eV)/\omega$, equals the amount of angular momentum $p_\phi = mc^2P_\phi/\omega$ transferred by the waves to the particle.

If the primary field is homogeneous ($\dot{\Omega} \equiv 0$), then we deduce from (33) and (35) that

$$P_\phi = \frac{|P|^2}{2\Omega} - \frac{\omega|C_2|^2}{2c^2} \quad .$$

Equation (42) then reduces to the constant:

$$\gamma - \chi(Z) - \frac{|P|^2}{2\Omega} = \text{constant} \quad . \tag{43}$$

I.5.3 Time-dependent primary field

We now suppose that the primary field is independent of the axial coordinate $Z(\partial\Omega/\partial Z = 0)$, but otherwise it may be time-dependent and as yet an arbitrary function of the radial coordinate $|r|$. In addition we assume that the axial electric field F_z depends on time only, and that the wave amplitude A is independent of z, so that the wave field given by (4) has the form

$$A_r(z,t) = h(t)e^{-ikz} \quad .$$

This may represent, e.g., a right-circularly polarized wave which is damped in time, or a wave whose amplitude and/or phase is modulated.

Under the restrictions mentioned we can integrate (40) which results in

$$P_z - NP_\phi + \int^t F_z(t')dt' = \text{constant} \quad . \tag{44}$$

Expressed in unnormalized quantities (44) states that, disregarding for the moment the term involving the axial electric field, due to the interaction between the wave and the particle, the gain in axial momentum divided by the wave number, i.e. $mu_z/k = mc^2/N\omega\ P_z$ equals the amount of angular momentum $p_\phi = mc^2P_\phi/\omega$ absorbed by the particle.

If the primary magnetic field is static and homogeneous, we have

$$P_\phi = \frac{|p|^2}{2\Omega} - \frac{\omega|C_2|^2}{2c^2} \quad .$$

Equation (44) then reduces to the constant

$$P_z - N\frac{|p|^2}{2\Omega} + \int^t F_z(t')dt' = \text{constant} \quad . \tag{45}$$

I.5.4 Homogeneous magnetostatic field

We next restrict ourselves to the homogeneous situation in which the magnetostatic field is homogeneous. Therefore, $\partial\Omega/\partial z$ and $\partial\Omega/\partial t$ vanish in (40) and (41).

Moreover, we assume that the wave amplitude A depends on z and t in the combination $a_1\omega t - a_2\frac{\omega}{c}z$, $a_{1,2}$ being as yet arbitrary constants. Hence, we suppose

$$\frac{e}{mc}A(z,t) = \frac{e}{mc}A(a_1\omega t - a_2\frac{\omega}{c}z) = g(\zeta) \quad , \tag{46a}$$

with

$$\zeta = a_1\omega t - a_2 Z = \int^s ds'\ (a_1\gamma - a_2P_z) \quad . \tag{46b}$$

If the amplitude A is real, the quantity a_1c/a_2 may be interpreted as the group velocity of the wave.

By multiplying (40) by a_1 and (41) by a_2 and subtracting these relations, we obtain after integration

$$a_2\gamma - a_1P_z - (a_2 - a_1N)\, P_\phi - \int^s ds'\, (a_1\gamma - a_2P_z)\, F_z = \text{constant} \quad . \tag{47}$$

As can be seen from (40) and (41), this relation still holds when Ω also depends on z and t in the combination $a_1\omega t - a_2 \frac{\omega}{c} z$.

The integration in the last term in the left-hand side can easily be performed when the normalized field F_z is constant, giving

$$\int^s ds'\, (a_1\gamma - a_2P_z)\, F_z = \zeta F_z \quad , \tag{48}$$

where ζ is given by (46b).

The integral in (47) can also be evaluated when F_z depends, like the normalized wave amplitude g, only on the function ζ. Indeed, in view of (46b) and the relation $F_z = -\,\partial\chi/\partial Z$, we then obtain:

$$\int^s ds'\, (a_1\gamma - a_2P_z)\, F_z = + a_2\chi(\zeta) \quad . \tag{49}$$

Since we only assumed the primary field to be independent of t and Z, the integral of motion (47) still holds in a primary field which is an arbitrary function of the radial coordinate $|r|$. If it is also independent of $|r|$, we may replace in (47) P_ϕ by $|P|^2/2\Omega$. The integral of motion then has a form which is equivalent to that given by LAIRD[6]. For a slow wave ($N \geq 1$) a Lorentz transformation can be used to express the integral (47) in terms of quantities measured in the reference system moving with the phase velocity $\frac{\omega}{k}$. When $a_2/a_1 = \omega/kc$ one then obtains the integral of motion in a form which is equivalent to that given by SONNERUP[7].

In the special case where the group velocity of the wave packet equals the phase velocity of the waves, i.e. if $a_1 = 1$, $a_2 = N$, (47) reduces to

$$N\gamma - P_z - \int^s ds'\, (\gamma - NP_z)\, F_z = \text{constant} \quad . \tag{50}$$

We emphasize that this relation is not only valid for a right-circularly polarized wave but also for a packet of arbitrary polarized plane waves. If, in addition, we have to do with free space propagation ($N = 1$), (50) constitutes the following first-order differential equation for $\zeta = \int^s ds'\, (\gamma - P_z)$, provided we assume that F_z is a function of ζ only:

$$\frac{d\zeta}{ds} - \int^{\zeta} d\zeta' \; F_z(\zeta') = \text{constant} \quad . \tag{51}$$

The solution of this equation, if obtainable, gives ζ as a function of the normalized proper time s and enables to express also the resonance function ψ given by (36), as well as the wave amplitude g, as functions of s. The problem of finding the motion of the particle is then reduced to the solution of the integral in the right-hand side of (34), Ω now being constant.

In the case of a vanishing axial electric field ($F_z \equiv 0$), (47) further reduces to[8)]

$$N(\gamma - \gamma_o) = P_z - P_{zo} \quad . \tag{52}$$

The kinetic energy of the particle then cannot change without a corresponding change of its axial momentum. The gain in total particle energy is due to the electric field of the wave, while that in axial momentum is caused by the radiation pressure, i.e. by the corresponding magnetic field B. The index of refraction $N = c \frac{B}{E}$ is a measure for the relative importance of these electric and magnetic fields. When $N^2 > 1$ the wave can be considered as more magnetic than electric and a gain in (normalized) axial momentum is larger than the associated change in total (normalized) energy. When $N^2 < 1$ the wave is more electric than magnetic and the increase in axial momentum is smaller than the increase in total energy. For $N^2 = 1$ both field contributions (properly normalized) are just equal.

Combining the relation (52) with the definition (36) for the resonance function we find, remembering the constant value of Ω,

$$\psi(s) = \psi_o + (1-N^2)\;(\gamma - \gamma_o) = \psi_o + \frac{(1-N^2)}{N^2}\;(P_z - P_{zo}) \quad . \tag{53}$$

Thus the resonance function proves to depend only on the kinetic energy or, alternatively, on the axial momentum of the particle. In the case of free space propagation (N = 1), the resonance function is constant.

I.5.5 Homogeneous background and magnetostatic field with a single circularly polarized wave

Finally, we shall consider the motion of an electron in a magnetic field, which is time-independent and spatially homogeneous, and in the presence of a single (g constant) right-circularly polarized wave propagating through a homogeneous background. We may suppose the amplitude g to be real by choosing a proper phase constant η. If the axial electric field is absent, the constants of motion (43) and (45), which now both hold, can be written in the following form:

$$\frac{|P|^2}{2\Omega} - (\gamma - \gamma_o) = \frac{|P_o|^2}{2\Omega} \quad , \tag{54}$$

$$N\,\frac{|P|^2}{2\Omega} - (P_z - P_{zo}) = N\,\frac{|P_o|^2}{2\Omega} \quad , \tag{55}$$

respectively.

From the above equations one can find again the relation (52) and the expression (53) for the resonance function.

For later use we need the constant of motion (54) in another form. With the aid of (31) this constant can be derived from the above equations, but it is simpler to do so directly from the equations of motion.

By differentiation of (34) with respect to s, remembering that Ω is constant, we get

$$\frac{dP}{ds} + i\psi P = i\Omega g \quad . \tag{56}$$

Multiplying this equation by the constant value of g and combining its real part with (30) (in which now $F_z = 0$, Ω and g are real constants), we get

$$\frac{d}{ds}\,\mathrm{Re}\; gP - \psi\,\frac{d\gamma}{ds} = 0 \quad .$$

According to (53) the resonance function ψ only depends on γ. Substitution of the expression (53) therefore results, after integration, in

$$\mathrm{Re}\; gP - \psi_o\,(\gamma - \gamma_o) - \frac{1}{2}\,(1-N^2)\,(\gamma - \gamma_o)^2 = \mathrm{Re}\; gP_o \quad . \tag{57}$$

With the aid of (52) we can also express Re P in terms of P_z instead of γ. In this form (57) and (55) then directly give the particle orbit in the "momentum space" (Re P, Im P, P_z), while eliminating P_z from these equations and combining the result with (39) we also obtain the particle orbit in the transverse coordinate plane (Re R, Im R). We shall use the constants of motion (54) or (55) and (57) in Chapter II. With the aid of these two equations the problem of finding the particle trajectory can be reduced to solving a first-order elliptic differential equation.

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CHAPTER II

ELECTRON MOTION IN A HOMOGENEOUS MEDIUM UNDER THE INFLUENCE OF A HOMOGENEOUS MAGNETIC FIELD AND A RIGHT-CIRCULARLY POLARIZED WAVE

II.1 Introduction

In this Chapter we shall consider the behaviour of an electron in the presence of a right-circularly polarized wave with constant amplitude (see I.4)

$$A_r(z,t) = A \exp i\left[\omega(t-t_o) - k\,(z-z_o) + \eta\right]$$

propagating along a time-independent and spatially homogeneous external magnetic field and through a homogeneous medium. The axial electric field is assumed to be zero ($F_z = 0$), while the constant wave amplitude A is taken as real.

In this homogeneous situation the equations of motion lead to one single first-order differential equation for $\gamma(s)$. In fact, by squaring (I.30) (remembering that $F_z = 0$, Ω and g are real constants) we find

$$\left[\frac{d\gamma}{ds}\right]^2 = g^2P_i^2 = g^2|P|^2 - g^2P_r^2 \quad .$$

Here and henceforth, P_r and P_i denote the real and imaginary parts, respectively, of the quantity P.
Next, using the integrals of motion (I.54) and (I.57), we get:

$$\left[\frac{d\gamma}{ds}\right]^2 = F(\gamma-\gamma_o) \tag{1}$$

with

$$F(\gamma-\gamma_o) \equiv g^2P_{io}^2 + 2\left[\Omega g^2 - gP_{ro}\psi_o\right](\gamma-\gamma_o) - \left[\psi_o^2 + (1-N^2)gP_{ro}\right](\gamma-\gamma_o)^2 - $$

$$- \psi_o(1-N^2)\;(\gamma-\gamma_o)^3 - \frac{1}{4}(1-N^2)^2\;(\gamma-\gamma_o)^4 \quad . \tag{2}$$

Apart from the introduction of the proper time and the difference in normalization (1) is equal to Eq. (2.14) of ROBERTS and BUCHSBAUM[1] and to Eq. (18) of SCHRAM[2]; it describes the motion in a one-dimensional pseudo-potential well, given by $F(\gamma-\gamma_o)$.

Since $\left[\frac{d\gamma}{ds}\right]^2$ must be non-negative, $F(\gamma-\gamma_o)$ is also non-negative during the motion. When both $N^2 = 1$ and $\psi_o = 0$, the case under consideration is called synchronous (DAVYDOVSKII[3]), $F(\gamma-\gamma_o)$ has only one zero and $F(\gamma-\gamma_o) \to \pm\infty$ when $\gamma \to \pm\infty$. However, in all other cases $F(\pm\infty) = -\infty$ when $\gamma \to \pm\infty$, while $F(0) \geq 0$ so that $F(\gamma-\gamma_o)$ has at least two real roots. In this general, non-synchronous, case the kinetic energy executes a periodic oscillation between the two real roots situated around $\gamma = \gamma_o$. Such a periodic motion is excluded in the synchronous case.

Since $F(\gamma-\gamma_o)$ is a polynomial of the fourth degree, we can express the normalized proper time s as a function of γ in terms of an elliptic integral of the first kind

$$s - s_o = \pm \int_{\gamma_o}^{\gamma} \frac{d\gamma'}{\sqrt{F(\gamma'-\gamma_o)}} \quad . \tag{3}$$

The solution can be inverted which results in a Jacobian elliptic function for γ. The dependence on the proper time of the generalized momenta, of the resonance function, and of the particle's position in the transverse plane can easily be found by substitution of the solution for γ into the integrals of motion and/or into the equations of motion. Remembering that $P_z = dZ/ds$, we find the axial position by integrating (I.52), viz.

$$Z = Z_o + (P_{zo} - N\gamma_o)\ (s-s_o) + N \int_{s_o}^{s} \gamma ds' \quad , \tag{4}$$

while, by integrating the differential equation (I.5), remembering that $s = \omega\tau$, we obtain the ordinary time t as a function of s or γ:

$$\omega(t-t_o) = \int_{s_o}^{s} \gamma(s')ds' = \pm \int_{\gamma_o}^{\gamma} \frac{\gamma' d\gamma'}{\sqrt{F(\gamma'-\gamma_o)}} \quad . \tag{5}$$

The last integral can be expressed as a combination of elliptic integrals of the first and third kind, but here the solution cannot be inverted in terms of known functions to find the dependence of γ on the time t.By choosing the proper time as an independent variable we avoid this difficulty and several features of the motion can be deduced without recourse to the complete solution.

The integrals of motion derived in Chapter I give directly the particle trajectories in the generalized momentum space (P_r, P_i, P_z)

and next in the complex R-plane. We shall show that these trajectories are closed and bounded except when simultaneously $\psi_o = 0$ and $N^2 = 1$. In this synchronous case the particle trajectory is not closed and unbounded.

The generalized momentum P and the position R with respect to the generalized guiding centre were defined with reference to a frame which rotates with respect to the rest frame with a normalized angular velocity equal to the Doppler-shifted frequency $\gamma - NP_z$ (see (I.27) and (I.28)). The vector potential is constant in this rotating frame. In view of (I.52) we can verify the relation:

$$\gamma - NP_z = \gamma_o - NP_{zo} + (1-N^2)\ (\gamma-\gamma_o)\ .$$

The angular velocity of the rotating coordinate system thus proves to be the superposition of an average value and a variation with the same periodicity as the energy oscillation. This angular velocity is only constant if $N^2 = 1$, or if there is no energy oscillation at all. This last situation arises when the initial conditions are such that $\gamma = \gamma_o$ is a multiple root of $F(\gamma-\gamma_o)$, since $d\gamma/ds$ then vanishes at all times. The function $F(\gamma-\gamma_o)$ has a double root in $\gamma = \gamma_o$ if, simultaneously,

$$P_{io} = 0, \qquad \Omega g - P_{ro}\psi_o = 0, \tag{6a}$$

while γ_o is a triple root of F if in addition ψ_o and P_{ro} satisfy the relation

$$\psi_o^2 + (1-N^2)g\ P_{ro} = 0,$$

i.e. if

$$P_{ro} = \left[\frac{\Omega^2 g}{N^2-1}\right]^{1/3}, \quad P_{io} = 0\ , \quad \psi_o = \left[\ (N^2-1)\Omega g^2\ \right]^{1/3}\ . \tag{6b}$$

If (6a) is satisfied then it follows from (I.56) that the time derivative of the generalized momentum vanishes. The contribution to the rate of change of P due to the wave field is then just cancelled by the contribution of the rotation of the particle with respect to the wave. Then the particle is at rest in the rotating frame.

If $P_{io} = 0$ then $\gamma = \gamma_o$ is a root of $F(\gamma-\gamma_o)$. Since $F(\gamma-\gamma_o)$ must be positive, γ_o is the maximum or the minimum of the extreme values between which γ oscillates. This depends on the sign of

$$\left(\frac{\partial F}{\partial \gamma}\right)_{\gamma\ =\ \gamma_o} = g[\ \Omega g\ -\ P_{ro}\psi_o\ ]\ .$$

If $\Omega g - P_{ro}\psi_o > 0$ then γ_o is a minimum and the particle will gain energy, while γ_o is a maximum if $\Omega g - P_{ro}\psi_o < 0$. The particle will always gain energy if $P_{ro}\psi_o < 0$. If the particle starts close to exact resonance with small initial transverse momentum, i.e. if $|P_{ro}|$ and $|\psi_o|$ are smaller than the values given in (6b) (see section II.5.2), then $|P_{ro}\psi_o| < \Omega g$. Hence, on the average the particle will always gain energy in this case.

Condition (6a) reduces to the Cerenkov-like condition[2]: $\gamma_o - NP_{zo} = 0$, if the particle starts with zero initial transverse velocity, i.e. $P_{io} = 0$, $P_{ro} = -g$. Hence, if the axial velocity initially equals the wave velocity, the particle will not experience an energy oscillation. However, if the axial velocity is less than the wave velocity, the particle will gain energy, while it will lose energy if the axial velocity exceeds the wave velocity. This last situation can only arise in the case of a slow wave ($N^2 > 1$).

In the rotating frame the wave vector A_r equals the real constant A, and the phase of P thus equals the phase difference between the generalized velocity $u_r - \frac{e}{m} A_r$ and the wave vector A_r. Hence, the trajectory in the (P_r, P_i, P_z)-space gives a direct insight into the behaviour of this phase difference. We could also obtain the particle trajectories in the rest frame[4], but we shall not perform the corresponding calculations because the relevant features of the motion are already contained in our knowledge of the trajectories in the rotating frame.

We shall use the concept of free and trapped particles. A particle is called free when its resonance function remains all the time above ($\psi > 0$) or below ($\psi < 0$) exact resonance, and it is called trapped when the resonance function either remains zero or changes periodically its sign during the motion.

II.2 Free-space propagation

We first treat this relatively simple case in which the index of refraction equals unity. Then the solution of (1) can be expressed in terms of harmonic functions. In this case the resonance function is constant $\psi = \psi_o$ (see (I.53)) and all particles are free except those which start at $s = s_o$ at exact resonance ($\psi_o = 0$). The rotating frame now has the constant angular velocity $\gamma_o - P_{zo}$.

We shall use the integrals of motion (I.55) and (I.57) in a somewhat different form. In view of the relation (I.52) we may replace $\gamma - \gamma_o$ by $P_z - P_{zo}$, and then obtain:

$$|P|^2 - 2\Omega(P_z - P_{zo}) = |P_o|^2 \quad , \tag{7}$$

$$P_r - \frac{\psi_o}{g}(P_z - P_{zo}) = P_{ro} \quad . \tag{8}$$

Introducing momentum space with coordinates (P_r, P_i, P_z), we consider the paraboloid given by (7) and the plane parallel to the P_i-axis given by (8). In this system the normalized wave amplitude g points into the direction of the P_r-axis. The particle moves along the intersection of both surfaces. The particle orbit is symmetric with respect to the plane $P_i = 0$; it is an ellipse except when the electron starts at exact resonance ($\psi_o = 0$); in this latter case the orbit is the parabola fixed by

$$P_i^2 - 2\Omega(P_z - P_{zo}) = P_{io}^2 \quad , \quad P_r = P_{ro} \; . \tag{9}$$

Hence, the particle trajectory in momentum space is closed and bounded except in the synchronous case.

The parabola I and the straight line II, shown in Fig. 1a, are the intersections of the plane $P_i = 0$ with the surfaces (7) and (8), respectively. The intersection points of these two curves correspond to the maximum and minimum values of P_z attained by the particle during its motion. In the (P_r, P_i) plane the electron moves along the circle (Fig. 1b) obtained from an elimination of P_z in (7) and (8), viz.

$$P_i^2 + \left(P_r - \frac{\Omega g}{\psi_o}\right)^2 = P_{io}^2 + \left(P_{ro} - \frac{\Omega g}{\psi_o}\right)^2 \quad . \tag{10}$$

If at the initial time $P_{io} = 0$ and $P_{ro} = \Omega g/\psi_o$ then the particle is at rest in the rotating frame; in the rest frame it runs with constant angular velocity $\gamma_o - P_{zo}$ along a circle with radius $\Omega g/|\psi_o|$.

For $\psi_o \to 0$ the circle in the (P_r, P_i) plane passes into a straight line parallel to the P_i-axis through the point $P_r = P_{ro}$.

By elimination of P_r from (7) and (8) the orbit in the (P_i, P_z) plane proves to be the ellipse (see Fig. 1c)

$$\frac{P_i^2}{P_{io}^2 + \left(P_{ro} - \frac{\Omega g}{\psi_o}\right)^2} + \frac{\left[P_z - P_{zo} + \frac{g}{\psi_o}\left(P_{ro} - \frac{\Omega g}{\psi_o}\right)\right]^2}{\frac{g^2}{\psi_o^2}\left[P_{io}^2 + \left(P_{ro} - \frac{\Omega g}{\psi_o}\right)^2\right]} = 1 \; . \tag{11}$$

For $\psi_o \to 0$ this ellipse transforms into the parabola (9).

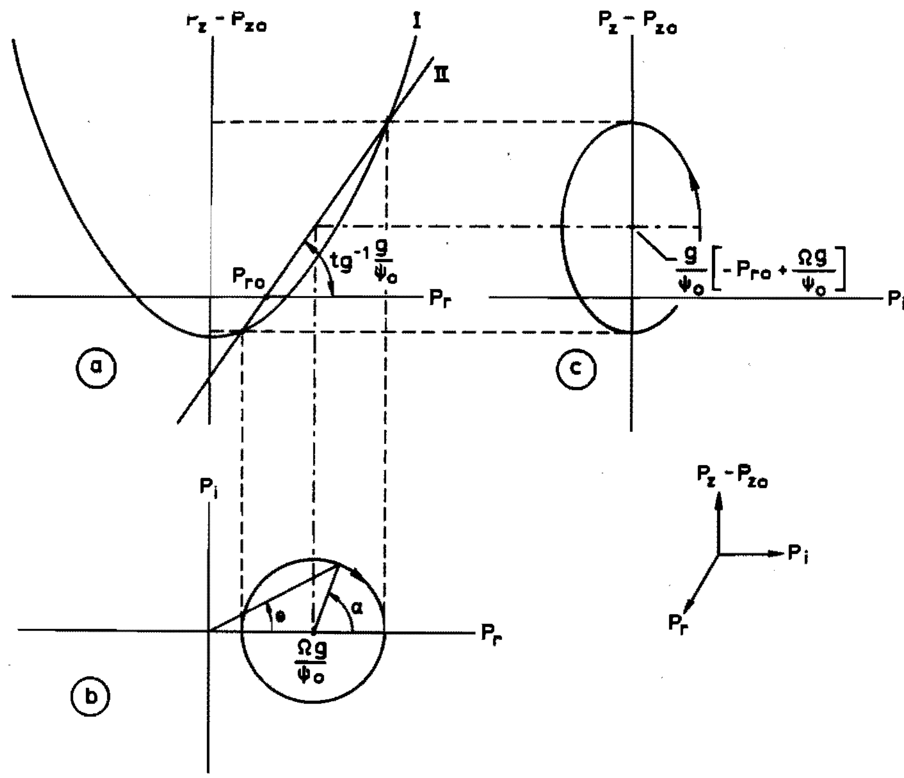


Fig. 1 Sketch of the trajectory in (P_r,P_i,P_z)-space in the case of free space propagation ($N^2 = 1$). The particle starts above resonance ($\psi_o > 0$).

We can express the proper time s as a function of the kinetic energy. For $N^2 = 1$, (3) becomes

$$s-s_o = \pm \int_{\gamma_o}^{\gamma} d\gamma' \frac{1}{\left\{g^2P_{io}^2 + g^2\left(P_{ro} - \frac{\Omega g}{\psi_o}\right)^2 - \psi_o^2\left[\gamma'-\gamma_o - \frac{g}{\psi_o}\left(\frac{\Omega g}{\psi_o} - P_{ro}\right)\right]^2\right\}^{\frac{1}{2}}} \,. \tag{12}$$

The sign in front of the integral has to be chosen such that $d\gamma/ds$ and P_i have the same sign, this being required by (I.30) (for $F_z = 0$, Ω and g are real constants). This also fixes the direction in which the particle moves in its orbit (see the arrows in Fig. 1).

Performing the integration in (12) results in

$$s-s_o = \frac{-1}{\psi_o} \arccos \frac{\frac{\psi_o}{g}(\gamma-\gamma_o) + \left(P_{ro} - \frac{\Omega g}{\psi_o}\right)}{\left[P_{io}^2 + \left(P_{ro} - \frac{\Omega g}{\psi_o}\right)^2\right]^{\frac{1}{2}}} + \frac{1}{\psi_o} \arccos \frac{P_{ro} - \Omega g/\psi_o}{\left[P_{io}^2 + \left(P_{ro} - \frac{\Omega g}{\psi_o}\right)^2\right]^{\frac{1}{2}}} . \tag{13}$$

Inverting this expression we obtain the following dependence of the kinetic energy and the axial momentum on the proper time

$$\begin{aligned}\gamma - \gamma_o &= P_z - P_{zo} \\ &= \frac{g}{\psi_o}\left[\frac{\Omega g}{\psi_o} - P_{ro}\right] + \frac{g}{\psi_o}\left[P_{io}^2 + \left(P_{ro} - \frac{\Omega g}{\psi_o}\right)^2\right]^{\frac{1}{2}} \cos\left[\alpha_o - \psi_o(s-s_o)\right] ,\end{aligned} \tag{14}$$

where α_o is the initial value of

$$\alpha = \operatorname{arctg} \frac{P_i}{P_r - \frac{\Omega g}{\psi_o}} \quad ; \tag{15}$$

the sign of α is defined by its representation in Fig. 1b.

With the aid of (8), (10) and (15) we deduce from (14)

$$\alpha = \alpha_o - \psi_o(s-s_o) \quad . \tag{16}$$

Therefore, the electron travels along the circle (10) in the (P_r, P_i) plane with the constant angular velocity $-\psi_o$ and completes one oscillation in the "proper time" interval

$$s_{osc} = \frac{2\pi}{|\psi_o|} \quad . \tag{17}$$

Note that in the linear approximation the same result is obtained for an ordinary time interval t_{osc}.

The trajectory in momentum space, which is given by (7) and (8), or also by (10) and (11), depends on both the wave amplitude and the resonance function according to the ratio g/ψ_o as well as on the associated quantities γ and P_z. However, the oscillation period in proper time is independent of the wave amplitude and only depends on the resonance function.

