Basic timed process algebra with non-existence ($BPA^{srt}_{\bot}$)

Basic Timed Process Algebra with Non-existence($BPA_{\perp}^{\mathrm{srt}}$)

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Abstract

Recently, a number of errors have been found in *Process Algebra for Hybrid Systems*, which is a well-known formalism for specification of hybrid systems. One of the most basic components of *Process Algebra for Hybrid Systems* is called *Basic Process Algebra with standard relative timing (srt) and Non existence* ($\perp$) abbreviated as $BPA_{\perp}^{srt}$. One of the errors in Process Algebra for Hybrid Systems can be traced down to this most basic component $BPA_{\perp}^{srt}$, therefore we think that fixing $BPA_{\perp}^{srt}$ is essential in rectifying the errors in Process Algebra for Hybrid Systems. Accordingly, in this report we present two proposals for correcting the process algebra $BPA_{\perp}^{\mathrm{srt}}$.

Keywords: Non-existence process constant, standard relative timing, time determinism, transition system, propositions, signals, flows, valuations.

1 Introduction

A hybrid system is one which deals with both discrete as well as continuous changes in the values of model variables. An embedded system where a digital micro controller controls a physical environment is a typical example of a hybrid system. With the advancing use of embedded systems in our daily lives, the interest in ensuring their correct behaviour and verifying certain properties is increasing. One way to gain insight and confidence about the behaviour of such systems is to use formal methods. Many formalisms (automata theory, process algebra and petri nets) have been extended with features to model hybrid systems. Examples are Hybrid Automata [17, 18], Process Algebra for Hybrid Systems ACP_{hs}^{srt} [1], Hybrid χ [20], HyPA [19] and Hybrid Petri Nets [23]. The work presented here is related to the Process Algebra for Hybrid Systems [1], abbreviated as ACP_{hs}^{srt}.

Recently, we discovered some errors in ACP_{hs}^{srt}. It turns out that in Process Algebra for Hybrid Systems, Choice is not associative (Axiom A2), Time determinism (Axiom SRT3) does not hold and a number of other less important axioms are also not sound.

The equational theory of Process Algebra for Hybrid Systems [1] is a blend of various process algebraic theories. Figure 1 shows a hierarchical structure of the theories ACP_{hs}^{srt} and BPA_{hs}^{srt}. BPA_{hs}^{srt} is Basic Process Algebra for Hybrid Systems—i.e. ACP_{hs}^{srt} without concurrency and communication. The discussion in this report does not involve concurrency and is therefore confined to Basic Process Algebra for Hybrid Systems.

BPA_{hs}^{srt} is constructed from some basic theories as follows:

Basic Process Algebra with standard relative timing (BPA^{srt}) [4] is extended with the Non-existence process from [3]. The resulting theory $BPA_{\perp}^{srt}$ is combined with Basic Process Algebra with propositional signals BPA_{ps} [3]. The succeeding algebra BPA_{ps}^{srt} is extended with two new operators needed to specify hybrid behaviour of processes. Then we have built the Basic Process Algebra for Hybrid Systems BPA_{hs}^{srt}. Integration and Recursion further increase the expressiveness of BPA_{hs}^{srt}.

There are two main errors in BPA_{hs}^{srt}. One is that the Choice is not associative. The other is that the axiom of Time Determinism (SRT3) does not hold. If we look at the hierarchical structure of BPA_{hs}^{srt} (see Figure 1), then the unsoundness of axiom SRT3 already appears in $BPA_{\perp}^{srt}$, i.e. there is a $BPA_{\perp}^{srt}$ process term violates axiom SRT3. On the other hand, the associativity of Choice only breaks down at the level of BPA_{ps}^{srt}. In the theories below BPA_{ps}^{srt}, we cannot find an example violating the associativity of Choice. The unsoundness of Time Determinism axiom and non-associativity of Choice propagate in theories above $BPA_{\perp}^{srt}$ and BPA_{ps}^{srt}, where we find more erroneous axioms.

The following examples exhibit violation of axiom SRT3 and non-associativity of choice. An introduction to the semantics of BPA_{hs}^{srt} is given in Appendix A. In the examples below we refer to the transition rules and the definitions of bisimulation given in this appendix.

Example 1 *(Counter-Example to SRT3)*

$$\sigma_{\mathsf{rel}}^{t}(\tilde{\tilde{a}}) + \sigma_{\mathsf{rel}}^{t}(\perp) \neq \sigma_{\mathsf{rel}}^{t}(\tilde{\tilde{a}} + \perp)$$

According to the semantics of BPA_{hs}^{srt} (Rule HS-13) the left-hand side of this inequation can perform a time-step,

$$\langle \sigma_{\mathsf{rel}}^{t}(\tilde{\tilde{a}}) + \sigma_{\mathsf{rel}}^{t}(\perp), \alpha \rangle \xmapsto{t,\rho} \langle \tilde{\tilde{a}}, \alpha' \rangle,$$

because one of the arguments ($\sigma_{\mathsf{rel}}^{t}(\tilde{\tilde{a}})$) can perform that time-step (Rule HS-6), while the other $\sigma_{\mathsf{rel}}^{t}(\perp)$ is consistent (Rule HS-37) but cannot perform this time-step (in particular, Rule HS-6 is not applicable because $\alpha \notin [\mathbf{s}(\perp)]$. On the other hand, the right-hand side of this inequation cannot perform a time-step of duration t, in particular because $\alpha' \notin [\mathbf{s}(\tilde{\tilde{a}} + \perp)]$, and hence Rule $HS - 6$ is not applicable. We conclude that the two processes are not ic-bisimilar (nor bisimilar).

Example 2 *(Non-Associativity of Choice)*

Let p, q, r be process terms, where,

$$\begin{aligned} p &= \sigma_{\mathsf{rel}}^{t}((l = 0) \curlywedge \tilde{\tilde{a}}) \\ q &= \sigma_{\mathsf{rel}}^{t}((l = 1) \curlywedge \tilde{\tilde{b}}) \\ r &= \sigma_{\mathsf{rel}}^{t}(\tilde{\tilde{c}}) \end{aligned}$$

$$(p + q) + r \neq p + (q + r)$$

Note, that for each of the three subprocesses, $\sigma_{\mathsf{rel}}^{t}((l = 0) \curlywedge \tilde{\tilde{a}})$, $\sigma_{\mathsf{rel}}^{t}((l = 1) \curlywedge \tilde{\tilde{b}})$ and $\sigma_{\mathsf{rel}}^{t}(\tilde{\tilde{c}})$, some time-step of duration t is possible, but the evolutions ρ that are visible during this time-step will end in different valuations of l for the first two. This observation shows that Rules HS-12 and HS-13 are not applicable to any combination of two of these three processes. Hence Rule HS-14 must be applied, which synchronizes the evolutions of the two alternatives. Applying rule HS-14 to $(\sigma_{\mathsf{rel}}^{t}((l = 0) \curlywedge \tilde{\tilde{a}}) + \sigma_{\mathsf{rel}}^{t}((l = 1) \curlywedge \tilde{\tilde{b}}))$, which occurs in the left-hand side of our *target inequality, is not possible because the end-valuations of the two processes is different, and hence there is no common ρ on which to synchronize. In other words, $(\sigma^t_{\mathsf{rel}}((l = 0) \curlywedge \tilde{\tilde{a}}) + \sigma^t_{\mathsf{rel}}((l = 1) \curlywedge \tilde{\tilde{b}}))$ cannot delay for a duration t, and using rule HS-13, we conclude that the left-hand side of the inequality can delay for a duration t as process term r and become process $\tilde{\tilde{c}}$.*

Regarding the right-hand side of the inequality, we find that rule HS-14 can be applied to $q + r$, resulting in a time-step with an evolution that ends in the valuation $(l = 1)$. Subsequently, HS-14 is not applicable to the right-hand side as a whole, because there is no common ρ on which p and $q+r$ can synchronize, and rules HS-12 and HS-13 are not applicable because both p and $q+r$ can delay individually.

Hence, the left-hand side can delay with duration t and become $\tilde{\tilde{c}}$ while the right-hand side cannot.

As a first step towards rectifying the mistakes in ACP^{srt}_{hs}, we set ourselves to correcting $BPA^{srt}_{\perp}$. In this report, two proposals for $BPA^{srt}_{\perp}$ are presented. The two proposals differ from each other in the sense that in the first proposal, time determinism holds for the process terms of BPA^{srt} and not for the non-existence process. In the second proposal, time determinism holds for all process terms of $BPA^{srt}_{\perp}$ including the Non-existence process. Both proposals are conservative extensions of BPA^{srt} [4] and $BPA_{\perp}$ [3].

The report is structured as follows: In the preliminaries section, we introduce the reader to Basic Timed Process Algebra with Non-existence ($BPA^{srt}_{\perp}$). In Section 3, we discuss what is wrong with the current presentation of the algebra $BPA^{srt}_{\perp}$ as it is put forward in [1]. We present our first proposal for a corrected $BPA^{srt}_{\perp}$ in Section 4. In this proposal, we leave the semantics intact and change the SRT3 axiom to fit the semantics. Before presenting our second proposal in Section 5, we discuss a number of possible attempts at preserving time determinism with all process terms in the theory. In Section 5.1, we review an instance from literature, where a timed process algebra has been combined with Non-existence process. In our second proposal of $BPA^{srt}_{\perp}$, we modify the semantics so that SRT3 holds for all process terms in the algebra. Section 6 discusses possibilities of extending our proposals for $BPA^{srt}_{\perp}$ to a hybrid process algebra.

2 Preliminaries

2.1 BPA

$BPA^{srt}_{\perp}$ is a combination of Basic Process Algebra (BPA) [2] extended with timing and Non-existence.

BPA can express sequential processes, i.e. processes that perform activities one after another. The set of closed process terms of BPA contains atomic actions from a set of actions 'A' representing independent activities; the deadlock process constant 'δ' representing absence of any activity; a binary operator sequential composition '.' to specify a process followed by another process; and

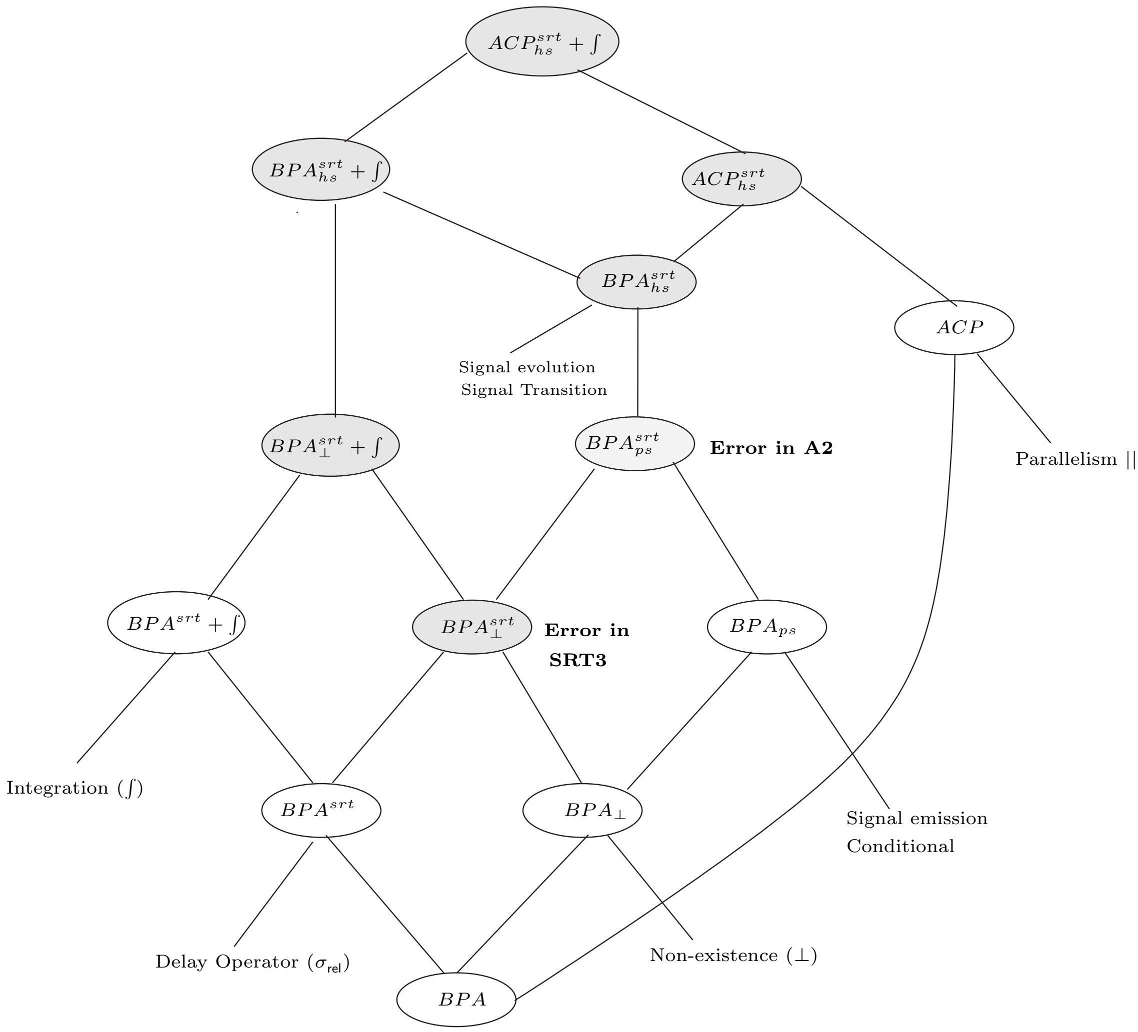

Figure 1: Hierarchical Structure of ACP_{hs}^{srt}

a binary operator alternative composition '+' to specify a choice between two processes.

The set Q of all closed terms of BPA, with $q \in Q$ is given in Table 1.

Table 1: BPA-Syntax Summary ($a \in A$)

$$q ::= a \mid \delta \mid q \cdot q \mid q + q$$

The axioms satisfied by all processes of BPA are given in Table 2.

Table 2: BPA-Axioms

$x + y = y + x$	$A1$
$x + (y + z) = (x + y) + z$	$A2$
$x + x = x$	$A3$
$(x + y) \cdot z = x \cdot z + y \cdot z$	$A4$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$	$A5$
$x + \delta = x$	$A6$
$\delta \cdot x = \delta$	$A7$

BPA is not sufficient to specify processes for which time plays an important role. For example controllers, communication protocols etc. (see [12, 13, 10, 9, 11]). In order to specify time related properties of such processes, BPA must be extended.

2.2 BPA^{srt}

There a number of ways in which timing can be added to BPA. The decisions to be made include whether the time domain is discrete or continuous and whether the time is recorded beginning at the start of a process or a record of time elapsed between events is kept.

Basic Process Algebra for Hybrid Systems is an extension of Basic Process Algebra with standard relative timing BPA^{srt} [4]. The word *standard* indicates that the time domain consists of real numbers—i.e. the time domain is continuous.

In BPA^{srt}, the actions are replaced by immediate actions. Immediate actions ($\tilde{\tilde{a}}, \tilde{\tilde{b}}$) are denoted by an action label $a, b \in A$ with a double tilde $\tilde{\tilde{}}$ on it. An action $\tilde{\tilde{a}}$ performs an action immediately and terminates in the current instance of time. The deadlock process δ of BPA is replaced by immediate deadlock process $\tilde{\tilde{\delta}}$. The process $\tilde{\tilde{\delta}}$ cannot perform an action nor can it flow to a later moment in time. The relative delay operator σ_{rel} adds a delay of non-negative duration before a process. A process $\sigma^0_{\mathsf{rel}}(p)$ behaves the same as process p. The relative undelayable timeout operator ν_{rel} (which we call the *now* operator), blocks the delay behaviour of a process. A process $\nu_{rel}(p)$ performs an action immediately if p can perform an action immediately otherwise $\nu_{rel}(p)$ behaves as a deadlock process.

The set P of BPA^{srt} process terms, with $p \in P$ is given in Table 3.

Table 3: BPA^{srt}- Syntax summary $(a \in A, r > 0)$

p	$::=$	$\tilde{\tilde{a}}$	undelayable action
	$\mid$	$\tilde{\tilde{\delta}}$	undelayable deadlock constant
	$\mid$	$\sigma^{0}_{\mathsf{rel}}(p)$	relative delay of zero duration
	$\mid$	$\sigma^{r}_{\mathsf{rel}}(p)$	relative delay
	$\mid$	$p+p$	alternative composition
	$\mid$	$p \cdot p$	sequential composition
	$\mid$	$\nu_{rel}(p)$	relative undelayable timeout operator/Now operator

The axioms of BPA^{srt} are the axioms of BPA (given in Table 2) extended with the following axioms (see Table 4):

Table 4: Additional axioms for BPA^{srt} $(a \in A, u, v \geq 0, r > 0)$

$x + \tilde{\tilde{\delta}} = x$	$A6SR$
$\tilde{\tilde{\delta}} \cdot x = \tilde{\tilde{\delta}}$	$A7SR$
$\sigma^{0}_{\mathsf{rel}}(x) = x$	$SRT1$
$\sigma^{u}_{\mathsf{rel}}(\sigma^{v}_{\mathsf{rel}}(x)) = \sigma^{u+v}_{\mathsf{rel}}(x)$	$SRT2$
$\sigma^{u}_{\mathsf{rel}}(x) + \sigma^{u}_{\mathsf{rel}}(y) = \sigma^{u}_{\mathsf{rel}}(x+y)$	$SRT3$
$\sigma^{u}_{\mathsf{rel}}(x) \cdot y = \sigma^{u}_{\mathsf{rel}}(x \cdot y)$	$SRT4$
$\nu_{rel}(\tilde{\tilde{a}}) = \tilde{\tilde{a}}$	$SRU1$
$\nu_{rel}(\sigma^{r}_{\mathsf{rel}}(x)) = \tilde{\tilde{\delta}}$	$SRU2$
$\nu_{rel}(x+y) = \nu_{rel}(x) + \nu_{rel}(y)$	$SRU3$
$\nu_{rel}(x \cdot y) = \nu_{rel}(x) \cdot y$	$SRU4$

Axioms A6 and A7 are replaced by axioms A6SR and A7SR which contain the immediate deadlock constant $\tilde{\tilde{\delta}}$ instead of deadlock δ. Axiom SRT1 expresses that adding a delay of zero time units does not alter the behaviour of a process. Axiom SRT2 expresses that consecutive delays can be added. Axiom SRT3 which is called the axiom of time factorization, expresses that a delay cannot make choices. We call it the *axiom of Time Determinism*, as *Time Determinism* is a more well-known term. Since $SRT3$ is central to this report, we explain it in more detail in Section 2.4. Axiom SRT4 reflects that time is counted relative to the last action performed. Axiom SRU1 and SRU2 reflect the fact that the relative undelayable timeout operator ν_{rel} when applied to a process does not change its action behaviour but blocks its initial delay. The now operator distributes over choice (SRU3) and it only effects the initial behaviour of a process (SRU4).

2.3 $BPA_{\perp}^{srt}$

In Process Algebra with Propositional Signals [3], a Basic Process Algebra with Non-existence ($BPA_{\perp}$) is introduced. $BPA_{\perp}$ is BPA extended with the Non-existence process ($\perp$). In [3], the states in a transition system are labelled by propositions. The propositions labelling a state are supposed to hold in that state and are called *signals emitted by the state.* A false proposition can never hold. Hence a process emitting the false signal cannot exist. Here comes the need to introduce a process constant called the non-existence process denoting a process that emits a false signal.

The behaviour of the Non-existence process ($\perp$) is described by the axioms given in Table 5.

Table 5: $BPA_{\perp}$-Additional axioms ($a \in A$)

$\perp + x = \perp$	$NE1$
$\perp \cdot x = \perp$	$NE2$
$a \cdot \perp = \delta$	$NE3$

The root state of a transition system of a Non-existence process ($\perp$) is called an *inconsistent* state. Axiom NE2 expresses that an inconsistent state can never be exited and axiom NE3 reflects that it is not possible to enter such a state from a consistent one.

The signature of $BPA_{\perp}$ is the signature of BPA extended with the Non-existence process. The set $Q_{\perp}$ of $BPA_{\perp}$ process terms, with $q_{\perp} \in Q_{\perp}$ is given in Table 6.

Table 6: $BPA_{\perp}$- Syntax summary ($a \in A$)

$$q_{\perp} \quad ::= \quad a \quad | \; \delta \quad | \; \perp \quad | \; q_{\perp} + \; q_{\perp} \quad | \; q_{\perp} \cdot \; q_{\perp}$$

The axioms of $BPA_{\perp}$ are the axioms of BPA extended with the axioms $NE1, NE2$ and $NE3$ defining the Non-existence process.

Now as depicted in Figure 1, the Basic Timed Process Algebra with Non-existence $BPA_{\perp}^{srt}$ is a combination of BPA^{srt} and $BPA_{\perp}$. The signature of $BPA_{\perp}^{srt}$ is the signature of BPA^{srt} (see Table 3), extended with the Non-existence process constant.

The set $P_{\perp}$ of $BPA_{\perp}^{srt}$ process terms, with $p_{\perp} \in P_{\perp}$ is given in Table 7.

The axioms $BPA_{\perp}^{srt}$ include the axioms for BPA^{srt}, extended with the axioms $NE1, NE2$ and $NE3SR$. The axiom $NE3SR$ is a modification of axiom $NE3$ where an action a is replaced by an undelayable action $\tilde{\tilde{a}}$.

$$\tilde{\tilde{a}} \cdot \perp = \tilde{\tilde{\delta}} \qquad NE3SR$$

Table 7: $BRA_{\perp}^{srt}$- Syntax summary ($a \in A, r > 0$)

$$p_{\perp} \quad ::= \quad \tilde{\tilde{a}} \quad | \; \tilde{\tilde{\delta}} \quad | \; \perp \quad | \; \sigma_{\mathsf{rel}}^{0}(p_{\perp}) \quad | \; \sigma_{\mathsf{rel}}^{r}(p_{\perp}) \quad | \; p_{\perp} + \; p_{\perp} \quad | \; p_{\perp} \cdot \; p_{\perp} \quad | \; \nu_{rel}(p_{\perp})$$

The axioms of $BPA_{\perp}^{srt}$ also include a new axiom representing the effect of the now operator ν_{rel} on non-existence.

$$\nu_{rel}(\perp) = \perp \qquad NESRU$$

A complete set of axioms of $BPA_{\perp}^{srt}$, taken from [1], is given in Table 8.

Table 8: Axioms of $BPA_{\perp}^{srt}$ as in [1] ($a \in A_{\delta}, u, v \geq 0, r > 0$)

$x + y = y + x$	$A1$	$\sigma_{\mathsf{rel}}^{0}(x) = x$	$SRT1$
$(x + y) + z = x + (y + z)$	$A2$	$\sigma_{\mathsf{rel}}^{u}(\sigma_{\mathsf{rel}}^{v}(x)) = \sigma_{\mathsf{rel}}^{u+v}(x)$	$SRT2$
$x + x = x$	$A3$	$\sigma_{\mathsf{rel}}^{u}(x) + \sigma_{\mathsf{rel}}^{u}(y) = \sigma_{\mathsf{rel}}^{u}(x + y)$	$SRT3$
$(x + y) \cdot z = x \cdot z + y \cdot z$	$A4$	$\sigma_{\mathsf{rel}}^{u}(x) \cdot y = \sigma_{\mathsf{rel}}^{u}(x \cdot y)$	$SRT4$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$	$A5$		
$x + \tilde{\tilde{\delta}} = x$	$A6SR$	$\nu_{rel}(\tilde{\tilde{a}}) = \tilde{\tilde{a}}$	$SRU1$
$\tilde{\tilde{\delta}} \cdot x = \tilde{\tilde{\delta}}$	$A7SR$	$\nu_{rel}(\sigma_{\mathsf{rel}}^{r}(x)) = \tilde{\tilde{\delta}}$	$SRU2$
		$\nu_{rel}(x + y) = \nu_{rel}(x) + \nu_{rel}(y)$	$SRU3$
$x + \perp = \perp$	$NE1$	$\nu_{rel}(x \cdot y) = \nu_{rel}(x) \cdot y$	$SRU4$
$\perp \cdot x = \perp$	$NE2$		
$\tilde{\tilde{a}} \cdot \perp = \tilde{\tilde{\delta}}$	$NE3SR$	$\nu_{rel}(\perp) = \perp$	$NESRU$

2.4 Time Determinism

The axiom of time determinism ($SRT3$) occupies a central place in the motivation behind this report. Therefore we pay special attention to it.

Time Determinism is an important property of a large class of timed systems. According to this property, choices between processes cannot be resolved while waiting.

Consider for example a computer application waiting for a key stroke from the user or a signal from the network. The behaviour of the application in the future depends on which action takes place first. No decision can be made while waiting.

Axiom SRT3 reflects that as long as all operands of a choice can idle, the decision of proceeding as one operand or the other is postponed.

$$\sigma_{\mathsf{rel}}^{u}(x) + \sigma_{\mathsf{rel}}^{u}(y) = \sigma_{\mathsf{rel}}^{u}(x + y) \qquad u \geq 0 (SRT3)$$

Time Determinism is a widely accepted property of timed systems. In many timed process algebras, counterparts to axiom SRT3 can be found. See, for example, "Algebra of Timed Processes" [27], Timed CCS [26] and Timed CSP [25].

In the field of hybrid systems, an interesting debate surrounding time determinism exists. In a hybrid system, two delay durations may be accompanied by different evolution of variables during the delay. In such a case, the debate is whether passage of time must resolve a choice, can resolve a choice or cannot resolve a choice between processes. To see arguments in favor of different approaches, see [1], [19] and [20].

In $BPA_{\perp}^{srt}$ as presented in Process Algebra for Hybrid Systems [1], the axiom of time determinism SRT3 is included in the set of axioms. But as we have shown in the Section *Introduction*, SRT3 does not hold in the semantics of [1]. In our first proposal of $BPA_{\perp}^{srt}$, (see Section 4), we replace SRT3 by a conditional axiom that exhibits time determinism in the absence of Non-existence process. The conditional axiom also holds in the semantics of [1]. Altering the axiomatization of $BPA_{\perp}^{srt}$ to fit the semantics is a straightforward solution, but due to the importance of SRT3, we also set our selves to the task of finding a semantics for $BPA_{\perp}^{srt}$, where axiom SRT3 holds. Our search results in our second proposal for $BPA_{\perp}^{srt}$, which is given in Section 5.

3 Violation of Time determinism

When Basic Process Algebra with standard relative timing (BPA^{srt}) is extended with Non-existence, one is faced with the issue of how to combine inconsistency with time.

The semantics of a timed process algebra allows processes to evolve by performing action steps as well as time steps. In the semantics of $BPA_{\perp}$, where only action steps are present, the following axiom holds:

$$a \cdot \perp = \delta \qquad\qquad (NE3)$$

This axiom reflects that a process cannot enter into an inconsistent state after performing an action. An inconsistent state is the root state of the Non-existence process constant.

This view is also adopted by Process Algebra for Hybrid Systems.

In BPA_{hs}^{srt}, a timed counterpart of axiom $NE3$ holds:

$$\tilde{\tilde{a}} \cdot \perp = \tilde{\tilde{\delta}} \qquad\qquad (NE3SR)$$

An introduction to the semantics of BPA_{hs}^{srt} is given in the Appendix A.

In addition to action steps, time steps are also included in the semantics of BPA_{hs}^{srt}. There, as in the case of an action step, a process cannot enter into an inconsistent state after doing a time step. It is mentioned on Page 222 of [1], that:

The process $\sigma^r_{\mathsf{rel}}(\perp)$, $(r > 0)$ is considered to be capable of idling (waiting), but only till arbitrarily close to the point of time that is reached after a period of time r. Thus, just like after performing an action, it is impossible to go on as $\perp$ after idling (waiting) for a period of time.

This characteristic of BPA^{srt}_{hs} is reflected in its transition rule HS-6 given below:

Let x be a process term, α, α' be any variable valuations, $r > 0$ and ρ be a state evolution, describing evolution of variables in the interval $[0, r]$.

$$\frac{\alpha' \in [\mathsf{s}(x)]}{\langle \sigma^r_{\mathsf{rel}}(x), \alpha \rangle \xmapsto{r,\rho} \langle x, \alpha' \rangle} \quad \text{HS-6}$$

(An Introduction to the semantics of BPA^{srt}_{hs} and the complete set of its transition rules is given in Appendix A.)

The rule states that a process $\sigma^r_{\mathsf{rel}}(x)$ can wait for r time units according to any state evolution ρ and become x. The only condition is that the valuation at the end of the delay must satisfy the signal emitted by x. The signal emitted by the Non-existence process $(\perp)$ is false, which cannot be satisfied by any valuation. Therefore, Rule HS-6 cannot be used to derive a time step of duration r for process $\sigma^r_{\mathsf{rel}}(\perp)$.

Since there are no other rules applicable (see Appendix A), therefore a time step of duration r for process $\sigma^r_{\mathsf{rel}}(\perp)$ cannot be derived. In BPA^{srt}_{hs}, the following predicate reflects this fact:

For any valuation α,

$$\langle \sigma^r_{\mathsf{rel}}(\perp), \alpha \rangle \not\xmapsto{r}$$

Rather surprisingly, a consequence of this choice in the semantics of BPA^{srt}_{hs} and its interaction with the rules for alternative composition is that the axiom of Time determinism does not hold. The axiom of time determinism does not hold when one of the process terms x or y is the Non-existence process or a process bisimilar to it. In Section 4, we show that SRT3 holds for all BPA^{srt} processes in the current semantics of [1].

Process Algebra for Hybrid Systems [1] adopts a uniform approach towards inconsistent states with regards to action and time steps. In the semantics of BPA^{srt}_{hs} an inconsistent state is unreachable by action or time steps. But a consequence of this choice together with the design of alternative composition is that the axiom $SRT3$ does not hold with the Non-existence process. In Section 4, we present an algebra $BPA^{srt}_{\perp}$, where we abandon axiom $SRT3$ for the non-existence process constant. In Section 5, we argue that in order to preserve SRT3 with Non-existence, the semantics of $BPA^{srt}_{\perp}$ needs to be modified. In that section, we also present the possibilities of modifications and our second proposal for $BPA^{srt}_{\perp}$. We prove that both our first and second proposals are conservative extensions of BPA^{srt} and $BPA_{\perp}$.

4 $BPA_{\perp}^{srt}$ with conditional Time Determinism

In this Section, we present a proposal for $BPA_{\perp}^{srt}$, in which general time determinism (i.e. time determinism in all cases including the Non-existence process) is **replaced** by conditional time determinism. The section is outlined as follows: first we introduce the conditional axiom that replaces the axiom of time determinism ($SRT3$) in this proposal. Then we give the semantics of this proposal in Section 4.2. A bisimulation is defined for this semantics and we show that bisimulation is a congruence relation. We prove that our first proposal for $BPA_{\perp}^{srt}$ is a conservative extension of $BPA_{\perp}$ and BPA^{srt}. In Section 4.3, the axioms that are sound in this proposal are presented. At the end of this section, we prove that for all $BPA_{\perp}^{srt}$ processes, the semantics of this proposal is equivalent to the semantics of BPA_{hs}^{srt}.

4.1 Axiom of Conditional Time Determinism

In our first proposal for $BPA_{\perp}^{srt}$, the axiom of time determinism holds for all processes that are not bisimilar to $\perp$. This includes all BPA^{srt} processes. The conditional axiom of time determinism, which we call SRTD, is given below:

$$\sigma_{\mathsf{rel}}^{u}(x) + \sigma_{\mathsf{rel}}^{u}(y) = \sigma_{\mathsf{rel}}^{u}(x + y)$$

where $u \geq 0$, x, y are $BPA_{\perp}^{srt}$ processes and none of them is bisimilar to $\perp$.

Later on in Section 4.2, we introduce a predicate `consistent` on process terms which holds for only those processes that are **not** bisimilar to $\perp$.

Then Axiom SRTD can be written as:

$$\begin{array}{ll} \sigma_{\mathsf{rel}}^{u}(x) + \sigma_{\mathsf{rel}}^{u}(y) = \sigma_{\mathsf{rel}}^{u}(x + y) & \qquad (SRTD) \\ \text{where } \langle \texttt{consistent } x \rangle \wedge \langle \texttt{consistent } y \rangle & \end{array}$$

The axiom SRTD reflects that a choice between two processes that do not enter into an inconsistent state at the end of their common delay is postponed till the end of their common delay.

If either of x or y in SRTD is a non-existence process, then the following axiom holds:

$$\sigma_{\mathsf{rel}}^{u+r}(x) + \sigma_{\mathsf{rel}}^{r}(\perp) = \sigma_{\mathsf{rel}}^{u+r}(x) \qquad (SRTD\perp)$$

where $r > 0$,$u \geq 0$ and x is a $BPA_{\perp}^{srt}$ process term.

The axiom SRTD$\perp$ expresses that a delay resolves a choice between two delaying processes, when one of them enters into inconsistency earlier than the other. In that case, the process entering the inconsistency earlier is **dropped** from the choice.

Next we introduce the semantics for this proposal of $BPA_{\perp}^{srt}$.

4.2 Semantics

For $BPA_{\perp}^{srt}$ process terms, the semantics of this proposal is the same as that of BPA_{hs}^{srt}. However, we have simplified it in order not to burden ourselves with unnecessary notations. In $BPA_{\perp}^{srt}$, there are no environment variables whose values need to be tracked, therefore in the semantics given below, variable valuations are not included in transitions and time steps do not contain variable trajectories.

The semantics consists of four relations. They are Action Relations; Time Relations; Termination Predicates; and Consistency Predicates.

The relations are defined below:

1. Action Relations:

   $\rightarrow \subseteq P \times A \times P$

   For $(x, a, x') \in \rightarrow$, we write:

$$\langle x \rangle \xrightarrow{a} \langle x' \rangle$$

2. Time Relations:

   $\mapsto \subseteq P \times \mathbb{R}^{>} \times P$

   For $(x, r, x') \in \mapsto$, we write:

$$\langle x \rangle \xmapsto{r} \langle x' \rangle$$

3. Termination Predicates:

   $\rightarrow_{\surd} \subseteq P \times A$

   For $(x, a) \in \rightarrow_{\surd}$, we write:

$$\langle x \rangle \xrightarrow{a} \surd$$

4. Consistency:

   $\texttt{Consistent} \subseteq P$

   For $(x) \in \texttt{Consistent}$, we write:

$$\langle \texttt{consistent } x \rangle$$

A predicate

$$\langle x \rangle \not\xmapsto{r}$$

stands for the predicate $\nexists x' \in P : \langle x \rangle \xmapsto{r} \langle x' \rangle$.

1. An action step $\langle x \rangle \xrightarrow{a} \langle x' \rangle$ represents that $\langle x \rangle$ can perform action a and proceed as term x';

2. A time step $\langle x \rangle \xmapsto{r} \langle x' \rangle$ represents that $\langle x \rangle$ can idle for r time units and proceed as term x';

3. A termination predicate $\langle x \rangle \xrightarrow{a} \surd$ represents that $\langle x \rangle$ can perform action a and terminate;

4. A predicate $\langle \texttt{consistent } x \rangle$ indicates that the process term x is consistent. The consistency predicate does not hold for the non-existence process constant and all process terms bisimilar to it. For example, $\bot, \sigma^0_{\mathsf{rel}}(\bot), \bot + x, \bot \cdot x$, etc. It holds for all process terms that are not bisimilar to the non-existence process. For example, $\tilde{\tilde{a}}, \tilde{\tilde{\delta}}, etc.$ This predicate is needed to distinguish between $\tilde{\tilde{\delta}}$ and $\bot$.

A set of transition rules for the signature of $BPA^{srt}_{\bot}$ is given in Table 9.

Next, we define a bisimulation on $BPA^{srt}_{\bot}$ process terms. Later on, in Section 4.3 we use this definition and prove that process terms that are derivably equal by the axioms are in fact bisimilar.

Definition 1 *(Bisimulation)*

A relation $R \subseteq P \times P$ on pairs of closed process terms of $BPA^{srt}_{\bot}$ is called a bisimulation relation if and only if the following conditions hold:

For all $a \in A, r > 0, x, y, z \in P$,

1.

$$((x, y) \in R \wedge \langle x \rangle \xrightarrow{a} \langle z \rangle) \implies \exists z' \in P : \langle y \rangle \xrightarrow{a} \langle z' \rangle \textit{ and } (z, z') \in R$$

2.

$$((x, y) \in R \wedge \langle y \rangle \xrightarrow{a} \langle z \rangle) \implies \exists z' \in P : \langle x \rangle \xrightarrow{a} \langle z' \rangle \textit{ and } (z', z) \in R$$

3.

$$((x, y) \in R \wedge \langle x \rangle \xmapsto{r} \langle z \rangle) \implies \exists z' \in P : \langle y \rangle \xmapsto{r} \langle z' \rangle \textit{ and } (z, z') \in R$$

4.

$$((x, y) \in R \wedge \langle y \rangle \xmapsto{r} \langle z \rangle) \implies \exists z' \in P : \langle x \rangle \xmapsto{r} \langle z' \rangle \textit{ and } (z', z) \in R$$

5.

$$(x, y) \in R \implies (\langle x \rangle \xrightarrow{a} \surd \iff \langle y \rangle \xrightarrow{a} \surd)$$

6.

$$(x, y) \in R \implies (\langle \texttt{consistent } x \rangle \iff \langle \texttt{consistent } y \rangle)$$

Two process terms x and y are called bisimilar written $\langle x \rangle \leftrightarrows \langle y \rangle$ if there exists a bisimulation relation R such that $(x, y) \in R$.

Theorem 1 *Bisimulation is a congruence for the signature of $BPA^{srt}_{\bot}$.*

<u>Proof</u> We use the theorem given in [16], to prove that bisimulation is a congruence for the signature of $BPA^{srt}_{\bot}$. The theorem used is given below:

Table 9: Semantics of Proposal 1 for $BPA_{\perp}^{\mathrm{srt}}$ $(a \in A, r, u > 0)$

$$\frac{}{\langle \mathtt{consistent}\ \tilde{\tilde{\delta}} \rangle} \quad \text{P1-}\underline{1}$$

$$\frac{}{\langle \mathtt{consistent}\ \tilde{\tilde{a}} \rangle} \quad \text{P1-}\underline{2} \qquad\qquad \frac{}{\langle \tilde{\tilde{a}} \rangle \xrightarrow{a} \surd} \quad \text{P1-}\underline{3}$$

$$\frac{\langle \mathtt{consistent}\ x \rangle}{\langle \mathtt{consistent}\ \sigma^{0}_{\mathsf{rel}}(x) \rangle} \quad \text{P1-}\underline{4} \qquad\qquad \frac{\langle x \rangle \xrightarrow{a} \surd}{\langle \sigma^{0}_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \surd} \quad \text{P1-}\underline{5}$$

$$\frac{\langle x \rangle \xrightarrow{a} \langle x' \rangle}{\langle \sigma^{0}_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \langle x' \rangle} \quad \text{P1-}\underline{6} \qquad\qquad \frac{\langle x \rangle \xmapsto{r} \langle x' \rangle}{\langle \sigma^{0}_{\mathsf{rel}}(x) \rangle \xmapsto{r} \langle x' \rangle} \quad \text{P1-}\underline{7}$$

$$\frac{}{\langle \mathtt{consistent}\ \sigma^{r}_{\mathsf{rel}}(x) \rangle} \quad \text{P1-}\underline{8} \qquad\qquad \frac{}{\langle \sigma^{r+u}_{\mathsf{rel}}(x) \rangle \xmapsto{u} \langle \sigma^{r}_{\mathsf{rel}}(x) \rangle} \quad \text{P1-}\underline{9}$$

$$\frac{\langle \mathtt{consistent}\ x \rangle}{\langle \sigma^{r}_{\mathsf{rel}}(x) \rangle \xmapsto{r} \langle x \rangle} \quad \text{P1-}\underline{10} \qquad\qquad \frac{\langle x \rangle \xmapsto{u} \langle x' \rangle}{\langle \sigma^{r}_{\mathsf{rel}}(x) \rangle \xmapsto{r+u} \langle x' \rangle} \quad \text{P1-}\underline{11}$$

$$\frac{\langle \mathtt{consistent}\ x \rangle}{\langle \mathtt{consistent}\ x \cdot y \rangle} \quad \text{P1-}\underline{12} \qquad\qquad \frac{\langle x \rangle \xrightarrow{a} \langle x' \rangle}{\langle x \cdot y \rangle \xrightarrow{a} \langle x' \cdot y \rangle} \quad \text{P1-}\underline{13}$$

$$\frac{\langle x \rangle \xrightarrow{a} \surd, \langle \mathtt{consistent}\ y \rangle}{\langle x \cdot y \rangle \xrightarrow{a} \langle y \rangle} \quad \text{P1-}\underline{14} \qquad\qquad \frac{\langle x \rangle \xmapsto{r} \langle x' \rangle}{\langle x \cdot y \rangle \xmapsto{r} \langle x' \cdot y \rangle} \quad \text{P1-}\underline{15}$$

$$\frac{\langle \mathtt{consistent}\ x \rangle, \langle \mathtt{consistent}\ y \rangle}{\langle \mathtt{consistent}\ x + y \rangle} \quad \text{P1-}\underline{16} \qquad\qquad \frac{\langle x \rangle \xrightarrow{a} \surd, \langle \mathtt{consistent}\ y \rangle}{\langle x + y \rangle \xrightarrow{a} \surd} \quad \text{P1-}\underline{17}$$

$$\frac{\langle y \rangle \xrightarrow{a} \surd, \langle \mathtt{consistent}\ x \rangle}{\langle x + y \rangle \xrightarrow{a} \surd} \quad \text{P1-}\underline{18} \qquad\qquad \frac{\langle x \rangle \xrightarrow{a} \langle x' \rangle, \langle \mathtt{consistent}\ y \rangle}{\langle x + y \rangle \xrightarrow{a} \langle x' \rangle} \quad \text{P1-}\underline{19}$$

$$\frac{\langle y \rangle \xrightarrow{a} \langle y' \rangle, \langle \mathtt{consistent}\ x \rangle}{\langle x + y \rangle \xrightarrow{a} \langle y' \rangle} \quad \text{P1-}\underline{20} \qquad\qquad \frac{\begin{array}{c}\langle x \rangle \xmapsto{r} \langle x' \rangle, \\ \langle y \rangle \xmapsto{r} \langle y' \rangle\end{array}}{\langle x + y \rangle \xmapsto{r} \langle x' + y' \rangle} \quad \text{P1-}\underline{21}$$

$$\frac{\langle x \rangle \xmapsto{r} \langle x' \rangle, \langle y \rangle \not\xmapsto{r}, \langle \mathtt{consistent}\ y \rangle}{\langle x + y \rangle \xmapsto{r} \langle x' \rangle} \quad \text{P1-}\underline{22}$$

$$\frac{\langle y \rangle \xmapsto{r} \langle y' \rangle, \langle x \rangle \not\xmapsto{r}, \langle \mathtt{consistent}\ x \rangle}{\langle x + y \rangle \xmapsto{r} \langle y' \rangle} \quad \text{P1-}\underline{23}$$

$$\frac{\langle \mathtt{consistent}\ x \rangle}{\langle \mathtt{consistent}\ \nu_{rel}(x) \rangle} \quad \text{P1-}\underline{24} \qquad\qquad \frac{\langle x \rangle \xrightarrow{a} \surd}{\langle \nu_{\mathrm{rel}}(x) \rangle \xrightarrow{a} \surd} \quad \text{P1-}\underline{25}$$

$$\frac{\langle x \rangle \xrightarrow{a} \langle x' \rangle}{\langle \nu_{\mathrm{rel}}(x) \rangle \xrightarrow{a} \langle x' \rangle} \quad \text{P1-}\underline{26}$$

Let $T = (\Sigma, D)$ be a well-founded, stratifiable term deduction system in panth format then strong bisimulation is a congruence for all function symbols occurring in Σ.

In our case, Σ is the signature of $BPA^{srt}_{\perp}$ and the set of deduction rules D is the set of rules given in Table 9. It is trivial to show that our term deduction system is well-founded and in PANTH format.

Below we give a function and that is a strict stratification for our term deduction system:

The function S when applied to a given transition returns the size of the source process term and when applied to a predicate returns the size of the process term on which the predicate is applied. For a process term p, the size of the process term is denoted by $|p|$.

For the given semantics, S is defined as follows:

$$\begin{array}{l} S(\langle \texttt{consistent}\ \ x\rangle) = |x| \\ S(\langle x\rangle \xrightarrow{a} \surd) = |x| \\ S(\langle x\rangle \xrightarrow{a} \langle x'\rangle) = |x| \\ S(\langle x\rangle \overset{r}{\longmapsto} \langle x'\rangle) = |x| \end{array}$$

The size of a process term is defined as follows:

$$\begin{array}{lcl} |\tilde{\tilde{a}}| & = & 1 \\ |\tilde{\tilde{\delta}}| & = & 1 \\ |\perp| & = & 1 \\ |\sigma^0_{\mathsf{rel}}(x)| & = & |x| + 1 \\ |\sigma^r_{\mathsf{rel}}(x)| & = & |x| + 1 \\ |x + y| & = & |x| + |y| \\ |x \cdot y| & = & |x| + |y| \\ |\nu_{\mathrm{rel}}(x)| & = & |x| + 1 \end{array}$$

⊠

4.3 Axioms

Table 10 contains the set of axioms that we present for this proposal.

In Theorem 2, we prove that process terms that are derivably equal by the axioms given in Table 10 are bisimilar.

Theorem 2 *For all closed terms t_1, t_2 of $BPA^{srt}_{\perp}$, we have,*

$$\textit{Proposal 1}\ \models t_1 = t_2 \implies t_1 \leftrightarrows t_2$$

Table 10: Proposal 1 $BPA_{\perp}^{srt}$- Axioms ($a \in A_{\delta}, u, v, v' \geq 0, r > 0$)

$x + y = y + x$	$A1$
$(x + y) + z = x + (y + z)$	$A2$
$x + x = x$	$A3$
$(x + y) \cdot z = x \cdot z + y \cdot z$	$A4$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$	$A5$
$x + \tilde{\tilde{\delta}} = x$	$A6SR$
$\tilde{\tilde{\delta}} \cdot x = \tilde{\tilde{\delta}}$	$A7SR$
$x + \perp = \perp$	$NE1$
$\perp \cdot x = \perp$	$NE2$
$\tilde{\tilde{a}} \cdot \perp = \tilde{\tilde{\delta}}$	$NE3SR$
$\sigma_{\mathsf{rel}}^{0}(x) = x$	$SRT1$
$\sigma_{\mathsf{rel}}^{u}(\sigma_{\mathsf{rel}}^{v}(x)) = \sigma_{\mathsf{rel}}^{u+v}(x)$	$SRT2$
$\sigma_{\mathsf{rel}}^{u}(x) + \sigma_{\mathsf{rel}}^{u}(y) = \sigma_{\mathsf{rel}}^{u}(x + y)$ if $\langle \texttt{consistent } x \rangle \wedge \langle \texttt{consistent } y \rangle$	$SRTD$
$\sigma_{\mathsf{rel}}^{r+u}(x) + \sigma_{\mathsf{rel}}^{r}(\perp) = \sigma_{\mathsf{rel}}^{r+u}(x)$	$SRTD\perp$
$\sigma_{\mathsf{rel}}^{u}(x) \cdot y = \sigma_{\mathsf{rel}}^{u}(x \cdot y)$	$SRT4$
$\nu_{rel}(\tilde{\tilde{a}}) = \tilde{\tilde{a}}$	$SRU1$
$\nu_{rel}(\sigma_{\mathsf{rel}}^{r}(x)) = \tilde{\tilde{\delta}}$	$SRU2$
$\nu_{rel}(x + y) = \nu_{rel}(x) + \nu_{rel}(y)$	$SRU3$
$\nu_{rel}(x \cdot y) = \nu_{rel}(x) \cdot y$	$SRU4$
$\nu_{rel}(\perp) = \perp$	$NESRU$

Proof The soundness proofs of the axioms are given in Appendix G ⊠

$BPA_{\perp}^{srt}$ is a combination of theories $BPA_{\perp}$ and BPA^{srt} (see Figure 1). Our first proposal for $BPA_{\perp}^{srt}$ is a conservative extension of $BPA_{\perp}$ and BPA^{srt}. By this we mean that Proposal 1 for $BPA_{\perp}^{srt}$ can express all process terms that can be expressed in $BPA_{\perp}$ or BPA^{srt}; the axioms of $BPA_{\perp}$ and that of BPA^{srt} are preserved in Proposal 1; and finally, Proposal 1 does not introduce any new equalities among the process terms of $BPA_{\perp}$ or those of BPA^{srt}. Theorem 4 asserts these observations.

Axiom $SRTD$ of Proposal 1 replaces the time determinism axiom $SRT3$ of BPA^{srt}. In Theorem 3, we claim that Axiom $SRTD$ covers all closed instances of Axiom $SRT3$ in BPA^{srt}. The proof of Theorem 4 uses this fact that replacing $SRT3$ by the conditional axiom $SRTD$ still covers all closed instances of BPA^{srt}.

