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Faraday's equations [1] read

$$\begin{aligned} \partial_t \underline{B} + \nabla \times \underline{E} &= \underline{0}\,, & \qquad \nabla \cdot \underline{E} &= 0\,, \\ \nabla \times \underline{H} &= \underline{j}\,, & \qquad \nabla \cdot \underline{B} &= 0\,. \end{aligned} \tag{0.1}$$

Here $\underline{B} = \mu \underline{H}$. Further, as a consequence, $\nabla \cdot \underline{j} = 0$. Note that, if we assume μ piecewise constant, Faraday's equations are, piecewise, two successive Poisson equations: Put $\underline{B} = \nabla \times \underline{A}$, with the gauge condition $\nabla \cdot \underline{A} = 0$. Then $\Delta \underline{A} = -\underline{j}$. After having recovered $\underline{B}$ we can, in principle, find $\underline{E}$ from the equation $\nabla \times \underline{E} = -\partial_t \underline{B}$. During this procedure the time t is just a parameter.

In Cartesian components Faraday's equations read:

$$\partial_t \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} + \begin{bmatrix} \partial_y E_z - \partial_z E_y \\ \partial_z E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \,, \qquad \begin{bmatrix} \partial_y H_z - \partial_z H_y \\ \partial_z H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{bmatrix} = \begin{bmatrix} j_x \\ j_y \\ j_z \end{bmatrix} \,,$$

$$\partial_x E_x + \partial_y E_y + \partial_z E_z = 0\,, \qquad \partial_x B_x + \partial_y B_y + \partial_z B_z = 0\,. \tag{0.2}$$

We now assume a layered structure. Our cartesian coordinates are taken such that there is translation invariance in the y-direction. The permeability depends only on x and z. So $\mu_0 = \mu(x, z)$. Typically, one could think of a flat air gap perpendicular to the x-axis, which lies in a ferro material. If we want to consider solutions which only depend on x and z, we assume $\underline{j}$ to depend on x and z only. Now Faraday's equations reduce to

$$\partial_t \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} + \begin{bmatrix} -\partial_z E_y \\ \partial_z E_x - \partial_x E_z \\ \partial_x E_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \,, \qquad \begin{bmatrix} -\partial_z H_y \\ \partial_z H_x - \partial_x H_z \\ \partial_x H_y \end{bmatrix} = \begin{bmatrix} j_x \\ j_y \\ j_z \end{bmatrix} \,,$$

$$\partial_x E_x + \partial_z E_z = 0\,, \qquad \partial_x B_x + \partial_z B_z = 0. \tag{0.3}$$

As a consequence $\partial_y j_y = 0$ and $\partial_x j_x + \partial_z j_z = 0$. In many practical cases we will have $j_x = j_z = 0$. Introduction of a 'stream function' $x, z \mapsto \Psi(x, z)$ with $H_z = \partial_x \Psi\,,\ H_x = -\partial_z \Psi$ leads to the 2-dimensional Poisson equation

$$(\partial_x \partial_x + \partial_z \partial_z)\Psi = -j_y. \tag{0.4}$$

[1] Non-historically speaking: These are Maxwell's equations with charge density 0 and the replacement current term ommitted

Next, a simple integration leads to H_y. Finally $\underline{E}$ can be solved from the 1st set of equations with the same technique.

In cylindrical coordinates Faraday's equations read

$$\partial_t \begin{bmatrix} B_r \\ B_\theta \\ B_z \end{bmatrix} + \begin{bmatrix} \frac{1}{r}\partial_\theta E_z - \partial_z E_\theta \\ \partial_z E_r - \partial_r E_z \\ \partial_r E_\theta + \frac{1}{r} E_\theta - \frac{1}{r}\partial_\theta E_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} \frac{1}{r}\partial_\theta H_z - \partial_z H_\theta \\ \partial_z H_r - \partial_r H_z \\ \partial_r H_\theta + \frac{1}{r} H_\theta - \frac{1}{r}\partial_\theta H_r \end{bmatrix} = \begin{bmatrix} j_r \\ j_\theta \\ j_z \end{bmatrix},$$

$$\partial_r E_r + \frac{1}{r} E_r + \frac{1}{r}\partial_\theta E_\theta + \partial_z E_z = 0\,, \qquad \partial_r B_r + \frac{1}{r} B_r + \frac{1}{r}\partial_\theta B_\theta + \partial_z B_z = 0. \tag{0.5}$$

We now assume a *rotationally* layered structure. Our cylindrical coordinates are taken such that there is rotational invariance in the θ-direction. The permeability depends only on r and z. So $\mu_0 = \mu(r, z)$. Typically, one could think of a cylindrical air gap, where the central axis coincides with the z-axis. The gap is surrounded by ferro material. If we want to consider solutions which only depend on r and z, we assume $\underline{j}$ to depend on r and z only. Now Faraday's equations reduce to

$$\partial_t \begin{bmatrix} rB_r \\ rB_\theta \\ rB_z \end{bmatrix} + \begin{bmatrix} -\partial_z(rE_\theta) \\ r(\partial_z E_r - \partial_r E_z) \\ \partial_r(rE_\theta) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} -\partial_z(rH_\theta) \\ \partial_z H_r - \partial_r H_z \\ \partial_r(rH_\theta) \end{bmatrix} = \begin{bmatrix} rj_r \\ j_\theta \\ rj_z \end{bmatrix},$$

$$\partial_r(rE_r) + \partial_z(rE_z) = 0\,, \qquad \partial_r(rB_r) + \partial_z(rB_z) = 0. \tag{0.6}$$

As a consequence $\partial_\theta j_\theta = 0$ and $\partial_r(rj_r) + \partial_z(rj_z) = 0$. In many practical cases we will have $j_r = j_z = 0$. Introduction of a 'stream function' $r, z \mapsto \Psi(r, z)$ with $H_z = \partial_r \Psi$, $H_r = -\partial_z \Psi$ leads to the 2-dimensional Poisson equation

$$(\partial_r\partial_r + \partial_z\partial_z)\Psi = -j_\theta. \tag{0.7}$$

This equation is exactly the same as (0.4)! Mathematically.
Next, a simple integration leads to rH_θ. Finally $\underline{E}$ can be solved from the 1st set of equations with the same technique.

This note has been inspired by Elena Lomonova and her students.

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