In view of (14) the gain (relative to the initial situation) in kinetic energy and in axial momentum, averaged over one oscillation, is given by

$$\langle P_z \rangle - P_{zo} = \langle \gamma \rangle - \gamma_o = \frac{g}{\psi_o} \left[\frac{\Omega g}{\psi_o} - P_{ro}\right] . \tag{18}$$

If $\psi_o P_{ro} < \Omega g$ we conclude that, on the average, the particle gains energy and axial momentum from the wave, while it loses on the average energy and momentum if $\psi_o P_{ro} > \Omega g$.

The maximum change in axial momentum and kinetic energy can be derived from (14) and is given by

$$P_{zmax} - P_{zmin} = \gamma_{max} - \gamma_{min} = 2\frac{g}{|\psi_o|} \left\{P_{io}^2 + \left(P_{ro} - \frac{\Omega g}{\psi_o}\right)^2\right\}^{\frac{1}{2}} . \tag{19}$$

At low initial momentum, i.e. if $|P_o| << \Omega g/|\psi_o|$, this total change and the averaged gain (18) are proportional to the square of the ratio g/ψ_o of the wave amplitude to the resonance function, whereas at high initial momentum, $|P_o| >> \Omega g/|\psi_o|$, they are linearly proportional to the ratio g/ψ_o.

Remembering that $d(\omega t)/ds = \gamma$ and $P_z = dZ/ds$ we obtain, after integration of (14), the following expressions for the time t and the axial position Z as functions of s

$$\begin{aligned}\omega(t-t_o) - \gamma_o(s-s_o) &= Z - Z_o - P_{zo}(s-s_o)\\ &= \frac{g}{\psi_o}\left[\frac{\Omega g}{\psi_o} - P_{ro}\right](s-s_o) + \frac{g}{\psi_o^2}\left[P_{io}^2 + \left(P_{ro} - \frac{\Omega g}{\psi_o}\right)^2\right]^{\frac{1}{2}}\\ &\quad \left[\sin\alpha_o - \sin\left(\alpha_o - \psi_o\,(s-s_o)\right)\right] .\end{aligned} \tag{20}$$

The oscillation time, that is the increase of t corresponding to an increase of s by s_{osc}, then equals

$$\omega t_{osc} = \left\{\gamma_o + \frac{g}{\psi_o}\left[\frac{\Omega g}{\psi_o} - P_{ro}\right]\right\} s_{osc} = \langle \gamma \rangle\, s_{osc} . \tag{21}$$

Further, the oscillation length, i.e. the distance which the particle travels in the axial direction during a time interval s_{osc}, is given by

$$z_{osc} = \frac{\omega}{c}\, z_{osc} = \left\{P_{zo} + \frac{g}{\psi_o}\left[\frac{\Omega g}{\psi_o} - P_{ro}\right]\right\} s_{osc} = \langle P_z \rangle\, s_{osc} , \tag{22}$$

where s_{osc} is given by (17), and $\langle\gamma\rangle$, $\langle P_z\rangle$ are given by (18). For $\psi_o \to 0$ the oscillation time t_{osc} and the oscillation length Z_{osc} tend to infinity much faster than s_{osc}.

The particle trajectory in the complex R-plane is similar to the one in the complex P-plane. In fact, a substitution of (I.39) into (10) results in

$$R_r^2 + \left(R_i + \frac{g}{\psi_o}\right)^2 = R_{ro}^2 + \left(R_{io} + \frac{g}{\psi_o}\right)^2 \quad , \tag{23}$$

where R_r and R_i denote the real and imaginary parts of R, respectively. The particle thus moves with the constant angular velocity $-\psi_o$ along a circle with radius $\left\{R_{ro}^2 + \left(R_{io} + \frac{g}{\psi_o}\right)^2\right\}^{\frac{1}{2}}$ and centered at the point $\left(0, -\frac{g}{\psi_o}\right)$ (see Fig. 2a). For $\psi_o = 0$ the circle transforms into a straight line parallel to the R_r-axis through the point $R_i = R_{io}$. The particle completes one single revolution in the proper time interval $s_{osc} = 2\pi/|\psi_o|$ mentioned in (17).

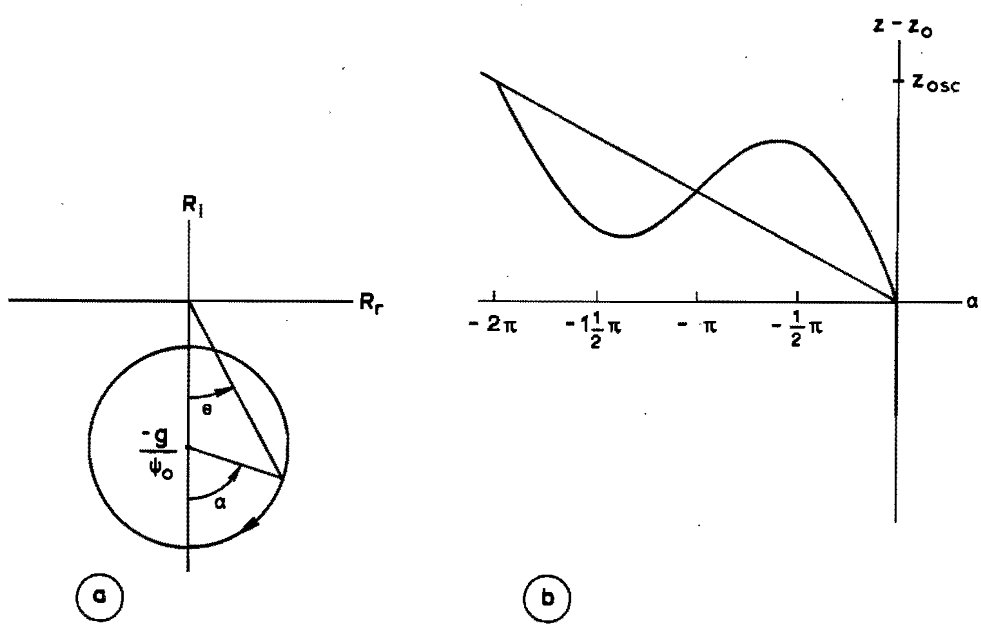


Fig. 2 Sketch of the particle trajectory in (R_r, R_i, Z)-space for $\alpha_o = 0$, $\psi_o > 0$, $\langle P_z \rangle > 0$, and for free-space propagation ($N^2 = 1$).

A combination of (16), (20) and (I.39) gives the axial position as a function of α (see Fig. 2b):

$$Z = Z_o + \frac{\langle P_z \rangle}{\psi_o}(\alpha_o - \alpha) + \frac{\Omega g}{\psi_o^2}\left[R_{ro}^2 + \left(R_{io} + \frac{g}{\psi_o}\right)^2\right]^{\frac{1}{2}} [\sin\alpha_o - \sin\alpha] \quad , \tag{24}$$

where $\langle P_z \rangle$ is given by (18).

Except for the synchronous case when $\psi_o = 0$, the motion in the R-plane thus proves to be closed and bounded, while the motion in the axial direction is the sum of a linearly increasing part and a periodic part. The axial motion is also bounded if $\langle P_z \rangle = 0$, i.e. if

$$P_{zo} = \frac{g}{\psi_o} \left[P_{ro} - \frac{\Omega g}{\psi_o} \right] \quad ,$$

which means that the initial axial momentum is then just cancelled by the average axial momentum imparted to the particle by the magnetic part of the Lorentz force.

The particle trajectory in momentum space for the synchronous case ($\psi_o = 0$) is given by (9), while the particle trajectory in the R-plane is a straight line parallel to the R_r-axis through the point $R_i = R_{io}$. The angular velocity of the rotating frame with respect to the rest frame is in this synchronous case just equal to the cyclotron frequency, $\gamma_o - P_{zo} = \Omega$ (see (I.36)).

Taking the limit $\psi_o \to 0$ in (14), taking into account (15) for $\alpha = \alpha_o$, leads to

$$\gamma - \gamma_o = P_z - P_{zo} = gP_{io}(s-s_o) + \frac{1}{2}\,\Omega g^2\, (s-s_o)^2 \quad ; \tag{25}$$

next, an integration yields:

$$\omega(t-t_o) = \gamma_o(s-s_o) + \frac{1}{2}\, gP_{io}(s-s_o)^2 + \frac{1}{6}\,\Omega g^2\, (s-s_o)^3 \quad . \tag{26}$$

The last expression can be inverted to obtain s as a function of t and then to find γ as a function of time from the first expression. However, we are only interested in asymptotic solutions. For large times, that is for

$$\omega g(t-t_o) >> \max \left\{ \frac{9}{2}\, \frac{|P_{io}|^3}{\Omega^2} \quad ; \quad \left\{ \frac{6\gamma_o^3}{\Omega} \right\}^{\frac{1}{2}} \right\} \quad , \tag{27}$$

we find from the last term in the right-hand side of (26)

$$\omega g(t-t_o) \simeq \frac{\Omega}{6}\, g^3 (s-s_o)^3 \tag{28}$$

and the asymptotic solution for γ becomes

$$\gamma \simeq \left[\frac{9}{2}\, \Omega g^2 \omega^2 (t-t_o)^2 \right]^{1/3} \quad . \tag{29}$$

From (8) we infer that P_r is constant if $\psi_o = 0$, while differentiating (25), remembering (I.30) ($F_z = 0$, Ω and g are real constants), it follows that

$$P_i = P_{io} + \Omega g(s-s_o) \quad . \tag{30}$$

Hence, the phase angle $\theta = \text{arctg } P_i/P_r$ tends to $\frac{\pi}{2}$ for $t,s \to \infty$. This means that the particle is accelerated in the direction of the electric field of the wave which is shifted in phase by $\frac{\pi}{2}$ with respect to the vector potential A_r. Moreover, P_i/γ and P_r/γ tend to zero when $t \to \infty$, hence, the transverse velocity in ordinary time $\frac{dr}{dt}$ goes to zero for large times. From (I.52) we observe that $\{P_z/\gamma\} -1 = \frac{\text{constant}}{\gamma}$ which goes to zero for $s \to \infty$. Thus, the axial velocity in ordinary time $dz/dt = cP_z/\gamma$ approaches the velocity of light.

II.3 General case of a homogeneous background differing from vacuum ($N^2 \neq 1$)

II.3.1 Description of the particle orbits in the generalized momentum space

We now drop the restriction imposed on the index of refraction in the last section and shall consider the general case of a constant N different from unity.

The resonance function ψ and the angular velocity $\gamma - NP_z$ of the rotating frame with respect to the rest frame are not constant now but periodic functions of the proper time.

We shall use the integrals of motion (I.54) and (I.57) in a modified form. A multiplication of (I.54) and (I.57) by $N^2 - 1$, while combining the result with (I.53) leads to the following expressions for these two integrals of motion

$$(N^2-1) \frac{|P|^2}{2\Omega} + \psi = B \quad , \tag{31}$$

$$(N^2-1) \; gP_r + \frac{1}{2} \psi^2 = C \quad . \tag{32}$$

The constants B and C are fixed by the initial conditions. With the aid of (I.31) and (I.36), the initial values γ_o and P_{zo} can be expressed in terms of P_o and ψ_o, or of B,C, and ψ_o.

We next define the quantity

$$G \equiv \left[|N^2-1| \;\; \Omega g^2\right]^{1/3} \quad , \tag{33}$$

and introduce the following variables in order that the integrals (31),

(32) and the differential equation (1) become independent of the parameters Ω,N, and g:

$$\bar{P} = \frac{|N^2-1|^{\frac{1}{2}}}{\Omega^{\frac{1}{2}}G^{\frac{1}{2}}} P \operatorname{sgn}(N^2-1) \quad ; \quad \bar{\psi} = \frac{\psi}{\operatorname{sgn}(N^2-1)G} \quad ; \quad \bar{s} = Gs \quad ,$$

$$\bar{B} = \frac{B}{\frac{3}{2}\operatorname{sgn}(N^2-1)G} \quad ; \quad \bar{C} = \frac{C}{\frac{3}{2}G^2} \quad . \tag{34}$$

The equations (31) and (32) then become:

$$\frac{1}{2}|\bar{P}|^2 + \bar{\psi} = \frac{3}{2}\bar{B} \quad , \tag{35}$$

$$\bar{P}_r + \frac{1}{2}\bar{\psi}^2 = \frac{3}{2}\bar{C} \quad . \tag{36}$$

The above normalization has been chosen such that for $\bar{P}_i = 0$, $\bar{P}_r = 1$, $\bar{\psi} = 1$, the polynomial $F(\gamma-\gamma_o)$ (see (2)) has a triple root (see (6b)), while the constants $\bar{B}$ and $\bar{C}$ then become equal to unity.

We introduce the Cartesian coordinate system ($\bar{P}_r$, $\bar{P}_i$, $\bar{\psi}$) and consider the paraboloid given by (35) and the parabolic cylinder parallel to the $\bar{P}_i$-axis given by (36). The particle moves along the intersection of these two surfaces. The paraboloid and the parabolic cylinder, thus also their intersection, are symmetric with respect to the plane $\bar{P}_i = 0$. Depending on the constant $\bar{B}$ and $\bar{C}$ the intersection consists of either one or two closed curves, according to the existence of one or two modes of oscillation. Both surfaces as well as the particle orbit are sketched in Fig. 3a for values of $\bar{B}$ and $\bar{C}$ leading to only one mode of oscillation, and in Fig. 3b for values involving two possible modes of oscillation. From these figures it is immediately clear that the particle orbit is always closed and bounded in the case of finite values of the constants $\bar{B}$ and $\bar{C}$. In view of the linear relation between ψ and P_z, given by (I.53), the particle orbit in the ($\bar{P}_r$, $\bar{P}_i$, $\bar{\psi}$)-space is simply related to the one in the (P_r, P_i, P_z)-space.

A multiplication of (I.30) (for $F_z = 0$, Ω and g real constants) by $1 - N^2$, while using (I.53) and (34) leads to

$$\frac{d\bar{\psi}}{d\bar{s}} = - \bar{P}_i \operatorname{sgn}(N^2-1) \quad . \tag{37}$$

The coefficient $\operatorname{sgn}(N^2-1)$ fixes the direction in which the particle moves along the intersection. The resonance function attains a maximum or minimum value at those points where $\bar{P}_i = 0$.

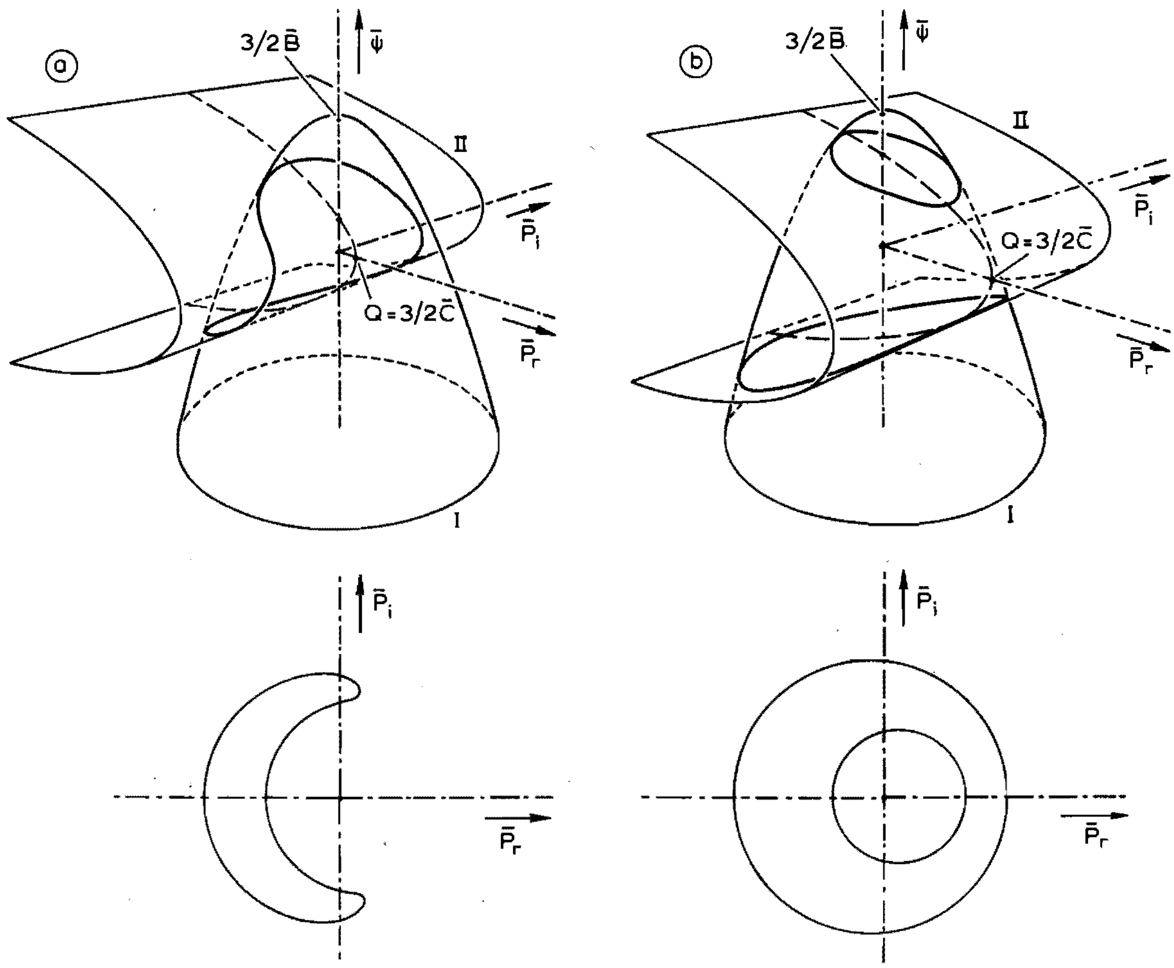


Fig. 3 Sketch of the paraboloid (35) (surface I) and the parabolic cylinder (36) (surface II). The particle moves along the intersection; a one mode of oscillation, b two possible modes of oscillation. The surface II cuts the $\bar{P}_r$-axis at the point Q.

The two parabolas shown in Figs. 4a and 4b are the intersections of both surfaces (35) and (36) with the plane $\bar{P}_i = 0$, curve I representing the intersection with the paraboloid (35) and curve II the one with the parabolic cylinder (36). In these figures the particle oscillates along the part of curve II that lies inside the parabola I. Since the extreme values of $\bar{\psi}$ are situated in the plane $\bar{P}_i = 0$ the intersection points represent the maximum and minimum values of $\bar{\psi}$ attained by the particle during its motion. In view of the linear relation between ψ and γ, P_z or $|P^2|/2\Omega$, the total kinetic energy, the axial momentum and the angular momentum also have extreme values at these intersection points.

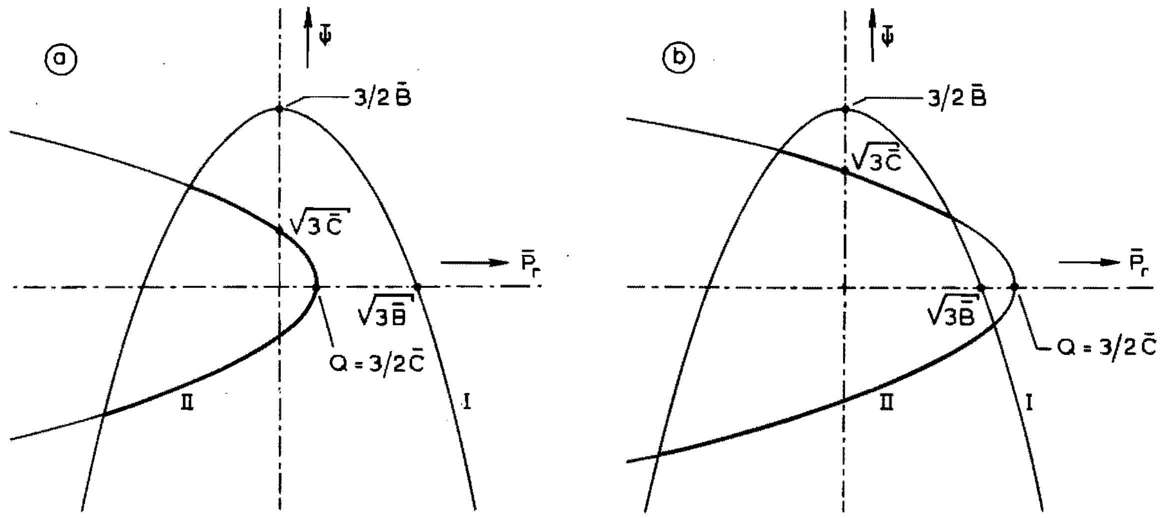


Fig. 4 Curves I and II are the intersections of the paraboloid (35) and the parabolic cylinder (36), respectively with the plane $\bar{P}_i = 0$; a one mode of oscillation, b two possible modes of oscillation.

The particle orbit cuts the plane $\bar{\Psi} = 0$ at the points

$$\bar{P}_r = \frac{3}{2}\,\bar{C} \quad ,$$

$$\bar{P}_i = \pm \left[3\bar{B} - \frac{9}{4}\,\bar{C}^2\right]^{\frac{1}{2}} \quad . \tag{38}$$

We have called a particle "trapped" when its resonance function changes periodically its sign during the motion. From (38) we then deduce that at least one of the two possible modes is trapped if

$$4\bar{B} - 3\bar{C}^2 \geq 0 \quad . \tag{39}$$

This means that in Fig. 4 the point Q on the $\bar{P}_r$-axis, at $\bar{P}_r = \frac{3}{2}\,\bar{C}$, lies between the points $\bar{P}_r = \pm\sqrt{3\bar{B}}$.

Eliminating $\bar{\Psi}$ from (35) and (36) we obtain the particle orbit in the complex $\bar{P}$-plane, viz.

$$\bar{P}_i^4 - 2\left[3\bar{B} - \bar{P}_r^2\right]\bar{P}_i^2 = Q_4(\bar{P}_r^2) \quad , \tag{40}$$

where Q_4 is the polynomial of the fourth degree

$$Q_4(\bar{P}_r) \equiv 4\left[3\bar{C} - 2\bar{P}_r\right] - \left[3\bar{B} - \bar{P}_r^2\right]^2 \quad . \tag{41}$$

The roots of Q_4 are the points at which the particle orbit in the $\bar{P}$-plane cuts the $\bar{P}_r$-axis ($\bar{P}_i = 0$). It depends on the values of the constants $\bar{B}$ and $\bar{C}$ whether the polynomial Q_4 has two or four real roots. Equation (40) then comprises one or two closed curves of the fourth degree, respectively. In the case of two modes of oscillation, the two possible particle orbits prove to be nested and do not intersect each other. The corresponding particle orbits in the R-plane can be found directly from (40) by applying (34) and (I.39). Apart from the scaling factor Ω and a rotation over an angle $\frac{\pi}{2}$, they are identical to the orbits in the P-plane.

A further elimination of $\bar{P}_r$ from (35) and (36) yields

$$4\bar{P}_i^2 = P_4(\bar{\psi}) \equiv 4\left[3\bar{B} - 2\bar{\psi}\right] - \left[3\bar{C} - \bar{\psi}^2\right]^2$$
$$\equiv -\left[\bar{\psi}^4 - 6\bar{C}\bar{\psi}^2 + 8\bar{\psi} + 9\bar{C}^2 - 12\bar{B}\right] \quad . \tag{42}$$

A combination of (42) and (37) leads to the following form of (1):

$$\left[\frac{d\bar{\psi}}{d\bar{s}}\right]^2 = \frac{1}{4}\, P_4(\bar{\psi}) \quad . \tag{43}$$

The dependence of $\bar{\psi}$ on the proper time $\bar{s}$ thus proves to be completely determined by the values of the constants $\bar{B}$ and $\bar{C}$. The polynomial P_4 must be non-negative and since $P_4(\pm\infty) = -\infty$ the function $\bar{\psi}$ oscillates between the two largest or between the two smallest real roots of P_4, depending on which of the two pairs is situated around $\bar{\psi} = \bar{\psi}_o$. The roots of P_4 are the extreme values of the function $\bar{\psi}$ attained by the particle during its motion. The corresponding values of $\bar{P}_r$ are the points where the particle orbit in the complex $\bar{P}$-plane cuts the $\bar{P}_r$-axis ($\bar{P}_i = 0$) and therefore, are the roots of the polynomial Q_4. Because the polynomials P_4 and Q_4 transform into each other by interchanging the constant $\bar{B}$ and $\bar{C}$, the roots of Q_4 can be found directly from the roots of P_4. The polynomials P_4 and Q_4 must have the same number of real roots. Hence, the criterion for zero, two, or four real roots, must be symmetric in $\bar{B}$ and $\bar{C}$. In Figs. 6 and 7 the roots of P_4 are plotted for some special choices of the constants $\bar{B}$ and $\bar{C}$.

II.3.2 The transition from one mode to two modes of oscillation

We are interested in the possible points of contact of the two surfaces (35) and (36) when $\bar{B}$ and $\bar{C}$ are chosen properly. These points mark the transition from no solution (all roots of P_4 complex, which involves the impossible situation connected with negative values of P_4 for all real $\bar{\psi}$) to a single mode of oscillation (only two real roots of P_4), or the transition from a single to two modes of oscillation (all roots of P_4 becoming real).

Both surfaces in question are of the second degree, and are symmetric with respect to the plane $\bar{P}_i = 0$. From this we conclude that their points of contact have to lie in the plane $\bar{P}_i = 0$, and that they are also the points of contact of the curves I and II (Fig. 4); the latter are the intersections of the surfaces (35) and (36) with the plane $\bar{P}_i = 0$. We obtain these points as follows.