Theorem 3 *The conditional Time Determinism axiom $SRTD$ of Table 10 covers all closed instances of axiom $SRT3$ for BPA^{srt} process terms.*

Proof The proof consists of the observation that for all BPA^{srt} process terms, the predicate ⟨`consistent` ⟩ holds. ⊠

Theorem 4 *(Conservative Extension)*

1. *Proposal 1 for $BPA_{\perp}^{srt}$ is a conservative extension of $BPA_{\perp}$.*
2. *Proposal 1 for $BPA_{\perp}^{srt}$ is a conservative extension of BPA_{srt}.*

Proof

1. $BPA_{\perp}$
   (a) If a and δ in the signature of $BPA_{\perp}$ (see Table 6) are mapped on to $\tilde{\tilde{a}}$ and $\tilde{\tilde{\delta}}$, then the signature of $BPA_{\perp}^{srt}$ (see Table 7) extends the signature of $BPA_{\perp}$ and the axioms of Proposal 1 (see Table 10) include the axioms of $BPA_{\perp}$.
   Hence, Proposal 1 for $BPA_{\perp}^{srt}$ is an extension of $BPA_{\perp}$.
   (b) All other axioms in Table 10, i.e. Axioms $SRT1$, $SRT2$, $SRTD$, $SRTD\perp$, $SRT4$, $SRU1 - SRU4$, $NESRU$, reason about process terms that are not included in the signature of $BPA_{\perp}$.
   Hence, Proposal 1 for $BPA_{\perp}^{srt}$ is a conservative extension of $BPA_{\perp}$.
2. BPA^{srt}
   (a) The signature of $BPA_{\perp}^{srt}$ (see Table 7) extends the signature of BPA^{srt} (see Table 3).
   Axioms of BPA^{srt} (Table 4) , i.e. $A1-A5, A6SR, A7SR, SRT1, SRT2, SRT4, SRU1-SRU4$ are included in the axioms of Proposal 1 (Table

10). Only Axiom $SRT3$ of BPA^{srt} is not present in Table 10. Theorem 3 proves that all closed instances of SRT3 for BPA^{srt} process terms are covered by axiom$SRTD$.

Hence, Proposal 1 for $BPA_{\perp}^{srt}$ is an extension of BPA_{srt}.

(b) All other axioms in Table 10, i.e. Axioms $NE1, NE2, NE3SR$, $SRTD\perp, NESRU$, reason about process terms that are not included in the signature of BPA^{srt}.

Hence, Proposal 1 for $BPA_{\perp}^{srt}$ is a conservative extension of BPA^{srt}.

⊠

4.4 Proposal 1 versus BPA_{hs}^{srt}

In this section, we discuss the relationship between the semantics of first proposal and the semantics of Process Algebra for Hybrid Systems [1]. We show that for all closed process terms of $BPA_{\perp}^{srt}$, the semantics of the first proposal is equivalent to the semantics of Basic Process Algebra for Hybrid Systems (reviewed in Section 5.1).

When two BPA_{hs}^{srt} processes x and y are ic-bisimilar, then we denote it by $x \underline{\leftrightarrows} y$. The definition of *IC-bisimulation* is given in the Appendix A.

Theorem 5 *Let x and y be closed process terms of $BPA_{\perp}^{srt}$. Then the following holds:*

$$(\textit{Proposal 1}) \quad x \leftrightarrows y \iff (BPA_{hs}^{srt}) \quad x \underline{\leftrightarrows} y$$

The set of closed process terms of $BPA_{\perp}^{srt}$ are neutral with respect to valuations. In a $BPA_{\perp}^{srt}$ process term (see the set of process terms $P_{\perp}$ in Table 7), there are no conditionals, signal emissions, signal evolutions or signal transitions. These operators of [1] allow a BPA_{hs}^{srt} process to behave differently in different valuations. If these operators are absent from a BPA_{hs}^{srt} process term, then it cannot differentiate between valuations. Hence, Proposal 1 being equivalent to the semantics of BPA_{hs}^{srt} for $BPA_{\perp}^{srt}$ terms means that if two terms x and y are bisimilar in Proposal 1, then they are bisimilar for all valuations in BPA_{hs}^{srt}. Not only that, but x and y are ic-bisimilar, since the target process term of a transition is again a $BPA_{\perp}^{srt}$ term–i.e. it is independent of valuations.

We prove the above theorem by proving the following:

Let x be a closed $BPA_{\perp}^{srt}$ process term.

> For every action step, time step , termination predicate and consistency predicate that is derivable for x in the semantics of Proposal 1, there exists a corresponding action step, time step , termination predicate and signal relation for x that is derivable in the semantics of BPA_{hs}^{srt}, and Vice Versa.

These conditions are formally stated in Theorem 6:

Theorem 6 *Let x, x' be closed process terms of $BPA_{\perp}^{srt}$, a be an action, r be a delay duration ($r > 0$), then the following holds:*

1. *(Proposal 1)* $\langle \texttt{consistent}\ x\rangle \iff (BPA_{hs}^{srt})\quad \forall\alpha : \alpha \in [\mathsf{s}(x)]$
2. *(Proposal 1)* $\langle x\rangle \xrightarrow{a} \surd \iff (BPA_{hs}^{srt})\quad \forall\alpha,\alpha' : \langle x,\alpha\rangle \xrightarrow{a} \langle\surd,\alpha'\rangle$
3. *(Proposal 1)* $\langle x\rangle \xrightarrow{a} \langle x'\rangle \iff (BPA_{hs}^{srt})\quad \forall\alpha,\alpha' : \langle x,\alpha\rangle \xrightarrow{a} \langle x',\alpha'\rangle$
4. *(Proposal 1)* $\langle x\rangle \xmapsto{r} \langle x'\rangle \iff (BPA_{hs}^{srt})\quad \forall\rho : \langle x,\alpha_0^{\rho}\rangle \xmapsto{r,\rho} \langle x',\alpha_r^{\rho}\rangle$

where, for some $t > 0$, α_t^{ρ} denotes a valuation that matches with the values assigned to variables by state evolution ρ at time t.

A consistency predicate in Proposal 1 is related to the set of signal relations with all valuations in BPA_{hs}^{srt}. An action and a termination step in Proposal 1 are related to the sets of action steps and termination steps in BPA_{hs}^{srt} with all possible pairs of source and target valuations and so on. Note that the non-derivability of a transition or predicate in Proposal 1 corresponds to non-derivability of a transition or predicate for all valuations in BPA_{hs}^{srt}.
Proof The proof of Theorem 6 is by structural induction on all closed process terms of $BPA_{\perp}^{srt}$. It is given in Appendix D. ⊠

Theorem 5 and Theorem 6 show that our Proposal 1 for $BPA_{\perp}^{srt}$ is equivalent to $BPA_{\perp}^{srt}$ presented in [1]. By Theorem 2, we conclude that $BPA_{\perp}^{srt}$ in [1] will be sound once the axiom SRT3 is replaced by the axiom $SRTD$ exhibiting conditional time determinism. The new axiom $SRTD\perp$ added to the axioms in [1] would reflect that a passage of time makes choices in the presence of the non-existence process constant in the semantics of [1].

4.5 Concluding Remarks

We have presented a proposal for $BPA_{\perp}^{srt}$, where we replace general time determinism by conditional time determinism. The conditional axiom $SRTD$ reflects that a passage of time does not resolve choices between process terms that are consistent now and cannot enter into an inconsistent state at the end of the delay. The behaviour of a choice between processes one of which can enter into an inconsistent state after a delay is expressed in the new axiom $SRTD\perp$. In the latter case, a passage of time makes choices in favor of the process term that stays consistent over time. The semantics of Proposal 1 coincides with that of [1] for $BPA_{\perp}^{srt}$ process terms. Removing the details like valuations and variable trajectories results in a concise and transparent semantics for $BPA_{\perp}^{srt}$.

5 Time Determinism in $BPA_{\perp}^{srt}$

In the previous chapter, we've shown that the time determinism problem of ACP_{hs}^{srt} can be solved partially, by letting time determinism apply only to consistent target processes. However, we feel there is also a need of a variant of $BPA_{\perp}^{srt}$ where the axiom of time determinism holds for all process terms including the Non-existence process. Following are our motivations for searching for a time deterministic $BPA_{\perp}^{srt}$:

1. First, in some schools of process algebra, time determinism is considered to be an essential property of all timed systems.

2. Secondly, it is clear that time determinism in all cases was the intention of the authors of Process Algebra for Hybrid Systems. On Page 222 of [1], the following equation is given as a derivable equation:

$$\sigma_{\mathsf{rel}}^{p+r}(x) + \sigma_{\mathsf{rel}}^{r}(\perp) = \sigma_{\mathsf{rel}}^{r}(\perp) \tag{1}$$

   where $r > 0, p \geq 0$. This equation can only be derived if the time determinism axiom holds for the non-existence process.

3. Thirdly, we found an instance in literature where a timed process algebra is combined with the non-existence process. It is Discrete Time Process Algebra, abbreviated as PA_{psc}^{drt} defined in [6]. The axiom of time determinism holds in PA_{psc}^{drt} for all processes including $\perp$.

To achieve this objective, we tried a number of approaches at finding a suitable semantics for $BPA_{\perp}^{srt}$ before we could reach a relatively satisfactory solution. The axiom of time determinism SRT3 (repeated below),

$$\sigma_{\mathsf{rel}}^{u}(x) + \sigma_{\mathsf{rel}}^{u}(y) = \sigma_{\mathsf{rel}}^{u}(x+y) \qquad u \geq 0 \qquad (SRT3)$$

reasons about two operators, the relative time delay operator and the alternative composition. We try to modify the semantics of these operators so that the axiom of time determinism holds in all cases.

This section is structured as follows: In Section 5.1, we give a brief account of choices in PA_{psc}^{drt} that enable time determinism to hold. In Sections 5.2, 5.3 and 5.4, three different semantics for $BPA_{\perp}^{srt}$ are given. Each semantics implements an idea for preserving time determinism with the Non-existence process. The final attempt namely "Testing for Future Inconsistency " constitutes the semantics of our second proposal of $BPA_{\perp}^{srt}$.

5.1 Time Determinism in PA_{drt}^{psc}

Discrete Time Process Algebra PA_{drt}^{psc} is introduced in [6]. It is an extension of Process Algebra with discrete relative timing (see [5]) and Process Algebra with Propositional Signals [3]. According to our knowledge, the process algebra PA_{drt}^{psc} is the first combination of a timed process algebra with non-existence.

The time domain in PA_{drt}^{psc} is discrete i.e. it is divided into slices or units. In [6], the authors give ample attention to the process $\sigma(\perp)$, which is non-existence process with a unit delay operator.

In PA_{drt}^{psc}, the axiom of time determinism is expressed as follows:

$$\sigma(x) + \sigma(y) = \sigma(x + y) \qquad \text{(DRT1)}$$

where, the operator $\sigma : P \rightarrow P$ adds a delay of a unit time to a process.

There are several factors in Discrete Time Process Algebra which ensure that time determinism holds with Non-existence process. We discuss them one by one below:

1. In the semantics of PA_{psc}^{drt}, a process term with the relative delay operator σ can unconditionally do a time step. Hence, the process term $\sigma(\perp)$ can delay for a unit time and become $\perp$. The following transition can be derived:

   $$\sigma(\perp) \xrightarrow{\sigma} \perp \qquad (2)$$

   (In the semantics of "Discrete Time Process Algebra", transition labels contain an extra symbol called valuation. We ignore this symbol in our discussion.)

2. Another factor contributing towards soundness of Time Determinism axiom is that a time step allows a process term to move into the next time slice and not further in time. For example,

   $$\sigma(\sigma(a)) \xrightarrow{\sigma} \sigma(a)$$

   is allowed. But no rule allows a time step to cross over multiple time slices. Therefore the transition,

   $$\sigma(\sigma(a)) \xrightarrow{\sigma,\sigma} a$$

   is not allowed.

   This property combined with the semantics of alternative composition (described next) contributes to time determinism in PA_{psc}^{drt}.

3. Finally, the alternative composition is defined as follows:

   Consider an alternative composition $p + q$. It can delay as follows:

   > If both p and q can do a unit delay to the next time slice, then $p + q$ can delay for a unit time such that the choice is retained after the delay.

   For example

   $$\sigma(\sigma(a)) + \sigma(\perp) \xrightarrow{\sigma} \sigma(a) + \perp$$

   > An alternative composition can proceed as one operand only if the other operand cannot perform the same time step. The root signal of the passive operand must be satisfied at the start.

Now the process term $\sigma(a) + \perp$ cannot delay further. Because the root signal of the right operand $\perp$ is false and it cannot be satisfied.

In PA_{drt}^{psc}, like $BPA_{\perp}$ (see Section), it is not possible to reach an inconsistent state by performing an action. In PA_{drt}^{psc}, a counterpart of axiom NE3 holds:

$$\underline{\underline{a}} \cdot \perp = \underline{\underline{\delta}} \tag{3}$$

(An undelayable action a and undelayable deadlock constants are denoted by $\underline{\underline{a}}$ and $\underline{\underline{\delta}}$ respectively in PA_{drt}^{psc}.)

In contrast to an action step, a process can delay for a unit time and then enter into an inconsistent state as shown in Transition 2. Hence, there is non-uniformity between action steps and time steps with regards to an inconsistent state. The authors of [6] explain this non-uniformity as follows:

Suppose, $\sigma(\perp)$ is not allowed to delay and is put equal to deadlock.

$$\sigma(\perp) = \underline{\underline{\delta}} \tag{4}$$

Then by using axioms of Basic Discrete Timed Process Algebra (given in Appendix C), Axiom NE1 and Equation 4, the following can be derived:

$$\begin{array}{rcll} \sigma(x) & = & \sigma(x) + \underline{\underline{\delta}} & \text{By DRT4A} \\ & = & \sigma(x) + \sigma(\perp) & \text{by Equation 4} \\ & = & \sigma(x + \perp) & \text{By axiom DRT1} \\ & = & \sigma(\perp) & \text{By Axiom NE1} \\ & = & \underline{\underline{\delta}} & \text{By Equation 4} \end{array}$$

Hence, allowing Equation 4 in PA_{drt}^{psc}, leads to undesirable results.

We see that certain choices in the semantics, help preserve the axiom of time determinism with the non-existence process in PA_{drt}^{psc}. Keeping in view this process algebra, we look for a time deterministic $BPA_{\perp}^{srt}$. The time domain in process algebra $BPA_{\perp}^{srt}$ is continuous which offers different challenges than the discrete time domain present in PA_{drt}^{psc}. Also, recall from Section 3, that in the semantics of [1], uniformity between action steps and time steps with regards to inconsistent states was intended. The combination of two goals, i.e. preserving axiom SRT3 unconditionally and a uniform approach towards inconsistency further complicates our task. In the next sections, we describe the attempts undertaken to construct a desired semantics for $BPA_{\perp}^{srt}$.

5.2 Modifying Alternative Composition

In this section, we present a semantics for $BPA_{\perp}^{srt}$ where the definition of alternative composition is modified so that the axiom of Time Determinism (SRT3) holds.

The semantics of $BPA_{\perp}^{srt}$ presented here has the same transition relations as defined for $BPA_{\perp}^{srt}$ with conditional time determinism (See Section 4).

They are: The Action Relation ($\rightarrow$); the Time Relation ($\mapsto$); the Termination Predicates ($\rightarrow_{\surd}$); and the Consistency Predicates (`Consistent`).

The transition rules for this semantics are in Table 11.
Important features of this semantics are given below:

1. An inconsistent state cannot be reached after a time step.

   Consider Rule AC- 9:
$$\frac{\langle \texttt{consistent } x \rangle}{\langle \sigma^r_{\mathsf{rel}}(x) \rangle \overset{r}{\longmapsto} \langle x \rangle}$$

   Hence, the following predicate holds:
$$\sigma^r_{\mathsf{rel}}(\perp) \overset{r}{\not\longmapsto}$$

2. An alternative composition $p+q$ is allowed to delay in one of the following ways:

   (a) If both p and q can delay for a non-zero duration, then they delay together for a duration less than or equal to their common duration. At the end of this time step, the choice between operands is unresolved.

   For example, a process term $\sigma^5_{\mathsf{rel}}(\tilde{\tilde{a}}) + \sigma^3_{\mathsf{rel}}(\tilde{\tilde{b}})$ can delay as follows:
$$\sigma^5_{\mathsf{rel}}(\tilde{\tilde{a}}) + \sigma^3_{\mathsf{rel}}(\tilde{\tilde{b}}) \overset{3}{\longmapsto} \sigma^2_{\mathsf{rel}}(\tilde{\tilde{a}}) + \tilde{\tilde{b}} \tag{5}$$

   (b) The term $p+q$ can delay as p only if q cannot do a time step of the same duration as p. The transition system of q has a consistent root. Also q cannot do time steps of any smaller durations.

   Similarly, $p+q$ can delay as q, if p satisfies the conditions mentioned above.

   For example, a process term $\sigma^2_{\mathsf{rel}}(\tilde{\tilde{a}}) + \tilde{\tilde{b}}$ can delay as follows:
$$\sigma^2_{\mathsf{rel}}(\tilde{\tilde{a}}) + \tilde{\tilde{b}} \overset{2}{\longmapsto} \tilde{\tilde{a}} \tag{6}$$

   (c) A time step for $p+q$ can also be a finite sequence of time steps, each of which has been obtained from one of the two ways described above.

   Hence, the process term $\sigma^5_{\mathsf{rel}}(\tilde{\tilde{a}}) + \sigma^3_{\mathsf{rel}}(\tilde{\tilde{b}})$ can also delay as follows:
$$\sigma^5_{\mathsf{rel}}(\tilde{\tilde{a}}) + \sigma^3_{\mathsf{rel}}(\tilde{\tilde{b}}) \overset{5}{\longmapsto} \tilde{\tilde{a}} \tag{7}$$

   Rules AC-19, AC-20 and AC-21 define the delay behaviour of an alternative composition.

   Rule AC-26 allows a time step that is a sequence of two time steps. Applying this rule a finite number of times allows one to derive a time step by appending multiple time steps.

Table 11: $BPA_{\perp}^{\mathrm{srt}}$-Modifying Choice ($a \in A, r, u > 0$)

$\dfrac{}{\langle \texttt{consistent}\ \tilde{\tilde{\delta}} \rangle}$	AC-1	$\dfrac{}{\langle \texttt{consistent}\ \tilde{\tilde{a}} \rangle}$	AC-2
$\dfrac{}{\langle \tilde{\tilde{a}} \rangle \xrightarrow{a} \surd}$	AC-3	$\dfrac{\langle x \rangle \xrightarrow{a} \langle x' \rangle}{\langle \sigma^{0}_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \langle x' \rangle}$	AC-4
$\dfrac{\langle x \rangle \xrightarrow{a} \surd}{\langle \sigma^{0}_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \surd}$	AC-5	$\dfrac{\langle x \rangle \overset{r}{\longmapsto} \langle x' \rangle}{\langle \sigma^{0}_{\mathsf{rel}}(x) \rangle \overset{r}{\longmapsto} \langle x' \rangle}$	AC-6
$\dfrac{\langle \texttt{consistent}\ x \rangle}{\langle \texttt{consistent}\ \sigma^{0}_{\mathsf{rel}}(x) \rangle}$	AC-7	$\dfrac{}{\langle \sigma^{r+u}_{\mathsf{rel}}(x) \rangle \overset{u}{\longmapsto} \langle \sigma^{r}_{\mathsf{rel}}(x) \rangle}$	AC-8
$\dfrac{\langle \texttt{consistent}\ x \rangle}{\langle \sigma^{r}_{\mathsf{rel}}(x) \rangle \overset{r}{\longmapsto} \langle x \rangle}$	AC-9	$\dfrac{}{\langle \texttt{consistent}\ \sigma^{r}_{\mathsf{rel}}(x) \rangle}$	AC-10
$\dfrac{\langle x \rangle \xrightarrow{a} \langle x' \rangle}{\langle x \cdot y \rangle \xrightarrow{a} \langle x' \cdot y \rangle}$	AC-11	$\dfrac{\langle x \rangle \xrightarrow{a} \surd, \langle \texttt{consistent}\ y \rangle}{\langle x \cdot y \rangle \xrightarrow{a} \langle y \rangle}$	AC-12
$\dfrac{\langle x \rangle \overset{r}{\longmapsto} \langle x' \rangle}{\langle x \cdot y \rangle \overset{r}{\longmapsto} \langle x' \cdot y \rangle}$	AC-13	$\dfrac{\langle \texttt{consistent}\ x \rangle}{\langle \texttt{consistent}\ x \cdot y \rangle}$	AC-14
$\dfrac{\langle x \rangle \xrightarrow{a} \langle x' \rangle, \langle \texttt{consistent}\ y \rangle}{\langle x + y \rangle \xrightarrow{a} \langle x' \rangle}$	AC-15	$\dfrac{\langle y \rangle \xrightarrow{a} \langle y' \rangle, \langle \texttt{consistent}\ x \rangle}{\langle x + y \rangle \xrightarrow{a} \langle y' \rangle}$	AC-16
$\dfrac{\langle x \rangle \xrightarrow{a} \surd, \langle \texttt{consistent}\ y \rangle}{\langle x + y \rangle \xrightarrow{a} \surd}$	AC-17	$\dfrac{\langle y \rangle \xrightarrow{a} \surd, \langle \texttt{consistent}\ x \rangle}{\langle x + y \rangle \xrightarrow{a} \surd}$	AC-18
$\dfrac{\langle x \rangle \overset{r}{\longmapsto} \langle x' \rangle,\ \langle y \rangle \overset{r}{\longmapsto} \langle y' \rangle}{\langle x + y \rangle \overset{r}{\longmapsto} \langle x' + y' \rangle}$	AC-19	$\dfrac{\langle x \rangle \overset{r}{\longmapsto} \langle x' \rangle, \langle \texttt{consistent}\ y \rangle,\ \langle y \rangle \overset{r}{\not\longmapsto}, (\forall s < r\ \ \langle y \rangle \overset{s}{\not\longmapsto})}{\langle x + y \rangle \overset{r}{\longmapsto} \langle x' \rangle}$	AC-20
$\dfrac{\langle y \rangle \overset{r}{\longmapsto} \langle y' \rangle, \langle \texttt{consistent}\ x \rangle,\ \langle x \rangle \overset{r}{\not\longmapsto}, (\forall s < r\ \ \langle x \rangle \overset{s}{\not\longmapsto})}{\langle x + y \rangle \overset{r}{\longmapsto} \langle y' \rangle}$	AC-21	$\dfrac{\langle \texttt{consistent}\ x \rangle, \langle \texttt{consistent}\ y \rangle}{\langle \texttt{consistent}\ x + y \rangle}$	AC-22
$\dfrac{\langle x \rangle \xrightarrow{a} \langle x' \rangle}{\langle \nu_{\mathrm{rel}}(x) \rangle \xrightarrow{a} \langle x' \rangle}$	AC-23	$\dfrac{\langle x \rangle \xrightarrow{a} \surd}{\langle \nu_{\mathrm{rel}}(x) \rangle \xrightarrow{a} \surd}$	AC-24
$\dfrac{\langle \texttt{consistent}\ x \rangle}{\langle \texttt{consistent}\ \nu_{rel}(x) \rangle}$	AC-25	$\dfrac{\langle x \rangle \overset{r}{\longmapsto} \langle x' \rangle, \langle x' \rangle \overset{s}{\longmapsto} \langle x'' \rangle}{\langle x \rangle \overset{r+s}{\longmapsto} \langle x'' \rangle}$	AC-26

In the semantics of $BPA_{\perp}^{srt}$ presented in Table 11, the axiom of time determinism holds. In order to prove the soundness of Axiom SRT3, we need a notion of bisimulation. The semantics in this section uses exactly the same relations as used in the semantics of first proposal for $BPA_{\perp}^{srt}$. Therefore, we use Definition 1 of Section 4.

Theorem 7 *Axiom SRT3 is sound in the semantics of Table 11.*

Proof The proof is given in Appendix E. ⊠

Consider the following equation derivable from Axiom SRT3:

$$\sigma_{\mathsf{rel}}^{u+r}(x) + \sigma_{\mathsf{rel}}^{r}(\perp) = \sigma_{\mathsf{rel}}^{r}(\perp) \tag{8}$$

where $u \geq 0$, $r > 0$.

In the semantics of $BPA_{\perp}^{srt}$ presented in this section, Equation 8 holds. The time steps derivable in this semantics for the left and right hand sides of Equation 8 are given below:

1. $\sigma_{\mathsf{rel}}^{r}(\perp)$:

   An inconsistent state is not reachable by a time step. Rule AC-9 cannot be applied to $\sigma_{\mathsf{rel}}^{r}(\perp)$. Since no other rules are applicable, hence we infer:

   $$\langle \sigma_{\mathsf{rel}}^{r}(\perp) \rangle \not\overset{r}{\longmapsto}$$

   For all $s, t > 0$, such that $r = s + t$, Rule AC-8 can derive the following time step for $\sigma_{\mathsf{rel}}^{r}(\perp)$:

   $$\langle \sigma_{\mathsf{rel}}^{s+t}(\perp) \rangle \overset{s}{\longmapsto} \langle \sigma_{\mathsf{rel}}^{t}(\perp) \rangle$$

   After each such time step, the resulting process term $\sigma_{\mathsf{rel}}^{t}(\perp)$ is again delayable. The time domain in $BPA_{\perp}^{srt}$ is dense. I.e. between any two real numbers, there exists an infinite number of real numbers. Hence, for any $t > 0$, there exist infinite numbers that are greater than zero and less than t. The process $\sigma_{\mathsf{rel}}^{r}(\perp)$ can do infinite time steps getting closer and closer to $\sigma_{\mathsf{rel}}^{0}(\perp)$ but not reaching it.

2. $\sigma_{\mathsf{rel}}^{u+r}(x) + \sigma_{\mathsf{rel}}^{r}(\perp)$:

   A time step for process term $\sigma_{\mathsf{rel}}^{u+r}(x) + \sigma_{\mathsf{rel}}^{r}(\perp)$ can be derived by Rule AC-19 and Rule AC-26. Applying Rule AC-19 on $\sigma_{\mathsf{rel}}^{u+r}(x) + \sigma_{\mathsf{rel}}^{r}(\perp)$, we can derive the following time steps:

   For all $s, t > 0$, such that $r = s + t$:

   $$\langle \sigma_{\mathsf{rel}}^{u+s+t}(x) + \sigma_{\mathsf{rel}}^{s+t}(\perp) \rangle \overset{s}{\longmapsto} \langle \sigma_{\mathsf{rel}}^{u+t}(x) + \sigma_{\mathsf{rel}}^{t}(\perp) \rangle$$

   Taking closure of successive time steps obtained by Rule AC-19 using Rule AC-26 results in a time step that has a target process term of the form $\sigma_{\mathsf{rel}}^{u+t}(x) + \sigma_{\mathsf{rel}}^{t}(\perp)$ for some $t > 0$.

Rule AC-20 (similarly Rule AC-21) is not applicable on $\sigma_{\mathsf{rel}}^{u+r}(x) + \sigma_{\mathsf{rel}}^{r}(\perp)$ as the condition that the right (left) alternative cannot delay at all is not satisfied. Also Rule AC-20 (similarly Rule AC-21) is not applicable on the target process terms of any transitions that have been derived by applying Rule AC-19 and Rule AC-26 on $\sigma_{\mathsf{rel}}^{u+r}(x) + \sigma_{\mathsf{rel}}^{r}(\perp)$.

We conclude that the left hand side and right hand side of Equation 8 are bisimilar in this semantics.

Contrary to the approach here, in Proposal 1 of $BPA_{\perp}^{srt}$ (see Section 4), a delay is allowed to resolve a choice between two delaying processes, when one of them enters into inconsistency earlier than the other. In that case, the process entering the inconsistency earlier is dropped from the choice. Axiom $SRTD\perp$ of Proposal 1, which is repeated below, reflects this fact:

$$\sigma_{\mathsf{rel}}^{u+r}(x) + \sigma_{\mathsf{rel}}^{r}(\perp) = \sigma_{\mathsf{rel}}^{r+u}(x) \qquad (SRTD\perp)$$

Considering the semantics of Table 11 a suitable semantics for a time deterministic $BPA_{\perp}^{srt}$, we investigate into further extending this semantics with other operators of BPA_{hs}^{srt} [1]. We find a problem in extending it with integration.

Integration represents an alternative composition over an infinite set of alternatives. It is briefly explained in Section 6.1. The set of transition rules for integration and axioms holding in BPA^{srt} [4] are given in the Appendix B.

The semantic rules for integration in BPA^{srt} are defined along the same lines as alternative composition. Following this approach, in the above semantics of $BPA_{\perp}^{srt}$, a rule for deriving a time step for an integral $\int_{u\in U} F(u)$ would allow it to delay for a duration that is less than or equal to a duration common among all delaying process terms in the set $\{F(p) \mid p \in U\}$. For example, for the above semantics, the following seems to be a good candidate of a transition rule allowing a process term $\int_{u\in U} F(u)$ to delay:

We call it Rule **<u>AC</u>-27**:

$$\frac{\begin{array}{l}\{F(q) \overset{r}{\longmapsto} F_1(q) \mid q \in U_1\}\\ \vdots \\ \{F(q) \overset{r}{\longmapsto} F_n(q) \mid q \in U_n\}\\ \{F(q) \overset{r}{\not\longmapsto}, \langle \texttt{consistent}\ \ F(q)\rangle,\\ \forall s < r, \quad F(q) \overset{s}{\not\longmapsto} \mid q \in U_{n+1}\}\end{array}}{\int_{u\in U} F(u) \overset{r}{\longmapsto} \int_{u\in U_1} F_1(u) + \ldots + \int_{u\in U_n} F_n(u)} \quad \begin{array}{l}\{U_1, \ldots U_n\} \text{ is a partition of}\\ U\backslash U_{n+1}, U_{n+1} \subset U\end{array}$$

The above Rule indicates that a process term $\int_{u\in U} F(u)$ can delay for a certain duration, if for a nonempty subset U' of U, all process terms $F(q)$, with $q \in U'$ can delay for that duration. The set U' is partitioned into n sets $\{U_1 \ldots U_n\}$. For each set U_i, $F(q)$ with $q \in U_i$ may evolve into a different process term after the delay. Whereas all process terms $F(q)$, with $q \in U\backslash U'$ have consistent roots and cannot perform a delay of that duration or any smaller delay.

Now consider the process term $\int_{u>0}\sigma^u_{\mathsf{rel}}(\tilde{\tilde{a}})$. In the set $\{\sigma^u_{\mathsf{rel}}(\tilde{\tilde{a}}) \mid u > 0\}$, all process terms are delayable, but we cannot determine a delay duration that is common among all members of the set as there does not exist a smallest real number greater than zero. Hence, when the semantics of $BPA^{srt}_\perp$ under discussion is extended with integration, then a time step for the process $\int_{u>0}\sigma^u_{\mathsf{rel}}(\tilde{\tilde{x}})$ cannot be derived. Infact, when we extend this semantics with integration, we find that the following equation holds:

$$\int_{u>0}\sigma^u_{\mathsf{rel}}(x) = \tilde{\tilde{\delta}} \tag{9}$$

for any process term x.

By modifying alternative composition as proposed above in $BPA^{srt}_\perp$, we can save the axiom of time determinism, but then we cannot add integration to this modified $BPA^{srt}_\perp$ as it has been added to BPA^{srt}.

Hence, we decide to look for other approaches for preserving time determinism with Non-existence process in $BPA^{srt}_\perp$.

5.3 Modifying the Relative Delay Operator σ^r_{rel}

Now we focus on changing the semantics of the other operator in the axiom of time determinism, i.e. the relative delay operator.

In the semantics presented here, we modify the semantics of the delay operator σ^r_{rel}, so that an inconsistent state is reachable by a time step. The semantics uses the same relations as the semantics for $BPA^{srt}_\perp$ with conditional time determinism (see Section 4), i.e. the Action Relation ($\rightarrow$), the Time Relation ($\mapsto$), the Termination Predicates ($\rightarrow_{\surd}$) and the Consistency Predicates (`Consistent`).

The transition rules for this semantics are given in Table 12.

We discuss the important features of this semantics below:

1. The semantics of the delay operator σ^r_{rel}, with $r > 0$, has been modified. Now a process term $\sigma^r_{\mathsf{rel}}(x)$ can delay unconditionally for r time units. Hence, the process term $\sigma^t_{\mathsf{rel}}(\perp)$ can delay for t seconds and become $\perp$. The following transition is derivable:

$$\sigma^r_{\mathsf{rel}}(\perp) \xmapsto{r} \perp$$

2. In order to preserve Axiom SRT3, the delay behaviour of an alternative composition is defined as follows:

   Consider an alternative composition, $p + q$. It can delay in one of the following ways:

   (a) If both p and q can delay for a given duration, then they delay together and the choice is retained at the end of their common delay.

   (b) The term $p + q$ can delay as p only if q cannot do a time step of the same duration as p; the transition system of q has a consistent root; also if q can do time steps of a **smaller** duration then all such time steps of q must end in processes with consistent states.

(c) Similarly, $p + q$ can behave also as q if p fulfills the conditions mentioned above for q. I.e. p cannot do a time step of the same duration as q; the transition system of p has a consistent root; also if p can do time steps of a **smaller** durations then all such time steps of p must end in processes with consistent states.

Rules RI-20, RI-21 and RI-22 define the delay behaviour of an alternative composition.

Table 12: $BPA_{\perp}^{\mathrm{srt}}$-Modifying Relative Delay $\sigma_{\mathsf{rel}}^{r}$ $(a \in A, r, u > 0)$

$\dfrac{}{\langle \mathtt{consistent}\ \tilde{\tilde{\delta}} \rangle}$	RI-$\underline{1}$	$\dfrac{}{\langle \mathtt{consistent}\ \tilde{\tilde{a}} \rangle}$	RI-$\underline{2}$
$\dfrac{}{\langle \tilde{\tilde{a}} \rangle \xrightarrow{a} \surd}$	RI-$\underline{3}$	$\dfrac{\langle x \rangle \xrightarrow{a} \langle x' \rangle}{\langle \sigma_{\mathsf{rel}}^{0}(x) \rangle \xrightarrow{a} \langle x' \rangle}$	RI-$\underline{4}$
$\dfrac{\langle x \rangle \xrightarrow{a} \surd}{\langle \sigma_{\mathsf{rel}}^{0}(x) \rangle \xrightarrow{a} \surd}$	RI-$\underline{5}$	$\dfrac{\langle x \rangle \stackrel{r}{\longmapsto} \langle x' \rangle}{\langle \sigma_{\mathsf{rel}}^{0}(x) \rangle \stackrel{r}{\longmapsto} \langle x' \rangle}$	RI-$\underline{6}$
$\dfrac{\langle \mathtt{consistent}\ x \rangle}{\langle \mathtt{consistent}\ \sigma_{\mathsf{rel}}^{0}(x) \rangle}$	RI-$\underline{7}$	$\dfrac{}{\langle \sigma_{\mathsf{rel}}^{r+u}(x) \rangle \stackrel{u}{\longmapsto} \langle \sigma_{\mathsf{rel}}^{r}(x) \rangle}$	RI-$\underline{8}$
$\dfrac{}{\langle \sigma_{\mathsf{rel}}^{r}(x) \rangle \stackrel{r}{\longmapsto} \langle x \rangle}$	RI-$\underline{9}$	$\dfrac{\langle x \rangle \stackrel{u}{\longmapsto} \langle x' \rangle}{\langle \sigma_{\mathsf{rel}}^{r}(x) \rangle \stackrel{r+u}{\longmapsto} \langle x' \rangle}$	RI-$\underline{10}$
$\dfrac{}{\langle \mathtt{consistent}\ \sigma_{\mathsf{rel}}^{r}(x) \rangle}$	RI-$\underline{11}$	$\dfrac{\langle x \rangle \xrightarrow{a} \langle x' \rangle}{\langle x \cdot y \rangle \xrightarrow{a} \langle x' \cdot y \rangle}$	RI-$\underline{12}$
$\dfrac{\langle x \rangle \xrightarrow{a} \surd, \langle \mathtt{consistent}\ y \rangle}{\langle x \cdot y \rangle \xrightarrow{a} \langle y \rangle}$	RI-$\underline{13}$	$\dfrac{\langle x \rangle \stackrel{r}{\longmapsto} \langle x' \rangle}{\langle x \cdot y \rangle \stackrel{r}{\longmapsto} \langle x' \cdot y \rangle}$	RI-$\underline{14}$
$\dfrac{\langle \mathtt{consistent}\ x \rangle}{\langle \mathtt{consistent}\ x \cdot y \rangle}$	RI-$\underline{15}$	$\dfrac{\langle x \rangle \xrightarrow{a} \langle x' \rangle, \langle \mathtt{consistent}\ y \rangle}{\langle x + y \rangle \xrightarrow{a} \langle x' \rangle}$	RI-$\underline{16}$
$\dfrac{\langle x \rangle \xrightarrow{a} \langle y' \rangle, \langle \mathtt{consistent}\ x \rangle}{\langle x + y \rangle \xrightarrow{a} \langle y' \rangle}$	RI-$\underline{17}$	$\dfrac{\langle x \rangle \xrightarrow{a} \surd, \langle \mathtt{consistent}\ y \rangle}{\langle x + y \rangle \xrightarrow{a} \surd}$	RI-$\underline{18}$
$\dfrac{\langle x \rangle \xrightarrow{a} \surd, \langle \mathtt{consistent}\ y \rangle}{\langle x + y \rangle \xrightarrow{a} \surd}$	RI-$\underline{19}$	$\dfrac{\langle x \rangle \stackrel{r}{\longmapsto} \langle x' \rangle, \quad \langle y \rangle \stackrel{r}{\longmapsto} \langle y' \rangle}{\langle x + y \rangle \stackrel{r}{\longmapsto} \langle x' + y' \rangle}$	RI-$\underline{20}$

Continued on Next Page…

Table 12 – Continued ($a \in A, r, u > 0$)

$$\frac{\langle x\rangle \overset{r}{\longmapsto} \langle x'\rangle, \langle \texttt{consistent}\ y\rangle, \langle y\rangle \overset{r}{\not\longmapsto}, (\forall y', \forall s < r\ \ \langle y\rangle \overset{s}{\longmapsto} \langle y'\rangle \implies \langle \texttt{consistent}\ y'\rangle)}{\langle x+y\rangle \overset{r}{\longmapsto} \langle x'\rangle}\ \text{RI-}\underline{21}$$

$$\frac{\langle y\rangle \overset{r}{\longmapsto} \langle y'\rangle, \langle \texttt{consistent}\ x\rangle, \langle x\rangle \overset{r}{\not\longmapsto}, (\forall x', \forall s < r\ \ \langle x\rangle \overset{s}{\longmapsto} \langle x'\rangle \implies \langle \texttt{consistent}\ x'\rangle)}{\langle x+y\rangle \overset{r}{\longmapsto} \langle y'\rangle}\ \text{RI-}\underline{22}$$

$$\frac{\langle \texttt{consistent}\ x\rangle, \langle \texttt{consistent}\ y\rangle}{\langle \texttt{consistent}\ x+y\rangle}\ \text{RI-}\underline{23} \qquad \frac{\langle x\rangle \xrightarrow{a} \langle x'\rangle}{\langle \nu_{\mathrm{rel}}(x)\rangle \xrightarrow{a} \langle x'\rangle}\ \text{RI-}\underline{24}$$

$$\frac{\langle x\rangle \xrightarrow{a} \surd}{\langle \nu_{\mathrm{rel}}(x)\rangle \xrightarrow{a} \surd}\ \text{RI-}\underline{25} \qquad \frac{\langle \texttt{consistent}\ x\rangle}{\langle \texttt{consistent}\ \nu_{rel}(x)\rangle}\ \text{RI-}\underline{26}$$

The transition rules defining alternative composition have a different format then the standard tyft format. In Rules RI-21 and RI-22, a universal quantifier on process terms is required to include all possible target terms of time transitions of all possible smaller durations. The rules come under the Mousavi-Reniers [14] UNTyft format.

The axiom of time determinism holds in the above semantics. The notion of bisimulation used here is the same as defined for Proposal 1 for $BPA_{\perp}^{srt}$ in Section 4.

Theorem 8 *Axiom SRT3 holds in the semantics of Table 12.*

Proof The proof is given in the Appendix F. $\boxtimes$

This appears to be a suitable semantics for a time deterministic $BPA_{\perp}^{srt}$. The only hunch is that making an inconsistent state reachable by a time step is not uniform with the approach adopted for action steps. Compare Rule 13 in Table 12 with Rule 9. From Rule 13 it is not possible to enter into an inconsistent state after performing an action. We find an answer to this dilemma in the next section.

5.4 Testing for Future Inconsistency

In this section, we introduce a semantics for $BPA_{\perp}^{srt}$, that adds a new predicate relation to the semantics for $BPA_{\perp}^{srt}$ with conditional time determinism (see

Section 4). We call this approach *Testing for future inconsistency*. The new predicate relation that is added to the semantics checks whether a process term can enter into an inconsistent state after a delay.

This semantics uses five relations. They are: The Action Relation ($\rightarrow$); the Time Relation ($\mapsto$); the Termination Predicate ($\rightarrow_{\surd}$); the Consistency Predicates (`Consistent`); and the Future Inconsistency Predicates ($\mapsto_{\perp}$).

The relation "Future Inconsistency" is defined on pairs of process terms and durations. It is denoted by $\mapsto_{\perp}$.

$$\mapsto_{\perp} \subseteq P \times \mathbb{R}^{>}$$

For $(x, r) \in \mapsto_{\perp}$, we write:

$$\langle x \rangle \overset{r}{\mapsto}_{\perp}$$

A future inconsistency predicate $\langle x \rangle \overset{r}{\mapsto}_{\perp}$ represents that if allowed to delay, the transition system of $\langle x \rangle$ would enter into an inconsistent state after r seconds.

For example, the following predicate holds:

$$\langle \sigma^{r}_{\mathsf{rel}}(\perp) \rangle \overset{r}{\mapsto}_{\perp}$$

The motivation for adding Future Inconsistency Predicates in the semantics comes from the previous section. Consider the rules in Table 12. When x is not bisimilar to the Non-existence process constant, then the transition $\sigma^{r}_{\mathsf{rel}}(\perp) \overset{r}{\mapsto} \perp$ is different from $\sigma^{r}_{\mathsf{rel}}(x) \overset{r}{\mapsto} x$. For example, the transition $\sigma^{r}_{\mathsf{rel}}(\perp) \overset{r}{\mapsto} \perp$ has a different effect on the definition of alternative composition. We might as well keep an inconsistent state unreachable after a time step and produce the same effect on alternative composition by adding a new predicate relation in the semantics. We call this Predicate Relation the *Future Inconsistency* predicate relation.

The two semantics of $BPA^{srt}_{\perp}$, one presented in this section and the other "Modifying the Relative Delay Operator σ_{rel}" (see Section 5.3) are claimed without proof to be isomorphic, with two process terms being bisimilar in one semantics if and only if they are bisimilar in the other.

This is the semantics we adopt for a time deterministic $BPA^{srt}_{\perp}$. The transition rules for this semantics are given in Table 13.

Some important features of this semantics regarding the operators of time determinism axiom are described below:

1. In this approach inconsistent states are unreachable. I.e., the following predicate holds:
$$\sigma^{r}_{\mathsf{rel}}(\perp) \overset{r}{\not\mapsto}$$
2. The delay behaviour of an alternative composition is defined as follows: Consider an alternative composition $p + q$. It can delay as follows:
    (a) If both p and q can delay for a given duration, then they delay together and the choice is retained at the end of their common delay.

(b) The term $p+q$ can delay as p only if q cannot do a time step of the same duration as p. The transition system of q has a consistent root. Also q cannot does not have an inconsistency predicate of duration less than or equal to that of the delay of p.

(c) Also allow $p+q$ to delay as q, if p satisfies the conditions mentioned above.

Rules P2- 24, P2-25 and P2-26 of Table 13. define the delay behaviour of an alternative composition.

As we see furthermore, the axiom of time determinism holds in this semantics.

Table 13: Semantics of Proposal 2 for $BPA^{\mathrm{srt}}_{\perp}(a \in A, r, u > 0)$

$\dfrac{}{\langle \texttt{consistent}\ \tilde{\tilde{\delta}} \rangle}$ P2-1	
$\dfrac{}{\langle \texttt{consistent}\ \tilde{\tilde{a}} \rangle}$ P2-2	$\dfrac{}{\langle \tilde{\tilde{a}} \rangle \xrightarrow{a} \surd}$ P2-3
$\dfrac{\langle x \rangle \xrightarrow{a} \langle x' \rangle}{\langle \sigma^{0}_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \langle x' \rangle}$ P2-4	$\dfrac{\langle x \rangle \xrightarrow{a} \surd}{\langle \sigma^{0}_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \surd}$ P2-5
$\dfrac{\langle x \rangle \stackrel{r}{\longmapsto} \langle x' \rangle}{\langle \sigma^{0}_{\mathsf{rel}}(x) \rangle \stackrel{r}{\longmapsto} \langle x' \rangle}$ P2-6	$\dfrac{\langle \texttt{consistent}\ x \rangle}{\langle \texttt{consistent}\ \sigma^{0}_{\mathsf{rel}}(x) \rangle}$ P2-7
$\dfrac{\langle x \rangle \stackrel{r}{\longmapsto} \perp}{\langle \sigma^{0}_{\mathsf{rel}}(x) \rangle \stackrel{r}{\longmapsto} \perp}$ P2-8	
$\dfrac{}{\langle \sigma^{r+u}_{\mathsf{rel}}(x) \rangle \stackrel{u}{\longmapsto} \langle \sigma^{r}_{\mathsf{rel}}(x) \rangle}$ P2-9	$\dfrac{\langle \texttt{consistent}\ x \rangle}{\langle \sigma^{r}_{\mathsf{rel}}(x) \rangle \stackrel{r}{\longmapsto} \langle x \rangle}$ P2-10
$\dfrac{\langle x \rangle \stackrel{u}{\longmapsto} \langle x' \rangle}{\langle \sigma^{r}_{\mathsf{rel}}(x) \rangle \stackrel{r+u}{\longmapsto} \langle x' \rangle}$ P2-11	$\dfrac{}{\langle \texttt{consistent}\ \sigma^{r}_{\mathsf{rel}}(x) \rangle}$ P2-12
$\dfrac{\neg \langle \texttt{consistent}\ x \rangle}{\langle \sigma^{r}_{\mathsf{rel}}(x) \rangle \stackrel{r}{\longmapsto} \perp}$ P2-13	$\dfrac{\langle x \rangle \stackrel{u}{\longmapsto} \perp}{\langle \sigma^{r}_{\mathsf{rel}}(x) \rangle \stackrel{r+u}{\longmapsto} \perp}$ P2-14
$\dfrac{\langle x \rangle \xrightarrow{a} \langle x' \rangle}{\langle x \cdot y \rangle \xrightarrow{a} \langle x' \cdot y \rangle}$ P2-15	$\dfrac{\langle x \rangle \xrightarrow{a} \surd, \langle \texttt{consistent}\ y \rangle}{\langle x \cdot y \rangle \xrightarrow{a} \langle y \rangle}$ P2-16
$\dfrac{\langle x \rangle \stackrel{r}{\longmapsto} \langle x' \rangle}{\langle x \cdot y \rangle \stackrel{r}{\longmapsto} \langle x' \cdot y \rangle}$ P2-17	$\dfrac{\langle \texttt{consistent}\ x \rangle}{\langle \texttt{consistent}\ x \cdot y \rangle}$ P2-18

Continued on Next Page...

