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SPECTRAL ANALYSIS OF INTEGRAL-DIFFERENTIAL OPERATORS APPLIED IN LINEAR ANTENNA MODELING

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Abstract. The current on a linear strip or wire solves an equation governed by a linear integral-differential operator that is the composition of the Helmholtz operator and an integral operator with logarithmically singular displacement kernel. We investigate the spectral behavior of this classical operator, particularly because various methods of analysis and solution rely on asymptotic properties of spectra, while no investigations of the spectrum of this operator seem to exist. In our approach, we first consider the composition of the second order differentiation operator and the integral operator with logarithmic displacement kernel. Employing the Weyl-Courant minimax principle and properties of the Čebysev polynomials of the first and second kind, we derive index-dependent bounds for the ordered sequence of eigenvalues of this operator and specify their ranges of validity. Additionally we derive bounds for the eigenvalues of the integral operator with logarithmic kernel. With slight modification our result extends to kernels that are the sum of the logarithmic displacement kernel and a real displacement kernel whose second derivative is square integrable. Employing this extension we derive bounds for the eigenvalues of the integral-differential operator of a linear strip with the complex kernel replaced by its real part. Finally, for specific geometry and frequency settings, we present numerical results for the eigenvalues of the considered operators using Ritz's methods with respect to finite bases.

Key words. Eigenvalue problems, integral-differential operators, logarithmic kernel, linear antennas, wire antennas, microstrip antennas

1. Introduction. Spectral analysis is one of the tools to obtain insight in the electromagnetic behavior of antennas and microwave components. The analysis of the eigenmodes of a rectangular waveguide that are obtained from Maxwell's equations by applying Sturm-Liouville theory to electric and magnetic scalar potentials is an example [1]. The eigenvalues corresponding to the eigenmodes are directly related to their cut-off frequencies, i.e., frequencies above which the modes propagate. A second example concerns the analysis of antenna arrays, where the eigenfunctions are standing waves that represent specific scan and resonant behavior of the array. The corresponding eigenvalues are characteristic impedances; they predict resonance phenomena, which are related to the occurrence of surface waves supported by the truncated periodic structure [2–4]. For overviews with other examples we refer to [5,6].

Apart from the electromagnetic insight, existing calculational methods rely on preknowledge of the spectrum [3, 7–12]. Generally, a sufficiently fast decay of the eigenvalues or their reciprocals is assumed to limit the numbers of eigenfunctions in the spectral transformations for single scatterers and arrays. In [3, 11, 12] these numbers are chosen on basis of physical insight or emperical rules derived from numerical results. In this respect we emphasize that only for relatively simple shapes, such as the rectangular waveguide or a loop antenna [13], the spectral transformation is analytically known. For problems that require numerical techniques to obtain this transformation, spectral analysis can provide a basis for its approximation by emperical and physical insight. In this paper we concentrate on properties of the spectrum of a linear antenna, where such techniques are required [3, 12].

One of the most common linear antennas is a straight, good conducting wire or strip of approximately half a wavelength, referred to as dipole. The wire diameter and

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the strip thickness and strip width are small with respect to the wavelength and the dipole length. For the indicated lengths, the linear antenna carries a sinusoidal current distribution of half a period. For larger lengths the dipole turns into a multipole that carries currents of more periods, while for much smaller lengths it turns into a monopole. Focusing first on a linear strip, we outline the derivation of an equation for the current, where we apply the classical assumptions that the electromagnetic field is time harmonic and that the metal is perfectly conducting. Since the strip thickness is much smaller than its width and the wavelength, we model the strip as an infinitely thin sheet. Then, introducing a magnetic vector potential we express the scattered electric field in terms of the current by Maxwell's equations. Invoking the condition that the total tangential electric field vanishes at the strip surface, we obtain an integral-differential equation, the electric field integral equation (EFIE), that relates the current to the tangential excitation field. Since the strip width is much smaller than the wavelength, we average the current and the tangential excitation field over the strip width. Thus we link the averaged current to the averaged tangential excitation field by the operator [3, Sec. 2.3.2]

$$\mathcal{Z}w = \frac{1}{2}\, iZ_0 k^2 \ell b \left(1 + \frac{1}{k^2\ell^2}\frac{d^2}{dx^2}\right)\mathcal{G}w\,, \tag{1.1}$$

where 2ℓ and $2b$ are the dipole length and width, k is the wave number, $Z_0 = \sqrt{\mu_0/\varepsilon_0}$ is the characteristic impedance of free space, x is the length coordinate normalized on ℓ, w is the width-averaged current, and $\mathcal{G}$ is the integral operator defined by

$$(\mathcal{G}w)(x) = \int_{-1}^{1} w(\xi) G(x-\xi)\, d\xi\,. \tag{1.2}$$

The displacement kernel G of this operator is defined by

$$G(x) = \frac{1}{2\pi k\ell}\int_0^2 (2-y)\frac{\exp\left(ik\ell\sqrt{x^2+\beta^2y^2}\right)}{\sqrt{x^2+\beta^2y^2}}\, dy\,, \qquad \beta = b/\ell\,. \tag{1.3}$$

and decomposes as

$$G(x) = -\frac{1}{\pi k\ell\beta}\log|x| + G_{\text{reg}}(x)\,, \tag{1.4}$$

where G_{reg} is even and once differentiable with a square integrable derivative [3, Sec. 2.3.2, App. A.1]. For linear wires, a similar expression for the integral-differential operator is obtained in the literature, where the current is averaged with respect to the wire circumference. The corresponding equation is called Pocklington's equation with exact kernel, which has a similar decomposition as (1.4) [14,15]. Recently the decomposition $F_1(z)\log|z| + F_2(z)$ has been proposed with F_1 and F_2 analytic functions on the real line [16]. In other modeling approaches for linear wire antennas the result is an equation with a continuous kernel, which is called the reduced kernel. Contrary to the equation with the exact kernel [17], the equation with the reduced kernel is driven by a compact operator and therefore ill-posed [18], [19]. For justifications of the approximations made in the derivations of both kernels we refer to [20] and [21].

Several investigations of spectra of integral operators related to the integral operator $\mathcal{G}$ can be found in the literature. Reade [22] derived upper and lower bounds for the integral operators generated by the kernels $\log|x-\xi|$ and $|x-\xi|^{-\alpha}$ with

$(x,\xi) \in [-1,1] \times [1,1]$ and $0 < \alpha < 1$. These kernels were considered earlier by Richter, who characterized the singularities in the solutions of the corresponding integral equations [23]. Asymptotic expressions for the eigenvalues of the slightly modified kernel $V(y)|x-\xi|^{-\alpha}$ were derived by Kac [24]. Estrada and Kanwal [25, Lemmas A.1 and A.2] proved two results for eigenvalue bounds of positive compact operators and applied them to kernels considered by Reade. Dostaníc [26] derived asymptotic expressions for the eigenvalues related to the kernel $|x-\xi|^{-\alpha}$ and put them in correspondence with the Riemann zeta function. Simíc [27] followed similar lines as Dostaníc to derive asymptotic upper and lower bounds for the singular values of the integral operator with kernel $\log^{\beta}|x-\xi|^{-1}$ with $0 < \xi < x < 1$ and $\beta > 0$.

No investigations seem to exist of spectra of integral-differential operators related to $\mathcal{Z}$, in particular the composition of the second-order differentiation operator and an integral operator with displacement kernel $\log|x-\xi|$. Given the aforementioned approximations of the spectral transformation in several methods of analysis and solution, particularly [3, 11, 12], the objective of our paper is the asymptotics of the eigenvalues of the integral-differential operator $\mathcal{Z}$ and related operators. In our approach we consider the integral operator $\mathcal{K}$ on the Hilbert space $\mathfrak{L}_2([-1,1])$,

$$(\mathcal{K}f)(x) = \int_{-1}^{1} \log|x-\xi|\, f(\xi)\, d\xi\,, \tag{1.5}$$

and the integral-differential operator $\frac{d^2}{dx^2}\mathcal{K}$ on the domain

$$\mathfrak{W} = \{ f \in \mathfrak{H}_{2,1}([-1,1]) \mid f(-1) = f(1) = 0 \}\,. \tag{1.6}$$

According to Reade [22], $\mathcal{K}$ is a compact self-adjoint operator with negative eigenvalues $\lambda_n(\mathcal{K})$ that satisfy

$$\frac{\pi}{4n} \le |\lambda_n(\mathcal{K})| \le \frac{\pi}{n-1}\,, \tag{1.7}$$

where the eigenvalues are indexed according to decreasing magnitude starting from $n = 1$. Validity of (1.7) for $n \ge n_0$ is not specified by Reade. In this paper we specify n_0 and put his result in a more general perspective. By this generalization we prove that $\frac{d^2}{dx^2}\mathcal{K}$, with domain $\mathfrak{W}$, extends to a positive self-adjoint operator with compact inverse and that the ordered sequence of eigenvalues $\lambda_n\left(\frac{d^2}{dx^2}\mathcal{K}\right)$, $n = 1, 2, \ldots$, satisfies $\lambda_n \ge \pi n$ for $n \ge 1$ and $\lambda_n \le \pi(4n-2)$ for $n \ge 2$. These results extend with a slight modification to integral operators $\tilde{\mathcal{K}}$ with displacement kernels of the form

$$\tilde{k}(x-\xi) = \log|x-\xi| + h(x-\xi)\,, \tag{1.8}$$

where h is real, even, and twice differentiable. Employing this modified result we derive bounds for the eigenvalues of the integral-differential operator $\mathcal{Z}$ with kernel G_{reg} replaced by its real part. In the last section of this paper we compare our analytic approach to numerical results for the eigenvalues of the considered operators.

2. Prerequisites. The Weyl-Courant minimax principle for positive, or nonnegative, compact operators is formulated in [28, Ch. 2, Sec. 1].

THEOREM 2.1. *Let $\mathcal{C}$ be a positive self-adjoint compact operator with eigenvalues $\lambda_1(\mathcal{C}) \ge \lambda_2(\mathcal{C}) \ge \ldots \ge 0$. Then,*

$$\lambda_n(\mathcal{C}) = \min_{\mathcal{F}} \|\mathcal{C} - \mathcal{F}\| \tag{2.1}$$

where the minimum is taken over all finite rank operators $\mathcal{F}$ with rank $\leq n-1$.
The principle has two consequences:

COROLLARY 2.2. *Let $\mathcal{C}$ be a compact positive self-adjoint operator and let $\mathcal{B}$ be a bounded operator. Then $\lambda_n(\mathcal{B}\mathcal{C}\mathcal{B}^*) \leq \|\mathcal{B}\|^2 \lambda_n(\mathcal{C})$.*

Proof. The chain of inequalities

$$\lambda_n(\mathcal{B}\mathcal{C}\mathcal{B}^*) = \min_{\mathcal{F}} \|\mathcal{B}\mathcal{C}\mathcal{B}^* - \mathcal{F}\| \leq \inf_{\mathcal{F}} \|\mathcal{B}\mathcal{C}\mathcal{B}^* - \mathcal{B}\mathcal{F}\mathcal{B}^*\| \leq \|\mathcal{B}\|^2 \min_{\mathcal{F}} \|\mathcal{C} - \mathcal{F}\| \tag{2.2}$$

proves the statement. □

COROLLARY 2.3. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be compact positive self-adjoint operators such that $\mathcal{C}_1 \geq \mathcal{C}_2$, i.e.,*

$$\forall f : \langle \mathcal{C}_1 f, f \rangle \geq \langle \mathcal{C}_2 f, f \rangle \,. \tag{2.3}$$

Let the eigenvalues of both operators be ordered as $\lambda_1(\mathcal{C}_N) \geq \lambda_2(\mathcal{C}_N) \geq \lambda_1(\mathcal{C}_{N-1}) \geq \ldots \geq 0$, $N = 1, 2$. Then, $\lambda_n(\mathcal{C}_1) \geq \lambda_n(\mathcal{C}_2)$ for all n.