The straight line

$$\bar{\psi} = c_1 \bar{P}_r + c_2 \tag{44}$$

is a tangent of curve I when the constants c_1 and c_2 satisfy the relation

$$c_2 = \frac{1}{2} c_1^2 + \frac{3}{2} \bar{B} \quad .$$

The coordinates of the point of contact then are

$$\bar{P}_r = -c_1 \quad , \quad \bar{\psi} = \frac{3}{2} \bar{B} - \frac{1}{2} c_1^2 \quad . \tag{45}$$

The straight line (44) is also a tangent of curve II of Fig. 4 when

$$c_2 = - \frac{1 + 3c_1^2\bar{C}}{2c_1} \quad ,$$

which involves the following coordinates of the point of contact:

$$\bar{P}_r = \frac{3c_1^2\bar{C} - 1}{2c_1^2} \quad , \quad \bar{\psi} = \frac{-1}{c_1} \quad . \tag{46}$$

The two points of contact (45) and (46) coincide when the two algebraic equations

$$2c_1^3 + 3\bar{C}c_1^2 - 1 = 0 \quad ,$$

$$c_1^3 - 3\bar{B}c_1 - 2 = 0 \quad , \tag{47}$$

have at least one real root in common. This is only possible when the constants $\bar{B}$ and $\bar{C}$ satisfy the relation

$$D \equiv (1 - \bar{B}\bar{C})^2 - 4(\bar{C}^2 - \bar{B})(\bar{B}^2 - \bar{C}) = 0 \quad . \tag{48a}$$

When this condition is satisfied the polynomial P_4 has at least two real and equal roots. The relation (48a) is symmetric in $\bar{B}$ and $\bar{C}$.

The equations

$$1 - \bar{B}\bar{C} = \pm\ 2\left[(\bar{C}^2 - \bar{B})\ (\bar{B}^2 - \bar{C})\right]^{\frac{1}{2}} \quad , \tag{48b}$$

connected with the condition D = 0, have been plotted in Fig. 5. The minus sign holds for the branches of the curve D = 0 in the first quadrant of the $(\bar{B},\bar{C})$-plane, while the plus sign holds for the other branch.

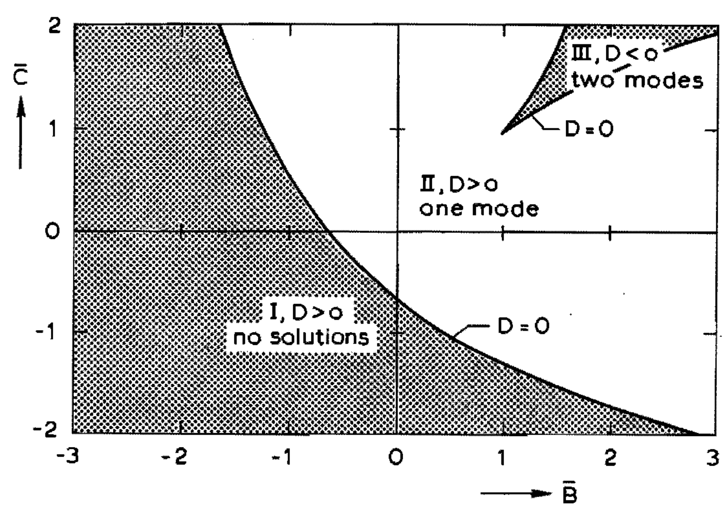


Fig. 5 Plot of the algebraic equation D = 0.

In region I of this figure,D is negative while a real solution of (43) cannot exist there, the constants $\bar{B}$ and $\bar{C}$ being such that P_4 has no real roots. This means that $\bar{B}$ and $\bar{C}$ correspond to initial values which do not fit each other. In region II, D is positive which involves a single mode of oscillation corresponding to the two real roots of P_4. In region III, D is again negative, and two modes of oscillation exist, which corresponds to four real roots. These conclusions follow from the consideration of the positions of the surfaces which represent the constants of motion (35) and (36); these conclusions can be verified by calculating the roots of P_4 with the method explained in the next subsection.

The common root of the equations (47) is given by

$$c_1 = \frac{1}{2}\,\frac{1 - \bar{B}\bar{C}}{\bar{C}^2 - \bar{B}} \quad ,$$

$$c_1 = -1 \text{ when } \bar{B} = \bar{C} = 1 \quad . \tag{49}$$

The coordinates of the point of contact of the tangenting curves I and II are given by (see (45) and (46))

$$\bar{P}_i = 0 \quad , \quad \bar{P}_r = -c_1 \quad , \quad \bar{\psi} = -\frac{1}{c_1} \quad , \tag{50}$$

c_1 being specified by (49), while $\bar{B}$ and $\bar{C}$ satisfy the relation (48). This point is also the point of contact of the paraboloid (35) and the parabolic cylinder (36). With the aid of (34) one can verify that (50) satisfies (6a), and also (6b) if $c_1 = -1$.

II.3.3 Properties of the roots of the polynomials associated with the particle orbits

In order to obtain the solution of (43) we have to calculate the roots of the polynomial P_4. We shall deduce a number of properties of the roots $\bar{\psi}_n$ and $\bar{P}_{nr}$ ($n = 1,2,3,4$) of P_4 and Q_4 respectively, and the various possible situations will be classified in the next subsection. Of course, we shall recover those results that can also be obtained by geometric considerations concerning the surfaces representing the constants of the motion in the $(\bar{P}_r, \bar{P}_i, \bar{\psi})$-space.

The polynomial P_4 as given by (42) can be decomposed as follows

$$P_4 = -(\bar{\psi}-\bar{\psi}_i)\;(\bar{\psi}-\bar{\psi}_j)\;(\bar{\psi}-\bar{\psi}_k)\;(\bar{\psi}-\bar{\psi}_\ell) \quad . \quad (i\neq j\neq k\neq \ell) \quad . \tag{51}$$

By comparing (51) with (42) it follows that the four roots have to satisfy the relations

$$\left.\begin{aligned}
&\bar{\psi}_i + \bar{\psi}_j + \bar{\psi}_k + \bar{\psi}_\ell = 0 \quad ,\\
&\bar{\psi}_i\bar{\psi}_j + \bar{\psi}_k\bar{\psi}_\ell + (\bar{\psi}_i + \bar{\psi}_j)\;(\bar{\psi}_k + \bar{\psi}_\ell) = -6\bar{C} \quad ,\\
&\bar{\psi}_i\bar{\psi}_j(\bar{\psi}_\ell + \bar{\psi}_k) + \bar{\psi}_k\bar{\psi}_\ell(\bar{\psi}_i + \bar{\psi}_j) = -8 \quad ,\\
&\bar{\psi}_i\bar{\psi}_j\bar{\psi}_k\bar{\psi}_\ell = 9\bar{C}^2 - 12\bar{B} \quad .
\end{aligned}\right\} \tag{52}$$

Eliminating $\bar{\psi}_i\bar{\psi}_j, \bar{\psi}_k\bar{\psi}_\ell$ and $\bar{\psi}_k + \bar{\psi}_\ell$ from these relations we obtain the polynomial of the third degree:

$$a^6 - 12\bar{C}\;a^4 + 48\bar{B}\;a^2 - 64 = 0 \tag{53a}$$

for a^2, with

$$a^2 \equiv (\bar{\psi}_i + \bar{\psi}_j)^2 \quad . \tag{53b}$$

In view of the first relation of (52), the three solutions a_n^2 ($n = 1,2,3$) represent the quantities

$$(\bar{\psi}_1 + \bar{\psi}_2)^2 = (\bar{\psi}_3 + \bar{\psi}_4)^2 \ ; \ (\bar{\psi}_1 + \bar{\psi}_3)^2 = (\bar{\psi}_2 + \bar{\psi}_4)^2 \ ; \ (\bar{\psi}_1 + \bar{\psi}_4)^2 = (\bar{\psi}_2 + \bar{\psi}_3)^2 . \tag{54}$$

By introducing $x = a^2 - 4\bar{C}$, (53a) becomes

$$x^3 - 48(\bar{C}^2 - \bar{B})x - 64\left[2\bar{C}^3 - 3\bar{B}\bar{C} + 1\right] = 0 \quad . \tag{55}$$

The three solutions of this equation are given by

$$x_1 = p_1^{1/3} + p_2^{1/3} \ ; \ x_2 = \varepsilon_1 p_1^{1/3} + \varepsilon_2 p_2^{1/3} \ ; \ x_3 = \varepsilon_2 p_1^{1/3} + \varepsilon_1 p_2^{1/3} \quad , \tag{56a}$$

where

$$p_{1,2} = 32\left[2\bar{C}^3 - 3\bar{B}\bar{C} + 1\right] \pm 32\sqrt{D} \ , \ \varepsilon_{1,2} = -\frac{1}{2} \pm \frac{i}{2}\sqrt{3} \quad , \tag{56b}$$

while the discriminant D is identical to the quantity defined by (48a), viz.

$$D \equiv \left[2\bar{C}^3 - 3\bar{B}\bar{C} + 1\right]^2 - 4\left[\bar{C}^2 - \bar{B}\right]^3 \quad . \tag{56c}$$

The two relations associated with $D = 0$, viz.

$$2\bar{C}^3 - 3\bar{B}\bar{C} + 1 = \pm 2\left[\bar{C}^2 - \bar{B}\right]^{3/2} \quad , \tag{57}$$

have been plotted in Fig. 5. The minus sign holds for the lower branch of the curve $D = 0$ in the first quadrant of the $(\bar{B},\bar{C})$-plane, while the plus sign holds for all other branches.

The equations (53) and (55) have the same number of real roots. This number depends on the sign of the discriminant D. Equation (53a) has one real and two complex roots if D is positive, three real roots if D is negative, three real roots of which two are equal if $D = 0$, and three equal real roots if, simultaneously, $2\bar{C}^3 - 3\bar{B}\bar{C} + 1 = 0$ and $\bar{C}^2 - \bar{B} = 0$, i.e. if $\bar{B} = \bar{C} = 1$.

In the following three possibilities are to be discussed. First, let the four roots of P_4 all be real, it then follows from (54) that all three solutions of (53) must be real and positive. From the above discussion we conclude that this is only possible if $D \leq 0$ and, since the solutions of (53) have to satisfy the relations

$$a_1^2 + a_2^2 + a_3^2 = 12\bar{C} \quad , \quad \frac{1}{a_1^2} + \frac{1}{a_2^2} + \frac{1}{a_3^2} = \frac{3}{4}\bar{B} \quad ,$$

the constants $\bar{B}$ and $\bar{C}$ must be positive in this case.

Secondly, if P_4 has two unequal real roots and two complex roots, it follows from the first of the relations (52) and from (54) that (53) has one real and two complex solutions. This situation is only possible if $D > 0$.

Thirdly, if P_4 has four complex roots, it follows from the first of the relations (52) and from (54) that (53) has one positive and two negative real roots. Hence, the discriminant D must be negative. Moreover, it is obvious from (53) that a negative solution for a^2 can only exist if $\bar{B}$ and $\bar{C}$ are not both positive.

With the aid of the relations (52) we can express the four roots of P_4 in terms of the single quantity a, satisfying (53a). In fact, we have

$$\bar{\psi}_{1,2} = -\frac{1}{2}a \pm \frac{1}{2}\left\{12\bar{C} - a^2 + \frac{16}{a}\right\}^{\frac{1}{2}} \;;\; \bar{\psi}_{3,4} = \frac{1}{2}a \pm \frac{1}{2}\left\{12\bar{C} - a^2 - \frac{16}{a}\right\}^{\frac{1}{2}} \quad . \qquad (58)$$

Since (53) has at least one positive solution for a^2 for arbitrary values of $\bar{B}$ and $\bar{C}$, we take in (58) for a the positive square root of such a positive solution.

By interchanging $\bar{B}$ and $\bar{C}$ the above relations and conclusions are also valid for the roots of the polynomial Q_4.

II.3.4 Classification of the various modes of oscillation

From the preceding discussion a number of conclusions about the roots of P_4 and Q_4 can be drawn for the several ranges for the values of the constants $\bar{B}$ and $\bar{C}$.

A. Properties connected with the polynomial P_4.

1. If the constants $\bar{B}$ and $\bar{C}$ are not both positive while $D < 0$, the polynomial P_4 has four complex roots. This means that if $\bar{B}$ and $\bar{C}$ are situated in region I of the $\bar{B},\bar{C}$-plane (Fig. 5) they correspond to initial values which do not fit each other, and no real solution of the equations of motion can exist.

2. If $D > 0$, i.e. if $\bar{B}$ and $\bar{C}$ lie in region II of Fig. 5, P_4 has two unequal real roots and two complex roots. In this case only one single mode of oscillation can occur. From (58) we conclude that the function

$\bar{\psi}$ then oscillates between the real roots $\bar{\psi}_1$ and $\bar{\psi}_2$ of P_4, that is between

$$\bar{\psi}_{1,2} = -\frac{1}{2}a \pm \frac{1}{2}\left[12\bar{C} - a^2 + \frac{16}{a}\right]^{\frac{1}{2}} , \tag{59}$$

where $a = -(\bar{\psi}_1 + \bar{\psi}_2)$ is the positive root of (53).

The root $\bar{\psi}_2$ is negative and from the last relation of (52) (or from the term $9\bar{C}^2 - 12\bar{B}$ in the right-hand side of (42)) it follows that $\bar{\psi}_1$ is also negative if $3\bar{C}^2 - 4\bar{B} > 0$. The function $\bar{\psi}$ then remains negative throughout the motion and the particle is free. If, on the other hand, $3\bar{C}^2 - 4\bar{B} < 0$, $\bar{\psi}_1$ has to be positive and the function $\bar{\psi}$ oscillates around $\bar{\psi} = 0$. In accordance with (39), the particle is trapped. The function $\bar{\psi}$ just reaches the value $\bar{\psi}_1 = 0$ if $3\bar{C}^2 - 4\bar{B} = 0$. For this case the real roots of P_4 as functions of $\bar{C}$ are graphically represented in Fig. 6a at the end of this subsection.

2a. If in particular the point $(\bar{B},\bar{C})$ lies on the boundary between the regions I and II of the $(\bar{B},\bar{C})$-plane (Fig. 5), so that $D = 0$ and $\bar{B},\bar{C}$ are not both positive, P_4 will have two equal real roots $\bar{\psi}_{1,2}$, the roots $\bar{\psi}_{3,4}$ being complex. This double root and the corresponding value of $\bar{P}_r$ depend on the integrals $\bar{B}$ and $\bar{C}$ according to (49) and (50). Indeed, it can easily be proved that if $D = 0$ and if $\bar{B}$ and $\bar{C}$ are not both positive (see (48b)), the relation for the existence of a double root, viz.

$$12\bar{C} - a^2 + \frac{16}{a} = 0$$

and (53) will have the common positive root

$$a = \frac{1 - \bar{B}\bar{C}}{\bar{B}^2 - \bar{C}} = 2\left[\frac{\bar{C}^2 - \bar{B}}{\bar{B}^2 - \bar{C}}\right]^{\frac{1}{2}} \tag{60}$$

We then obtain from (58) the following four roots of P_4

$$\bar{\psi}_{1,2} = -\frac{1}{2}a \quad , \quad \bar{\psi}_{3,4} = \frac{1}{2}a \pm \sqrt{\frac{-8}{a}} , \tag{61}$$

where a is given by (60).

3. If $D < 0$, while $\bar{B}$ and $\bar{C}$ are both positive, i.e. if $\bar{B}$ and $\bar{C}$ lie in region III of Fig. 5, P_4 will have four real roots. In this case two modes of oscillation are possible. It depends on the initial value $\bar{\psi}_o$ whether the function $\bar{\psi}$ will oscillate between the two largest or between the two smallest roots of P_4. From (59) we infer that at least one positive and one negative root occur, viz.

$$\bar{\Psi}_3 = \frac{1}{2}\, a + \frac{1}{2}\left[12\bar{C} - a^2 - \frac{16}{a}\right]^{\frac{1}{2}} \text{ and } \bar{\Psi}_2 = -\frac{1}{2}\, a - \frac{1}{2}\left[12\bar{C} - a^2 + \frac{16}{a}\right]^{\frac{1}{2}} \quad .$$

Because the modulus of the positive root is smaller than the modulus of the negative root, and since the sum of the four roots vanishes, there must exist at least one second positive root. From the last relation of (52) we then conclude that P_4 has two positive and two negative roots if $3\bar{C}^2 - 4\bar{B} > 0$, and three positive and one negative root if $3\bar{C}^2 - 4\bar{B} < 0$. In the first case two free-particle oscillation modes do exist, while in the latter case one free and one trapped-particle oscillation mode have to occur in accordance with (39). For the trapped-particle mode the function $\bar{\Psi}$ oscillates between the two smallest roots of P_4, and just reaches the value $\bar{\Psi} = 0$ when $3\bar{C}^2 - 4\bar{B} = 0$ (see Fig. 6b).

3a. If the point $(\bar{B},\bar{C})$ lies on the boundary between regions II and III of the $(\bar{B},\bar{C})$-plane (Fig. 5), so that $D = 0$ and $\bar{B},\bar{C}$ are both positive, P_4 will have four real roots, two of which are equal. This double root $\bar{\Psi}_{3,4}$ and the corresponding values of $\bar{P}_r$, expressed in $\bar{B}$ and $\bar{C}$, are given by (49) and (50). Again one can prove that if $D = 0$ and $\bar{B}$ and $\bar{C}$ are both positive (see (48b)), the equation

$$12\bar{C} - a^2 - \frac{16}{a} = 0$$

and (53) have the common positive root

$$a = \frac{\bar{B}\bar{C} - 1}{\bar{B}^2 - \bar{C}} = 2\left[\frac{\bar{C}^2 - \bar{B}}{\bar{B}^2 - \bar{C}}\right]^{\frac{1}{2}} \quad . \tag{62}$$

For this value of a we obtain from (58) the following four roots of P_4:

$$\bar{\Psi}_{1,2} = -\frac{1}{2}\, a \pm \sqrt{\frac{8}{a}} \quad , \quad \bar{\Psi}_3 = \bar{\Psi}_4 = \frac{1}{2}\, a \quad . \tag{63}$$

We conclude from (62) that $a > 2$ for $(\bar{B},\bar{C})$ on the upper branch of the curve $D = 0$, that is the branch for which $\bar{C} > \bar{B} > 1$. From (63) it then follows that $\bar{\Psi}_{3,4} > \bar{\Psi}_{1,2}$. This means that, when we pass from region II to region III (Fig. 5) by crossing the upper branch of the curve $D = 0$, the two new appearing real roots $\bar{\Psi}_{3,4}$ are both larger than $\bar{\Psi}_{1,2}$. On the other hand, when we cross the lower branch of the curve $D = 0$ (for which $\bar{B} > \bar{C} > 1$ and $a < 2$), the two new appearing roots $\bar{\Psi}_{3,4}$ lie between $\bar{\Psi}_{1,2}$, i.e. $\bar{\Psi}_1 > \bar{\Psi}_{3,4} > \bar{\Psi}_2$.

When the constants $\bar{B}$ and $\bar{C}$ are both equal to unity, P_4 has three equal roots. From (53) or (55) we now easily obtain the positive solution $a = 2$. Substituting this value into (58) we find the following roots of P_4:

$$\bar{\psi}_1 = \bar{\psi}_3 = \bar{\psi}_4 = 1 \quad , \quad \bar{\psi}_2 = -3 \quad , \tag{64a}$$

in accordance with (49) and (50). The corresponding values of $\bar{P}_r$ are the roots of Q_4 and can be found immediately from (35) and (36) ($\bar{P}_{ni} = 0$):

$$\bar{P}_{1r} = \bar{P}_{3r} = \bar{P}_{4r} = 1 \quad , \quad \bar{P}_{2r} = -3 \quad . \tag{64b}$$

B. Properties connected with the polynomial Q_4.

By interchanging $\bar{B}$ and $\bar{C}$ in all relevant equations and using analogous arguments we can draw a number of conclusions about the roots $\bar{P}_{nr}$ of Q_4 (see (41)) and about the particle orbit in the $\bar{P}$-plane. The four roots of Q_4 can be represented as follows (compare (58)):

$$(\bar{P}_{1,2})_r = -\frac{1}{2}\,b \pm \frac{1}{2}\left[12\bar{B}-b^2 + \frac{16}{b}\right]^{\frac{1}{2}} \; , \; (\bar{P}_{3,4})_r = \frac{1}{2}\,b \pm \frac{1}{2}\left[12\bar{B}-b^2 - \frac{16}{b}\right]^{\frac{1}{2}} \; , \tag{65}$$

b being an (always existing) positive root of the equation (compare (53))

$$b^6 - 12\bar{B}b^4 + 48\bar{C}b - 64 = 0 \quad . \tag{66}$$

The $\bar{P}_{nr}$ are the intersections of the particle orbit in the $\bar{P}$-plane with the $\bar{P}_r$-axis ($\bar{P}_i = 0$). All roots (65) of Q_4 are complex when the point $(\bar{B},\bar{C})$ lies in region I of the $(\bar{B},\bar{C})$-plane of Fig. 5. This corresponds again to initial values which do not fit each other. The polynomial Q_4 has two real roots $(\bar{P}_{1,2})_r$ and two complex roots $(\bar{P}_{3,4})_r$ when the point $(\bar{B},\bar{C})$ lies in region II. The root $\bar{P}_{2r}$ is always negative here and the sign of $\bar{P}_{1r}$ depends on the last term on the right of (41). If $3\bar{B}^2 - 4\bar{C} < 0$, the two roots $(\bar{P}_{1,2})_r$ have opposite signs, and the particle orbit in the $\bar{P}$-plane encloses the origin. In the opposite case $3\bar{B}^2 - 4\bar{C} > 0$ both roots are negative, and the particle orbit does not enclose the origin. In the latter case, therefore, the phase angle between the generalized momentum and the wave vector is restricted to a certain interval smaller than 2π.

On the boundary between regions I and II of the $(\bar{B},\bar{C})$-plane the two real roots $(\bar{P}_{1,2})_r$ become just equal (compare (60) and (61)), and are given by

$$(\bar{P}_{1,2})_r = -\frac{1}{2}\,b \quad , \tag{67a}$$

with

$$b = \frac{1 - \bar{B}\bar{C}}{\bar{C}^2 - \bar{B}} = 2\left[\frac{\bar{B}^2 - \bar{C}}{\bar{C}^2 - \bar{B}}\right]^{\frac{1}{2}} \quad . \tag{67b}$$

In region III of the $(\bar{B},\bar{C})$-plane all four roots $\bar{P}_{nr}$ are real. At least one of them is negative and at least two are positive. The number of negative roots depends again on the sign of the last term on the right of (41), thus leading to two different situations.

If $3\bar{B}^2 - 4\bar{C} < 0$, the polynomial Q_4 has one negative and three positive roots. Since the two possible orbits in the $\bar{P}$-plane are nested, one mode exists for which $\bar{P}_r$ oscillates between the negative root and the largest positive root, while for the other mode $\bar{P}_r$ oscillates between the two smallest positive roots. The first particle orbit encloses the origin of the complex $\bar{P}$-plane, but the second one does not. This means for this latter orbit that the phase angle is again restricted to an interval smaller than 2π.

In the other case, for which $3\bar{B}^2 - 4\bar{C} > 0$, the polynomial Q_4 has two negative and two positive roots. One mode now exists for which $\bar{P}_r$ oscillates between the smallest and largest root of Q_4, and another one for which $\bar{P}_r$ oscillates between the intermediate roots. Both particle orbits enclose the origin of the $\bar{P}$-plane. In the transitional case $3\bar{B}^2 - 4\bar{C} = 0$, one particle orbit just passes through the origin of the $\bar{P}$-plane. For this case the dependence of the corresponding roots of P_4 on the integrals of motion $\bar{B},\bar{C}$ is graphically represented in Fig. 6b.

On the boundary between regions II and III, Q_4 has four real roots of which at least two are equal. These roots are given by (compare (62) and (63))

$$(\bar{P}_{1,2})_r = -\frac{1}{2}\,b \pm\sqrt{\frac{8}{b}} \quad , \quad \bar{P}_{r3} = \bar{P}_{r4} = \frac{1}{2}\,b \quad , \tag{68a}$$

where

$$b = \frac{\bar{B}\bar{C} - 1}{\bar{C}^2 - \bar{B}} = 2\left[\frac{\bar{B}^2 - \bar{C}}{\bar{C}^2 - \bar{B}}\right]^{\frac{1}{2}} \quad . \tag{68b}$$

From (68) we conclude that, when we pass from region II to region III by crossing the upper branch of the curve $D = 0$ (where $\bar{C} > \bar{B} > 1$ and $b < 2$), the two new appearing real roots $(\bar{P}_{3,4})_r$ lie between $(\bar{P}_{1,2})_r$, i.e. $\bar{P}_{1r} > (\bar{P}_{3,4})_r > \bar{P}_{2r}$. On the other hand, when we cross the lower branch of the curve $D = 0$ (where $\bar{B} > \bar{C} > 1$ and $b > 2$), the two new appearing real roots $(\bar{P}_{3,4})_r$ are both larger than $(\bar{P}_{1,2})_r$, i.e. $(\bar{P}_{3,4})_r > \bar{P}_{1r} > \bar{P}_{2r}$. For $\bar{B} = \bar{C} = 1$ the three equal real roots of Q_4 are given by (64b).

The different cases mentioned above are listed in table 1 and the corresponding regions of the $(\bar{B},\bar{C})$-plane are indicated in Fig. 8.

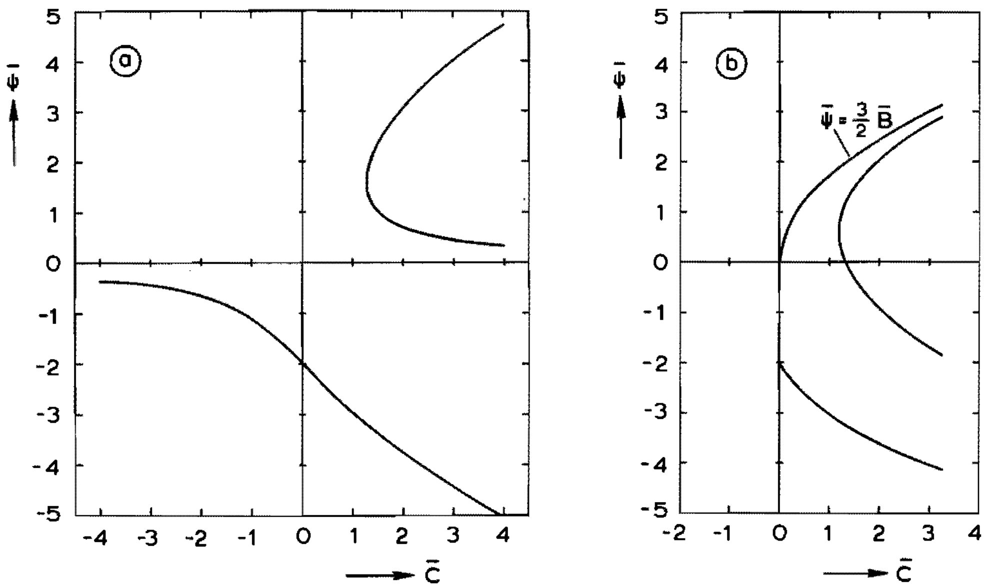


Fig. 6 Plot of the real roots of the polynomial P_4 given by (42) as functions of the constant $\bar{C}$;
<u>a</u> $3\bar{C}^2 - 4\bar{B} = 0$, $\bar{\Psi} = 0$ is a real root of P_4,
<u>b</u> $3\bar{B}^2 - 4\bar{C} = 0$, $\bar{\Psi} = 3/2\ \bar{B}$ is the real root of P_4 that corresponds to the root $\bar{P}_r = 0$ of Q_4.