Table 13 – Continued ($a \in A, r, u > 0$)

$$\frac{\langle x\rangle \overset{r}{\longmapsto}_{\perp}}{\langle x \cdot y\rangle \overset{r}{\longmapsto}_{\perp}} \quad \text{P2-}\underline{19}$$

$$\frac{\langle x\rangle \xrightarrow{a} \langle x'\rangle, \langle \texttt{consistent}\ \ y\rangle}{\langle x + y\rangle \xrightarrow{a} \langle x'\rangle} \quad \text{P2-}\underline{20}$$

$$\frac{\langle y\rangle \xrightarrow{a} \langle y'\rangle, \langle \texttt{consistent}\ \ x\rangle}{\langle x + y\rangle \xrightarrow{a} \langle y'\rangle} \quad \text{P2-}\underline{21}$$

$$\frac{\langle x\rangle \xrightarrow{a} \surd, \langle \texttt{consistent}\ \ y\rangle}{\langle x + y\rangle \xrightarrow{a} \surd} \quad \text{P2-}\underline{22}$$

$$\frac{\langle y\rangle \xrightarrow{a} \surd, \langle \texttt{consistent}\ \ x\rangle}{\langle x + y\rangle \xrightarrow{a} \surd} \quad \text{P2-}\underline{23}$$

$$\frac{\begin{array}{c}\langle x\rangle \overset{r}{\longmapsto} \langle x'\rangle, \\ \langle y\rangle \overset{r}{\longmapsto} \langle y'\rangle\end{array}}{\langle x + y\rangle \overset{r}{\longmapsto} \langle x' + y'\rangle} \quad \text{P2-}\underline{24}$$

$$\frac{\begin{array}{l}\langle x\rangle \overset{r}{\longmapsto} \langle x'\rangle, \langle \texttt{consistent}\ \ y\rangle, \\ \langle y\rangle \not\overset{r}{\longmapsto}, \forall s \le r(\langle y\rangle \not\overset{s}{\longmapsto}_{\perp})\end{array}}{\langle x + y\rangle \overset{r}{\longmapsto} \langle x'\rangle} \quad \text{P2-}\underline{25}$$

$$\frac{\begin{array}{l}\langle y\rangle \overset{r}{\longmapsto} \langle y'\rangle, \langle \texttt{consistent}\ \ x\rangle, \\ \langle x\rangle \not\overset{r}{\longmapsto}, \forall s \le r(\langle x\rangle \not\overset{s}{\longmapsto}_{\perp})\end{array}}{\langle x + y\rangle \overset{r}{\longmapsto} \langle y'\rangle} \quad \text{P2-}\underline{26}$$

$$\frac{\langle \texttt{consistent}\ \ x\rangle, \langle \texttt{consistent}\ \ y\rangle}{\langle \texttt{consistent}\ \ x + y\rangle} \quad \text{P2-}\underline{27}$$

$$\frac{\begin{array}{l}\langle x\rangle \overset{r}{\longmapsto}_{\perp}, \langle \texttt{consistent}\ \ y\rangle, \\ \forall s < r(\langle y\rangle \not\overset{s}{\longmapsto}_{\perp})\end{array}}{\langle x + y\rangle \overset{r}{\longmapsto}_{\perp}} \quad \text{P2-}\underline{28}$$

$$\frac{\begin{array}{l}\langle y\rangle \overset{r}{\longmapsto}_{\perp}, \langle \texttt{consistent}\ \ x\rangle, \\ \forall s < r(\langle x\rangle \not\overset{s}{\longmapsto}_{\perp})\end{array}}{\langle x + y\rangle \overset{r}{\longmapsto}_{\perp}} \quad \text{P2-}\underline{29}$$

$$\frac{\langle x\rangle \xrightarrow{a} \langle x'\rangle}{\langle \nu_{\mathrm{rel}}(x)\rangle \xrightarrow{a} \langle x'\rangle} \quad \text{P2-}\underline{30}$$

$$\frac{\langle x\rangle \xrightarrow{a} \surd}{\langle \nu_{\mathrm{rel}}(x)\rangle \xrightarrow{a} \surd} \quad \text{P2-}\underline{31}$$

$$\frac{\langle \texttt{consistent}\ \ x\rangle}{\langle \texttt{consistent}\ \ \nu_{\mathrm{rel}}(x)\rangle} \quad \text{P2-}\underline{32}$$

Next we define a bisimulation on $BPA_{\perp}^{srt}$ process terms that relates two process terms when they have exactly the same behaviour in the semantics presented above. Later on, when we present the set of axioms, we use this definition and prove that process terms that are set equal by the axioms are infact semantically similar. The definition of bisimulation for the semantics presented in this section is obtained by adding a comparison of future inconsistency predicates in the bisimulation definition of $BPA_{\perp}^{srt}$ with conditional time determinism (see Definition 1).

It is defined as follows:

Definition 2 *A relation $R \subseteq P \times P$ is called a bisimulation relation if and only if the following conditions hold:*

For all $a \in A, r > 0, x, y, z \in P$,

1.
$$((x, y) \in R \wedge \langle x \rangle \xrightarrow{a} \langle z \rangle) \implies \exists z' \in P : \langle y \rangle \xrightarrow{a} \langle z' \rangle \text{ and } (z, z') \in R$$

2.
$$((x, y) \in R \wedge \langle y \rangle \xrightarrow{a} \langle z \rangle) \implies \exists z' \in P : \langle x \rangle \xrightarrow{a} \langle z' \rangle \text{ and } (z', z) \in R$$

3.
$$((x, y) \in R \wedge \langle x \rangle \overset{r}{\longmapsto} \langle z \rangle) \implies \exists z' \in P : \langle y \rangle \overset{r}{\longmapsto} \langle z' \rangle \text{ and } (z, z') \in R$$

4.
$$((x, y) \in R \wedge \langle y \rangle \overset{r}{\longmapsto} \langle z \rangle) \implies \exists z' \in P : \langle x \rangle \overset{r}{\longmapsto} \langle z' \rangle \text{ and } (z', z) \in R$$

5.
$$(x, y) \in R \implies (\langle x \rangle \xrightarrow{a} \surd \iff \langle y \rangle \xrightarrow{a} \surd)$$

6.
$$(x, y) \in R \implies (\langle x \rangle \overset{r}{\longmapsto}_{\perp} \iff \langle y \rangle \overset{r}{\longmapsto}_{\perp})$$

7.
$$(x, y) \in R \implies (\langle \texttt{consistent } x \rangle \iff \langle \texttt{consistent } y \rangle)$$

Two process terms x and y are called bisimilar to each other written as $\langle x \rangle \leftrightarrows \langle y \rangle$ if there exists a bisimulation relation R such that $(x, y) \in R$.

Theorem 9 *Bisimulation is a congruence for the signature of $BPA_{\perp}^{srt}$.*

<u>Proof</u> This theorem is proven on the same lines as Theorem 1 using congruence theorem in [16].

It is trivial to show that the given term deduction system with the set of deduction rules given in Table 13 is well-founded and all the transition rules are in PANTH format.

A stratification S which is an extension of the stratification function S given in the proof of Theorem 1. S is a strict stratification for our term deduction system.

The stratification S_2 is defined as follows:

$$\begin{array}{lcl} S(\langle \texttt{consistent } x\rangle) & = & S(\langle \texttt{consistent } x\rangle) \\ S(\langle x\rangle \xrightarrow{a} \surd) & = & S(\langle x\rangle \xrightarrow{a} \surd) \\ S(\langle x\rangle \xrightarrow{a} \langle x'\rangle) & = & S(\langle x\rangle \xrightarrow{a} \langle x'\rangle) \\ S(\langle x\rangle \overset{r}{\longmapsto} \langle x'\rangle) & = & S(\langle x\rangle \overset{r}{\longmapsto} \langle x'\rangle) \\ S(\langle x\rangle \overset{r}{\longmapsto}_{\perp}) & = & |x| \end{array}$$

⊠

Table 14 consists a list of axioms we present for this proposal.

Theorem 10 *(Soundness of Proposal 2)*
For all closed terms t_1, t_2 of $BPA_{\perp}^{srt}$, we have,

$$\textit{Proposal 2} \models t_1 = t_2 \implies t_1 \leftrightarrows t_2$$

Proof The soundness proofs of the axioms are given in Appendix H ⊠

Table 14: Proposal 2 $BPA_{\perp}^{srt}$-Axioms ($a \in A_{\delta} u, v \geq 0$)

$x + y = y + x$	$A1$	$\sigma^0_{\mathsf{rel}}(x) = x$	$SRT1$
$(x + y) + z = x + (y + z)$	$A2$	$\sigma^u_{\mathsf{rel}}(\sigma^v_{\mathsf{rel}}(x)) = \sigma^{u+v}_{\mathsf{rel}}(x)$	$SRT2$
$x + x = x$	$A3$	$\sigma^u_{\mathsf{rel}}(x) + \sigma^u_{\mathsf{rel}}(y) = \sigma^u_{\mathsf{rel}}(x + y)$	$SRT3$
$(x + y) \cdot z = x \cdot z + y \cdot z$	$A4$	$\sigma^u_{\mathsf{rel}}(x) \cdot y = \sigma^u_{\mathsf{rel}}(x \cdot y)$	$SRT4$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$	$A5$		
$x + \tilde{\tilde{\delta}} = x$	$A6SR$	$\nu_{rel}(\tilde{\tilde{a}}) = \tilde{\tilde{a}}$	$SRU1$
$\tilde{\tilde{\delta}} \cdot x = \tilde{\tilde{\delta}}$	$A7SR$	$\nu_{rel}(\sigma^r_{\mathsf{rel}}(x)) = \tilde{\tilde{\delta}}$	$SRU2$
		$\nu_{rel}(x + y) = \nu_{rel}(x) + \nu_{rel}(y)$	$SRU3$
$x + \perp = \perp$	$NE1$	$\nu_{rel}(x \cdot y) = \nu_{rel}(x) \cdot y$	$SRU4$
$\perp \cdot x = \perp$	$NE2$		
$\tilde{\tilde{a}} \cdot \perp = \tilde{\tilde{\delta}}$	$NE3SR$	$\nu_{rel}(\perp) = \perp$	$NESRU$

Theorem 11 *Our Proposal 2 for $BPA_{\perp}^{srt}$ is a conservative extension of BPA^{srt} and BPA.*

Proof Trivial ⊠

This concludes our research for our possible second proposal for $BPA_{\perp}^{srt}$, where the time determinism axiom SRT3 holds unconditionally.

5.5 Concluding Remarks

Observing the importance of time determinism for timed systems, we set to finding a time deterministic proposal of $BPA_{\perp}^{srt}$. We have presented three attempts of preserving time determinism axiom in $BPA_{\perp}^{srt}$. Each attempt proposes a modification or addition to the semantics of $BPA_{\perp}^{srt}$ with conditional time determinism (see Section 4). The first attempt (Section 5.2) proposing a modification in alternative composition was given up due to its limitations regarding extension with integration.

Second attempt ("Modifying Relative Delay Operator $\sigma_{\mathsf{rel}}^{r}$", Section 5.3) although suitable for extension with integration was dropped in favor of the third semantics ("Testing for future Inconsistency", Section 5.4) which treats both actions and delays before an inconsistent state uniformly.

Next, we examine the options of extending the time deterministic $BPA_{\perp}^{srt}$ and $BPA_{\perp}^{srt}$ with conditional time determinism with other operators of hybrid process algebra.

6 Extensions of $BPA_{\perp}^{srt}$

$BPA_{\perp}^{srt}$ can only describe a subset of processes expressible by Process Algebra for Hybrid Systems. We are naturally interested in analyzing the possibilities of extensions of $BPA_{\perp}^{srt}$ towards BPA_{hs}^{srt} and ACP_{hs}^{srt}. In this section we throw some light on our insights regarding this matter. The section is outlined as follows: We dedicate a separate section to discuss addition of the operator integration (Section 6.1); afterwards, we discuss extending $BPA_{\perp}^{srt}$ to represent hybrid processes. This section comprises of ideas and suggestions. The implementation of these ideas is left as future work.

6.1 Integration

The addition of integration to Basic Timed Process Algebra BPA^{srt} enables it to model processes that can perform actions at any instance in a time interval. Recognizing the importance of integration, we first consider adding integration to $BPA_{\perp}^{srt}$.

Integration provides for alternative composition over a set of alternatives that can be infinite. Let F be a function from non-negative reals to processes in BPA^{srt}, then an integral of $F(u)$ over an interval U, represented by $\int_{u \in U} F(u)$, behaves like one of the process terms in the set $\{F(p) \mid p \in U\}$.

The set of transition rules and axioms for integration in BPA^{srt} [4] are given in the Appendix B.

The semantics of integration is defined on the same principle as that of alternative composition. We discuss how our two proposals of $BPA_{\perp}^{srt}$ can be extended with integration below:

1. In our first proposal of $BPA_{\perp}^{srt}$, i.e. the proposal with conditional time determinism, the transition rules for integration are similar to the rules of

integration given for the process algebra BPA^{srt} [4]. The only difference is that here (in integration in $BPA^{srt}_{\perp}$), we add consistency checking of all process terms constituting an integral expression.

Thus adding integration to the first proposal is straightforward. The Rules for Integration are given in Table 15.

Table 15: Proposal 1 Rules for Integration ($a \in A, p, q, \geq 0, r > 0$)

$$\frac{\langle F(p)\rangle \xrightarrow{a} \langle x'\rangle, \{\langle \texttt{consistent}\ \ F(q)\rangle \mid q \in U\}}{\langle \int_{u\in U} F(u)\rangle \xrightarrow{a} \langle x'\rangle}\ p \in U \quad \text{P1-}\underline{27}$$

$$\frac{\langle F(p)\rangle \xrightarrow{a} \langle \surd\rangle, \{\langle \texttt{consistent}\ \ F(q)\rangle \mid q \in U\}}{\langle \int_{u\in U} F(u)\rangle \xrightarrow{a} \langle \surd\rangle}\ p \in U \quad \text{P1-}\underline{28}$$

$$\frac{\begin{array}{l}\{\langle F(q)\rangle \overset{r}{\longmapsto} \langle F_1(q)\rangle \mid q \in U_1\},\\ \vdots \\ \{\langle F(q)\rangle \overset{r}{\longmapsto} \langle F_n(q)\rangle \mid q \in U_n\},\\ \{\langle F(q)\rangle \overset{r}{\not\longmapsto}, \langle \texttt{consistent}\ \ F(q)\rangle \\ \mid q \in U_{n+1}\}\end{array}}{\langle \int_{u\in U} F(u)\rangle \overset{r}{\longmapsto} \langle \int_{u\in U_1} F_1(u) + \ldots + \int_{u\in U_n} F_n(u)\rangle}\ \begin{array}{l}\{U_1, \ldots U_n\}\\ \text{partition of } U\backslash U_{n+1},\\ \text{and } U_{n+1} \subset U\end{array} \quad \text{P1-}\underline{29}$$

$$\frac{\{\langle \texttt{consistent}\ \ F(q)\rangle \mid q \in U\}}{\langle \texttt{consistent}\ \ \int_{u\in U} (F(u))\rangle} \quad \text{P1-}\underline{30}$$

With the integration available, we can derive interesting equalities concerning the non-existence process constant and relative delay operator. In the semantics of our first proposal of $BPA^{srt}_{\perp}$ with integration, the following equality holds:

$$\sigma^{t}_{\mathsf{rel}}(\perp) = \int_{u<t} \sigma^{u}_{\mathsf{rel}}(\tilde{\tilde{\delta}}) \tag{10}$$

Equation 10 states that it is not possible to enter into inconsistency after a delay. A process term $\sigma^{t}_{\mathsf{rel}}(\perp)$ deadlocks at some time before time instance t.

2. Adding integration to our second proposal turns out to be more complex.

   In the second proposal for $BPA^{srt}_{\perp}$, a future inconsistency relation has been added in the semantics. A future inconsistency predicate with duration t for a process x, represents that if allowed to delay x would enter into inconsistency after time t. A process term $\sigma^{t}_{\mathsf{rel}}(\perp)$ has a future inconsistency predicate with duration t. It is written as follows:

$$\sigma^{t}_{\mathsf{rel}}(\perp) \overset{t}{\longmapsto}_{\perp}$$

Integration rules can be defined for the second proposal of $BPA^{srt}_{\perp}$ in the same way as the rules for the alternative composition. Then, in the second proposal an integral expression $\int_{u \in U} F(u)$ would be allowed to delay for a duration r only if none of the process terms $\{F(q) \mid q \in U\}$ have a future inconsistency predicate of duration shorter than or equal to r (See Rules P2-25 and Rules P2-26 in Table 13). Similarly, the term $\int_{u \in U} F(u)$ would have a future inconsistency predicate that has the shortest duration among future inconsistency predicates for all constituent terms $\{F(q) \mid q \in U\}$ (See Rules P2-28 and Rules P2-29 in Table 13). This straightforward extension of our second proposal for $BPA^{srt}_{\perp}$ does not give a satisfactory result. We explain it below:

Consider the following equation:

$$\int_{t>0} \sigma^t_{\mathsf{rel}}(\perp) = \tilde{\tilde{\delta}} \tag{11}$$

Equation 11 holds in the second proposal of $BPA^{srt}_{\perp}$. Consider the set of process terms $\{\sigma^t_{\mathsf{rel}}(\perp) \mid t > 0\}$ that constitute the integral $\int_{t>0} \sigma^t_{\mathsf{rel}}(\perp)$. Since the smallest real number greater than zero does not exist, therefore a future inconsistency predicate for $\int_{t>0} \sigma^t_{\mathsf{rel}}(\perp)$ cannot be derived. This sets the process term $\int_{t>0} \sigma^t_{\mathsf{rel}}(\perp)$ bisimilar to immediate deadlock.

On close investigation of our equational system we find out that this poses a problem. Equation 11, together with axiom SRT3, NE1 and some standard axioms of integration allows the following derivation:

$$\begin{array}{rcll}
\int_{t>0} \sigma^t_{\mathsf{rel}}(\perp) & = & \int_{t>0} \sigma^t_{\mathsf{rel}}(\perp + x) & \text{By NE1} \\
& = & \int_{t>0} (\sigma^t_{\mathsf{rel}}(\perp) + \sigma^t_{\mathsf{rel}}(x)) & \text{By SRT3} \\
& = & \int_{t>0} \sigma^t_{\mathsf{rel}}(\perp) + \int_{t>0} \sigma^t_{\mathsf{rel}}(x) & \text{By INT11} \\
& = & \tilde{\tilde{\delta}} + \int_{t>0} \sigma^t_{\mathsf{rel}}(x) & \text{By Equation 11} \\
& = & \int_{t>0} \sigma^t_{\mathsf{rel}}(x) & \text{By A6SR}
\end{array}$$

The process term $\int_{t>0} \sigma^t_{\mathsf{rel}}(\perp)$ can be proven equal to $\int_{t>0} \sigma^t_{\mathsf{rel}}(x)$, with x being any process term. So, when the semantics of Section 5.4 is straightforwardly extended with integration, the time determinism axiom leads to unsound derivations once more. This cannot be allowed. Hence, the semantics must differentiate between $\int_{t>0} \sigma^t_{\mathsf{rel}}(\perp)$ and immediate deadlock $\tilde{\tilde{\delta}}$. [1]

A solution can be to add an extra relation to the semantics once more, that holds for process term $\int_{t>0} \sigma^t_{\mathsf{rel}}(\perp)$. A unary relation $\twoheadrightarrow_{\perp} \subseteq P$ that includes processes that enter into inconsistency with a delay of shortest possible duration, i.e. "immediately after now."

[1] Note that this problem will also arise in the semantics described in Section 5.3. Adding the integration on the same lines as the rules for alternative composition, requires that during a delay for $\int_{t>0} \sigma^t_{\mathsf{rel}}(\perp)$, none of the members of the set $\{\sigma^t_{\mathsf{rel}}(\perp) \mid t > 0\}$ become inconsistent. For this we need to know the smallest number greater than zero.

We agree that the resulting semantics will be complex with six relations. The relations Future inconsistency and Consistency can actually be combined. The Consistency relation can be removed from the semantics. Instead, a future inconsistency predicate of duration 0 holds for a process term x, whenever $\neg\langle \texttt{consistent}\ x\rangle$ was holding. I.e.

$$\begin{array}{l}
\bot \overset{0}{\longmapsto}_{\bot} \\
\sigma^0_{\mathsf{rel}}(\bot) \overset{0}{\longmapsto}_{\bot} \\
\bot \cdot x \overset{0}{\longmapsto}_{\bot} \\
\bot + x \overset{0}{\longmapsto}_{\bot}
\end{array}$$

Extending the second proposal of $BPA^{srt}_{\bot}$ with integration in this way is left as future work.

We extended our first proposal for $BPA^{srt}_{\bot}$ with integration defined by rules given in Table 15. We expect that the axioms (excluding the time determinism axiom INT10SR) that hold for BPA^{srt} with integration (see Appendix B) also hold for our first proposal of $BPA^{srt}_{\bot}$. Formulating and proving axioms of integration is left as future work.

While considering the proposal of Section 5.4 (which constitutes our second proposal of $BPA^{srt}_{\bot}$), we find that extending it with integration is not that simple. We propose a solution for adding integration to it and leave it as future work.

6.2 Flow Determinism

A Process Algebra for Hybrid Systems is inherently more complex than a timed process algebra. Hence an extension of $BPA^{srt}_{\bot}$ to a basic hybrid process algebra requires a thorough research. Below, we present our views on extending the two proposals of $BPA^{srt}_{\bot}$ to a hybrid setting:

1. An extension of our first proposal of $BPA^{srt}_{\bot}$ to a hybrid process algebra is quite obvious. The signature of $BPA^{srt}_{\bot}$ is extended with extra operators i.e. conditional operator, signal emission operator , signal evolution operator and signal transition operator defined in [1]. The semantics is also extended with details like valuations of model variables and their trajectories during delays that are necessary to describe hybrid behaviour of processes. New rules are added to the semantic to define the behaviour of new operators.

   Here, we need a careful approach, because with the extended signature, comes a chance of repeating the mistakes of Process Algebra for Hybrid Systems. As mentioned before, in Process Algebra for Hybrid Systems, alternative composition is not associative.

   This error can be corrected by modifying the semantics of alternative composition, so that too much emphasis on duration of delays is replaced by an equal emphasis on durations and trajectories of model variables during delays.

The delay bahaviour of alternative composition as it is **currently** defined in Process Algebra for Hybrid Systems is narrated below:

Consider an alternative composition $p + q$.

> If p and q can delay together for a given duration of delay with the same trajectory of variables, then $p+q$ delays so that at the end of delay, choice is retained.
>
> The process term $p + q$ can also delay for a given duration and proceed as one of the process terms p or q only if the process term left behind cannot delay for the same duration with **any** trajectory of variables.

The behaviour described latter should be modified as follows:

> The process term $p + q$ can also delay for a given duration and proceed as one of the process terms p or q only if the process term left behind cannot delay for the same duration with the **same** trajectory of variables.

Consider the transition rules HS-12 and HS-13, describing the delay behaviour of an alternative composition in Appendix A, Table 16. These rules will be replaced by the following two rules:

$$\frac{\langle x, \alpha\rangle \xmapsto{r,\rho} \langle x', \alpha'\rangle, \langle y, \alpha\rangle \not\xmapsto{r,\rho}, \alpha \in [\mathsf{s}(y)]}{\langle x + y, \alpha\rangle \xmapsto{r,\rho} \langle x', \alpha'\rangle}$$

$$\frac{\langle y, \alpha\rangle \xmapsto{r,\rho} \langle y', \alpha'\rangle, \alpha \in [\mathsf{s}(x)], \langle x, \alpha\rangle \not\xmapsto{r,\rho}}{\langle x + y, \alpha\rangle \xmapsto{r,\rho} \langle y', \alpha'\rangle}$$

Note that the negative formulaes $\langle y, \alpha\rangle \not\xmapsto{r}$ and $\langle y, \alpha\rangle \not\xmapsto{r}$ in rules 12 and 13 have been replaced by $\langle y, \alpha\rangle \not\xmapsto{r,\rho}$ and $\langle x, \alpha\rangle \not\xmapsto{r,\rho}$ here. A predicate $\langle x, \alpha\rangle \not\xmapsto{r,\rho}$ represents that there does not exists a x', such that

$$\langle x, \alpha\rangle \xmapsto{r,\rho} \langle x', \alpha_r^\rho\rangle$$

is derivable.

With this modification, the alternative composition will remain associative with the new operators added to $BPA_\perp^{srt}$. We expect that some axioms that are currently not sound (for example, HSE7, INT11, HSINT7) will hold with the new definition of alternative composition. On the other hand some of the unsound axioms (for example HSE13, HSSCRM) will be dropped. With regards to time determinism, we expect the following axiom to hold:

$$\sigma^r_{\mathsf{rel}}(x) + \sigma^r_{\mathsf{rel}}(y) = \sigma^r_{\mathsf{rel}}(x + y) + \sigma^r_{\mathsf{rel}}(\neg s_\rho(y) \wedge\!\!\!\blacktriangle\, x) + \sigma^r_{\mathsf{rel}}(\neg s_\rho(x) \wedge\!\!\!\blacktriangle\, y)$$

where $r > 0$.

The operator s_ρ defined in [24] returns the signal emitted by a process.

2. The semantics of the second proposal for $BPA^{srt}_\perp$ is more elaborate than that of the first proposal. Extending it to a hybrid setting also turns out to be more involved than the first.

   In addition to the relations of first proposal, the semantics of second proposal contains a future inconsistency relation. When we extend this proposal with the operators for hybrid processes, then for each operator we need to define rules for deriving future inconsistency predicates ($\mapsto_\perp$) and inconsistency immediately after now predicates ($\twoheadrightarrow_\perp$). This needs further research. An idea is that inconsistency is absence of a solution to a given set of constraints on variable trajectories. We cannot say more at this point.

Studying the possibilities of extensions of $BPA^{srt}_\perp$ to a hybrid process algebra, we find that extending $BPA^{srt}_\perp$ with conditional time determinism with operators of BPA^{srt}_{hs} is much simpler than the time deterministic $BPA^{srt}_\perp$.

7 Conclusions

This report is related to Process Algebra for Hybrid Systems [1] which is a well-known formalism for specification of hybrid systems. Recently, a number of errors have been uncovered in this algebra. This report is an initial development towards correcting these errors.

Process Algebra for Hybrid Systems, denoted by ACP^{srt}_{hs}, has a hierarchical structure. It is built from the most basic algebra BPA [2] in several layers to a process algebra ACP^{srt}_{hs} for description of hybrid systems. The hierarchical structure of ACP^{srt}_{hs} is shown in the figure 1. We summarize it below:

The algebra *Basic Process Algebra* BPA is extended with operators and constants to describe properties of timed systems. This forms *Basic Process Algebra with standard relative timing* BPA^{srt}. The algebra BPA^{srt} is extended with the Non-existence process from [3] to form $BPA^{srt}_\perp$. When $BPA^{srt}_\perp$ is combined with operators of Basic Process Algebra with Propositional Signals BPA_{ps}, the resulting theory is BPA^{srt}_{ps}. Adding operators for description of hybrid behaviour of processes gives us BPA^{srt}_{hs}. The algebra BPA^{srt}_{hs}, called *Basic Process Algebra for Hybrid Systems*, is ACP^{srt}_{hs} without parallelism and concurrency. ACP^{srt}_{hs} is obtained from BPA^{srt}_{hs} by adding parallelism to it.

This report is confined to the discussion of Process Algebra for Hybrid Systems without parallelism i.e. BPA^{srt}_{hs}.

As shown in the Figure 1, the errors found in ACP_{hs}^{srt} appear at two levels. First at the level of $BPA_{\perp}^{srt}$ and secondly at the level of BPA_{ps}^{srt}. The error in algebra $BPA_{\perp}^{srt}$ is the unsoundness of axiom of Time Determinism ($SRT3$). The error in BPA_{ps}^{srt} and its derived theories ($BPA_{hs}^{srt}, ACP_{hs}^{srt}$, etc.) is that the Choice is non-associative and related to this error, a number of other axioms turn out to be unsound.

As a first step towards making corrections in BPA_{hs}^{srt}, we present in this report two proposals for correcting the algebra $BPA_{\perp}^{srt}$.

In our first proposal, we replace axiom of Time Determinism (SRT3) by a conditional axiom (SRTD). The conditional axiom represents time determinism in cases when the target process terms are not bisimilar to Non-existence. Also a new axiom ($SRTD\perp$) is added to $BPA_{\perp}^{srt}$ [1], that reflects that passage of time makes choices in the presence of Non-existence process. These axioms ($SRTD$, $SRTD\perp$) also hold in the semantics of [1]. In fact we show that for all $BPA_{\perp}^{srt}$ processes, our first proposal is equivalent to the process algebra BPA_{hs}^{srt}.

Due to the importance of Time Determinism, we also search for a variant of $BPA_{\perp}^{srt}$, where the axiom SRT3 holds for all process terms including the Non-existence process. Finding a time deterministic $BPA_{\perp}^{srt}$ turns out to be non-trivial. To achieve our goal, we try modifying the semantics of the operators reasoned about in SRT3, i.e. alternative composition and the relative delay operator. Finally, we adopt an approach that we call "Testing for Future Inconsistency" (Section 5.4) for our second proposal of $BPA_{\perp}^{srt}$. The axiom of time determinism SRT3 holds in this proposal.

Both our first and second proposals of $BPA_{\perp}^{srt}$ are conservative extensions of BPA^{srt} and BPA.

Lastly, we consider extensions of $BPA_{\perp}^{srt}$ with the operators from [1] to describe hybrid behaviour of processes.

It appears that extending the first proposal is simple. When extending the signature of $BPA_{\perp}^{srt}$ with operators of BPA_{hs}^{srt}, we need to modify the alternative composition of [1] so that it is associative.

Extending the second proposal to a hybrid process algebra needs more research. For the second proposal, we need to determine what is the meaning of "Future Inconsistency" ($\mapsto_{\perp}$) and "Inconsistency immediately after now" ($\twoheadrightarrow_{\perp}$) in a hybrid environment.

In our work on Process Algebra for Hybrid Systems, tracing the error of time determinism back to $BPA_{\perp}^{srt}$ was a major step. After isolating the error, the rest of the road map was evident. The reason for presenting both the conditional time determinism approach and the general time determinism approach for $BPA_{\perp}^{srt}$ is to clarify the choices available in each case. An interested researcher can then make his own comparison and decide according to the problem at hand. The idea of "Testing for Future Inconsistency" which constitutes our second proposal is new and the research on it is in progress. There exist better possibilities for the implementation of the idea (one of them proposed by Jos Baeten) than given in this report. Realizing these suggestions is left as future work.

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A Introduction to BPA_{hs}^{srt} Semantics

In this Section, we give a brief introduction to the semantics of Basic Process Algebra for Hybrid Systems.

To describe the behaviour of a hybrid process, we need to keep account of the values of model variables. The values of these variables may change gradually over an interval of time or suddenly when an action is performed. Assume a set V of model variables and a set A of actions. Let the set $\dot{V} = \{\dot{v} \mid v \in V\}$ denote the derivatives of all variables $v \in V$. A mapping of variables from the set $V \cup \dot{V}$ to the set of real numbers is called a valuation. We denote the set of all possible valuations by S, i.e. $S = V \cup \dot{V} \rightarrow R$. A *valuation* has been mentioned as a *state* in [1].

A function of the type $[0,t] \rightarrow (V \rightarrow R)$ gives the evolution of variables in a duration $[0,t]$, $t \in R^{>0}$.

We define a set $\mathcal{D}$ for pairs of time durations and state evolution functions possible during a delay of that duration.

$$\mathcal{D} = \{(t, \rho) \mid t \in R^{\geq 0} \wedge \rho \in [0,t] \mapsto (V \mapsto R)\}$$

Let $\rho \in ([0,t] \rightarrow (V \rightarrow R))$. We use the notation $\rho \trianglerighteq r$, with $0 < r < t$, for the state evolution ρ shifted to left by r time units. The duration of $\rho \trianglerighteq r$ is $t - r$.

$$\begin{array}{ll} (\rho \trianglerighteq r)(0) = \rho(r) & \\ (\rho \trianglerighteq r)(s) = \rho(r+s), & s > 0 \\ \rho \trianglerighteq r \in ([0,t-r] \rightarrow (V \rightarrow R)) & \end{array}$$

The semantics of BPA_{hs}^{srt} uses four different relations. They are:

1. Action step Relation
2. Action termination Relation
3. Time step Relation; and
4. Signal Relation.

As we are dealing with hybrid processes, the sources and targets of transitions include valuations of variables to reflect how variables vary during an action or delay.

The relations are defined below:

(Let P be the set of all closed process terms of BPA_{hs}^{srt}, S be the set of all valuations, A be the set of all actions and $\mathcal{D}$ be the set of pairs of all possible time durations and variable evolutions during them.)

1. Action Step Relation: A process term can perform an action and become another process term.

   The Action Step relation is of type $P \times S \times A \times P \times S$.

   For a tuple $(x, \alpha, a, x', \alpha')$ in the Action Step Relation, we write:

   $$\langle x, \alpha \rangle \xrightarrow{a} \langle x', \alpha' \rangle$$

   This transition represents that in valuation α, x performs action a and then proceeds as process term x'. The new values of variables after the action are given by valuation α'.

2. Action Termination Relation: A process term can perform an action and terminate.

   The Action Termination Relation is of type $P \times S \times A \times S$.

   For a tuple (x, α, a, α') in the Action Termination Relation, we write:

   $$\langle x, \alpha \rangle \xrightarrow{a} \langle \surd, \alpha' \rangle$$

   This transition represents that in valuation α, x performs action a and terminates. The new values of variables after the action are given by valuation α'.

3. Time Step Relation: A process term can idle for some time and become another process term.

   The Time Step Relation is of type $P \times S \times \mathcal{D} \times P \times S$.

   For a tuple, $(x, \alpha, (t, \rho), x', \alpha')$ in the Time Step Relation, we write:

   $$\langle x, \alpha \rangle \stackrel{t,\rho}{\longmapsto} \langle x', \alpha' \rangle$$

   The above time step represents that in valuation α, x idles for t time units and then proceeds as process term x'. The values of variables during idling evolve according to the trajectory ρ. The values of variables at the end of delay are given by valuation α'. The valuations α and α' match with the values assigned by the trajectory ρ at instance 0 and r.

4. Signal Relation: The signal emitted by a process term holds in a given valuation.

   Signal Relation is of type $S \times P$. For a tuple (α, x) in the Signal Relation, we write:

   $$\alpha \in [\mathsf{s}(x)]$$

   The above predicate indicates that the signal emitted by process term x holds in valuation α.

Some operators (namely signal emission and signal evolution) in Process Algebra for Hybrid Systems associate propositions with process terms. These propositions then constitute the signal emitted by that process term. The rules stating when *signal emitted* by a process terms holds in a valuation are given in Table 18.

A predicate $\langle x, \alpha \rangle \not\stackrel{t}{\mapsto}$ represents the following:

$$\nexists \rho \in [0, t] \to (V \to R), x' \in \mathcal{P}, \alpha' \in S : \langle x, \alpha \rangle \xmapsto{t,\rho} \langle x', \alpha' \rangle$$

A.1 Bisimulation

There are two kinds of bisimulation equivalences for defined in [1]. One is called *bisimulation* and the other is called *Interference Compatible bisimulation* or *ic-bisimulation.*

The behaviour of a hybrid process is specified in a valuation of model variables. Each action and time step of a process may modify the valuation.

In bisimulation equivalence, the initial behaviour of two processes is compared in a given valuation and for subsequent steps, the behaviour of two processes is compared in the valuation obtained at the end of the previous step.

It is defined as follows:

Definition 3 *A bisimulation is a symmetric binary relation $B \subseteq (P \times S) \times (P \times S)$ on pairs of closed process terms and valuations called configurations. For all configurations, $\langle x, \alpha \rangle$,$\langle y, \alpha \rangle$ with $(\langle x, \alpha \rangle, \langle y, \alpha \rangle) \in B$ the following conditions hold:*

- *for all actions $a \in A$, process terms $x' \in P$, valuations $\alpha' \in S$, if there is an action step $\langle x, \alpha \rangle \xrightarrow{a} \langle x', \alpha' \rangle$, then there exists a $y' \in P$ and an action step $\langle y, \alpha \rangle \xrightarrow{a} \langle y', \alpha' \rangle$. Also $(\langle x', \alpha' \rangle, \langle y', \alpha' \rangle) \in B$;*
- *for all actions $a \in A$, valuations $\alpha' \in S$, if there is a termination step $\langle x, \alpha \rangle \xrightarrow{a} \langle \surd, \alpha' \rangle$, then there also exists the termination step $\langle y, \alpha \rangle \xrightarrow{a} \langle \surd, \alpha' \rangle$;*
- *for all delays $(r, \rho) \in \mathcal{D}$, process terms $x' \in P$, valuations $\alpha' \in S$, if there exists a time step $\langle x, \alpha \rangle \xmapsto{r,\rho} \langle x', \alpha' \rangle$, then there exists a $y' \in P$ and a time step $\langle y, \alpha \rangle \xmapsto{r,\rho} \langle y', \alpha' \rangle$. Also $(\langle x', \alpha' \rangle, \langle y', \alpha' \rangle) \in B$;*
- *if the signal relation $\alpha \in [\mathsf{s}(x)]$ holds then the signal relation $\alpha \in [\mathsf{s}(y)]$ also holds.*

Two configurations $\langle x, \alpha \rangle$ and $\langle y, \alpha \rangle$ are bisimulation equivalent *or* bisimilar *written as $\langle x, \alpha \rangle \leftrightarroweq \langle y, \alpha \rangle$, if there exists a bisimulation relation B such that $(\langle x, \alpha \rangle, \langle y, \alpha \rangle) \in B$.*

Additionally, two process terms x and y are bisimulation equivalent *or* bisimilar *written as $x \leftrightarroweq y$, if for all valuations α, there exists a bisimulation relation B such that $(\langle x, \alpha \rangle, \langle y, \alpha \rangle) \in B$.*

The above definition is sufficient when only sequential processes are considered. Bisimulation is not a congruence when parallel processes are studied. In case of parallelism, the valuation can be modified by a third process in parallel.

Consider the following example:

$$
\begin{array}{lcl}
X & = & = (v^\bullet = 1) \ulcorner\!\!\blacktriangledown \tilde{\tilde{a}} \cdot (v = 1) :\rightarrow \tilde{\tilde{b}} \\
Y & = & = (v^\bullet = 1) \ulcorner\!\!\blacktriangledown \tilde{\tilde{a}} \cdot \tilde{\tilde{b}}
\end{array}
$$

The two processes are bisimilar but when they are placed parallel with a process Z, they behave differently.

$$Z = (v^\bullet = 0) \ulcorner\!\!\blacktriangledown \tilde{\tilde{c}}$$

For an equivalence on processes to be a congruence with respect to parallel operator, the equivalence definition must cater for interferences by parallel processes.

In *Ic-bisimulation*, the initial behaviour of two processes is compared in all possible valuations and for subsequent steps, the same policy is adopted. I.e. at each stage the behaviour of two processes is compared in all possible valuations.

Its is defined as follows:

Definition 4 *An ic-bisimulation is a symmetric binary relation $B \subseteq P \times P$ on pairs of closed terms. For all pairs, (x, y) with $(x, y) \in B$ the following conditions hold:*

For all valuations α:

- *for all actions $a \in A$, process terms $x' \in P$, valuations $\alpha' \in S$, if there is an action step $\langle x, \alpha \rangle \xrightarrow{a} \langle x', \alpha' \rangle$, then there exists a $y' \in P$ and an action step $\langle y, \alpha \rangle \xrightarrow{a} \langle y', \alpha' \rangle$. Also $(x', y') \in B$;*
- *for all actions $a \in A$, valuations $\alpha' \in S$, if there is a termination step $\langle x, \alpha \rangle \xrightarrow{a} \langle \surd, \alpha' \rangle$, then there also exists the termination step $\langle y, \alpha \rangle \xrightarrow{a} \langle \surd, \alpha' \rangle$;*
- *for all delays $(r, \rho) \in \mathcal{D}$, process terms $x' \in P$, valuations $\alpha' \in S$, if there exists a time step $\langle x, \alpha \rangle \xmapsto{r,\rho} \langle x', \alpha' \rangle$, then there exists a $y' \in P$ and a time step $\langle y, \alpha \rangle \xmapsto{r,\rho} \langle y', \alpha' \rangle$. Also $(x', y') \in B$;*
- *if the signal relation $\alpha \in [\mathsf{s}(x)]$ holds then the signal relation $\alpha \in [\mathsf{s}(y)]$ also holds.*

Two process terms x and y ic- bisimulation equivalent *or* ic-bisimilar *written as $x \leftrightarroweq y$, if there exists a bisimulation relation B such that $(x, y) \in B$.*

A.2 Transition Rules for BPA_{hs}^{srt}

We have for all closed terms x and x', for all $\alpha, \alpha' : V \cup \dot{V} \rightarrow \mathbb{R}$, $a \in A$, $r, s \in \mathbb{R}^{>}$ and $\rho \in \epsilon_r$,$\rho' \in \epsilon_{r+s}$ the following transition rules:

Table 16: $BPA^{\mathrm{srt}}_{\mathrm{hs}}$-Transition Rules ($a \in A, r, s > 0$)

$$\frac{}{\langle \tilde{\tilde{a}}, \alpha\rangle \xrightarrow{a} \langle \surd, \alpha'\rangle} \quad \text{HS-}\underline{1}$$

$$\frac{\langle x, \alpha\rangle \xrightarrow{a} \langle x', \alpha'\rangle}{\langle \sigma^0_{\mathsf{rel}}(x), \alpha\rangle \xrightarrow{a} \langle x', \alpha'\rangle} \quad \text{HS-}\underline{2}$$

$$\frac{\langle x, \alpha\rangle \xrightarrow{a} \langle \surd, \alpha'\rangle}{\langle \sigma^0_{\mathsf{rel}}(x), \alpha\rangle \xrightarrow{a} \langle \surd, \alpha'\rangle} \quad \text{HS-}\underline{3}$$

$$\frac{\langle x, \alpha\rangle \xmapsto{r,\rho} \langle x', \alpha'\rangle}{\langle \sigma^0_{\mathsf{rel}}(x), \alpha\rangle \xmapsto{r,\rho} \langle x', \alpha'\rangle} \quad \text{HS-}\underline{4}$$

$$\frac{}{\langle \sigma^{r+s}_{\mathsf{rel}}(x), \alpha\rangle \xmapsto{r,\rho} \langle \sigma^{r}_{\mathsf{rel}}(x), \alpha'\rangle} \quad \text{HS-}\underline{5}$$

$$\frac{\alpha' \in [\mathsf{s}(x)]}{\langle \sigma^{r}_{\mathsf{rel}}(x), \alpha\rangle \xmapsto{r,\rho} \langle x, \alpha'\rangle} \quad \text{HS-}\underline{6}$$

$$\frac{\langle x, \alpha'\rangle \xmapsto{s,\rho' \trianglerighteq r} \langle x', \alpha''\rangle}{\langle \sigma^{r}_{\mathsf{rel}}(x), \alpha\rangle \xmapsto{r+s,\rho'} \langle x', \alpha''\rangle} \quad \text{HS-}\underline{7}$$

$$\frac{\langle x, \alpha\rangle \xrightarrow{a} \langle x', \alpha'\rangle, \alpha \in [\mathsf{s}(y)]}{\langle x + y, \alpha\rangle \xrightarrow{a} \langle x', \alpha'\rangle} \quad \text{HS-}\underline{8}$$

$$\frac{\langle y, \alpha\rangle \xrightarrow{a} \langle y', \alpha'\rangle, \alpha \in [\mathsf{s}(x)]}{\langle x + y, \alpha\rangle \xrightarrow{a} \langle y', \alpha'\rangle} \quad \text{HS-}\underline{9}$$

$$\frac{\langle x, \alpha\rangle \xrightarrow{a} \langle \surd, \alpha'\rangle, \alpha \in [\mathsf{s}(y)]}{\langle x + y, \alpha\rangle \xrightarrow{a} \langle \surd, \alpha'\rangle} \quad \text{HS-}\underline{10}$$

$$\frac{\langle y, \alpha\rangle \xrightarrow{a} \langle \surd, \alpha'\rangle, \alpha \in [\mathsf{s}(x)]}{\langle x + y, \alpha\rangle \xrightarrow{a} \langle \surd, \alpha'\rangle} \quad \text{HS-}\underline{11}$$

$$\frac{\begin{array}{c}\langle x, \alpha\rangle \xmapsto{r,\rho} \langle x', \alpha'\rangle, \\ \langle y, \alpha\rangle \not\xmapsto{r}, \alpha \in [\mathsf{s}(y)]\end{array}}{\langle x + y, \alpha\rangle \xmapsto{r,\rho} \langle x', \alpha'\rangle} \quad \text{HS-}\underline{12}$$

$$\frac{\begin{array}{c}\langle y, \alpha\rangle \xmapsto{r,\rho} \langle y', \alpha'\rangle, \\ \alpha \in [\mathsf{s}(x)], \langle x, \alpha\rangle \not\xmapsto{r}\end{array}}{\langle x + y, \alpha\rangle \xmapsto{r,\rho} \langle y', \alpha'\rangle} \quad \text{HS-}\underline{13}$$

$$\frac{\begin{array}{c}\langle x, \alpha\rangle \xmapsto{r,\rho} \langle x', \alpha'\rangle, \\ \langle y, \alpha\rangle \xmapsto{r,\rho} \langle y', \alpha'\rangle\end{array}}{\langle x + y, \alpha\rangle, \xmapsto{r,\rho} \langle x' + y', \alpha'\rangle} \quad \text{HS-}\underline{14}$$

$$\frac{\langle x, \alpha\rangle \xrightarrow{a} \langle x', \alpha'\rangle}{\langle x \cdot y, \alpha\rangle \xrightarrow{a} \langle x' \cdot y, \alpha'\rangle} \quad \text{HS-}\underline{15}$$

$$\frac{\langle x, \alpha\rangle \xrightarrow{a} \langle \surd, \alpha'\rangle, \alpha' \in [\mathsf{s}(y)]}{\langle x \cdot y, \alpha\rangle \xrightarrow{a} \langle y, \alpha'\rangle} \quad \text{HS-}\underline{16}$$

$$\frac{\langle x, \alpha\rangle \xmapsto{r,\rho} \langle x', \alpha'\rangle}{\langle x \cdot y, \alpha\rangle \xmapsto{r,\rho} \langle x' \cdot y, \alpha'\rangle} \quad \text{HS-}\underline{17}$$

$$\frac{\langle x, \alpha\rangle \xrightarrow{a} \langle x', \alpha'\rangle}{\langle \psi :\rightarrow x, \alpha\rangle \xrightarrow{a} \langle x', \alpha'\rangle} \quad \alpha \models \psi \quad \text{HS-}\underline{18}$$

$$\frac{\langle x, \alpha\rangle \xrightarrow{a} \langle \surd, \alpha'\rangle}{\langle \psi :\rightarrow x, \alpha\rangle \xrightarrow{a} \langle \surd, \alpha'\rangle} \quad \alpha \models \psi \quad \text{HS-}\underline{19}$$

$$\frac{\langle x, \alpha\rangle \xmapsto{r,\rho} \langle x', \alpha'\rangle}{\langle \psi :\rightarrow x, \alpha\rangle \xmapsto{r,\rho} \langle x', \alpha'\rangle} \quad \alpha \models \psi \quad \text{HS-}\underline{20}$$

Continued on Next Page. . .