Proof. Let $\mathcal{P}_n$ be the orthogonal projection onto the linear span of the eigenvectors corresponding to $\lambda_1(\mathcal{C}_1), ..., \lambda_{n-1}(\mathcal{C}_1)$. Since $\mathcal{C}_1$ and $\mathcal{P}_n$ commute, $\mathcal{C}_1(\mathcal{I} - \mathcal{P}_n)$ is self-adjoint. Then, its spectral radius equals its norm [30, pp. 391,394] and its norm equals its numerical radius [30, p. 466]. Consequently,

$$\lambda_n(\mathcal{C}_1) = \|\mathcal{C}_1(\mathcal{I} - \mathcal{P}_n)\| = \max_{f, \|f\|=1} \langle \mathcal{C}_1(\mathcal{I} - \mathcal{P}_n) f, f \rangle \,, \tag{2.4}$$

where we write the maximum instead of the supremum, because $\mathcal{C}_1(\mathcal{I}-\mathcal{P}_n)$ is compact. By decomposing $f = \mathcal{P}_n f + (\mathcal{I} - \mathcal{P}_n) f$, we readily observe that $\langle \mathcal{C}_1(\mathcal{I} - \mathcal{P}_n) f, f \rangle = \langle \mathcal{C}_1(\mathcal{I} - \mathcal{P}_n) f, (\mathcal{I} - \mathcal{P}_n) f \rangle$. Then, substituting this result in (2.4) and subsequently applying the assumption $\mathcal{C}_1 \geq \mathcal{C}_2$, we derive

$$\begin{aligned} \lambda_n(\mathcal{C}_1) = \max_{f, \|f\|=1} \langle \mathcal{C}_1(\mathcal{I} - \mathcal{P}_n) f, (\mathcal{I} - \mathcal{P}_n) f \rangle &\geq \max_{f, \|f\|=1} \langle \mathcal{C}_2(\mathcal{I} - \mathcal{P}_n) f, (\mathcal{I} - \mathcal{P}_n) f \rangle = \\ = \|(\mathcal{I} - \mathcal{P}_n)\mathcal{C}_2(\mathcal{I} - \mathcal{P}_n)\| &= \|\mathcal{C}_2 - (\mathcal{P}_n\mathcal{C}_2 + \mathcal{C}_2\mathcal{P}_n - \mathcal{P}_n\mathcal{C}_2\mathcal{P}_n)\| \,. \end{aligned} \tag{2.5}$$

Since $\mathcal{P}_n\mathcal{C}_2 + \mathcal{C}_2\mathcal{P}_n - \mathcal{P}_n\mathcal{C}_2\mathcal{P}_n$ has finite rank n, it follows from Theorem 2.1 that $\lambda_n(\mathcal{C}_1) \geq \lambda_n(\mathcal{C}_2)$. □

As we noted in the introduction, our techniques are closely related to the ones used by Reade, who employs properties of the Čebysev polynomials. Since we want to keep the paper self-contained, we introduce these properties starting with the definition of the Čebysev polynomials $\{T_n\}_{n=0}^{\infty}$ and $\{U_n\}_{n=0}^{\infty}$:

$$T_n(\cos\theta) = \cos n\theta \,, \qquad U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta} \,, \qquad n = 0, 1, 2, \ldots . \tag{2.6}$$

The polynomials satisfy the orthogonality relations

$$\int_{-1}^{1} T_n(x)\, T_m(x) \frac{1}{\sqrt{1-x^2}}\, dx = \begin{cases} \pi, & (n,m) = (0,0), \\ \dfrac{\pi}{2}\, \delta_{nm}, & (n,m) \neq (0,0), \end{cases} \tag{2.7}$$

and

$$\int_{-1}^{1} U_n(x)\, U_m(x) \sqrt{1-x^2}\, dx = \frac{\pi}{2}\, \delta_{nm} \,. \tag{2.8}$$

Correspondingly we introduce two complete orthogonal sequences in $\mathfrak{L}_2\,([-1,1])$:

$$\hat{T}_n(x) = (1-x^2)^{-\frac{1}{4}}\, T_n(x)\,, \quad \hat{U}_n(x) = (1-x^2)^{\frac{1}{4}}\, U_n(x)\,, \quad n = 0,1,\dots. \tag{2.9}$$

Further, for $\nu \in \mathbb{R}$ we introduce the self-adjoint multiplication operator $\mathcal{M}_\nu$ in $\mathfrak{L}_2([-1,1])$ by $\mathcal{M}_\nu f = (1-x^2)^\nu f$. For $\nu \geq 0$ the operator is bounded with $\|\mathcal{M}_\nu\| = 1$, i.e., the essential supremum of the function $(1-x^2)^\nu$ on the interval $[-1,1]$. From the goniometric formulas

$$\cos n\theta \sin\theta = \frac{1}{2}(\sin(n+1)\theta - \sin(n-1)\theta), \tag{2.10}$$

$$\sin\theta \sin(n+1)\theta = \frac{1}{2}(\cos n\theta - \cos(n+2)\theta), \tag{2.11}$$

we derive the relations

$$\mathcal{M}_{\frac{1}{2}}\hat{T}_0 = \hat{U}_0, \quad \mathcal{M}_{\frac{1}{2}}\hat{T}_1 = \frac{1}{2}\hat{U}_1\,, \quad \mathcal{M}_{\frac{1}{2}}\hat{T}_n = \frac{1}{2}\left(\hat{U}_n - \hat{U}_{n-2}\right), \quad n = 2,3,\dots\,, \tag{2.12}$$

$$\mathcal{M}_{\frac{1}{2}}\hat{U}_n = \frac{1}{2}\left(\hat{T}_n - \hat{T}_{n+2}\right), \quad n = 0,1,\dots\,. \tag{2.13}$$

3. Asymptotic behavior of eigenvalues of integral operators described by Čebysev polynomial expansions. In this section we study the asymptotics of the eigenvalues of the integral operators on the Hilbert space $\mathfrak{L}_2([-1,1])$ related to the following two types of kernels

$$k_1(x,\xi) = \sum_{n=0}^{\infty} \alpha_n T_n(x) T_n(\xi) \tag{3.1}$$

and

$$k_2(x,\xi) = \sum_{n=0}^{\infty} \alpha_n U_n(x) U_n(\xi) \sqrt{1-x^2}\sqrt{1-\xi^2}\,. \tag{3.2}$$

Here the sequence (α_n) satisfies $\alpha_n \downarrow 0$ as $n \to \infty$, $\alpha_0 \geq \alpha_1 \geq \ldots \geq 0$. The symmetric kernels k_1 and k_2 correspond to the integral operators $\mathcal{K}_1$ and $\mathcal{K}_2$

$$(\mathcal{K}_{1,2}\, f)(x) = \int_{-1}^{1} k_{1,2}(x,\xi)\, f(\xi)\, d\xi\,. \tag{3.3}$$

We define the positive self-adjoint compact operators $\hat{\mathcal{K}}_1$ and $\hat{\mathcal{K}}_2$ on $\mathfrak{L}_2([-1,1])$ by

$$\hat{\mathcal{K}}_1 f = \sum_{n=0}^{\infty} \alpha_n \langle f, \hat{T}_n\rangle_{\mathfrak{L}_2} \hat{T}_n\,, \qquad \hat{\mathcal{K}}_2 f = \sum_{n=0}^{\infty} \alpha_n \langle f, \hat{U}_n\rangle_{\mathfrak{L}_2} \hat{U}_n\,. \tag{3.4}$$

Employing (2.7), (2.8), and (2.9), we find

$$\hat{\mathcal{K}}_1\hat{T}_0 = \pi\alpha_0\hat{T}_0, \qquad \hat{\mathcal{K}}_1\hat{T}_n = \frac{\pi}{2}\alpha_n\hat{T}_n\,, \qquad n = 1,2,\dots\,, \tag{3.5}$$

$$\hat{\mathcal{K}}_2\hat{U}_n = \frac{\pi}{2}\alpha_n\hat{U}_n\,, \qquad n = 0,1,\dots\,. \tag{3.6}$$

The operators $\hat{\mathcal{K}}_1$ and $\hat{\mathcal{K}}_2$ are related to the operators $\mathcal{K}_1$ and $\mathcal{K}_2$ according to

$$\mathcal{K}_1 = \mathcal{M}_{\frac{1}{4}}\hat{\mathcal{K}}_1\mathcal{M}_{\frac{1}{4}}\,, \quad \mathcal{K}_2 = \mathcal{M}_{\frac{1}{4}}\hat{\mathcal{K}}_2\mathcal{M}_{\frac{1}{4}}\,. \tag{3.7}$$

Since $\mathcal{M}_{\frac{1}{4}}$ is self-adjoint and bounded, $\mathcal{K}_1$ and $\mathcal{K}_2$ are compact, positive, and self-adjoint. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ denote the bounded operators on $\mathfrak{L}_2([-1,1])$ defined by $\mathcal{S}_1\hat{T}_n = \hat{T}_{n+2}$ and $\mathcal{S}_2\hat{U}_n = \hat{U}_{n+2}$, with adjoints $\mathcal{S}_1^*$ and $\mathcal{S}_2^*$ that satisfy

$$\mathcal{S}_1^*\hat{T}_0 = \mathcal{S}_1^*\hat{T}_1 = 0, \qquad \mathcal{S}_1^*\hat{T}_2 = \frac{1}{2}\hat{T}_0, \qquad \mathcal{S}_1^*\hat{T}_n = \hat{T}_{n-2}\,,\ n \geq 3, \tag{3.8}$$

$$\mathcal{S}_2^*\hat{U}_0 = \mathcal{S}_2^*\hat{U}_1 = 0, \qquad \mathcal{S}_2^*\hat{U}_n = \hat{U}_{n-2}\,,\ n \geq 2. \tag{3.9}$$

Employing the relations (2.12) – (2.13) we obtain

$$\begin{aligned}
&\mathcal{M}_{\frac{1}{2}}\hat{\mathcal{K}}_1\mathcal{M}_{\frac{1}{2}}f = \\
&= \alpha_0\langle f,\hat{U}_0\rangle_{\mathfrak{L}_2}\hat{U}_0 + \frac{\alpha_1}{4}\langle f,\hat{U}_1\rangle_{\mathfrak{L}_2}\hat{U}_1 + \frac{1}{4}\sum_{n=2}^{\infty}\alpha_n\langle f,\hat{U}_n-\hat{U}_{n-2}\rangle_{\mathfrak{L}_2}(\hat{U}_n-\hat{U}_{n-2}) = \\
&= \frac{3\alpha_0}{4}\langle f,\hat{U}_0\rangle_{\mathfrak{L}_2}\hat{U}_0 + \frac{1}{4}(\mathcal{I}-\mathcal{S}_2{}^*)\hat{\mathcal{K}}_2(\mathcal{I}-\mathcal{S}_2)\,f
\end{aligned} \tag{3.10}$$

and, similarly,

$$\mathcal{M}_{\frac{1}{2}}\hat{\mathcal{K}}_2\mathcal{M}_{\frac{1}{2}} = \frac{1}{4}(\mathcal{I}-\mathcal{S}_1)\hat{\mathcal{K}}_1(\mathcal{I}-\mathcal{S}_1{}^*). \tag{3.11}$$