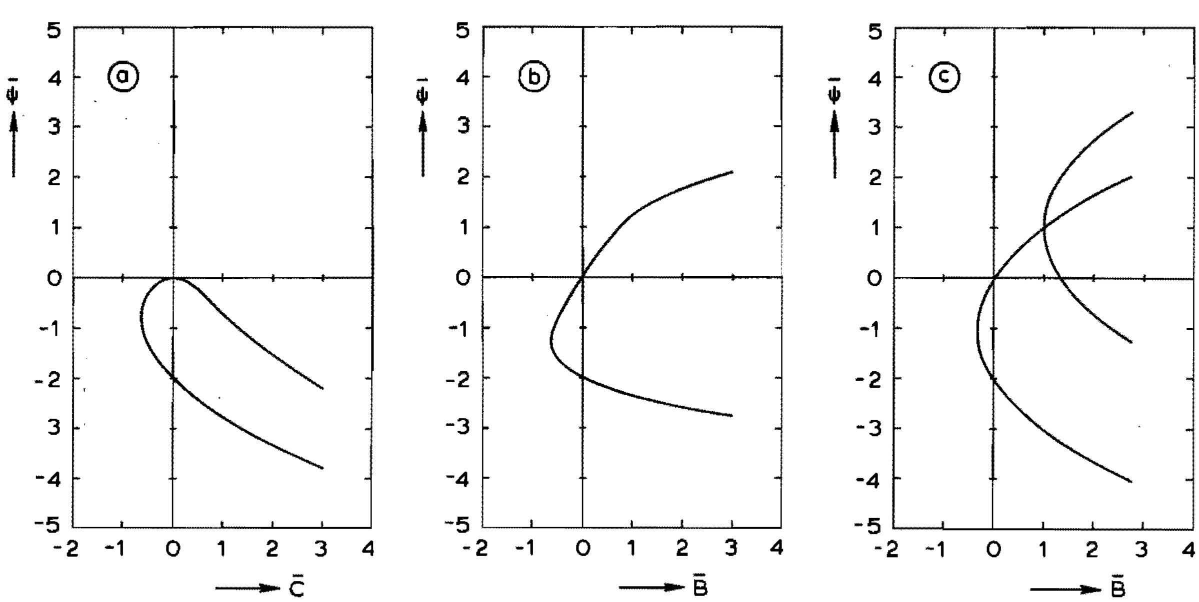


Fig. 7 Plot of the real roots of the polynomial P_4 as functions of the constants $\bar{B}$ and $\bar{C}$;
<u>a</u> $\bar{B} = 0$, <u>b</u> $\bar{C} = 0$, <u>c</u> $\bar{B} = \bar{C}$.

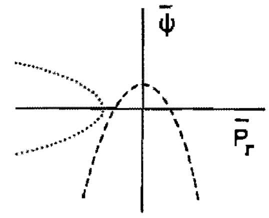
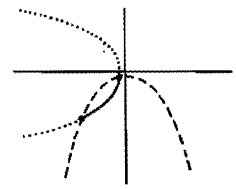
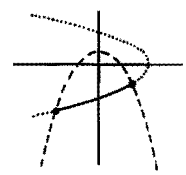
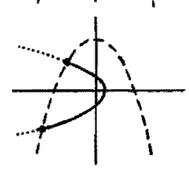
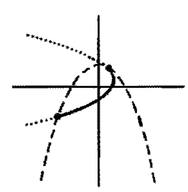
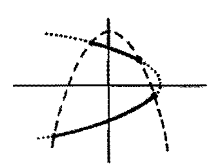
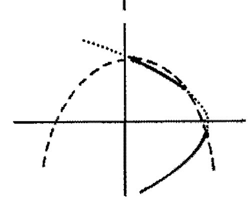
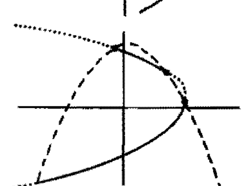
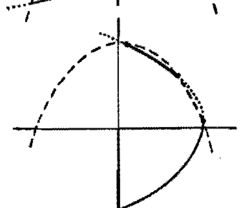
	region	$3\bar{C}^2 - 4\bar{B}$	$3\bar{B}^2 - 4\bar{C}$		
1	I $D < 0$	> 0	> 0		no solutions
2	II $D > 0$	> 0	> 0		one free particle mode, not encircling the origin
3	$D > 0$	> 0	< 0		one free particle mode, encircling the origin
4	$D > 0$	< 0	> 0		one trapped particle mode, not encircling the origin
5	$D > 0$	< 0	< 0		one trapped particle mode, encircling the origin
6	III $D < 0$	> 0	> 0		two free particle modes, both encircling the origin
7	$D < 0$	> 0	< 0		two free particle modes of which one encircles the origin
8	$D < 0$	< 0	> 0		one free and one trapped particle mode both encircling the origin
9	$D < 0$	< 0	< 0		one free particle mode which does not encircle the origin and one trapped particle mode which encircles the origin

Table 1 The graphs are analogous to those represented in Fig. 4.

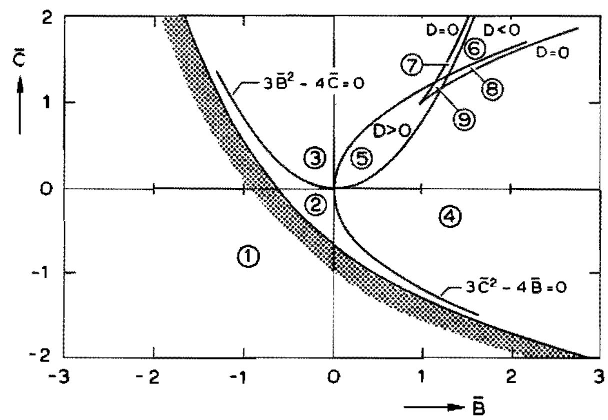


Fig. 8 Regions of the $(\bar{B},\bar{C})$-plane corresponding to the cases listed in table 1.

For some special cases we shall calculate the positive roots a^2 and b^2 of (53) and (66), and the roots $\bar{\psi}_n$ and $\bar{P}_{nr}$ of both polynomials P_4 and Q_4, respectively, as functions of the constants $\bar{B}$ and $\bar{C}$, or also as functions of the initial conditions $\bar{P}_o$ and $\bar{\psi}_o$.

II.4 Calculation of the oscillation times and the oscillation length

II.4.1 Solution of the equations of motion

We shall now solve equation (43) in order to obtain $\bar{\psi}$ as a function of $\bar{s}$, and thus, in view of the definitions (33) and (34), the dependence of the resonance function ψ on the proper time s. With the aid of this solution we can find the s-dependence of all quantities that are relevant to the motion of the particle. From the definition (I.36) of ψ we directly obtain the Doppler-shifted frequency $\gamma - NP_z$, while the total kinetic energy $mc^2\gamma$ and the axial momentum P_z follow from the linear relations between ψ and γ, and between ψ and P_z, respectively (see (I.53)). A substitution of $\psi(s)$, thus derived, into the constants of motion (31) and (32) leads to expressions for the transverse momentum P(s) and, in view of (I.39), also for the transverse position R(s).

Since $P_4(\psi)$ is a polynomial of the fourth degree, we can express $\bar{s}$ as a function of $\bar{\psi}$ in terms of an elliptic integral of the first kind

$$\bar{s} = \pm 2 \int^{\bar{\psi}} \frac{d\bar{\psi}'}{[P_4(\bar{\psi}')]^{\frac{1}{2}}} \quad . \tag{69}$$

This solution can be inverted which results in an expression for $\bar{\psi}(\bar{s})$ in terms of a Jacobian elliptic function. Next, substituting the corresponding $\psi(s)$ into (I.53), and integrating this equation with respect to s (remembering that $\gamma = \omega \frac{dt}{ds}$ and $P_z = \frac{dZ}{ds}$), we find the ordinary time t and the axial position $Z = \frac{\omega}{c} z$ as functions of the proper time s with the aid of the following formula:

$$\int^{s} \psi(s')ds' = \psi_o s + (1-N^2)\left[\omega t-\gamma_o s\right] = \psi_o s + \frac{(1-N^2)}{N}\left[\frac{\omega}{c} z - P_{zo} s\right] \quad . \qquad (70)$$

The integral in the left-hand side of these relations can be expressed as a combination of elliptic integrals in $\psi(s)$ of the first and third kind. This enables us to find t and z in terms of either s or of ψ. However, these relations cannot be inverted to obtain the explicit dependence of s or ψ on the time t in terms of known functions. Hence, we cannot get simple expressions for the dependence of the relevant quantities on the ordinary time t. On the other hand, by performing the integration in (70) over one period of oscillation we, nevertheless, can obtain relations between the oscillation period in ordinary time t_{osc}, the oscillation length z_{osc}, and the oscillation period in proper time s_{osc}.

In the next subsections we shall express the solution of (43), the oscillation times, and the oscillation length in terms of the roots of the polynomial P_4. The details of the calculation are given in the Appendix.

II.4.2 The solution in the single-mode domain

Inside region II of the $(\bar{B},\bar{C})$-plane (see Fig. 5), the polynomial P_4 has the two real roots $\bar{\psi}_{i,j}$, and the two conjugate complex roots $\bar{\psi}_{k,\ell}$, all given by (58); $\bar{\psi}_{k,\ell}$ just become real on the boundary with region III. In this $\bar{B},\bar{C}$ domain the function $\bar{\psi}$ therefore oscillates between $\bar{\psi}_i$ and $\bar{\psi}_j$. For this single mode the solution of (43) reads as follows in terms of the Jacobian elliptic function cn(u,k) (see Appendix):

$$\bar{\psi}(\bar{s}) = \frac{\bar{\psi}_i R_j + \bar{\psi}_j R_i - (\bar{\psi}_i R_j - \bar{\psi}_j R_i)\,\mathrm{cn}\{\tfrac{1}{2}\sqrt{R_i R_j}(\bar{s}-\bar{s}_j),k\}}{R_i + R_j + (R_i - R_j)\,\mathrm{cn}\{\tfrac{1}{2}\sqrt{R_i R_j}(\bar{s}-\bar{s}_j),k\}} \;, \quad (\bar{\psi}_j \le \bar{\psi} \le \bar{\psi}_i) \qquad (71)$$

with

$$R^2_{i,j} = (\bar{\psi}_{i,j} - \bar{\psi}_k)\,(\bar{\psi}_{i,j} - \bar{\psi}_\ell) \quad , \quad (i \neq j \neq k \neq \ell) \qquad (72)$$

and the modulus k given by

$$k^2 = \frac{1}{2} - \frac{(\bar\psi_i - \bar\psi_k)(\bar\psi_j - \bar\psi_\ell) + (\bar\psi_i - \bar\psi_\ell)(\bar\psi_j - \bar\psi_k)}{4R_iR_j}$$

$$= \frac{(\bar\psi_i - \bar\psi_j)^2 - (R_i - R_j)^2}{4R_iR_j} \quad . \tag{73}$$

The constant $\bar{s}_j$ has been chosen such that $\bar\psi = \bar\psi_j$ holds at $\bar{s} = \bar{s}_j$. Inside region II the value of k^2 is situated in the range of $0 < k^2 < 1$, while $k^2 = 1$ on the lower branch of the boundary with region III, and $k^2 = 0$ on the upper branch of the boundary with region III as well as on the boundary with region I.

Inside and on the boundaries of region II both quantities $R^2_{i,j}$ are positive, except at the point $\bar{B} = \bar{C} = 1$ where $R_i^2 = 0$, $\bar\psi_i$ being there a triple root of P_4. On the boundary between regions I and II, where $\bar\psi_i = \bar\psi_j$, R_i and R_j are equal.

The oscillation period $\bar{s}_{osc}$ of the periodic function $\bar\psi$ may be found from the property that the period of the Jacobian elliptic function is $4K(k)$, K being the complete elliptic integral of the first kind, whose modulus is again given by (73). In view of the argument of the above elliptic function we thus find

$$\bar{s}_{osc} = \frac{8K(k)}{\sqrt{R_iR_j}} \quad . \tag{74}$$

In view of (33) and (34) we obtain the oscillation proper time s_{osc} by dividing (74) by G.

Approaching the lower branch of the boundary with region III, the modulus k tends to unity. Since $K(1) = \infty$, the oscillation period $\bar{s}_{osc}$ then tends to infinity.

The oscillation period in ordinary time t_{osc} and the oscillation length $Z_{osc} = \frac{\omega}{c} z_{osc}$ can be expressed in terms of s_{osc}. For that purpose we integrate (71) over one period of oscillation. Combining the result with (70) and (74) we get (see Appendix):

$$|N^2-1|\left[\omega t_{osc} - \gamma_o s_{osc}\right] = \frac{|N^2-1|}{N}\left[Z_{osc} - P_{zo}s_{osc}\right] =$$

$$= \left[\bar\psi_o - \frac{\bar\psi_i R_j + \bar\psi_j R_i}{R_i + R_j}\right]\bar{s}_{osc} - 4\pi \operatorname{sgn}(R_i - R_j)\left[1 - \Lambda_o(\beta,k)\right] \quad , \tag{75a}$$

$\Lambda_o(\beta,k)$ being Heuman's lambda function[5] (see Appendix, Eq. (22)), whose argument is given by

$$\sin\beta = \frac{2\sqrt{R_iR_j}}{R_i + R_j} \quad . \tag{75b}$$

The initial values γ_o and P_{zo} may not be chosen arbitrarily, but are fixed by the choice of the initial point ($\bar{P}_{io}$, $\bar{P}_{ro}$, $\bar{\psi}_o$) on the particle orbit. They may equally well be considered as functions of $\bar{B},\bar{C}$ and $\bar{\psi}_o$.

Figure 9 shows the Heuman function plotted as a function of its argument β and of the modulus k.

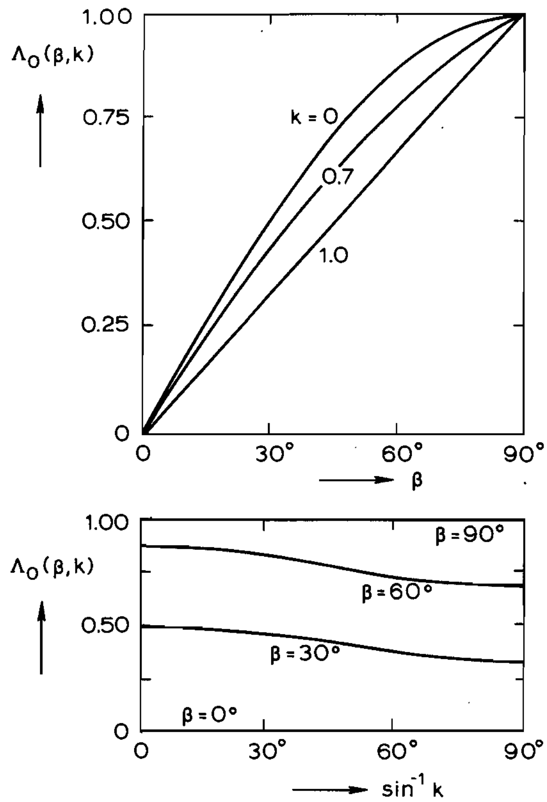


Fig. 9 $\Lambda_o(\beta,k)$ plotted as a function of its argument β and of the modulus k.

II.4.3 The solution in the two-mode domain

This domain was characterized by P_4 having four real roots. This situation occurs when the constants of motion $\bar{B},\bar{C}$ are such that the corresponding point situates inside or on the boundary of region III of Fig. 5. The function $\bar{\psi}$ then oscillates either between the two smallest, or between the two largest roots, that is, e.g. between $\bar{\psi}_i$ and $\bar{\psi}_j$. The solution of (43) now reads as follows in terms of the Jacobian elliptic function $cd(u,k_1)$ (see Appendix):

$$\bar{\psi}(\bar{s}) = \frac{\bar{\psi}_i R_j + \bar{\psi}_j R_i + (\bar{\psi}_j R_i - \bar{\psi}_i R_j)\,\mathrm{cd}\left[\tfrac{1}{2}\{R_i R_j(1-k^2)\}^{\frac{1}{2}}(\bar{s}-\bar{s}_j),k_1\right]}{R_i + R_j + (R_i - R_j)\,\mathrm{cd}\left[\tfrac{1}{2}\{R_i R_j(1-k^2)\}^{\frac{1}{2}}(\bar{s}-\bar{s}_j),k_1\right]} \quad ,$$

$$(\bar{\psi}_j \le \bar{\psi} \le \bar{\psi}_i) \quad , \qquad (76)$$

with $R^2_{i,j}$ and k^2 once again given by (72) and (73), respectively, and the modulus k_1 of the Jacobian elliptic function $\mathrm{cd}(u,k_1)$ by:

$$k_1^2 = \frac{-k^2}{1-k^2} = \frac{(\bar{\psi}_i - \bar{\psi}_j)^2 - (R_i - R_j)^2}{(\bar{\psi}_i - \bar{\psi}_j)^2 - (R_i + R_j)^2} \quad . \qquad (77)$$

Inside region III the value of k_1^2 lies in the range of $0 < k_1^2 < 1$, while $k_1^2 = 0$ on the upper branch and $k_1^2 = 1$ on the lower branch of the boundary with region II. The value of k_1^2 at the point $\bar{B} = \bar{C} = 1$ depends on the path along which this point is approached.

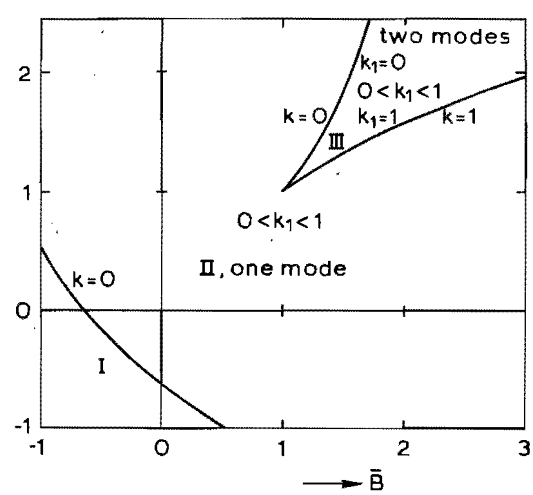


Fig. 10 The ranges of k and k_1 in the $(\bar{B},\bar{C})$-plane. The curves represent the algebraic equation $D = 0$.

Inside this region both quantities $R^2_{i,j}$ are positive, since all roots of P_4 are real and different there, and $\bar{\psi}_{i,j}$ are both larger or both smaller than $\bar{\psi}_{k,\ell}$. On the upper branch of the boundary which separates regions II and III, $R_i^2 \neq R_j^2$ or $R_i^2 = R_j^2$ holds according to $\bar{\psi}_{i,j} < \bar{\psi}_k = \bar{\psi}_\ell$, or $\bar{\psi}_{k,\ell} < \bar{\psi}_i = \bar{\psi}_j$, respectively. On the lower branch of this boundary either R_i^2 or R_j^2 vanishes according to $\bar{\psi}_j < \bar{\psi}_i = \bar{\psi}_k < \bar{\psi}_\ell$, or $\bar{\psi}_k < \bar{\psi}_\ell = \bar{\psi}_j < \bar{\psi}_i$, holds respectively. At the point $\bar{B} = \bar{C} = 1$ either of the relations $\bar{\psi}_j < \bar{\psi}_i = \bar{\psi}_k = \bar{\psi}_\ell$ or $\bar{\psi}_k < \bar{\psi}_\ell = \bar{\psi}_j = \bar{\psi}_i$ is valid, so that at this point $R_i^2 = 0$, $R_j^2 \neq 0$ or $R_i^2 = R_j^2 = 0$, respectively.

The oscillation period of the Jacobian elliptic function $cd(u_1,k_1)$ is $4K(k_1)$, hence, in view of the argument of the function cd in (76),

$$\bar{s}_{osc} = \frac{8K(k_1)}{[R_iR_j(1-k^2)]^{\frac{1}{2}}} = \frac{16K(k_1)}{[(R_i+R_j)^2 - (\bar{\psi}_i - \bar{\psi}_j)^2]^{\frac{1}{2}}} \quad . \tag{78}$$

It follows from (72) that $R_iR_j = R_kR_\ell$. Equation (73) then shows that the values of k^2, thus also the values of k_1^2, are equal for both oscillation modes. Hence, both oscillation modes have the same period $\bar{s}_{osc}$ in proper time.

When approaching the lower branch of the boundary with region II, the modulus k_1 will tend to unity. Since $K(1) = \infty$, the oscillation periods then tend to infinity.

An integration of (76) over one oscillation period, and a combination of the result with (70) and (78) again leads to relations between the oscillation time t_{osc}, the oscillation length Z_{osc} and s_{osc}, viz. (see Appendix):

$$|N^2-1|\left[\omega t_{osc} - \gamma_o s_{osc}\right] = \frac{|N^2-1|}{N}\left[Z_{osc} - P_{zo}s_{osc}\right] =$$

$$= \left[\bar{\psi}_o - \frac{\bar{\psi}_iR_j + \bar{\psi}_jR_i}{R_i+R_j}\right]\bar{s}_{osc} - 4\pi\,\mathrm{sgn}(R_i - R_j)\left[1-\Lambda_o(\beta_1,k_1)\right]; \tag{79a}$$

here the argument of Heuman's lambda function $\Lambda_o(\beta_1,k_1)$ is given by

$$\sin\beta_1 = \frac{2\left[R_iR_j(1-k^2)\right]^{\frac{1}{2}}}{R_i+R_j} \quad . \tag{79b}$$

Again the initial values γ_o and P_{zo} may not be chosen arbitrarily, but are fixed by the choice of the initial position ($\bar{P}_{io}$, $\bar{P}_{ro}$, $\bar{\psi}_o$) on the particle orbit. They may equally well be considered as functions of $\bar{B},\bar{C}$ and $\bar{\psi}_o$.

II.5 Solutions for special values of the constants $\bar{B}$ and $\bar{C}$

In the next subsections we shall calculate the roots of P_4, the oscillation times s_{osc}, t_{osc} and the oscillation length $Z_{osc} = \frac{\omega}{c}\, z_{osc}$ for various values of the constants $\bar{B}$ and $\bar{C}$.

For that purpose we want to express the quantities $R_{i,j}^2$, the modulus k^2 and the coefficients of (74), (75), (78) and (79) in terms of $\bar{\psi}_i$ and $\bar{\psi}_j$. With the aid of (52) and (53) we obtain from (72) and (73) the following expressions, which are applicable in both the single-mode and the two-mode domains:

$$R^2_{i,j} = (\bar{\psi}_i + \bar{\psi}_j)^2 - \frac{8}{\bar{\psi}_i + \bar{\psi}_j} \pm (\bar{\psi}^2_i - \bar{\psi}^2_j) \quad , \tag{80}$$

$$R^2_i R^2_j = 3\left[(\bar{\psi}_i + \bar{\psi}_j)^2 - 4\bar{C}\right]^2 - 48(\bar{C}^2 - \bar{B}) \quad , \tag{81}$$

$$\frac{\bar{\psi}_i R_j + \bar{\psi}_j R_i}{R_i + R_j} = \frac{R_i R_j}{2(\bar{\psi}_i + \bar{\psi}_j)} + \frac{4}{(\bar{\psi}_i + \bar{\psi}_j)^2} \quad , \tag{82}$$

$$k^2 = \frac{1}{2} - \frac{3}{4}\,\frac{(\bar{\psi}_i + \bar{\psi}_j)^2 - 4\bar{C}}{R_i R_j} \quad . \tag{83}$$

If the point $(\bar{B},\bar{C})$ is situated on the boundary between the regions I and II, or on the upper branch of the boundary between II and III, one oscillation mode will exist for which $\bar{\psi}_i = \bar{\psi}_j = \bar{\psi}_o$. For this mode we obtain from (80):

$$R^2_i = R^2_j = 4\,\bar{\psi}^2_o\left[1 - \frac{1}{\bar{\psi}^3_o}\right] \quad .$$

Since $k = k_1 = 0$ on the specified boundaries, we obtain the following expression for the oscillation period $\left(K(o) = \frac{\pi}{2}\right)$, from (74) or (78),

$$\bar{s}_{osc} = \frac{2\pi}{|\bar{\psi}_o|\,\left[1 - \frac{1}{\bar{\psi}^3_o}\right]^{\frac{1}{2}}} \quad . \tag{84}$$

It follows from (75b) and (79b) that $\beta = \beta_1 = \frac{\pi}{2}$. Since $\Lambda_o(\frac{\pi}{2}, o) = 1$, we then find from (75a) or (79a) the relations $\omega t_{osc} = \gamma_o s_{osc}$, $z_{osc} = \frac{\omega}{c} z_{osc} = P_{zo} s_{osc}$. These results are trivial, since, for the mode under consideration, no energy oscillation will occur.

In subsection II.3.2 we have shown that if $\bar{P}_{io} = 0$ and $\bar{P}_{ro}\bar{\psi}_o = 1$, the polynomial P_4 has a double root. If, in addition, $\bar{\psi}_o < 0$ then the corresponding point $(\bar{B},\bar{C})$ is situated on the boundary between the regions I and II, while this point lies on the upper branch of the boundary between the regions II and III if $\bar{\psi}_o > 1$. For $0 < \bar{\psi}_o < 1$, the point $(\bar{B},\bar{C})$ lies on the lower branch of the boundary between the regions II and III. In that case (84) does not hold.