Table 16 – Continued ($a \in A, r, s > 0$)

$\dfrac{\langle x,\alpha\rangle \xrightarrow{a} \langle x',\alpha'\rangle}{\langle \psi \wedge\!\!\blacktriangle\, x,\alpha\rangle \xrightarrow{a} \langle x',\alpha'\rangle}\ \alpha \models \psi$ HS-<u>21</u>	$\dfrac{\langle x,\alpha\rangle \xrightarrow{a} \langle \surd,\alpha'\rangle}{\langle \psi \wedge\!\!\blacktriangle\, x,\alpha\rangle \xrightarrow{a} \langle \surd,\alpha'\rangle}\ \alpha \models \psi$ HS-<u>22</u>
$\dfrac{\langle x,\alpha\rangle \xmapsto{r,\rho} \langle x',\alpha'\rangle}{\langle \psi \wedge\!\!\blacktriangle\, x,\alpha\rangle \xmapsto{r,\rho} \langle x',\alpha'\rangle}\ \alpha \models \psi$ HS-<u>23</u>	$\dfrac{\langle x,\alpha\rangle \xrightarrow{a} \langle x',\alpha'\rangle}{\langle \phi \curvearrowright_V x,\alpha\rangle \xrightarrow{a} \langle x',\alpha'\rangle}\ \alpha \models \phi$ HS-<u>24</u>
$\dfrac{\langle x,\alpha\rangle \xrightarrow{a} \langle \surd,\alpha'\rangle}{\langle \phi \curvearrowright_V x,\alpha\rangle \xrightarrow{a} \langle \surd,\alpha'\rangle}\ \alpha \models \phi$ HS-<u>25</u>	
$\dfrac{\langle x,\alpha\rangle \xmapsto{r,\rho} \langle x',\alpha'\rangle}{\langle \phi \curvearrowright_V x,\alpha\rangle \xmapsto{r,\rho} \langle \phi \curvearrowright_V x',\alpha'\rangle}\ \alpha \xmapsto{r,\rho} \alpha' \models_V \phi$ HS-<u>26</u>	
$\dfrac{\langle x,\alpha\rangle \xrightarrow{a} \langle x',\alpha'\rangle}{\langle \chi \ulcorner\!\blacktriangledown\, x,\alpha\rangle \xrightarrow{a} \langle x',\alpha'\rangle}\ \alpha \to \alpha' \models \chi$ HS-<u>27</u>	
$\dfrac{\langle x,\alpha\rangle \xrightarrow{a} \langle \surd,\alpha'\rangle}{\langle \chi \ulcorner\!\blacktriangledown\, x,\alpha\rangle \xrightarrow{a} \langle \surd,\alpha'\rangle}\ \alpha \to \alpha' \models \chi$ HS-<u>28</u>	
$\dfrac{\langle x,\alpha\rangle \xmapsto{r,\rho} \langle x',\alpha'\rangle}{\langle \chi \ulcorner\!\blacktriangledown\, x,\alpha\rangle \xmapsto{r,\rho} \langle x',\alpha'\rangle}\ \alpha \models {}^{\circ}\chi$ HS-<u>29</u>	$\dfrac{\langle x,\alpha\rangle \xrightarrow{a} \langle x',\alpha'\rangle}{\langle \nu_{rel}(x),\alpha\rangle \xrightarrow{a} \langle x',\alpha'\rangle}$ HS-<u>30</u>
$\dfrac{\langle x,\alpha\rangle \xrightarrow{a} \langle \surd,\alpha'\rangle}{\langle \nu_{rel}(x),\alpha\rangle \xrightarrow{a} \langle \surd,\alpha'\rangle}$ HS-<u>31</u>	

Table 17: $BPA^{\mathrm{srt}}_{\mathrm{hs}}$-Rules for Integration $(a \in A, p, q, \geq 0, r > 0)$

$$\frac{\langle F(p), \alpha\rangle \xrightarrow{a} \langle x', \alpha'\rangle, \{\alpha \in [\mathsf{s}(F(q))] \mid q \in U\}}{\langle \int_{u \in U} F(u), \alpha\rangle \xrightarrow{a} \langle x', \alpha'\rangle} \quad p \in U \quad \text{HS-}\underline{32}$$

$$\frac{\langle F(p), \alpha\rangle \xrightarrow{a} \langle \surd, \alpha'\rangle, \{\alpha \in [\mathsf{s}(F(q))] \mid q \in U\}}{\langle \int_{u \in U} F(u), \alpha\rangle \xrightarrow{a} \langle \surd, \alpha'\rangle} \quad p \in U \quad \text{HS-}\underline{33}$$

$$\frac{\begin{array}{l}\{\langle F(q), \alpha\rangle \xmapsto{r,\rho} \langle F_1(q), \alpha'\rangle \mid q \in U_1\},\\ \ldots \\ \{\langle F(q), \alpha\rangle \xmapsto{r,\rho} \langle F_n(q), \alpha'\rangle \mid q \in U_n\},\\ \{\langle F(q), \alpha\rangle \not\xmapsto{r}, \alpha \in [\mathsf{s}(F(q))] \mid q \in U_{n+1}\}\end{array}}{\langle \int_{u \in U} F(u), \alpha\rangle \xmapsto{r,\rho} \langle \int_{u \in U_1} F_1(u) + \ldots + \int_{u \in U_n} F_n(u), \alpha'\rangle} \quad \begin{array}{l}\{U_1, \ldots U_n\}\\ \text{partition of } U \backslash U_{n+1}, U_{n+1} \subset U\end{array} \quad \text{HS-}\underline{34}$$

Table 18: $BPA^{\mathrm{srt}}_{\mathrm{hs}}$-Rules for $\alpha \in [\mathsf{s}(\_)]$ $(a \in A_\delta)$

$$\frac{}{\alpha \in [\mathsf{s}(\tilde{\tilde{a}})]} \quad \text{HS-}\underline{35}$$

$$\frac{\alpha \in [\mathsf{s}(x)]}{\alpha \in [\mathsf{s}(\sigma^0_{\mathsf{rel}}(x))]} \quad \text{HS-}\underline{36}$$

$$\frac{r > 0}{\alpha \in [\mathsf{s}(\sigma^r_{\mathsf{rel}}(x))]} \quad \text{HS-}\underline{37}$$

$$\frac{\alpha \in [\mathsf{s}(x)], \alpha \in [\mathsf{s}(y)]}{\alpha \in [\mathsf{s}(x + y)]} \quad \text{HS-}\underline{38}$$

$$\frac{\alpha \in [\mathsf{s}(x)]}{\alpha \in [\mathsf{s}(x \cdot y)]} \quad \text{HS-}\underline{39}$$

$$\frac{\alpha \in [\mathsf{s}(x)]}{\alpha \in [\mathsf{s}(\psi :\rightarrow x)]} \quad \text{HS-}\underline{40}$$

$$\frac{}{\alpha \in [\mathsf{s}(\psi :\rightarrow x)]} \quad \alpha \not\models \psi \quad \text{HS-}\underline{41}$$

$$\frac{\alpha \in [\mathsf{s}(x)]}{\alpha \in [\mathsf{s}(\psi \mathbin{\wedge\!\blacktriangle} x)]} \quad \alpha \models \psi \quad \text{HS-}\underline{42}$$

$$\frac{\alpha \in [\mathsf{s}(x)]}{\alpha \in [\mathsf{s}(\phi \mathbin{\curvearrowright\!\blacktriangledown}_V x)]} \quad \alpha \models \phi \quad \text{HS-}\underline{43}$$

$$\frac{\alpha \in [\mathsf{s}(x)]}{\alpha \in [\mathsf{s}(\chi \mathbin{\ulcorner\!\blacktriangledown} x)]} \quad \text{HS-}\underline{44}$$

$$\frac{}{\alpha \in [\mathsf{s}(\chi \mathbin{\ulcorner\!\blacktriangledown} x)]} \quad \alpha \not\models {}^\circ\chi \quad \text{HS-}\underline{45}$$

$$\frac{\alpha \in [\mathsf{s}(x)]}{\alpha \in [\mathsf{s}(\nu_{rel}(x))]} \quad \text{HS-}\underline{46}$$

$$\frac{\{\alpha \in [\mathsf{s}(F(q))] \mid q \in U\}}{\alpha \in [\mathsf{s}(\int_{u \in U} F(u))]} \quad \text{HS-}\underline{47}$$

B BPA^{srt} with Integration

Table 19: Rules for Integration for BPA^{srt} from [4] ($a \in A, p, q, \geq 0, r > 0$)

$$\frac{\langle F(p)\rangle \xrightarrow{a} \langle x'\rangle}{\langle \int_{u\in U} F(u)\rangle \xrightarrow{a} \langle x'\rangle}\ p \in U \quad \text{RI-}\underline{27}$$

$$\frac{\langle F(p)\rangle \xrightarrow{a} \langle \surd\rangle}{\langle \int_{u\in U} F(u)\rangle \xrightarrow{a} \langle \surd\rangle}\ p \in U \quad \text{RI-}\underline{28}$$

$$\frac{\begin{array}{l}\{\langle F(q)\rangle \overset{r}{\longmapsto} \langle F_1(q)\rangle \mid q \in U_1\},\\ \ldots\\ \{\langle F(q)\rangle \overset{r}{\longmapsto} \langle F_n(q)\rangle \mid q \in U_n\},\\ \{\langle F(q)\rangle \overset{r}{\not\longmapsto} \mid q \in U_{n+1}\}\end{array}}{\langle \int_{u\in U} F(u)\rangle \overset{r}{\longmapsto} \langle \int_{u\in U_1} F_1(u) + \ldots + \int_{u\in U_n} F_n(u)\rangle}\ \begin{array}{l}\{U_1, \ldots U_n\}\\ \text{partition of } U\backslash U_{n+1}, U_{n+1} \subset U\end{array} \quad \text{RI-}\underline{29}$$

The following axioms have been taken from [1]. Here we only give the axioms of integration regarding BPA^{srt} process terms and leave other axioms of integration dealing with operators of BPA^{srt}_{ps} and BPA^{srt}_{hs}.

Table 20: Axioms for Integration in BPA^{srt} $(p \geq 0)$

$\int_{u \in U} F(u) = \int_{u' \in U} F(w)$	INT1
$\int_{u \in \emptyset} F(u) = \tilde{\tilde{\delta}}$	INT2
$\int_{u \in \{p\}} F(u) = F(p)$	INT3
$\int_{u \in U \cup U'} F(u) = \int_{u \in U} F(u) + \int_{u \in U'} F(u)$	INT4
$U \neq \emptyset \implies \int_{u \in U} x = x$	INT5
$(\forall u \in U \bullet F(u) = G(u)) \implies \int_{u \in U} F(u) = \int_{u \in U} G(u)$	INT6
U, U' unbounded $\implies \int_{u \in U} \sigma^u_{\mathsf{rel}}(\tilde{\tilde{\delta}}) = \int_{u \in U'} \sigma^u_{\mathsf{rel}}(\tilde{\tilde{\delta}})$	INT8SR
$\sup U = p, p \in U \implies \int_{u \in U} \sigma^u_{\mathsf{rel}}(\tilde{\tilde{\delta}}) = \sigma^p_{\mathsf{rel}}(\tilde{\tilde{\delta}})$	INT9SR
$\int_{u \in U} (\sigma^p_{\mathsf{rel}}(F(u))) = \sigma^p_{\mathsf{rel}}(\int_{u \in U} F(u))$	INT10SR
$\int_{u \in U} (F(u) + G(u)) = \int_{u \in U} F(u) + \int_{u \in U} G(u)$	INT11
$\int_{u \in U} (F(u) \cdot x) = (\int_{u \in U} F(u)) \cdot x$	INT12
$\int_{u \in U} \nu_{rel}(F(u)) = \nu_{rel}(\int_{u \in U} F(u))$	INT13

C Axioms of BPA_{drt}

The axioms of Basic Process Algebra with discrete relative timing are given below:

Table 21: Axioms of BPA^{-}_{drt}-ID as in [6] ($a \in A_{\delta}$)

$x + y = y + x$	$A1$	$\sigma(x) + \sigma(y) = \sigma(x + y)$	$DRT1$
$(x + y) + z = x + (y + z)$	$A2$	$\sigma(x) \cdot y = \sigma(x \cdot y)$	$DRT2$
$x + x = x$	$A3$	$\underline{\underline{\delta}} \cdot x = \underline{\underline{\delta}}$	$DRT3$
$(x + y) \cdot z = x \cdot z + y \cdot z$	$A4$	$x + \underline{\underline{\delta}} = x$	$DRT4A$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$	$A5$		
		$\nu_{rel}(\underline{\underline{a}}) = \underline{\underline{a}}$	$DCS1$
		$\nu_{rel}(x + y) = \nu_{rel}(x) + \nu_{rel}(y)$	$DCS2$
		$\nu_{rel}(x \cdot y) = \nu_{rel}(x) \cdot y$	$DCS3$
		$\nu_{rel}(\sigma(x)) = \underline{\underline{\delta}}$	$DCS4$

D Theorem 6

We prove the four conditions given in Theorem 6 one by one. The proof is by structural induction on all closed terms of $BPA_{\perp}^{srt}$.

Let p, p' be closed process terms of $BPA_{\perp}^{srt}$, a be an action, r be a delay duration.

Theorem 6.1

$$\text{(Proposal 1)} \quad \langle \texttt{consistent } p \rangle \iff (BPA_{hs}^{srt}) \quad \forall \alpha, \quad \alpha \in [\mathsf{s}(p)]$$

Proof

First we prove the above statement for all constants in $BPA_{\perp}^{srt}$.

1. $p = \tilde{\tilde{a}}$.

   By Rule P1-2:
   $$\text{(Proposal 1)} \quad \langle \texttt{consistent } \tilde{\tilde{a}} \rangle$$

   By Rule HS-35, for all α:

   $$(BPA_{hs}^{srt}) \quad \forall \alpha, \quad \alpha \in [\mathsf{s}(\tilde{\tilde{a}})]$$

   Hence the left right implication is proved.

2. $p = \tilde{\tilde{\delta}}$.

   By Rule P1-1:
   $$\text{(Proposal 1)} \quad \langle \texttt{consistent } \tilde{\tilde{\delta}} \rangle$$

   By Rule HS-35, for all α:

   $$(BPA_{hs}^{srt}) \quad \forall \alpha, \quad \alpha \in [\mathsf{s}(\tilde{\tilde{\delta}})]$$

   Hence the left right implication is proved.

3. $p = \perp$.

   BPA_{hs}^{srt} : A signal relation for $\perp$ cannot be derived.

   Proposal 1: A consistency predicate for $\perp$ cannot be derived.

   Hence the left right implication is proved.

Next, we prove the given statement for operators $\sigma_{\mathsf{rel}}^0, \sigma_{\mathsf{rel}}^r, \cdot, +, \nu_{rel}$, by structural induction. We give the complete proof for σ_{rel}^0. For proofs of other operators we only mention the rules that have been used.

1. $p = \sigma^0_{\mathsf{rel}}(x)$.
   Suppose
   $$(\text{Proposal 1}) \quad \langle \texttt{consistent } \sigma^0_{\mathsf{rel}}(x)\rangle$$
   This can only be derived by Rule P1-4.
   Then from the premise of the rule:
   $$(\text{Proposal 1}) \quad \langle \texttt{consistent } x\rangle$$
   By Induction, for all α:
   $$(BPA^{srt}_{hs}) \quad \alpha \in [\mathsf{s}(x)]$$
   Apply Rule HS-36, for all α:
   $$(BPA^{srt}_{hs}) \quad \alpha \in [\mathsf{s}(\sigma^0_{\mathsf{rel}}(x))]$$
   Vice Versa

   Suppose, for all α,
   $$(BPA^{srt}_{hs}) \quad \alpha \in [\mathsf{s}(\sigma^0_{\mathsf{rel}}(x))]$$
   This can only be derived by Rule 36.
   Then from the premise of the rule:
   For all α:
   $$(BPA^{srt}_{hs}) \quad \alpha \in [\mathsf{s}(x)]$$
   By Induction:
   $$(\text{Proposal 1}) \quad \langle \texttt{consistent } x\rangle$$
   Apply Rule P1-4:
   $$(\text{Proposal 1}) \quad \langle \texttt{consistent } \sigma^0_{\mathsf{rel}}(x)\rangle$$
2. $p = \sigma^r_{\mathsf{rel}}(x)$.
   By Rule P1-8:
   $$(\text{Proposal 1}) \quad \langle \texttt{consistent } \sigma^r_{\mathsf{rel}}(x)\rangle$$
   By Rule HS-37, for all α:
   $$(BPA^{srt}_{hs}) \quad \forall\alpha, \quad \alpha \in [\mathsf{s}(\sigma^r_{\mathsf{rel}}(x))]$$
   Hence the left right implication is proved.
3. $p = x + y$.
   The proof is by induction using Rules P1-16 and HS-38.

4. $p = x \cdot y$.

   The proof is by induction using Rules P1-12 and HS-39.

5. $p = \nu_{rel}(x)$.

   The proof is by induction using Rules P1-24 and HS-46.

$\boxtimes$

Theorem 6.2

$$\text{(Proposal 1)} \quad \langle x \rangle \xrightarrow{a} \surd \iff (BPA^{srt}_{hs}) \quad \forall \alpha, \alpha' : \quad \langle x, \alpha \rangle \xrightarrow{a} \langle \surd, \alpha' \rangle$$

Proof First we prove the above statement for all constants in $BPA^{srt}_{\perp}$.

1. $p = \tilde{\tilde{a}}$

   From Rule P1-3:

   $$\text{Proposal 1)} \quad \langle \tilde{\tilde{a}} \rangle \xrightarrow{a} \surd$$

   From Rule HS-1:

   For all α, α'

   $$(BPA^{srt}_{hs}) \quad \langle \tilde{\tilde{a}}, \alpha \rangle \xrightarrow{a} \langle \surd, \alpha' \rangle$$

   Hence the left right implication is proved.

2. $p = \tilde{\tilde{\delta}}$

   BPA^{srt}_{hs} : A termination step for $\tilde{\tilde{\delta}}$ cannot be derived.

   Proposal 1: A termination step for $\tilde{\tilde{\delta}}$ cannot be derived.

   Hence the left right implication is proved.

3. $p = \perp$

   BPA^{srt}_{hs} : A termination step for $\perp$ cannot be derived.

   Proposal 1: A termination step for $\perp$ cannot be derived.

   Hence the left right implication is proved.

Next, we prove the given statement for operators $\sigma^0_{\mathsf{rel}}, \sigma^r_{\mathsf{rel}}, \cdot, +, \nu_{rel}$, by structural induction. We give the complete proof for σ^0_{rel} and for other operators only mention the rules applied.

1. $p = \sigma^0_{\mathsf{rel}}(x)$

   Suppose,

   $$\text{(Proposal 1)} \quad \langle \sigma^0_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \surd$$

   This can only be derived from Rule P1-5. Hence the following must hold:

   $$\text{(Proposal 1)} \quad \langle x \rangle \xrightarrow{a} \surd$$

By Induction, for all α, α', the following holds:

$$(BPA_{hs}^{srt}) \quad \langle x, \alpha \rangle \xrightarrow{a} \langle \surd, \alpha' \rangle$$

Apply rule HS-3:
For all α, α':

$$(BPA_{hs}^{srt}) \quad \langle \sigma_{\mathsf{rel}}^{0}(x), \alpha \rangle \xrightarrow{a} \langle \surd, \alpha' \rangle$$

<u>Vice Versa</u>
Suppose, for all α, α':

$$(BPA_{hs}^{srt}) \quad \langle \sigma_{\mathsf{rel}}^{0}(x), \alpha \rangle \xrightarrow{a} \langle \surd, \alpha' \rangle$$

This can only be derived from Rule HS-3. Hence the following must hold:
For all α, α':

$$(BPA_{hs}^{srt}) \quad \langle x, \alpha \rangle \xrightarrow{a} \langle \surd, \alpha' \rangle$$

By Induction:

$$(\text{Proposal 1}) \quad \langle x \rangle \xrightarrow{a} \surd$$

Apply Rule P1-5, we get:

$$(\text{Proposal 1}) \quad \langle \sigma_{\mathsf{rel}}^{0}(x) \rangle \xrightarrow{a} \surd$$

Hence the left right implication is proved.

2. $p = \sigma_{\mathsf{rel}}^{r}(x)$
   BPA_{hs}^{srt} : An termination step for $\sigma_{\mathsf{rel}}^{r}(x)$ cannot be derived.
   Proposal 1: An termination step for $\sigma_{\mathsf{rel}}^{r}(x)$ cannot be derived.
   Hence the left right implication is proved.

3. $p = x + y$.
   Suppose,

$$(\text{Proposal 1}) \quad \langle x + y \rangle \xrightarrow{a} \surd \tag{12}$$

   The above Transition can be derived from two rules.

   - Rule P1-17
     Then from the premise of the rule, the following must hold:

$$\begin{array}{l} (\text{Proposal 1}) \quad \langle x \rangle \xrightarrow{a} \surd \\ \text{Proposal 1} \quad \langle \texttt{consistent } y \rangle \end{array}$$

     By Induction for all α, α':

$$(BPA_{hs}^{srt}) \quad \langle x, \alpha \rangle \xrightarrow{a} \langle \surd, \alpha' \rangle$$

     By Theorem 6.1, for all α:

$$(BPA_{hs}^{srt}) \quad \alpha \in [\mathsf{s}(y)]$$

Apply rule HS-10 on the above transitions and relations:
For all α, α':

$$(BPA_{hs}^{srt}) \quad \langle x+y, \alpha \rangle \xrightarrow{a} \langle \surd, \alpha' \rangle$$

- Rule P1-18
  Same as above.

<u>Vice Versa</u>

Suppose, for all α, α':

$$(BPA_{hs}^{srt}) \quad \langle x+y, \alpha \rangle \xrightarrow{a} \langle \surd, \alpha' \rangle \tag{13}$$

The above Transition can be derived from two rules. We discuss these rules one by one:

- Rule HS-10
  Then from the premise of the rule, the following must hold:
  For all α, α':
  $$\begin{array}{ll} (BPA_{hs}^{srt}) & \langle x, \alpha \rangle \xrightarrow{a} \langle \surd, \alpha' \rangle \\ (BPA_{hs}^{srt}) & \alpha \in [\mathsf{s}(y)] \end{array}$$
  By Induction:
  $$(\text{Proposal 1}) \quad \langle x \rangle \xrightarrow{a} \surd$$
  By Theorem 6.1:
  $$\text{Proposal 1} \quad \langle \texttt{consistent } y \rangle$$
  Apply rule P1-17 on the above transitions and relations:
  $$(\text{Proposal 1}) \quad \langle x+y \rangle \xrightarrow{a} \surd$$
- Rule HS-11
  Same as above.

Hence, left right implication is proved.

4. $p = x \cdot y$
   BPA_{hs}^{srt} : A termination step for $x \cdot y$ cannot be derived.
   Proposal 1: A termination step for $x \cdot y$ cannot be derived.
   Hence, left right implication is proved.

5. $p = \nu_{rel}(x)$.

   Suppose,

$$\text{(Proposal 1)} \quad \langle \nu_{rel}(x) \rangle \xrightarrow{a} \surd$$

   This can only be derived from Rule P1-25. Hence the following must hold:

$$\text{(Proposal 1)} \quad \langle x \rangle \xrightarrow{a} \surd$$

   By Induction, for all α, α', the following holds:

$$(BPA_{hs}^{srt}) \quad \langle x, \alpha \rangle \xrightarrow{a} \langle \surd, \alpha' \rangle$$

   Apply rule HS-31:

   For all α, α':

$$(BPA_{hs}^{srt}) \quad \langle \nu_{rel}(x), \alpha \rangle \xrightarrow{a} \langle \surd, \alpha' \rangle$$

   Vice Versa

   Suppose, for all α, α':

$$(BPA_{hs}^{srt}) \quad \langle \nu_{rel}(x), \alpha \rangle \xrightarrow{a} \langle \surd, \alpha' \rangle$$

   This can only be derived from Rule HS-31. Hence the following must hold:

   For all α, α':

$$(BPA_{hs}^{srt}) \quad \langle x, \alpha \rangle \xrightarrow{a} \langle \surd, \alpha' \rangle$$

   By Induction:

$$\text{(Proposal 1)} \quad \langle x \rangle \xrightarrow{a} \surd$$

   Apply Rule P1-25, we get:

$$\text{(Proposal 1)} \quad \langle \nu_{rel}(x) \rangle \xrightarrow{a} \surd$$

   Hence the left right implication is proved.

$\boxtimes$

Theorem 6.3

$$\text{(Proposal 1)} \quad \langle x \rangle \xrightarrow{a} \langle x' \rangle \iff (BPA_{hs}^{srt}) \quad \forall \alpha, \alpha' : \quad \langle x, \alpha \rangle \xrightarrow{a} \langle x', \alpha' \rangle$$

Proof First we prove the above statement for all constants in $BPA_{\perp}^{srt}$.

1. $p = \tilde{\tilde{a}}$

   BPA_{hs}^{srt} : An action step for $\tilde{\tilde{a}}$ cannot be derived.

   Proposal 1: An action step for $\tilde{\tilde{a}}$ cannot be derived.

   Hence, left right implication is proved.

2. $p = \tilde{\tilde{\delta}}$

   BPA_{hs}^{srt} : A action step for $\tilde{\tilde{\delta}}$ cannot be derived.

   Proposal 1: A action step for $\tilde{\tilde{\delta}}$ cannot be derived.

   Hence, left right implication is proved.

3. $p = \perp$

   BPA_{hs}^{srt} : A action step for $\perp$ cannot be derived.

   Proposal 1: A action step for $\perp$ cannot be derived.

   Hence, left right implication is proved.

Next, we prove the given statement for operators $\sigma_{\mathsf{rel}}^0, \sigma_{\mathsf{rel}}^r, \cdot, +, \nu_{rel}$, by structural induction.

1. $p = \sigma_{\mathsf{rel}}^0(x)$

   Suppose,

   $$\text{(Proposal 1)} \quad \langle \sigma_{\mathsf{rel}}^0(x) \rangle \xrightarrow{a} \langle p' \rangle$$

   This can only be derived from Rule P1-6. Hence the following must hold:

   $$\text{(Proposal 1)} \quad \langle x \rangle \xrightarrow{a} \langle p' \rangle$$

   By Induction, for all α, α', the following holds:

   $$(BPA_{hs}^{srt}) \quad \langle x, \alpha \rangle \xrightarrow{a} \langle p', \alpha' \rangle$$

   Apply rule HS-2:

   For all α, α':

   $$(BPA_{hs}^{srt}) \quad \langle \sigma_{\mathsf{rel}}^0(x), \alpha \rangle \xrightarrow{a} \langle p', \alpha' \rangle$$

   <u>Vice Versa</u>

   Suppose, for all α, α':

   $$(BPA_{hs}^{srt}) \quad \langle \sigma_{\mathsf{rel}}^0(x), \alpha \rangle \xrightarrow{a} \langle p', \alpha' \rangle$$

   This can only be derived from Rule HS-2. Hence the following must hold:

   For all α, α':

   $$(BPA_{hs}^{srt}) \quad \langle x, \alpha \rangle \xrightarrow{a} \langle p', \alpha' \rangle$$

   By Induction:

   $$\text{(Proposal 1)} \quad \langle x \rangle \xrightarrow{a} \langle p' \rangle$$

   Apply Rule P1-6, we get:

   $$\text{(Proposal 1)} \quad \langle \sigma_{\mathsf{rel}}^0(x) \rangle \xrightarrow{a} \langle p' \rangle$$

   Hence the left right implication is proved.

2. $p = \sigma^r_{\mathsf{rel}}(x)$

   BPA^{srt}_{hs} : An action step for $\sigma^r_{\mathsf{rel}}(x)$ cannot be derived.

   Proposal 1: An action step for $\sigma^r_{\mathsf{rel}}(x)$ cannot be derived.

   Hence, left right implication is proved.

3. $p = x + y$.

   Suppose,

$$\text{(Proposal 1)} \quad \langle x + y \rangle \xrightarrow{a} \langle p' \rangle \tag{14}$$

   The above Transition can be derived from two rules.

   - Rule P1-19
     Then from the premise of the rule, the following must hold:

$$\begin{array}{ll} \text{(Proposal 1)} & \langle x \rangle \xrightarrow{a} \langle p' \rangle \\ \text{Proposal 1} & \langle \texttt{consistent}\ \ y \rangle \end{array}$$

     By Induction for all α, α':

$$(BPA^{srt}_{hs}) \quad \langle x, \alpha \rangle \xrightarrow{a} \langle p', \alpha' \rangle$$

     By Theorem 6.1, for all α:

$$(BPA^{srt}_{hs}) \quad \alpha \in [\mathsf{s}(y)]$$

     Apply rule HS-8 on the above transitions and relations:
     For all α, α':

$$(BPA^{srt}_{hs}) \quad \langle x + y, \alpha \rangle \xrightarrow{a} \langle p', \alpha' \rangle$$

   - Rule P1-20
     Same as above.

   Vice Versa

   Suppose, for all α, α':

$$(BPA^{srt}_{hs}) \quad \langle x + y, \alpha \rangle \xrightarrow{a} \langle p', \alpha' \rangle \tag{15}$$

   The above Transition can be derived from two rules. We discuss these rules one by one:

   - Rule HS-8
     Then from the premise of the rule, the following must hold:
     For all α, α':

$$\begin{array}{ll} (BPA^{srt}_{hs}) & \langle x, \alpha \rangle \xrightarrow{a} \langle p', \alpha' \rangle \\ (BPA^{srt}_{hs}) & \alpha \in [\mathsf{s}(y)] \end{array}$$

By Induction:

$$(\text{Proposal 1}) \quad \langle x\rangle \xrightarrow{a} \langle p'\rangle$$

By Theorem 6.1:

$$\text{Proposal 1} \quad \langle \texttt{consistent } y\rangle$$

Apply rule P1-19 on the above transitions and relations:

$$(\text{Proposal 1}) \quad \langle x+y\rangle \xrightarrow{a} \langle p'\rangle$$

- Rule HS-9
  Same as above.

4. $p = x \cdot y$
   Suppose,

$$(\text{Proposal 1}) \quad \langle x \cdot y\rangle \xrightarrow{a} \langle p'\rangle \tag{16}$$

This can be derived from two rules:

- Rule P1-15
  Then, for some process term x', $p' = x' \cdot y$, and the following must be derivable:

$$(\text{Proposal 1}) \quad \langle x\rangle \xrightarrow{a} \langle x'\rangle$$

  By Induction, for all α, α':

$$(BPA_{hs}^{srt}) \quad \langle x, \alpha\rangle \xrightarrow{a} \langle x', \alpha'\rangle$$

  Apply rule HS-15 on the above transition:
  For all α, α':

$$(BPA_{hs}^{srt}) \quad \langle x \cdot y, \alpha\rangle \xrightarrow{a} \langle x' \cdot y, \alpha'\rangle$$

- Rule P1-16
  If Transition 16 is derived from this rule, then, $p' = y$, and the following must be derivable:

$$\begin{array}{ll}(\text{Proposal 1}) & \langle x\rangle \xrightarrow{a} \surd \\ (\text{Proposal 1}) & \langle \texttt{consistent } x\rangle\end{array}$$

  By Theorem 6.2, for all α, α':

$$(BPA_{hs}^{srt}) \quad \langle x, \alpha\rangle \xrightarrow{a} \langle \surd, \alpha'\rangle$$

  By Theorem 6.1, for all α:

$$(BPA_{hs}^{srt}) \quad \alpha \in [\mathsf{s}(y)]$$

  Apply rule HS-16 on the above transition:
  For all α, α':

$$(BPA_{hs}^{srt}) \quad \langle x \cdot y, \alpha\rangle \xrightarrow{a} \langle y, \alpha'\rangle$$

Vice Versa
Suppose, for all α, α':

$$(BPA_{hs}^{srt}) \quad \langle x \cdot y, \alpha \rangle \xrightarrow{a} \langle p', \alpha' \rangle \tag{17}$$

This can be derived from two rules:

- Rule HS-15
  If Transition 17 is derived from this rule, then for some process term x', $p' = x' \cdot y$, and the following must be derivable:
  For all α, α':
  $$(BPA_{hs}^{srt}) \quad \langle x, \alpha \rangle \xrightarrow{a} \langle x', \alpha' \rangle$$
  By Induction:
  $$(\text{Proposal 1}) \quad \langle x \rangle \xrightarrow{a} \langle x' \rangle$$
  Apply rule P1-13:
  $$(\text{Proposal 1}) \quad \langle x \cdot y \rangle \xrightarrow{a} \langle x' \cdot y \rangle$$

- Rule HS-16
  If Transition 17 is derived from this rule, then $p' = y$, and the following must be derivable:
  For all α, α':
  $$\begin{array}{ll} (BPA_{hs}^{srt}) & \langle x, \alpha \rangle \xrightarrow{a} \langle \surd, \alpha' \rangle \\ (BPA_{hs}^{srt}) & \alpha' \in [\mathsf{s}(y)] \end{array}$$
  By Theorem 6.2:
  $$(\text{Proposal 1}) \quad \langle x \rangle \xrightarrow{a} \surd$$
  By Theorem 6.1:
  $$(\text{Proposal 1}) \quad \alpha \in [\mathsf{s}(y)]$$
  Apply rule P1-14:
  $$(\text{Proposal 1}) \quad \langle x \cdot y \rangle \xrightarrow{a} \langle y \rangle$$

Hence the left right implication is proved.

5. $p = \nu_{rel}(x)$.
   Suppose,
   $$(\text{Proposal 1}) \quad \langle \nu_{rel}(x) \rangle \xrightarrow{a} \langle p' \rangle$$
   This can only be derived from Rule P1-26. Hence the following must hold:
   $$(\text{Proposal 1}) \quad \langle x \rangle \xrightarrow{a} \langle p' \rangle$$
   By Induction, for all α, α', the following holds:
   $$(BPA_{hs}^{srt}) \quad \langle x, \alpha \rangle \xrightarrow{a} \langle p', \alpha' \rangle$$

Apply rule HS-30:

For all α, α':

$$(BPA_{hs}^{srt}) \quad \langle \nu_{rel}(x), \alpha \rangle \xrightarrow{a} \langle p', \alpha' \rangle$$

Vice Versa

Suppose, for all α, α':

$$(BPA_{hs}^{srt}) \quad \langle \nu_{rel}(x), \alpha \rangle \xrightarrow{a} \langle p', \alpha' \rangle$$

This can only be derived from Rule HS-30. Hence the following must hold:

For all α, α':

$$(BPA_{hs}^{srt}) \quad \langle x, \alpha \rangle \xrightarrow{a} \langle p', \alpha' \rangle$$

By Induction:

$$(\text{Proposal 1}) \quad \langle x \rangle \xrightarrow{a} \langle p' \rangle$$

Apply Rule P1-26, we get:

$$(\text{Proposal 1}) \quad \langle \nu_{rel}(x) \rangle \xrightarrow{a} \langle p' \rangle$$

Hence the left right implication is proved.

⊠

Theorem 6.4

$$(\text{Proposal 1}) \quad \langle x \rangle \xmapsto{r} \langle x' \rangle \iff (BPA_{hs}^{srt}) \quad \forall \rho, \quad \langle x, \alpha_0^{\rho} \rangle \xmapsto{r,\rho} \langle x', \alpha_r^{\rho} \rangle$$

Proof First we prove the above statement for all constants in $BPA_{\perp}^{srt}$.

1. $p = \tilde{\tilde{a}}$

   BPA_{hs}^{srt} : A time step for $\tilde{\tilde{a}}$ cannot be derived.

   Proposal 1: A time step for $\tilde{\tilde{a}}$ cannot be derived.

   Hence, left right implication is proved.

2. $p = \tilde{\tilde{\delta}}$

   BPA_{hs}^{srt} : A time step for $\tilde{\tilde{\delta}}$ cannot be derived.

   Proposal 1: A time step for $\tilde{\tilde{\delta}}$ cannot be derived.

   Hence, left right implication is proved.

3. $p = \perp$

   BPA_{hs}^{srt} : A time step for $\perp$ cannot be derived.

   Proposal 1: A time step for $\perp$ cannot be derived.

   Hence, left right implication is proved.

Next, we prove the given statement for operators $\sigma^0_{\mathsf{rel}}, \sigma^r_{\mathsf{rel}}, \cdot, +, \nu_{rel}$, by structural induction.

1. $p = \sigma^0_{\mathsf{rel}}(x)$

   Suppose, for all ρ:

$$(BPA^{srt}_{hs}) \quad \langle \sigma^0_{\mathsf{rel}}(x), \alpha^\rho_0 \rangle \xrightarrow{r,\rho} \langle p', \alpha^\rho_r \rangle$$

   This can only be derived from Rule HS-4. Hence the following must hold:
   For all ρ:

$$(BPA^{srt}_{hs}) \quad \langle x, \alpha^\rho_0 \rangle \xrightarrow{r,\rho} \langle p', \alpha^\rho_r \rangle$$

   By Induction:

$$(\text{Proposal 1}) \quad \langle x \rangle \xmapsto{r} \langle p' \rangle$$

   Apply Rule P1-7, we get:

$$(\text{Proposal 1}) \quad \langle \sigma^0_{\mathsf{rel}}(x) \rangle \xmapsto{r} \langle p' \rangle$$

   <u>Vice Versa</u>
   Suppose,

$$(\text{Proposal 1}) \quad \langle \sigma^0_{\mathsf{rel}}(x) \rangle \xmapsto{r} \langle p' \rangle$$

   This can only be derived from Rule P1-7. Hence the following must hold:

$$(\text{Proposal 1}) \quad \langle x \rangle \xmapsto{r} \langle p' \rangle$$

   By Induction, for all ρ, the following holds:

$$(BPA^{srt}_{hs}) \quad \langle x, \alpha^\rho_0 \rangle \xrightarrow{r,\rho} \langle p', \alpha^\rho_r \rangle$$

   Apply rule HS-4:

$$(BPA^{srt}_{hs}) \quad \langle \sigma^0_{\mathsf{rel}}(x), \alpha^\rho_0 \rangle \xrightarrow{r,\rho} \langle p', \alpha^\rho_r \rangle$$

   Hence the left right implication is proved.

2. $p = \sigma^r_{\mathsf{rel}}(x)$

   Suppose, for all ρ:

$$(BPA^{srt}_{hs}) \quad \langle \sigma^r_{\mathsf{rel}}(x), \alpha^\rho_0 \rangle \xrightarrow{t,\rho} \langle p', \alpha^\rho_t \rangle \tag{18}$$

   - Case $t < r$:
     Let $r = u + t$, for some $u > 0$. Then Transition 18 must be derived from Rule HS-5. Then $p' = \sigma^u_{\mathsf{rel}}(x)$. Rule HS-5 can always be applied. Hence for all ρ:

$$(BPA^{srt}_{hs}) \quad \langle \sigma^{u+t}_{\mathsf{rel}}(x), \alpha^\rho_0 \rangle \xrightarrow{t,\rho} \langle \sigma^u_{\mathsf{rel}}(x), \alpha^\rho_t \rangle$$

     In Proposal 1, by Rule P1-9, the following is derivable:

$$\text{Proposal} \quad \langle \sigma^{u+t}_{\mathsf{rel}}(x) \rangle \xmapsto{t} \langle \sigma^u_{\mathsf{rel}}(x) \rangle$$

- Case $t = r$:
  This can only be derived from Rule HS- 6. Then $p' = x$ in Transition 18. Rewriting Transition 18:
  For all ρ:
  $$(BPA_{hs}^{srt}) \quad \langle \sigma_{\mathsf{rel}}^{t}(x), \alpha_0^{\rho} \rangle \xmapsto{t,\rho} \langle x, \alpha_t^{\rho} \rangle$$
  From the premise of rule HS-6, for all α_t^{ρ}:
  $$\alpha_t^{\rho} \in [\mathsf{s}(x)]$$
  Since there is no restriction on ρ and hence on α_t^{ρ}, therefore we have:
  For all α
  $$(BPA_{hs}^{srt}) \quad \alpha \in [\mathsf{s}(x)]$$
  By Theorem 6.1:
  $$\text{Proposal 1} \quad \langle \texttt{consistent } x \rangle$$
  Then by Rule P1-10, the following is derivable:
  $$\text{Proposal1} \quad \langle \sigma_{\mathsf{rel}}^{t}(x) \rangle \xmapsto{t} \langle x \rangle$$

- Case $t > r$.
  Let $t = r + r_1$, for some $r_1 > 0$. Rewriting Transition 18:
  For all ρ:
  $$(BPA_{hs}^{srt}) \quad \langle \sigma_{\mathsf{rel}}^{r}(x), \alpha_0^{\rho} \rangle \xmapsto{r+r_1,\rho} \langle p', \alpha_t^{\rho} \rangle \tag{19}$$
  This can only be derivable from Rule HS-7. Hence, the premise of the rule must hold. From Premise of the Rule HS-7, Transition 19 can only be derived if the following holds:
  $$(BPA_{hs}^{srt}) \quad \langle x, \alpha_r^{\rho} \rangle \xmapsto{r_1,\rho \trianglerighteq r} \langle p', \alpha_t^{\rho} \rangle$$
  For the definition of symbol $\rho \trianglerighteq r$, see Appendix A. Briefly, $\rho \trianglerighteq r$ denotes the state evolution ρ after r time units have elapsed. If the time interval of ρ is $[0, r+r_1]$, then the time interval of $\rho \trianglerighteq r$ is $[0, r_1]$. As there are no restrictions on ρ, therefore there are no restrictions on $\rho \trianglerighteq r$.
  By Structural Induction:
  $$\text{Proposal1} \quad \langle x \rangle \xmapsto{r_1} \langle p' \rangle$$
  Apply Rule P1-11, the following holds:
  $$\text{Proposal1} \quad \langle \sigma_{\mathsf{rel}}^{r}(x) \rangle \xmapsto{r+r_1} \langle p' \rangle$$

<u>Vice versa</u>

Suppose,

$$(\text{Proposal 1}) \quad \langle \sigma^{r}_{\mathsf{rel}}(x) \rangle \overset{t}{\longmapsto} \langle p' \rangle \tag{20}$$

- Case $t < r$:
  Let $r = u + t$, for some $u > 0$. The Transition 20 must have been derived from Rule P1-9 and $p' = \sigma^{u}_{\mathsf{rel}}(x)$.

  $$(\text{Proposal 1}) \quad \langle \sigma^{u+t}_{\mathsf{rel}}(x) \rangle \overset{t}{\longmapsto} \langle \sigma^{u}_{\mathsf{rel}}(x) \rangle$$

  In BPA^{srt}_{hs}, by Rule HS-5, the following can be derived for all ρ:

  $$(BPA^{srt}_{hs}) \quad \langle \sigma^{u+t}_{\mathsf{rel}}(x), \alpha^{\rho}_{0} \rangle \xrightarrow{t,\rho} \langle \sigma^{u}_{\mathsf{rel}}(x), \alpha^{\rho}_{t} \rangle$$

- Case $t = r$:
  This can only be derived from Rule P1-10. Then $p' = x$ in Transition 20.

  $$(\text{Proposal 1}) \quad \langle \sigma^{t}_{\mathsf{rel}}(x) \rangle \overset{t}{\longmapsto} \langle x \rangle$$

  From the premise of the rule,

  $$\langle \texttt{consistent} \;\; x \rangle$$

  By Theorem6.1, for all α:

  $$BPA^{srt}_{hs} \quad \alpha \in [\mathsf{s}(x)]$$

  Then by Rule HS-6, the following is derivable:
  For all ρ:

  $$(BPA^{srt}_{hs}) \quad \langle \sigma^{t}_{\mathsf{rel}}(x), \alpha^{\rho}_{0} \rangle \xrightarrow{t,\rho} \langle x, \alpha^{\rho}_{t} \rangle$$

- Case $t > r$.
  Let $t = r + r_1$, for some $r_1 > 0$. Rewriting Transition 20:

  $$(\text{Proposal 1}) \quad \langle \sigma^{r}_{\mathsf{rel}}(x) \rangle \xrightarrow{r+r_1} \langle p' \rangle$$

  This can only be derivable from Rule P1-11. Hence, the premise of the rule must hold:

  $$(\text{Proposal 1}) \quad \langle x \rangle \xrightarrow{r_1} \langle p' \rangle$$

  By Structural Induction, for all ρ

  $$(BPA^{srt}_{hs}) \quad \langle x, \alpha^{\rho}_{0} \rangle \xrightarrow{r_1,\rho} \langle p', \alpha^{\rho}_{r_1} \rangle$$

  Apply Rule HS-7, the following holds:

  $$(BPA^{srt}_{hs}) \quad \langle \sigma^{r}_{\mathsf{rel}}(x), \alpha^{\rho'}_{0} \rangle \xrightarrow{r+r_1,\rho'} \langle p', \alpha^{\rho'}_{t} \rangle$$

where $\rho = \rho' \trianglerighteq r$. For the definition of the state evolution $\rho = \rho' \trianglerighteq r$, see Appendix A. Briefly, $\rho' \trianglerighteq r$ is the state evolution ρ after the passage of r time units. Since there is no restrictions on ρ, hence there is no restriction on ρ'. Hence, we can write:
For all ρ':

$$(BPA^{srt}_{hs}) \quad \langle \sigma^r_{\mathsf{rel}}(x), \alpha_0^{\rho'} \rangle \xmapsto{r+r_1, \rho'} \langle p', \alpha_t^{\rho'} \rangle$$

Hence, left right implication is proved.

3. $p = x + y$.

Suppose,

$$\text{(Proposal 1)} \quad \langle x + y \rangle \xmapsto{r} \langle p' \rangle \tag{21}$$

The above Transition can be derived from three rules. We discuss these rules one by one:

- Rule P1-21:
  Then for some process term x', y', p' in Transition 21 is $x' + y'$. Rewriting Transition 21:

  $$\text{(Proposal 1)} \quad \langle x + y \rangle \xmapsto{r} \langle x' + y' \rangle \tag{22}$$

  From the premise of Rule P1-21:

  $$\begin{array}{ll} \text{(Proposal 1)} & \langle x \rangle \xmapsto{r} \langle x' \rangle \\ \text{(Proposal 1)} & \langle y \rangle \xmapsto{r} \langle y' \rangle \end{array}$$

  By Induction, for all ρ:

  $$\begin{array}{ll} (BPA^{srt}_{hs}) & \langle x, \alpha_0^{\rho} \rangle \xmapsto{r, \rho} \langle x', \alpha_r^{\rho} \rangle \\ (BPA^{srt}_{hs}) & \langle y, \alpha_0^{\rho} \rangle \xmapsto{r, \rho} \langle y', \alpha_r^{\rho} \rangle \end{array}$$

  Apply rule HS-14 on the above transitions:
  For all ρ:

  $$(BPA^{srt}_{hs}) \quad \langle x + y, \alpha_0^{\rho} \rangle \xmapsto{r, \rho} \langle x' + y', \alpha_r^{\rho} \rangle$$

- Rule P1-22
  Then from the premise of the rule, the following must hold:

  $$\begin{array}{ll} \text{(Proposal 1)} & \langle x \rangle \xmapsto{r} \langle p' \rangle \\ \text{Proposal 1} & \langle \texttt{consistent } y \rangle \\ \text{Proposal 1} & \langle y \rangle \not\xmapsto{r} \end{array}$$

  By Induction for all ρ:

  $$\begin{array}{ll} (BPA^{srt}_{hs}) & \langle x, \alpha_0^{\rho} \rangle \xmapsto{r, \rho} \langle p', \alpha_r^{\rho} \rangle \\ (BPA^{srt}_{hs}) & \langle y, \alpha_0^{\rho} \rangle \not\xmapsto{r} \end{array}$$

By Theorem 6.1, for all α:

$$(BPA_{hs}^{srt}) \quad \alpha \in [\mathsf{s}(y)]$$

Apply rule HS-12 on the above transitions and relations:
For all ρ:

$$(BPA_{hs}^{srt}) \quad \langle x+y, \alpha_0^{\rho}\rangle \xmapsto{r,\rho} \langle p', \alpha_r^{\rho}\rangle$$

- Rule P1-23
  Same as above.

Vice Versa

Suppose, for all ρ:

$$(BPA_{hs}^{srt}) \quad \langle x+y, \alpha_0^{\rho}\rangle \xmapsto{r} \langle p', \alpha_r^{\rho}\rangle \tag{23}$$

The above Transition can be derived from three rules. We discuss these rules one by one:

- Rule HS-14:
  Then for some process term x', y', p' in Transition 23 is $x' + y'$. Rewriting Transition 23:
  For all ρ:
  $$(BPA_{hs}^{srt}) \quad \langle x+y, \alpha_0^{\rho}\rangle \xmapsto{r} \langle x'+y', \alpha_r^{\rho}\rangle \tag{24}$$
  From the premise of Rule HS-14:
  For all ρ:
  $$(BPA_{hs}^{srt}) \quad \langle x, \alpha_0^{\rho}\rangle \xmapsto{r} \langle x', \alpha_r^{\rho}\rangle$$
  $$(BPA_{hs}^{srt}) \quad \langle y, \alpha_0^{\rho}\rangle \xmapsto{r} \langle y', \alpha_r^{\rho}\rangle$$
  By Induction:
  $$(\text{Proposal 1}) \quad \langle x\rangle \xmapsto{r} \langle x'\rangle$$
  $$(\text{Proposal 1}) \quad \langle y,\rangle \xmapsto{r} \langle y'\rangle$$
  Apply rule P1-21 on the above transitions:
  $$(\text{Proposal 1}) \quad \langle x+y\rangle \xmapsto{r} \langle x'+y'\rangle$$

- Rule HS-12
  Then from the premise of the rule, the following must hold:
  For all ρ
  $$(BPA_{hs}^{srt}) \quad \langle x, \alpha_0^{\rho}\rangle \xmapsto{r,\rho} \langle p', \alpha_r^{\rho}\rangle$$
  $$(BPA_{hs}^{srt}) \quad \alpha_0^{\rho} \in [\mathsf{s}(y)]$$
  $$(BPA_{hs}^{srt}) \quad \langle y, \alpha_0^{\rho}\rangle \not\xmapsto{r}$$
  By Induction:
  $$(\text{Proposal 1}) \quad \langle x\rangle \xmapsto{r} \langle p'\rangle$$
  $$\text{Proposal 1} \quad \langle y\rangle \not\xmapsto{r}$$

By Theorem 6.1:

$$\text{Proposal 1} \quad \langle \texttt{consistent} \ y \rangle$$

Apply rule P1-22 on the above transitions and relations:

$$(\text{Proposal 1}) \quad \langle x + y \rangle \xmapsto{r} \langle p' \rangle$$

- Rule HS-13
  Same as above.

Hence, left right implication is proved.

4. $p = x \cdot y$

   Suppose,

   $$(\text{Proposal 1}) \quad \langle x \cdot y \rangle \xmapsto{r} \langle p' \rangle$$

   This can only be derived from Rule P1-15. Then, for some process term x', $p' = x' \cdot y$, and the following must be derivable:

   $$(\text{Proposal 1}) \quad \langle x \rangle \xmapsto{r} \langle x' \rangle$$

   By Induction, for all ρ:

   $$(BPA_{hs}^{srt}) \quad \langle x, \alpha_0^{\rho} \rangle \xmapsto{r,\rho} \langle x', \alpha_r^{\rho} \rangle$$

   Apply rule HS-17 on the above transition:

   For all ρ:

   $$(BPA_{hs}^{srt}) \quad \langle x \cdot y, \alpha_0^{\rho} \rangle \xmapsto{r,\rho} \langle x' \cdot y, \alpha_r^{\rho} \rangle$$

   Vice Versa

   Suppose, for all ρ:

   $$(BPA_{hs}^{srt}) \quad \langle x \cdot y, \alpha_0^{\rho} \rangle \xmapsto{r,\rho} \langle p', \alpha_r^{\rho} \rangle$$

   This can only be derived from rule HS-17. Hence for some process term x', $p' = x' \cdot y$, and the following must be derivable:

   For all ρ:

   $$(BPA_{hs}^{srt}) \quad \langle x, \alpha_0^{\rho} \rangle \xmapsto{r,\rho} \langle x', \alpha_r^{\rho} \rangle$$

   By Induction:

   $$(\text{Proposal 1}) \quad \langle x \rangle \xmapsto{r} \langle x' \rangle$$

   Apply rule P1-15:

   $$(\text{Proposal 1}) \quad \langle x \cdot y \rangle \xmapsto{r} \langle x' \cdot y \rangle$$

   Hence, left right implication is proved.