Let $\mathcal{P}_{1,N}$ and $\mathcal{P}_{2,N}$ denote the orthogonal projections onto the linear spans of $\left\{\hat{T}_n \,|\, n=0,\dots,2N\right\}$ and $\left\{\hat{U}_n \,|\, n=0,\dots,2N\right\}$, respectively. Then $\hat{\mathcal{K}}_2 \geq \frac{\pi}{2}\alpha_{2N}\mathcal{P}_{2,N}$ and $\hat{\mathcal{K}}_1 \geq \frac{1}{2}\pi\alpha_{2N}\mathcal{Q}_{1,N}$, where $\mathcal{Q}_{1,N}f = \mathcal{P}_{1,N}f + \frac{1}{\pi}(f,\hat{T}_0)_{\mathfrak{L}_2}\hat{T}_0$ so that

$$\mathcal{M}_{\frac{1}{2}}\hat{\mathcal{K}}_1\mathcal{M}_{\frac{1}{2}} \geq \frac{\pi}{8}\alpha_{2N}(\mathcal{I}-\mathcal{S}_2{}^*)\mathcal{P}_{2,N}(\mathcal{I}-\mathcal{S}_2) =: \frac{\pi}{8}\alpha_{2N}\mathcal{A}_N \tag{3.12}$$

and

$$\mathcal{M}_{\frac{1}{2}}\hat{\mathcal{K}}_2\mathcal{M}_{\frac{1}{2}} \geq \frac{\pi}{8}\alpha_{2N}(\mathcal{I}-\mathcal{S}_1)\mathcal{Q}_{1,N}(\mathcal{I}-\mathcal{S}_1{}^*) =: \frac{\pi}{8}\alpha_{2N}\mathcal{B}_N\,. \tag{3.13}$$

By Corollary 2.3 it then follows that

$$\lambda_n(\mathcal{M}_{\frac{1}{2}}\hat{\mathcal{K}}_1\mathcal{M}_{\frac{1}{2}}) \geq \frac{\pi}{8}\alpha_{2N}\lambda_n(\mathcal{A}_N)\,, \qquad \lambda_n(\mathcal{M}_{\frac{1}{2}}\hat{\mathcal{K}}_2\mathcal{M}_{\frac{1}{2}}) \geq \frac{\pi}{8}\alpha_{2N}\lambda_n(\mathcal{B}_N)\,, \tag{3.14}$$

where all eigenvalues are indexed according to decreasing magnitude.

Since $\mathcal{A}_N\hat{U}_n = 0$ for $n > 2N$ and

$$\mathcal{A}_N\hat{U}_n = \begin{cases} 2\hat{U}_n - \hat{U}_{n+2}, & n = 0,1, \\ 2\hat{U}_n - \hat{U}_{n+2} - \hat{U}_{n-2}, & n = 2,\dots,2N-2, \\ \hat{U}_n - \hat{U}_{n-2}, & n = 2N-1, 2N, \end{cases} \tag{3.15}$$

its matrix A_N with respect to the orthonormal basis $\{\sqrt{2/\pi}\,\hat{U}_n \,|\, n=0,\dots,2N\}$ with $N \geq 2$ is given by

$$\begin{pmatrix} 2 & 0 & -1 & & & & \\ 0 & 2 & 0 & -1 & & & \\ -1 & 0 & 2 & 0 & -1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 2 & 0 & -1 \\ & & & -1 & 0 & 1 & 0 \\ & & & & -1 & 0 & 1 \end{pmatrix} \tag{3.16}$$

To determine the eigenvalues of A_N, Reade [22] applies the following strategy. First, grouping the even and odd indices of the functions in the basis, he observes that this matrix is similar with the diagonal block matrix with the blocks C_N and C_{N+1} on its diagonal, where C_N corresponds to the odd indices and C_{N+1} corresponds to the even indices. The $N \times N$ matrix C_N is tridiagonal with diagonal $(2, 2, \ldots, 2, 1)$ and codiagonals $(-1, -1, \ldots, -1)$. The characteristic polynomial of the block matrix is

$$\chi_{A_N}(\lambda) = q_N(\lambda) q_{N+1}(\lambda), \tag{3.17}$$

where q_N is the characteristic polynomial of C_N. By taking lower determinants of the first row of C_N it follows that

$$q_N(\lambda) = (2 - \lambda) q_{N-1}(\lambda) - q_{N-2}(\lambda), \tag{3.18}$$

and by taking lower determinants of the last row of C_N it follows that

$$q_N(\lambda) = (1 - \lambda) p_{N-1}(\lambda) - p_{N-2}(\lambda) = p_N(\lambda) - p_{N-1}(\lambda). \tag{3.19}$$

where p_N is the characteristic polynomial of the tridiagonal matrix with diagonal $(2, 2, \ldots, 2)$ and codiagonals $(1, 1, \ldots, 1)$. By taking lower determinants of the first row of this matrix it follows that p_N satisfies

$$p_{N+1}(\lambda) = (2 - \lambda) p_N(\lambda) - p_{N-1}(\lambda), \qquad . \tag{3.20}$$

This recurrence relation, with initial conditions $p_0 = 1$ and $p_1(\lambda) = 2 - \lambda$, is the same as the recurrence relation of the Čebysev polynomials U_N with argument $1 - \lambda/2$. Thus, $p_N(\lambda) = U_N(1-\lambda/2)$. Then, the eigenvalues of C_N follow by substitution of this expression in (3.19) by which $q_N(\lambda) = \cos((2N+1)\theta/2)/\cos(\theta/2)$ with $\lambda = 2(1-\cos\theta)$, see the corollary of Lemma 3 in [22] for details. Then, for $N \geq 1$, the eigenvalues of A_N are those of C_N and those of C_{N+1},

$$\nu^{(1)}_{A_N,m} = 4\cos^2 \frac{\pi m}{2N+1}\,, \qquad m = 1, 2, \ldots, N\,, \tag{3.21}$$

$$\nu^{(2)}_{A_N,m} = 4\cos^2 \frac{\pi m}{2N+3}\,, \qquad m = 1, 2, \ldots, N+1\,, \tag{3.22}$$

We follow similar lines to calculate the eigenvalues of $\mathcal{B}_N$. In detail, since $\mathcal{B}_N \hat{T}_n = 0$ for $n > 2N + 2$ and

$$\mathcal{B}_N \hat{T}_n = \begin{cases} 2(\hat{T}_0 - \hat{T}_2)\,, & n = 0, \\ \hat{T}_1 - \hat{T}_3\,, & n = 1, \\ 2\hat{T}_n - \hat{T}_{n+2} - \hat{T}_{n-2}\,, & n = 2, \ldots, 2N, \\ \hat{T}_n - \hat{T}_{n-2}\,, & n = 2N+1, 2N+2, \end{cases} \tag{3.23}$$

its matrix B_N with respect to the orthonormal basis

$$\left\{ \sqrt{1/\pi}\, \hat{T}_0 \right\} \cup \left\{ \sqrt{2/\pi}\, \hat{T}_n \,|\, n = 1, \ldots, 2N+2 \right\} \tag{3.24}$$

with $N \geq 1$ is given by

$$\begin{pmatrix} 2 & 0 & -\sqrt{2} & & & & & \\ 0 & 1 & 0 & -1 & & & & \\ -\sqrt{2} & 0 & 2 & 0 & -1 & & & \\ & -1 & 0 & 2 & 0 & -1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & -1 & 0 & 2 & 0 & -1 \\ & & & & -1 & 0 & 1 & 0 \\ & & & & & -1 & 0 & 1 \end{pmatrix} \tag{3.25}$$

Grouping the even and odd indices of the functions in the basis, we observe that the matrix is simlar with the diagonal block matrix with the blocks D_{N+2} and E_{N+1} on its diagonal, where the $(N+2) \times (N+2)$ tridiagonal matrix D_{N+2} with diagonal $(2, 2, \ldots, 2, 1)$ and codiagonals $(-\sqrt{2}, -1, -1, \ldots, -1)$ corresponds to the even indices and where the $(N+1) \times (N+1)$ tridiagonal matrix E_{N+1} with diagonal $(1, 2, 2, \ldots, 2, 1)$ and codiagonals $(-1, -1, \ldots, -1)$ corresponds to the odd indices. By taking lower determinants of the first rows of the matrices we derive that the characteristic polynomials of D_{N+2} and E_{N+1} satisfy

$$\chi_{D_{N+2}}(\lambda) = (2-\lambda) q_{N+1}(\lambda) - 2q_N(\lambda), \qquad \chi_{E_{N+1}}(\lambda) = (1-\lambda) q_N(\lambda) - q_{N-1}(\lambda). \tag{3.26}$$

By (3.18) these relations can be written as

$$\chi_{D_{N+2}}(\lambda) = q_{N+2}(\lambda) - q_N(\lambda), \qquad \chi_{E_{N+1}}(\lambda) = q_{N+1}(\lambda) - q_N(\lambda). \tag{3.27}$$

Substituting $q_N(\lambda) = \cos((2N+1)\theta/2)/\cos(\theta/2)$ in these expressions and applying goniometric identities we obtain

$$\chi_{D_{N+2}}(\lambda) = -4 \sin \frac{(2N+3)\theta}{2} \sin \frac{\theta}{2}, \qquad \chi_{E_{N+1}}(\lambda) = -\frac{2 \sin(N+1)\theta \, \sin \frac{\theta}{2}}{\cos \frac{\theta}{2}}. \tag{3.28}$$

Employing $\lambda = 2(1 - \cos\theta)$ we calculate the eigenvalues of D_{N+2} and E_{N+1} and therewith we obtain the eigenvalues of B_N for $N \geq 1$,

$$\nu^{(1)}_{B_N,m} = 4 \sin^2 \frac{m\pi}{2N+3}, \qquad m = 0, ..., N+1, \tag{3.29}$$

$$\nu^{(2)}_{B_N,m} = 4 \sin^2 \frac{m\pi}{2(N+1)}, \qquad m = 0, ..., N. \tag{3.30}$$

We note that two eigenvalues are zero.