II.5.1 The situation of small values of the integrals of motion

We first consider a particle whose initial conditions are such that the integrals of motion $\bar{B}$ and $\bar{C}$ satisfy the inequalities

$$|\bar{B}| < 1 \quad , \quad |\bar{C}| < 1 \quad . \tag{85}$$

The corresponding point then lies in region II of the $(\bar{B},\bar{C})$-plane (Fig. 5). For this range of values of $\bar{B}$ and $\bar{C}$ the polynomial P_4 (Q_4, respectively) has only two real roots. It follows from (35) and (36) that the relations (85) are certainly satisfied if

$$|\bar{P}_o| < 1 \quad \text{and} \quad |\bar{\psi}_o| < 1 \quad . \tag{86a}$$

This means that the initial transverse momentum is small and the particle starts close to exact resonance. In view of (37) these relations read as follows in the original, non-normalized quantities:

$$|P_o|^2 < \left[\frac{\Omega^2 g}{N^2-1}\right]^{2/3} \quad , \quad |\psi_o| < |(N^2-1)\Omega g^2|^{1/3} \quad . \tag{86b}$$

In most experimental situations $g << 1$, and $|N^2-1| = O(1)$, so that the first inequality holds for relatively cold particles. If $N^2-1 \to 0$ the second inequality can only be satisfied for values of ψ_o in a small interval around exact resonance $\psi_o = 0$. With the aid of (85) and (56) we obtain the following approximate expression for the positive root of (53), which is explicit up to the second order in $\bar{B}$ and $\bar{C}$,

$$a^2 = 4\left[1 - \bar{B} + \bar{C} + \bar{C}^2 - \bar{B}\bar{C}\right] + O(\bar{B}^3, \bar{C}^3, \bar{B}^2\bar{C}) \quad . \tag{87}$$

Substituting this expression into (58) we find the following approximations, correct up to second order in $\bar{B}$ and $\bar{C}$, for the two real roots of P_4

$$\bar{\psi}_1 = \frac{3}{2}\bar{B} - \frac{9}{8}\bar{C}^2$$

$$\bar{\psi}_2 = -2\left[1 + \frac{1}{4}\bar{B} + \frac{1}{2}\bar{C} - \frac{3}{16}\bar{C}^2 - \frac{1}{8}\bar{B}^2 - \frac{1}{4}\bar{B}\bar{C}\right] \quad . \tag{88}$$

The root $\bar{\psi}_2$ is always negative for the assumed small values of $\bar{B}$ and $\bar{C}$, while the sign of $\bar{\psi}_1$ depends on $\mathrm{sgn}(4\bar{B} - 3\bar{C}^2)$. Both roots are negative if $4\bar{B} - 3\bar{C}^2 < 0$. Then the particle is free since the resonance function ψ either remains below or above exact resonance ($\bar{\psi} = 0$). When $4\bar{B} - 3\bar{C}^2 > 0$ the root $\bar{\psi}_1$ will be positive and the resonance function oscillates around $\bar{\psi} = 0$; hence, the particle is then trapped.

The roots (88) are the maximum and minimum values of $\bar{\psi}$, respectively, attained by the particle during its motion. In view of (42) $\bar{P}_i$ vanishes at these extrema of $\bar{\psi}$, while the corresponding values of $\bar{P}_r$ are the real roots of Q_4. According to the symmetry in the definitions (41) and (42) of Q_4 and P_4, the latter roots can be found by interchanging $\bar{B}$ and $\bar{C}$ in (88), giving:

$$\bar{P}_{1r} = \frac{3}{2}\,\bar{C} - \frac{9}{8}\,\bar{B}^2 \quad ,$$

$$\bar{P}_{2r} = -2\left[1 + \frac{1}{4}\,\bar{C} + \frac{1}{2}\,\bar{B} - \frac{3}{16}\,\bar{B}^2 - \frac{1}{8}\,\bar{C}^2 - \frac{1}{4}\,\bar{B}\bar{C}\right] \quad . \tag{89}$$

The second root is negative, while the sign of $\bar{P}_{1r}$ depends on $\mathrm{sgn}(4\bar{C} - 3\bar{B}^2)$. The particle orbit in the $\bar{P}$-plane encircles the origin if $\bar{P}_{1r} > 0$, that is if $4\bar{C} - 3\bar{B}^2 > 0$, and it does not if $\bar{P}_{1r} < 0$, and consequently $4\bar{C} - 3\bar{B}^2 < 0$.

The relationship between $\bar{\psi}_{i,j}$ and the corresponding extreme values of the resonance function ψ, the normalized kinetic energy γ, the axial momentum P_z, the generalized angular momentum $|P|^2/2\Omega$ and the radial position in the R-plane are contained in (34), (I.53), (31) and (I.39) from which formulae we deduce

$$\gamma_{i,j} - \gamma_o = \frac{P_{zi,j} - P_{zo}}{N} = \frac{|P|^2_{i,j} - |P_o|^2}{2\Omega} = \frac{\Omega}{2}\left[|R|^2_{i,j} - |R_o|^2\right] =$$

$$= \frac{\psi_{i,j} - \psi_o}{1-N^2} = \frac{G}{|N^2-1|}\left[-\bar{\psi}_{i,j} + \bar{\psi}_o\right] \quad . \tag{90}$$

With the aid of (35) and (36), $\bar{B},\bar{C}$ can be expressed in terms of the initial values $\bar{P}_o$ and $\bar{\psi}_o$, we then obtain the roots (88) in terms of $\bar{P}_o$ and $\bar{\psi}_o$. The extreme values of the normalized kinetic energy can be derived up to second order in the initial values $\bar{P}_o$ and $\bar{\psi}_o$ from (88), (90), (35) and (36) yielding:

$$\gamma_{min} = \gamma_o - \frac{1}{2}\left[\frac{\Omega g^2}{(N^2-1)^2}\right]^{1/3} \bar{P}^2_{io} = \gamma_o - \frac{P^2_{io}}{2\Omega} \quad ,$$

$$\gamma_{max} = \gamma_o + 2\left[\frac{\Omega g^2}{(N^2-1)^2}\right]^{1/3} \left\{1 + \frac{1}{3}\,\bar{P}_{ro} + \frac{2}{3}\,\bar{\psi}_o + \frac{1}{12}\,\bar{P}^2_{io} - \frac{1}{9}\,\bar{P}_{ro}\bar{\psi}_o\right\} \quad . \tag{91}$$

Remembering that, in view of (I.30) for $F_z = \partial g/\partial t = \partial\Omega/\partial t = 0$, $d\gamma/ds = gP_i$, we conclude that, when $P_{io} < 0$, the energy first decreases until $\gamma = \gamma_{min}$, and next raises to its maximum value $\gamma = \gamma_{max}$. On the contrary, the kinetic energy initially increases when $P_{io} > 0$.

The maximum value γ_{max} tends to infinity when $N^2 \to 1$. This is in accordance with the results of subsection II.2, since in this limit the above approximation is only valid if the particle starts at exact resonance $\psi_o = 0$. Not only the kinetic energy but all maximum quantities mentioned in (90) tend to infinity for $N^2 \to 1$.

The oscillation times s_{osc} and t_{osc}, and the oscillation length $Z_{osc} = \frac{\omega}{c} z_{osc}$ can be calculated from (74) and (75). For simplicity we only consider the case $\bar{P}_o$, $\bar{\psi}_o \to 0$, i.e. $\bar{B},\bar{C} \to 0$. From (87) we then obtain $\bar{\psi}_i = 0$ and $\bar{\psi}_j = -2$. A substitution of these values into (80), (82) and (83) results in:

$$R_i^2 = 4 \;,\; R_j^2 = 12 \;;\; \frac{\bar{\psi}_i R_j + \bar{\psi}_j R_i}{R_i + R_j} = 1 - 3^{\frac{1}{2}} = -0.732 \quad ,$$

$$k^2 = \frac{1}{2} - \frac{1}{4}\sqrt{3} = 0.0670 \quad .$$

From (75b) we next obtain

$$\sin\beta = 3^{\frac{1}{4}}\left[3^{\frac{1}{2}} - 1\right] = 0.9634 \quad .$$

For the above values of k and β, $K(k) = 1.598$ and $\Lambda_o(\beta,k) = 0.95$. Finally, we get from (74) and (75)

$$s_{osc} = 4.86\ G^{-1} = 4.86\left[|N^2-1|\Omega g^2\right]^{-1/3} \quad , \tag{92a}$$

$$\omega t_{osc} = 4.86\ \gamma_o\left[|N^2-1|\Omega g^2\right]^{-1/3} + \frac{3.56 + 0.2\pi}{|N^2-1|} \tag{92b}$$

$$\frac{\omega}{c} z_{osc} = 4.86\ P_{zo}\left[|N^2-1|\Omega g^2\right]^{-1/3} + (3.56 + 0.2\pi)\ \frac{N}{|N^2-1|} \quad . \tag{92c}$$

These expressions have been derived for $\bar{\psi}_o = 0$ and $\bar{P}_o = 0$. They remain approximately valid if $\bar{P}_o$ and $\bar{\psi}_o$ satisfy the inequalities (86). The results (92) agree with those obtained by SCHRAM[2]. Neglecting the difference between ωt_{osc} and $\gamma_o s_{osc}$, and the one between $\frac{\omega}{c} z_{osc}$ and $P_{zo} s_{osc}$, (92) also agrees with the results derived by ROBERTS and BUCHSBAUM[1]. These differences are described by the second term on the right-hand side of (92b) and (92c), respectively. These terms become important under the following conditions

$$\gamma_o^3\ \frac{|N^2-1|^2}{\Omega g^2} \lesssim 1 \quad ; \quad \left(\frac{P_{zo}}{N}\right)^3\ \frac{|N^2-1|^2}{\Omega g^2} \lesssim 1 \quad . \tag{93}$$

Since in general $g \ll 1$, these are very strong conditions on the index of refraction. The total change in kinetic energy, and the one in axial momentum then become large.

II.5.2 The case of a particle initially far from exact resonance

Next, we consider the situation in which the integrals of motion satisfy the inequalities

$$27\bar{C}^3 >> 1 \quad , \quad \frac{4|\bar{B}|}{3\bar{C}^2} << 1 \quad . \tag{94}$$

The corresponding point of the $(\bar{B},\bar{C})$-plane (Fig. 8) then lies in the upper half plane outside the two parabolas $4|\bar{B}| = 3\bar{C}^2$. We then only have to do with free-particle orbits for which $\bar{C}$ is positive and sufficiently large. A particle which starts with initial conditions such that (94) is satisfied can never reach exact resonance. It follows from (35) and (36) that the relations (94) are in any case satisfied if, simultaneously,

$$|\bar{\psi}_o| >> 2 \quad ; \quad \bar{\psi}_o^2 >> 2|\bar{P}_o| \quad ; \tag{95a}$$

this involves that the initial distance ψ_o to exact resonance should be large, since in view of (37) and (36),

$$|\psi_o| > 2|(N^2-1)\Omega g^2|^{1/3} \quad , \quad \psi_o^2 > 2|(N^2-1)gP_o| \quad . \tag{95b}$$

These inequalities can also be satisfied in the limit $N^2 \to 1$ and for vanishing wave amplitude $(g \to 0)$.

If (94) holds, we obtain the following positive solution of (53):

$$a^2 = 12\bar{C}\left[1 - \frac{\bar{B}}{3\bar{C}^2} - \frac{\bar{B}^2}{9\bar{C}^4} - \frac{2}{27}\frac{\bar{B}^3}{\bar{C}^6} + \frac{1}{27\bar{C}^3} + \frac{1}{27}\frac{\bar{B}}{\bar{C}^5} + O\left(\frac{\bar{B}^4}{\bar{C}^8}, \frac{\bar{B}^2}{\bar{C}^7}, \frac{1}{\bar{C}^6}\right)\right] \quad . \tag{96a}$$

With the aid of (35), (36) and (95a) we can express (96a) in terms of the initial values $\bar{P}_o$ and $\bar{\psi}_o$. We then find the following expression which is correct up to third order in the small quantities $|\bar{P}_o|/|\bar{\psi}_o|^2$ and $\bar{\psi}_o^{-3}$,

$$a^2 = 4\bar{\psi}_o^2\left[1 + \frac{2\bar{P}_{ro}}{\bar{\psi}_o^2} - \frac{2}{\bar{\psi}_o^3}\left\{1 - \frac{2\bar{P}_{ro}}{\bar{\psi}_o^2} + \frac{4\bar{P}_{ro}^2}{\bar{\psi}_o^4}\right\} - \right.$$
$$\left. - \frac{|\bar{P}_o|^2}{\bar{\psi}_o^4}\left\{1 - \frac{2\bar{P}_{ro}}{\bar{\psi}_o^2} + \frac{4}{\bar{\psi}_o^3}\right\} - \frac{3}{\bar{\psi}_o^6}\left\{1 - \frac{20}{3}\frac{\bar{P}_{ro}}{\bar{\psi}_o^2} + \frac{10}{3}\frac{1}{\bar{\psi}_o^3}\right\}\right] \quad . \tag{96b}$$

Substituting (96b) into (58), we find that the function $\bar{\psi}$ oscillates between the following two real roots of P_4 that are situated above and below the initial value $\bar{\psi}_o$, viz.:

$$\bar{\psi}_{max,min} - \bar{\psi}_o = \left[- \frac{1 - \bar{\psi}_o \bar{P}_{ro}}{\bar{\psi}_o^2} \left\{1 + \frac{2(1 - \frac{1}{2} \bar{P}_{ro}\bar{\psi}_o)}{\bar{\psi}_o^3}\right\} - \frac{\bar{P}_{io}^2}{2\bar{\psi}_o^3}\right] \pm$$

$$\pm \frac{1}{\bar{\psi}_o^2} \left[(1 - \bar{\psi}_o \bar{P}_{ro})^2 \left\{1 + \frac{4(1 - \frac{1}{2} \bar{P}_{ro}\bar{\psi}_o)}{\bar{\psi}_o^3}\right\} + \right.$$

$$\left. + \bar{P}_{io}^2 \bar{\psi}_o^2 \left\{1 + \frac{3}{\bar{\psi}_o^3} - \frac{2\bar{P}_{ro}}{\bar{\psi}_o^2}\right\}\right]^{\frac{1}{2}} , \tag{97}$$

$\bar{\psi}_o$ and $\bar{P}_o$ have to satisfy (95). The total relative change $(\bar{\psi}_{max} - \bar{\psi}_{min})/\bar{\psi}_o$ thus proves to be of first order in the small quantities $|\bar{P}_o|/\bar{\psi}_o^2$ and $1/\bar{\psi}_o^3$. Equation (97) represents a double root of P_4 when, simultaneously, $\bar{P}_{ro}\bar{\psi}_o = 1$ and $\bar{P}_{io} = 0$, that is when $P_{ro}\psi_o = \Omega g$ and $P_{io} = 0$. This is in agreement with (6a); no energy oscillation whatever then occurs.

The remaining two roots of P_4 are given by

$$- \bar{\psi}_o \left[1 - \frac{1 - \bar{\psi}_o \bar{P}_{ro}}{\bar{\psi}_o^3}\right] \pm 2 \left[\frac{1}{\bar{\psi}_o} \left[1 + \frac{|\bar{P}_o^2|}{4\bar{\psi}_o}\right]\right]^{\frac{1}{2}} .$$

These further roots are real if $\bar{\psi}_o$ is sufficiently large, that is if

$$\bar{\psi}_o \geq - \frac{1}{4} |\bar{P}_o|^2 , \tag{98a}$$

otherwise they are complex. This means that if in addition to (95b), the initial values P_o and ψ_o satisfy the inequality

$$\left[\text{sgn}(N^2-1)\right] \psi_o \geq - \frac{1}{4}|N^2-1| \frac{|P_o|^2}{\Omega} , \tag{98b}$$

the polynomial P_4 has four real roots.

When we pass from region II to region III of the $(\bar{B},\bar{C})$-plane by crossing the upper branch of the boundary, all four roots of P_4 become real. In this case the roots (97), between which $\bar{\psi}$ oscillates, are the two largest or the two smallest roots of P_4. This depends on the sign of the initial value $\bar{\psi}_o$.

The relationship between the roots (97) and the corresponding extreme values of the resonance function ψ, the normalized kinetic energy γ, the axial momentum P_z, the generalized angular momentum $\frac{|P|^2}{2\Omega}$ and the radial position in the R-plane is again given by (90). By combining (97) and (90), remembering (33) and (34), we obtain, e.g., for the extreme values of γ the following expressions in terms of the initial transverse momentum P_o and the initial distance ψ_o to exact resonance:

$$\gamma_{max,min} - \gamma_o = \frac{g}{\psi_o^2}\left[(\Omega g-P_{ro}\psi_o)\left\{1+g(N^2-1)\,\frac{2\Omega g-P_{ro}\psi_o}{\psi_o^3}\right\} + \frac{1}{2}\,g(N^2-1)\,\frac{P_{io}^2}{\psi_o}\right] \pm$$

$$\pm \frac{g}{\psi_o^2}\left[(\Omega g-P_{ro}\psi_o)^2\left\{1+2(N^2-1)g\,\frac{2\Omega g-P_{ro}\psi_o}{\psi_o^3}\right\} +\right.$$

$$\left.+ P_{io}^2\psi_o^2\left\{1+g(N^2-1)\,\frac{3\Omega g-2P_{ro}\psi_o}{\psi_o^3}\right\}\right]^{\frac{1}{2}} \quad . \tag{99}$$

In view of (95), the terms containing $g(N^2-1)$ are small. Neglecting them we obtain the result that can also be derived from (14).

The roots (97) can be used to calculate the oscillation times and the oscillation length. Neglecting small terms we then reobtain the expressions given in section II.2. Hence, the results that are exact for $N^2 = 1$ remain approximately valid if the particle starts far enough from exact resonance.

II.5.3 The case of a large initial transverse momentum

Finally, we consider the case in which the integrals of motion satisfy the relations

$$27\bar{B}^3 \gg 1 \quad , \quad \frac{4|\bar{C}|}{3\bar{B}^2} \ll 1 \quad . \tag{100}$$

The corresponding point in the $(\bar{B},\bar{C})$-plane then lies in the right half plane of Fig. 8 outside the two parabolas $\pm\, 4\bar{C} = 3\bar{B}^2$. Thus (100) includes all trapped-particle orbits ($3\bar{C}^2 \leq 4\bar{B}$) for which $\bar{B}$ is sufficiently large. It follows from (35) and (36) that the relations (100) are in any case satisfied if, simultaneously,

$$\frac{1}{2}\,|\bar{P}_o|^2 \gg |\bar{\psi}_o| \quad , \quad |\bar{P}_o| \gg 2 \quad ; \tag{101a}$$

this means that the initial transverse momentum should be large according to the equivalent relations:

$$|N^2-1|\,\frac{|P_o|^2}{2\Omega} \gg |\psi_o| \quad ; \quad |P_o| \gg 2\left|\frac{\Omega^2 g}{N^2-1}\right|^{1/3} \quad . \tag{101b}$$

These inequalities can only be satisfied for $N^2 \neq 1$.

For $\bar{B}$ and $\bar{C}$ satisfying (100) we obtain the following approximate positive solution directly starting from (53),

$$a^2 \simeq \frac{4}{3\bar{B}}\left[1 + \frac{\bar{C}}{3\bar{B}^2} + \frac{2\bar{C}^2}{9\bar{B}^4} - \frac{1}{27\bar{B}^3} - \frac{5\bar{C}}{81\bar{B}^5} + O\left(\frac{\bar{C}^3}{\bar{B}^6}, \frac{\bar{C}^2}{\bar{B}^7}, \frac{1}{\bar{B}^6}\right)\right] \quad . \tag{102}$$

A substitution of (102) into (58) will give the four roots of the polynomial P_4 in terms of the integrals of motion.

We shall restrict ourselves to trapped-particle orbits. We may then suppose that the particle starts at exact resonance ($\psi_o = 0$). A combination of (35) and (36) for $\bar{\psi}_o = 0$ with (102) then leads to:

$$a = \frac{2}{|\bar{P}_o|}\left[1 + \frac{\bar{P}_{ro}}{|\bar{P}_o|^4} + O(\bar{P}_o^{-6})\right] \quad . \tag{103}$$

By substituting this expression into (58), we obtain the following four roots of the polynomial P_4:

$$-\frac{1}{|\bar{P}_o|}\left[1 + \frac{\bar{P}_{ro}}{|\bar{P}_o|^4}\right] \pm \left[2|\bar{P}_o| + 2\bar{P}_{ro} - \frac{|\bar{P}_o| + 2\bar{P}_{ro}}{|\bar{P}_o|^3}\right]^{\frac{1}{2}} \quad , \tag{104a}$$

$$\frac{1}{|\bar{P}_o|}\left[1 + \frac{\bar{P}_{ro}}{|\bar{P}_o|^4}\right] \pm \left[-2|\bar{P}_o| + 2\bar{P}_{ro} - \frac{|\bar{P}_o| - 2\bar{P}_{ro}}{|\bar{P}_o|^3}\right]^{\frac{1}{2}} \quad , \tag{104b}$$

with $\bar{P}_o$ satisfying (101).

The roots (104a) are always real, while the roots (104b) are complex except when the initial phase $\theta_o = \arg \bar{P}_o$ lies in a small interval around $\theta_o = 0$, viz.

$$\cos \theta_o > 1 - \tfrac{1}{2}\,|\bar{P}_o|^{-3} \quad . \tag{105}$$

For θ_o outside this interval the function $\bar{\psi}(\bar{s})$ oscillates between the real roots $\bar{\psi}_{i,j} = \bar{\psi}_{max,min}$, given by (104a), which lie around the initial value $\bar{\psi}_o = 0$.

When $\cos \theta_o \to 1$, the point $(\bar{B},\bar{C})$ passes from region II to III by crossing the lower branch of the boundary, the two new real roots (104b) then being situated between the real roots (104a). In region III the particle will oscillate between the smallest of the roots (104a) and the smallest of the roots (104b), since there these roots are situated around the initial value $\bar{\psi}_o = 0$. Hence, by crossing the lower branch of the boundary, the oscillation "amplitude" $\bar{\psi}_{max} - \bar{\psi}_{min}$ decreases, and the same holds for the total change in kinetic energy, in axial momentum and in angular momentum.

The relations between the extreme values (104a) of the function $\bar{\psi}$ and the corresponding values of the kinetic energy, the axial momentum, etc. are again given by (90) taking account of (104a), (35) and (36). We obtain e.g., the following maximum and minimum values of the normalized kinetic energy

$$\gamma_{max,min} - \gamma_o = \frac{\Omega g}{|N^2-1|}\frac{1}{|P_o|}\left[1 + \frac{\Omega^2 g}{|N^2-1|}\frac{P_{ro}\ \mathrm{sgn}(N^2-1)}{|P_o|^4}\right] \pm$$

$$\pm\left[\frac{g}{|N^2-1|}\right]^{\frac{1}{2}}\left[2|P_o|+2P_{ro}\ \mathrm{sgn}(N^2-1) - \frac{\Omega^2 g}{|N^2-1|}\frac{|P_o|+2P_{ro}\ \mathrm{sgn}(N^2-1)}{|P_o|^3}\right]^{\frac{1}{2}} . \tag{106}$$

With the aid of the roots (104a) we are able to calculate the oscillation times and the oscillation length. We only consider the case for which the roots (104b) are complex. A substitution of (104a) into (80) - (83), neglecting corrections of order $|\bar{P}_o|^{-3}$, results in

$$R_{i,j}^2 = 4|\bar{P}_o| \mp \frac{4(2|\bar{P}_o| + 2\bar{P}_{ro})^{\frac{1}{2}}}{|\bar{P}_o|} , \tag{107}$$

$$R_i^2 R_j^2 = 16|\bar{P}_o|^2 \quad ; \quad \frac{\bar{\Psi}_i R_j + \bar{\Psi}_j R_i}{R_i + R_j} = \frac{\bar{P}_{ro}}{|\bar{P}_o|^2} , \tag{108}$$

$$k^2 = \frac{1}{2} + \frac{1}{2}\frac{\bar{P}_{ro}}{|\bar{P}_o|} . \tag{109}$$

The argument β of Heuman's lambda functions follows from (75b), (107) and (108), viz.

$$\sin\beta = 1 \quad , \quad \beta = \frac{\pi}{2} . \tag{110}$$

Since $\Lambda_o(\pi/2,k) = 1$, we obtain, from a combination of (107) - (110) with (74) and (75), the following expressions for the oscillation times and the oscillation length

$$s_{osc} = \bar{s}_{osc}\, G^{-1} = \frac{4K(k)}{|\bar{P}_o|^{\frac{1}{2}} G} = \frac{4K(k)}{\left\{g|N^2-1|\ |P_o|\right\}^{\frac{1}{2}}} , \tag{111}$$

$$\omega t_{osc} - \gamma_o s_{osc} = \frac{1}{N}\left[z_{osc} - P_{zo} s_{osc}\right]$$

$$= -\frac{\bar{P}_{ro}}{|\bar{P}_o|^2\ |N^2-1|}\, G\, s_{osc} = -\frac{\Omega g\, P_{ro}}{(N^2-1)\ |P_o|^2}\, s_{osc} , \tag{112}$$

with k given by (109).
Since we have assumed that $|\bar{P}_o| > 2$, and because in general $G << 1$, it follows from (112) that in most cases the difference between ωt_{osc} and $\gamma_o s_{osc}$ and that between $\frac{\omega}{c} z_{osc}$ and $P_{zo} s_{osc}$ can be neglected.

When $\cos\theta \to 1$, the point $(\bar{B},\bar{C})$ approaches the lower branch of the boundary between the regions II and III. Since then $k^2 \to 1$, the oscillation times and the oscillation length become large. In that case it follows from (104a) that the oscillation amplitude $\bar{\psi}_{max} - \bar{\psi}_{min}$, hence also the corresponding total change in kinetic energy, in axial momentum and in angular momentum, become large.

II.6 The influence of inhomogeneities on particle motion near cyclotron resonance

In this Chapter we have considered the motion of an electron in the combination of a homogeneous magnetostatic field and a single, right-circularly polarized wave. In the singular case in which the refractive index equals unity ($N^2 = 1$), while the particle starts at exact resonance, the motion is non-oscillatory and unbounded (synchronous case). In all other situations the motion proves to be periodic. The particle behaviour is completely determined by the constants $\bar{B}$ and $\bar{C}$ given by (35) and (36), respectively, and by the initial value $\bar{\psi}_o$ of the normalized resonance function.

However, when the system is inhomogeneous, for instance when the primary field is space or time-dependent or when axial electric fields are present, this oscillatory or synchronous motion will be disturbed. The inhomogeneity can be inherent to the system as is the case in a mirror-confined e(lectron) c(yclotron) r(esonance) plasma, but it can also be imposed externally, for instance by modulating the high-frequency field.

In the "disturbed" case, the quantities B and C given by (31) and (32), respectively, are not constant, but depend on space and time. The resonance function ψ is then not only changed by the high-frequency field, but it is also influenced by the inhomogeneities.