5. $p = \nu_{rel}(x)$.
   BPA_{hs}^{srt} : A time step for $\nu_{rel}(x)$ cannot be derived.
   Proposal 1: A time step for $\nu_{rel}(x)$ cannot be derived.
   Hence, left right implication is proved.

$\boxtimes$

E Theorem 7

Thoerem 7

Axiom SRT3 is sound in the semantics of Section 5.2.

$$\sigma^{v}_{\mathsf{rel}}(x) + \sigma^{v}_{\mathsf{rel}}(y) \leftrightarroweq \sigma^{v}_{\mathsf{rel}}(x+y) \qquad (SRT3)$$

where $v \geq 0$

Proof

We prove the soundness of Axiom SRT3 in two steps.

Case $u = 0$

From Rules AC-4, AC-5, AC-6 and AC-7, it easy to prove the following holds in the semantics of $BPA^{srt}_{\perp}$ with modified Alternative Composition (Section 5.2):

For any process term x,

$$\sigma^{0}_{\mathsf{rel}}(x) \leftrightarroweq x$$

Since Bisimulation is a congruence therefore, then it becomes trivial to prove that:

$$\sigma^{0}_{\mathsf{rel}}(x) + \sigma^{0}_{\mathsf{rel}}(y) \leftrightarroweq \sigma^{0}_{\mathsf{rel}}(x+y)$$

Case $u > 0$

Let $\mathcal{I}$ be the following relation:

$$\mathcal{I} = \{(p,p) \mid p \in P\}$$

Let R be the following relation:

$$R = \{(\sigma^{t}_{\mathsf{rel}}(x) + \sigma^{t}_{\mathsf{rel}}(y)), \sigma^{t}_{\mathsf{rel}}(x+y) \mid 0 < t \leq v, x, y \in P\}$$

We prove that $R \cup \mathcal{I}$ is a bisimulation relation:

For all $a \in A, r > 0, z \in P$:

1. $$\langle \sigma^{t}_{\mathsf{rel}}(x) + \sigma^{t}_{\mathsf{rel}}(y)\rangle \xrightarrow{a} \langle z\rangle \implies \begin{array}{l} \exists z' \in P : \langle \sigma^{t}_{\mathsf{rel}}(x+y)\rangle \xrightarrow{a} \langle z'\rangle \\ (z, z') \in R \cup \mathcal{I} \end{array}$$

   Trivial.

2. $$\langle \sigma^{t}_{\mathsf{rel}}(x+y)\rangle \xrightarrow{a} \langle z\rangle \implies \begin{array}{l} \exists z' \in P : \langle \sigma^{t}_{\mathsf{rel}}(x) + \sigma^{t}_{\mathsf{rel}}(y)\rangle \xrightarrow{a} \langle z'\rangle \\ (z', z) \in R \cup \mathcal{I} \end{array}$$

   Trivial.

3. $$\langle \sigma^{t}_{\mathsf{rel}}(x) + \sigma^{t}_{\mathsf{rel}}(y)\rangle \xrightarrow{a} \surd \iff \langle \sigma^{t}_{\mathsf{rel}}(x+y)\rangle \xrightarrow{a} \surd$$

   Trivial.

4.

$$\langle \texttt{consistent}\ \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y)\rangle \iff \langle \texttt{consistent}\ \sigma^t_{\mathsf{rel}}(x+y)\rangle$$

Trivial.

5.

$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y)\rangle \xmapsto{r} \langle z\rangle \implies \begin{array}{l}\exists z' \in P : \langle \sigma^t_{\mathsf{rel}}(x+y)\rangle \xmapsto{r} \langle z'\rangle \\ (z, z') \in R \cup \mathcal{I}\end{array}$$

Suppose,

$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y)\rangle \xmapsto{r} \langle z\rangle \tag{25}$$

This can be derived from Rules AC-19, AC-20, AC-21 and AC-26.

(a) Rule AC-19

Then $z = z_1 + z_2$.
Rewriting Transition 25:

$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y)\rangle \xmapsto{r} \langle z_1 + z_2\rangle \tag{26}$$

From premise of the rule, the following holds:

$$\langle \sigma^t_{\mathsf{rel}}(x)\rangle \xmapsto{r} \langle z_1\rangle \tag{27}$$

$$\langle \sigma^t_{\mathsf{rel}}(y)\rangle \xmapsto{r} \langle z_2\rangle \tag{28}$$

We distinguish between three cases for different values of r:

i. **Case $r < t$**

Let for some $0 < r_1 < t$,

$$t = r + r_1 \tag{29}$$

Rewriting Transitions 26, 27 and 28:

$$\langle \sigma^{r+r_1}_{\mathsf{rel}}(x) + \sigma^{r+r_1}_{\mathsf{rel}}(y)\rangle \xmapsto{r} \langle z_1 + z_2\rangle \tag{30}$$

$$\langle \sigma^{r+r_1}_{\mathsf{rel}}(x)\rangle \xmapsto{r} \langle z_1\rangle \tag{31}$$

$$\langle \sigma^{r+r_1}_{\mathsf{rel}}(y)\rangle \xmapsto{r} \langle z_2\rangle \tag{32}$$

Then Transitions 31 and 32 can be derived from Rules AC-8 and AC-26.
That gives us four cases:

A. Transitions 31 and 32 are derived from Rule AC-8.
B. Transition 31 is derived from Rule AC-8 and Transition 32 is derived Rule AC-26.

C. Transition 31 is derived from Rule AC-26 and Transition 32 is derived Rule AC-8.

D. Transitions 31 and 32 are derived from Rule AC-26.

We prove that in all four cases, the target process terms z_1 and z_2 are as follows:

$$z_1 = \sigma_{\mathsf{rel}}^{r_1}(x) \text{ and } z_2 = \sigma_{\mathsf{rel}}^{r_1}(y)$$

In case Rule AC-8 is used to derive Transition 31 (or Transition 32), it is easy to see that $z_1 = \sigma_{\mathsf{rel}}^{r_1}(x)$ ($z_2 = \sigma_{\mathsf{rel}}^{r_1}(y)$).
Below we argue the cases when Rule AC-26 is used to derive one or both of the Transitions 31 and 32.

A. **Transition 31 by Rule AC-26**
Suppose this rule is used to derive Transition 31. By Rule AC-26, we can combine successive time transitions into a single time transition. For a derivable time transition, the process of applying Rule AC-26 must be finite. Hence, we can say that there exists an $n > 1$ such that Transition 31 is obtained by combining n successive transitions. Each of the n transitions has been derived by rules other than Rule AC-26. Note that n is taken to be greater than 1 because one application of Rule AC-26 joins two successive transitions.
Let $s_1, \ldots, s_n$ denote the durations of the constituent time transitions and let $p_1, \ldots, p_{n-1}$ denote the intermediate process terms.
Splitting Transition 31 into n transitions:

$$\langle \sigma_{\mathsf{rel}}^{r+r_1}(x) \rangle \xmapsto{s_1} \langle p_1 \rangle \xmapsto{s_2} \ldots \langle p_{n-1} \rangle \xmapsto{s_n} \langle z_1 \rangle \tag{33}$$

and

$$s_1 + \ldots + s_n = r \tag{34}$$

From 29 and 34 we infer that:

$$t = s_1 + \ldots + s_n + r_1$$

Rewriting Transition 33 by replacing t by the sum of durations:

$$\langle \sigma_{\mathsf{rel}}^{s_1+\ldots+s_n+r_1}(x) \rangle \xmapsto{s_1} \langle p_1 \rangle \xmapsto{s_2} \ldots \langle p_{n-1} \rangle \xmapsto{s_n} \langle z_1 \rangle$$

Consider the first transition

$$\langle \sigma_{\mathsf{rel}}^{s_1+\ldots+s_n+r_1}(x) \rangle \xmapsto{s_1} \langle p_1 \rangle \tag{35}$$

For a process term $\sigma_{\mathsf{rel}}^{u}(x)$, with $u > 0$, only two rules (other than Rule AC- 26) are applicable. A time step of duration

$v < u$ can only be derived by Rule AC-8 and a time step of duration u can only be derived from Rule AC-9. Transition 35 can only be derived from Rule AC-8 as $s_1 < (s_1 + \ldots + s_n + r_1)$. From the rule we infer that:

$$p_1 = \sigma_{\mathsf{rel}}^{s_2+\ldots+s_n+r_1}(x)$$

Rewriting Transition 33 by replacing p_1 by $\sigma_{\mathsf{rel}}^{s_2+\ldots+s_n+r_1}(x)$:

$$\langle\sigma_{\mathsf{rel}}^{s_1+\ldots+s_n+r_1}(x)\rangle \xmapsto{s_1} \langle\sigma_{\mathsf{rel}}^{s_2+\ldots+s_n+r_1}(x)\rangle \xmapsto{s_2} \ldots \langle p_{n-1}\rangle \xmapsto{s_n} \langle z_1\rangle$$

Again the second transition

$$\langle\sigma_{\mathsf{rel}}^{s_2+\ldots+s_n+r_1}(x)\rangle \xmapsto{s_2} \langle p_2\rangle$$

can only be derived from Rule AC-8 as $s_2 < (s_2+\ldots+s_n+r_1)$. From the rule we infer that:

$$p_2 = \sigma_{\mathsf{rel}}^{s_3+\ldots+s_n+r_1}(x)$$

Continuing the same reasoning, we infer that all the n time steps of Transition 33 have been derived from Rule AC-8 and the target of the $(n-1)$ time step in Transition 33 is as follows:

$$p_{n-1} = \sigma_{\mathsf{rel}}^{s_n+r_1}(x)$$

Rewriting the n^{th} transition of Transition 33:

$$\langle\sigma_{\mathsf{rel}}^{s_n+r_1}(x)\rangle \xmapsto{s_n} \langle z_1\rangle$$

The above transition is derived from Rule AC-8. Then z_1 must be of the following form:

$$z_1 = \sigma_{\mathsf{rel}}^{r_1}(x) \tag{36}$$

B. **Transition 32 by Rule AC-26**
By reasoning given for Transition 31, we can say that there exists an $m > 1$ such that Transition 32 is obtained by combining m successive transitions. Each of the m transitions has been derived by rules other than Rule AC-26. Let $u_1, \ldots, u_m$ denote the durations of the constituent time transitions and let $q_1, \ldots, q_{m-1}$ denote the intermediate process terms.
Splitting Transition 32 into m transitions:

$$\langle\sigma_{\mathsf{rel}}^{r+r_1}(y)\rangle \xmapsto{u_1} \langle q_1\rangle \xmapsto{u_2} \ldots \langle q_{m-1}\rangle \xmapsto{u_m} \langle z_2\rangle \tag{37}$$

and

$$u_1 + \ldots u_m = r \tag{38}$$

From 29 and 38, we infer that:

$$t = u_1 + \ldots + u_m + r_1$$

Rewriting Transition 37 by replacing t by the sum of durations:

$$\langle \sigma_{\mathsf{rel}}^{u_1+\ldots+u_m+r_1}(y) \rangle \xmapsto{u_1} \langle q_1 \rangle \xmapsto{u_2} \ldots \langle q_{m-1} \rangle \xmapsto{u_m} \langle z_2 \rangle$$

By the same reasoning as applied for Transition 31, we can infer the following:

- All the constituent transitions of Transition 37 have been derived by Rule AC-8.
- The intermediate process terms $q_1, \ldots, q_{m-1}$, and the final process term are as follows:

$$q_1 = \sigma_{\mathsf{rel}}^{u_2+\ldots+u_m+r_1}(y) \tag{39}$$
$$q_2 = \sigma_{\mathsf{rel}}^{u_3+\ldots+u_m+r_1}(y) \tag{40}$$
$$\vdots$$
$$q_{m-1} = \sigma_{\mathsf{rel}}^{u_m+r_1}(y) \tag{41}$$
$$z_2 = \sigma_{\mathsf{rel}}^{r_1}(y) \tag{42}$$

Putting the values of z_1 and z_2 from equations 36 and 42 in Transition 30, we get:

$$\langle \sigma_{\mathsf{rel}}^{r+r_1}(x) + \sigma_{\mathsf{rel}}^{r+r_1}(y) \rangle \xmapsto{r} \langle \sigma_{\mathsf{rel}}^{r_1}(x) + \sigma_{\mathsf{rel}}^{r_1}(y) \rangle \tag{43}$$

Again, Rule 8 can derive the following:

$$\langle \sigma_{\mathsf{rel}}^{r+r_1}(x+y) \rangle \xmapsto{r} \langle \sigma_{\mathsf{rel}}^{r_1}(x+y) \rangle \tag{44}$$

Consider Transitions 43 and 44. For $0 < r_1 < t$, the pair $(\sigma_{\mathsf{rel}}^{r_1}(x) + \sigma_{\mathsf{rel}}^{r_1}(y), \sigma_{\mathsf{rel}}^{r_1}(x+y)) \in R$.

ii. **Case $r = t$**

Then Transitions 27 and 28 can only be derived from Rules AC-9 and AC-26.
Rewriting Transitions 26, 27 and 28:

$$\langle \sigma_{\mathsf{rel}}^{r}(x) + \sigma_{\mathsf{rel}}^{r}(y) \rangle \xmapsto{r} \langle z_1 + z_2 \rangle \tag{45}$$
$$\langle \sigma_{\mathsf{rel}}^{r}(x) \rangle \xmapsto{r} \langle z_1 \rangle \tag{46}$$
$$\langle \sigma_{\mathsf{rel}}^{r}(y) \rangle \xmapsto{r} \langle z_2 \rangle \tag{47}$$

Then Transitions 46 and 47 can be derived from Rules AC-9 and AC-26.
That again gives us four cases:

A. Transitions 46 and 47 are derived from Rule AC-9.
B. Transition 46 is derived from Rule AC-9 and Transition 47 is derived Rule AC-26.
C. Transition 46 is derived from Rule AC-26 and Transition 47 is derived Rule AC-9.
D. Transitions 46 and 47 are derived from Rule AC-26.

We prove that in all four cases, the target process terms z_1 and z_2 are as follows:

$$z_1 = x \text{ and } z_2 = y$$

And the following holds:

$$\langle \texttt{consistent } z_1 \rangle \text{ and } \langle \texttt{consistent } y \rangle$$

In case Rule AC-9 is used to derive Transition 46 (or Transition 47), it is easy to see that $z_1 = x$ ($z_2 = y$). From the premise of the rule $\langle \texttt{consistent } x \rangle$ ($\langle \texttt{consistent } y \rangle$) holds.
Below we argue the cases when Rule AC-26 is used to derive one or both of the Transitions 46 and 47.

A. **Transition 46 by Rule AC-26**
By reasoning given for Transition 31, we can say that there exists an $n > 1$ such that Transition 46 is obtained by combining n successive transitions. Each of the n transitions has been derived by rules other than Rule AC-26. Let $s_1, \ldots, s_n$ denote the durations of the constituent time transitions and let $p_1, \ldots, p_{n-1}$ denote the intermediate process terms.
Splitting Transition 46 into n transitions:

$$\langle \sigma^r_{\mathsf{rel}}(x) \rangle \xmapsto{s_1} \langle p_1 \rangle \xmapsto{s_2} \ldots \langle p_{n-1} \rangle \xmapsto{s_n} \langle z_2 \rangle \tag{48}$$

and

$$s_1 + \ldots s_n = r \tag{49}$$

From 49 and the fact that we are considering the case for $r = t$, we infer that:

$$t = s_1 + \ldots + s_n$$

Rewriting Transition 48 by replacing t by the sum of durations:

$$\langle \sigma^{s_1 + \ldots s_n}_{\mathsf{rel}}(x) \rangle \xmapsto{s_1} \langle p_1 \rangle \xmapsto{s_2} \ldots \langle p_{n-1} \rangle \xmapsto{s_n} \langle z_2 \rangle \tag{50}$$

Now , for a process term $\sigma^u_{\mathsf{rel}}(x)$, with $u > 0$, only two rules (other than Rule AC- 26) are applicable. A time step of duration $v < u$ can only be derived by Rule AC-8 and a time step of duration u can only be derived from Rule AC-9. Consider the first time step of Transition 50:

$$\langle \sigma_{\mathsf{rel}}^{s_1+\ldots+s_n}(x)\rangle \xmapsto{s_1} \langle p_1\rangle \tag{51}$$

We know that $n > 1$, as Rule AC-26 applied once joins two transitions. For $n > 1$, $s_1 < (s_1 + \ldots + s_n)$, therefore Transition 51 can only be derived from Rule AC-8. From the rule we infer that:

$$p_1 = \sigma_{\mathsf{rel}}^{s_2+\ldots+s_n}(x)$$

Rewriting Transition 50 by replacing p_1 by $\sigma_{\mathsf{rel}}^{s_2+\ldots+s_n}(x)$:

$$\langle \sigma_{\mathsf{rel}}^{s_1+\ldots+s_n}(x)\rangle \xmapsto{s_1} \langle \sigma_{\mathsf{rel}}^{s_2+\ldots+s_n}(x)\rangle \xmapsto{s_2} \ldots \langle p_{n-1}\rangle \xmapsto{s_n} \langle z_1\rangle$$

The n^{th} transition will be the final one with its target equal to z_1.

$$\langle p_{n-1}\rangle \xmapsto{s_n} \langle z_1\rangle \tag{52}$$

Extending the reasoning given above for process term p_1 to other intermediate process terms, we infer the following:

$$\begin{array}{lcll} p_2 & = & \sigma_{\mathsf{rel}}^{s_3+\ldots+s_n}(x) & \text{if } n > 2 \\ p_3 & = & \sigma_{\mathsf{rel}}^{s_4+\ldots+s_n}(x) & \text{if } n > 3 \\ \vdots & & & \\ p_{n-1} & = & \sigma_{\mathsf{rel}}^{s_n}(x) & \end{array}$$

Putting the value of p_{n-1} in Transition 52, we get:

$$\langle \sigma_{\mathsf{rel}}^{s_n}(x)\rangle \xmapsto{s_n} \langle z_1\rangle \tag{53}$$

As the delay duration is equal to the duration of the relative delay operator, therefore the above transition can only be derived from Rule AC-9. Then z_1 is equal to x,

$$z_1 = x \tag{54}$$

And from the premise of Rule AC-9

$$\langle \texttt{consistent}\ x\rangle \tag{55}$$

B. **Transition 47 by Rule AC-26**
By similar reasoning as given above for Transition 46, we infer that the following holds:

$$z_2 = y \tag{56}$$

$$\langle \texttt{consistent}\ x\rangle \tag{57}$$

Putting the values of z_1 and z_2 from equations 54 and 56 in Transition 45, we get:

$$\langle \sigma^r_{\mathsf{rel}}(x) + \sigma^r_{\mathsf{rel}}(y) \rangle \xmapsto{r} \langle x + y \rangle \tag{58}$$

Again, using Predicates 55 and 57, Rule AC-9 can derive the following:

$$\langle \sigma^r_{\mathsf{rel}}(x + y) \rangle \xmapsto{r} \langle x + y \rangle \tag{59}$$

Consider Transitions 58 and 59. The pair $(x + y, x + y) \in \mathcal{I}$.

iii. **Case $r > t$**

If $r > t$, then Transitions 27 and 28 can only be derived from Rule AC-26.

A. **Transition 27 by Rule AC-26**
By reasoning given above for derivation of Transitions 31, 32, 46 and 47 using Rule AC-26, we say that there exists $n > 1$, such that Transition 27 is obtained from Rule 26 by combining n successive transitions. Each of the n transitions has been derived by rules other than Rule AC-26. Let $s_1, \ldots, s_n$ denote the durations of the constituent time transitions and let $p_1, \ldots, p_{n-1}$ denote the intermediate process terms.
Splitting Transition 27 into n transitions:

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xmapsto{s_1} \langle p_1 \rangle \xmapsto{s_2} \ldots \langle p_{n-1} \rangle \xmapsto{s_n} \langle z_1 \rangle \tag{60}$$

and

$$s_1 + \ldots + s_n = r \tag{61}$$

From 61 and the fact that we are considering the case with $r > t$, we infer that:

$$s_1 + \ldots + s_n > t \tag{62}$$

In each of the constituent transitions of Transition 60, a single rule other than Rule AC-26 has been applied.
There are only two rules applicable on $\sigma^t_{\mathsf{rel}}(x)$, Rule AC-8 and Rule AC-9.
Consider the first time step of Transition 60.

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xmapsto{s_1} \langle p_1 \rangle \tag{63}$$

Two cases arise. In Transition 63, $s_1 < t$ or $s_1 = t$. The case $s_1 > t$ does not arise because no rule (other than Rule AC-26) can derive that.

- Case $s_1 = t$:
  Then Rule AC-9 has been applied to derive Transition 63. Then $p_1 = x$. Rewriting Transition 63, we get:

  $$\langle \sigma_{\mathsf{rel}}^{s_1}(x)\rangle \xmapsto{s_1} \langle x\rangle \tag{64}$$

  Note $s_1 = t$.
  Putting Transition 64 in Transition 60, we get:

  $$\langle \sigma_{\mathsf{rel}}^{s_1}(x)\rangle \xmapsto{s_1} \langle x\rangle \xmapsto{s_2} \ldots \langle p_{n-1}\rangle \xmapsto{s_n} \langle z_1\rangle \tag{65}$$

- Case $s_1 < t$:
  Then Rule AC-8 has been applied to derive Transition 63. Let

  $$t = s_1 + u \tag{66}$$

  . Rewriting Transition 63, we get:

  $$\langle \sigma_{\mathsf{rel}}^{s_1+u}(x)\rangle \xmapsto{s_1} \langle \sigma_{\mathsf{rel}}^{u}(x)\rangle \tag{67}$$

  Putting Transition 67 in Transition 60, we get:

  $$\langle \sigma_{\mathsf{rel}}^{t}(x)\rangle \xmapsto{s_1} \langle \sigma_{\mathsf{rel}}^{u}(x)\rangle \xmapsto{s_2} \langle p_2\rangle \ldots \langle p_{n-1}\rangle \xmapsto{s_n} \langle z_1\rangle \tag{68}$$

  Again there are only two rules applicable on $\sigma_{\mathsf{rel}}^{u}(x)$, Rule AC-8 and Rule AC-9 and only two cases are possible:

  $$s_2 < u \text{ or } s_2 = u$$

  – If $s_2 = u$, then $p_2 = x$. Then from 66,

  $$t = s_1 + s_2 \tag{69}$$

  Rewriting Transition 68 by putting the value of p_2, we get:

  $$\langle \sigma_{\mathsf{rel}}^{t}(x)\rangle \xmapsto{s_1} \langle \sigma_{\mathsf{rel}}^{u}(x)\rangle \xmapsto{s_2} \langle x\rangle \ldots \langle p_{n-1}\rangle \xmapsto{s_n} \langle z_1\rangle \tag{70}$$

  where $t = s_1 + s_2$.

  – If $s_2 < u$, then the second time step in 68 is derived by Rule AC-8. Let $u = s_2 + v$. Rewriting second time step in Transition 68, we get:

  $$\langle \sigma_{\mathsf{rel}}^{s_2+v}(x)\rangle \xmapsto{s_2} \langle \sigma_{\mathsf{rel}}^{v}(x)\rangle \tag{71}$$

  Rewriting Transition 68 by putting the value of p_2, we get:

  $$\langle \sigma_{\mathsf{rel}}^{t}(x)\rangle \xmapsto{s_1} \langle \sigma_{\mathsf{rel}}^{u}(x)\rangle \xmapsto{s_2} \langle \sigma_{\mathsf{rel}}^{v}(x)\rangle \ldots \langle p_{n-1}\rangle \xmapsto{s_n} \langle z_1\rangle \tag{72}$$

  where $t = s_1 + u = s_1 + s_2 + v$.

Consider the instantiations 65, 70 and 72 of Transition 60. We notice a pattern. Till the duration t is covered by the constituent time steps, no rules other than Rule AC-8 and Rule AC-9 are applicable. The time step in which the duration t is covered, is derived from Rule AC-9.
From this observation we infer that there exists a j, such that the sum of delays of first j transitions of Transition 60 equals t. I.e.,

$$s_1 + \ldots + s_j = t \tag{73}$$

From 62, we know that the duration of Transition 60 is greater than t. The extra duration $(s_1 + \ldots + s_n) - t$ in Transition 60 must be due to the delay of x as operator σ^t_{rel} in front of x caters for a delay of first j time steps. Then j must be smaller than n, as at least one transition is required to cover the delay of x.
Now $(s_1 + \ldots + s_n) = r$ in Transition 60. From 73:

$$r - t = (s_1 + \ldots + s_n) - (s_1 + \ldots + s_j) = s_{j+1} + \ldots + s_n \tag{74}$$

We partition Transition 60 into time transitions of durations '$s_1 + \ldots + s_j$' and '$s_{j+1} + \ldots + s_n$', we get:

$$\langle \sigma^{s_1+\ldots+s_j}_{\mathsf{rel}}(x) \rangle \xmapsto{s_1+\ldots+s_j} \langle x \rangle \tag{75}$$

$$\langle x \rangle \xmapsto{s_{j+1}+\ldots+s_n} \langle z_1 \rangle \tag{76}$$

The j^{th} transition in Transition 61 is obtained by Rule AC-9. From the premise of the rule, the following holds:

$$\langle \texttt{consistent}\ x \rangle \tag{77}$$

B. **Transition 28 by Rule AC-26**
We can apply the same reasoning for Transition 28 as given for Transition 27. There exists $m > 1$, such that Transition 28 is obtained from Rule 26 by combining m successive transitions.
Let $u_1, \ldots, u_m$ denote the durations of the constituent time transitions and let $q_1, \ldots, q_{m-1}$ denote the intermediate process terms.
Splitting Transition 28 into m transitions:

$$\langle \sigma^t_{\mathsf{rel}}(y) \rangle \xmapsto{u_1} \langle q_1 \rangle \xmapsto{u_2} \ldots \langle q_{m-1} \rangle \xmapsto{u_m} \langle z_2 \rangle \tag{78}$$

and

$$u_1 + \ldots + u_m = r \tag{79}$$

From 79 and the fact that we are considering the case with $r > t$, we infer that:

$$u_1 + \ldots + u_m > t$$

Now, there are only two rules applicable on $\sigma^t_{\mathsf{rel}}(y)$, Rule AC-8 and Rule AC-9. Together the application of rules Rule AC-8 and Rule AC-9 can cover a duration t for a process term $\sigma^t_{\mathsf{rel}}(y)$. The extra duration $(u_1 + \ldots + u_m) - t$ in Transition 78 is covered by a delay of y.
By reasoning given for Transition 60, there exists a k, with $1 \leq k < m$ such that:

$$u_1 + \ldots + u_k = t \tag{80}$$

Then from 79 and the fact that $k < m$, we infer:

$$u_{k+1} + \ldots + u_m = r - t \tag{81}$$

We partition Transition 78 into time transitions of durations '$u_1 + \ldots + u_k$' and '$u_{k+1} + \ldots + u_m$', we get:

$$\langle \sigma^{u_1+\ldots+u_m}_{\mathsf{rel}}(y) \rangle \xmapsto{u_1+\ldots+u_k} \langle y \rangle \tag{82}$$

$$\langle y \rangle \xmapsto{u_{k+1}+\ldots+u_m} \langle z_2 \rangle \tag{83}$$

The k^{th} transition in Transition 82 is obtained by Rule AC-9. From the premise of the rule, the following holds:

$$\langle \texttt{consistent}\ \ y \rangle \tag{84}$$

From Predicates 77 and 84, we infer that:

$$\langle \texttt{consistent}\ \ x + y \rangle$$

Then, Rule AC-9 can derive the following transition for the duration $s_1 + \ldots + s_j$ defined in 73.

$$\langle \sigma^t_{\mathsf{rel}}(x + y) \rangle \xmapsto{s_1+\ldots+s_j} \langle x + y \rangle \tag{85}$$

From 74 and 81, we infer that:

$$s_{j+1} + \ldots + s_n = u_{k+1} + \ldots + u_m$$

Apply Rule 19 on Transitions 76 and 83:

$$\langle x + y \rangle \xmapsto{s_{j+1}+\ldots+s_n} \langle z_1 + z_2 \rangle \tag{86}$$

Apply Rule AC-26 on Transitions 85 and 86:

$$\langle \sigma^t_{\mathsf{rel}}(x+y)\rangle \xmapsto{s_1+\ldots+s_n} \langle z_1+z_2\rangle$$

From 61,

$$r = s_1 + \ldots + s_n$$

I.e.,

$$\langle \sigma^t_{\mathsf{rel}}(x+y)\rangle \xmapsto{r} \langle z_1+z_2\rangle \tag{87}$$

Consider Transitions 26 and 87. The pair $(z_1+z_2, z_1+z_2) \in \mathcal{I}$.

(b) Rule AC-20

$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y)\rangle \xmapsto{r} \langle z\rangle \tag{25}$$

If Transition 25, given above, is derived from this rule, then the following must hold:

$$\langle \sigma^t_{\mathsf{rel}}(x)\rangle \xmapsto{r} \langle z\rangle \tag{88}$$

$$\langle \texttt{consistent}\ \sigma^t_{\mathsf{rel}}(y)\rangle \tag{89}$$

$$\langle \sigma^t_{\mathsf{rel}}(y)\rangle \not\xmapsto{r} \tag{90}$$

$$\forall s < r, \quad \langle \sigma^t_{\mathsf{rel}}(y)\rangle \not\xmapsto{s} \tag{91}$$

Consider Predicate 91, for $s < t$. Let $t = s + s_1$, from some $s_1 > 0$. Then, the following transition is always derivable from Rule AC-8:

$$\langle \sigma^{s+s_1}_{\mathsf{rel}}(y)\rangle \xmapsto{s} \langle \sigma^{s_1}_{\mathsf{rel}}(y)\rangle$$

Hence, Predicate Predicate 91 doesn't hold.
We conclude that Rule AC-20 cannot be used to derive Transition 25.

(c) Rule AC-21

Rule AC-21 is not applicable due to the same reasons as Rule AC-20.

(d) Rule AC-26

Suppose, Transition 25 (repeated below) is derived from this rule:

$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y)\rangle \xmapsto{r} \langle z\rangle \tag{25}$$

By reasoning give above, we can say that there exists an $n > 1$ such that Transition 25 is obtained by combining n successive transitions. Let $s_1, \ldots, s_n$ denote the durations of the constituent time transitions and let $p_1, \ldots, p_{n-1}$ denote the intermediate process terms.

Splitting Transition 25 into n transitions:

$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y) \rangle \xmapsto{s_1} \langle p_1 \rangle \xmapsto{s_2} \ldots \langle p_{n-1} \rangle \xmapsto{s_n} \langle z \rangle \tag{92}$$

and

$$s_1 + \ldots + s_n = r \tag{93}$$

We distinguish between three cases:

i. <u>**Case** $s_1 + \ldots + s_n < t$</u>

Let

$$t = s_1 + \ldots + s_n + r_1 \tag{94}$$

for some r_1, with $0 < r_1 < t$.
Rewriting Transition 92:

$$\langle \sigma^{s_1+\ldots+s_n+r_1}_{\mathsf{rel}}(x) + \sigma^{s_1+\ldots+s_n+r_1}_{\mathsf{rel}}(y) \rangle \xmapsto{s_1} \langle p_1 \rangle \ldots \langle p_{n-1} \rangle \xmapsto{s_n} \langle z \rangle \tag{95}$$

A time step for an alternative composition can be derived from Rules AC-19, AC-20 and AC-21.
Consider the first constituent time step of Transition 95:

$$\langle \sigma^{s_1+\ldots+s_n+r_1}_{\mathsf{rel}}(x) + \sigma^{s_1+\ldots+s_n+r_1}_{\mathsf{rel}}(y) \rangle \xmapsto{s_1} \langle p_1 \rangle \tag{96}$$

Transition 96 can only be derived from Rule AC-19 as Rules AC-20 and AC-21 are not applicable. The application of Rule AC-20 (Rule AC-21) requires that the right (left) alternative is undelayable.
From the premise of Rule AC-19, for some $p', p'' \in P$:

$$p_1 = p' + p''$$

and the following holds:

$$\langle \sigma^{s_1+\ldots+s_n+r_1}_{\mathsf{rel}}(x) \rangle \xmapsto{s_1} \langle p' \rangle \tag{97}$$

$$\langle \sigma^{s_1+\ldots+s_n+r_1}_{\mathsf{rel}}(y) \rangle \xmapsto{s_1} \langle p'' \rangle \tag{98}$$

Transitions 97 and 98 can only be derived from Rules AC-8.
Then,

$$p' = \sigma^{s_2+\ldots+s_n+r_1}_{\mathsf{rel}}(x) \text{ and } p'' = \sigma^{s_2+\ldots+s_n+r_1}_{\mathsf{rel}}(y)$$

By similar reasoning, all n time steps of composite Time Transition 95 have been derived from Rule AC-19 as Rules AC-20 and AC-21 are not applicable. The final process term z is as follows:

$$\sigma^{r_1}_{\mathsf{rel}}(x) + \sigma^{r_1}_{\mathsf{rel}}(y) \tag{99}$$

Rewriting Transition 25:

$$\langle \sigma^{t}_{\mathsf{rel}}(x) + \sigma^{t}_{\mathsf{rel}}(y)\rangle \xmapsto{r} \langle \sigma^{r_1}_{\mathsf{rel}}(x) + \sigma^{r_1}_{\mathsf{rel}}(y)\rangle \tag{100}$$

From 93 and 94:

$$t = r + r_1$$

By Rule AC-8, the following is derivable:

$$\langle \sigma^{r+r_1}_{\mathsf{rel}}(x + y)\rangle \xmapsto{r} \langle \sigma^{r_1}_{\mathsf{rel}}(x + y)\rangle \tag{101}$$

Consider the target terms in Transitions 100 and 101. For $0 < r_1 < t$, the pair $(\sigma^{r_1}_{\mathsf{rel}}(x) + \sigma^{r_1}_{\mathsf{rel}}(y), \sigma^{r_1}_{\mathsf{rel}}(x + y))$ is in R.

ii. **Case** $s_1 + \ldots + s_n = t$

Replacing the value t in Transition 92 by the sum $s_1 + \ldots + s_n$:

$$\langle \sigma^{s_1+\ldots+s_n}_{\mathsf{rel}}(x) + \sigma^{s_1+\ldots+s_n}_{\mathsf{rel}}(y)\rangle \xmapsto{s_1} \langle p_1\rangle \xmapsto{s_2} \ldots \langle p_{n-1}\rangle \xmapsto{s_n} \langle z\rangle \tag{102}$$

A time step for an alternative composition can be derived from Rules AC-19, AC-20 and AC-21.
Consider the first constituent time step of Transition 102:

$$\langle \sigma^{s_1+\ldots+s_n}_{\mathsf{rel}}(x) + \sigma^{s_1+\ldots+s_n}_{\mathsf{rel}}(y)\rangle \xmapsto{s_1} \langle p_1\rangle \tag{103}$$

Transition 103 can only be derived from Rule AC-19 as Rules AC-20 and AC-21 are not applicable. The application of Rule AC-20 (Rule AC-21) requires that the right (left) alternative is undelayable.
From the premise of Rule AC-19, for some $p', p'' \in P$:

$$p_1 = p' + p''$$

and the following holds:

$$\langle \sigma^{s_1+\ldots+s_n}_{\mathsf{rel}}(x)\rangle \xmapsto{s_1} \langle p'\rangle \tag{104}$$

$$\langle \sigma^{s_1+\ldots+s_n}_{\mathsf{rel}}(y)\rangle \xmapsto{s_1} \langle p''\rangle \tag{105}$$

Note that $n > 1$ and for all i, $s_i > 0$. Therefore, Transitions 104 and 105 can only be derived from Rules AC-8.
Then,

$$p' = \sigma^{s_2+\ldots+s_n}_{\mathsf{rel}}(x) \text{ and } p'' = \sigma^{s_2+\ldots+s_n}_{\mathsf{rel}}(y)$$

Rewriting Transition 103 by putting the value of p_1:

$$\langle \sigma^{s_1+\ldots+s_n}_{\mathsf{rel}}(x) + \sigma^{s_1+\ldots+s_n}_{\mathsf{rel}}(y)\rangle \xmapsto{s_1} \langle \sigma^{s_2+\ldots+s_n}_{\mathsf{rel}}(x) + \sigma^{s_2+\ldots+s_n}_{\mathsf{rel}}(y)\rangle \tag{106}$$

Now consider the second constituent time step of Transition 102:

$$\langle \sigma_{\mathsf{rel}}^{s_2+\ldots+s_n}(x) + \sigma_{\mathsf{rel}}^{s_2+\ldots+s_n}(y)\rangle \xrightarrow{s_2} \langle p_2\rangle \tag{107}$$

Transition 107 is again only be derived from Rule AC-19 as Rules AC-20 and AC-21 are not applicable on the source of the above transition.
Till the duration t on process term $\sigma_{\mathsf{rel}}^{t}(x) + \sigma_{\mathsf{rel}}^{t}(y)$ is covered, all the constituent transitions of Transition 102 are obtained by Rule AC-19. Other rules for deriving delay of an alternative composition, Rules AC-20 and AC-21, require that the passive operand must be undelayable, which is not satisfied till *atleast* the relative delay operator disappears from a process term $\sigma_{\mathsf{rel}}^{t}(x)$. We are considering the case where $t = s_1 + \ldots s_n$, which is the total duration of Transition 102. Therefore, the duration t is only covered in the last time transition.
The last time step is as follows:

$$\langle \sigma_{\mathsf{rel}}^{s_n}(x) + \sigma_{\mathsf{rel}}^{s_n}(y)\rangle \xrightarrow{s_n} \langle z\rangle \tag{108}$$

From Rule AC-19, for some $z_1, z_2 \in P$, $z = z_1 + z_2$.
From premise of Rule AC-19 the following holds:

$$\langle \sigma_{\mathsf{rel}}^{s_n}(x)\rangle \xrightarrow{s_n} \langle z_1\rangle \tag{109}$$

$$\langle \sigma_{\mathsf{rel}}^{s_n}(y)\rangle \xrightarrow{s_n} \langle z_2\rangle \tag{110}$$

Rewriting Transition 25 by replacing z by $z_1 + z_2$:

$$\langle \sigma_{\mathsf{rel}}^{t}(x) + \sigma_{\mathsf{rel}}^{t}(y)\rangle \xrightarrow{t} \langle z\rangle \tag{111}$$

Transitions 109 and 110 can only be derived from Rule AC-9.
Then, $z_1 = x$ and $z_2 = y$.
Rule AC-9 requires:

$$\langle \texttt{consistent}\ \ x\rangle \text{ and } \langle \texttt{consistent}\ \ y\rangle$$

which implies:

$$\langle \texttt{consistent}\ \ x + y\rangle \tag{112}$$

Rewriting Transition 111 by putting in the values of z_1 and z_2:

$$\langle \sigma_{\mathsf{rel}}^{t}(x) + \sigma_{\mathsf{rel}}^{t}(y)\rangle \xrightarrow{t} \langle x + y\rangle \tag{113}$$

From Predicate 112, Rule AC-9 becomes applicable to derive the following time step.

$$\langle \sigma_{\mathsf{rel}}^{s_1+\ldots+s_n}(x + y)\rangle \xrightarrow{s_1+\ldots+s_n} \langle x + y\rangle \tag{114}$$

The pair $(x + y, x + y)$ is in $\mathcal{I}$.

iii. **Case $s_1 + \ldots + s_n > t$**

Repeating Transition 25 with r replaced by the sum of durations $s_1 + \ldots + s_n$ from 93.

$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y) \rangle \xmapsto{s_1+\ldots+s_n} \langle z \rangle \tag{115}$$

As explained in the last case for $s_1 + \ldots + s_n = t$, the first k time steps of the above composite Transition, such that $s_1 + \ldots + s_k \leq t$, can only be derived from Rule AC-19.
We distinguish between two cases: In one case, there exists a j, with $1 \leq j < n$ such that $s_1 + \ldots + s_j = t$. In the second case, for all i with $1 \leq i \leq n$, $s_1 + \ldots + s_i$ is either strictly less than t or $s_1 + \ldots + s_i$ is strictly greater than t.
The second case arises due to the following reason:
All transitions in 115 have been derived from rules other than Rule AC-26. The rules allowing a delay of alternative composition are Rules AC-19, AC-20 and AC-21. The premise of these rules contain time transitions which may have been derived by Rule AC-26. The following is an example exhibiting the second case:
Consider the process term $\sigma^1_{\mathsf{rel}}(\sigma^1_{\mathsf{rel}}(\tilde{\tilde{a}})) + \sigma^2_{\mathsf{rel}}(b)$. Both process terms can delay for 2 time units. Hence Rule AC-19 can be applied to derive the following transition:

$$\langle \sigma^1_{\mathsf{rel}}(\sigma^1_{\mathsf{rel}}(\tilde{\tilde{a}})) + \sigma^2_{\mathsf{rel}}(\tilde{\tilde{b}}) \rangle \xmapsto{2} \langle \tilde{\tilde{a}} + \tilde{\tilde{b}} \rangle$$

But one of the prerequisites of the rule, (Transition 116) is derived from Rule AC-26.

$$\langle \sigma^1_{\mathsf{rel}}(\sigma^1_{\mathsf{rel}}(\tilde{\tilde{a}})) \rangle \xmapsto{1+1} \langle \tilde{\tilde{a}} \rangle \tag{116}$$

$$\langle \sigma^2_{\mathsf{rel}}(\tilde{\tilde{b}}) \rangle \xmapsto{2} \langle \tilde{\tilde{b}} \rangle \tag{117}$$

In the derivation of time transitions of duration greater than t with a source process term of the form $\sigma^t_{\mathsf{rel}}(z)$ (For example Transition 60), there always exists a j such that $s_1 + \ldots s_j = t$. Because the rules for the process term $\sigma^t_{\mathsf{rel}}(z)$ do not contain any delay transitions in their premises.

A. **Case 1**

Suppose, there exists a j with $1 \leq j < n$ such that

$$s_1 + \ldots + s_j = t \tag{118}$$

Rewriting Transition 115:

$$\langle \sigma^{s_1+\ldots+s_j}_{\mathsf{rel}}(x) + \sigma^{s_1+\ldots+s_j}_{\mathsf{rel}}(y) \rangle \xmapsto{s_1+\ldots+s_n} \langle z \rangle \tag{119}$$

Then the j^{th} time step is as follows:

$$\langle \sigma_{\mathsf{rel}}^{s_j}(x) + \sigma_{\mathsf{rel}}^{s_j}(y) \rangle \xmapsto{s_j} \langle p \rangle \tag{120}$$

for some process term p.
Only Rule AC-19 can derive Transition 120. From the premise of the rule, $p = p' + p''$ and the following holds:

$$\langle \sigma_{\mathsf{rel}}^{s_j}(x) \rangle \xmapsto{s_j} \langle p' \rangle \tag{121}$$

$$\langle \sigma_{\mathsf{rel}}^{s_j}(y) \rangle \xmapsto{s_j} \langle p'' \rangle \tag{122}$$

The above transitions can only be derived from Rule AC-9. Then $p = x + y$ and

$$\langle \texttt{consistent}\ \ x \rangle \text{ and } \langle \texttt{consistent}\ \ y \rangle$$

which implies:

$$\langle \texttt{consistent}\ \ x + y \rangle \tag{123}$$

Rewriting Transition 120:

$$\langle \sigma_{\mathsf{rel}}^{s_j}(x) + \sigma_{\mathsf{rel}}^{s_j}(y) \rangle \xmapsto{s_j} \langle x + y \rangle \tag{124}$$

Partitioning Transition 119 into two transitions of durations $s_1 + \ldots + s_j$ and $s_{j+1} + \ldots s_n$ respectively:

$$\langle \sigma_{\mathsf{rel}}^{s_1 + \ldots + s_j}(x) + \sigma_{\mathsf{rel}}^{s_1 + \ldots + s_j}(y) \rangle \xmapsto{s_1 + \ldots + s_j} \langle x + y \rangle \tag{125}$$

$$\langle x + y \rangle \xmapsto{s_{j+1} + \ldots + s_n} \langle z \rangle \tag{126}$$

From Predicate 123, Rule AC-9 can be applied to derive the following:

$$\langle \sigma_{\mathsf{rel}}^{t}(x + y) \rangle \xmapsto{t} \langle x + y \rangle \tag{127}$$

Apply Rule AC-26 on time steps 126 and 127. We get:

$$\langle \sigma_{\mathsf{rel}}^{t}(x + y) \rangle \xmapsto{t + s_{j+1} + \ldots + s_n} \langle z \rangle \tag{128}$$

From 118, $t = s_1 + \ldots + s_j$.
Consider target terms in Transitions 115 and 128. The pair (z, z) is in $\mathcal{I}$.

B. **Case 2**

In the second case, for all i with $1 \le i \le n$, $s_1 + \ldots + s_i$ is either strictly less than t or $s_1 + \ldots + s_i$ is strictly greater than t.

