Monotonicity and Supermodularity Results

for the Erlang Loss System

K.B. Öner, G.P. Kiesmüller, G.J. van Houtum*

Department of Technology Management, Eindhoven University of Technology,

P.O. Box 513, 5600 MB Eindhoven, the Netherlands

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Abstract

For the Erlang loss system with s servers and offered load a, we show that: (i) the load carried by the last server is strictly increasing in a; (ii) the carried load of the whole system is strictly supermodular on $\{(s,a)|s = 0, 1, \dots \text{ and } a > 0\}$.

1. Introduction

Consider the Erlang loss system, also denoted as $M/G/s/s$ queue, with arrival rate $\lambda > 0$, mean service time μ^{-1}, $(\mu > 0)$, and s parallel servers ($s \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$). Its steady-state probability that all servers are busy is equal to

$$B(s,a) = \frac{\frac{a^s}{s!}}{\sum_{i=0}^{s} \frac{a^i}{i!}}, \tag{1}$$

where $a = \lambda\mu^{-1}$ (> 0) is the offered load. The formula in (1) is called the *Erlang loss formula* or *Erlang B formula*, and it was first derived by Erlang [2] for deterministic service times. Later, Sevastyanov [7] showed that $B(s,a)$ is insensitive to the service time distribution; that is, equation (1) is valid for any service time distribution with mean μ^{-1}. The Erlang loss formula occurs in many different applications and its analytical properties are useful for e.g. solving design problems (see [1]).

In the literature, the following properties are known for $B(s,a)$ and related quantities. Karush [4] showed that $B(s,a)$ is strictly convex and decreasing as a function of $s \in \mathbb{N}_0$ (see also Remark 2 in [5]). Harel [3] investigated $B(s,a)$ as a function of the traffic intensity

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Corresponding author, e-mail: g.j.v.houtum@tue.nl, tel: + 31 40 2472230

$\rho = \frac{\lambda}{s\mu}$, service rate μ, and arrival rate λ. He showed that, for each fixed $s \in \mathbb{N}$, there exists a ρ^* such that $B(s,a)$ is strictly convex and increasing in ρ for all $\rho < \rho^*$ and strictly concave and increasing in ρ for all $\rho > \rho^*$. Hence, equivalently, for each fixed $s \in \mathbb{N}$, there exists an a^* such that $B(s,a)$ is strictly convex and increasing in a for all $a < a^*$ and strictly concave and increasing in a for all $a > a^*$. For $s = 0$, $B(s,a) = 1$ for all a, i.e., then $B(s,a)$ is a constant function of a (or ρ). Harel also showed that $B(s,a)$ is strictly convex and decreasing in μ for a fixed λ and $s \in \mathbb{N}$.

The *carried load* $A(s,a)$ is defined as the time-average amount of work carried out by the Erlang loss system, and is equal to

$$A(s,a) = a\,[1 - B(s,a)], \quad s \in \mathbb{N}_0. \tag{2}$$

By the above result of Karush for $B(s,a)$, $A(s,a)$ is strictly concave and increasing in s. Yao and Shanthikumar [9] showed that, the throughput $\lambda[1 - B(s,a)]$ is concave and increasing in λ for a fixed μ. Hence, equivalently, $A(s,a)$ is concave and increasing in a.

The *load carried by the last server* of a system with s servers is defined as the extra load that can be handled in comparison to a system with $s - 1$ servers. This load carried by the last server is denoted by $F_B(s,a)$, and it holds that

$$\begin{aligned} F_B(s,a) &= A(s,a) - A(s-1,a) \\ &= a\,[B(s-1,a) - B(s,a)], \quad s \in \mathbb{N}. \end{aligned} \tag{3}$$

Because of the strict concavity of $A(s,a)$ as a function of s, $F_B(s,a)$ is strictly decreasing in s. The first main result of this paper concerns a monotonicity property for $F_B(s,a)$ as a function of the offered load a.

Theorem 1. *For each $s \in \mathbb{N}$, $F_B(s,a)$ is strictly increasing as a function of $a \in (0, \infty)$.*

The proof of Theorem 1 is lengthy and therefore postponed to Section 2. As $A(s,a) = \sum_{i=1}^{s} F_B(i,a)$, Theorem 1 implies that, for each fixed $s \in \mathbb{N}$, $A(s,a)$ is strictly increasing in a. (For $s = 0$, $A(s,a) = 0$ for all a, i.e., then $B(s,a)$ is a constant function of a.)