If the disturbance is large enough, the change of the resonance function due to the wave-field will be small compared to the one due to the inhomogeneities. In that case the interaction between the wave and the particle is too weak to keep the particle near resonance, and the particle will be pulled through resonance in a time short compared to the undisturbed oscillation time. The rate of change of the transverse momentum will still be governed by the high-frequency field, while the time spent by the particle near resonance will be controlled by the disturbances of the homogeneous situation. Consequently the energy gain or loss will be determined by both the wave field and the magnitude of the perturbation[6-9].

Dependent on the initial conditions, especially on the initial phase, it is possible that the particle is kept closer to exact resonance for a time longer than it would be in the homogeneous situation. As a consequence the energy gain will then be larger. This mechanism could explain the occurrence of very energetic particles in e.c.r. plasmas. It may be possible that, in spite of resonance conditions changing along the orbit, the particle still performs a periodic motion. An example would be a particle moving under resonance conditions close to the midplane of a mirror field. When this particle does not penetrate deeply into the mirror region, the change of the resonance function due to the high-frequency field can be of the same order or exceed the variation in the cyclotron frequency during the motion back and forth between the mirrors. Then the particle will remain close to resonance and will execute an energy oscillation with a period which is much larger than the bounce time[10,11].

In the other limit of weak inhomogeneities the time-dependence of the resonance function is only slightly modified, and a quasi-oscillatory motion will occur. In this case the quantities $\bar{B}$ and $\bar{C}$ are not constants but slowly varying. Therefore, the point $(\bar{B},\bar{C})$ moves along some path in the $(\bar{B},\bar{C})$-plane. As an important consequence, the character of the motion may be changed; the particle can pass from trapped to untrapped, or vice versa.

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APPENDIX

A. Determination of the dependence of the resonance function on the proper time

We suppose that the function $\bar{\psi}(\bar{s})$ oscillates between either the two largest or the two smallest real roots, $\bar{\psi}_i$ and $\bar{\psi}_j$, say, of P_4, (with $\bar{\psi}_j \leq \psi_i$). It depends on the value of the constants of motion $\bar{B}$ and $\bar{C}$ whether the polynomial P_4 has two or four real roots.

A formal integration of (II.43) yields

$$\bar{s} - \bar{s}_j = \pm 2 \int_{\bar{\psi}_j}^{\bar{\psi}} \frac{d\bar{\psi}'}{\sqrt{P_4(\bar{\psi}')}} \quad , \tag{1}$$

P_4 being defined by (II.51) and (II.52).

We introduce the angle θ defined by

$$\cos\theta = \frac{(\bar{\psi}_i - \bar{\psi})R_j - (\bar{\psi} - \bar{\psi}_j)R_i}{(\bar{\psi}_i - \bar{\psi})R_j + (\bar{\psi} - \bar{\psi}_j)R_i} \quad , \tag{2}$$

with

$$R^2_{i,j} = (\bar{\psi}_{i,j} - \bar{\psi}_k)\ (\bar{\psi}_{i,j} - \bar{\psi}_\ell) \quad . \quad (i \neq j \neq k \neq \ell) . \tag{3}$$

The quantities $R^2_{i,j}$ are real and non-negative if P_4 has at least two real roots.

Inverting (2) we find

$$\bar{\psi} = \frac{\bar{\psi}_i R_j + \bar{\psi}_j R_i + (\bar{\psi}_j R_i - \bar{\psi}_i R_j)\cos\theta}{R_i + R_j + (R_i - R_j)\cos\theta} \quad . \tag{4}$$

In order to express the integral on the right-hand side of (1) in terms of θ, we differentiate (4) to θ which leads to

$$\frac{d\bar{\psi}}{d\theta} = \frac{2R_iR_j(\bar{\psi}_i - \bar{\psi}_j)}{\left[R_i + R_j + (R_i - R_j)\cos\theta\right]^2}\sin\theta \quad .$$

Further, a substitution of (4) into (II.51), while making use of (3), results in the following expression for P_4:

$$P_4 = \frac{4R_i^3R_j^3(\bar{\psi}_i - \bar{\psi}_j)^2}{\left[R_i + R_j + (R_i + R_j)\cos\theta\right]^4}\left[1 - k^2\sin^2\theta\right]\sin^2\theta \quad ,$$

with

$$k^2 = \frac{1}{2} - \frac{(\bar{\psi}_j - \bar{\psi}_\ell)(\bar{\psi}_i - \bar{\psi}_k) + (\bar{\psi}_i - \bar{\psi}_\ell)(\bar{\psi}_j - \bar{\psi}_k)}{4R_iR_j}$$

$$= \frac{(\bar{\psi}_i - \bar{\psi}_j)^2 - (R_i - R_j)^2}{4R_iR_j} \quad . \tag{5}$$

With the aid of the above expressions, (1) reduces to the following form

$$\bar{s}-\bar{s}_j = \frac{2}{\sqrt{R_iR_j}}\int_0^\theta d\theta' \frac{1}{\left[1-k^2\sin^2\theta'\right]^{\frac{1}{2}}} = \frac{2}{\sqrt{R_iR_j}}F(\theta,k) \quad ; \tag{6}$$

$F(\theta,k)$ represents the elliptic integral of the first kind whose modulus is given by (5).

Making use of one of the inverse functions associated with the elliptic integral (5), we obtain from (4) $\bar{\psi}$ as the following function of $\bar{s}$

$$\bar{\psi} = \frac{\bar{\psi}_iR_j + \bar{\psi}_jR_i + (\bar{\psi}_jR_i - \bar{\psi}_iR_j)\,\mathrm{cn}\{\tfrac{1}{2}\sqrt{R_iR_j}(\bar{s}-\bar{s}_j),k\}}{R_i + R_j + (R_i - R_j)\,\mathrm{cn}\{\tfrac{1}{2}\sqrt{R_iR_j}(\bar{s}-\bar{s}_j),k\}} \quad ; \tag{7}$$

$\mathrm{cn}\{\tfrac{1}{2}\sqrt{R_iR_j}(\bar{s}-\bar{s}_j),k\}$ here marks the Jacobian elliptic function defined by

$$\cos\theta = \mathrm{cn}F(\theta,k) \quad . \tag{8}$$

Applying (6) over one period of oscillation, that is taking $\theta = 2\pi$, we obtain the oscillation proper time $\bar{s}_{osc}$ (which also results by multiplying the result for $\theta = \pi/2$ by 4),

$$\bar{s}_{osc} = \frac{8K(k)}{\sqrt{R_iR_j}} \quad ; \tag{9}$$

$K(k) = F(\frac{\pi}{2},k)$ here represents the complete elliptic integral of the first kind.

We also consider the integration over $\bar{s}$ of $\bar{\psi}$, of which the role as a phase is apparent from (I.34). We need this quantity in order to be able to derive the relations between the oscillation period in ordinary time t_{osc}, the oscillation length $Z_{osc} = \frac{\omega}{c} z_{osc}$, and the oscillation period in proper time s_{osc}. The resulting "phase integral" becomes

$$\oint \bar{\Psi} d\bar{s} = \frac{2}{\sqrt{R_i R_j}} \int_0^{2\pi} d\theta \, \frac{\bar{\Psi}_i R_j + \bar{\Psi}_j R_i + (\bar{\Psi}_j R_i - \bar{\Psi}_i R_j)\cos\theta}{R_i + R_j + (R_i - R_j)\cos\theta} \, \frac{1}{\left[1 - k^2 \sin^2\theta\right]^{\frac{1}{2}}}$$

$$= \frac{\bar{\Psi}_j R_i - \bar{\Psi}_i R_j}{R_i - R_j} \, \frac{8K(k)}{\sqrt{R_i R_j}} + 4 \, \frac{\bar{\Psi}_i - \bar{\Psi}_j}{\sqrt{R_i R_j}} \, \frac{R_i + R_j}{R_i - R_j} \, \Pi(\alpha^2, k) \quad , \tag{10}$$

$\Pi(\alpha^2,k)$ being the complete elliptic integral of the third kind defined by

$$\Pi(\alpha^2, k) = \int_0^{\pi/2} d\theta \, \frac{1}{\left[1 - \alpha^2 \sin^2\theta\right]\left[1 - k^2 \sin^2\theta\right]^{\frac{1}{2}}} \quad , \tag{11}$$

with

$$\alpha^2 = - \frac{(R_i - R_j)^2}{4 R_i R_j} \quad . \tag{12}$$

B. Discussion of the ranges for the squared modulus k^2

The expressions (7), (9) and (10) hold for all values of k. However, the elliptic integrals are only tabulated for $0 \leq k^2 \leq 1$. Therefore, we shall investigate the range of values of k^2 in region II and III of the $(\bar{B},\bar{C})$-plane. If k^2 lies outside $0 \leq k^2 \leq 1$, we need a modulus transformation in order to express $\bar{\Psi}, \bar{s}_{osc}$ and $\oint \bar{\Psi} d\bar{s}$ into tabulated functions.

From (3) and (5) we obtain the following expressions for $k^2(k^2 - 1)$,

$$k^2(k^2 - 1) = \frac{(\bar{\Psi}_i - \bar{\Psi}_j)^2 \, (\bar{\Psi}_\ell - \bar{\Psi}_k)^2}{16 R_i^2 R_j^2} \quad . \tag{13}$$

For the single mode of region II of the $(\bar{B},\bar{C})$-plane the roots $\bar{\Psi}_k$ and $\bar{\Psi}_\ell$ are conjugate complex. Hence, $k^2(k^2 - 1) < 0$ in this region, which is only possible if $0 < k^2 < 1$. Since $\bar{\Psi}_i = \bar{\Psi}_j$ holds on the boundary separating regions I and II, it follows from (3) and (5) that $k^2 = 0$ there. We next consider the two branches of the boundary between regions II and III. In view of the discussion following (II.63) the two new real roots appearing on the upper branch are both larger than $\bar{\Psi}_i$ and $\bar{\Psi}_j$, i.e. $\bar{\Psi}_{i,j} < \bar{\Psi}_k = \bar{\Psi}_\ell$, while those on the lower branch situate between $\bar{\Psi}_i$ and $\bar{\Psi}_j$, i.e. $\psi_j < \bar{\Psi}_k = \bar{\Psi}_\ell < \bar{\Psi}_i$. We then conclude from (3) and (5), taking into account that R_i and R_j are defined as the positive roots of (3), that $k^2 = 0$ on the upper branch, and that $k^2 = 1$ on the lower branch of this boundary. At the point $\bar{B} = \bar{C} = 1$ the value of k depends on the path along which we approach this point.

Inside and on the boundary of region III all four roots of P_4 are real. We conclude from the discussion following (II.63) that on the upper branch of the boundary with region II one of the following relations between the roots of P_4 holds

$$\bar{\psi}_{k,\ell} < \bar{\psi}_i = \bar{\psi}_j \quad \text{or} \quad \bar{\psi}_{i,j} < \bar{\psi}_k = \bar{\psi}_\ell \quad ,$$

while on the lower branch the roots satisfy one of the relations

$$\bar{\psi}_j < \bar{\psi}_i = \bar{\psi}_k < \bar{\psi}_\ell \quad \text{or} \quad \bar{\psi}_k < \bar{\psi}_\ell = \bar{\psi}_j < \bar{\psi}_i \quad .$$

It then follows from (13), (3) and (5) that $k^2 < 0$ inside region III, while $k^2 = 0$ on the upper branch, and $k^2 = -\infty$ on the lower branch of the boundary with region II.

Since $k^2 \leq 0$ in region III we there apply the following modulus transformation[1)]

$$k_1^2 = \frac{-k^2}{1-k^2} \quad , \quad \sin\theta_1 = \left[\frac{1-k^2}{1-k^2\sin^2\theta}\right]^{\frac{1}{2}} \sin\theta \quad ; \tag{14}$$

we then have $0 < k_1^2 < 1$ inside region III, while $k_1^2 = 0$ on the upper branch, and $k_1^2 = 1$ on the lower branch of the boundary with region II.

The conclusions stated above about k^2 and k_1^2 are indicated in Fig. 10.

C. Transformation to tabulated elliptic functions in region III

In order to perform this transformation we derive from (14),

$$\cos\theta = \frac{\cos\theta_1}{\left[1-k_1^2\sin^2\theta_1\right]^{\frac{1}{2}}} \equiv \operatorname{cd} F(\theta_1,k_1) \quad , \tag{15}$$

cd $F(\theta_1,k_1)$ being a Jacobian elliptic function. An application of (13) to (6) and (11) leads to the following expressions

$$\bar{s}-\bar{s}_j = \frac{2F(\theta,k)}{\left[R_iR_j\right]^{\frac{1}{2}}} = \frac{2F(\theta_1,k_1)}{\left[R_iR_j(1-k^2)\right]^{\frac{1}{2}}} \quad . \tag{16}$$

$$\Pi(\alpha^2,k) = \frac{1}{(1-k^2)^{\frac{1}{2}}}\left[\frac{k^2}{k^2-\alpha^2}K(k_1) + \frac{\alpha^2}{\alpha^2-k^2}\Pi(\alpha_1^2,k_1)\right] \quad , \tag{17a}$$

with α defined by (12) and α_1 by

$$\alpha_1^2 = \frac{\alpha^2 - k^2}{1-k^2} = - \frac{(\bar{\psi}_i - \bar{\psi}_j)^2}{(R_i + R_j)^2 - (\bar{\psi}_i - \bar{\psi}_j)^2} \quad . \tag{17b}$$

A substitution of (15) - (17) into (4), (9) and (10), while making use of (5) and (12), yields the following expressions in terms of tabulated elliptic functions

$$\bar{\psi}(\bar{s}) = \frac{\bar{\psi}_i R_j + \bar{\psi}_j R_i + (\bar{\psi}_j R_i - \bar{\psi}_i R_j) \mathrm{cd}\left[\tfrac{1}{2}\{R_i R_j (1-k^2)\}^{\frac{1}{2}} (\bar{s} - \bar{s}_j), k_1\right]}{R_i + R_j + (R_i - R_j) \mathrm{cd}\left[\tfrac{1}{2}\{R_i R_j (1-k^2)\}^{\frac{1}{2}} (\bar{s} - \bar{s}_j), k_1\right]} \quad , \tag{18}$$

$$\bar{s}_{osc} = \frac{8K(k_1)}{\left[R_i R_j (1-k^2)\right]^{\frac{1}{2}}} \quad , \tag{19}$$

$$\oint \bar{\psi} d\bar{s} = \frac{-8(\bar{\psi}_i + \bar{\psi}_j)}{\left[R_i R_j (1-k^2)\right]^{\frac{1}{2}}} \left\{ \tfrac{1}{2} K(k_1) - \Pi(\alpha_1^2, k_1) \right\} \quad . \tag{20}$$

In obtaining (20) we made use of the relation

$$R_i^2 - R_j^2 = 2(\bar{\psi}_i^2 - \bar{\psi}_j^2) \quad , \tag{21}$$

which follows from (3) and the first of relation (II.52).

D. Simplified expressions for the phase integral

We can find expressions for $\oint \bar{\psi} d\bar{s}$ which are much simpler than (10) and (20). Such expressions result from the following relation between the complete elliptic integrals of the second and third kind, and Heuman's lambda function Λ_o 2,3):

$$\Pi(p^2, q) = \frac{K(q)}{1-p^2} + \frac{\pi p^2 \left[\Lambda_o(\beta, q) - 1\right]}{2\left[p^2(1-p^2)(p^2-q^2)\right]^{\frac{1}{2}}} \quad , \quad p^2 \leq 0 \quad ; \tag{22a}$$

here, q represents the modulus of Λ_o, while its argument is given by

$$\sin \beta = \frac{1}{\left[1-p^2\right]^{\frac{1}{2}}} \quad . \tag{22b}$$

The Heuman function has been plotted in Fig. 9 as a function of its argument and modulus.

In region II, where $0 \leq k^2 \leq 1$ and $\alpha^2 \leq 0$, we substitute (22) into (10), taking $p^2 = \alpha^2$ and $q = k$. Remembering (5), (12) and (21), this leads to

$$\oint \bar{\Psi} d\bar{s} = \frac{\bar{\Psi}_i R_j + \bar{\Psi}_j R_i}{R_i + R_j} \frac{8K(k)}{[R_i R_j]^{\frac{1}{2}}} + 4\pi \mathrm{sgn}(R_i - R_j) \left[1 - \Lambda_o(\beta, k)\right] , \tag{23a}$$

the argument of $\Lambda_o(\beta,k)$ here being given by

$$\sin \beta = \frac{1}{[1-\alpha^2]^{\frac{1}{2}}} = \frac{2[R_i R_j]^{\frac{1}{2}}}{R_i + R_j} . \tag{23b}$$

In region III, where $0 \le k_1^2 \le 1$ and $\alpha_1^2 \le 0$, we again substitute (22) into (20) but now we take $p^2 = \alpha_1^2$, $q = k_1$. With the aid of (5), (14), (17b) and (21) we then obtain

$$\oint \bar{\Psi} d\bar{s} = \frac{\bar{\Psi}_i R_j + \bar{\Psi}_j R_i}{R_i + R_j} \frac{8K(k_1)}{[R_i R_j (1-k^2)]^{\frac{1}{2}}} + 4\pi \mathrm{sgn}(R_i - R_j) \left[1 - \Lambda_o(\beta_1, k_1)\right] , \tag{24a}$$

the argument β_1 then being given by

$$\sin \beta_1 = \frac{1}{[1-\alpha_1^2]^{\frac{1}{2}}} = \frac{2[R_i R_j (1-k^2)]^{\frac{1}{2}}}{R_i R_j} . \tag{24b}$$

Performing the integration in (II.70) over one period of oscillation, remembering (II.33) and (II.34), leads to

$$|N^2-1| \left[\omega t_{osc} - \gamma_o s_{osc}\right] = \frac{|N^2-1|}{N} \left[Z_{osc} - P_{zo} s_{osc}\right] = \oint (\bar{\Psi}_o - \bar{\Psi}) d\bar{s} .$$

A substitution of (23) and (24) into this relation leads to (II.75) for region II and to (II.79) for region III, respectively.

REFERENCES

1 P.F. Byrd and M.D. Friedman, Handbook of Elliptic Integrals for Engineers and Physicists (1954), p. 38.

2 loc. cit. p. 226.

3 loc. cit. p. 35 - 37.

CHAPTER III

SERIES SOLUTIONS AND THE ADIABATIC INVARIANT OF THE HELMHOLTZ EQUATION

III.1 Introduction

We want to discuss the solutions of the Helmholtz equation, to be represented here as follows

$$\frac{d^2u}{d\tau^2} + f^2(x)u = 0 \quad , \quad \frac{dx}{d\tau} = \varepsilon \quad ; \tag{1}$$

the parameter ε is assumed to be positive constant.

Many problems in physics lead to this equation. It plays a role in very different questions like the propagation of a plane wave through a stratified medium, the motion of a charged particle in a varying magnetic field, or the reflection of a spinless particle by a one-dimensional potential barrier.

The function $f(x)$ does not depend explicitly on τ, but only through the linear dependence of x on τ. The introduction of the additional variable x is convenient to express that the rate of change of f may be much smaller than that of the corresponding solution u. In the limit $\varepsilon = 0$, x and hence, $f(x)$ become independent of τ. In the following we shall suppose throughout that $f(x)$ vanishes nowhere and that its derivatives of any order exist and are continuous.

We shall discuss two methods for solving (1). In the first one we look for solutions of the form

$$F^{-\frac{1}{2}} \exp\left\{i \int^{\tau} F d\tau'\right\} \quad .$$

We then obtain from (1) a nonlinear second-order differential equation for F. This equation could be solved by iteration, obtaining a series in powers of ε then used as a smallness parameter. However, the resulting series will, in general, not converge, but will only be asymptotic.

Using this method, KULSRUD[1] has shown that in the case of a real function $f(x)$ varying smoothly from a constant value $f(-\infty)$ for $x \to -\infty$ to another value $f(+\infty)$ for $x \to +\infty$ (vanishing derivatives at $x = \pm\infty$), the "energy of the oscillator" (1) divided by its "frequency" f, viz.

$$I(\tau) = \frac{1}{2f}\left[f^2u^2 + \left(\frac{du}{d\tau}\right)^2\right] \tag{2}$$

constitutes an "adiabatic invariant" associated with (1). The property that $I(\tau)$ is an adiabatic invariant to all orders means that its total change $I(+\infty) - I(-\infty)$ vanishes in all orders of the smallness parameter ε; this implies that in the limit of a very slow transition ($\varepsilon \to 0$), this change vanishes more rapidly than any power of ε. This does not mean that I is a rigorous constant. It is an example of a class of adiabatic invariants discussed by LENARD[2].

In the second method the solution of (1) is obtained with the aid of a series expansion. In his discussion of the scattering of a plane wave by a stratified medium, BREMMER[3] describes an exact solution of the wave equation (1). The stratified medium is approximated by adjacent thin homogeneous layers with different indices of refraction. Taking into account all internal reflections, these being a consequence of the jumps in the refractive index, and passing to the limit for infinitesimal layers, Bremmer obtained a series which is an exact solution of (1). The subsequent terms of this series may be interpreted as due to an increasing number of reflections. The first term represents the WKB approximation; it constitutes the forward wave resulting when all internally reflected waves are neglected.

A similar approach to this scattering problem has been performed by SLUIJTER[4]. In his work the stratified medium is approximated by thin inhomogeneous layers inside each of which the WKB approximation is identical to the exact solution. The internal reflections are then due to the introduced discontinuities of the first derivative of the index of refraction.

Considering the initial value problem instead of the scattering problem, BROER[5] and BROER and VAN VROONHOVEN[6] also found a series solution of (1). They show that their series is equivalent to the Bremmer series.

In the next sections we shall extend both methods. By applying consecutive Liouville transformations (see equation (3)) to (1) we obtain a solution in the form of an asymptotic series and a remainder. This rest term is related to the solution of a modified Helmholtz equation and can be expressed as an infinite series related to the Bremmer series. With the aid of this solution we shall arrive at an expression for the change of $I(\tau)$.

III.2 Liouville transformation

III.2.1 General remarks

We shall apply to (1) the following Liouville transformation

$$\bar{u} = u\, F^{\frac{1}{2}}(x,\varepsilon) \quad ,$$

$$\bar{\tau} = \int^{\tau} d\tau'\; F\{x(\tau'),\varepsilon\} \quad ,$$

$$\bar{x} = \int^{x} dx'\; F(x',\varepsilon) \quad , \tag{3}$$

where $F(x,\varepsilon)$ is a yet unspecified function of x and the smallness parameter ε.

Then we obtain an equation for $\bar{u}$, which again has the reduced form

$$\frac{d^2\bar{u}}{d\bar{\tau}^2} + \bar{f}^2(\bar{x},\varepsilon)\;\bar{u} = 0 \quad , \quad \frac{d\bar{x}}{d\bar{\tau}} = \varepsilon \quad , \tag{4a}$$

with $\bar{f}^2$ given by

$$\bar{f}^2(\bar{x},\varepsilon) = \frac{1}{F^2}\left[f^2 + \varepsilon^2 F^{\frac{1}{2}} \frac{d^2}{dx^2} F^{-\frac{1}{2}}\right] \quad . \tag{4b}$$

If we choose F such that it constitutes a solution of the non-linear equation

$$F^2(x,\varepsilon) = f^2(x) + \varepsilon^2 F^{\frac{1}{2}} \frac{d^2}{dx^2} F^{-\frac{1}{2}} \quad , \tag{5}$$

we find from (3) - (5) the complete solution of (1) in the form

$$u(\tau) = \frac{C_1}{F^{\frac{1}{2}}\{x(\tau),\varepsilon\}} \exp\left\{ i \int^{\tau} F\{x(\tau'),\varepsilon\}d\tau' \right\} +$$

$$+ \frac{C_2}{F^{\frac{1}{2}}\{x(\tau),\varepsilon\}} \exp\left\{ -i \int^{\tau} F\{x(\tau'),\varepsilon\}d\tau' \right\} \quad , \tag{6}$$

where $C_{1,2}$ are arbitrary constants. The solution of (1) for $\varepsilon = 0$ is trivial since then f^2 becomes a constant. From the form of the solution (6) it is then required that the solution of (5) for $\varepsilon = 0$ should be given by

$$F^2(x,o) = f^2(x) \quad . \tag{7}$$

It follows from (4) that the system described by (1) has an exact invariant (an expression not depending on $\bar{\tau}$) reading:

$$\bar{I} = \frac{1}{2}\left[\bar{u}^2 + \left(\frac{d\bar{u}}{d\bar{\tau}}\right)^2\right]$$

$$= \frac{1}{2}\left[u^2F + \left\{\frac{1}{F^{\frac{1}{2}}}\frac{du}{d\tau} + \varepsilon\frac{u}{F}\frac{d}{dx}F^{\frac{1}{2}}\right\}^2\right] \quad , \tag{8}$$

provided that $\bar{f}^2 = 1$, so that F constitutes a solution of (5). This even holds if f is complex, so that also F, $\bar{\tau}$ and $\bar{x}$ become complex. This invariant is equivalent to that obtained by LEWIS[7], when applying KRUSKAL'S[8] asymptotic theory to the classical oscillator with time-dependent frequency.

In general we cannot find explicit expressions for the invariant (8), since we cannot solve (5) for arbitrary functions f(x). In his treatment, Lewis calculates the solution of (5) for some functions f(x) for which the solutions of (1) are known.

On the other hand, we can solve (5) by iteration, which leads to a series in powers of the smallness parameter ε, the first term of which constitutes the well-known WKB approximation. The consecutive terms give the higher-order corrections to this WKB solution. However, in general the series thus obtained will not converge, but only be asymptotic.