THEOREM 3.1. *Let $\mathcal{K}_1$ be the integral operator defined on the Hilbert space $\mathfrak{L}_2([-1,1])$ by (3.3) with kernel k_1 defined by (3.1), where the sequence (α_n) satisfies $\alpha_n \downarrow 0$ as $n \to \infty$, $\alpha_n \geq 0$ for all n, $\alpha_n \geq \alpha_{n+1}$ for $n \geq N_0 \geq 1$, and $\alpha_n \geq \alpha_{N_0}$ for $n < N_0$. The operator $\mathcal{K}_1$ is compact and positive with eigenvalues $\lambda_n(\mathcal{K}_1)$, $n = 1, 2, \ldots$, that satisfy*

$$\lambda_n(\mathcal{K}_1) \leq \frac{\pi}{2} \alpha_{n-1}, \qquad n \geq N_0 + 1, \tag{3.31}$$

$$\lambda_n(\mathcal{K}_1) \geq \frac{\pi}{4} \alpha_{2n}, \qquad n \geq \max\left(\lceil N_0/2 \rceil, 2\right), \tag{3.32}$$

where the eigenvalues are indexed according to decreasing magnitude.

Proof. Since $\mathcal{K}_1 = \mathcal{M}_{\frac{1}{4}}\hat{\mathcal{K}}_1\mathcal{M}_{\frac{1}{4}}$, we obtain by Corollary 2.2 and by (3.5)

$$\lambda_n(\mathcal{K}_1) \le \lambda_n(\hat{\mathcal{K}}_1) = \frac{\pi}{2}\,\alpha_{n-1}\,, \qquad n \ge N_0 + 1. \tag{3.33}$$

Employing Corollary 2.2 we also find

$$\lambda_n(\mathcal{K}_1) \ge \lambda_n(\mathcal{M}_{\frac{1}{2}}\hat{\mathcal{K}}_1\mathcal{M}_{\frac{1}{2}}) \tag{3.34}$$

To derive a lower bound we recall the first inequality in (3.14), which is derived for monotonically decreasing sequences (α_n). It can be straightforwardly verified that the inequality is also valid for the sequences (α_n) in this theorem if the additional requirement $N \ge \lceil N_0/2 \rceil$ is added. Then, since $\mathcal{A}_N$ is defined for $N \ge 2$, it follows from (3.14) and (3.34) that

$$\lambda_n(\mathcal{K}_1) \ge \frac{\pi}{8}\,\alpha_{2N}\lambda_n(\mathcal{A}_N)\,, \tag{3.35}$$

where we can select an appropriate $N \ge \max(\lceil N_0/2 \rceil, 2)$. The eigenvalues $\lambda_n(\mathcal{A}_N)$, $n = 1, 2, \ldots, 2N+1$, are given by (3.21) and (3.22) for $N \ge 2$. Indexing these eigenvalues according to decreasing magnitude, we find that $\lambda_1(\mathcal{A}_N)$ is $\nu^{(2)}_{A_N,1}$. To determine the eigenvalue with index n of $\mathcal{A}_N$ we invoke the property

$$\nu^{(2)}_{A_N,m} > \nu^{(1)}_{A_N,m} > \nu^{(2)}_{A_N,m+1}\,. \tag{3.36}$$

Then,

$$\lambda_n(\mathcal{A}_N) = \begin{cases} \nu^{(1)}_{A_N,n/2} = 4\cos^2\left(\dfrac{\pi}{4}\dfrac{n}{N+\frac{1}{2}}\right), & n \text{ even}, \\[2ex] \nu^{(2)}_{A_N,(n+1)/2} = 4\cos^2\left(\dfrac{\pi}{4}\dfrac{n+1}{N+\frac{3}{2}}\right), & n \text{ odd}. \end{cases} \tag{3.37}$$

Considering the eigenvalues with index $n \ge \max(\lceil N_0/2 \rceil, 2)$ and choosing $N = n$, we obtain from (3.37) that $\lambda_n(\mathcal{A}_n) \ge 4\cos^2(\pi/4) = 2$. Consequently from (3.35) with $N = n$ follows the inequality (3.32). We note that Reade chooses $N = n+1$ in his specific analysis of the eigenvalues of the integral operator with logarithmic kernel, see [22, p. 143], since he indexes the eigenvalues starting from $n = 0$. We index the eigenvalues starting from $n = 1$ because of the application of the Weyl-Courant minimax principle in the form (2.1). ☐

THEOREM 3.2. *Let $\mathcal{K}_2$ be the integral operator defined on the Hilbert space $\mathfrak{L}_2([-1,1])$ by (3.3) with kernel k_2 defined by (3.2), where the sequence (α_n) satisfies $\alpha_n \downarrow 0$ as $n \to \infty$, $\alpha_n \ge 0$ for all n, $\alpha_n \ge \alpha_{n+1}$ for $n \ge N_1 \ge 1$, and $\alpha_n \ge \alpha_{N_1}$ for $n < N_1$. The operator $\mathcal{K}_2$ is compact and positive with eigenvalues $\lambda_n(\mathcal{K}_2)$, $n = 1, 2, \ldots$, that satisfy*

$$\lambda_n(\mathcal{K}_2) \le \frac{\pi}{2}\,\alpha_{n-1}\,, \qquad n \ge N_1, \tag{3.38}$$

$$\lambda_n(\mathcal{K}_2) \ge \frac{\pi}{4}\,\alpha_{2(n-1)}\,, \qquad n \ge \left\lceil \frac{N_1}{2} \right\rceil + 1\,, \tag{3.39}$$

where the eigenvalues $\lambda_n(\mathcal{K}_2)$ are indexed according to decreasing magnitude.

Proof. Since $\mathcal{K}_2 = \mathcal{M}_{\frac{1}{4}}\hat{\mathcal{K}}_2\mathcal{M}_{\frac{1}{4}}$, we obtain by Corollary 2.2 and by (3.6)

$$\lambda_n(\mathcal{K}_2) \leq \lambda_n(\hat{\mathcal{K}}_2) = \frac{\pi}{2}\,\alpha_{n-1}\,, \qquad n \geq N_1. \tag{3.40}$$

To derive a lower bound we first recall the second inequality in (3.14), which is derived for monotonically decreasing sequences (α_n). It can be straightforwardly verified that the inequality is also valid for the sequences (α_n) in this theorem if the additional requirement $N \geq \lceil N_1/2 \rceil$ is added. Then, employing also Corollary 2.2, we obtain

$$\lambda_n(\mathcal{K}_2) \geq \lambda_n(\mathcal{M}_{\frac{1}{2}}\hat{\mathcal{K}}_2\mathcal{M}_{\frac{1}{2}}) \geq \frac{\pi}{8}\,\alpha_{2N}\lambda_n(\mathcal{B}_N), \tag{3.41}$$

where we can select an appropriate $N \geq \lceil N_1/2 \rceil$. The eigenvalues $\lambda_n(\mathcal{B}_N)$, $n = 1, 2, \ldots, 2N+3$, are given by (3.29) and (3.30) for $N \geq 1$. Indexing these eigenvalues according to decreasing magnitude, we find that $\lambda_1(\mathcal{B}_N)$ is $\nu^{(1)}_{B_N,N+1}$. To determine the eigenvalue with index n of $\mathcal{B}_N$ we invoke the property

$$\nu^{(1)}_{B_N,m+1} > \nu^{(2)}_{B_N,m} > \nu^{(1)}_{B_N,m}\,. \tag{3.42}$$

Then,

$$\lambda_n(\mathcal{B}_N) = \begin{cases} \nu^{(2)}_{B_N,N-(n-2)/2} = 4\sin^2\left(\dfrac{\pi}{4}\,\dfrac{2N+2-n}{N+1}\right), & n \text{ even}, \\[2ex] \nu^{(1)}_{B_N,N+1-(n-1)/2} = 4\sin^2\left(\dfrac{\pi}{4}\,\dfrac{2N+3-n}{N+\frac{3}{2}}\right), & n \text{ odd}. \end{cases} \tag{3.43}$$

Considering the eigenvalues with index $n \geq \lceil N_1/2 \rceil + 1$ and choosing $N = n-1$ we obtain from (3.43) that $\lambda_n(\mathcal{B}_{n-1}) \geq 4\sin^2(\pi/4) = 2$. Consequently from (3.41) with $N = n-1$ follows the inequality (3.39). □

4. Application: asymptotic behavior of an integral-differential operator with logarithmically singular kernel. Since

$$\log|x-\xi| = -\sum_{n=0}^{\infty} \gamma_n T_n(\xi) T_n(x) \tag{4.1}$$

with $\gamma_0 = \log 2$ and $\gamma_n = 2/n$ [22, Lemma 1], the operator $\mathcal{K}$ on $\mathfrak{L}_2([-1,1])$ defined by (1.5) is compact, negative, and self-adjoint. Applying Theorem 3.1 with $N_0 = 3$ (since $\gamma_2 > \gamma_0 > \gamma_3$), we specify the index n in the inequalities (1.7) obtained by Reade:

Corollary 4.1. *The (negative) eigenvalues $\lambda_n(\mathcal{K})$, $n = 1, 2, \ldots,$ of the compact self-adjoint operator $\mathcal{K}$ defined by (1.5) on the Hilbert space $\mathfrak{L}_2([-1,1])$ satisfy*

$$\frac{\pi}{4n} \leq |\lambda_n(\mathcal{K})|\,, \quad n \geq 2, \qquad\qquad |\lambda_n(\mathcal{K})| \leq \frac{\pi}{n-1}\,, \quad n \geq 4, \tag{4.2}$$

where the eigenvalues are indexed according to decreasing magnitude.

Next we apply the results of Section 3 to the derivation of the asymptotics of the eigenvalues of the operator $\frac{d^2}{dx^2}\mathcal{K}$. For that we do some auxiliary work. The Hilbert transform $\mathcal{V}$ on the Hilbert space $\mathfrak{L}_2(\mathbb{R})$ is defined by

$$(\mathcal{V}w)(x) = P.V. \int_{-\infty}^{\infty} \frac{w(\xi)}{x-\xi}\,d\xi \tag{4.3}$$

The operator is bounded and its adjoint satisfies $\mathcal{V}^* = -\mathcal{V}$. The Fourier transformation $\mathcal{F}$ on $\mathfrak{L}_2(\mathbb{R})$ defined by

$$(\mathcal{F}w)(y) = \int_{-\infty}^{\infty} w(x)\, e^{-iyx}\, dx \tag{4.4}$$

and the Hilbert transform satisfy the relation

$$((\mathcal{F} \circ \mathcal{V})w)(y) = -\pi i\, \mathrm{sign}(y)\, (\mathcal{F}w)(y)\,. \tag{4.5}$$

The well-known identity $-\mathcal{V}^2 = \mathcal{V}^*\mathcal{V} = \pi^2\mathcal{I}$ follows by composing $\mathcal{F}$ and $\mathcal{V}^2$, applying twice (4.5), and taking the inverse Fourier transform of the result.