Via Theorem 1, we obtain that $A(s,a)$ is strictly supermodular, which is the second main result of this paper. $A(s,a)$ is defined on the set $X = \{(s,a)|s \in \mathbb{N}_0 \text{ and } a \in (0, \infty)\}$, for which we can use the regular '$\leq$' ordering; i.e., for elements $(s,a), (s',a') \in X$, we say that $(s,a) \leq (s',a')$ if and only if $s \leq s'$ and $a \leq a'$. Then the set X is a so-called lattice, and thus the definitions of (strictly) supermodular and submodular functions apply; see p. 43 of Topkis [8].

Theorem 2. *$A(s,a)$ is strictly supermodular on X.*

Proof. Let $s^-, s^+ \in \mathbb{N}_0$ with $s^- < s^+$ and $a^-, a^+ \in (0, \infty)$ with $a^- < a^+$. We must show that

$$A(s^+, a^-) + A(s^-, a^+) < A(s^-, a^-) + A(s^+, a^+). \tag{4}$$

By Theorem 1, we find that

$$
\begin{aligned}
A(s^+, a^-) - A(s^-, a^-) &= \sum_{s=s^-+1}^{s^+} [A(s, a^-) - A(s-1, a^-)] = \sum_{s=s^-+1}^{s^+} F_B(s, a^-) \\
&< \sum_{s=s^-+1}^{s^+} F_B(s, a^+) = \sum_{s=s^-+1}^{s^+} [A(s, a^+) - A(s-1, a^+)] \\
&= A(s^+, a^+) - A(s^-, a^+),
\end{aligned}
$$

which implies (4). $\qquad\square$

Theorems 1 and 2 may be relevant for design problems with the offered load a (or the arrival rate λ when μ is fixed) and the number of servers s as decision variables. To demonstrate this relevance, we exploit Theorem 2 in a simple optimization problem for an Erlang loss system in Section 3.

The main motivation for deriving Theorems 1 and 2 came from a component reliability problem studied in Öner *et al.* [6]. In that paper, a model has been developed for the effect of the reliability level of a single component of a complex capital good on the life cycle costs for the whole installed base of that capital good. In that model, in order to distinguish between fast and slow repair, also the spare parts stock is modeled explicitly; one has fast repair when a spare part is available upon failure of a component, and otherwise repair will take somewhat longer (in that case the failed part itself is repaired as quick as possible). In the resulting optimization problem, one has the reliability level and the spare parts stock as decision variables. These variables play a similar role as the arrival rate λ and the number of servers s of the Erlang loss system. Theorem 1 is applied in the derivation of an efficient optimization procedure.

The rest of this paper consists of Section 2 with the proof of Theorem 1 and Section 3 with an application of Theorem 2.

2. Proof of Theorem 1

$F_B(s, a)$ can be rewritten as

$$F_B(s, a) = a \left[\frac{\frac{a^{s-1}}{(s-1)!}}{\sum_{i=0}^{s-1} \frac{a^i}{i!}} - \frac{\frac{a^s}{s!}}{\sum_{i=0}^{s} \frac{a^i}{i!}} \right] = \frac{\frac{a^s}{s!} \sum_{i=0}^{s-1} (s-i) \frac{a^i}{i!}}{\left(\sum_{i=0}^{s-1} \frac{a^i}{i!} \right) \left(\sum_{i=0}^{s} \frac{a^i}{i!} \right)}, \quad s \in \mathbb{N}. \tag{5}$$

The numerator of the righthand side of equation (5) can be rewritten as

$$\sum_{i=0}^{s-1} \frac{s-i}{s!i!} a^{s+i} = \sum_{i=s}^{2s-1} \frac{2s-i}{s!(i-s)!} a^i.$$

After multiplying the terms in the denominator of equation (5) and by taking the terms with the same power of a together, we find that the denominator is equal to

$$\left(\sum_{i=0}^{s-1} \frac{a^i}{i!}\right)\left(\sum_{i=0}^{s} \frac{a^i}{i!}\right) = \sum_{i=0}^{s-1}\sum_{j=0}^{i} \frac{a^i}{i!(i-j)!} + \sum_{i=s}^{2s-1}\sum_{j=0}^{2s-1-i} \frac{a^i}{(s-j)![i-(s-j)]!}.$$