Let $1 \leq j \leq (n-1)$, such that:

$$s_1 + \ldots + s_j < t \tag{129}$$
$$s_1 + \ldots + s_{j+1} > t \tag{130}$$

Let

$$t = s_1 + \ldots + s_j + r_1 \tag{131}$$

Rewriting Transition 115 by writing t as a sum of durations:

$$\langle \sigma_{\mathsf{rel}}^{s_1+\ldots+s_j+r_1}(x) + \sigma_{\mathsf{rel}}^{s_1+\ldots+s_j+r_1}(y) \rangle \xmapsto{s_1+\ldots+s_n} \langle z \rangle \tag{132}$$

Partitioning the above Time transition into two transitions. One of duration $s_1+\ldots+s_j$ and the other of duration $s_{j+1}+\ldots s_n$.
Let for some process term p:

$$\langle \sigma_{\mathsf{rel}}^{s_1+\ldots+s_j+r_1}(x) + \sigma_{\mathsf{rel}}^{s_1+\ldots+s_j+r_1}(y) \rangle \xmapsto{s_1+\ldots+s_j} \langle p \rangle \tag{133}$$
$$\langle p \rangle \xmapsto{s_{j+1}+\ldots+s_n} \langle z \rangle \tag{134}$$

It is easy to prove that $p = \sigma_{\mathsf{rel}}^{r_1}(x) + \sigma_{\mathsf{rel}}^{r_1}(y)$.
The process term p is the source of the Transition $j+1$ of Transition 134. Let the target of $j+1$ time step be q. Partitioning Transition 134 into two transitions of durations s_{j+1} and $s_{j+2} \ldots + s_n$.

$$\langle \sigma_{\mathsf{rel}}^{r_1}(x) + \sigma_{\mathsf{rel}}^{r_1}(y) \rangle \xmapsto{s_{j+1}} \langle q \rangle \tag{135}$$
$$\langle q \rangle \xmapsto{s_{j+2}+\ldots+s_n} \langle z \rangle \tag{136}$$

Again on process term $\sigma_{\mathsf{rel}}^{r_1}(x) + \sigma_{\mathsf{rel}}^{r_1}(y)$, only Rule AC-19 is applicable.
Then, for some $q_1, q_2 \in P$, q in Transition 135 is:

$$q = q_1 + q_2 \tag{137}$$

From the premise of the rule, the following holds:

$$\langle \sigma_{\mathsf{rel}}^{r_1}(x) \rangle \xmapsto{s_{j+1}} \langle q_1 \rangle \tag{138}$$
$$\langle \sigma_{\mathsf{rel}}^{r_1}(y) \rangle \xmapsto{s_{j+1}} \langle q_2 \rangle \tag{139}$$

Rewriting Transitions 133, 135 and 136:

$$\langle \sigma_{\mathsf{rel}}^{s_1+\ldots+s_j+r_1}(x) + \sigma_{\mathsf{rel}}^{s_1+\ldots+s_j+r_1}(y) \rangle \xmapsto{s_1+\ldots+s_j} \langle \sigma_{\mathsf{rel}}^{r_1}(x) + \sigma_{\mathsf{rel}}^{r_1}(y) \rangle \tag{140}$$
$$\langle \sigma_{\mathsf{rel}}^{r_1}(x) + \sigma_{\mathsf{rel}}^{r_1}(y) \rangle \xmapsto{s_{j+1}} \langle q_1 + q_2 \rangle \tag{141}$$
$$\langle q_1 + q_2 \rangle \xmapsto{s_{j+2}+\ldots+s_n} \langle z \rangle \tag{142}$$

From 129, 130 and 131, we know that $s_{j+1} > r_1$. Then Transitions 138 and 139 can only be derived from Rule AC-26.
Let for some $m > 1$:

$$s_{j+1} = u_1 + \ldots + u_m \tag{143}$$

Rewriting Transitions 138 and 139:

$$\langle \sigma^{r_1}_{\mathsf{rel}}(x) \rangle \xmapsto{u_1 + \ldots + u_m} \langle q_1 \rangle \tag{144}$$
$$\langle \sigma^{r_1}_{\mathsf{rel}}(y) \rangle \xmapsto{u_1 + \ldots + u_m} \langle q_2 \rangle \tag{145}$$

By applying the same reasoning as applied for Transition 60, there exists a k, with $1 \leq k < m$, such that:

$$u_1 + \ldots + u_k = r_1 \tag{146}$$

The following constituents of Transitions 144 and 145 have been derived by applying Rule AC-9.

$$\langle \sigma^{u_k}_{\mathsf{rel}}(x) \rangle \xmapsto{u_k} \langle x \rangle \tag{147}$$
$$\langle \sigma^{u_k}_{\mathsf{rel}}(y) \rangle \xmapsto{u_k} \langle y \rangle \tag{148}$$

We can infer from above transitions that the following holds:

$$\langle \texttt{consistent } x + y \rangle \tag{149}$$

Transition 144 is obtained by combining m time steps in a sequence. All intermediate time transitions are derivable, other wise Rule AC-26 could not be applied.
We partition Transition 144 into time transitions of durations '$u_1 + \ldots + u_k$' and '$u_{k+1} + \ldots + u_m$', we get:

$$\langle \sigma^{u_1 + \ldots + u_k}_{\mathsf{rel}}(x) \rangle \xmapsto{u_1 + \ldots + u_k} \langle x \rangle \tag{150}$$
$$\langle x \rangle \xmapsto{u_{k+1} + \ldots + u_m} \langle q_1 \rangle \tag{151}$$

Similarly, partitioning Transition 145 into time transitions of durations '$u_1 + \ldots + u_k$' and '$u_{k+1} + \ldots + u_m$', we get:

$$\langle \sigma^{u_1 + \ldots + u_k}_{\mathsf{rel}}(y) \rangle \xmapsto{u_1 + \ldots + u_k} \langle y \rangle \tag{152}$$
$$\langle y \rangle \xmapsto{u_{k+1} + \ldots + u_m} \langle q_2 \rangle \tag{153}$$

Apply Rule AC-19 on Transitions 151 and 153:

$$\langle x + y \rangle \xmapsto{u_{k+1} + \ldots + u_m} \langle q_1 + q_2 \rangle \tag{154}$$

Using Predicate 149, Rule AC-9 can derive the following:

$$\langle \sigma_{\mathsf{rel}}^{r_1}(x+y)\rangle \xmapsto{r_1} \langle x+y\rangle \tag{155}$$

Apply Rule AC-26 on Transitions 155 and 154:

$$\langle \sigma_{\mathsf{rel}}^{r_1}(x+y)\rangle \xmapsto{r_1+u_{k+1}+\ldots+u_m} \langle q_1+q_2\rangle \tag{156}$$

By 143, 146 and the fact that $k < m$, $s_{j+1} = r_1 + u_{k+1} + \ldots + u_m$.
Rewriting Transition 156:

$$\langle \sigma_{\mathsf{rel}}^{r_1}(x+y)\rangle \xmapsto{s_{j+1}} \langle q_1+q_2\rangle \tag{157}$$

Apply Rule AC-26 on Transitions 142 and 157.

$$\langle \sigma_{\mathsf{rel}}^{r_1}(x+y)\rangle \xmapsto{s_{j+1}+\ldots+s_n} \langle z\rangle \tag{158}$$

For the sum of durations $s_1+\ldots+s_j$, (from 129) Rule AC-8 can derive the following:

$$\langle \sigma_{\mathsf{rel}}^{s_1+\ldots+s_j+r_1}(x+y)\rangle \xmapsto{s_1+\ldots+s_j} \langle \sigma_{\mathsf{rel}}^{r_1}(x+y)\rangle \tag{159}$$

Again, Apply Rule AC-26 on Transitions 158 and 159:

$$\langle \sigma_{\mathsf{rel}}^{s_1+\ldots+s_j+r_1}(x+y)\rangle \xmapsto{s_1+\ldots++s_n} \langle z\rangle \tag{160}$$

Consider the target process terms in Transitions 159 and 159. The pair (z, z) is in R.

6.
$$\langle \sigma_{\mathsf{rel}}^{t}(x+y)\rangle \xmapsto{r} \langle z\rangle \implies \begin{array}{l}\exists z' \in P : \langle \sigma_{\mathsf{rel}}^{t}(x) + \sigma_{\mathsf{rel}}^{t}(y)\rangle \xmapsto{r} \langle z'\rangle \\ (z', z) \in R\end{array}$$

Suppose,

$$\langle \sigma_{\mathsf{rel}}^{t}(x+y)\rangle \xmapsto{r} \langle z\rangle \tag{161}$$

We distinguish between three cases for different values of r.

(a) Case $r < t$

Let $t = r + r_1$, for some r_1 with $0 < r_1 < t$.
Then Transition 161 can only be derived from Rules AC-8 or AC-26. It is easy to argue that in case of both rules, the process z in Transition 161 is of the form $\sigma_{\mathsf{rel}}^{r_1}(x+y)$.

From Rule AC-8 the following can be derived:

$$\langle \sigma_{\mathsf{rel}}^{r+r_1}(x) \rangle \xmapsto{r} \langle \sigma_{\mathsf{rel}}^{r_1}(x) \rangle \tag{162}$$
$$\langle \sigma_{\mathsf{rel}}^{r+r_1}(y) \rangle \xmapsto{r} \langle \sigma_{\mathsf{rel}}^{r_1}(y) \rangle \tag{163}$$

Apply Rule AC-19 on the above transitions:

$$\langle \sigma_{\mathsf{rel}}^{r+r_1}(x) + \sigma_{\mathsf{rel}}^{r+r_1}(y) \rangle \xmapsto{r} \langle \sigma_{\mathsf{rel}}^{r_1}(x) + \sigma_{\mathsf{rel}}^{r_1}(y) \rangle \tag{164}$$

Consider the target process terms in Transitions 161 and 164. For $0 < r_1 < t$, the pair $(\sigma_{\mathsf{rel}}^{r_1}(x) + \sigma_{\mathsf{rel}}^{r_1}(y), \sigma_{\mathsf{rel}}^{r_1}(x+y))$ is in R.

(b) Case $r = t$

Then Transition 161 can only be derived from Rules AC-9 or AC-26. It is easy to argue that in case of Rule AC-26 (transitive closure), the last rule applied is AC-9 and the process z in Transition 161 is of the form $x + y$.
From premise of Rule AC-9, the following holds:

$$\langle \texttt{consistent}\ \ x + y \rangle$$

which can only hold, if:

$$\langle \texttt{consistent}\ \ x \rangle \text{ and } \langle \texttt{consistent}\ \ y \rangle$$

Then Rule AC-9 can also be applied to derive the following transitions:

$$\langle \sigma_{\mathsf{rel}}^{r}(x) \rangle \xmapsto{r} \langle x \rangle$$
$$\langle \sigma_{\mathsf{rel}}^{r}(y) \rangle \xmapsto{r} \langle y \rangle$$

Apply Rule AC-19 on the above transitions:

$$\langle \sigma_{\mathsf{rel}}^{r+r_1}(x) + \sigma_{\mathsf{rel}}^{r+r_1}(y) \rangle \xmapsto{r} \langle x + y \rangle \tag{165}$$

Consider the target process terms in Transitions 161 and 165. The pair $(x + y, x + y)$ is in R.

(c) Case $r > t$

Transition 161 for $r > t$ can only be derived from Rule AC-26.
Let for some $n > 0$,

$$r = s_1 + \ldots s_n$$

Rewriting Transition 161:

$$\langle \sigma_{\mathsf{rel}}^{t}(x + y) \rangle \xmapsto{s_1 + \ldots + s_n} \langle z \rangle \tag{166}$$

Transition 166 has been obtained by applying Rule AC-26 on n time transitions. The constituent transitions have been each obtained from application of a single rule other than Rule AC- 26. The duration $s_1 + \ldots + s_n$ of Transitions 166 is greater than t. Hence, there exists a j, with $1 \leq j < n$ such that:

$$s_1 + \ldots + s_j = t \tag{167}$$

Rewriting Transition 166, (replacing t by the sum $s_1 + \ldots + s_j$):

$$\langle \sigma_{\mathsf{rel}}^{s_1+\ldots+s_j}(x+y)\rangle \xmapsto{s_1+\ldots+s_n} \langle z \rangle \tag{168}$$

The jth constituent of Transition 168 has been derived by applying Rule AC-9.

$$\langle \sigma_{\mathsf{rel}}^{s_j}(x) + \sigma_{\mathsf{rel}}^{s_j}(y)\rangle \xmapsto{s_j} \langle x+y \rangle \tag{169}$$

Rule AC-9 can only be used to derive the above transitions, if $\langle \texttt{consistent}\ x+y\rangle$, which implies:

$$\langle \texttt{consistent}\ x \rangle \tag{170}$$

$$\langle \texttt{consistent}\ y \rangle \tag{171}$$

We partition Transition 168 into time transitions of durations '$s_1 + \ldots + s_j$' and '$s_{j+1} + \ldots + s_n$', we get:

$$\langle \sigma_{\mathsf{rel}}^{s_1+\ldots+s_j}(x+y)\rangle \xmapsto{s_1+\ldots+s_j} \langle x+y \rangle \tag{172}$$

$$\langle x+y \rangle \xmapsto{s_{j+1}+\ldots+s_n} \langle z \rangle \tag{173}$$

From Predicates 170 and 171, Rule AC-9 can derive the following:

$$\langle \sigma_{\mathsf{rel}}^{s_1+\ldots+s_j}(x)\rangle \xmapsto{s_1+\ldots+s_j} \langle x \rangle \tag{174}$$

$$\langle \sigma_{\mathsf{rel}}^{s_1+\ldots+s_j}(y)\rangle \xmapsto{s_1+\ldots+s_j} \langle y \rangle \tag{175}$$

Apply Rule AC-19 on the above transitions:

$$\langle \sigma_{\mathsf{rel}}^{s_1+\ldots+s_j}(x) + \sigma_{\mathsf{rel}}^{s_1+\ldots+s_j}(y)\rangle \xmapsto{s_1+\ldots+s_j} \langle x+y \rangle \tag{176}$$

Apply Rule AC-26 on Transitions 176 and 173:

$$\langle \sigma_{\mathsf{rel}}^{s_1+\ldots+s_j}(x) + \sigma_{\mathsf{rel}}^{s_1+\ldots+s_j}(y)\rangle \xmapsto{s_1+\ldots+s_n} \langle z \rangle \tag{177}$$

Consider the target process terms in Transitions 177 and 161. The pair (z, z) is in R.

$\boxtimes$

F Theorem 8

<u>*Thoerem 8*</u>

Axiom SRT3 is sound in the semantics of Section 5.3.

$$\sigma^u_{\mathsf{rel}}(x) + \sigma^u_{\mathsf{rel}}(y) = \sigma^u_{\mathsf{rel}}(x+y) \qquad (SRT3)$$

<u>**Proof**</u>

We prove the soundness of Axiom SRT3 in two steps.

<u>Case $u = 0$</u>

From Rules RI-4, RI-5, RI-6 and RI-7, it easy to prove the following holds in the semantics of $BPA^{srt}_{\perp}$ with modified Relative Delay Operator (Section 5.3):

For any process term x,

$$\sigma^0_{\mathsf{rel}}(x) \leftrightarroweq x$$

Since Bisimulation is a congruence therefore, then it becomes trivial to prove that:

$$\sigma^0_{\mathsf{rel}}(x) + \sigma^0_{\mathsf{rel}}(y) \leftrightarroweq \sigma^0_{\mathsf{rel}}(x+y)$$

<u>Case $u > 0$</u>

Let $\mathcal{I}$ be the following relation:

$$\mathcal{I} = \{(p,p) \mid p \in P\}$$

Let R be the following relation:

$$R = \{(\sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(x)), \sigma^t_{\mathsf{rel}}(x+y) \mid 0 < t \leq u, x, y \in P\}$$

We prove that $R \cup \mathcal{I}$ is a bisimulation relation:

For all $a \in A, r > 0, z \in P$:

1.
$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y) \rangle \xrightarrow{a} \langle z \rangle \implies \begin{array}{l} \exists z' \in P : \langle \sigma^t_{\mathsf{rel}}(x+y) \rangle \xrightarrow{a} \langle z' \rangle \\ (z, z') \in R \end{array}$$

   Trivial.

2.
$$\langle \sigma^t_{\mathsf{rel}}(x+y) \rangle \xrightarrow{a} \langle z \rangle \implies \begin{array}{l} \exists z' \in P : \langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y) \rangle \xrightarrow{a} \langle z' \rangle \\ (z', z) \in R \end{array}$$

   Trivial.

3.
$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y) \rangle \xrightarrow{a} \surd \iff \langle \sigma^t_{\mathsf{rel}}(x+y) \rangle \xrightarrow{a} \surd$$

   Trivial.

4.

$$\langle \texttt{consistent}\ \ \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y)\rangle \iff \langle \texttt{consistent}\ \ \sigma^t_{\mathsf{rel}}(x+y)\rangle$$

Trivial.

5.

$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y)\rangle \overset{r}{\longmapsto} \langle z\rangle \implies \begin{array}{l}\exists z' \in P : \langle \sigma^t_{\mathsf{rel}}(x+y)\rangle \overset{r}{\longmapsto} \langle z'\rangle \\ (z, z') \in R\end{array}$$

Suppose,

$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y)\rangle \overset{r}{\longmapsto} \langle z\rangle \tag{178}$$

This can be derived from Rules RI-20, RI-21, RI-22.

(a) Rule RI-20

Then $z = z_1 + z_2$. Rewriting Transtion 178:

$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(x)\rangle \overset{r}{\longmapsto} \langle z_1 + z_2\rangle \tag{179}$$

And the following must hold:

$$\langle \sigma^t_{\mathsf{rel}}(x)\rangle \overset{r}{\longmapsto} \langle z_1\rangle \tag{180}$$

$$\langle \sigma^t_{\mathsf{rel}}(y)\rangle \overset{r}{\longmapsto} \langle z_2\rangle \tag{181}$$

We distinguish between three cases:

i. **Case** $r < t$
Let $t = r + r_1$, for some $0 < r_1 < t$.
Then Transtions 180 and 181 can only be derived from Rule 8.

$$\langle \sigma^{r+r_1}_{\mathsf{rel}}(x)\rangle \overset{r}{\longmapsto} \langle \sigma^{r_1}_{\mathsf{rel}}(x)\rangle$$
$$\langle \sigma^{r+r_1}_{\mathsf{rel}}(y)\rangle \overset{r}{\longmapsto} \langle \sigma^{r_1}_{\mathsf{rel}}(y)\rangle$$

Rule 8 can derive the following:

$$\langle \sigma^{r+r_1}_{\mathsf{rel}}(x+y)\rangle \overset{r}{\longmapsto} \langle \sigma^{r_1}_{\mathsf{rel}}(x+y)\rangle \tag{182}$$

For $0 < r_1 < t$, the pair $(\sigma^{r_1}_{\mathsf{rel}}(x) + \sigma^{r_1}_{\mathsf{rel}}(y), \sigma^{r_1}_{\mathsf{rel}}(x+y)) \in R$.

ii. **Case** $r = t$
Proof is similar to above. Rule 9 is used which has no conditions like Rule 8.

iii. **Case** $r > t$
Let $r = t + t_1$, for $t_1 > 0$. Then Transtions 180 and 181 can only be derived from Rule 10.
From the premise of the rule, the following holds:

$$\langle x \rangle \xmapsto{t_1} \langle z_1 \rangle$$
$$\langle y \rangle \xmapsto{t_1} \langle z_2 \rangle$$

Apply Rule 20 on above transitions:

$$\langle x + y \rangle \xmapsto{t_1} \langle z_1 + z_2 \rangle$$

Apply Rule 10 on above transition:

$$\langle \sigma^t_{\mathsf{rel}}(x + y) \rangle \xmapsto{t+t_1} \langle z_1 + z_2 \rangle$$

The pair $(z_1 + z_2, z_1 + z_2) \in \mathcal{I}$.

(b) Rule RI-21

If Transition 178 given below is derived from this rule:

$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(x) \rangle \xmapsto{r} \langle z \rangle$$

Then the following must holds:

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xmapsto{r} \langle z \rangle \tag{183}$$
$$\langle \texttt{consistent } \sigma^t_{\mathsf{rel}}(y) \rangle \tag{184}$$
$$\langle \sigma^t_{\mathsf{rel}}(y) \rangle \not\xmapsto{r} \tag{185}$$
$$\forall y', \forall s < r(\langle \sigma^t_{\mathsf{rel}}(y) \rangle \xmapsto{s} \langle y' \rangle \implies \langle \texttt{consistent } y' \rangle) \tag{186}$$

We distinguish between three cases:

i. **Case** $r < t$
Not Applicable. Transition 185 cannot be derived.

ii. **Case** $r = t$
Not Applicable. Transition 185 cannot be derived.

iii. **Case** $r > t$
Let $r = t + t_1$, for some $t_1 > 0$.
Then Transition 183 can only be derived from Rule RI-10. From the premise,

$$\langle x \rangle \xmapsto{t_1} \langle z \rangle \tag{187}$$

Transition 185 implies that Rule RI-10 is not applicable. Therefore the premise must not hold:

$$\langle y \rangle \not\xmapsto{t_1} \tag{188}$$

In 186, for $s = t$, a time transition for $\sigma^t_{\mathsf{rel}}(y)$ can only be derived from Rule 9. Then $y' = y$ and y is consistent.

$$\langle \texttt{consistent}\ y \rangle \tag{189}$$

In 186, for $s > t$, a time transition for $\sigma^t_{\mathsf{rel}}(y)$ can only be derived from Rule 10.
Let $s = v + t$, for $0 < v < t_1$.
Then, the following must hold:

$$\forall y', \forall v < t_1 (\langle y \rangle \xmapsto{v} \langle y' \rangle \implies \langle \texttt{consistent}\ y' \rangle) \tag{190}$$

Combine Transitions 187,188,189 and 190 and apply Rule RI-21:

$$\langle x + y \rangle \xmapsto{t_1} \langle z \rangle \tag{191}$$

Now apply Rule RI-10 on the above trnaistion:

$$\langle \sigma^t_{\mathsf{rel}}(x + y) \rangle \xmapsto{t+t_1} \langle z \rangle \tag{192}$$

Consider Transitions 178 and 192. The pair $(z, z) \in R$.

(c) Rule RI-22

Same as Rule RI-21

6.
$$\langle \sigma^t_{\mathsf{rel}}(x + y) \rangle \xmapsto{r} \langle z \rangle \implies \begin{array}{l} \exists z' \in P : \langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y) \rangle \xmapsto{r} \langle z' \rangle \\ (z', z) \in R \end{array}$$

Suppose,

$$\langle \sigma^t_{\mathsf{rel}}(x + y) \rangle \xmapsto{r} \langle z \rangle \tag{193}$$

We distinguish between three cases:

(a) **Case** $r < t$
Let $t = r + r_1$, for $0 < r_1 < t$.
Rule RI-8 can derive the following transitions:

$$\begin{array}{c} \langle \sigma^{r+r_1}_{\mathsf{rel}}(x) \rangle \xmapsto{r} \langle \sigma^{r_1}_{\mathsf{rel}}(x) \rangle \\ \langle \sigma^{r+r_1}_{\mathsf{rel}}(y) \rangle \xmapsto{r} \langle \sigma^{r_1}_{\mathsf{rel}}(y) \rangle \end{array}$$

Apply Rule RI-20 on above transitions:

$$\langle \sigma^{r+r_1}_{\mathsf{rel}}(x) + \sigma^{r+r_1}_{\mathsf{rel}}(y) \rangle \xmapsto{r} \langle \sigma^{r_1}_{\mathsf{rel}}(x) + \sigma^{r_1}_{\mathsf{rel}}(y) \rangle \tag{194}$$

(b) **Case** $r = t$

Similar to above. Rule RI-9 is used.

(c) **Case** $r > t$

Let $r = t + t_1$, for $t_1 > 0$.

Transition 193 can only be derived from Rule RI-10. Then the following must hold:

$$\langle x + y \rangle \xmapsto{t_1} \langle z \rangle \tag{195}$$

The above transition can be derived from three rules:

i. Rule RI-20

If Transition 195 is derived from this rule, then $z = z_1 + z_2$. From the premise of the rule, the following holds:

$$\langle x \rangle \xmapsto{t_1} \langle z_1 \rangle$$
$$\langle y \rangle \xmapsto{t_1} \langle z_2 \rangle$$

Apply Rule RI-10 on the above transitions:

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xmapsto{t+t_1} \langle z_1 \rangle$$
$$\langle \sigma^t_{\mathsf{rel}}(y) \rangle \xmapsto{t+t_1} \langle z_2 \rangle$$

Apply Rule RI-20:

$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y) \rangle \xmapsto{t+t_1} \langle z_1 + z_2 \rangle$$

ii. Rule RI-21

If Transition 195 is derived from this rule, then:

$$\langle x \rangle \xmapsto{t_1} \langle z \rangle \tag{196}$$
$$\langle \texttt{consistent}\ y \rangle \tag{197}$$
$$\langle y \rangle \not\xmapsto{t_1} \tag{198}$$
$$\forall y', \forall s < t_1 (\langle y \rangle \xmapsto{s} \langle y' \rangle \implies \langle \texttt{consistent}\ y' \rangle) \tag{199}$$

Apply Rule RI-10 on Transition 196:

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xmapsto{t+t_1} \langle z \rangle \tag{200}$$

From Rule RI-11,

$$\langle \texttt{consistent}\ \sigma^t_{\mathsf{rel}}(y) \rangle \tag{201}$$

From Predicate 198, Rule RI-10 is not applicable. Then,

$$\langle \sigma^t_{\mathsf{rel}}(y) \rangle \not\xmapsto{t+t_1} \tag{202}$$

We know that y is consistent from Predicate 197.
From Predicate 197, Rules RI-8, RI-11 and RI-9:

$$\forall y', \forall s \leq t (\langle \sigma^t_{\mathsf{rel}}(y) \rangle \xmapsto{s} \langle y' \rangle \implies \langle \texttt{consistent } y' \rangle) \tag{203}$$

By Rule RI-10:

$$\langle y \rangle \xmapsto{s} \langle y' \rangle \implies \langle \sigma^t_{\mathsf{rel}}(y) \rangle \xmapsto{t+s} \langle y' \rangle$$

Using 199,

$$\forall y', \forall s < t_1 \langle \sigma^t_{\mathsf{rel}}(y) \rangle \xmapsto{t+s} \langle y' \rangle \implies \langle \texttt{consistent } y' \rangle \tag{204}$$

Join 203 and 204:

$$\forall y', \forall s < t + t_1 \qquad \langle \sigma^t_{\mathsf{rel}}(y) \rangle \xmapsto{s} \langle y' \rangle \implies \langle \texttt{consistent } y' \rangle \tag{205}$$

Join Transitions 200, 201, 202 and 205 and apply Rule RI-21:

$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y) \rangle \xmapsto{t+t_1} \langle z \rangle \tag{206}$$

iii. Rule RI-22

Same as the the Rule RI-21.

⊠

G Soundness Proofs for Proposal 1

Let $\mathcal{I}$ be a binary relation on process terms defined as follows:

$$\mathcal{I} = \{(x, x) \mid x \in P\}$$

It is obvious that $\mathcal{I}$ is a bisimulation relation. We will use the relation $\mathcal{I}$ frequently in the proofs. We prove that the axioms given in Table 10 hold in the semantics given in Section 4.

The proofs of the soundness theorem use the following two theorems.

G.1 Theorem : Sources of Transitions are Consistent

Theorem 12 *For all closed terms p the following holds:*
For all $p', p'' \in P$, $a, b \in A$, $r, s > 0$:

$$(\langle p\rangle \xrightarrow{a} \langle p'\rangle) \vee (\langle p\rangle \overset{r}{\longmapsto} \langle p''\rangle) \vee (\langle p\rangle \xrightarrow{b} \surd) \\ \Longrightarrow \langle \mathtt{consistent}\ p\rangle$$

Proof We prove the above theorem by structural induction on a process term $p \in P$. The base case of the structural induction comprises of constant process terms, i.e. all undelayable actions in $\mathcal{A}$, the deadlock process term δ and the inconsistent process $\perp$.

Base Case

1. $p = \tilde{\tilde{a}}$.

   From Rule P1-2, $\langle \mathtt{consistent}\ \tilde{\tilde{a}}\rangle$. Hence all conditions of the theorem are trivially satisfied.

2. $p = \tilde{\tilde{\delta}}$

   From Rule P1-1, $\langle \mathtt{consistent}\ \tilde{\tilde{\delta}}\rangle$. Hence all conditions of the theorem are trivially satisfied.

3. $p = \perp$

   There are no rules for an inconsistent process $\perp$ in the semantics of $BPA^{srt}_{\perp}$. Hence all conditions of the theorem are trivially satisfied (as the left hand sides of the implications do not hold.)

By Induction Hypothesis

1. $p = \sigma^0_{\mathsf{rel}}(x)$, for a closed term x. We show that if p can perform an action or a time step or a termination predicate holds for p, then $\langle \mathtt{consistent}\ p\rangle$ holds.

(a) *Action Step:*
Suppose,

$$\langle \sigma^0_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \langle p' \rangle$$

It can only be derived from Rule P1-6. From the premise of the rule,

$$\langle x \rangle \xrightarrow{a} \langle p' \rangle$$

By Induction on the above action step, we get:

$$\langle \texttt{consistent}\ x \rangle$$

Apply Rule P1-4. We get:

$$\langle \texttt{consistent}\ \sigma^0_{\mathsf{rel}}(x) \rangle$$

Hence proved.

(b) *Time Step:*
Suppose,

$$\langle \sigma^0_{\mathsf{rel}}(x) \rangle \overset{r}{\longmapsto} \langle p' \rangle$$

It can only be derived from Rule P1-7. From the premise of the rule,

$$\langle x \rangle \overset{r}{\longmapsto} \langle p' \rangle$$

By Induction on the above action step, we get:

$$\langle \texttt{consistent}\ x \rangle$$

Apply Rule P1-4. We get:

$$\langle \texttt{consistent}\ \sigma^0_{\mathsf{rel}}(x) \rangle$$

Hence proved.

(c) *Termination Predicate:*
Suppose,

$$\langle \sigma^0_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \surd$$

It can only be derived from Rule P1-5. From the premise of the rule,

$$\langle x \rangle \xrightarrow{a} \surd$$

By Induction on the above action step, we get:

$$\langle \texttt{consistent}\ x \rangle$$

Apply Rule P1-4. We get:

$$\langle \texttt{consistent}\ \sigma^0_{\mathsf{rel}}(x) \rangle$$

Hence proved.

2. $p = \sigma^t_{\mathsf{rel}}(x) \qquad t > 0$
   From Rule P1-8, for a process term $\sigma^t_{\mathsf{rel}}(x)$, with $t > 0$, the following holds:

   $$\langle \texttt{consistent} \;\; \sigma^t_{\mathsf{rel}}(x) \rangle$$

   Hence all conditions of the theorem are trivially proved.

3. $p = x \cdot y$.
   We prove the four conditions of the theorem one by one.

   (a) *Action Step:*
   Suppose,

   $$\langle x \cdot y \rangle \xrightarrow{a} \langle p' \rangle \tag{207}$$

   It can only be derived from Rule P1-13 or Rule P1-14.

   - *Rule P1-13*
     Then for some process term p'', $p' = p'' \cdot y$. From the premise of the rule,

     $$\langle x \rangle \xrightarrow{a} \langle p'' \rangle$$

     By Induction on the above action step, we get:

     $$\langle \texttt{consistent} \;\; x \rangle$$

     Apply Rule P1-12. We get:

     $$\langle \texttt{consistent} \;\; x \cdot y \rangle$$

     Hence proved.
   - *Rule P1-14*
     Then, $p' = y$. From the premise of the rule,

     $$\langle x \rangle \xrightarrow{a} \surd$$

     By Induction on the above predicate, we get:

     $$\langle \texttt{consistent} \;\; x \rangle$$

     Apply Rule P1-12. We get:

     $$\langle \texttt{consistent} \;\; x \cdot y \rangle$$

     Hence proved.

(b) *Time Step:*
Suppose,

$$\langle x \cdot y \rangle \overset{r}{\longmapsto} \langle p' \rangle$$

It can only be derived from Rule P1-15. From the premise of the rule,

$$\langle x \rangle \overset{r}{\longmapsto} \langle p' \rangle$$

By Induction on the above time step, we get:

$$\langle \texttt{consistent}\ \ x \rangle$$

Apply Rule P1-12. We get:

$$\langle \texttt{consistent}\ \ x \cdot y \rangle$$

Hence proved.

(c) *Termination Predicate:*
Suppose,

$$\langle x \cdot y \rangle \xrightarrow{a} \surd$$

There are no rules to derive a termination predicate for a sequential composition. Hence the left hand side of the implication does not hold and the implication is trivially satisfied.

4. $p = x + y$.

We prove the four conditions of the theorem one by one.

(a) *Action Step:*
Suppose,

$$\langle x + y \rangle \xrightarrow{a} \langle p' \rangle$$

It can only be derived from Rule P1-19 or Rule P1-20.

- *Rule P1-19*
From the premise of the rule,

$$\langle x \rangle \xrightarrow{a} \langle p' \rangle \tag{208}$$
$$\langle \texttt{consistent}\ \ y \rangle \tag{209}$$

By Induction on Transition 208, we get:

$$\langle \texttt{consistent}\ \ x \rangle \tag{210}$$

Apply Rule P1-16 on Predicates 209 and 210. We get:

$$\langle \texttt{consistent}\ \ x + y \rangle$$

Hence proved.

- *Rule P1-20*
  From the premise of the rule,

$$\langle y \rangle \xrightarrow{a} \langle p' \rangle \tag{211}$$
$$\langle \texttt{consistent } x \rangle \tag{212}$$

  By Induction on Transition 211, we get:

$$\langle \texttt{consistent } y \rangle \tag{213}$$

  Apply Rule P1-16 on Predicates 212 and 213. We get:

$$\langle \texttt{consistent } x + y \rangle$$

  Hence proved.

(b) *Time Step:*
Suppose,

$$\langle x + y \rangle \overset{r}{\longmapsto} \langle p' \rangle$$

It can only be derived from Rule P1-21 or Rule P1-22 or Rule P1-23.

- *Rule P1-21*
  Then for some process terms x_1, y_1, $p' = x_1 + y_1$. From the premise of the rule the following holds:

$$\langle x \rangle \overset{r}{\longmapsto} \langle x_1 \rangle \tag{214}$$
$$\langle y \rangle \overset{r}{\longmapsto} \langle y_1 \rangle \tag{215}$$

  By Induction on the above time steps, we get:

$$\langle \texttt{consistent } x \rangle$$
$$\langle \texttt{consistent } y \rangle$$

  Apply Rule P1-16 on the above Predicates. We get:

$$\langle \texttt{consistent } x + y \rangle$$

  Hence proved.

- *Rule P1-22*
  From the premise of the rule the following holds:

$$\langle x \rangle \overset{r}{\longmapsto} \langle p' \rangle \tag{216}$$
$$\langle \texttt{consistent } y \rangle \tag{217}$$
$$\langle y \rangle \overset{r}{\not\longmapsto} \tag{218}$$
$$\tag{219}$$

By Induction on time step 216, we get:

$$\langle \texttt{consistent}\ \ x \rangle \tag{220}$$

Apply Rule P1-16 on Predicates 220 and 217. We get:

$$\langle \texttt{consistent}\ \ x + y \rangle$$

Hence proved.

- *Rule P1-23*
  From the premise of the rule the following holds:

$$\langle y \rangle \overset{r}{\longmapsto} \langle p' \rangle \tag{221}$$
$$\langle \texttt{consistent}\ \ x \rangle \tag{222}$$
$$\langle x \rangle \overset{r}{\not\longmapsto} \tag{223}$$
$$\tag{224}$$

  By Induction on time step 221, we get:

$$\langle \texttt{consistent}\ \ y \rangle \tag{225}$$

  Apply Rule P1-16 on Predicates 225 and 222. We get:

$$\langle \texttt{consistent}\ \ x + y \rangle$$

  Hence proved.

(c) *Termination Predicate:*
Suppose,

$$\langle x + y \rangle \xrightarrow{a} \surd$$

It can only be derived from Rule P1-17 or Rule P1-18.

- *Rule P1-17*
  From the premise of the rule,

$$\langle x \rangle \xrightarrow{a} \surd \tag{226}$$
$$\langle \texttt{consistent}\ \ y \rangle \tag{227}$$

  By Induction on Predicate 226, we get:

$$\langle \texttt{consistent}\ \ x \rangle \tag{228}$$

  Apply Rule P1-16 on Predicates 227 and 228. We get:

$$\langle \texttt{consistent}\ \ x + y \rangle$$

  Hence proved.

- *Rule P1-18*
  From the premise of the rule,

$$\langle y \rangle \xrightarrow{a} \surd \tag{229}$$

$$\langle \texttt{consistent} \ \ x \rangle \tag{230}$$

  By Induction on Predicate 229, we get:

$$\langle \texttt{consistent} \ \ y \rangle \tag{231}$$

  Apply Rule P1-16 on Predicates 230 and 231. We get:

$$\langle \texttt{consistent} \ \ x + y \rangle$$

  Hence proved.

5. $p = \nu_{\mathsf{rel}}(x)$

   - *Action Step:*
     Suppose,

$$\langle \nu_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \langle p' \rangle$$

     It can only be derived from Rule P1-26. From the premise of the rule,

$$\langle x \rangle \xrightarrow{a} \langle p' \rangle$$

     By Induction on the above action step, we get:

$$\langle \texttt{consistent} \ \ x \rangle$$

     Apply Rule P1-24. We get:

$$\langle \texttt{consistent} \ \ \nu_{\mathsf{rel}}(x) \rangle$$

     Hence proved.

   - *Time Step:*
     No rule allows a derivation of a time step for the now operator.

   - *Termination Predicate:*
     Suppose,

$$\langle \nu_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \surd$$

     It can only be derived from Rule P1-26. From the premise of the rule,

$$\langle x \rangle \xrightarrow{a} \surd$$

By Induction on the above predicate, we get:

$$\langle \texttt{consistent}\ x\rangle$$

Apply Rule P1-24. We get:

$$\langle \texttt{consistent}\ \nu_{\mathsf{rel}}(x)\rangle$$

Hence proved.

$\boxtimes$

G.2 Theorem : Time Determinism

Theorem 13 *For all closed terms p, durations $r > 0$ the following holds:*

$$\langle p\rangle \xmapsto{r} \langle p_1\rangle \wedge \langle p\rangle \xmapsto{r} \langle p_2\rangle$$
$$\implies p_1 \equiv p_2$$

Proof We prove the above theorem by structural induction on a process term $p \in P$. The base case of the structural induction comprises of constant process terms, i.e. all undelayable actions in $\mathcal{A}$, the deadlock process term δ and the inconsistent process $\perp$.

Base Case

1. $p = \tilde{\tilde{a}}$.

   There are no rules to derive a time step for an undelayable action.

2. $p = \tilde{\tilde{\delta}}$

   There are no rules to derive a future Inconsistency predicate for the deadlock constant.

3. $p = \perp$

   There are no rules for an inconsistent process $\perp$.

By Induction Hypothesis

1. $p = \sigma^0_{\mathsf{rel}}(x)$, for a closed term x.

   Suppose,

$$\langle \sigma^0_{\mathsf{rel}}(x)\rangle \xmapsto{r} \langle p_1\rangle \tag{232}$$
$$\langle \sigma^0_{\mathsf{rel}}(x)\rangle \xmapsto{r} \langle p_2\rangle \tag{233}$$

Only Rule P1-7 allows derivation of a time step for the operator σ^0_{rel}. From the premise of the rule,

$$\langle x \rangle \overset{r}{\longmapsto} \langle p_1 \rangle \tag{234}$$
$$\langle x \rangle \overset{r}{\longmapsto} \langle p_2 \rangle \tag{235}$$

By Induction on the above predicate, we get:

$$p_1 \equiv p_2$$

Proved.

2. $p = \sigma^t_{\mathsf{rel}}(x) \qquad t > 0$

   Suppose,

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \overset{r}{\longmapsto} \langle p_1 \rangle \tag{236}$$
$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \overset{r}{\longmapsto} \langle p_2 \rangle \tag{237}$$

   We distinguish between three cases depending on the duration r.

   (a) Case $r < t$

   Let $t = r + r_1$, for some $r_1 > 0$.
   Only Rule P1-9 can derive time steps 236 and 237. Then the target process terms in both time steps is $\sigma^{r_1}_{\mathsf{rel}}(x)$. I.e.,

$$p_1 = \sigma^{r_1}_{\mathsf{rel}}(x) \wedge p_2 = \sigma^{r_1}_{\mathsf{rel}}(x)$$

   Hence

$$p_1 \equiv p_2$$

   Proved.

   (b) Case $r = t$
   Rewriting time steps 236 and 237, we get:

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \overset{t}{\longmapsto} \langle p_1 \rangle \tag{238}$$
$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \overset{t}{\longmapsto} \langle p_2 \rangle \tag{239}$$

   Only Rule P1-10 can derive time steps 238 and 239. Then the target process terms in both time steps is x. Hence,

$$p_1 \equiv p_2$$

   Proved.

(c) Case $r > t$
Let $r = u + t$, for $u > 0$.
Rewriting time steps 236 and 237, we get:

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xmapsto{t+u} \langle p_1 \rangle \tag{240}$$

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xmapsto{t+u} \langle p_2 \rangle \tag{241}$$

Only Rule P1-11 can derive time steps 240 and 241. From the premise of the rule, the following must hold:

$$\langle x \rangle \xmapsto{u} \langle p_1 \rangle \tag{242}$$

$$\langle x \rangle \xmapsto{u} \langle p_2 \rangle \tag{243}$$

By Induction,

$$p_1 \equiv p_2$$

Proved.

3. $p = x \cdot y$.
Suppose,

$$\langle x \cdot y \rangle \xmapsto{r} \langle p_1 \rangle \tag{244}$$

$$\langle x \cdot y \rangle \xmapsto{r} \langle p_2 \rangle \tag{245}$$

The above time steps can only be derived from Rule P1-15.
Then for some process term p_1', $p_1 = p_1' \cdot y$.
Rewriting Transition 244:

$$\langle x \cdot y \rangle \xmapsto{r} \langle p_1' \cdot y \rangle \tag{246}$$

Also for some process term p_2', $p_2 = p_2' \cdot y$.
Rewriting Transition 245:

$$\langle x \cdot y \rangle \xmapsto{r} \langle p_2' \cdot y \rangle \tag{247}$$

From the premise of Rule P1-15, Transitions 246 and 247 can only be derived if the following holds:

$$\langle x \rangle \xmapsto{r} \langle p_1' \rangle \tag{248}$$

$$\langle x \rangle \xmapsto{r} \langle p_2' \rangle \tag{249}$$

By Induction

$$p_1' \equiv p_2'$$

Hence,

$$p_1' \cdot y \equiv p_2' \cdot y \text{ I.e. } p_1 \equiv p_2$$

Proved.

4. $p = x + y$.

Suppose,

$$\langle x + y\rangle \overset{r}{\longmapsto} \langle p_1\rangle \tag{250}$$
$$\langle x + y\rangle \overset{r}{\longmapsto} \langle p_2\rangle \tag{251}$$

Rule P1-21, Rule P1-22 or Rule P1-23 can be used to derive the above time steps. We discuss these rules one by one. We show both transitions are derived by the same rule and that only one rule is applicable at a time.

(a) *Rule P1-21*

Suppose Transition 250 is derived from this rule. Then for some process terms x_1, y_1,

$$p_1 = x_1 + y_1 \tag{252}$$

From the premise of the rule the following holds:

$$\langle x\rangle \overset{r}{\longmapsto} \langle x_1\rangle \tag{253}$$
$$\langle y\rangle \overset{r}{\longmapsto} \langle y_1\rangle \tag{254}$$

From Transition 253, ($\langle x\rangle \overset{r}{\longmapsto} \langle x_1\rangle$), Rule P1-23 becomes inapplicable to derive a time step for $x + y$.
From Transition 254, ($\langle y\rangle \overset{r}{\longmapsto} \langle y_1\rangle$), Rule P1-22 becomes inapplicable to derive a time step for $x + y$.
Therefore Transition 251 can also be only derived by Rule P1-21. From the premise of the rule, for some process terms x_2, y_2,

$$p_2 = x_2 + y_2 \tag{255}$$

and the following must hold:

$$\langle x\rangle \overset{r}{\longmapsto} \langle x_2\rangle \tag{256}$$
$$\langle y\rangle \overset{r}{\longmapsto} \langle y_2\rangle \tag{257}$$

Apply Induction Hypothesis on Transitions 253 and 256, and on Transitions 254 and 257. We get:

$$x_1 \equiv x_2$$
$$y_1 \equiv y_2$$

which implies

$$x_1 + y_1 \equiv x_2 + y_2$$

From Statements 252 and 255,

$$p_1 \equiv p_2$$

Proved.

(b) *Rule P1-22*

Suppose Transition 250 is derived from this rule. From the premise of the rule the following holds:

$$\langle x \rangle \xmapsto{r} \langle p_1 \rangle \tag{258}$$

$$\langle \texttt{consistent} \ \ y \rangle \tag{259}$$

$$\langle y \rangle \not\xmapsto{r} \tag{260}$$

From Transition 258, ($\langle x \rangle \xmapsto{r} \langle p_1 \rangle$), Rule P1-23 becomes inapplicable to derive a time step for $x + y$.

From Transition 260, ($\langle y \rangle \not\xmapsto{r}$), Rule P1-21 becomes inapplicable to derive a time step for $x + y$.

Hence Transition 251 can only be derived from Rule P1-22.

From the premise of the rule, in addition to Predicates 259 and 260, the following holds :

$$\langle x \rangle \xmapsto{r} \langle p_2 \rangle \tag{261}$$

Apply Induction Hypothesis on Transition 258 and Transition 261, we get:

$$p_1 \equiv p_2$$

Proved.

(c) *Rule P1-23*

Suppose Transition 250 is derived from this rule. From the premise of the rule the following holds:

$$\langle y \rangle \xmapsto{r} \langle p_1 \rangle \tag{262}$$

$$\langle \texttt{consistent} \ \ x \rangle \tag{263}$$

$$\langle x \rangle \not\xmapsto{r} \tag{264}$$

From Transition 262, ($\langle y \rangle \xmapsto{r} \langle p_1 \rangle$), Rule P1-22 becomes inapplicable to derive a time step for $x + y$.

From Transition 264, ($\langle x \rangle \not\xmapsto{r}$), Rule P1-21 becomes inapplicable to derive a time step for $x + y$.

Hence Transition 251 can only be derived from Rule P1-23.

From the premise of the rule, in addition to Predicates 263 and 264 the following holds:

$$\langle y \rangle \xmapsto{r} \langle p_2 \rangle \tag{265}$$

Apply Induction Hypothesis on Transition 262 and Transition 265, we get:

$$p_1 \equiv p_2$$

Proved.

5. $p = \nu_{\mathsf{rel}}(x)$

   There are no rules to derive a time step for the now operator. Hence the theorem trivially holds.

$\boxtimes$

G.3 Axiom A1 (Commutativity)

$$x + y = y + x$$

We need to prove, $x + y \leftrightarrows y + x$
Let R be a binary relation on process terms defined as follows:

$$R = \{(x + y, y + x) \mid x, y \in P\}$$

The relation $R \cup \mathcal{I}$ is a bisimulation relation.
The proof is trivial and therefore left.

G.4 Axiom A2 (Associativity)

$$x + (y + z) = (x + y) + z$$

We need to prove,

$$x + (y + z) \leftrightarrows (x + y) + z$$

Let

$$R = \{((x + y) + z, x + (y + z)) \mid x, y, z \in P\}$$

be a binary relation on process terms.

We prove that the relation $R \cup \mathcal{I}$ is the witness relation for bisimilarity of $x+(y+z)$ and $(x+y)+z$. We show that all pairs in R satisfy the conditions of bisimulation. For $(x, x) \in \mathcal{I}$, it is trivial that all properties of bisimulation are satisfied.

For all $a \in A, x, y, z, p \in P, r > 0$, the following holds:

1. $\langle (x+y)+z \rangle \xrightarrow{a} \langle p \rangle \implies \exists p' \in P : \langle x+(y+z) \rangle \xrightarrow{a} \langle p' \rangle$ and $(p, p') \in R \cup \mathcal{I}$.

   Suppose,

$$\langle (x + y) + z \rangle \xrightarrow{a} \langle p \rangle \tag{266}$$

   The above transition can only be derived using Rules P1-19 or P1 20.

(a) Rule P1-19
Then we must have:

$$\langle x + y \rangle \xrightarrow{a} \langle p \rangle \tag{267}$$

$$\langle \texttt{consistent}\ \ z \rangle \tag{268}$$

Again Transition 267 can be obtained using rules P1-19 or P1 20.

i. *Rule P1 19*

Then in Transition 267, the left most process term must perform the action and the other process term must be consistent. We have:

$$\langle x \rangle \xrightarrow{a} \langle p \rangle \tag{269}$$

$$\langle \texttt{consistent}\ \ y \rangle \tag{270}$$

From Predicates 268 and 270:

$$\langle \texttt{consistent}\ \ y \rangle \wedge \langle \texttt{consistent}\ \ z \rangle$$

Hence

$$\langle \texttt{consistent}\ \ y + z \rangle$$

Using Rule P1-19:

$$\langle x + (y + z) \rangle \xrightarrow{a} \langle p \rangle \tag{271}$$

Consider the target process terms in Transitions 266 and 271. The pair (p, p) is in $\mathcal{I}$.

ii. *Rule P1 20*

Then in Transition 267, the right most process term must perform the action and the other process term must be consistent. We have:

$$\langle y \rangle \xrightarrow{a} \langle p \rangle, \tag{272}$$

$$\langle \texttt{consistent}\ \ x \rangle \tag{273}$$

Apply Rule P1-19 on Transition 272 using Predicate 268. We get:

$$\langle y + z \rangle \xrightarrow{a} \langle p \rangle$$

Taking $\langle \texttt{consistent}\ \ x \rangle$ from Predicate 273, apply Rule P1-20 on the above transition, we get:

$$\langle x + (y + z) \rangle \xrightarrow{a} \langle p \rangle \tag{274}$$

Consider the target process terms in Transitions 266 and 274. The pair (p, p) is in $\mathcal{I}$.

(b) Rule P1-20

If transition 266 is derived using rule P1 20, then the process term z must perform the action, i.e.:

$$\langle z\rangle \xrightarrow{a} \langle p\rangle, \tag{275}$$
$$\langle \texttt{consistent}\ \ x+y\rangle \tag{276}$$

$\langle \texttt{consistent}\ \ x+y\rangle$ only holds if:

$$\langle \texttt{consistent}\ \ x\rangle \tag{277}$$
$$\langle \texttt{consistent}\ \ y\rangle \tag{278}$$

Apply Rule P1 20 on Transition 275 using Predicate 278:

$$\langle y+z\rangle \xrightarrow{a} \langle p\rangle$$

Again apply Rule P1 20 on the above transition using Predicate 277:

$$\langle x+(y+z)\rangle \xrightarrow{a} \langle p\rangle \tag{279}$$

Consider the target process terms in Transitions 266 and 279. The pair (p, p) is in $\mathcal{I}$.

2. $\langle x+(y+z)\rangle \xrightarrow{a} \langle p\rangle \implies \exists p' \in P : \langle (x+y)+z\rangle \xrightarrow{a} \langle p'\rangle$ and $(p', p) \in R\cup\mathcal{I}$.
Suppose,

$$\langle x+(y+z)\rangle \xrightarrow{a} \langle p\rangle \tag{280}$$

The above transition can only be derived from rules P1-19 or P1-20.

(a) Rule P1-19

If Rule P1-19 is used to derive Transition 280, then the left most process term, i.e. x must perform action a and the other process term $y+z$ must be consistent. Therefore,

$$\langle x\rangle \xrightarrow{a} \langle p\rangle \tag{281}$$
$$\langle \texttt{consistent}\ \ y+z\rangle \tag{282}$$

The predicate $\langle \texttt{consistent}\ \ y+z\rangle$ can only hold if:

$$\langle \texttt{consistent}\ \ y\rangle \tag{283}$$
$$\langle \texttt{consistent}\ \ z\rangle \tag{284}$$

Apply Rule P1-19 on Transition 281 and Predicate 283:

$$\langle x+y\rangle \xrightarrow{a} \langle p\rangle$$

Again apply Rule P1-19 on the above transition with Predicate 284:

$$\langle (x + y) + z \rangle \xrightarrow{a} \langle p \rangle \tag{285}$$

Consider the target process terms in Transitions 280 and 285. The pair (p, p) is in $\mathcal{I}$.