To illustrate this asymptotic character, we consider the situation in which (1) describes the propagation of a plane wave through an unbounded, stratified medium, whose refractive index $f\{x(\tau)\}$ changes smoothly from the constant value $f(-\infty)$ for $\tau \to -\infty$ to the other value $f(+\infty)$ for $\tau \to +\infty$ (see Fig. 1). Moreover, we restrict ourselves to the simple case in which the incident wave arrives from $\tau = -\infty$, while imposing the boundary condition that the reflected wave should vanish for $\tau \to +\infty$. For $\tau \to \pm\infty$ the solution of (1) then has to approach the asymptotic solutions

$$u(\tau) = a_1 \exp\{if(-\infty)\tau\} + a_2 \exp\{-if(-\infty)\tau\} \quad , \quad \tau \to -\infty \quad , \tag{9a}$$

$$u(\tau) = a_3 \exp\{if(+\infty)\tau\} \quad , \quad \tau \to +\infty \quad . \tag{9b}$$

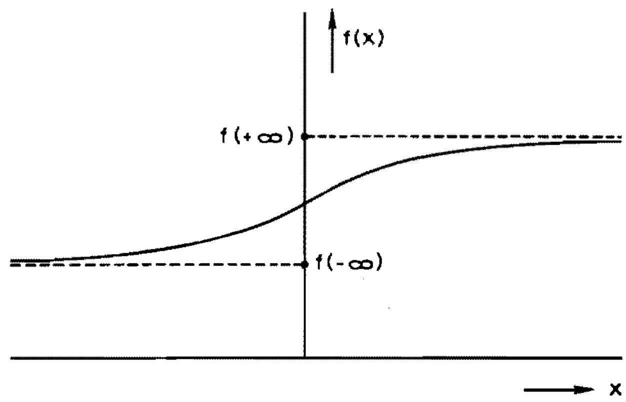

Fig. 1 Schematic picture of the index of refraction f(x) as a function of x.

Let (5) have a positive, single-valued and integrable solution $F(x,\varepsilon)$, which satisfies the boundary conditions

$$F\{x(\tau),\varepsilon\} \rightarrow f(\pm\infty) \quad \text{for} \quad \tau \rightarrow \pm\infty \quad ;$$

we then conclude from (9) and (6) that the amplitude C_2 must vanish, $C_2 \equiv 0$. This would mean that no reflected wave could occur. Obviously, this physically incorrect result implies that (5) cannot have solutions that satisfy the stated conditions, so that the series obtained from (5) by iteration cannot converge for all x. An analogous conclusion has been drawn by HERTWECK and SCHLÜTER[9)] in their discussion of the change of the magnetic moment of a charged particle in a slowly varying field.

On account of the above remarks we shall proceed along a somewhat different way.

III.2.2 A hierarchy of Liouville transformations

III.2.2.1 Definition of the transformations

An obvious change of variables for treating (1) is given by

$$u_1 = u\, f^{\frac{1}{2}} \quad , \quad \tau_1 = \int^{\tau} f\{x(\tau')\}\, d\tau' \quad , \quad x_1 = \int^{x} f(x')dx' \quad . \qquad (10a)$$

Equation (1) then becomes (compare (4) for F replaced by f)

$$\frac{d^2u_1}{d\tau_1^2} + f_1^2(x_1,\varepsilon)\, u_1 = 0 \quad , \quad \frac{dx_1}{d\tau_1} = \varepsilon \quad , \tag{10b}$$

in which

$$f_1^2\,(x_1,\varepsilon) = 1 + \varepsilon^2 f^{-3/2}(x)\, \frac{d^2}{dx^2}\, f^{-1/2}(x) \quad . \tag{10c}$$

An analogous transformation can be applied to (10b), and next again to the resulting equation, and so on (POLISHCHUCK[10]).

The hierarchy thus obtained has been used by LITTLEWOOD[11] to show the adiabaticity of the quantity I of (2). However, in the following the successive Liouville transformations will be used to obtain corrections to the WKB approximation, which is the solution of (1) to first order in ε, and to show the asymptotic character of the abovementioned iteration procedure.

Thus applying in succession the transformations

$$u_n = u_{n-1}\, f_{n-1}^{1/2} \quad , \quad n \geq 1$$

$$\tau_n = \int^{\tau_{n-1}} f_{n-1} d\tau'_{n-1} \quad , \quad x_n = \int^{x_{n-1}} f_{n-1} dx'_{n-1} \quad ,$$

$$u_o = u \quad , \quad \tau_o = \tau \quad , \quad x_o = x \quad , \quad f_o(x_o,\varepsilon) = f(x) \quad . \tag{11a}$$

We obtain a sequence of second order differential equations which all have the reduced form:

$$\frac{d^2u_n}{d\tau_n^2} + f_n^2(x_n,\varepsilon)u_n = 0 \quad , \quad \frac{dx_n}{d\tau_n} = \varepsilon \quad , \quad (n \geq 0) \quad . \tag{11b}$$

Each f_n is then related to f_{n-1} by the recurrence formula

$$f_n^2(x_n,\varepsilon) = 1 + \varepsilon^2 f_{n-1}^{-3/2}\, \frac{d^2}{dx_{n-1}^2}\, f_{n-1}^{-1/2} \quad , \quad (n \geq 1) \quad . \tag{11c}$$

III.2.2.2 Expressions for the hierarchy in terms of the original variables

In view of the chain of transformations (11a), the quantity f_n^2, which is defined as a function of x_n and ε, can equally well be considered as a function of x and ε. Therefore, it proves to be more convenient to express the preceding relations in terms of the functions $F_n(x,\varepsilon)$, defined by

$$F_n(x,\varepsilon) \equiv \prod_{k=0}^{n} f_k(x,\varepsilon) \quad , \quad (n \geq 0) \quad . \tag{12}$$

According to (12) and (11a), the relation between u_n and u, τ_n and τ, and the one between x_n and x can also be given by

$$u_n = u(\tau)\, F_{n-1}^{\frac{1}{2}}(x,\varepsilon) \quad , \quad n \geq 1 \quad ,$$

$$\tau_n = \int^{\tau} F_{n-1}\{x(\tau'),\varepsilon\}d\tau' \quad , \quad x_n = \int^{x} F_{n-1}(x',\varepsilon)dx' \quad ,$$

$$u_o = u(\tau) \quad , \quad \tau_o = \tau \quad , \quad x_o = x \quad , \quad F_o(x_o,\varepsilon) = f(x) \quad . \tag{13}$$

Moreover, the relation (11c) can also be expressed in terms of the functions F_n instead of f_n. In fact, substituting $f_\gamma = F_\gamma/F_{\gamma-1}$ for $\gamma = n$, $n-1$ in (11c), while applying the relation $dx_{n-1} = F_{n-2}dx$, we first find

$$F_n^2 - \varepsilon^2 F_{n-1}^{\frac{1}{2}} \frac{d^2}{dx^2} F_{n-1}^{-\frac{1}{2}} = F_{n-1}^2 - \varepsilon^2 F_{n-2}^{\frac{1}{2}} \frac{d^2}{dx^2} F_{n-2}^{-\frac{1}{2}} \quad ;$$

the following equation then results after a summation over n in:

$$F_n^2 = \left[F_1^2 - \varepsilon^2 F_o^{\frac{1}{2}} \frac{d^2 F_o^{-\frac{1}{2}}}{dx^2}\right] + \varepsilon^2 F_{n-1}^{\frac{1}{2}} \frac{d^2}{dx^2} F_{n-1}^{-\frac{1}{2}} \quad .$$

Reducing this expression with the aid of (11c) for $n = 1$, remembering that $F_1 = f_1 f_o$ and $F_o = f_o = f$, we find

$$F_n^2(x,\varepsilon) = f^2 + \varepsilon^2 F_{n-1}^{\frac{1}{2}} \frac{d^2}{dx^2} F_{n-1}^{-\frac{1}{2}} \qquad (n \geq 1) \tag{14a}$$

$$= f^2 + \varepsilon^2\left\{-\frac{1}{4}\frac{1}{F_{n-1}^2}\frac{d^2F_{n-1}^2}{dx^2} + \frac{5}{16}\left[\frac{1}{F_{n-1}^2}\frac{dF_{n-1}^2}{dx}\right]^2\right\} \quad . \tag{14b}$$

Dividing both sides of (14a) by F_{n-1}^2, and combining (13) and (11c), respectively, we find the following alternative expressions for $f_n^2(x,\varepsilon)$ $(n \geq 1)$

$$f_n^2(x,\varepsilon) = \frac{1}{F_{n-1}^2}\left[f^2 + \varepsilon^2 F_{n-1}^{\frac{1}{2}} \frac{d^2}{dx^2} F_{n-1}^{-\frac{1}{2}}\right] \quad , \tag{15a}$$

$$f_n^2(x,\varepsilon) \equiv 1 + \frac{\varepsilon^2}{F_{n-1}^2}\left[-\frac{1}{4}\frac{1}{f_{n-1}^2}\frac{d^2 f_{n-1}^2}{dx^2} + \frac{3}{16}\left\{\frac{1}{f_{n-1}^2}\frac{df_{n-1}^2}{dx}\right\}^2 + \right.$$

$$\left. + \frac{1}{8}\left\{\frac{1}{f_{n-1}^2}\frac{df_{n-1}^2}{dx}\right\}\left\{\frac{1}{F_{n-1}^2}\frac{dF_{n-1}^2}{dx}\right\}\right] \quad . \tag{15b}$$

The expressions for F_n^2 and f_n^2, which can be obtained from (14) and (15) by repeated differentiations, contain $f^2(x)$ and its first 2n derivatives. The assumption that $f^2(x)$ nowhere vanishes, and that its first 2n derivatives exist and are bounded, ensures that, for values of ε^2 small enough, f_n^2 and F_n^2 also exist and are also bounded, nowhere vanishing, functions of x and ε. We should stress that F_n^2 and f_n^2 are not simple finite series in positive powers of ε^2, but expressible as rational functions of ε^2.

III.2.2.3 Asymptotic character of the sequence $\{F_n\}$

The relations (11b), (13), (14a) and (15a) strongly resemble the equations (4a), (3), (5) and (4b), respectively.
This might suggest that, when taking the limit $n \to \infty$, we have found a recipe for obtaining an exact solution of (5), and thus of (1). However, it is not allowed to take this limit since, in general, the sequences of functions $\{F_n\}$ and $\{f_n\}$ do not converge for all values of x. Hence, in general it is not allowed to deduce from (14) an equation for a possible limit of F_n for $n \to \infty$ for all values of x.

As an example, we consider the reflection problem mentioned at the end of subsection 2.1. In that case the sequence $\{F_n\}$ will certainly converge in the limit $n \to \infty$ for $\tau \to \pm\infty$, since all derivatives of $f\{x(\tau)\}$ vanish for $\tau \to \pm\infty$; $F_n \to f$ is then bounded while f_n tends to unity. In the intermediate region, however, the sequence does not converge for all values of τ. The solution of (1) for $\tau \to -\infty$ then cannot be linked with that for $\tau \to +\infty$ (Stokes phenomenon).

By induction we obtain from (14b) and (15b), respectively the following estimates for F_n^2 and f_n^2,

$$F_n^2(x,\varepsilon) = f^2(x) + O(\varepsilon^2) \quad , \quad f_n^2(x,\varepsilon) = 1 + O(\varepsilon^{2n}) \quad . \tag{16}$$

In the limit $\varepsilon = 0$ the functions f_n^2 and F_n^2 satisfy the relations

$$f_n^2(x,0) = 1 \quad , \quad F_n(x,0) = f(x) \quad . \tag{17}$$

In general it does not follow from the estimates (16) that, for small but finite values of ε, and for increasing values of n, the quantities f_n^2-1 become zero to a higher and higher degree of accuracy. Both estimates are meant in an asymptotic sense; they refer to the limit $\varepsilon \to 0$ for a fixed value of n only. This can be illustrated by the following example.

We suppose that for large x, f(x) is given by

$$f^2(x) = a + b\,x^{-\alpha} \quad (\alpha > 0) \quad , \quad x \to \infty \quad .$$

By repeated differentiation we then obtain from (14b) and (15b) the following expression for f_n^2:

$$f_n^2 = 1 + \frac{b}{a}\left[-\frac{\varepsilon^2}{4a}\right]^n (\alpha + 2n - 1)!\; x^{-\alpha-2n} + O(x^{-2\alpha-2n}) \quad .$$

Due to the factor $(\alpha + 2n - 1)!$ the quantities f_n^2-1 will only become arbitrarily small for increasing values of n if ε depends on n in a proper way.

III.2.3 Introduction of higher-order invariants

On the analogy of the function I given by (2), which was associated with the original equation (1), we introduce quantities I_n associated similarly with (11b). Starting from $I_o = I$ as given by (2), these I_n will be defined by the two following relations which are equivalent in view of (13)

$$\begin{aligned} I_n(\tau) &= \frac{1}{2f_n}\left[f_n^2\, u_n^2 + \left(\frac{du_n}{d\tau_n}\right)^2\right] \\ &= \frac{1}{2f_n}\left[f_n^2\, F_{n-1} u^2 + \left\{\frac{1}{F_{n-1}^{\frac{1}{2}}}\frac{du}{d\tau} + \varepsilon\,\frac{u}{F_{n-1}}\frac{dF_{n-1}^{\frac{1}{2}}}{dx}\right\}^2\right] , \end{aligned}$$

$$(n \geq 1) \quad . \tag{18}$$

Let the system vary slowly from one steady state to a different steady state such that $f(x) = a$ for $x < x_1$, $f(x) = b$ for $x > x_1$, with $x_1 = x(\tau_1)$ and $x_2 = x(\tau_2)$. This excludes periodic functions f(x). It then follows from repeated applications of (14b) and (15b) that

$$\begin{aligned} F_n(x,\varepsilon) &= a \quad , \quad f_n(x,\varepsilon) = 1 \quad \text{for} \quad x < x_1 \quad , \\ & \qquad\qquad\qquad\qquad\qquad\qquad (n \geq 1) \\ F_n(x,\varepsilon) &= b \quad , \quad f_n(x,\varepsilon) = 1 \quad \text{for} \quad x > x_2 \quad . \end{aligned} \tag{19}$$

From (18) we then deduce that the functions $I_n(\tau)$ all approach the same limit viz.

$$I_n(\tau) = I(\tau) \quad \text{for} \quad \tau < \tau_1 \quad \text{and} \quad \tau > \tau_2 \quad . \tag{20}$$

In the following sections we shall prove that the functions I_n form a hierarchy of adiabatic invariants.

III.3 A splitting of the n-th order solution

We want to convert, for each n, the second-order differential equation (11b) into two coupled equations of the first order. An infinite number of such conversions is possible, but we shall choose a special one.

To that end we split u_n into the functions v_n and w_n, defined by

$$\begin{aligned} v_n &\equiv \frac{1}{2} u_n - \frac{i}{2f_n} \frac{du_n}{d\tau_n} , \\ & \qquad\qquad\qquad\qquad (n \geq 0) \\ w_n &\equiv \frac{1}{2} u_n + \frac{i}{2f_n} \frac{du_n}{d\tau_n} . \end{aligned} \tag{21}$$

The splitting (21) is unique in the sense that, if f_n is real, it also splits the energy and the Wronskian associated with (11b) into contributions depending on v_n and w_n only[6]. For n = 0 it is identical to the splitting used by BREMMER[3]. Replacing in (21) f_n by 1, we obtain for n = 1 the splitting used by SLUIJTER[4]. It can easily be verified that, in view of (11b), the functions v_n, w_n indeed satisfy the two coupled first-order equations contained in:

$$\frac{d}{d\tau_n} \begin{pmatrix} v_n \\ w_n \end{pmatrix} = \begin{pmatrix} if_n - \frac{\varepsilon}{2f_n} \frac{df_n}{dx_n} & \frac{\varepsilon}{2f_n} \frac{df_n}{dx_n} \\ \frac{\varepsilon}{2f_n} \frac{df_n}{dx_n} & -if - \frac{\varepsilon}{2f_n} \frac{df_n}{dx_n} \end{pmatrix} \begin{pmatrix} v_n \\ w_n \end{pmatrix} . \tag{22}$$

A relation connecting v_o, w_o with v_n, w_n can be found as follows. We apply to (21) for n = 0, the transformation $u_n = u_o F_{n-1}^{\frac{1}{2}}$, $d\tau_n = F_{n-1} d\tau_o$. This yields a relation between v_o, w_o, u_n and $du_n/d\tau_n$. The latter two quantities can be expressed in terms of v_n and w_n with the aid of (21), which leads to the relation in question, viz.

$$\begin{pmatrix} v_o \\ \\ w_o \end{pmatrix} = A_n \begin{pmatrix} v_n \\ \\ w_n \end{pmatrix} \quad , \tag{23a}$$

in which $(n \geq 1)$

$$A_n(x,\varepsilon) \equiv \frac{1}{2F_{n-1}^{\frac{1}{2}}} \begin{pmatrix} 1 + \frac{F_n}{f} + \frac{i\varepsilon}{2fF_{n-1}} \frac{dF_{n-1}}{dx} & 1 - \frac{F_n}{f} + \frac{i\varepsilon}{2fF_{n-1}} \frac{dF_{n-1}}{dx} \\ 1 - \frac{F_n}{f} - \frac{i\varepsilon}{2fF_{n-1}} \frac{dF_{n-1}}{dx} & 1 + \frac{F_n}{f} - \frac{i\varepsilon}{2fF_{n-1}} \frac{dF_{n-1}}{dx} \end{pmatrix} , \tag{23b}$$

while $A_o(x,\varepsilon) = E$, E denoting the identity matrix.

If the system varies from the steady state $f(x) = a$ for $x < x_1$ to the other steady state $f(x) = b$ for $x > x_2$, it follows from (19) and (23b) that

$$A_n(x,\varepsilon) = f^{-\frac{1}{2}}(x)\ E \quad \text{for} \quad x < x_1 \quad \text{and} \quad x > x_2 \quad , \quad (n \geq 1)\ . \tag{24}$$

In order to get new equations depending on a simpler matrix, which only has off-diagonal elements, we pass from v_n, w_n to the functions $\bar{v}_n$, $\bar{w}_n$ defined by

$$\begin{pmatrix} v_n \\ \\ w_n \end{pmatrix} \equiv B_n \begin{pmatrix} \bar{v}_n \\ \\ \bar{w}_n \end{pmatrix} \quad , \tag{25a}$$

with the square matrix

$$B_n(x,\varepsilon) \equiv f_n^{-\frac{1}{2}} \begin{pmatrix} \exp\{\frac{i}{\varepsilon} \int^x F_n dx'\} & 0 \\ 0 & \exp\{-\frac{i}{\varepsilon} \int^x F_n dx'\} \end{pmatrix} . \tag{25b}$$

It then follows from (23), (25) and (13) that the functions $\bar{v}_n$, $\bar{w}_n$ have to satisfy the equations

$$\frac{d}{d\tau} \begin{pmatrix} \bar{v}_n \\ \\ \bar{w}_n \end{pmatrix} = \varepsilon\ L_n(x,\varepsilon) \begin{pmatrix} \bar{v}_n \\ \\ \bar{w}_n \end{pmatrix} \quad , \quad \frac{dx}{d\tau} = \varepsilon \quad , \tag{26a}$$

the matrix L_n being defined by

$$L_n(x,\varepsilon) \equiv l_n(x,\varepsilon) \begin{pmatrix} 0 & \exp\{\frac{-2i}{\varepsilon} \int^x F_n dx'\} \\ \exp\{\frac{2i}{\varepsilon} \int^x F_n dx'\} & 0 \end{pmatrix} , \tag{26b}$$

with

$$l_n(x,\varepsilon) \equiv \frac{1}{2f_n}\frac{df_n}{dx} \quad . \tag{26c}$$

From (16) and (26c) we conclude that $l_n = O(\varepsilon^{2n})$. Hence, by each consecutive Liouville transformation and corresponding splitting (21), the order of the coefficient in the right-hand side of the first relation (26a) is raised by a factor ε^2. The above procedure resembles in principle very much the one used by VAN KAMPEN[12]. However, we have made use of the freedom to choose our splitting such that $u = F_{n-1}^{-\frac{1}{2}}\,(v_n + w_n)$ satisfies the original equation (1).

In the next section we shall express the solution of (26) by an infinite series of multiple integrals. The solution of (11b) will then be obtained, since $F_n dx = \varepsilon f_n d\tau_n$, in the form:

$$u_n = v_n + w_n = \frac{\bar{v}_n}{f_n^{\frac{1}{2}}}\exp\left\{+i\int^{\tau_n} f_n d\tau_n'\right\} + \frac{\bar{w}_n}{f_n^{\frac{1}{2}}}\exp\left\{-i\int^{\tau_n} f_n d\tau_n'\right\} , \tag{27}$$

or, applying the relations $u = u_n\,F_{n-1}^{-\frac{1}{2}}$ and $f_n d\tau_n = F_n/\varepsilon\; dx$ following from (13), the solution of (1) in the form

$$u(\tau) = \frac{\bar{v}_n(\tau)}{F_n^{\frac{1}{2}}(x,\varepsilon)}\exp\left\{\frac{i}{\varepsilon}\int^{x} F_n(x',\varepsilon)dx'\right\} +$$

$$+ \frac{\bar{w}_n(\tau)}{F_n^{\frac{1}{2}}(x,\varepsilon)}\exp\left\{-\frac{i}{\varepsilon}\int^{x} F_n(x',\varepsilon)dx'\right\} \quad ,$$

$$x = x_a + \varepsilon(\tau - \tau_a) \quad . \tag{28}$$

The quantities $\frac{1}{F_n^{\frac{1}{2}}}\exp\left\{\pm i\int^{\tau} F_n d\tau'\right\}$ contain the first n corrections with respect to the smallness parameter ε to the WKB solutions of (1); therefore, we shall call them the n-th order WKB solutions of (1). Hence, the functions $\bar{v}_n$, $\bar{w}_n$ are just the amplitudes of the first-order WKB solutions of the equation (11b) for the n-th Liouville function u_n, as well as those of the n-th order WKB solutions of the original equation (1).

The quantities I_n given by (2) and (18) can be expressed in terms of $\bar{v}_n$ and $\bar{w}_n$. Making use of (21) and (25) we find

$$I_n = 2\; f_n\; v_n\; w_n = 2\;\bar{v}_n\;\bar{w}_n \quad . \tag{29}$$

III.4 Series solution of the equation for the WKB amplitudes

The solution of (26a) can be expressed as follows in terms of a matrix ${}_nU^x_{x_a}$, which describes the evolution of the system from an initial value τ_a of τ up to some arbitrary τ:

$$\begin{pmatrix} \bar{v}_n(\tau) \\ \\ \bar{w}_n(\tau) \end{pmatrix} = {}_nU^x_{x_a} \begin{pmatrix} \bar{v}_n(\tau_a) \\ \\ \bar{w}_n(\tau_a) \end{pmatrix} , \tag{30}$$

with $x = x_a + \varepsilon(\tau - \tau_a)$.

The evolution matrix ${}_nU^x_{x_a}$ (also called the matrizant) then has to satisfy a matrix differential equation and a normalization condition, viz.:

$$\frac{\partial}{\partial x}\, {}_nU^x_{x'} = L_n(x,\varepsilon)\, {}_nU^x_{x'} \quad ; \quad {}_nU^x_x = E \quad . \tag{31}$$

Solving (31), we obtain the solution in the form of a series of multiple integrals,

$${}_nU^x_{x_a} = E + \sum_{k=1}^{\infty} \int_{x_a}^{x} ds_1 \int_{x_a}^{s_1} ds_2 \ldots \int_{x_a}^{s_{k-1}} ds_k\, L_n(s_1)L_n(s_2)\ldots L_n(s_k) \quad . \tag{32}$$

This equation constitutes the Neumann-Liouville expansion for the equivalent integral equation

$${}_nU^x_{x_a} = E + \int_{x_a}^{x} L_n(x',\varepsilon)\, {}_nU^{x'}_{x_a}\, dx' \quad .$$

The multiple integrals in (32) cannot be reduced to single integrals, since the matrix L_n, given by (26b) does not commute for different values of its argument, i.e.

$$L_n(s_1)\, L_n(s_2) \neq L_n(s_2)\, L_n(s_1) \quad .$$

In view of (26b) the expansion (32) involves the following expressions for the elements of ${}_nU^x_{x_a}$:

$${}_nU^x_{x_a 11} = 1 + \sum_{k=2,4,6..}^{\infty} \int_{x_a}^{x} ds_1 \ldots \int_{x_a}^{s_{k-1}} ds_k\, l_n(s_1)\ldots l_n(s_k) \times$$

$$\times \exp\left\{\frac{2i}{\varepsilon} \sum_{p=1}^{k} (-1)^p \int^{s_p} F_n\, ds'_p\right\} , \tag{33a}$$

$$ {}_nU^x_{x_a}12 = \sum_{k=1,3,5..}^{\infty} \int_{x_a}^{x} ds_1 \ldots \int_{x_a}^{s_{k-1}} ds_k \; l_n(s_1)\ldots l_n(s_k) \times $$

$$ \times \exp\left\{\frac{2i}{\varepsilon} \sum_{p=1}^{k} (-1)^p \int^{s_p} F_n \, ds'_p\right\} \quad . \tag{33b} $$

The expressions for ${}_nU^x_{x_a}22$ and ${}_nU^x_{x_a}21$ are obtained from (33a) and (33b) respectively, by replacing i by -i.

By combining (33) and (30) we finally obtain the exact solution for (26) for the n-th order WKB amplitudes $\bar{v}_n(\tau)$ and $\bar{w}_n(\tau)$ in terms of multiple integrals.

In order to obtain estimates of the WKB amplitudes $\bar{v}_n$ and $\bar{w}_n$, while restricting ourselves to real f and therefore, also to real F_n, we yet define the non-negative functions m_n and M_n by the relations

$$ m_n(x,x_a) = \max \left| \int_{x_a}^{x_1} dx' l_n(x',\varepsilon) \exp\left\{- \frac{2i}{\varepsilon} \int^{x'} F_n(x'',\varepsilon)dx''\right\}\right| \tag{34a} $$

$$ \text{for } x_a \leq x_1 \leq x \quad , $$

$$ M_n(x,x_a) = \int_{x_a}^{x} \left|l_n(x',\varepsilon)\right| \, d\left|x'\right| \quad . \tag{34b} $$

Due to the exponential, m_n will generally be much smaller than M_n.

The moduli of the series (33a) and (33b), respectively, are then majorized by the following series ($m_n \leq M_n$)

$$ 1 + m_n M_n + m_n \frac{M_n^3}{3!} + \ldots \quad ; \quad m_n + m_n \frac{M_n^2}{2!} + m_n \frac{M_n^4}{4!} + \ldots \quad . \tag{35} $$

Either of the series (33) is thus in any case absolutely and uniformly convergent on the interval (x,x_a) if

$$ M_n(x,x_a) < \infty \quad . \tag{36} $$

This constitutes a rather weak restriction. It even admits a vanishing of $f^2 - f(+\infty)$ for $x \to \infty$, which may be as slow as $x^{-\alpha}(\alpha > 0)$.