Hence, $F_B(s,a)$ may be written as

$$F_B(s,a) = \frac{\sum_{i=0}^{2s-1} p_i a^i}{\sum_{i=0}^{2s-1} q_i a^i}, \quad s \in \mathbb{N}, \tag{6}$$

with

$$p_i = \begin{cases} 0 & \text{for } i = 0, 1, \ldots, s-1; \\ \frac{2s-i}{s!(i-s)!} & \text{for } i = s, s+1, \ldots, 2s-1, \end{cases}$$

$$q_i = \begin{cases} \sum_{j=0}^{i} \frac{1}{i!(i-j)!} & \text{for } i = 0, 1, \ldots, s-1; \\ \sum_{j=0}^{2s-1-i} \frac{1}{(s-j)![i-(s-j)]!} & \text{for } i = s, s+1, \ldots, 2s-1. \end{cases}$$

Notice that $p_i > 0$ for $i = s, s+1, \ldots, 2s-1$, and $q_i > 0$ for $i = 0, 1, \ldots, 2s-1$.

Below, in Lemma 1, we show a basic property for the coefficients p_i and q_i. Next, it follows from Lemma 2 and equation (6) that $F_B(s,a)$ is strictly increasing in a. That completes the proof.

Lemma 1. *For all $s \in \mathbb{N}$, it holds that*

$$\frac{p_0}{q_0} \leqslant \frac{p_1}{q_1} \leqslant \frac{p_2}{q_2} \leqslant \ldots \leqslant \frac{p_{s-1}}{q_{s-1}} < \frac{p_s}{q_s} < \frac{p_{s+1}}{q_{s+1}} < \ldots < \frac{p_{2s-1}}{q_{2s-1}}.$$

Proof. The proof is trivial for $s = 1$. In the rest of the proof, we assume that $s \geq 2$. $p_i = 0$ for $i = 0, 1, \ldots, s-1$, $p_s = \frac{1}{(s-1)!} > 0$, and $q_i > 0$ for all i. Hence,

$$\frac{p_0}{q_0} \leqslant \frac{p_1}{q_1} \leqslant \frac{p_2}{q_2} \leqslant \ldots \leqslant \frac{p_{s-1}}{q_{s-1}} < \frac{p_s}{q_s}.$$

Next, we define $\hat{p}_i = p_{2s-1-i}$ and $\hat{q}_i = q_{2s-1-i}$ for $i = 0, 1, \ldots, s-1$. Then,

$$\hat{p}_i = \frac{1+i}{s!(s-1-i)!} \quad \text{for } i = 0, 1, \ldots, s-1,$$

$$\hat{q}_i = \sum_{j=0}^{i} \frac{1}{(s-j)!(s-i+j-1)!} \quad \text{for } i = 0, 1, \ldots, s-1.$$

Below, we show that

$$\frac{\hat{q}_0}{\hat{p}_0} < \frac{\hat{q}_1}{\hat{p}_1} < \frac{\hat{q}_2}{\hat{p}_2} < \ldots < \frac{\hat{q}_{s-1}}{\hat{p}_{s-1}}, \tag{7}$$

which is equivalent to
$$\frac{p_s}{q_s} < \frac{p_{s+1}}{q_{s+1}} < \frac{p_{s+2}}{q_{s+2}} < \ldots < \frac{p_{2s-1}}{q_{2s-1}},$$
and thus completes the proof.

It holds that $\frac{\hat{q}_0}{\hat{p}_0} = 1$, and for $i = 1, \ldots, s-1$,

$$
\begin{aligned}
\frac{\hat{q}_i}{\hat{p}_i} &= \frac{1}{1+i} \sum_{j=0}^{i} \frac{s!(s-1-i)!}{(s-j)!(s-i+j-1)!} \\
&= \frac{1}{1+i} \left[1 + \sum_{j=1}^{i} \frac{s \cdot (s-1) \cdot \ldots \cdot (s-j+1)}{(s-i+j-1) \cdot (s-i+j) \cdot \ldots \cdot (s-i)} \right] \\
&= \frac{1}{1+i} \left[1 + \sum_{j=0}^{i-1} \frac{s \cdot (s-1) \cdot \ldots \cdot (s-j)}{(s-i+j) \cdot (s-i+j-1) \cdot \ldots \cdot (s-i)} \right].
\end{aligned}
$$