(b) <u>Rule P1-20</u>

If Rule P1-20 is used to derive Transition 280, then the right most process term, i.e. $(y+z)$ must perform action a and the other process term x must be consistent. Therefore,

$$\langle y + z \rangle \xrightarrow{a} \langle p \rangle \tag{286}$$

$$\langle \texttt{consistent}\ \ x \rangle \tag{287}$$

Transition 286 can only be obtained by using rules P1-19 or P1 20.

i. *Rule P1 19:*

Premise of P1-19:

$$\langle y \rangle \xrightarrow{a} \langle p \rangle \tag{288}$$

$$\langle \texttt{consistent}\ \ z \rangle \tag{289}$$

Apply rule P1-20 on Transition 288 using Predicate 287::

$$\langle x + y \rangle \xrightarrow{a} \langle p \rangle \tag{290}$$

Again applying P1-19 on the above transition using Predicate 289::

$$\langle (x + y) + z \rangle \xrightarrow{a} \langle p \rangle \tag{291}$$

Consider the target process terms in Transitions 280 and 291. The pair (p, p) is in $\mathcal{I}$.

ii. *Rule P1 20:*

Suppose Transition 286 has been derived from Rule P1-20. Then from the premise of Rule P1 20:

$$\langle z \rangle \xrightarrow{a} \langle p \rangle \tag{292}$$

$$\langle \texttt{consistent}\ \ y \rangle \tag{293}$$

Again by Rule P1 20, for any process term q with $\langle \texttt{consistent}\ \ q \rangle$:

$$\langle p_1 + z \rangle \xrightarrow{a} \langle p \rangle \tag{294}$$

From Predicates 287 and 293:

$$\langle \texttt{consistent}\ \ x \rangle \wedge \langle \texttt{consistent}\ \ y \rangle$$

which implies:$\langle \texttt{consistent}\ x+y\rangle$.
Put $q = x + y$ in Transition 294, we get the desired transition:

$$\langle (x+y)+z\rangle \xrightarrow{a} \langle p\rangle$$

And $(p, p) \in \mathcal{I}$.

3. $\langle x+(y+z)\rangle \xrightarrow{a} \langle\surd\rangle \iff \langle (x+y)+z\rangle \xrightarrow{a} \langle\surd\rangle$.
Reasoning similar to above applies.

4. $\langle (x+y)+z\rangle \xmapsto{r} \langle p\rangle \implies \exists p' \in P : \langle x+(y+z)\rangle \xmapsto{r} \langle p'\rangle$ and $(p, p') \in R\cup\mathcal{I}$.
Suppose,

$$\langle (x+y)+z\rangle \xmapsto{r} \langle p\rangle \tag{295}$$

The above time transition can be derived from rules P1-21, P1-22 or P1-refprop1rule:alt:delayoneright.

(a) Rule P1-21
Then for some process terms p_1, p_2, the process term p in transition 295 must be of the following form:

$$p = p_1 + p_2$$

Rewriting Transition 295:

$$\langle (x+y)+z\rangle \xmapsto{r} \langle p_1 + p_2\rangle \tag{296}$$

From the premise of the rule, following must be derivable:

$$\langle x+y\rangle \xmapsto{r} \langle p_1\rangle \tag{297}$$
$$\langle z\rangle \xmapsto{r} \langle p_2\rangle \tag{298}$$

Again Transition 297 can be obtained from rules P1-21, P1-22 or P1-23.

i. *Rule P1-21*
Then $p_1 = q_1 + q_2$, for some $q_1, q_2 \in P$.
Rewriting Transitions 296 and 297. We get:

$$\langle (x+y)+z\rangle \xmapsto{r} \langle (q_1+q_2)+p_2\rangle \tag{299}$$
$$\langle x+y\rangle \xmapsto{r} \langle q_1+q_2\rangle \tag{300}$$

From the premise of the rule, the following is derivable:

$$\langle x\rangle \xmapsto{r} \langle q_1\rangle \tag{301}$$
$$\langle y\rangle \xmapsto{r} \langle q_2\rangle \tag{302}$$

Apply Rule P1-21 on Transitions 298 and 302, we get:

$$\langle y + z \rangle \stackrel{r}{\longmapsto} \langle q_2 + p_2 \rangle \tag{303}$$

Again apply Rule P1-21 on Transitions 301 and 303, we get:

$$\langle x + (y + z) \rangle \stackrel{r}{\longmapsto} \langle q_1 + (q_2 + p_2) \rangle \tag{304}$$

Consider the target process terms in Transitions 295 and 304. The pair $((q_1 + q_2) + p_2, q_1 + (q_2 + p_2))$ is in R.

ii. <u>*Rule P1-22*</u>

Transition 297 can also be derived using Rule P1 22. Then the following must be derivable:

$$\langle x \rangle \stackrel{r}{\longmapsto} \langle p_1 \rangle \tag{305}$$

$$\langle y \rangle \stackrel{r}{\not\longmapsto} \tag{306}$$

$$\langle \texttt{consistent}\ \ y \rangle \tag{307}$$

Combine Transition 298 and Predicates 306 and 307. Apply Rule P1-23, we get:

$$\langle y + z \rangle \stackrel{r}{\longmapsto} \langle p_2 \rangle \tag{308}$$

Now apply Rule P1-21 on Transitions 308 and 305. We get:

$$\langle x + (y + z) \rangle \stackrel{r}{\longmapsto} \langle p_1 + p_2 \rangle \tag{309}$$

Consider the target process terms in Transitions 309 and 296. The pair $(p_1 + p_2, p_1 + p_2)$ is in $\mathcal{I}$.

iii. <u>*Rule P1-23*</u>

Transition 297 can also be obtained by Rule P1-23. Then:

$$\langle y \rangle \stackrel{r}{\longmapsto} \langle p_1 \rangle \tag{310}$$

$$\langle x \rangle \stackrel{r}{\not\longmapsto} \tag{311}$$

$$\langle \texttt{consistent}\ \ x \rangle \tag{312}$$

From Transition 298, process term z can delay as follows:

$$\langle z \rangle \stackrel{r}{\longmapsto} \langle p_2 \rangle \qquad (298)$$

Apply Rule P1-21 on Transitions 298 and 310. We get:

$$\langle y + z \rangle \stackrel{r}{\longmapsto} \langle p_1 + p_2 \rangle \tag{313}$$

On Transition 313 and Predicates 311 and 312, apply Rule P1-23:

$$\langle x + (y + z) \rangle \stackrel{r}{\longmapsto} \langle p_1 + p_2 \rangle \tag{314}$$

Consider the target process terms in Transitions 314 and 296. The pair $(p_1 + p_2, p_1 + p_2)$ is in $\mathcal{I}$.

(b) Rule P1-22

$$\langle (x+y)+z \rangle \overset{r}{\longmapsto} \langle p \rangle \qquad (295)$$

Transition 295 can also be obtained by applying Rule P1-22. Then from the premise of the rule the following must be derivable:

$$\langle x+y \rangle \overset{r}{\longmapsto} \langle p \rangle \tag{315}$$

$$\langle z \rangle \overset{r}{\not\longmapsto} \tag{316}$$

$$\langle \texttt{consistent}\ z \rangle \tag{317}$$

Again Transition 315 can be derived by rules P1-21, P1-22 or P1-23.

i. *Rule P1-21*

If Transition 315 is derived from Rule P1 21, then for some $p_1, p_2 \in P$, $p = p_1 + p_2$. Rewriting Transitions 295 and 315, we get:

$$\langle (x+y)+z \rangle \overset{r}{\longmapsto} \langle p_1 + p_2 \rangle \tag{318}$$

$$\langle x+y \rangle \overset{r}{\longmapsto} \langle p_1 + p_2 \rangle \tag{319}$$

And the following must be derivable:

$$\langle x \rangle \overset{r}{\longmapsto} \langle p_1 \rangle \tag{320}$$

$$\langle y \rangle \overset{r}{\longmapsto} \langle p_2 \rangle \tag{321}$$

Apply Rule P1-22 on Transition 321 and Predicates 316 and 317. We get:

$$\langle y+z \rangle \overset{r}{\longmapsto} \langle p_2 \rangle \tag{322}$$

Combine Transition 320 and Transition 322 and apply Rule P1-21. We get:

$$\langle x+(y+z) \rangle \overset{r}{\longmapsto} \langle p_1 + p_2 \rangle \tag{323}$$

Consider the target process terms in Transitions 318 and 323. The pair $(p_1 + p_2, p_1 + p_2)$ is in $\mathcal{I}$.

ii. *Rule P1-22*

If Transition 315 is derived from Rule P1-22, then the following must be derivable:

$$\langle x \rangle \overset{r}{\longmapsto} \langle p \rangle \tag{324}$$

$$\langle y \rangle \overset{r}{\not\longmapsto} \tag{325}$$

$$\langle \texttt{consistent}\ y \rangle \tag{326}$$

Combine Predicates 316 and 325, Predicates 317 and 326. We can infer:

$$\langle y + z \rangle \not\overset{r}{\mapsto} \tag{327}$$

$$\langle \texttt{consistent}\ \ y + z \rangle \tag{328}$$

Combine Transition 324 and Predicates 327 and 328. Apply Rule P1-22. We get:

$$\langle x + (y + z) \rangle \overset{r}{\mapsto} \langle p \rangle \tag{329}$$

Consider the target process terms in Transitions 295 and 329. The pair (p, p) is in $\mathcal{I}$.

iii. <u>*Rule P1-23*</u>

If Transition 315 is derived from Rule P1 23, then the following must be derivable:

$$\langle y \rangle \overset{r}{\mapsto} \langle p \rangle \tag{330}$$

$$\langle x \rangle \not\overset{r}{\mapsto} \tag{331}$$

$$\langle \texttt{consistent}\ \ x \rangle \tag{332}$$

Combine Transition 330 and Predicates 316 and 317. Apply Rule P1-22. We get:

$$\langle y + z \rangle \overset{r}{\mapsto} \langle p \rangle \tag{333}$$

Combine Predicates 331, 332 and Transition 333 and apply Rule P1-23. We get:

$$\langle x + (y + z) \rangle \overset{r}{\mapsto} \langle p \rangle \tag{334}$$

Consider the target process terms in Transitions 295 and 334. The pair (p, p) is in $\mathcal{I}$.

(c) <u>Rule P1-23</u>

$$\langle (x + y) + z \rangle \overset{r}{\mapsto} \langle p \rangle \qquad (295)$$

Transition 295 can also be obtained by applying Rule P1-23. Then from the premise of the rule the following must be derivable:

$$\langle z \rangle \overset{r}{\mapsto} \langle p \rangle \tag{335}$$

$$\langle x + y \rangle \not\overset{r}{\mapsto} \tag{336}$$

$$\langle \texttt{consistent}\ \ x + y \rangle \tag{337}$$

Predicate 336 can only hold if none of the process terms x and y can do a time step with duration r. Therefore the following holds:

$$\langle x \rangle \not\xmapsto{r} \tag{338}$$

$$\langle y \rangle \not\xmapsto{r} \tag{339}$$

From predicate 337, we can infer the following:

$$\langle \texttt{consistent}\ x \rangle \tag{340}$$

$$\langle \texttt{consistent}\ y \rangle \tag{341}$$

Combine Transition 335 and Predicates 339 and 341 and apply rule P1-23. We get:

$$\langle y + z \rangle \xmapsto{r} \langle p \rangle \tag{342}$$

Again combine Transition 342 and Predicates 338 and 340 and apply rule P1-23. We get:

$$\langle x + (y + z) \rangle \xmapsto{r} \langle p \rangle \tag{343}$$

Consider the target process terms in Transitions 295 and 343. The pair (p, p) is in $\mathcal{I}$.

5. $\langle x+(y+z) \rangle \xmapsto{r} \langle p \rangle \implies \exists p' \in P : \langle (x+y)+z \rangle \xmapsto{r} \langle p' \rangle$ and $(p', p) \in R \cup \mathcal{I}$.
Suppose,

$$\langle x + (y + z) \rangle \xmapsto{r} \langle p \rangle \tag{344}$$

The above time transition can be derived from rules P1-21, P1-22 or P1-refprop1rule:alt:delayoneright.

(a) Rule P1-21
Then for some process terms p_1, p_2, the process term p (in Transition 344) can be written as:

$$p = p_1 + p_2 \tag{345}$$

Rewriting Transition 344:

$$\langle x + (y + z) \rangle \xmapsto{r} \langle p_1 + p_2 \rangle \tag{346}$$

From the premise of Rule P1-21, the following is derivable:

$$\langle x \rangle \xmapsto{r} \langle p_1 \rangle \tag{347}$$

$$\langle y + z \rangle \xmapsto{r} \langle p_2 \rangle \tag{348}$$

Again Transition 348 can be obtained from rules P1-21, P1-22 or P1-23.

i. *Rule P1-21*
Then $p_2 = q_1 + q_2$, for some $q_1, q_2 \in P$.
Rewriting Transitions 344 and 348:

$$\langle x + (y + z)\rangle \overset{r}{\longmapsto} \langle p_1 + (q_1 + q_2)\rangle \tag{349}$$
$$\langle y + z\rangle \overset{r}{\longmapsto} \langle q_1 + q_2\rangle \tag{350}$$

and the following is derivable:

$$\langle y\rangle \overset{r}{\longmapsto} \langle q_1\rangle \tag{351}$$
$$\langle z\rangle \overset{r}{\longmapsto} \langle q_2\rangle \tag{352}$$

Apply Rule P1-21 on Transitions 347 and 351, we get:

$$\langle x + y\rangle \overset{r}{\longmapsto} \langle p_1 + q_1\rangle \tag{353}$$

Again apply Rule P1-21 on Transitions 353 and 352, we get:

$$\langle (x + y) + z\rangle \overset{r}{\longmapsto} \langle (p_1 + q_1) + q_2\rangle \tag{354}$$

Consider the target process terms in Transitions 349 and 354. The pair $((p_1 + q_1) + q_2, p_1 + (q_1 + q_2))$ is in R.

ii. *Rule P1-22*

If Transition 348 is derived using Rule P1 22. Then the following must be hold:

$$\langle y\rangle \overset{r}{\longmapsto} \langle p_2\rangle \tag{355}$$
$$\langle z\rangle \overset{r}{\not\longmapsto} \tag{356}$$
$$\langle \texttt{consistent}\ z\rangle \tag{357}$$

Combine Transitions 347 and 355 and apply Rule P1-21, we get:

$$\langle x + y\rangle \overset{r}{\longmapsto} \langle p_1 + p_2\rangle \tag{358}$$

Now apply Rule P1-22 on Transition 358 and Predicates 356 and 357. We get:

$$\langle (x + y) + z\rangle \overset{r}{\longmapsto} \langle p_1 + p_2\rangle \tag{359}$$

Consider the target process terms in Transitions 346 and 359. The pair $(p_1 + p_2, p_1 + p_2)$ is in $\mathcal{I}$.

iii. *Rule P1-23*

Transition 348 can also be obtained by Rule P1-23. Then:

$$\langle z\rangle \overset{r}{\longmapsto} \langle p_2\rangle \tag{360}$$
$$\langle y\rangle \overset{r}{\not\longmapsto} \tag{361}$$
$$\langle \texttt{consistent}\ y\rangle \tag{362}$$

From Transition 347, process term x can delay as follows:

$$\langle x \rangle \overset{r}{\longmapsto} \langle p_1 \rangle \qquad (347)$$

Apply Rule P1-22 on Transition 347 and Predicates 361 and 362. We get:

$$\langle x + y \rangle \overset{r}{\longmapsto} \langle p_1 \rangle \tag{363}$$

Joining Transitions 360 and 363 and apply Rule P1-21:

$$\langle (x + y) + z \rangle \overset{r}{\longmapsto} \langle p_1 + p_2 \rangle \tag{364}$$

Consider the target process terms in Transitions 346 and 364. The pair $(p_1 + p_2, p_1 + p_2)$ is in $\mathcal{I}$.

(b) Rule P1-22

Transition 344 can also be derived from rule P1 22. Then from the premise of the rule:

$$\langle x \rangle \overset{r}{\longmapsto} \langle p \rangle \tag{365}$$

$$\langle y + z \rangle \overset{r}{\not\longmapsto} \tag{366}$$

$$\langle \texttt{consistent}\ \ y + z \rangle \tag{367}$$

Predicate 367 implies:

$$\langle \texttt{consistent}\ \ y \rangle \tag{368}$$

$$\langle \texttt{consistent}\ \ z \rangle \tag{369}$$

Predicate 366 can only hold if **none of the rules** P1-21, P1-22 or P1-23 can be applied to derive a time transition for $\langle y + z \rangle$ with duration r.

Rule P1-21 cannot be applied only if $\langle y \rangle$ and $\langle z \rangle$ cannot both do a time transition with delay r. Suppose one of them can do the time step and the other cannot. Then if $\langle y \rangle$ can delay, then rule P1-22 can be used to derive a time transition for $\langle y + z \rangle$. If $\langle z \rangle$ can delay, then rule P1-23 can be used to derive a time transition for $\langle y + z \rangle$. Hence predicate 366 only holds if none of the process term x, y can do a time transition with delay $(r,)$.

Therefore,

$$\langle y \rangle \overset{r}{\not\longmapsto} \tag{370}$$

$$\langle z \rangle \overset{r}{\not\longmapsto} \tag{371}$$

Apply Rule P1-22 on Transition 365, Predicate 370 and Predicate 368, we get:

$$\langle x + y \rangle \overset{r}{\longmapsto} \langle p \rangle \tag{372}$$

Again apply Rule P1-22 on Transition 372 and Predicates 371 and 369, we get:

$$\langle (x + y) + z \rangle \overset{r}{\longmapsto} \langle p \rangle \tag{373}$$

Consider the target process terms in Transitions 344 and 373. The pair (p, p) is in $\mathcal{I}$.

(c) Rule P1-23

Transition 344 can also be derived from rule P1 23. Then from the premise of the rule:

$$\langle y + z \rangle \overset{r}{\longmapsto} \langle p \rangle \tag{374}$$

$$\langle x \rangle \overset{r}{\not\longmapsto} \tag{375}$$

$$\langle \texttt{consistent } x \rangle \tag{376}$$

Again Transition 374 can be derived by rules P1-21, P1-22 or P1-23.

i. *Rule P1-21*

If Transition 374 is derived from Rule P1 21, then for some $p_1, p_2 \in P$, $p = p_1 + p_2$. Rewriting Transitions 344 and 374

$$\langle x + (y + z) \rangle \overset{r}{\longmapsto} \langle p_1 + p_2 \rangle \tag{377}$$

$$\langle y + z \rangle \overset{r}{\longmapsto} \langle p_1 + p_2 \rangle \tag{378}$$

And the following must be derivable:

$$\langle y \rangle \overset{r}{\longmapsto} \langle p_1 \rangle \tag{379}$$

$$\langle z \rangle \overset{r}{\longmapsto} \langle p_2 \rangle \tag{380}$$

Combine Predicates 375, 376 and Transition 379 and apply Rule P1-23. We get:

$$\langle x + y \rangle \overset{r}{\longmapsto} \langle p_1 \rangle \tag{381}$$

Combine Transitions 380 and Transitions 381 and apply Rule P1-21. We get:

$$\langle (x + y) + z \rangle \overset{r}{\longmapsto} \langle p_1 + p_2 \rangle \tag{382}$$

Consider the target process terms in Transitions 377 and 382. The pair $(p_1 + p_2, p_1 + p_2)$ is in $\mathcal{I}$.

ii. *Rule P1-22*

If Transition 374 is derived from Rule P1 22, then the following must be derivable:

$$\langle y \rangle \overset{r}{\longmapsto} \langle p \rangle \tag{383}$$

$$\langle z \rangle \overset{r}{\not\longmapsto} \tag{384}$$

$$\langle \texttt{consistent } z \rangle \tag{385}$$

Combine Predicates 375 and 376 and Transition 383 and apply Rule P1-23. We get:

$$\langle x + y\rangle \overset{r}{\longmapsto} \langle p\rangle \tag{386}$$

Combine Predicates 384 and 385 and Transition 386 and apply Rule P1-22. We get:

$$\langle (x + y) + z\rangle \overset{r}{\longmapsto} \langle p\rangle \tag{387}$$

Consider the target process terms in Transitions 374 and 387. The pair (p, p) is in $\mathcal{I}$.

iii. <u>*Rule P1-23*</u>

If Transition 374 is derived from Rule P1 23, then the following must be derivable:

$$\langle z\rangle \overset{r}{\longmapsto} \langle p\rangle \tag{388}$$

$$\langle y\rangle \overset{r}{\not\longmapsto} \tag{389}$$

$$\langle \texttt{consistent}\ \ y\rangle \tag{390}$$

Combine Predicates 375 and 390. None of the rules for delay of an alternative composition can be applied. Hence the following predicate holds:

$$\langle x + y\rangle \overset{r}{\not\longmapsto} \tag{391}$$

Combine Predicates 376 and 390. The following predicate holds:

$$\langle \texttt{consistent}\ \ x + y\rangle \tag{392}$$

Combine Transition 388, Predicates 391 and 392. Apply Rule P1-23. We get:

$$\langle (x + y) + z\rangle \overset{r}{\longmapsto} \langle p\rangle \tag{393}$$

Consider the target process terms in Transitions 374 and 393. The pair (p, p) is in $\mathcal{I}$.

6.

$$\langle \texttt{consistent}\ \ (x + y) + z\rangle \iff \langle \texttt{consistent}\ \ x + (y + z)\rangle$$

<u>Left Implication</u>

Suppose

$$\in [\mathsf{s}((x + y) + z)]$$

This can only be derived from Rule P1-16. From the premise of the rule, the following must hold:

$$\langle \texttt{consistent}\ \ (x + y))\rangle \tag{394}$$

$$\langle \texttt{consistent}\ \ z\rangle \tag{395}$$

Again Predicate 394 can only be derived from Rule P1-16. Then the following holds:

$$\langle \texttt{consistent}\ \ x \rangle \tag{396}$$

$$\langle \texttt{consistent}\ \ y \rangle \tag{397}$$

Combine predicates 397 and 395 and apply Rule P1-16:

$$\langle \texttt{consistent}\ \ y + z \rangle \tag{398}$$

Again combine Predicate 396 and Predicate 398 apply Rule P1-16. We get:

$$\langle \texttt{consistent}\ \ x + (y + z) \rangle$$

Left Implication

Suppose

$$\langle \texttt{consistent}\ \ x + (y + z) \rangle$$

By similar reasoning, as given above, the following can be derived.

$$\langle \texttt{consistent}\ \ (x + y) + z \rangle$$

G.5 Axiom A3 (Idempotency)

$$x + x = x$$

We need to prove, $x + x \leftrightarrows x$.
Let R be a binary relation on process terms defined as follows:

$$R = \{(x + x, x) \mid x \in P\}$$

We prove that the relation $R \cup \mathcal{I}$ satisfies all conditions of a bisimulation relation. Below, we only prove that all pairs in R satisfy the conditions of bisimulation relation.

1.
$$\langle x + x \rangle \xrightarrow{a} \langle p \rangle \implies \begin{array}{l} \exists z \in P : \langle x \rangle \xrightarrow{a} \langle z \rangle \\ \text{and } (p, z) \in R \cup \mathcal{I}. \end{array}$$

Suppose,

$$\langle x + x \rangle \xrightarrow{a} \langle p \rangle \tag{399}$$

The above action step can be derived from either Rule P1-19 or Rule P1-20. The premise of each rule requires that the following holds:

$$\langle x \rangle \xrightarrow{a} \langle p \rangle$$

2.
$$\langle x\rangle \xrightarrow{a} \langle p\rangle \implies \begin{array}{l}\exists z \in P : \langle x+x\rangle \xrightarrow{a} \langle z\rangle \\ \text{and } (z,p) \in R \cup \mathcal{I}.\end{array}$$

Suppose,
$$\langle x\rangle \xrightarrow{a} \langle p\rangle \tag{400}$$

Then from Theorem 12, the following holds:
$$\langle \texttt{consistent}\ x\rangle \tag{401}$$

Apply Rule P1-19 or Rule P1-20 on 400 and 401. We get:
$$\langle x+x\rangle \xrightarrow{a} \langle p\rangle \tag{402}$$

Consider the target process terms in Transitions 402 and 400. The pair (p,p) is in R.

3.
$$\langle x+x\rangle \overset{r}{\longmapsto} \langle p\rangle \implies \begin{array}{l}\exists z \in P : \langle x\rangle \overset{r}{\longmapsto} \langle z\rangle \\ \text{and } (p,z) \in R \cup \mathcal{I}.\end{array}$$

Suppose,
$$\langle x+x\rangle \overset{r}{\longmapsto} \langle p\rangle \tag{403}$$

The premises (namely $\langle x\rangle \overset{r}{\longmapsto} \langle p\rangle$ and $x \overset{r}{\not\longmapsto}$) of Rule P1-22 and Rule P1-23 cannot be satisfied. Hence Transition 403 can only be derived from Rule P1-21.

From Premise of Rule P1-21, for some process terms x_1 and y_1, $p = x_1+y_1$. Rewriting Transition 400, we get:
$$\langle x+x\rangle \overset{r}{\longmapsto} \langle x_1+y_1\rangle \tag{404}$$

Also from the premise of Rule P1-21 the following holds:
$$\langle x\rangle \overset{r}{\longmapsto} \langle x_1\rangle \tag{405}$$
$$\langle x\rangle \overset{r}{\longmapsto} \langle y_1\rangle \tag{406}$$

By Theorem 13, $x_1 \equiv y_1$. Hence rewriting Transition 404, we get:
$$\langle x+x\rangle \overset{r}{\longmapsto} \langle x_1+x_1\rangle \tag{407}$$

Consider the target process terms in Transitions 405 and 407. The pair (x_1+x_1, x_1) is in R.

4.
$$\langle x\rangle \overset{r}{\longmapsto} \langle p\rangle \implies \begin{array}{l}\exists z \in P : \langle x+x\rangle \overset{r}{\longmapsto} \langle z\rangle \\ \text{and } (p,z) \in R \cup \mathcal{I}.\end{array}$$

Suppose,
$$\langle x\rangle \overset{r}{\longmapsto} \langle p\rangle \tag{408}$$

Then by Rule P1-21, the following holds:

$$\langle x + x \rangle \overset{r}{\longmapsto} \langle p + p \rangle \tag{409}$$

Consider the target process terms in Transitions 408 and 409. The pair $(p + p, p)$ is in R.

5.

$$\langle x \rangle \xrightarrow{a} \surd \iff \langle x + x \rangle \xrightarrow{a} \surd$$

Left Implication

Suppose,

$$\langle x \rangle \xrightarrow{a} \surd \tag{410}$$

Then from Theorem 12, the following holds:

$$\langle \texttt{consistent}\ \ x \rangle \tag{411}$$

Apply Rule P1-17 or Rule P1-18 on 410 and 411. We get:

$$\langle x + x \rangle \xrightarrow{a} \surd$$

Right Implication

Suppose,

$$\langle x + x \rangle \xrightarrow{a} \surd \tag{412}$$

The above predicate can be derived from either Rule P1-17 or Rule P1-18. The premise of each rule requires that the following holds:

$$\langle x \rangle \xrightarrow{a} \surd$$

Proved.

6.

$$\langle \texttt{consistent}\ \ x + x \rangle \iff \langle \texttt{consistent}\ \ x \rangle$$

Left Implication

Suppose,

$$\langle \texttt{consistent}\ \ x + x \rangle$$

This predicate can only be derived from Rule P1-16. From the premise of the rule,

$$\langle \texttt{consistent}\ \ x \rangle$$

Right Implication

Suppose,

$$\langle \texttt{consistent}\ \ x \rangle$$

Apply Rule P1-16. We get:

$$\langle \texttt{consistent}\ \ x + x \rangle$$

G.6 Axiom A4 (Right Distributivity)

$$(x + y) \cdot z = x \cdot z + y \cdot z$$

We need to prove, $(x + y) \cdot z \leftrightarrows x \cdot z + y \cdot z$.
Let R be a binary relation on process terms defined as follows:

$$R \quad = \quad \{ \quad ((x + y) \cdot z, x \cdot z + y \cdot z) \mid x, y, z \in P\}$$

We show that the relation $R \cup \mathcal{I}$ is a bisimulation relation. Below we prove that all pairs in R satisfy the conditions of bisimulation.
For all $a \in A, r \in R^{>0}, x, y, z, p \in P$, the following holds:

1.
$$\langle (x + y) \cdot z\rangle \xrightarrow{a} \langle p\rangle \implies \begin{array}{l} \exists p' \in P : \langle x \cdot z + y \cdot z\rangle \xrightarrow{a} \langle p'\rangle \\ \text{and } (p, p') \in R \cup \mathcal{I} \end{array}$$

Suppose,

$$\langle (x + y) \cdot z\rangle \xrightarrow{a} \langle p\rangle \tag{413}$$

An action transition for a sequential composition can be derived only from rules P1- 13 or P1- 14. We discuss them one by one:

(a) Rule P1- 13
Then for some process term p', $p = p' \cdot z$. Rewriting Transition 413, we get:

$$\langle (x + y) \cdot z\rangle \xrightarrow{a} \langle p' \cdot z\rangle \tag{414}$$

From the premise of Rule P1- 13, the following holds:

$$\langle x + y\rangle \xrightarrow{a} \langle p'\rangle \tag{415}$$

The above transition can be derived from Rules P1- 19 or P1- 20.

i. *Rule P1- 19*
If Transition 415 is derived from this rule, then from the premise of the rule the following holds:

$$\langle x\rangle \xrightarrow{a} \langle p'\rangle \tag{416}$$

$$\langle \texttt{consistent}\ y\rangle \tag{417}$$

Apply Rule P1- 13 on Transition 416, we get:

$$\langle x \cdot z\rangle \xrightarrow{a} \langle p' \cdot z\rangle \tag{418}$$

From Rule P1- 19, for any term q with $\langle \texttt{consistent}\ q\rangle$, the following can be derived:

$$\langle x \cdot z + q\rangle \xrightarrow{a} \langle p' \cdot z\rangle \tag{419}$$

From Predicate 417, we can infer by using Rule P1-12, $\langle$`consistent` $y \cdot z\rangle$. Then q in Transition 419 can be $y \cdot z$. Hence we get:

$$\langle x \cdot z + y \cdot z\rangle \xrightarrow{a} \langle p' \cdot z\rangle \tag{420}$$

Consider the target process terms in Transitions 414 and 420. The pair $(p' \cdot z, p' \cdot z)$ is in $\mathcal{I}$.

ii. *Rule P1- 20*
If Transition 415 is derived from this rule, then from the premise of the rule, the following holds:

$$\langle y\rangle \xrightarrow{a} \langle p'\rangle \tag{421}$$

$$\langle \texttt{consistent}\ \ x\rangle \tag{422}$$

Similar reasoning as given above for Rule 19 applies here too.

(b) Rule P1- 14
If this rule is used to derive transition 413 Then, $p = z$. Rewriting Transition 413, we get:

$$\langle (x + y) \cdot z\rangle \xrightarrow{a} \langle z\rangle \tag{423}$$

And from the premise of Rule P1- 14, the following holds:

$$\langle x + y\rangle \xrightarrow{a} \surd \tag{424}$$

$$\langle \texttt{consistent}\ \ z\rangle \tag{425}$$

The Transition 424 can be derived from Rules P1- 17 or P1- 18.

i. *Rule P1- 17*
If Transition 424 is derived from this rule, then from the premise of the rule the following holds:

$$\langle x\rangle \xrightarrow{a} \surd \tag{426}$$

$$\langle \texttt{consistent}\ \ y\rangle \tag{427}$$

From Predicate 425, we have $\langle$`consistent` $z\rangle$. Apply Rule P1- 14 on Transition 426 using process term z, we get:

$$\langle x \cdot z\rangle \xrightarrow{a} \langle z\rangle \tag{428}$$

From Rule P1- 19, for any term q with $\langle$`consistent` $q\rangle$, the following can be derived:

$$\langle x \cdot z + q\rangle \xrightarrow{a} \langle z\rangle \tag{429}$$

From Predicate 427, we infer $\langle$`consistent` $y \cdot z\rangle$. Then q in Transition 429 can be $y \cdot z$. Hence we get:

$$\langle x \cdot z + y \cdot z\rangle \xrightarrow{a} \langle z\rangle \tag{430}$$

Consider the target process terms in Transitions 423 and 430. The pair (z, z) is in $\mathcal{I}$.

ii. *Rule P1- 18*
Similar reasoning as above applies.

2.
$$\langle x \cdot z + y \cdot z \rangle \xrightarrow{a} \langle p \rangle \implies \begin{array}{l} \exists p' \in P : \langle (x + y) \cdot z \rangle \xrightarrow{a} \langle p' \rangle \\ \text{and } (p', p) \in R \cup \mathcal{I} \end{array}$$

Suppose,

$$\langle x \cdot z + y \cdot z \rangle \xrightarrow{a} \langle p \rangle \tag{431}$$

The above transition can be derived from Rules P1- 19 or P1- 20.

(a) Rule P1- 19
If Transition 431 is derived from this rule, then from the premise of the rule the following holds:

$$\langle x \cdot z \rangle \xrightarrow{a} \langle p \rangle \tag{432}$$
$$\langle \texttt{consistent } y \cdot z \rangle \tag{433}$$

Transition 432 can be derived from Rule P1- 13 or Rule P1- 14. We discuss the two rules one by one.

i. *Rule P1- 13*
Then for some process term p', $p = p' \cdot z$. Rewriting Transition 431 and 432, we get:

$$\langle x \cdot z + y \cdot z \rangle \xrightarrow{a} \langle p' \cdot z \rangle \tag{434}$$
$$\langle x \cdot z \rangle \xrightarrow{a} \langle p' \cdot z \rangle \tag{435}$$

From premise of the Rule P1- 13, the following holds:

$$\langle x \rangle \xrightarrow{a} \langle p' \rangle$$

Apply Rule P1- 19 on the above transition. Then for any term q, with $\langle \texttt{consistent } q \rangle$, the following holds:

$$\langle x + q \rangle \xrightarrow{a} \langle p' \rangle \tag{436}$$

From Predicate 433, it can be inferred that $\langle \texttt{consistent } y \rangle$. Hence q can be replaced by y in Transition 436.

$$\langle x + y \rangle \xrightarrow{a} \langle p' \rangle \tag{437}$$

Apply Rule P1- 13 on the above Transition, we get:

$$\langle (x + y) \cdot z \rangle \xrightarrow{a} \langle p' \cdot z \rangle \tag{438}$$

ii. *Rule P1- 14*
If this rule is used to derive Transition 432, then $p = z$. Rewriting Transitions 431 and 432:

$$\langle x \cdot z + y \cdot z \rangle \xrightarrow{a} \langle z \rangle \tag{439}$$
$$\langle x \cdot z \rangle \xrightarrow{a} \langle z \rangle \tag{440}$$

From Premise of Rule P1- 14, the following holds:

$$\langle x \rangle \xrightarrow{a} \surd \tag{441}$$
$$\langle \texttt{consistent} \ \ z \rangle \tag{442}$$

Let q b a term with $\langle \texttt{consistent} \ \ q \rangle$. Apply Rule P1- 17 on Transition 441, we get:

$$\langle x + q \rangle \xrightarrow{a} \surd \tag{443}$$

Consider Predicate 433, $\langle \texttt{consistent} \ \ y \cdot z \rangle$. From the predicate, it can be inferred that:

$$\langle \texttt{consistent} \ \ y \rangle$$

Replace q by y in Transition 443:

$$\langle x + y \rangle \xrightarrow{a} \surd \tag{444}$$

From Predicate 442, $\langle \texttt{consistent} \ \ z \rangle$. Using term z, apply Rule 14 on Transition 444, we get:

$$\langle (x + y) \cdot z \rangle \xrightarrow{a} \langle z \rangle \tag{445}$$

Consider the target process terms in Transitions 439 and 445. The pair (z, z) is in $\mathcal{I}$.

(b) Rule P1- 20
Similar reasoning as given above applies.

3.
$$\langle (x + y) \cdot z \rangle \overset{r}{\longmapsto} \langle p \rangle \implies \quad \exists z' \in P : \langle x \cdot z + y \cdot z \rangle \overset{r}{\longmapsto} \langle z' \rangle \text{ and } (p, z') \in R \cup \mathcal{I}$$

Suppose,
$$\langle (x + y) \cdot z \rangle \overset{r}{\longmapsto} \langle p \rangle \tag{446}$$

A time transition for a sequential composition can be derived only from rule P1- 15. Then, for some $p' \in P$, p must be equal to $p' \cdot z$. Rewriting Transition 446:
$$\langle (x + y) \cdot z \rangle \overset{r}{\longmapsto} \langle p' \cdot z \rangle \tag{447}$$

And the following must hold from premise of Rule P1- 15:

$$\langle x + y \rangle \overset{r}{\longmapsto} \langle p' \rangle \tag{448}$$

The above transition can be derived from Rules P1- 21, P1- 22 or P1- 23. We discuss them one by one:

(a) Rule P1- 21
Then for some process term x_1, x_2, $p' = x_1 + x_2$. Re-writing Transitions 447 and 448:

$$\langle (x+y)\cdot z\rangle \xmapsto{r} \langle (x_1 + x_2)\cdot z\rangle \tag{449}$$
$$\langle x+y\rangle \xmapsto{r} \langle x_1 + x_2\rangle \tag{450}$$

From premise of Rule P1- 21, the following must hold:

$$\langle x\rangle \xmapsto{r} \langle x_1\rangle$$
$$\langle y\rangle \xmapsto{r} \langle x_2\rangle$$

Apply Rule P1- 15 on the above transitions, we get:

$$\langle x\cdot z\rangle \xmapsto{r} \langle x_1\cdot z\rangle \tag{451}$$
$$\langle y\cdot z\rangle \xmapsto{r} \langle x_2\cdot z\rangle \tag{452}$$

Apply Rule P1- 21 on the above two transitions, we get:

$$\langle x\cdot x + y\cdot z\rangle \xmapsto{r} \langle x_1\cdot z + x_2\cdot z\rangle \tag{453}$$

Consider Transitions 449 and 453. The pair of their target process terms $((x_1 + x_2)\cdot z, x_1\cdot z + x_2\cdot z)$ is in R.

(b) Rule P1- 22
If Transition 448 is derived from this rule, the from the premise of the rule, the following must hold:

$$\langle x\rangle \xmapsto{r} \langle p'\rangle \tag{454}$$
$$\langle \texttt{consistent}\ \ y\rangle \tag{455}$$
$$\langle y\rangle \not\xmapsto{r} \tag{456}$$

Apply Rule P1- 15 on Transition 454.

$$\langle x\cdot z\rangle \xmapsto{r} \langle p'\cdot z\rangle \tag{457}$$

We can infer from Predicate 456 the following:

$$\langle y\cdot z\rangle \not\xmapsto{r} \tag{458}$$

We can infer from Predicate 455 by using Rule P1-12:

$$\langle \texttt{consistent}\ \ y\cdot z\rangle \tag{459}$$

Join Transitions (Predicates) 457, 458 and 459 and apply Rule P1-22. We get:

$$\langle x\cdot z + y\cdot z\rangle \xmapsto{r} \langle p'\cdot z\rangle \tag{460}$$

Consider Transitions 447 and 460. The pair of their target process terms $(p'\cdot z, p'\cdot z)$ is in $\mathcal{I}$.

(c) Rule P1- 23
If Transition 448 is derived from this rule, then from the premise of the rule, the following must hold:

$$\langle y \rangle \xmapsto{r} \langle p' \rangle \tag{461}$$

$$\langle \texttt{consistent}\ x \rangle \tag{462}$$

$$\langle x \rangle \not\xmapsto{r} \tag{463}$$

Similar reasoning as given above for Rule P1-22 should be applied here.

4.

$$\langle x \cdot z + y \cdot z \rangle \xmapsto{r} \langle p \rangle \implies \quad \exists z' \in P : \langle (x+y) \cdot z \rangle \xmapsto{r} \langle z' \rangle \text{ and } (p, z') \in R \cup \mathcal{I}$$

Suppose,

$$\langle x \cdot z + y \cdot z \rangle \xmapsto{r} \langle p \rangle \tag{464}$$

The above transition can be derived from Rule P1-21, Rule P1-22 or Rule P1-23. We discuss them one by one:

(a) Rule P1-21
Then for some process terms x', y', $p = x' + y'$. Rewriting Transition 464,

$$\langle x \cdot z + y \cdot z \rangle \xmapsto{r} \langle x' + y' \rangle \tag{465}$$

And from the premise of Rule P1-21, the following holds:

$$\langle x \cdot z \rangle \xmapsto{r} \langle x' \rangle \tag{466}$$

$$\langle y \cdot z \rangle \xmapsto{r} \langle y' \rangle \tag{467}$$

A time step for a sequential composition can only be derived from Rule P1-15. Then for some process terms x_1 and y_1, $x' = x_1 \cdot z$ and $y' = y_1 \cdot z$. Rewriting Transition 465:

$$\langle x \cdot z + y \cdot z \rangle \xmapsto{r} \langle x_1 \cdot z + y_1 \cdot z \rangle \tag{468}$$

From premise of Rule P1-15, the following must hold:

$$\langle x \rangle \xmapsto{r} \langle x_1 \rangle \tag{469}$$

$$\langle y \rangle \xmapsto{r} \langle y_1 \rangle \tag{470}$$

Apply Rule P1-21 on above transitions, we get:

$$\langle x + y \rangle \xmapsto{r} \langle x_1 + y_1 \rangle \tag{471}$$

Apply Rule P1-15 on above transition. We get:

$$\langle (x + y) \cdot z \rangle \xmapsto{r} \langle (x_1 + y_1) \cdot z \rangle \tag{472}$$

Consider Transitions 468 and 472. The pair of their target process terms $(x_1 \cdot z + y_1 \cdot z, (x_1 + y_1) \cdot z)$ is in R.

(b) Rule P1-22

If Transition 464 is derived from this rule, the from the premise of the rule, the following must hold:

$$\langle x \cdot z \rangle \xmapsto{r} \langle p \rangle \tag{473}$$
$$\langle \texttt{consistent}\ y \cdot z \rangle \tag{474}$$
$$\langle y \cdot z \rangle \not\xmapsto{r} \tag{475}$$

Transition 473 can only be derived from Rule P1- 15. Then for some process term p', $p = p' \cdot z$. Rewriting Transition 464 and Transition 473, we get:

$$\langle x \cdot z + y \cdot z \rangle \xmapsto{r} \langle p' \cdot z \rangle \tag{476}$$
$$\langle x \cdot z \rangle \xmapsto{r} \langle p' \cdot z \rangle \tag{477}$$

From the premise of Rule P1-15, the following must hold:

$$\langle x \rangle \xmapsto{r} \langle p' \rangle \tag{478}$$

Predicate 475 can hold if Rule P1-15 cannot apply. Hence, its premise must not hold:

$$\langle y \rangle \not\xmapsto{r} \tag{479}$$

Predicate 474 can only be derived from Rule P1-12. Hence, its premise must hold:

$$\langle \texttt{consistent}\ y \rangle \tag{480}$$

Join Transitions (Predicates) 478, 479 and 480 and apply Rule P1-22. We get:

$$\langle x + y \rangle \xmapsto{r} \langle p' \rangle \tag{481}$$

Apply Rule 15 on above transition, we get:

$$\langle (x + y) \cdot z \rangle \xmapsto{r} \langle p' \cdot z \rangle \tag{482}$$

Consider Transitions 476 and 482. The pair of their target process terms $(p' \cdot z, p' \cdot z)$ is in $\mathcal{I}$.

(c) Rule P1-23
Same reasoning as given for Rule P1-22 applies here.

5.
$$\langle (x + y) \cdot z \rangle \xrightarrow{a} \surd \iff \langle x \cdot z + y \cdot z \rangle \xrightarrow{a} \surd$$

Left Implication
Suppose,

$$\langle (x + y) \cdot z \rangle \xrightarrow{a} \surd \tag{483}$$

For a sequential composition, a termination predicate cannot be derived from any rules. Hence our supposition is wrong and the left implication trivially holds.

Right Implication

Suppose,

$$\langle x \cdot z + y \cdot z \rangle \xrightarrow{a} \surd \tag{484}$$

The above predicate can be derived from Rule P1-17 or Rule P1-18.

(a) Rule P1-17

If Predicate 484 is derived from this rule, then the following must hold:

$$\langle x \cdot z \rangle \xrightarrow{a} \surd \tag{485}$$

$$\langle \texttt{consistent}\ \ y \cdot z \rangle \tag{486}$$

Predicate 485 cannot be derived from any rules. Hence Predicate 484 cannot be derived from this rule.

(b) Rule P1-17

If Predicate 484 is derived from this rule, then the following must hold:

$$\langle y \cdot z \rangle \xrightarrow{a} \surd \tag{487}$$

$$\langle \texttt{consistent}\ \ x \cdot z \rangle \tag{488}$$

Predicate 487 cannot be derived from any rules. Hence Predicate 484 cannot be derived from this rule.

The Predicate 484 cannot be derived from any rules. Hence the right implication holds trivially.

6.

$$\langle \texttt{consistent}\ \ (x + y) \cdot z \rangle \iff \langle \texttt{consistent}\ \ x \cdot z + y \cdot z \rangle$$

Left Implication

Suppose,

$$\langle \texttt{consistent}\ \ (x + y) \cdot z \rangle \tag{489}$$

This is only derivable from Rule P1- 12. From the premise of the rule, the following must hold:

$$\langle \texttt{consistent}\ \ x + y \rangle \tag{490}$$

This can only be derived from Rule P1- 16. Then from the premise of the rules, the following hold:

$$\langle \texttt{consistent}\ \ x\rangle \tag{491}$$
$$\langle \texttt{consistent}\ \ y\rangle \tag{492}$$

Apply Rule P1- 12 on Predicate 491 with process term z. We get:

$$\langle \texttt{consistent}\ \ x \cdot z\rangle \tag{493}$$

Apply Rule 12 on Predicate 492 also with process term z. We get:

$$\langle \texttt{consistent}\ \ y \cdot z\rangle \tag{494}$$

Apply Rule P1-16 on Predicates 493 and 494, we get:

$$\langle \texttt{consistent}\ \ x \cdot z + y \cdot z\rangle$$

Hence left implication is proved.

Right Implication

Suppose,

$$\langle \texttt{consistent}\ \ x \cdot z + y \cdot z\rangle$$

The above predicate can only be derived from Rule P1- 16.

From the premise of the rule, the following must hold:

$$\langle \texttt{consistent}\ \ x \cdot z\rangle \tag{495}$$
$$\langle \texttt{consistent}\ \ y \cdot z\rangle \tag{496}$$

The above predicates can only be derived from Rule P1- 12. Then the following must hold:

$$\langle \texttt{consistent}\ \ x\rangle \tag{497}$$
$$\langle \texttt{consistent}\ \ y\rangle \tag{498}$$

Apply Rule P1- 16 on the above predicates, we get:

$$\langle \texttt{consistent}\ \ x + y\rangle$$

Apply Rule P1-12 on the above predicate, we get:

$$\langle \texttt{consistent}\ \ (x + y) \cdot z\rangle$$

Hence right implication is proved.

G.7 Axiom A5

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

We need to prove, $(x \cdot y) \cdot z \leftrightarrows x \cdot (y \cdot z)$.

Let R be a binary relation on process terms defined as follows:

$$R \quad = \quad \{ \quad ((x \cdot y) \cdot z, x \cdot (y \cdot z)) \mid x, y, z \in P\}$$

We prove that the relation $R \cup \mathcal{I}$ is a bisimulation relation. Below we show that all pairs in R satisfy the conditions of bisimulation.