On $\mathfrak{L}_2([-1,1])$ we introduce the finite Hilbert transform $\mathcal{H}$ by $\mathcal{H}f = (\mathcal{V}w_f)|_{[-1,1]}$ with w_f the natural extension of $f \in \mathfrak{L}_2([-1,1])$ to $\mathfrak{L}_2(\mathbb{R})$. We conclude that $\mathcal{H}$ is bounded with $\|\mathcal{H}\| \le \pi$. The Čebysev polynomials satisfy the relations [29, p. 261]

$$\frac{1}{\pi} P.V. \int_{-1}^{1} \frac{1}{x-\xi} \frac{1}{\sqrt{1-\xi^2}}\, T_n(\xi)\, d\xi = -U_{n-1}(x), \tag{4.6}$$

$$\frac{1}{\pi} P.V. \int_{-1}^{1} \frac{1}{x-\xi} \sqrt{1-\xi^2}\, U_{n-1}(\xi)\, d\xi = T_n(x), \tag{4.7}$$

for $-1 \le x \le 1$ and $n = 1, 2, \ldots$. Note that from (4.7) we conclude $\|\mathcal{H}\| = \pi$. We write the relations (4.6) and (4.7) as

$$\mathcal{H}\mathcal{M}_{-\frac{1}{4}}\hat{T}_n = -\pi\mathcal{M}_{-\frac{1}{4}}\hat{U}_{n-1}, \qquad \mathcal{H}\mathcal{M}_{\frac{1}{4}}\hat{U}_{n-1} = \pi\mathcal{M}_{\frac{1}{4}}\hat{T}_n. \tag{4.8}$$

The first relation inspires us introducing $\hat{\mathcal{H}}$ on $\mathfrak{L}_2([-1,1])$ by

$$\hat{\mathcal{H}}g = -2\sum_{n=0}^{\infty} \langle g, \hat{T}_{n+1}\rangle_{\mathfrak{L}_2} \hat{U}_n\,, \tag{4.9}$$

such that $\hat{\mathcal{H}}\hat{T}_n = -\pi\hat{U}_{n-1}$ for $n = 1, 2, \ldots$ and $\hat{\mathcal{H}}\hat{T}_0 = 0$. Applying $\hat{\mathcal{H}}$ to $\mathcal{M}_{\frac{1}{4}}f$ with $f \in \mathfrak{L}_2([-1,1])$ and multiplying by $\mathcal{M}_{-\frac{1}{4}}$ we obtain

$$\mathcal{M}_{-\frac{1}{4}}\hat{\mathcal{H}}\mathcal{M}_{\frac{1}{4}}f = -2\sum_{n=0}^{\infty} \langle f, T_{n+1}\rangle_{\mathfrak{L}_2} U_n \tag{4.10}$$

by straightforwardly employing the definitions of $\mathcal{M}_\nu$, $\hat{T}_n$, and $\hat{U}_n$. Instead, the action of the operator $\mathcal{M}_{-\frac{1}{4}}\hat{\mathcal{H}}\mathcal{M}_{\frac{1}{4}}$ can also be calculated as

$$\mathcal{M}_{-\frac{1}{4}}\hat{\mathcal{H}}\mathcal{M}_{\frac{1}{4}}f = -2\sum_{n=0}^{\infty} \langle f, \mathcal{M}_{\frac{1}{4}}\hat{T}_{n+1}\rangle_{\mathfrak{L}_2} U_n = -\frac{2}{\pi}\sum_{n=0}^{\infty} \langle f, \mathcal{H}\mathcal{M}_{\frac{1}{4}}\hat{U}_n\rangle_{\mathfrak{L}_2} U_n =$$
$$\frac{2}{\pi}\sum_{n=0}^{\infty} \langle \mathcal{H}f, \mathcal{M}_{\frac{1}{2}}U_n\rangle_{\mathfrak{L}_2} U_n = \mathcal{H}f\,, \tag{4.11}$$

where the second equality is obtained by invoking the second relation of (4.8) and the third equality by invoking the adjoint $\mathcal{H}^* = -\mathcal{H}$. Combining (4.10) and (4.11) we conclude that $\mathcal{H} = \mathcal{M}_{-\frac{1}{4}}\hat{\mathcal{H}}\mathcal{M}_{\frac{1}{4}}$ and

$$\mathcal{H}f = -2\sum_{n=0}^{\infty} \langle f, T_{n+1}\rangle_{\mathfrak{L}_2} U_n \tag{4.12}$$

with convergence in $\mathfrak{L}_2([-1,1])$. Since

$$\mathcal{K}f = -\log 2\,\langle f, T_0\rangle_{\mathfrak{L}_2} T_0 - 2\sum_{n=0}^{\infty} \frac{1}{n+1}\langle f, T_{n+1}\rangle_{\mathfrak{L}_2} T_{n+1} \tag{4.13}$$

and $\frac{dT_{n+1}}{dx} = (n+1)U_n$ for $n = 0, 1, 2, \ldots$, we observe that for all $f \in \mathfrak{L}_2([-1,1])$

$$\frac{d}{dx}\mathcal{K}f = -2\sum_{n=0}^{\infty}\langle f, T_{n+1}\rangle_{\mathfrak{L}_2} U_n = \mathcal{H}f \tag{4.14}$$

and thus $\mathcal{K}f \in \mathfrak{H}_{2,1}([-1,1])$. By straighfoward partial integration we derive for $f \in \mathfrak{H}_{2,1}([-1,1])$

$$(\mathcal{K}f)(x) = \int_{-1}^{1} f(\xi)\frac{d}{d\xi}\left(-\int_0^{x-\xi} \log|t|\,dt\right) d\xi = -f(1)\int_{-1}^{x-1} \log|t|\,dt + \\ + f(-1)\int_0^{x+1} \log|t|\,dt + \int_{-1}^{1}\int_0^{x-\xi} \log|t|\,dt\,\frac{df}{dx}(\xi)\,d\xi\,. \tag{4.15}$$

From this expression we observe that $\mathcal{K}$ satisfies $\frac{d}{dx}(\mathcal{K}f) = \mathcal{K}\left(\frac{df}{dx}\right)$ for all $f \in \mathfrak{W}$, where $\mathfrak{W}$ is the dense subspace of $\mathfrak{L}_2([-1,1])$ defined by (1.6). Employing this property and (4.14) we derive for $f \in \mathfrak{W}$

$$\frac{d^2}{dx^2}(\mathcal{K}f) = \mathcal{H}\left(\frac{df}{dx}\right) = -2\sum_{n=0}^{\infty}\left\langle \frac{df}{dx}, T_{n+1}\right\rangle_{\mathfrak{L}_2} U_n = 2\sum_{n=0}^{\infty}(n+1)\langle f, U_n\rangle_{\mathfrak{L}_2} U_n\,. \tag{4.16}$$

We introduce the compact positive self-adjoint operator $\hat{\mathcal{T}}$ on $\mathfrak{L}_2([-1,1])$ by

$$\hat{\mathcal{T}} f = \frac{2}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{n+1}\langle f, \hat{U}_n\rangle_{\mathfrak{L}_2}\hat{U}_n \tag{4.17}$$

and correspondingly we introduce $\mathcal{T}$ on $\mathfrak{L}_2([-1,1])$ by $\mathcal{T} = \mathcal{M}_{\frac{1}{4}}\hat{\mathcal{T}}\mathcal{M}_{\frac{1}{4}}$. Then for all $f \in \mathfrak{W}$

$$\frac{d^2}{dx^2}(\mathcal{K}f) = 2\sum_{n=0}^{\infty}(n+1)\langle \mathcal{M}_{-\frac{1}{4}}f, \hat{U}_n\rangle_{\mathfrak{L}_2}\mathcal{M}_{-\frac{1}{4}}\hat{U}_n = \mathcal{M}_{-\frac{1}{4}}\hat{\mathcal{T}}^{-1}\mathcal{M}_{-\frac{1}{4}}f = \mathcal{T}^{-1}f\,. \tag{4.18}$$

Thus we show that the unbounded operator $\frac{d^2}{dx^2}\mathcal{K}$ extends to a positive self-adjoint operator, given by $\mathcal{T}^{-1}$, with domain the range of the compact self-adjoint operator $\mathcal{T}$. From (4.17) and the definition of $\mathcal{T}$ it follows that the kernel of $\mathcal{T}$ equals

$$K_{\mathcal{T}}(x,\xi) = \frac{2}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{n+1}U_n(x)U_n(\xi)\sqrt{1-x^2}\sqrt{1-\xi^2}\,. \tag{4.19}$$

Applying Theorem 3.2 with $N_1 = 1$ we obtain

COROLLARY 4.2. *The eigenvalues $\lambda_n(\mathcal{T})$, $n = 1, 2, \ldots$, of the compact positive self-adjoint operator $\mathcal{T}$ defined on the Hilbert space $\mathfrak{L}_2([-1,1])$ satisfy*

$$\frac{1}{\pi(4n-2)} \le \lambda_n(\mathcal{T})\,, \quad n \ge 2, \qquad\qquad \lambda_n(\mathcal{T}) \le \frac{1}{\pi n}\,, \quad n \ge 1, \tag{4.20}$$

where the eigenvalues are indexed according to decreasing magnitude.

THEOREM 4.3. *Let $\mathcal{K}$ be the compact self-adjoint operator on the Hilbert space $\mathfrak{L}_2([-1,1])$ defined by (1.5). Then, the differential-integral operator $\frac{d^2}{dx^2}\mathcal{K}$ defined on $\mathfrak{W}$ according to*

$$\left(\frac{d^2}{dx^2}\mathcal{K}\,f\right)(x) = P.V.\int_{-1}^{1}\frac{1}{x-\xi}\,\frac{df}{dx}(\xi)\,d\xi \tag{4.21}$$

extends to a positive self-adjoint operator with domain $\operatorname{ran}(\mathcal{T})$*, where $\mathcal{T}$ is the integral operator defined by the kernel $K_{\mathcal{T}}$ in (4.19). The eigenvalues $\lambda_n(\mathcal{T}^{-1})$, $n = 1, 2, \ldots$, of the self-adjoint extension $\mathcal{T}^{-1}$ of $\frac{d^2}{dx^2}\mathcal{K}$ satisfy*

$$\pi n \le \lambda_n(\mathcal{T}^{-1})\,, \quad n \ge 1, \qquad\qquad \lambda_n(\mathcal{T}^{-1}) \le \pi(4n-2)\,, \quad n \ge 2, \tag{4.22}$$

where the eigenvalues are indexed according to increasing magnitude.

THEOREM 4.4. *Let $\widetilde{\mathcal{K}}$ be the integral operator on the Hilbert space $\mathfrak{L}_2([-1,1])$ defined by the displacement kernel*

$$\tilde{k}(x-\xi) = \log|x-\xi| + h(x-\xi)\,, \tag{4.23}$$

where h is real, even and twice differentiable with square integrable second derivative. Then, the operator $\frac{d^2}{dx^2}\widetilde{\mathcal{K}} = \frac{d^2}{dx^2}\mathcal{K} + \widetilde{\mathcal{H}}$ extends to a self-adjoint operator with a discrete spectrum of eigenvalues that satisfy

$$\pi n - \|\widetilde{\mathcal{H}} - \gamma\mathcal{I}\|_{\mathfrak{L}_2} + \gamma \le \lambda_n\left(\frac{d^2}{dx^2}\widetilde{\mathcal{K}}\right) \le \pi(4n-2) + \|\widetilde{\mathcal{H}} - \gamma\mathcal{I}\|_{\mathfrak{L}_2} + \gamma\,, \tag{4.24}$$

with $\gamma = \inf_{f,\|f\|=1}\langle\widetilde{\mathcal{H}}f, f\rangle_{\mathfrak{L}_2}$, $n \ge 1$ for the first inequality, and $n \ge 2$ for the second inequality.