This equation may be written as

$$\frac{\hat{q}_i}{\hat{p}_i} = \frac{1}{1+i} \left(1 + \sum_{j=0}^{i-1} \prod_{k=0}^{j} a_{i,k} \right), \tag{8}$$

where $a_{i,k} = \frac{s-k}{s-i+k}$ for $i = 1, 2, \ldots, s-1$ and $k = 0, 1, \ldots, i-1$. By (8), $\frac{\hat{q}_1}{\hat{p}_1} = \frac{1}{2}(1 + \frac{s}{s-1})$, and we find that $\frac{\hat{q}_1}{\hat{p}_1} > 1 = \frac{\hat{q}_0}{\hat{p}_0}$. Via (8), we can also prove that $\frac{\hat{q}_i}{\hat{p}_i} < \frac{\hat{q}_{i+1}}{\hat{p}_{i+1}}$ for $1 \leq i \leq s-2$, where we distinguish the cases with even i and odd i. The proof is similar for both cases; we treat the case with even i.

Let $1 \leq i \leq s-2$ and i is even; notice that this case is only relevant for $s \geq 4$. It holds that $a_{i,\frac{i}{2}} = 1$ and $a_{i,\frac{i}{2}-r} = 1/a_{i,\frac{i}{2}+r}$ for $1 \leq r \leq \frac{i}{2}-1$. That is, the terms $a_{i,1}, a_{i,2}, \ldots, a_{i,\frac{i}{2}-1}$ are reciprocals of the terms $a_{i,i-1}, a_{i,i-2}, \ldots, a_{i,\frac{i}{2}+1}$ and the pairs of reciprocals vanish against each other when they occur in the products $\prod_{k=0}^{j} a_{i,k}$. We find that

$$\prod_{k=0}^{j} a_{i,k} = \prod_{k=0}^{\frac{i}{2}-(j-\frac{i}{2})-1} a_{i,k} = \prod_{k=0}^{i-j-1} a_{i,k} \qquad \text{for } j = \frac{i}{2}, \frac{i}{2}+1, \ldots, i-1,$$

and

$$\sum_{j=\frac{i}{2}}^{i-1} \prod_{k=0}^{j} a_{i,k} = \sum_{j=0}^{\frac{i}{2}-1} \prod_{k=0}^{j} a_{i,k}.$$

Thus, equation (8) for $\frac{\hat{q}_i}{\hat{p}_i}$ can be rewritten as

$$\frac{\hat{q}_i}{\hat{p}_i} = \frac{1}{1+i} \left(1 + 2 \sum_{j=0}^{\frac{i}{2}-1} \prod_{k=0}^{j} a_{i,k} \right). \tag{9}$$

Similarly, we can show that equations (8) for $\frac{\hat{q}_{i+1}}{\hat{p}_{i+1}}$ can be rewritten as

$$\frac{\hat{q}_{i+1}}{\hat{p}_{i+1}} = \frac{1}{2+i} \left(1 + 2 \sum_{j=0}^{\frac{i}{2}-1} \prod_{k=0}^{j} a_{i+1,k} + \prod_{k=0}^{\frac{i}{2}} a_{i+1,k} \right).$$

$\frac{\hat{q}_{i+1}}{\hat{p}_{i+1}}$ is equal to the weighted average of the following terms:

- 1 with weight $\frac{1}{2+i}$,

- $\prod_{k=0}^{j} a_{i+1,k}$ with weight $\frac{2}{2+i}$, $j = 0, 1, \ldots, \frac{i}{2} - 1$,

- $\prod_{k=0}^{\frac{i}{2}} a_{i+1,k}$ with weight $\frac{1}{2+i}$.

Because $a_{i+1,k} > 1$ for all $k = 1, \ldots, \frac{i}{2}$, it holds that $\prod_{k=0}^{\frac{i}{2}} a_{i+1,k} > \prod_{k=0}^{j} a_{i+1,k}$ for $j = 0, 1, \ldots, \frac{i}{2} - 1$ and $\prod_{k=0}^{\frac{i}{2}} a_{i+1,k} > 1$, and thus

$$\frac{\hat{q}_{i+1}}{\hat{p}_{i+1}} > \frac{1}{1+i} \left(1 + 2 \sum_{j=0}^{\frac{i}{2}-1} \prod_{k=0}^{j} a_{i+1,k} \right). \tag{10}$$

As $a_{i,k} < a_{i+1,k}$ for all $k = 0, 1, \ldots, i-1$, combining (9) and (10) shows that $\frac{\hat{q}_i}{\hat{p}_i} < \frac{\hat{q}_{i+1}}{\hat{p}_{i+1}}$. $\quad\square$

