

## On a partial symmetry of Faraday's equations

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On a partial symmetry of  
Faraday's equations

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# On a Partial Symmetry of Faraday's Equations

J. de GRAAF 15-6-2007

Faraday's equations <sup>1</sup> read

$$\begin{aligned}\partial_t \underline{B} + \nabla \times \underline{E} &= \underline{0}, & \nabla \cdot \underline{E} &= 0, \\ \nabla \times \underline{H} &= \underline{j}, & \nabla \cdot \underline{B} &= 0.\end{aligned}\tag{0.1}$$

Here  $\underline{B} = \mu \underline{H}$ . Further, as a consequence,  $\nabla \cdot \underline{j} = 0$ . Note that, if we assume  $\mu$  piecewise constant, Faraday's equations are, piecewise, two successive Poisson equations: Put  $\underline{B} = \nabla \times \underline{A}$ , with the gauge condition  $\nabla \cdot \underline{A} = 0$ . Then  $\Delta \underline{A} = -\underline{j}$ . After having recovered  $\underline{B}$  we can, in principle, find  $\underline{E}$  from the equation  $\nabla \times \underline{E} = -\partial_t \underline{B}$ . During this procedure the time  $t$  is just a parameter.

In Cartesian components Faraday's equations read:

$$\begin{aligned}\partial_t \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} + \begin{bmatrix} \partial_y E_z - \partial_z E_y \\ \partial_z E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, & \begin{bmatrix} \partial_y H_z - \partial_z H_y \\ \partial_z H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{bmatrix} &= \begin{bmatrix} j_x \\ j_y \\ j_z \end{bmatrix}, \\ \partial_x E_x + \partial_y E_y + \partial_z E_z &= 0, & \partial_x B_x + \partial_y B_y + \partial_z B_z &= 0.\end{aligned}\tag{0.2}$$

We now assume a layered structure. Our cartesian coordinates are taken such that there is translation invariance in the  $y$ -direction. The permeability depends only on  $x$  and  $z$ . So  $\mu_0 = \mu(x, z)$ . Typically, one could think of a flat air gap perpendicular to the  $x$ -axis, which lies in a ferro material. If we want to consider solutions which only depend on  $x$  and  $z$ , we assume  $\underline{j}$  to depend on  $x$  and  $z$  only. Now Faraday's equations reduce to

$$\begin{aligned}\partial_t \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} + \begin{bmatrix} -\partial_z E_y \\ \partial_z E_x - \partial_x E_z \\ \partial_x E_y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, & \begin{bmatrix} -\partial_z H_y \\ \partial_z H_x - \partial_x H_z \\ \partial_x H_y \end{bmatrix} &= \begin{bmatrix} j_x \\ j_y \\ j_z \end{bmatrix}, \\ \partial_x E_x + \partial_z E_z &= 0, & \partial_x B_x + \partial_z B_z &= 0.\end{aligned}\tag{0.3}$$

As a consequence  $\partial_y j_y = 0$  and  $\partial_x j_x + \partial_z j_z = 0$ . In many practical cases we will have  $j_x = j_z = 0$ . Introduction of a 'stream function'  $x, z \mapsto \Psi(x, z)$  with  $H_z = \partial_x \Psi$ ,  $H_x = -\partial_z \Psi$  leads to the 2-dimensional Poisson equation

$$(\partial_x \partial_x + \partial_z \partial_z) \Psi = -j_y.\tag{0.4}$$

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<sup>1</sup>Non-historically speaking: These are Maxwell's equations with charge density 0 and the replacement current term omitted

Next, a simple integration leads to  $H_y$ . Finally  $\underline{E}$  can be solved from the 1st set of equations with the same technique.

In cylindrical coordinates Faraday's equations read

$$\partial_t \begin{bmatrix} B_r \\ B_\theta \\ B_z \end{bmatrix} + \begin{bmatrix} \frac{1}{r} \partial_\theta E_z - \partial_z E_\theta \\ \partial_z E_r - \partial_r E_z \\ \partial_r E_\theta + \frac{1}{r} E_\theta - \frac{1}{r} \partial_\theta E_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{r} \partial_\theta H_z - \partial_z H_\theta \\ \partial_z H_r - \partial_r H_z \\ \partial_r H_\theta + \frac{1}{r} H_\theta - \frac{1}{r} \partial_\theta H_r \end{bmatrix} = \begin{bmatrix} j_r \\ j_\theta \\ j_z \end{bmatrix},$$

$$\partial_r E_r + \frac{1}{r} E_r + \frac{1}{r} \partial_\theta E_\theta + \partial_z E_z = 0, \quad \partial_r B_r + \frac{1}{r} B_r + \frac{1}{r} \partial_\theta B_\theta + \partial_z B_z = 0. \quad (0.5)$$

We now assume a *rotationally* layered structure. Our cylindrical coordinates are taken such that there is rotational invariance in the  $\theta$ -direction. The permeability depends only on  $r$  and  $z$ . So  $\mu_0 = \mu(r, z)$ . Typically, one could think of a cylindrical air gap, where the central axis coincides with the  $z$ -axis. The gap is surrounded by ferro material. If we want to consider solutions which only depend on  $r$  and  $z$ , we assume  $\underline{j}$  to depend on  $r$  and  $z$  only. Now Faraday's equations reduce to

$$\partial_t \begin{bmatrix} rB_r \\ rB_\theta \\ rB_z \end{bmatrix} + \begin{bmatrix} -\partial_z(rE_\theta) \\ r(\partial_z E_r - \partial_r E_z) \\ \partial_r(rE_\theta) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -\partial_z(rH_\theta) \\ \partial_z H_r - \partial_r H_z \\ \partial_r(rH_\theta) \end{bmatrix} = \begin{bmatrix} rj_r \\ j_\theta \\ rj_z \end{bmatrix},$$

$$\partial_r(rE_r) + \partial_z(rE_z) = 0, \quad \partial_r(rB_r) + \partial_z(rB_z) = 0. \quad (0.6)$$

As a consequence  $\partial_\theta j_\theta = 0$  and  $\partial_r(rj_r) + \partial_z(rj_z) = 0$ . In many practical cases we will have  $j_r = j_z = 0$ . Introduction of a 'stream function'  $r, z \mapsto \Psi(r, z)$  with  $H_z = \partial_r \Psi$ ,  $H_r = -\partial_z \Psi$  leads to the 2-dimensional Poisson equation

$$(\partial_r \partial_r + \partial_z \partial_z) \Psi = -j_\theta. \quad (0.7)$$

**This equation is exactly the same as (0.4)!** Mathematically.

Next, a simple integration leads to  $rH_\theta$ . Finally  $\underline{E}$  can be solved from the 1st set of equations with the same technique.

This note has been inspired by Elena Lomonova and her students.

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