Proof. We use Corollary A.2 with $\mathcal{A}$ the self-adjoint extension of $\frac{d^2}{dx^2}\mathcal{K}$ and $\mathcal{D}$ the bounded self-adjoint operator $\widetilde{\mathcal{H}} - \gamma\mathcal{I}$, where $\widetilde{\mathcal{H}}$ is the integral operator generated by the symmetric kernel $\frac{d^2h}{dx^2}(x-\xi)$. Then, $\mathcal{A} + \mathcal{D}$ has a compact inverse and its eigenvalues $\lambda_n(\mathcal{A} + \mathcal{D})$ satisfy

$$\lambda_n(\mathcal{A}) - \|\mathcal{D}\|_{\mathfrak{L}_2} \le \lambda_n(\mathcal{A} + \mathcal{D}) \le \lambda_n(\mathcal{A}) + \|\mathcal{D}\|_{\mathfrak{L}_2}\,. \tag{4.25}$$

Moreover, we conclude that $\frac{d^2}{dx^2}\widetilde{\mathcal{K}}$ extends to the self-adjoint operator $\mathcal{A} + \mathcal{D} + \gamma\mathcal{I}$ with a discrete spectrum of eigenvalues that satisfy

$$\lambda_n\left(\frac{d^2}{dx^2}\widetilde{\mathcal{K}}\right) = \lambda_n(\mathcal{A} + \mathcal{D}) + \gamma\,. \tag{4.26}$$

From (4.22), (4.25), and (4.26) it follows that these eigenvalues satisfy (4.24). □

Theorem 4.4 can be applied to the operator $\mathcal{Z}$ in (1.1) with the integral kernel G replaced by its real part. To this end we first specify the regular part of G by decomposing it as $G_{\text{reg}} = G_1 + G_2 + G_3$, where

$$G_1(x) = \frac{1}{\pi k\ell}\int_0^2 \frac{1}{\sqrt{x^2+\beta^2y^2}}\,dy + \frac{1}{\pi k\ell\beta}\log|x| = \\ = \frac{1}{\pi k\ell\beta}\log\left(2\beta + \sqrt{4\beta^2 + x^2}\right), \tag{4.27}$$

$$G_2(x) = \frac{1}{\pi k\ell}\int_0^2 \frac{\exp\left(ik\ell\sqrt{x^2+\beta^2y^2}\right)-1}{\sqrt{x^2+\beta^2y^2}}\,dy = \frac{1}{\pi k\ell}\sum_{n=0}^{\infty}\frac{(ik\ell)^{n+1}}{(n+1)!}\,Q_n(x)\,, \quad (4.28)$$

$$Q_n(x) = \int_{y=0}^2 \left(x^2+\beta^2y^2\right)^{\frac{n}{2}}\,dy\,, \quad (4.29)$$

and

$$G_3(x) = -\frac{1}{2\pi k\ell}\int_0^2 \frac{y\,\exp\left(ik\ell\sqrt{x^2+\beta^2y^2}\right)}{\sqrt{x^2+\beta^2y^2}}\,dy = -\frac{1}{2\pi i k^2\ell^2\beta^2}\left[\exp\left(ik\ell\sqrt{x^2+\beta^2y^2}\right)-\exp(ik\ell|x|)\right]\,. \quad (4.30)$$

For decompositions of the thin-wire kernel we refer to [16], where Taylor expansions as in G_2 are employed to arrive at the aforementioned kernel decomposition $F_1(z)\log|z|+F_2(z)$. Next, we write the action of $\mathcal{Z}$ as

$$\mathcal{Z}w = -\frac{iZ_0}{2\pi k\ell}\left(\frac{d^2}{dx^2}\mathcal{K}w - \pi k\ell\beta\,\frac{d^2}{dx^2}\mathcal{G}_{\text{reg}}w - \pi k^3\ell^3\beta\,\mathcal{G}w\right)\,, \quad (4.31)$$

where $\mathcal{G}_{\text{reg}}$ is the integral operator induced by the kernel G_{reg}. The second derivative of G_1 is square integrable. By term-wise differentiation of the series expansion of G_2, it can be straightforwardly shown that the second derivative of G_2 is also square integrable and has a logarithmic singularity. Decomposing G_3 as

$$G_3(x) = -\frac{1}{2\pi i k^2\ell^2\beta^2}\left(-ik\ell|x| + g_3(x)\right)\,, \quad (4.32)$$

we observe that g_3 is two times continuously differentiable. Moreover, the composition of the second derivative and the integral operator induced by the displacement kernel $|x|$ equals twice the identity operator. The action of $\mathcal{Z}$ can thus be written as

$$\frac{1}{Z_1}\,(\mathcal{Z}w)(x) = \frac{d^2}{dx^2}(\mathcal{K}w)(x) - \frac{1}{\beta}\,w(x) + \int_{-1}^1 \frac{d^2}{dx^2}h(x-\xi)\,w(\xi)\,d\xi\,, \quad (4.33)$$

where $Z_1 = -iZ_0/2\pi k\ell$ and

$$\frac{d^2h}{dx^2} = -\pi k\ell\beta\left(\frac{d^2G_1}{dx^2} + \frac{d^2G_2}{dx^2} - \frac{1}{2\pi i k^2\ell^2\beta^2}\frac{d^2g_3}{dx^2}\right) - \pi k^3\ell^3\beta G\,. \quad (4.34)$$

For the second derivative of G_2 and the evaluation of Q_n we refer to Appendix B.

Let $\widetilde{\mathcal{Z}}$ be the operator $\mathcal{Z}$ with the kernel G replaced by its real part or, equivalently, with h in (4.33) replaced by $\operatorname{Re} h$. Applying Theorem 4.4 to the kernel $\tilde{k} = \log|\cdot| + \operatorname{Re} h$, we obtain

$$\pi n - \|\widetilde{\mathcal{H}}\|_{\mathfrak{L}_2} - \frac{1}{\beta} \le \lambda_n\left(\frac{1}{Z_1}\,\widetilde{\mathcal{Z}}\right) \le \pi(4n-2) + \|\widetilde{\mathcal{H}}\|_{\mathfrak{L}_2} - \frac{1}{\beta}\,, \quad (4.35)$$

for the same values of n as in Theorem 4.4, where the operator $\widetilde{\mathcal{H}}$ is generated by the kernel $\operatorname{Re}\frac{d^2h}{dx^2}$ and where we employed that $\|\widetilde{\mathcal{H}}-\gamma\mathcal{I}\|_{\mathfrak{L}_2}\pm\gamma \le \|\widetilde{\mathcal{H}}\|_{\mathfrak{L}_2}$ in Theorem 4.4. In our numerical results we replace $\|\widetilde{\mathcal{H}}\|_{\mathfrak{L}_2}$ by its upper bound,

$$\|\widetilde{\mathcal{H}}\|_{\mathfrak{L}_2} \le \int_{-2}^2 \left|\operatorname{Re}\frac{d^2h}{dx^2}(\xi)\right|\,d\xi\,. \quad (4.36)$$

5. Numerical Results. To validate the theorems derived in the previous section, we compute the eigenvalues of the operators $\mathcal{K}$ and $\frac{d^2}{dx^2}\mathcal{K}$ by employing a projection method. For a specified set of independent functions that belong to $\mathfrak{W}$, we compute the matrix $G^{-1}Z$, where G is the Gram matrix of the set of functions with respect to the classical inner product in $\mathfrak{L}_2([-1,1])$ and Z are the matrices of inner products generated by $\langle\cdot\,,\mathcal{K}\,\cdot\rangle_{\mathfrak{L}_2}$ and $\langle\cdot\,,\frac{d^2}{dx^2}\mathcal{K}\,\cdot\rangle_{\mathfrak{L}_2}$. On $\mathfrak{W}$ the second inner product can be rewritten to $-\langle\frac{d}{dx}\,\cdot\,,\mathcal{K}\frac{d}{dx}\,\cdot\rangle_{\mathfrak{L}_2}$. We define two sets of functions in $\mathfrak{W}$. The first one is the Fourier basis $\cos((2n-1)\pi x/2)$, $\sin n\pi x$, where $n = 1, 2, \ldots, N$. The second one is a set of uniformly distributed, piecewise linear splines,

$$\Lambda_n(x) = \Lambda\left(\frac{x - x_n}{\Delta}\right), \tag{5.1}$$

where $\Lambda(x) = (1 - |x|)1_{[1,1]}(x)$, $\Delta = 2/(N+1)$, $x_n = -1 + n\Delta$, and $n = 1, 2, \ldots, N$. We calculate the matrix Z by rewriting its entries as the inner product of the kernel and the convolution of the two basis functions. Next we calculate the contribution of the logarithmic part of the integrand analytically and we compute the contribution of the regular part by a composite Simpson rule, see [3, Secs. 3.3, 3.4]. For the Fourier basis the Gram matrix G is the identity and for the splines it is a tridiagonal matrix with $2\Delta/3$ on its diagonal and $\Delta/6$ on its two codiagonals.

Figure 5.1 shows the absolute eigenvalues of $\mathcal{K}$ computed with both sets of functions together with the upper and lower bounds of Corollary 4.1. Similarly Figure 5.2 shows the eigenvalues of $\frac{d^2}{dx^2}\mathcal{K}$ together with the upper and lower bounds of Theorem 4.3. For both results we employed $N = 20$ for the Fourier basis and $N = 40$ for the piecewise linear splines. We clearly observe that the (absolute) computed eigenvalues satisfy the derived bounds for both operators. Moreover, for the operator $\frac{d^2}{dx^2}\mathcal{K}$, we observe that the eigenvalues obtained with the two bases start to deviate for eigenvalue indices $n \gtrsim 20$. In this respect we demonstrated in [3, Sec. 5.2] that if the eigenvalues of a dipole are generated by a set of P uniformly distribited linear splines and by the first P functions in the Fourier basis, the first $\lfloor P/2 \rfloor$ eigenvalues match.