Lemma 2. *Let* $n \in \mathbb{N}$ *and* $f(x) = \frac{P(x)}{Q(x)}$, $x \geq 0$, *where*

$$
\begin{aligned}
P(x) &= \sum_{i=0}^{n} u_i x^i \text{ and } u_i \geq 0 \text{ for all } i \in \{0, 1, \ldots, n\}, \\
Q(x) &= \sum_{i=0}^{n} v_i x^i \text{ and } v_i > 0 \text{ for all } i \in \{0, 1, \ldots, n\},
\end{aligned}
$$

and

$$\frac{u_0}{v_0} \leq \frac{u_1}{v_1} \leq \frac{u_2}{v_2} \leq \ldots \leq \frac{u_n}{v_n}. \tag{11}$$

Then $f(x)$ *is increasing. If, in addition,* $\frac{u_i}{v_i} < \frac{u_{i+1}}{v_{i+1}}$ *for some* $i \in \{0, 1, \ldots, n-1\}$, *then* $f(x)$ *is strictly increasing.*

Proof. The derivative of $f(x)$ is equal to

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{N(x)}{Q(x)^2},$$

where

$$N(x) = \left(\sum_{i=1}^{n} i u_i x^{i-1} \right) \left(\sum_{i=0}^{n} v_i x^i \right) - \left(\sum_{i=1}^{n} i v_i x^{i-1} \right) \left(\sum_{i=0}^{n} u_i x^i \right).$$

$N(x)$ can be rewritten as

$$N(x) = \sum_{i=0}^{n-1} A_i x^i + \sum_{i=0}^{n-1} B_i x^{n+i},$$

where the factors A_i and B_i are defined by

$$
\begin{aligned}
A_i &:= \sum_{j=0}^{i} (j+1)(u_{j+1} v_{i-j} - v_{j+1} u_{i-j}), \quad i = 0, 1, \ldots, n-1, \\
B_i &:= \sum_{j=i}^{n-1} (j+1)(u_{j+1} v_{n+i-j} - v_{j+1} u_{n+i-j}), \quad i = 0, 1, \ldots, n-1.
\end{aligned}
$$

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Below, we show that $A_i \geq 0$ and $B_i \geq 0$ for all $i = 0, 1, \ldots, n-1$, which implies that $N(x) \geq 0$ for all $x \geq 0$ and thus $f(x)$ is increasing.

For $n = 1$ and $n = 2$, it is trivial to show that $A_i \geq 0$ for $i = 0, 1, \ldots, n-1$. For $n \geq 3$, the proof is as follows. It is easily verified that $A_0 \geq 0$ and $A_1 \geq 0$. Let i be even and $2 \leq i \leq n-1$. A_i can be written as

$$A_i = \sum_{j=0}^{\frac{i}{2}-1}(j+1)(u_{j+1}v_{i-j} - v_{j+1}u_{i-j}) + \sum_{j=\frac{i}{2}}^{i-1}(j+1)(u_{j+1}v_{i-j} - v_{j+1}u_{i-j})$$
$$+ (i+1)(u_{i+1}v_0 - v_{i+1}u_0).$$

By substituting $m = i - j - 1$ for the second sum in this expression, we find

$$A_i = \sum_{j=0}^{\frac{i}{2}-1}(j+1)(u_{j+1}v_{i-j} - v_{j+1}u_{i-j}) + \sum_{m=0}^{\frac{i}{2}-1}(i-m)(u_{i-m}v_{m+1} - v_{i-m}u_{m+1})$$
$$+ (i+1)(u_{i+1}v_0 - v_{i+1}u_0)$$
$$= \sum_{j=0}^{\frac{i}{2}-1}(i-1-2j)(u_{i-j}v_{j+1} - v_{i-j}u_{j+1}) + (i+1)(u_{i+1}v_0 - v_{i+1}u_0). \qquad (12)$$

By (11), $u_{i-j}v_{j+1} - v_{i-j}u_{j+1} \geq 0$ for all $j = 0, 1, \ldots, \frac{i}{2} - 1$, and $u_{i+1}v_0 - v_{i+1}u_0 \geq 0$, and hence $A_i \geq 0$.

For odd i and $2 \leq i \leq n-1$, A_i can be written as

$$A_i = \sum_{j=0}^{\frac{i-3}{2}}(i-1-2j)(u_{i-j}v_{j+1} - v_{i-j}u_{j+1}) + (i+1)(u_{i+1}v_0 - v_{i+1}u_0), \qquad (13)$$

and, by (11), we find that also then $A_i \geq 0$.