Finally, we want to express the splitting v_o, w_o of the original equation (1) in terms of ${}_nU^x_{x_a}$. This can be achieved as follows. Combining (23a) and (25a), we find the relations

$$ \begin{pmatrix} v_o(\tau) \\ w_o(\tau) \end{pmatrix} = A_n(x,\varepsilon) \; B_n(x,\varepsilon) \begin{pmatrix} \bar{v}_n(\tau) \\ \bar{w}_n(\tau) \end{pmatrix} \quad , $$

$$\begin{pmatrix} \bar{v}_n(\tau) \\ \\ \bar{w}_n(\tau) \end{pmatrix} = B_n^{-1}(x,\varepsilon)\; A_n^{-1}(x,\varepsilon) \begin{pmatrix} v_o(\tau) \\ \\ w_o(\tau) \end{pmatrix} ,$$

where the superscript -1 indicates the inverse. A combination of these relations with (30) leads to the result

$$\begin{pmatrix} v_o(\tau) \\ \\ w_o(\tau) \end{pmatrix} = A_n(x,\varepsilon)\; B_n(x,\varepsilon)\; {}_nU^{x}_{x_a}\; B_n^{-1}(x_a,\varepsilon)\; A_n^{-1}(x_a,\varepsilon) \begin{pmatrix} v_o(\tau_a) \\ \\ w_o(\tau_a) \end{pmatrix} . \qquad (37)$$

The solution of (1) can be represented by the formula (28) with $\bar{v}_n$, $\bar{w}_n$ given by (30), as well as by $u(\tau) = v_o(\tau) + w_o(\tau)$, with v_o, w_o given by (37).

III.5 Equivalence of the series solution with the Bremmer type solution

For the sake of convenience we shall use the terminology referring to the situation in which equation (1) describes the propagation of a plane wave through an unbounded stratified medium. The refractive index of this medium changes smoothly from the constant value $f(-\infty)$ for $\tau \to -\infty$ to the other value $f(+\infty)$ for $\tau \to +\infty$. The incident wave arrives from $\tau = -\infty$, and we impose the boundary condition that the reflected wave should vanish for $\tau \to +\infty$. We shall prove that in this case the series solution obtained in the previous section is equivalent to the Bremmer type solution.

For that purpose we introduce the operators $O_n^{\pm}$, $P_n^{\pm}$ and $Q_n^{\pm}$ fixed by the relations:

$$O_n^{\pm}\, h(x) = \int_{-\infty}^{x} dx'\, l_n(x',\varepsilon)\, \exp\left\{\pm \frac{2i}{\varepsilon} \int^{x'} F_n(x'',\varepsilon)dx''\right\} h(x') ,$$

$$P_n^{\pm}\, h(x) = \int_{x}^{\infty} dx'\, l_n(x',\varepsilon)\, \exp\left\{\pm \frac{2i}{\varepsilon} \int^{x'} F_n(x'',\varepsilon)dx''\right\} h(x') ,$$

$$Q_n^{\pm}\, h(x) = \int_{-\infty}^{\infty} dx\, l_n\, (x,\varepsilon)\, \exp\left\{\pm \frac{2i}{\varepsilon} \int^{x} F_n(x',\varepsilon)dx'\right\} h(x) , \qquad (38)$$

with l_n and F_n defined by (26c) and (12), respectively.

In view of (31) and (32) we shall interpret the evolution matrix ${}_nU^{x}_{x_a}$ as an operator acting on a matrix with constant elements. Because of the boundary condition at $x = -\infty$ we take $x_a = -\infty$. According to (33) the elements of ${}_nU^{x}_{-\infty}$ can then be expressed as follows in terms of the operator $O_n^{\pm}$,

$$
\begin{aligned}
{}_nU^x_{-\infty 11} &= 1 + O_n^- O_n^+ + O_n^- O_n^+ O_n^- O_n^+ + \ldots \equiv \frac{1}{1 - O_n^- O_n^+} \; , \\
{}_nU^x_{-\infty 22} &= 1 + O_n^+ O_n^- + O_n^+ O_n^- O_n^+ O_n^- + \ldots \equiv \frac{1}{1 - O_n^+ O_n^-} \; , \\
{}_nU^x_{-\infty 12} &= O_n^- + O_n^- O_n^+ O_n^- + \ldots \equiv O_n^- \frac{1}{1 - O_n^+ O_n^-} \; , \\
{}_nU^x_{-\infty 21} &= O_n^+ + O_n^+ O_n^- O_n^+ + \ldots \equiv O_n^+ \frac{1}{1 - O_n^- O_n^+} \; .
\end{aligned}
\tag{39}
$$

The fractions on the right are only meant as convenient abbreviations.

A substitution of (39) into (30), applied for x_a, $\tau_a \to -\infty$, leads to the following expressions for the n-th order WKB amplitudes,

$$
\bar{v}_n(\tau) = \left[\frac{1}{1 - O_n^- O_n^+} + R O_n^- \frac{1}{1 - O_n^+ O_n^-} \right] A \; , \tag{40a}
$$

$$
\bar{w}_n(\tau) = \left[O_n^+ \frac{1}{1 - O_n^- O_n^+} + R \frac{1}{1 - O_n^+ O_n^-} \right] A \; , \tag{40b}
$$

where we have introduced the parameters $A \equiv \bar{v}_n(-\infty)$ and $R \equiv \bar{w}_n(-\infty)/\bar{v}_n(-\infty)$. By repeated application of (14) we find that $F_n\{x(\tau),\varepsilon\} \to f(\pm\infty)$ for $x \to \pm\infty$. We then conclude from (28) that $\bar{w}_n(+\infty) = 0$ because of the boundary condition of a vanishing reflected wave for $\tau \to +\infty$, and that $\bar{v}_n(-\infty)$ is the amplitude of the incident WKB wave. Thus A and R represent the WKB amplitude of the incident wave and the reflection coefficient, respectively.

In view of the vanishing of $\bar{w}_n(\infty)$ an application of (30) for $x, \tau \to +\infty$ and $x_a, \tau_a \to -\infty$ leads to the relation

$$
R \; {}_nU^{+\infty}_{-\infty 22} \, A = - \, {}_nU^{+\infty}_{-\infty 21} \, A \; .
$$

With the aid of (39) and (38) this formula can also be expressed as follows, taking into account that the substitution of $x \to \infty$ into the product of successive operators prescribes to replace only the first operator $O_n^{\pm}$ by $Q_n^{\pm}$,

$$
Q_n^+ \frac{1}{1 - O_n^- O_n^+} A = - \left[R + R Q_n^+ O_n^- \frac{1}{1 - O_n^+ O_n^-} \right] A \; . \tag{41}
$$

This formula will be used below.

Each individual term of the Bremmer series for the solution u_n of the equation (11b) for the n-th order Liouville transformation can be derived from repeated applications of the equations (8) of Ref. 3. The resulting series then proves to be representable by:

$$\bar{u}_n(\tau)\uparrow = \left[1 - O_n^- P_n^+ + O_n^- P_n^+ O_n^- P_n^+ - \dots\right] A = \frac{1}{1 + O_n^- P_n^+} A \quad , \tag{42a}$$

$$\bar{u}_n(\tau)\downarrow = \left[- P_n^+ + P_n^+ O_n^- P_n^+ - \dots\right] A = - P_n^+ \frac{1}{1 + O_n^- P_n^+} A \quad ; \tag{42b}$$

here the quantities $\bar{u}_n\uparrow$ and $\bar{u}_n\downarrow$ are the WKB amplitudes of the wave $u\uparrow$ propagating towards $\tau \to +\infty$ and the wave $u\downarrow$ propagating towards $\tau \to -\infty$, respectively ($u = u\uparrow + u\downarrow$),

$$u(\tau)\uparrow = \frac{\bar{u}_n(\tau)\uparrow}{F_n(x,\varepsilon)^{\frac{1}{2}}} \exp\left\{\frac{i}{\varepsilon}\int^x F_n(x',\varepsilon)dx'\right\} \quad ,$$

$$u(\tau)\downarrow = \frac{\bar{u}_n(\tau)\downarrow}{F_n(x,\varepsilon)^{\frac{1}{2}}} \exp\left\{-\frac{i}{\varepsilon}\int^x F_n(x',\varepsilon)dx'\right\} \quad . \tag{43}$$

To obtain the representation (42) and (43) we made use of the relations $u_n = uF_{n-1}^{\frac{1}{2}}$ and $f_n d\tau_n = F_n/\varepsilon \, dx$ following from (13), in order to express all quantities in terms of our original variables x and τ, instead of x_n and τ_n.

It can easily be shown that the vectors $(\bar{u}_n\uparrow, \bar{u}_n\downarrow)$ and $(\bar{v}_n, \bar{w}_n)$ satisfy the same vectorial differential equations (26a). Hence, they should be identical, since they are also subject to the same boundary conditions at $x = \pm\infty$. However, we shall prove directly that the amplitudes $\bar{u}_n\uparrow$, $\bar{u}_n\downarrow$ are identical to $\bar{v}_n$, $\bar{w}_n$, respectively. To that end we shall show that the right-hand sides of (40a) and (42a) and also those of (40b), (42b) are equal.

By applying the operator $1 + O_n^- P_n^+$ to the right of (40a), while making use of the relation $P_n^+ = Q_n^+ - O_n^+$, we find the expression,

$$\left[1 + O_n^- P_n^+\right] \left[\frac{1}{1 - O_n^- O_n^+} + RO_n^- \frac{1}{1 - O_n^+ O_n^-}\right] =$$

$$= O_n^- Q_n^+ \left[\frac{1}{1 - O_n^- O_n^+} + RO_n^- \frac{1}{1 - O_n^+ O_n^-}\right] + 1 + RO_n^- \quad .$$

A substitution of (41) into the first term on the right leads to the identity:

$$\left[1 + O_n^- P_n^+\right] \left[\frac{1}{1 - O_n^- O_n^+} + O_n^- \frac{1}{1 - O_n^+ O_n^-} R\right] A = A \quad .$$

Thus the application of the operator $1 + O_n^- P_n^+$ to (40a) yields

$$\left[1 + O_n^- P_n^+\right] \bar{v}_n(\tau) = A \quad , \tag{44}$$

or

$$\bar{v}_n(\tau) = \frac{1}{1 + O_n^- P_n^+} A \quad , \tag{45}$$

provided that the series, thus defined, converges. Hence, in view of (42a), $\bar{u}_n(\tau)\uparrow = \bar{v}_n(\tau)$ holds indeed. In a similar way we can deduce that $\bar{u}_n\downarrow = \bar{w}_n$.

As a consequence, we conclude that the solution (40) is equivalent to the Bremmer type solution (42). Obviously, this conclusion can only be drawn if both series expansions (40) and (42) are convergent. In view of (36) a sufficient condition for the convergence of the series expansions occurring in (40) is given by

$$\int_{-\infty}^{+\infty} dx \left| \frac{1}{2f_n} \frac{df_n}{dx} \right| < \infty \quad , \tag{46}$$

while, applying the derivation of ATKINSON[13], we obtain the much more restrictive condition for the convergence of the Bremmer solution (42):

$$\int_{-\infty}^{+\infty} dx \left| \frac{1}{2f_n} \frac{df_n}{dx} \right| \leq \tfrac{1}{2}\pi \quad . \tag{47}$$

III.6 Conclusions concerning the asymptotic behaviour of the solutions

The majorizing expansions (35) lead to an estimate of the magnitude of the elements ${}_nU^x_{x_a}$, given by (33), viz.:

$$\begin{aligned} \left|{}_nU^x_{x_a}{}_{11}-1\right| &= \left|{}_nU^x_{x_a}{}_{22}-1\right| \leq m_n(x,x_a)\ \sinh M_n(x,x_a) \\ \left|{}_nU^x_{x_a}{}_{12}\right| &= \left|{}_nU^x_{x_a}{}_{21}\right| \leq m_n(x,x_a)\ \cosh M_n(x,x_a) \quad . \end{aligned} \tag{48}$$

These upper estimates are rather pessimistic since they are based on a neglect of all effects of phase interferences occurring in (33).

Since $\max l_n(x,\varepsilon) = O(\varepsilon^{2n})$, it follows from (34), remembering that $x - x_a = \varepsilon(\tau - \tau_a)$, that

$$m_n(x,x_a) \leq M_n(x,x_a) = O(\varepsilon^{2n}|x-x_a|) = O(\varepsilon^{2n+1}|\tau-\tau_a|) \quad . \tag{49}$$

By combining this relation with (48) and (30), we obtain the following estimates for the WKB amplitudes $\bar{v}_n$ and $\bar{w}_n$:

$$\bar{v}_n(\tau) - \bar{v}_n(\tau_a) = O(\varepsilon^{2n+1}|\tau-\tau_a|) \quad ,$$

$$\bar{w}_n(\tau) - \bar{w}_n(\tau_a) = O(\varepsilon^{2n+1}|\tau-\tau_a|) \quad . \tag{50}$$

Hence, the amplitudes of n-th order WKB solutions of (1) are constants, up to a special order ε^p ($p \geq 1$), for a bounded interval of the independent variable, viz. for the interval $|\tau-\tau_a| = O(\varepsilon^{p-2n-1})$.

A substitution of (49) into (28) leads to the solution of (1) in the following form

$$u(\tau) = \frac{\bar{v}_n(\tau_a)}{F_n^{\frac{1}{2}}(x,\varepsilon)} \exp\left\{ \frac{i}{\varepsilon} \int^x F_n(x',\varepsilon)dx' \right\} + $$
$$+ \frac{\bar{w}_n(\tau_a)}{F_n^{\frac{1}{2}}(x,\varepsilon)} \exp\left\{ - \frac{i}{\varepsilon} \int^x F_n(x',\varepsilon)dx' \right\} + O(\varepsilon^{2n+1}|\tau-\tau_a|) . \tag{51}$$

The function F_n can be derived from (14) by 2n repeated differentiations. By developing the resulting expressions into powers of ε^2, we obtain F_n up to order ε^{2n}. Hence, for a given finite interval of the independent variable τ, equation (51) yields the solution of the original equation (1) to any desired power in the smallness parameter when taking n sufficiently large.

We now again consider the situation discussed in subsection 2.3, and thus assume that the system slowly passes from one steady state into another in a finite interval of τ:

$$f(x) = a \text{ for } x \leq x_1 \quad , \quad f(x) = b \text{ for } x \geq x_2 \quad .$$

We then conclude from (19) and (26) that the n-th order WKB amplitudes are constant on $\tau \leq \tau_1$ and $\tau \geq \tau_2$, since $l_n(x,\varepsilon) = 0$ for $x \leq x_1$ and $x \geq x_2$. From (34) we now again find (49), however, with $\tau - \tau_a$ replaced by $\tau_1 - \tau_2$ on the right-hand side. Hence, (50) transforms into

$$\bar{v}_n(\tau) - \bar{v}_n(\tau \geq \tau_2) = O(\varepsilon^{2n+1}|\tau_1 - \tau_2|) \quad ,$$

$$\bar{w}_n(\tau) - \bar{w}_n(\tau \geq \tau_2) = O(\varepsilon^{2n+1}|\tau_1 - \tau_2|) \quad . \tag{52}$$

Introducing the difference,

$$\Delta I_k \equiv I_k(\tau \leq \tau_1) - I_k(\tau \geq \tau_2) \quad ,$$

we infer from (20) that $\Delta I_k = \Delta I_n$, so that a substitution of (52) into (29) yields

$$\Delta I_k = \Delta I_n = 2\left[\bar{v}_n(\tau \le \tau_1)\ \bar{w}_n(\tau \le \tau_1) - \bar{v}_n(\tau \ge \tau_2)\ \bar{w}_n(\tau \ge \tau_2)\right]$$

$$= O(\varepsilon^{2n+1}|\tau_1 - \tau_2|) \quad , \quad (0 \le k \le n) \quad . \tag{53}$$

Since we assumed that $\tau_1 - \tau_2$ is bounded, this means that the total change of either of the quantities $I_k(\tau)$ defined by (2) and (18), vanishes to all orders in the smallness parameter. These quantities are thus adiabatic constants of motion.

Let the original equation (1) again describe the propagation of a plane wave through a stratified medium. Outside the interval $\tau_1 \le \tau \le \tau_2$ the amplitudes $\bar{v}_n$, $\bar{w}_n$ are constants. There, we may then identify (compare (28)) $\bar{v}_n$ with the WKB amplitude of the incoming wave, and $\bar{w}_n$ with that of the reflected wave. The interpretation of the total solution as being composed inside the mentioned interval of two waves v_n, w_n propagating in opposite directions, is not completely justified. This is due to the fact that the amplitudes $\bar{v}_n(\tau)$ and $\bar{w}_n(\tau)$ are complex functions containing an additional phase factor depending on τ. However, since the amplitudes prove to be constant up to any order in the smallness parameter, the indefiniteness of the interpretation in question is connected with the neglected terms.

Requiring a vanishing of the reflected wave for $\tau \ge \tau_2$, i.e. $\bar{w}_n(\tau \ge \tau_2) = 0$, it follows from (52) that

$$\bar{w}_n(\tau) = O(\varepsilon^{2n+1}|\tau_1 - \tau_2|) \quad . \tag{54}$$

The reflection from the stratified medium thus proves to be zero to any order in the parameter of smallness. Following a different approach this result has also been obtained by VAN KAMPEN[12)].

III.7 Application to the inhomogeneous Helmholtz equation

The method described in the preceding sections can also be applied to inhomogeneous second-order differential equations. In order to show this we shall consider the equation

$$\frac{d^2u}{d\tau^2} + f^2(x)\ u = k(\tau) \quad , \quad \frac{dx}{d\tau} = \varepsilon \quad . \tag{55}$$

This equation, e.g., describes the motion of a charged particle in the combination of a slowly varying magnetic field and the field of a high-frequency electromagnetic wave (compare (16) of Chapter I).

Applying the transformation (13) to (55), we find, instead of (11b), the following equations:

$$\frac{d^2u_n}{d\tau_n^2} + f_n^2\, u_n = k_n \quad , \quad \frac{dx_n}{d\tau_n} = \varepsilon \quad , \tag{56a}$$

with

$$k_n \equiv F_n^{-3/2}\, k \quad , \tag{56b}$$

the functions f_n and F_n again being given by (14) and (15), respectively.

With the aid of the splitting (21) and the definition (25), we now find, instead of (26a), the following set of coupled first-order differential equations:

$$\frac{d}{d\tau}\begin{bmatrix}\bar{v}_n \\ \bar{w}_n\end{bmatrix} = \varepsilon\, L_n(x,\varepsilon)\begin{bmatrix}\bar{v}_n \\ \bar{w}_n\end{bmatrix} + \frac{k(\tau)}{2F_n^{\frac{1}{2}}(x,\varepsilon)}\begin{bmatrix} -i\,\exp\left\{\frac{-i}{\varepsilon}\int^{x} F_n dx'\right\} \\ i\,\exp\left\{\frac{i}{\varepsilon}\int^{x} F_n dx'\right\}\end{bmatrix} \quad , \tag{57a}$$

$$\frac{dx}{d\tau} = \varepsilon \quad , \tag{57b}$$

the matrix L_n being given by (26b).

The general solution of (57) can be expressed as follows in terms of the evolution matrix ${}_nU^{x}_{x_a}$ given by (32):

$$\begin{bmatrix}\bar{v}_n(\tau) \\ \bar{w}_n(\tau)\end{bmatrix} = {}_nU^{x}_{x_a}\begin{bmatrix}\bar{v}_n(\tau_a) \\ \bar{w}_n(\tau_a)\end{bmatrix} + \int_{\tau_a}^{\tau} d\tau'\,\frac{k(\tau')}{2F_n^{\frac{1}{2}}(x',\varepsilon)}\,{}_nU^{x}_{x'}\begin{bmatrix} -i\,\exp\left\{\frac{-i}{\varepsilon}\int^{x'} F_n dx''\right\} \\ i\,\exp\left\{\frac{i}{\varepsilon}\int^{x'} F_n dx''\right\}\end{bmatrix} \quad , \tag{58a}$$

with

$$x = x_a + \varepsilon(\tau-\tau_a) \quad . \tag{58b}$$

The solution of (55) can still be represented by the formula (28), but now with the amplitudes $\bar{v}_n(\tau)$ and $\bar{w}_n(\tau)$ given by (58a).

Combining (49) with (48), we find

$${}_nU^{x}_{x_a} = O(\varepsilon^{2n+1}|\tau-\tau_a|) \quad . \tag{59}$$

A substitution of (59) into (58) results in expressions for the amplitudes $\bar{v}_n(\tau)$ and $\bar{w}_n(\tau)$, that are correct up to any order in the smallness parameter. However, due to the external force $k(\tau)$, the quantity $I(\tau)$ defined by (2), is not an adiabatic constant anymore.

A combination of (59), (58a) and (28) leads to the following form for the solution of (55),

$$u(\tau) = \left[\bar{v}_n(\tau_a) - i \int_{\tau_a}^{\tau} d\tau' \frac{k(\tau')}{2F_n^{\frac{1}{2}}(x',\varepsilon)} \exp\left\{\frac{-i}{\varepsilon} \int^{x'} F_n dx''\right\}\right] \frac{\exp\left\{\frac{i}{\varepsilon} \int^{x} F_n dx'\right\}}{F_n^{\frac{1}{2}}(x,\varepsilon)} +$$

$$+ \left[\bar{w}_n(\tau_a) + i \int_{\tau_a}^{\tau} d\tau' \frac{k(\tau')}{2F_n^{\frac{1}{2}}(x',\varepsilon)} \exp\left\{\frac{i}{\varepsilon} \int^{x'} F_n dx''\right\}\right] \frac{\exp\left\{\frac{-i}{\varepsilon} \int^{x} F_n dx'\right\}}{F_n^{\frac{1}{2}}(x,\varepsilon)} +$$

$$+ O(\varepsilon^{2n+1}|\tau-\tau_a|) \quad . \tag{60}$$

When the force $k(\tau)$ is a harmonic function of τ, e.g. $k(\tau) \exp i \int^{\tau} \omega(\tau')d\tau'$, the right-hand side of (60) leads to resonant contributions near the points where $\omega \pm F_n = 0$. Since $F_n(x,\varepsilon) = f(x) + O(\varepsilon^2)$, the shift of such resonances for small but finite values of ε with respect to those for $\varepsilon = 0$, will be of second-order in the smallness parameter. In addition we notice that, in the case of finite ε, also an amplitude modification of $k(\tau)$ occurs that is due to the factor $F_n^{\frac{1}{2}}$ in the denominator of the τ' integral.

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C U R R I C U L U M   V I T A E

De auteur van dit proefschrift studeerde van 1957 tot 1965 elektrotechniek aan de Technische Hogeschool te Delft. Zijn afstudeerwerk betrof het berekenen van de verstrooiing van elektromagnetische golven aan een obstakel in een golfpijp. In 1965 trad hij in dienst van de Stichting voor Fundamenteel Onderzoek der Materie. Na aanvankelijk werkzaam te zijn geweest bij het FOM-Instituut voor Plasmafysica te Jutphaas, ging hij in 1970 over naar de Werkgroep TN-1 die is ondergebracht in hetzelfde instituut. In deze werkgroep werd het werk verricht dat in dit proefschrift is beschreven.

S T E L L I N G E N

I

De constanten van de beweging die in hoofdstuk II van dit proefschrift worden gebruikt zijn vorm-invariant voor Lorentz-transformaties. Dit in tegenstelling tot die welke in de literatuur worden gebruikt.

II

In de literatuur betreffende cyclotronresonantie wordt het begrip "gevangen deeltje" verschillend gedefinieerd. Het verdient aanbeveling om naar analogie met elektrostatische golven een deeltje te beschouwen als "gevangen" indien het periodiek om resonantie oscilleert.

R.F. Lutomirski and R.N. Sudan, Phys. Rev. 147, 156 (1966).
P. Palmadesso and G. Schmidt, Phys. Fluids 14, 1411 (1971).
M.J. Laird, J. Plasma Phys. 8, 255 (1972).

III

Juist bij Tokamak-experimenten, waarin de plasmastroom wordt geïnduceerd d.m.v. een luchtkerntransformator en waarbij de ruimte tussen de primaire windingen en de secundaire (plasma)winding grotendeels wordt gevuld door het koper van de magneetspoelen voor het opwekken van het toroïdale veld, is een goede voorionisatie van groot belang. Indien dan een initiële vertraging in de aflevering van de primaire flux mogelijk is, zal door de snellere stijging van de stroom na de opbouw van het plasma een nuttiger gebruik gemaakt worden van de aangeboden primaire flux.

IV

Ionentemperaturen beneden 1 eV kunnen in een heliumplasma met een nauwkeurigheid beter dan 0.1 eV bepaald worden uit de verhoudingen van maxima en minimum in het Doppler-verbrede profiel van de He II 4687 Å-lijn.

V

De energie-constanten die door Davidson & Hammer zijn afgeleid van de Vlasov-vergelijking gelden met de nodige veranderingen ook voor de relativistische Vlasov-vergelijking.

R.C. Davidson and D.A. Hammer, Phys. Fluids 15, 1282 (1972).

VI

Indien het hoogfrequente veld afkomstig is van een circulair gepolariseerde golf waarvan de amplitude tijdafhankelijk is, b.v. een gedempte niet-lineaire "whistler", dan beschrijft één van de in de vorige stelling genoemde constanten dat de toename van de gegeneraliseerde azimutale impulsdichtheid evenredig is met de toename van de axiale impulsdichtheid.

VII

Bij de beschrijving in het MHD-model van het stationaire evenwicht van een cilinder-symmetrische plasmakolom, wordt het radiële elektrische veld onbepaald indien de traagheidstermen ab initio worden verwaarloosd.

VIII

De fysische gronden waarop Roberts & Potter besluiten tot de afwezigheid van krachtvrije stromen in het gebied buiten een dichte plasmakolom zijn ontoereikend.

K.V. Roberts and D.E. Potter, in Methods in Computational Physics vol. 9, Academic Press (1970).

IX

De ontwerpers van de toekomstige organisatie van het wetenschappelijk onderzoek hebben een te beperkte opvatting van hun verantwoordelijkheid indien zij voorstellen doen over de wijze van financiering van het onderzoek en een daarmee gekoppeld doorstromingsbeleid t.a.v. het wetenschappelijk personeel zonder aandacht te besteden aan de maatschappelijke gevolgen voor de onderzoekers.

Nota van de Gespreksgroep Universitair Onderzoek (1972).

X

De voortdurende en minutieuze zelfportrettering door Paul Léautaud is een bewijs van zijn groot talent.

Eindhoven, 6 februari 1973 T.J. Schep