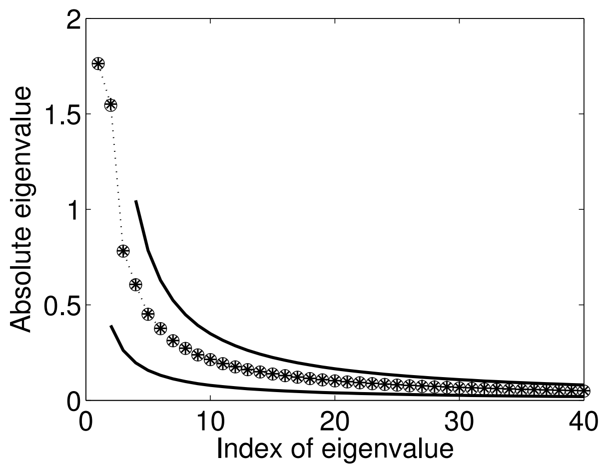

FIG. 5.1. *Eigenvalues of $\mathcal{K}$ obtained with piecewise linear splines ($\circ$, $N = 40$) and with the Fourier basis ($*$, $N = 20$). The bounds (4.2) are depicted by solid lines.*

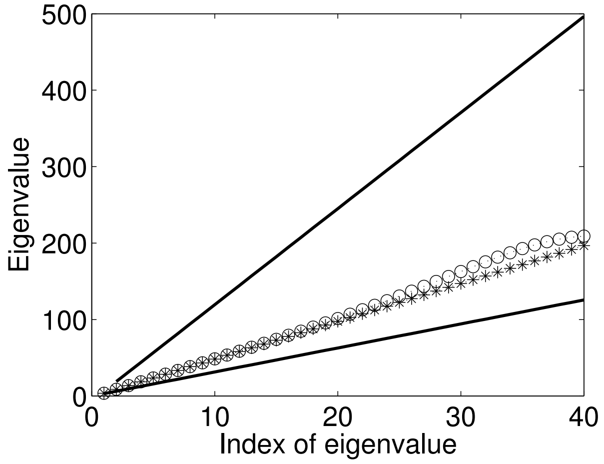

FIG. 5.2. *Eigenvalues of* $\frac{d^2}{dx^2}\mathcal{K}$ *obtained with piecewise linear splines* ($\circ$, $N = 40$) *and the Fourier basis* ($*$, $N = 20$). *The bounds (4.22) are depicted by solid lines.*

Next we consider the operator $\mathcal{Z}$ for the current on a strip. First we choose the frequency such that the dipole is half a wavelength long, $2\ell = \lambda/2$, and that it is narrow with respect to the wavelength, $\beta = 1/50$. Figure 5.3 shows the real part of the eigenvalues of the operator $\mathcal{Z}/Z_1$, the eigenvalues of $\mathcal{Z}/Z_1$ with the kernel G replaced by its real part (i.e., $\widetilde{\mathcal{Z}}/Z_1$), the eigenvalues of $\frac{d^2}{dx^2}\mathcal{K}-\mathcal{I}/\beta$, and the upper and lower bounds obtained from (4.35) and (4.36). Note that in this numerical example $\|\widetilde{\mathcal{H}}\|_{\mathfrak{L}_2} \leq 48.5$. For all three operators, the eigenvalues are computed by the Fourier basis with $N = 20$. We observe that the real parts of the eigenvalues of $\mathcal{Z}/Z_1$ and the eigenvalues of $\mathcal{Z}/Z_1$ with G replaced by its real part are the same. We also observe that for $n \gtrsim 20$ the eigenvalues of $\frac{d^2}{dx^2}\mathcal{K}-\mathcal{I}/\beta$ match the real parts of the eigenvalues of $\mathcal{Z}/Z_1$. The first observation demonstrates that the real parts of the eigenvalues are determined by the real part of the integral kernel and suggests that a similar conclusion is valid for the imaginary parts. The second observation is explained by the boundedness of the integral operator with kernel $\frac{d^2h}{dx^2}$ in (4.33) and of its real counter part $\widetilde{\mathcal{H}}$ with kernel $\mathrm{Re}\,\frac{d^2h}{dx^2}$. These explanations suggest that the imaginary parts of the eigenvalues of $\mathcal{Z}/Z_1$ are only siginificant for the lower eigenvalues and that the eigenvalues of $\mathcal{Z}/Z_1$ with complex kernel G also satisfy the bounds in Figure 5.3, which is confirmed by numerical results. Physically, this observation indicates that only a limited number of eigenfunctions are radiative, since the real parts of the eigenvalues of $\mathcal{Z}/Z_1$, or the imaginary parts of the eigenvalues of $\mathcal{Z}$, correspond to the reactive energy of the eigenfunctions of the dipole, while the imaginary parts correspond to the radiated energy of these eigenfunctions.

As second example we consider a much shorter dipole, $2\ell = \lambda/15$, with the same width by which $\beta = 3/20$. Analogously to Figure 5.3, Figure 5.4 shows the three curves of eigenvalues and the upper and lower bounds. Note that in this numerical example $\|\widetilde{\mathcal{H}}\|_{\mathfrak{L}_2} \leq 5.7$. The real parts of the eigenvalues $\mathcal{Z}/Z_1$ are not only the same as the eigenvalues of $\mathcal{Z}/Z_1$ with G replaced by its real part, but also approximately the same as the eigenvalues of $\frac{d^2}{dx^2}\mathcal{K}-\mathcal{I}/\beta$, except for significant deviations in the smallest eigenvalues. The imaginary parts of the eigenvalues of $\mathcal{Z}/Z_1$ are a factor 10^3 or

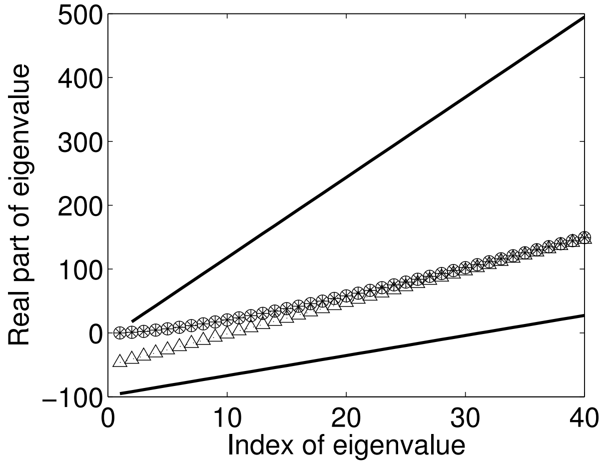

FIG. 5.3. *For a dipole of half a wavelength: real part of the eigenvalues of $\mathcal{Z}/Z_1$ ($*$), eigenvalues of $\mathcal{Z}/Z_1$ with G replaced by its real part ($\circ$), and eigenvalues of $\frac{d^2}{dx^2}\mathcal{K} - \mathcal{I}/\beta$ ($\triangle$), computed by the Fourier basis ($N = 20$). The bounds given by (4.35), combined with (4.36), are depicted by solid lines. Parameter values: $2\ell = \lambda/2$, $\beta = 1/50$.*

more smaller than their real parts, which means physically that all eigenfunctions are reactive. Based on these observations we may consider to compute the smallest eigenvalues of $\mathcal{Z}/Z_1$ from Rayleigh-Ritz quotients applied to the first few eigenfunctions of $\frac{d^2}{dx^2}\mathcal{K}$ and to approximate the other eigenvalues by the eigenvalues of $\frac{d^2}{dx^2}\mathcal{K} - \mathcal{I}/\beta$. This approach facilitates a rapid eigenvalue computation since the eigenvalues and eigenfunctions of $\frac{d^2}{dx^2}\mathcal{K}$ do not depend on the geometrical parameters.

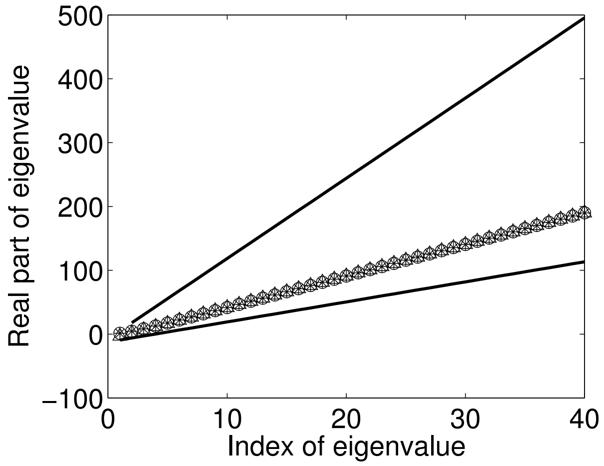

FIG. 5.4. *For a dipole of one fifteenth wavelength: real part of the eigenvalues of $\mathcal{Z}/Z_1$ ($*$), eigenvalues of $\mathcal{Z}/Z_1$ with G replaced by its real part ($\circ$), and eigenvalues of $\frac{d^2}{dx^2}\mathcal{K} - \mathcal{I}/\beta$ ($\triangle$), computed by the Fourier basis ($N = 20$). The bounds given by (4.35), combined with (4.36), are depicted by solid lines. Parameter values: $2\ell = \lambda/15$, $\beta = 3/20$.*

6. Conclusion. In our investigation of the spectral behavior of the integral-differential operator that governs the time-harmonic current on a linear strip or wire, we came across the problem of deriving explicit bounds for the ordered sequence of eigenvalues of the composition of the second-order differentiation operator and the integral operator with logarithimic displacement kernel. To tackle this problem we used methods of an earlier work by J. B. Reade who employs the Weyl-Courant minimax principle and explicit properties of the Čebysev polynomials of the first and second kind. By his methods Reade was able to derive explicit index-dependent bounds for the ordered sequence of eigenvalues of the integral operator $\mathcal{K}$ with logarithmic displacement kernel. In this paper we modified and extended Reade's result to integral operators with kernels described by arbitrary expansions of $T_m(x)T_m(\xi)$ and $U_m(x)U_m(\xi)$. In particular we showed that the upper and lower bounds $\pi/(n-1)$ and $\pi/4n$ derived by Reade for the absolute values of the eigenvalues λ_n of $\mathcal{K}$ $(n = 1, 2, \dots)$ are valid for $n \geq 4$ and $n \geq 2$, respectively. Further, for the integral-differential operator $\frac{d^2}{dx^2}\mathcal{K}$ we proved that its eigenvalues are bounded from below by πn for $n \geq 1$ and from above by $\pi(4n-2)$ for $n \geq 2$. We extended this result to kernels that are the sum of the logarithmic displacement kernel and a real displacement kernel whose second derivative is square integrable. Subsequently we applied this extension to the integral-differential operator corresponding to a linear strip, where we replaced the complex integral kernel by its real part. For this operator we found lower and upper bounds expressed in terms of the bounds πn and $\pi(4n-2)$, a uniform shift, and the norm of the integral operator corresponding to the regular part of the kernel. Numerically we showed how well the eigenvalues of the considered operators, computed by Ritz's methods, fit the analytically derived bounds. Although our analysis does not provide bounds for the complex kernel corresponding to a linear strip, the absolute values of the computed eigenvalues satisfy the bounds derived for the real part of the kernel. Moreover, for the larger eigenvalue indices, the computed eigenvalues for the complex kernel match those computed for the real kernel.

Appendix A.