The proof of $B_i \geq 0$ for all $i = 0, 1, \ldots, n-1$ goes along similar lines. This completes the proof that $f(x)$ is increasing.

Finally, note that the term $(i+1)(u_{i+1}v_0 - v_{i+1}u_0)$ occurs in both formula (12) for A_i for even i and in formula (13) for A_i for odd i. Hence, if $\frac{u_i}{v_i} < \frac{u_{i+1}}{v_{i+1}}$ for some $i \in \{0, 1, \ldots, n-1\}$, then $u_{i+1}v_0 - v_{i+1}u_0 > 0$ and thus $A_i > 0$ for that i, which implies that $N(x) > 0$ for all $x > 0$ and $f(x)$ is strictly increasing for all $x \geq 0$. $\qquad \square$

3. Application

Consider an Erlang loss system (e.g., a call center), with arrival rate λ, average service time μ^{-1} (> 0), and $s \in \mathbb{N}_0$ parallel servers. The arrival rate depends on the intensity of advertisements activities; $\lambda \in [\lambda_l, \lambda_u]$, where $0 < \lambda_l < \lambda_u$. One earns a fixed revenue r (> 0) for each served customer, and costs consist of advertisement costs and costs for the servers.

The advertisement costs to obtain an arrival rate λ are given by a function $K(\lambda)$, which is assumed to be increasing and convex on $[\lambda_l, \lambda_u]$. These costs are made per time unit. The cost per server per time unit is c (> 0). The average profit per time unit is denoted by the function $P(s, \lambda)$, and is equal to

$$P(s, \lambda) = rA(s, a) - K(\lambda) - cs, \qquad s \in \mathbb{N}_0, \ \lambda \in [\lambda_l, \lambda_u], \tag{14}$$

where $a = \lambda\mu^{-1}$ is the offered load and $C(s, a)$ is the carried load by the system (cf. the definitions in Section 1).

By Theorem 2, we know that $A(s, a)$ is supermodular in (s, λ), where $(s, \lambda) \in X' = \{(s, \lambda)|s \in \mathbb{N}_0 \text{ and } \lambda \in [\lambda_l, \lambda_u]\}$. As the second and third term on the righthand side of equation (14) only depend on λ and s, respectively, they are also supermodular on X', and this implies that $P(s, \lambda)$ is supermodular on X'. Therefore we obtain the following monotonicity results for optimal solutions.

Suppose that $s \in \mathbb{N}_0$ is fixed and that we are interested in the optimization of λ. $P(s, \lambda)$ is concave in λ, and hence $P(s, \lambda)$ is maximized by

$$\lambda^*(s) := \begin{cases} \lambda_l \text{ if } P(s, \lambda) \text{ is strictly decreasing on } [\lambda_l, \lambda_u]; \\ \lambda_u \text{ if } P(s, \lambda) \text{ is strictly increasing on } [\lambda_l, \lambda_u]; \\ \text{the smallest } \lambda \text{ for which } \frac{\mathrm{d}}{\mathrm{d}\lambda}P(s, \lambda) = 0 \text{ otherwise.} \end{cases}$$

Because of the supermodularity of $P(s, \lambda)$, it holds that $\lambda^*(s)$ is increasing as a function of s. Similarly, we may assume that $\lambda \in [\lambda_l, \lambda_u]$ is fixed and that we want to optimize s. $P(s, \lambda)$ is strictly concave in s, and hence $P(s, \lambda)$ is maximized by

$$s^*(\lambda) := \text{the smallest } s \text{ for which } P(s + 1, \lambda) - P(s, \lambda) \leq 0.$$

Because of the supermodularity of $P(s, \lambda)$, $s^*(\lambda)$ is increasing as a function of λ.

Finally, suppose that we want to optimize both s and λ. Then the above properties can be exploited to obtain the following efficient optimization procedure. First, determine $s_l = s^*(\lambda_l)$ and $s_u = s^*(\lambda_u)$. Notice that there is an optimal solution (s^*, λ^*) with $s^* \in \{s|s_l \leq s \leq s_u\}$. Next, determine $\lambda^*(s)$ for each $s = s_l, s_l + 1, \ldots, s_u$. Finally, an optimal solution (s^*, λ^*) is found as a best solution among the set $\{(s, \lambda^*(s))|s_l \leq s \leq s_u\}$.

Acknowledgements

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