LEMMA A.1. *Let $\mathcal{A}$ be an invertible positive self-adjoint operator and let the operator $\mathcal{B}$ be such that $\mathcal{B} \geq \mathcal{A}$. Then, $\mathcal{B}$ is invertible and $\mathcal{B}^{-1} \leq \mathcal{A}^{-1}$.*

Proof. By the Spectral Theorem $A^{1/2}$ exists and is positive and invertible. Define the positive self-adjoint operator $\mathcal{P}$ by $\mathcal{P} = \mathcal{A}^{-1/2}(\mathcal{B}-\mathcal{A})\mathcal{A}^{-1/2}$. Then, $\mathcal{B} = \mathcal{A}^{1/2}(\mathcal{I}+\mathcal{P})\mathcal{A}^{1/2}$. It follows from the Spectral Theorem for unbounded self-adjoint operators [31] that $\mathcal{I}+\mathcal{P}$ is invertible and $(\mathcal{I}+\mathcal{P})^{-1} \leq \mathcal{I}$. We derive

$$\mathcal{B}^{-1} = \mathcal{A}^{-1/2}(\mathcal{I}+\mathcal{P})^{-1}\mathcal{A}^{-1/2} \leq \mathcal{A}^{-1/2}\mathcal{I}\mathcal{A}^{-1/2} = \mathcal{A}^{-1}\,, \tag{A.1}$$

where the inequality follows from $(\mathcal{I}+\mathcal{P})^{-1} \leq \mathcal{I}$ and $\mathcal{A}^{-1/2}$ being self-adjoint. □

COROLLARY A.2. *Let $\mathcal{A}$ be a positive self-adjoint operator with compact inverse and let $\mathcal{D}$ be a bounded self-adjoint operator such that $\mathcal{A}+\mathcal{D}$ is invertible (with bounded inverse). Then $\mathcal{A}+\mathcal{D}$ has a compact inverse and its eigenvalues $\lambda_n(\mathcal{A}+\mathcal{D})$ satisfy*

$$\lambda_n(\mathcal{A}) - \|\mathcal{D}\| \leq \lambda_n(\mathcal{A}+\mathcal{D}) \leq \lambda_n(\mathcal{A}) + \|\mathcal{D}\|\,. \tag{A.2}$$

Proof. Since $\mathcal{A}+\mathcal{D} = \mathcal{A}(\mathcal{I}+\mathcal{A}^{-1}\mathcal{D})$ and since $\mathcal{A}+\mathcal{D}$ and $\mathcal{A}$ are invertible, $\mathcal{I}+\mathcal{A}^{-1}\mathcal{D}$ has a bounded inverse. Then, since the operator $(\mathcal{A}+\mathcal{D})^{-1}$ is the product of the bounded operator $(\mathcal{I}+\mathcal{A}^{-1}\mathcal{D})^{-1}$ and the compact operator $\mathcal{A}^{-1}$, it is compact. If $\mathrm{dom}(\mathcal{A})$ is the domain of definition of $\mathcal{A}$, then $\mathcal{A}+\mathcal{D}$ is self-adjoint on $\mathrm{dom}(\mathcal{A})$. By the Cauchy-Schwarz inequality we have $|\langle \mathcal{D}f, f\rangle| \le \|\mathcal{D}\|\langle f, f\rangle$ and thus $\pm\mathcal{D} \le \|\mathcal{D}\|\mathcal{I}$. Consequently, $\mathcal{A} \le \mathcal{A} + \mathcal{D} + \|\mathcal{D}\|\mathcal{I} \le \mathcal{A} + 2\|\mathcal{D}\|\mathcal{I}$. From Corollary A.1 it follows that

$$(\mathcal{A} + 2\|\mathcal{D}\|\mathcal{I})^{-1} \le (\mathcal{A} + \mathcal{D} + \|\mathcal{D}\|\mathcal{I})^{-1} \le \mathcal{A}^{-1}\,. \tag{A.3}$$

Since $(\mathcal{A} + \mathcal{D} + \|\mathcal{D}\|\mathcal{I})^{-1}$ and $(\mathcal{A} + 2\|\mathcal{D}\|\mathcal{I})^{-1}$ are positive and compact, it follows by Corollary 2.3 that

$$\lambda_n\left((\mathcal{A} + 2\|\mathcal{D}\|\mathcal{I})^{-1}\right) \le \lambda_n\left((\mathcal{A} + \mathcal{D} + \|\mathcal{D}\|\mathcal{I})^{-1}\right) \le \lambda_n(\mathcal{A}^{-1}). \tag{A.4}$$

Thus, $\lambda_n(\mathcal{A}) \le \lambda_n(\mathcal{A} + \mathcal{D} + \|\mathcal{D}\|\mathcal{I}) \le \lambda_n(\mathcal{A} + 2\|\mathcal{D}\|\mathcal{I})$, from which (A.2) follows. □

Appendix B.

The second derivative of G_2 is given by

$$\frac{d^2G_2}{dx^2} = \frac{k\ell}{2\pi\beta}\left[\log|x| + 1 - \log\left(2\beta + \sqrt{4\beta^2 + x^2}\right) + \right.$$
$$\left. - \frac{x^2}{\sqrt{4\beta^2 + x^2}\left(2\beta + \sqrt{4\beta^2 + x^2}\right)}\right] + \frac{k^3\ell^3}{8\pi\beta}\, x^2\left[-\log|x| + \log\left(2\beta + \sqrt{4\beta^2 + x^2}\right)\right] +$$
$$+ \frac{1}{\pi k\ell}\sum_{n=0}^{\infty}\frac{(ik\ell)^{n+3}}{(n+3)(n+1)!}\left(1 - \frac{k^2\ell^2}{n+5}\, x^2\right) Q_n(x)\,. \tag{B.1}$$

The function $Q_n(x)$ given by (4.29) can be evaluated by the recurrence relation

$$Q_n(x) = \frac{2}{n+1}\left(x^2 + 4\beta^2\right)^{\frac{n}{2}} + \frac{n}{n+1}\, x^2 Q_{n-2}(x) \tag{B.2}$$

with initial conditions $Q_0(x) = 2$ and

$$Q_1(x) = \sqrt{x^2 + 4\beta^2} + \frac{x^2}{2\beta}\left[-\log|x| + \log\left(2\beta + \sqrt{4\beta^2 + x^2}\right)\right]. \tag{B.3}$$

REFERENCES

[1] N. Marcuvitz, *Waveguide Handbook*, McGraw-Hill, 1951.

[2] S. Li and R. W. Scharstein, *Eigenmodes and scattering by an axial array of tape rings*, IEEE Trans. Antennas Propagat., Vol. 54, No. 7, July 2006, pp. 2096–2104.

[3] D. J. Bekers, *Finite antenna arrays: an eigencurrent approach*, Ph.D. Thesis, Eindhoven, The Netherlands: Tech. Univ. Eindhoven, Dec. 2004 [Online]. Available: http://alexandria.tue.nl/extra2/200411410.pdf

[4] D. J. Bekers, S. J. L. van Eijndhoven, A. A. F. van de Ven, P-P. Borsboom, and A. G. Tijhuis, *Eigencurrent analysis of resonant behavior in finite antenna arrays*, IEEE Trans. Microwave Theory Tech., Vol. 54, No. 6, Part II, June 2006, pp. 2821–2829.

[5] V. Shestopalov, Y. Shestopalov, *Spectral theory and excitation of open structures*, Peregrinus, London, 1996.

[6] E. M. Kartchevski, A. I. Nosich, and G. W. Hanson, Mathematical Analysis of the Generalized Natural Modes of an Inhomogeneous Optical Fiber, SIAM J. Appl. Math., Vol. 65, No. 6, 2005, pp. 2033–2048.

[7] R. F. Harrington and J. Mautz, *Theory of characteristic modes for conducting bodies*, IEEE Trans. Antennas Propagat., Vol. AP-19, No. 5, Sept. 1971, pp. 622–628.

[8] A. G. Ramm, *Mathematical Foundations of the Singularity and Eigenmode Expansion Methods (SEM and EEM)*, J. Math. Ann. Appl. 86, 1982, pp. 562–591.

[9] T. F. Jablonski and M. J. Sowinski, *Analysis of dielectric guiding structures by the iterative eigenfunction expansion method*, IEEE Trans. Microwave Theory Tech., IEEE Trans. Microwave Theory Tech., Vol. 37, No. 1, 1989, pp. 63-70.

[10] B. E., Fischer, A. E. Yagle, and J. L. Volakis, *On the eigen-decomposition of electromagnetic systems and the frequency dependence of the associated eigenvalues*, Antennas and Propagat. Symp., Washington D.C., 3–8 June 2005.
[11] V. Lancellotti, B. P. de Hon, A. G. Tijhuis, *An Eigencurrent Approach to the Analysis of Electrically Large 3-D Structures Using Linear Embedding via Greens Operators*, IEEE Trans. Antennas Propagat., Vol. 57, No. 11, Nov. 2009, pp. 3575–3585.
[12] D. J. Bekers, S. J. L. Eijndhoven, and A. G. Tijhuis, *An eigencurrent approach for the analysis of finite antenna arrays*, IEEE Trans. Antennas Propagat., Vol. 57, No. 12, Dec. 2009, pp. 3772–3782.
[13] J. E. Storer, *Impedance of thin-wire loop antennas*, AIEE Transactions (Part I. Comm. and Electr.), Vol. 75, Nov. 1956, pp. 606–619.
[14] L. W. Pearson, *Separation of the logarithmic singularity in the exact kernel of the cylindrical antenna integral equation*, IEEE Trans. Antennas Propagat., Vol. 23, No. 2, March 1975, pp. 256–258.
[15] D. S. Jones, *Note on the integral equation for a straight wire antenna*, IEE Proc., Vol. 128, Pt. H, No. 2, April 1981, pp. 114–116.
[16] O. P. Bruno and M. C. Haslam, *Regularity Theory andD Superalgebraic Solvers for Wire Antenna Problems*, SIAM J. Sci. Comput., Vol. 29, No. 4, 2007, pp. 1375–1402.
[17] B. P. Rynne, *The well-posedness of the integral equations for thin wire antennas*, IMA J. Appl. Math. 49, 1992, pp. 35–44.
[18] P. J. Davies, D. B. Duncan, and S. A. Funken, *Accurate and efficient algorithms for frequency domain scattering from a thin wire*, J. Comp. Physics 168, 2001, pp. 155-183.
[19] M. C. van Beurden and A. G. Tijhuis, *Analysis and regularization of the thin-wire integral equation with reduced kernel*, IEEE Trans. Antennas Propagat., Vol. 55, No. 1, Jan. 2007, pp. 120–129.
[20] A. G. Tijhuis, P. Zhongqui, and A. R. Bretones, *Transient excitation of a straight thin wire segment: A new look at an old problem*, IEEE Trans. Antennas Propag., Vol. 40, No. 10, Oct. 1992, pp. 1132–1146.
[21] X. Claeys, *On the theoretical justification of Pocklington's Equation*, Math. Models Meth. Appl. Sci., Vol. 19, No. 8, 2009, pp. 1325–1355.
[22] J. B. Reade, *Asymptotic behaviour of eigen-values of certain integral equations*, Proc. Edinburgh Math. Soc., 22, 1979, pp. 137–144.
[23] G. R. Richter, *On weakly singular Fredholm integral equations with displacement kernels*, J. Math. An. Appl. 55, 1976, pp. 32–42.
[24] M. Kac, *Distribution of eigenvalues of certain integral operators*, Mich. Math. J. 3, 1956, pp. 141–148.
[25] R. Estrada and R. P. Kanwal, *Integral equations with logarithmic kernels*, IMA J. Appl. Math., 43 (1989), pp. 133–155.
[26] M. R. Dostanic, *Spectral properties of the operator of Riesz potential type*, Proc. Amer. Math. Soc., Vol. 126, No. 8, Aug. 1998, pp. 2291–2297.
[27] S. Simíc, *An Estimation of the singular values of integral operator with logarithmic kernel*, Facta Universitatis, Ser. Math. Inform., 21 (2006), pp. 49–55.
[28] I. C. Gohberg and M. G. Krein, *Introduction to the theory of linear nonselfadjoint operators*, Amer. Math. Soc., Trans. of Math. Monographs, vol. 18, 1979.
[29] W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and theorems for the special functions of mathematical physics*, Die Grundlehren der mathematischen Wissenschaften, Band 52, Springer, New York, 1966.
[30] E. Kreyszig, *Introductory functional analysis with applications*, Wiley, New York, 1978.
[31] J. Weidmann, *Linear Operators in Hilbert spaces*, Springer, New York, 1980.

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