For all $a \in A, r \in R^{>0}, x, y, z, p \in P$, the following holds:

1. $$\begin{aligned} \langle (x \cdot y) \cdot z \rangle \xrightarrow{a} \langle p \rangle \implies \quad & \exists z' \in P : \langle x \cdot (y \cdot z) \rangle \xrightarrow{a} \langle z' \rangle \\ & \text{and } (p, z') \in R \cup \mathcal{I} \end{aligned}$$

   Suppose,
   $$\langle (x \cdot y) \cdot z \rangle \xrightarrow{a} \langle p \rangle \tag{499}$$
   An action transition for a sequential composition can be derived only from rules P1- 13 or P1- 14. We discuss them one by one:

   (a) Rule P1- 13
   Then for some process term p', $p = p' \cdot z$. Rewriting Transition 499, we get:
   $$\langle (x \cdot y) \cdot z \rangle \xrightarrow{a} \langle p' \cdot z \rangle \tag{500}$$
   From the premise of Rule P1- 13, the following holds:
   $$\langle x \cdot y \rangle \xrightarrow{a} \langle p' \rangle \tag{501}$$
   The above transition can be derived from Rules P1- 13 or P1- 14.

   i. *Rule P1- 13*
   If Transition 501 is derived from this rule, then for some process term p'', $p' = p'' \cdot y$. Rewriting Transitions 500 and 501, we get:
   $$\langle (x \cdot y) \cdot z \rangle \xrightarrow{a} \langle (p'' \cdot y) \cdot z \rangle \tag{502}$$
   $$\langle x \cdot y \rangle \xrightarrow{a} \langle p'' \cdot y \rangle \tag{503}$$
   From premise of Rule P1- 13 the following holds:
   $$\langle x \rangle \xrightarrow{a} \langle p'' \rangle \tag{504}$$
   Apply Rule P1- 13 on Transition 504. For any process term q we get:
   $$\langle x \cdot q \rangle \xrightarrow{a} \langle p'' \cdot q \rangle$$
   The term q can be $y \cdot z$.
   $$\langle x \cdot (y \cdot z) \rangle \xrightarrow{a} \langle p'' \cdot (y \cdot z) \rangle \tag{505}$$
   Consider the target process terms in Transitions 502 and 505. The pair $((p'' \cdot y) \cdot z, p'' \cdot (y \cdot z))$ is in R.

ii. *Rule P1- 14*
If Transition 501 is derived from this rule, then for some process term, $p' = y$. Rewriting Transitions 500 and 501, we get:

$$\langle (x \cdot y) \cdot z \rangle \xrightarrow{a} \langle y \cdot z \rangle \tag{506}$$
$$\langle x \cdot y \rangle \xrightarrow{a} \langle y \rangle \tag{507}$$

From premise of Rule P1- 14 the following holds:

$$\langle x \rangle \xrightarrow{a} \surd \tag{508}$$
$$\langle \texttt{consistent} \ \ y \rangle \tag{509}$$

From Predicate 509 by Rule P1- 12, the following holds:

$$\langle \texttt{consistent} \ \ y \cdot z \rangle$$

Apply Rule P1- 14 on Transition 508. For any process term q with $\langle \texttt{consistent} \ \ q \rangle$, have:

$$\langle x \cdot q \rangle \xrightarrow{a} \langle q \rangle$$

The process term q can be $y \cdot z$.

$$\langle x \cdot (y \cdot z) \rangle \xrightarrow{a} \langle y \cdot z \rangle \tag{510}$$

Consider the target process terms in Transitions 506 and 510. The pair $(y \cdot z, y \cdot z)$ is in $\mathcal{I}$.

(b) Rule P1- 14
If Transition 499 is derived from this rule, then $p = z$. Rewriting Transition 499, we get:

$$\langle (x \cdot y) \cdot z \rangle \xrightarrow{a} \langle z \rangle \tag{511}$$

From premise of Rule P1- 14 the following holds:

$$\langle x \cdot y \rangle \xrightarrow{a} \surd \tag{512}$$
$$\langle \texttt{consistent} \ \ z \rangle \tag{513}$$

The transition 512 cannot be derived. (No termination transition for a sequential composition can be derived.)
Hence Rule P1- 14 cannot be used to derive Transition 499.

2.
$$\langle x \cdot (y \cdot z) \rangle \xrightarrow{a} \langle p \rangle \implies \begin{array}{l} \exists z' \in P : \langle (x \cdot y) \cdot z \rangle \xrightarrow{a} \langle z' \rangle \\ \text{and } (z', p) \in R \cup \mathcal{I} \end{array}$$

Suppose,

$$\langle x \cdot (y \cdot z) \rangle \xrightarrow{a} \langle p \rangle \tag{514}$$

An action transition for a sequential composition can be derived only from rules P1- 13 or P1- 14. We discuss them one by one:

(a) Rule P1- 13

If this rule is used to derive Transition 514, then for some process term p', $p = p' \cdot (y \cdot z)$. Rewriting Transition 514, we get:

$$\langle x \cdot (y \cdot z) \rangle \xrightarrow{a} \langle p' \cdot (y \cdot z) \rangle \tag{515}$$

From premise of the rule, the following must hold:

$$\langle x \rangle \xrightarrow{a} \langle p' \rangle \tag{516}$$

Apply Rule P1- 13 on the above transitions **twice**. We get:

$$\langle (x \cdot y) \cdot z \rangle \xrightarrow{a} \langle (p' \cdot y) \cdot z \rangle \tag{517}$$

Consider the target process terms in Transitions 515 and 517. The pair $(p' \cdot (y \cdot z), (p' \cdot y) \cdot z)$ is in R.

(b) Rule P1- 14

If Transition 514 is derived from this rule, then $p = y \cdot z$. Rewriting Transition 514, we get:

$$\langle x \cdot (y \cdot z) \rangle \xrightarrow{a} \langle y \cdot z \rangle \tag{518}$$

From premise of Rule P1- 14 the following holds:

$$\langle x \rangle \xrightarrow{a} \surd \tag{519}$$

$$\langle \texttt{consistent}\ y \cdot z \rangle \tag{520}$$

A consistency predicate (Predicate 520) for a sequential composition can only be derived from Rule P1-12. From the premise of the rule, the following must hold:

$$\langle \texttt{consistent}\ y \rangle \tag{521}$$

Apply Rule P1- 14 on Transition 519 using Predicate 521, we get:

$$\langle x \cdot y \rangle \xrightarrow{a} \langle y \rangle$$

Apply Rule P1- 13 on the above transition. We get:

$$\langle (x \cdot y) \cdot z \rangle \xrightarrow{a} \langle y \cdot z \rangle \tag{522}$$

Consider the target process terms in Transitions 518 and 522. The pair $(y \cdot z, y \cdot z)$ is in $\mathcal{I}$.

3.
$$\langle (x \cdot y) \cdot z \rangle \xmapsto{r} \langle p \rangle \implies \begin{array}{l} \exists z' \in P : \langle x \cdot (y \cdot z) \rangle \xmapsto{r} \langle z' \rangle \\ \text{and } (p, z') \in R \cup \mathcal{I} \end{array}$$

Suppose,

$$\langle (x \cdot y) \cdot z \rangle \xmapsto{r} \langle p \rangle \tag{523}$$

A time step for a sequential composition can be derived only from rule P1- 15. Then for some p', $p = p' \cdot z$. Rewriting Transition 523:

$$\langle (x \cdot y) \cdot z \rangle \xmapsto{r} \langle p' \cdot z \rangle \tag{524}$$

From the premise of the rule, the following holds:

$$\langle x \cdot y \rangle \xmapsto{r} \langle p' \rangle \tag{525}$$

Again the above transition can only be derived from Rule P1- 15. Then for some p'', $p' = p'' \cdot y$. Rewriting Transition 524 and Transition 525:

$$\langle (x \cdot y) \cdot z \rangle \xmapsto{r} \langle (p'' \cdot y) \cdot z \rangle \tag{526}$$

$$\langle x \cdot y \rangle \xmapsto{r} \langle p'' \cdot y \rangle \tag{527}$$

From the premise of the rule, the following holds:

$$\langle x \rangle \xmapsto{r} \langle p'' \rangle \tag{528}$$

Apply Rule P1- 15 on Transition 528. For any process term q we get:

$$\langle x \cdot q \rangle \xmapsto{r} \langle p'' \cdot q \rangle$$

The term q can be $y \cdot z$.

$$\langle x \cdot (y \cdot z) \rangle \xmapsto{r} \langle p'' \cdot (y \cdot z) \rangle \tag{529}$$

Consider the target process terms in Transitions 526 and 529. The pair $((p'' \cdot y) \cdot z, p'' \cdot (y \cdot z))$ is in R.

4.
$$\langle x \cdot (y \cdot z) \rangle \xmapsto{r} \langle p \rangle \implies \begin{array}{l} \exists z' \in P : \langle (x \cdot y) \cdot z \rangle \xmapsto{r} \langle z' \rangle \\ \text{and } (z', p) \in R \cup \mathcal{I} \end{array}$$

Suppose,

$$\langle x \cdot (y \cdot z) \rangle \xmapsto{r} \langle p \rangle \tag{530}$$

A time step for a sequential composition can be derived only from rule P1- 15. Then, for some process term p', $p = p' \cdot (y \cdot z)$. Rewriting Transition 530, we get:

$$\langle x \cdot (y \cdot z) \rangle \xmapsto{r} \langle p' \cdot (y \cdot z) \rangle \tag{531}$$

From the premise of the rule, the following must hold:

$$\langle x \rangle \xmapsto{r} \langle p' \rangle \tag{532}$$

Apply Rule P1- 15 on the above time step with process term y , we get:

$$\langle x \cdot y \rangle \overset{r}{\longmapsto} \langle p' \cdot y \rangle$$

Again apply Rule P1- 15 on the above time step with process term z , we get:

$$\langle (x \cdot y) \cdot z \rangle \overset{r}{\longmapsto} \langle (p' \cdot y) \cdot z \rangle \tag{533}$$

Consider the target process terms in Transitions 531 and 533. The pair $(p' \cdot (y \cdot z), (p' \cdot y) \cdot z)$ is in R.

5.
$$\langle (x \cdot y) \cdot z \rangle \xrightarrow{a} \surd \iff \langle x \cdot (y \cdot z) \rangle \xrightarrow{a} \surd$$

Left Implication

Suppose,

$$\langle (x \cdot y) \cdot z \rangle \xrightarrow{a} \surd \tag{534}$$

A termination predicate for a sequential composition cannot be derived from any rules. Hence the above predicate doesn't hold.

Right Implication

Suppose,

$$\langle x \cdot (y \cdot z) \rangle \xrightarrow{a} \surd \tag{535}$$

A termination predicate for a sequential composition cannot be derived from any rules. Hence the above predicate doesn't hold.

6.
$$\langle \texttt{consistent}\ (x \cdot y) \cdot z \rangle \iff \langle \texttt{consistent}\ x \cdot (y \cdot z) \rangle$$

Left Implication

Suppose,

$$\langle \texttt{consistent}\ (x \cdot y) \cdot z \rangle$$

The above predicate can only be derived from Rule P1-12. Then from the premise of the rule,

$$\langle \texttt{consistent}\ x \cdot y \rangle$$

Again, the above predicate can only be derived from Rule P1-12. Hence,

$$\langle \texttt{consistent}\ x \rangle$$

Apply Rule P1-12 on the above predicate. For any process term q, the following holds:

$$\langle \texttt{consistent}\ x \cdot q \rangle$$

The process term q can be $y \cdot z$. Hence we have,

$$\langle \texttt{consistent}\ x \cdot (y \cdot z) \rangle$$

Right Implication
Suppose,

$$\langle \texttt{consistent} \;\; x \cdot (y \cdot z) \rangle$$

The above predicate can only be derived from Rule P1-12. Then from the premise of the rule,

$$\langle \texttt{consistent} \;\; x \rangle$$

Apply Rule P1-12 on the above predicate. For any process term q, the following holds:

$$\langle \texttt{consistent} \;\; x \cdot q \rangle$$

The process term q can be y. Hence we have,

$$\langle \texttt{consistent} \;\; x \cdot y \rangle$$

By repeating the same reasoning,

$$\langle \texttt{consistent} \;\; (x \cdot y) \cdot z \rangle$$

G.8 Axiom A6SR

$$x + \tilde{\tilde{\delta}} = x$$

We need to prove, $x + \tilde{\tilde{\delta}} \leftrightarrows x$.
Let R be a binary relation on process terms defined as follows:

$$R = \{(x + \tilde{\tilde{\delta}}, x) \mid x \in P\}$$

The relation $R \cup \mathcal{I}$ is a bisimulation relation.

G.9 Axiom A7SR

$$\tilde{\tilde{\delta}} \cdot x = \tilde{\tilde{\delta}}$$

We need to prove, $\tilde{\tilde{\delta}} \cdot x \leftrightarrows \tilde{\tilde{\delta}}$.
Let R be a binary relation on process terms defined as follows:

$$R = \{(\tilde{\tilde{\delta}} \cdot x, \tilde{\tilde{\delta}}) \mid x \in P\}$$

The relation $R \cup \mathcal{I}$ is a bisimulation relation.

G.10 Axiom NE1

$$x + \perp = \perp$$

We need to prove, $x + \perp \leftrightarroweq \perp$.
Let R be a binary relation on process terms defined as follows:

$$R = \{(x + \perp, \perp) \mid x \in P\}$$

The relation $R \cup \mathcal{I}$ is a bisimulation relation.

G.11 Axiom NE2

$$\perp \cdot x = \perp$$

We need to prove, $\perp \cdot x \leftrightarroweq \perp$.
Let R be a binary relation on process terms defined as follows:

$$R = \{(\perp \cdot x, \perp) \mid x \in P\}$$

The relation $R \cup \mathcal{I}$ is a bisimulation relation.

G.12 Axiom NE3SR

$$\tilde{\tilde{a}} \cdot \perp = \tilde{\tilde{\delta}}$$

We need to prove, $\tilde{\tilde{a}} \cdot \perp \leftrightarroweq \tilde{\tilde{\delta}}$.
Let R be a binary relation on process terms defined as follows:

$$R = \{(\tilde{\tilde{a}} \cdot \perp, \tilde{\tilde{\delta}}), (\tilde{\tilde{\delta}}, \tilde{\tilde{a}} \cdot \perp) \mid a \in A\}$$

The relation $R \cup \mathcal{I}$ is a bisimulation relation.

G.13 Axiom SRT1

$$\sigma^0_{\mathsf{rel}}(x) = x$$

We need to prove, $\sigma^0_{\mathsf{rel}}(x) \leftrightarroweq x$.
Let R be a binary relation on process terms defined as follows:

$$R = \{(\sigma^0_{\mathsf{rel}}(x), x) \mid x \in P\}$$

Then $R \cup \mathcal{I}$ is a bisimulation relation that witnesses $\sigma^0_{\mathsf{rel}}(x) \leftrightarroweq x$.

G.14 Axiom SRT2

$$\sigma^{v}_{\mathsf{rel}}(\sigma^{u}_{\mathsf{rel}}(x)) = \sigma^{v+u}_{\mathsf{rel}}(x)$$

where $v, u \geq 0$.

We need to prove, $\sigma^{v}_{\mathsf{rel}}(\sigma^{u}_{\mathsf{rel}}(x)) \leftrightarroweq \sigma^{v+u}_{\mathsf{rel}}(x)$.

We prove this axiom in four steps:

Case $v = 0, u = 0$

Proof Trivial using Axiom SRT1.

Case $v = 0, u > 0$

Proof Trivial using Axiom SRT1.

Case $v > 0, u = 0$

Proof Trivial using Axiom SRT1.

Case $v > 0, u > 0$

Let R be a binary relation on process terms defined as follows:

$$R = \quad \{ \quad (\sigma^{t}_{\mathsf{rel}}(\sigma^{u}_{\mathsf{rel}}(x)), \sigma^{t+u}_{\mathsf{rel}}(x)), \mid x \in P, 0 < t \leq v\}$$

We prove that the relation $R \cup \mathcal{I}$ satisfies all conditions of bisimulation.

For all $a \in A, r > 0, x, y \in P$, the following holds:

1. $$\langle \sigma^{t}_{\mathsf{rel}}(\sigma^{u}_{\mathsf{rel}}(x))\rangle \xrightarrow{a} \langle y\rangle \implies \begin{array}{l}\exists z' \in P : \langle \sigma^{t+u}_{\mathsf{rel}}(x)\rangle \xrightarrow{a} \langle z'\rangle \\ \text{and } (p, z') \in R \cup \mathcal{I}.\end{array}$$

   Suppose,
   $$\langle \sigma^{t}_{\mathsf{rel}}(\sigma^{u}_{\mathsf{rel}}(x))\rangle \xrightarrow{a} \langle y\rangle$$

   A process term with relative delay operator with duration greater than 0 cannot perform an action step. Hence our supposition doesn't hold.

2. $$\langle \sigma^{t+u}_{\mathsf{rel}}(x)\rangle \xrightarrow{a} \langle y\rangle \implies \begin{array}{l}\exists z' \in P : \langle \sigma^{t}_{\mathsf{rel}}(\sigma^{u}_{\mathsf{rel}}(x))\rangle \xrightarrow{a} \langle z'\rangle \\ \text{and } (p, z') \in R \cup \mathcal{I}.\end{array}$$

   Suppose,
   $$\langle \sigma^{t+u}_{\mathsf{rel}}(x)\rangle \xrightarrow{a} \langle y\rangle$$

   A process term with relative delay operator with duration greater than 0 cannot perform an action step. Hence our supposition doesn't hold.

3. $$\langle \sigma^{t}_{\mathsf{rel}}(\sigma^{u}_{\mathsf{rel}}(x))\rangle \overset{r}{\longmapsto} \langle y\rangle \implies \begin{array}{l}\exists z' \in P : \langle \sigma^{t+u}_{\mathsf{rel}}(x)\rangle \overset{r}{\longmapsto} \langle z'\rangle \\ \text{and } (p, z') \in R \cup \mathcal{I}.\end{array}$$

   Suppose,
   $$\langle \sigma^{t}_{\mathsf{rel}}(\sigma^{u}_{\mathsf{rel}}(x))\rangle \overset{r}{\longmapsto} \langle y\rangle \tag{536}$$

   We distinguish between three cases for different values of r.

(a) Case $r < t$

Let $t = r + r_1$ for some r_1 with $0 < r_1 < t$.
Then Transition 536 is derived from Rule P1-9 and $y = \sigma_{\mathsf{rel}}^{r_1}(\sigma_{\mathsf{rel}}^{u}(x))$.
Rewriting Transition 536:

$$\langle \sigma_{\mathsf{rel}}^{r+r_1}(\sigma_{\mathsf{rel}}^{u}(x)) \rangle \xmapsto{r} \langle \sigma_{\mathsf{rel}}^{r_1}(\sigma_{\mathsf{rel}}^{u}(x)) \rangle \tag{537}$$

By Rule P1-9 the following can be derived:

$$\langle \sigma_{\mathsf{rel}}^{r+r_1+u}(x) \rangle \xmapsto{r} \langle \sigma_{\mathsf{rel}}^{r_1+u}(x) \rangle \tag{538}$$

Consider the target process terms in Transitions 537 and 538. The pair $(\sigma_{\mathsf{rel}}^{r_1}(\sigma_{\mathsf{rel}}^{u}(x)), \sigma_{\mathsf{rel}}^{r_1+u}(x))$, where $0 < r_1 < t$ is in R.

(b) Case $r = t$

Then Transition 536 is derived from Rule P1-10. Then $y = \sigma_{\mathsf{rel}}^{u}(x)$.
Rewriting Transition 536:

$$\langle \sigma_{\mathsf{rel}}^{t}(\sigma_{\mathsf{rel}}^{u}(x)) \rangle \xmapsto{t} \langle \sigma_{\mathsf{rel}}^{u}(x) \rangle \tag{539}$$

By Rule P1-9 the following can be derived:

$$\langle \sigma_{\mathsf{rel}}^{t+u}(x) \rangle \xmapsto{t} \langle \sigma_{\mathsf{rel}}^{u}(x) \rangle \tag{540}$$

Consider the target process terms in Transitions 539 and 540. The pair $(\sigma_{\mathsf{rel}}^{u}(x), \sigma_{\mathsf{rel}}^{u}(x))$ is in $\mathcal{I}$.

(c) Case $r > t$

Let $r = t + s$, for some $s > 0$. Rewriting Transition 536,

$$\langle \sigma_{\mathsf{rel}}^{t}(\sigma_{\mathsf{rel}}^{u}(x)) \rangle \xmapsto{t+s} \langle y \rangle \tag{541}$$

The above transition can only be derived from Rule P1-11. From the premise of the rule, the following holds:

$$\langle \sigma_{\mathsf{rel}}^{u}(x) \rangle \xmapsto{s} \langle y \rangle \tag{542}$$

We distinguish between three cases depending on different values of the duration s of the time step.

i. Case $s < u$

Let $u = s + s_1$, for some s_1 with $0 < s_1 < s$.
Then Transition 542 can only be derived from Rule P1-9. Then $y = \sigma_{\mathsf{rel}}^{s_1}(x)$. Rewriting Transitions 541 and 542, we get:

$$\langle \sigma_{\mathsf{rel}}^{t}(\sigma_{\mathsf{rel}}^{u}(x)) \rangle \xmapsto{t+s} \langle \sigma_{\mathsf{rel}}^{s_1}(x) \rangle \tag{543}$$
$$\langle \sigma_{\mathsf{rel}}^{u}(x) \rangle \xmapsto{s} \langle \sigma_{\mathsf{rel}}^{s_1}(x) \rangle \tag{544}$$

From Rule P1-9, the following can be derived:

$$\langle \sigma_{\mathsf{rel}}^{t+u}(x) \rangle \xmapsto{t+s} \langle \sigma_{\mathsf{rel}}^{s_1}(x) \rangle \tag{545}$$

Consider the target process terms in Transitions 543 and 545. The pair $(\sigma_{\mathsf{rel}}^{s_1}(x), \sigma_{\mathsf{rel}}^{s_1}(x))$ is in $\mathcal{I}$.

ii. Case $s = u$

Then Transition 542 can only be derived from Rule P1-10. Then $y = x$. Rewriting Transitions 541 and 542, we get:

$$\langle \sigma_{\mathsf{rel}}^{t}(\sigma_{\mathsf{rel}}^{u}(x)) \rangle \xmapsto{t+u} \langle x \rangle \tag{546}$$

$$\langle \sigma_{\mathsf{rel}}^{u}(x) \rangle \xmapsto{u} \langle x \rangle \tag{547}$$

From the premise of Rule P1-10, the following must hold:

$$\langle \texttt{consistent}\ x \rangle$$

Applying Rule P1-10 on process term $\sigma_{\mathsf{rel}}^{t+u}(x)$, the following can be derived:

$$\langle \sigma_{\mathsf{rel}}^{t+u}(x) \rangle \xmapsto{t+u} \langle x \rangle \tag{548}$$

Consider the target process terms in Transitions 546 and 548. The pair (x, x) is in $\mathcal{I}$.

iii. Case $s > u$

Let $s = u + t_1$, for some $t_1 > 0$. Rewriting Transitions 541 and 542, we get:

$$\langle \sigma_{\mathsf{rel}}^{t}(\sigma_{\mathsf{rel}}^{u}(x)) \rangle \xmapsto{t+u+t_1} \langle y \rangle \tag{549}$$

$$\langle \sigma_{\mathsf{rel}}^{u}(x) \rangle \xmapsto{u+t_1} \langle y \rangle \tag{550}$$

Transition 550 can only be derived from Rule P1-11. Then from the premise of the rule the following must hold:

$$\langle x \rangle \xmapsto{t_1} \langle y \rangle \tag{551}$$

Apply Rule P1-11 on the above transition. For any $m > 0$, the following is derivable:

$$\langle \sigma_{\mathsf{rel}}^{m}(x) \rangle \xmapsto{m+t_1} \langle y \rangle$$

In the above transition, m can be $t + u$. Hence, we get:

$$\langle \sigma_{\mathsf{rel}}^{t+u}(x) \rangle \xmapsto{t+u+t_1} \langle y \rangle \tag{552}$$

Consider the target process terms in Transition 549 and Transition 552. The pair (y, y) is in $\mathcal{I}$.

4.
$$\langle \sigma_{\mathsf{rel}}^{t+u}(x)\rangle \xmapsto{r} \langle y\rangle \implies \begin{array}{l}\exists z' \in P : \langle \sigma_{\mathsf{rel}}^{t}(\sigma_{\mathsf{rel}}^{u}(x))\rangle \xmapsto{r} \langle z'\rangle \\ \text{and } (p, z') \in R \cup \mathcal{I}.\end{array}$$

Suppose,

$$\langle \sigma_{\mathsf{rel}}^{t+u}(x)\rangle \xmapsto{r} \langle y\rangle \tag{553}$$

We distinguish between three cases for different values of r.

(a) Case $r < (t + u)$

Again we distinguish between three cases:

i. *Case* $r < t$
Let $t = r + r_1$, for some r_1 such that, $0 < r_1 < t$.
Then Transition 553 can only be derived from Rule P1-9. Then $y = \sigma_{\mathsf{rel}}^{r_1+u}(x)$. Rewriting Transition 553, we get:

$$\langle \sigma_{\mathsf{rel}}^{r+r_1+u}(x)\rangle \xmapsto{r} \langle \sigma_{\mathsf{rel}}^{r_1+u}(x)\rangle \tag{554}$$

Then from Rule P1-9, the following can be derived:

$$\langle \sigma_{\mathsf{rel}}^{r+r_1}(\sigma_{\mathsf{rel}}^{u}(x))\rangle \xmapsto{r} \langle \sigma_{\mathsf{rel}}^{r_1}(\sigma_{\mathsf{rel}}^{u}(x))\rangle \tag{555}$$

Consider the target process terms in Transitions 554 and 555. For $0 < r_1 < t$, the pair $(\sigma_{\mathsf{rel}}^{r_1}(\sigma_{\mathsf{rel}}^{u}(x)), \sigma_{\mathsf{rel}}^{r_1+u}(x))$ is in R.

ii. *Case* $r = t$
Then Transition 553 can only be derived from Rule P1-9. Then $y = \sigma_{\mathsf{rel}}^{u}(x)$. Rewriting Transition 553, we get:

$$\langle \sigma_{\mathsf{rel}}^{t+u}(x)\rangle \xmapsto{t} \langle \sigma_{\mathsf{rel}}^{u}(x)\rangle \tag{556}$$

From Rule P1-10, the following can be derived:

$$\langle \sigma_{\mathsf{rel}}^{t}(\sigma_{\mathsf{rel}}^{u}(x))\rangle \xmapsto{t} \langle \sigma_{\mathsf{rel}}^{u}(x)\rangle \tag{557}$$

Consider the target process terms in Transitions 556 and 557. The pair $(\sigma_{\mathsf{rel}}^{u}(x), \sigma_{\mathsf{rel}}^{u}(x))$ is in $\mathcal{I}$.

iii. *Case* $r > t$
Let $r = t + s$ for some $s > 0$.
Note that $s < u$ because of our assumption that $r < (t+u)$. Let $u = s + s_1$ for some s_1 such that $0 < s_1 < u$.
Rewriting Transition 553, we get:

$$\langle \sigma_{\mathsf{rel}}^{t+s+s_1}(x)\rangle \xmapsto{t+s} \langle \sigma_{\mathsf{rel}}^{s_1}(x)\rangle \tag{558}$$

By Rule P1-9, the following can be derived:

$$\langle \sigma_{\mathsf{rel}}^{s+s_1}(x)\rangle \xmapsto{s} \langle \sigma_{\mathsf{rel}}^{s_1}(x)\rangle \tag{559}$$

Apply Rule P1-11 on the above transition. We get:

$$\langle \sigma^t_{\mathsf{rel}}(\sigma^{s+s_1}_{\mathsf{rel}}(x))\rangle \xmapsto{t+s} \langle \sigma^{s_1}_{\mathsf{rel}}(x)\rangle \tag{560}$$

Consider the target process terms in Transitions 558 and 560. The pair $(\sigma^{s_1}_{\mathsf{rel}}(x), \sigma^{s_1}_{\mathsf{rel}}(x))$ is in $\mathcal{I}$.

(b) Case $r = (t + u)$

Then Transition 553 can only be derived from Rule P1-10 and $y = x$. Rewriting Transition 553, we get:

$$\langle \sigma^{t+u}_{\mathsf{rel}}(x)\rangle \xmapsto{t+u} \langle x\rangle \tag{561}$$

From the premise of the rule, the following holds:

$$\langle \texttt{consistent}\ x\rangle$$

Apply Rule P1-10 on the above predicate, we get:

$$\langle \sigma^{u}_{\mathsf{rel}}(x)\rangle \xmapsto{u} \langle x\rangle \tag{562}$$

Apply Rule P1-11 on the above transition. We get:

$$\langle \sigma^t_{\mathsf{rel}}(\sigma^{u}_{\mathsf{rel}}(x))\rangle \xmapsto{t+u} \langle x\rangle \tag{563}$$

Consider the target process terms in Transitions 561 and 563. The pair (x, x) is in $\mathcal{I}$.

(c) Case $r > (t + u)$

Let $r = t+u+t_1$, for some $t_1 > 0$. Rewriting Transition 553, we get:

$$\langle \sigma^{t+u}_{\mathsf{rel}}(x)\rangle \xmapsto{t+u+t_1} \langle y\rangle \tag{564}$$

From the premise of the rule the following must hold:

$$\langle x\rangle \xmapsto{t_1} \langle y\rangle \tag{565}$$

Apply Rule P1-11 on the above transition. We get:

$$\langle \sigma^{u}_{\mathsf{rel}}(x)\rangle \xmapsto{u+t_1} \langle y\rangle \tag{566}$$

Again apply Rule P1-11 on the above transition. We get:

$$\langle \sigma^t_{\mathsf{rel}}(\sigma^{u}_{\mathsf{rel}}(x))\rangle \xmapsto{t+u+t_1} \langle y\rangle \tag{567}$$

Consider the target process terms in Transitions 564 and 567. The pair (y, y) is in $\mathcal{I}$.

5.

$$\langle \sigma^t_{\mathsf{rel}}(\sigma^{u}_{\mathsf{rel}}(x))\rangle \xrightarrow{a} \surd \iff \langle \sigma^{t+u}_{\mathsf{rel}}(x)\rangle \xrightarrow{a} \surd$$

Trivial. Both process terms cannot perform an action.

6.

$$\langle \texttt{consistent}\ \sigma^t_{\mathsf{rel}}(\sigma^{u}_{\mathsf{rel}}(x))\rangle \iff \langle \texttt{consistent}\ \sigma^{t+u}_{\mathsf{rel}}(x)\rangle$$

Trivial. Both are consistent.

G.15 Axiom SRTD (Conditional Time Determinism)

$$\begin{array}{ll}\sigma^v_{\mathsf{rel}}(x) + \sigma^v_{\mathsf{rel}}(y) \leftrightarrows \sigma^v_{\mathsf{rel}}(x+y) & (SRT3) \\ \text{where } \langle \texttt{consistent } x\rangle \wedge \langle \texttt{consistent } y\rangle & \end{array}$$

where $v \geq 0$.

Proof

We prove the soundness of Axiom SRTD in two steps.

Case $v = 0$

By Axiom SRT1, we know that for any process term x,

$$\sigma^0_{\mathsf{rel}}(x) \leftrightarrows x$$

Since Bisimulation is a congruence therefore, then it becomes trivial to prove that:

$$\sigma^0_{\mathsf{rel}}(x) + \sigma^0_{\mathsf{rel}}(y) \leftrightarrows \sigma^0_{\mathsf{rel}}(x+y)$$

Case $v > 0$

Let R be the following relation:

$$R = \{(\sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y)), \sigma^t_{\mathsf{rel}}(x+y) \mid 0 < t \leq v, x, y \in P\}$$

We prove that $R \cup \mathcal{I}$ is a bisimulation relation:

For all $a \in A, r > 0, z \in P$:

1. $$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y)\rangle \xrightarrow{a} \langle z\rangle \implies \begin{array}{l}\exists z' \in P : \langle \sigma^t_{\mathsf{rel}}(x+y)\rangle \xrightarrow{a} \langle z'\rangle \\ (z, z') \in R \cup \mathcal{I}\end{array}$$

   Trivial.

2. $$\langle \sigma^t_{\mathsf{rel}}(x+y)\rangle \xrightarrow{a} \langle z\rangle \implies \begin{array}{l}\exists z' \in P : \langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y)\rangle \xrightarrow{a} \langle z'\rangle \\ (z', z) \in R \cup \mathcal{I}\end{array}$$

   Trivial.

3. $$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y)\rangle \xrightarrow{a} \surd \iff \langle \sigma^t_{\mathsf{rel}}(x+y)\rangle \xrightarrow{a} \surd$$

   Trivial.

4. $$\langle \texttt{consistent } \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y)\rangle \iff \langle \texttt{consistent } \sigma^t_{\mathsf{rel}}(x+y)\rangle$$

   Trivial.

5.
$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y)\rangle \xmapsto{r} \langle z\rangle \implies \begin{array}{l}\exists z' \in P : \langle \sigma^t_{\mathsf{rel}}(x+y)\rangle \xmapsto{r} \langle z'\rangle \\ (z, z') \in R \cup \mathcal{I}\end{array}$$

Suppose,

$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y)\rangle \xmapsto{r} \langle z\rangle \tag{568}$$

This can be derived from Rules P1-21, P1-22 and P1-23.

(a) Rule P1-21
If Transition 568 is derived from this rule, then

$$z = z_1 + z_2$$

And the following holds:

$$\langle \sigma^t_{\mathsf{rel}}(x)\rangle \xmapsto{r} \langle z_1\rangle \tag{569}$$

$$\langle \sigma^t_{\mathsf{rel}}(y)\rangle \xmapsto{r} \langle z_2\rangle \tag{570}$$

We distinguish between three cases:

i. Case $r < t$

Let $t = r + r_1$. Then both Transitions 569 and 570 are derived from Rule P1-9. Then

$$z_1 = \sigma^{r_1}_{\mathsf{rel}}(x) \text{ and } z_2 = \sigma^{r_1}_{\mathsf{rel}}(y)$$

Rewriting Transition 568:

$$\langle \sigma^{r+r_1}_{\mathsf{rel}}(x) + \sigma^{r+r_1}_{\mathsf{rel}}(y)\rangle \xmapsto{r} \langle \sigma^{r_1}_{\mathsf{rel}}(x) + \sigma^{r_1}_{\mathsf{rel}}(y)\rangle \tag{571}$$

By Rule P1-9, the following can be derived:

$$\langle \sigma^{r+r_1}_{\mathsf{rel}}(x+y)\rangle \xmapsto{r} \langle \sigma^{r_1}_{\mathsf{rel}}(x+y)\rangle \tag{572}$$

ii. Case $r = t$

Then both Transitions 569 an d570 are derived from Rule P1-10. Then

$$z_1 = x \text{ and } z_2 = y$$

From premise of Rule P1-10, the following holds:

$$\langle \texttt{consistent}\ \ x\rangle \text{ and } \langle \texttt{consistent}\ \ y\rangle$$

Apply Rule P1-16 on the above predicates, we get:

$$\langle \texttt{consistent}\ \ x+y\rangle$$

Then Rule P1-10 becomes applicable to derive the following:

$$\langle \sigma^r_{\mathsf{rel}}(x+y)\rangle \xmapsto{r} \langle x+y\rangle \tag{573}$$

iii. Case $r > t$

Let $r = t + t_1$.
Rewriting Transitions 569 and 570:

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xmapsto{t+t_1} \langle z_1 \rangle \tag{574}$$

$$\langle \sigma^t_{\mathsf{rel}}(y) \rangle \xmapsto{t+t_1} \langle z_2 \rangle \tag{575}$$

The above transitions can only be derived from Rule P1-11.
From the premise of the rule, the following holds:

$$\langle x \rangle \xmapsto{t_1} \langle z_1 \rangle \tag{576}$$

$$\langle y \rangle \xmapsto{t_1} \langle z_2 \rangle \tag{577}$$

Apply Rule P1-21 on the above transitions:

$$\langle x + y \rangle \xmapsto{t_1} \langle z_1 + z_2 \rangle \tag{578}$$

Apply Rule P1-11 on the above transitions, we get:

$$\langle \sigma^{t+t_1}_{\mathsf{rel}}(x + y) \rangle \xmapsto{t+t_1} \langle z_1 + z_2 \rangle \tag{579}$$

(b) Rule P1-22
Suppose, Transition 568 (which is repeated below) is derived from this rule.

$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y) \rangle \xmapsto{r} \langle z \rangle \tag{568}$$

Then from the premise of the rule, the following holds:

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xmapsto{r} \langle z \rangle \tag{580}$$

$$\langle \texttt{consistent } \sigma^t_{\mathsf{rel}}(y) \rangle \tag{581}$$

$$\langle \sigma^t_{\mathsf{rel}}(y) \rangle \not\xmapsto{r} \tag{582}$$

Again we distinguish between three cases:

i. Case $r < t$

For $r < t$, Predicate 582 can not be derived. We conclude that Rule P1-22 cannot be used to derive Transition 568 for $r < t$.

ii. Case $r = t$

Then Predicate 582can only be derived if y is inconsistent. I.e.

$$\neg \langle \texttt{consistent } y \rangle$$

The axiom SRTD is conditional and for inconsistent y, we do not need to derive a corresponding transition of $\sigma^t_{\mathsf{rel}}(x + y)$.

iii. Case $r > t$

Let $r = t + t_1$. Rewriting Transitions 580, 581 and 582:

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xmapsto{t+t_1} \langle z \rangle \tag{583}$$
$$\langle \texttt{consistent } \sigma^t_{\mathsf{rel}}(y) \rangle \tag{584}$$
$$\langle \sigma^t_{\mathsf{rel}}(y) \rangle \not\xmapsto{t+t_1} \tag{585}$$

Then Transition 583 can only be derived by Rule P1-11. From the premise of the rule, the following holds:

$$\langle x \rangle \xmapsto{t_1} \langle z \rangle \tag{586}$$

Predicate 585 can hold only if Rule P1-11 is not appllicable on process term $\sigma^t_{\mathsf{rel}}(y)$. Hence, the premise of the rule must not hold. I.e:

$$\langle y \rangle \not\xmapsto{t_1} \tag{587}$$

Since, we are proving the axiom for consistent process terms, hence:

$$\langle \texttt{consistent } y \rangle \tag{588}$$

Apply Rule P1-22 on Transitions 586,587 and 588, we get:

$$\langle x + y \rangle \xmapsto{t_1} \langle z \rangle \tag{589}$$

Apply Rule P1-11 on the above transition, we get:

$$\langle \sigma^t_{\mathsf{rel}}(x + y) \rangle \xmapsto{t+t_1} \langle z \rangle \tag{590}$$

(c) Rule P1-23

Same reasoning as applied for Rule P1-22.

6.
$$\langle \sigma^t_{\mathsf{rel}}(x + y) \rangle \xmapsto{r} \langle z \rangle \implies \begin{array}{l} \exists z' \in P : \langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y) \rangle \xmapsto{r} \langle z' \rangle \\ (z', z) \in R \end{array}$$

Suppose,

$$\langle \sigma^t_{\mathsf{rel}}(x + y) \rangle \xmapsto{r} \langle z \rangle \tag{591}$$

We distinguish between three cases for different values of r.

(a) Case $r < t$
Let $t = r + r_1$. Then Transition 591 is derived from Rule P1-9. Then

$$z = \sigma_{\mathsf{rel}}^{r_1}(x + y)$$

Rewriting Transition 591:

$$\langle \sigma_{\mathsf{rel}}^{r+r_1}(x + y) \rangle \xmapsto{r} \langle \sigma_{\mathsf{rel}}^{r_1}(x + y) \rangle \tag{592}$$

By Rule P1-9, the following can be derived:

$$\langle \sigma_{\mathsf{rel}}^{r+r_1}(x) \rangle \xmapsto{r} \langle \sigma_{\mathsf{rel}}^{r_1}(x) \rangle \tag{593}$$

$$\langle \sigma_{\mathsf{rel}}^{r+r_1}(y) \rangle \xmapsto{r} \langle \sigma_{\mathsf{rel}}^{r_1}(y) \rangle \tag{594}$$

Apply Rule P1-21 on the above transitions, we get:

$$\langle \sigma_{\mathsf{rel}}^{r+r_1}(x) + \sigma_{\mathsf{rel}}^{r+r_1}(y) \rangle \xmapsto{r} \langle \sigma_{\mathsf{rel}}^{r_1}(x) + \sigma_{\mathsf{rel}}^{r_1}(y) \rangle \tag{595}$$

(b) Case $r = t$
Then Transition 591 is derived from Rule P1-10. Then

$$z = x$$

From the premise of the rule, the folllowing holds:

$$\langle \texttt{consistent}\ \ x + y \rangle$$

which implies

$$\langle \texttt{consistent}\ \ x \rangle \text{ and } \langle \texttt{consistent}\ \ y \rangle$$

Then Rule P1-10 becomes applicable to derive the following:

$$\langle \sigma_{\mathsf{rel}}^{r}(x) \rangle \xmapsto{r} \langle x \rangle \tag{596}$$

$$\langle \sigma_{\mathsf{rel}}^{r}(y) \rangle \xmapsto{r} \langle y \rangle \tag{597}$$

Apply Rule P1-21 on the above transitions, we get:

$$\langle \sigma_{\mathsf{rel}}^{r}(x) + \sigma_{\mathsf{rel}}^{r}(y) \rangle \xmapsto{r} \langle x + y \rangle \tag{598}$$

(c) Case $r > t$
Let $r = t + t_1$.
Rewriting Transition 591:

$$\langle \sigma_{\mathsf{rel}}^{t}(x + y) \rangle \xmapsto{t+t_1} \langle z \rangle \tag{599}$$

The above transition can only be derived from Rule P1-11. From the premise of the rule, the following holds:

$$\langle x + y \rangle \xmapsto{t_1} \langle z \rangle \tag{600}$$

Transition 600 can be derived from three rules. Rules P1-21, P1-22 and P1-23.

i. **Rules P1-21**

Then z in Transitions 599 and 600 is as follows:

$$z = z_1 + z_2$$

From the premise of the rule, the following holds:

$$\langle x \rangle \xmapsto{t_1} \langle z_1 \rangle \tag{601}$$

$$\langle y \rangle \xmapsto{t_1} \langle z_2 \rangle \tag{602}$$

Apply Rule P1-11 on the above transitions:

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xmapsto{t+t_1} \langle z_1 \rangle \tag{603}$$

$$\langle \sigma^t_{\mathsf{rel}}(y) \rangle \xmapsto{t+t_1} \langle z_2 \rangle \tag{604}$$

Apply Rule P1-21 on the above transitions:

$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y) \rangle \xmapsto{t+t_1} \langle z_1 + z_2 \rangle \tag{605}$$

ii. **Rules P1-22**

If Transition 600 is derived from this rule, then from the premise of the rule, the following holds:

$$\langle x \rangle \xmapsto{t_1} \langle z_1 \rangle \tag{606}$$

$$\langle \texttt{consistent}\ y \rangle \tag{607}$$

$$\langle y \rangle \not\xmapsto{t_1} \tag{608}$$

Apply Rule P1-11 on Transition 606:

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xmapsto{t+t_1} \langle z_1 \rangle \tag{609}$$

From Predicate 608, Rule P1-11 cannot be applied on $\sigma^t_{\mathsf{rel}}(y)$. Since, this is the only rule allowing a delay of length greater than t, hence the following predicate holds:

$$\langle \sigma^t_{\mathsf{rel}}(y) \rangle \not\xmapsto{t+t_1} \tag{610}$$

From Rule P1-8, the following holds:

$$\langle \texttt{consistent}\ \sigma^t_{\mathsf{rel}}(y) \rangle \tag{611}$$

for $t > 0$.

Apply Rule P1-22 on transitions 609, 610 and 611. We get:

$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y) \rangle \xmapsto{t+t_1} \langle z_1 \rangle \tag{612}$$

iii. **Rules P1-23**

Same reasoning as given for Rules P1-22 applies.

$\boxtimes$

G.16 Axiom SRTD$\perp$

$$\sigma_{\mathsf{rel}}^{u+r}(x) + \sigma_{\mathsf{rel}}^{r}(\perp) = \sigma_{\mathsf{rel}}^{u+r}(x)$$

where $u \geq, r > 0$.

We need to prove, $\sigma_{\mathsf{rel}}^{u+r}(x) + \sigma_{\mathsf{rel}}^{r}(\perp) \leftrightarrows \sigma_{\mathsf{rel}}^{u+r}(x)$.

Let R be a binary relation on process terms defined as follows:

$$R = \quad \{ \quad (\sigma_{\mathsf{rel}}^{u+s}(x) + \sigma_{\mathsf{rel}}^{s}(\perp), \sigma_{\mathsf{rel}}^{u+s}(x)) \mid x, \in P, 0 < s \leq r\}$$

We prove that the relation $R \cup \mathcal{I}$ is a bisimulation relation. Below we prove the conditions that all pairs in R must satisfy in order for $R \cup \mathcal{I}$ to be a bisimulation relation

For all $a \in A$, $t \in R^{>}$,$p \in P$, the following holds:

1. $$\langle \sigma_{\mathsf{rel}}^{u+s}(x) + \sigma_{\mathsf{rel}}^{s}(\perp) \rangle \xrightarrow{a} \langle p \rangle \implies \begin{array}{l} \exists p' \in P : \langle \sigma_{\mathsf{rel}}^{u+s}(x) \rangle \xrightarrow{a} \langle p' \rangle \\ \text{and } (p, p') \in R \cup \mathcal{I} \end{array}$$

   Trivial. The left hand side of the implication does not hold.

2. $$\langle \sigma_{\mathsf{rel}}^{u+s}(x) \rangle \xrightarrow{a} \langle p \rangle \implies \begin{array}{l} \exists p' \in P : \langle \sigma_{\mathsf{rel}}^{u+s}(x) + \sigma_{\mathsf{rel}}^{s}(\perp) \rangle \xrightarrow{a} \langle p' \rangle \\ \text{and } (p', p) \in R \cup \mathcal{I} \end{array}$$

   Trivial. The left hand side of the implication does not hold.

3. $$\langle \sigma_{\mathsf{rel}}^{u+s}(x) + \sigma_{\mathsf{rel}}^{s}(\perp) \rangle \overset{t}{\longmapsto} \langle p \rangle \implies \begin{array}{l} \exists p' \in P : \langle \sigma_{\mathsf{rel}}^{u+s}(x) \rangle \overset{t}{\longmapsto} \langle p' \rangle \\ \text{and } (p, p') \in R \cup \mathcal{I} \end{array}$$

   Suppose,

   $$\langle \sigma_{\mathsf{rel}}^{u+s}(x) + \sigma_{\mathsf{rel}}^{s}(\perp) \rangle \overset{t}{\longmapsto} \langle p \rangle \tag{613}$$

   This can only be derived from three rules.

   (a) Rule P1-21

   Then $p = p_1 + p_2$.
   Rewriting Transition 613:

   $$\langle \sigma_{\mathsf{rel}}^{u+s}(x) + \sigma_{\mathsf{rel}}^{s}(\perp) \rangle \overset{t}{\longmapsto} \langle p_1 + p_2 \rangle \tag{614}$$

   From the premise of the rule, the following must hold:

   $$\langle \sigma_{\mathsf{rel}}^{u+s}(x) \rangle \overset{t}{\longmapsto} \langle p_1 \rangle \tag{615}$$

   $$\langle \sigma_{\mathsf{rel}}^{s}(\perp) \rangle \overset{t}{\longmapsto} \langle p_2 \rangle \tag{616}$$

   We distinguish between three cases for different values of t.

i. Case $t < s$

Let $s = t + t_1$, for some $0 < t_1 < s$.
Then Transitions 615 and 616 are derived from Rule P1-9. And,

$$p_1 = \sigma_{\mathsf{rel}}^{u+t_1}(x) \text{ and } p_2 = \sigma_{\mathsf{rel}}^{t_1}(\perp)$$

Rewriting Transition 614:

$$\langle \sigma_{\mathsf{rel}}^{u+t+t_1}(x) + \sigma_{\mathsf{rel}}^{t+t_1}(\perp)\rangle \overset{t}{\longmapsto} \langle \sigma_{\mathsf{rel}}^{u+t_1}(x) + \sigma_{\mathsf{rel}}^{t_1}(\perp)\rangle \tag{617}$$

From Rule P1-9, the following can be derived:

$$\langle \sigma_{\mathsf{rel}}^{u+t+t_1}(x)\rangle \overset{t}{\longmapsto} \langle \sigma_{\mathsf{rel}}^{u+t_1}(x)\rangle \tag{618}$$

Consider target process terms in Transition 617 and 618. For $t_1 > 0$, the pair $(\sigma_{\mathsf{rel}}^{u+t_1}(x) + \sigma_{\mathsf{rel}}^{t_1}(\perp), \sigma_{\mathsf{rel}}^{u+t_1}(x))$ is in R.

ii. Case $t = s$

Then 616 must be derived from Rule P1-10 And,

$$p_2 = \perp$$

which is not possible.
Hence, Rule P1-21 cannot be used to derive Transition 613 when $s = t$.

iii. Case $t > s$
Let $t = s + s_1$.
Then Transition 616 can only be derived from Rule P1-11, which requires that $\perp$ can delay for s_1 time units. Again this is impossible. Hence, Rule P1-21 cannot be used to derive Transition 613 when $t > s$.

(b) Rule P1-22

If Transition 613 is derived from this rule, then the following must hold:

$$\langle \sigma_{\mathsf{rel}}^{u+s}(x)\rangle \overset{t}{\longmapsto} \langle p\rangle \tag{619}$$

$$\langle \sigma_{\mathsf{rel}}^{s}(\perp)\rangle \overset{t}{\not\longmapsto} \tag{620}$$

$$\langle \texttt{consistent } \sigma_{\mathsf{rel}}^{s}(\perp)\rangle \tag{621}$$

We distinguish between three cases for different values of t.

i. Case $t < s$
Predicate 620 cannot be derived for $t < s$. Because from Rule P1-9, a process term $\sigma_{\mathsf{rel}}^{s}(z)$ can always delay for a duration shorter than s.

ii. Cases $t = s$
Rewriting Transitions 619 and 620.

$$\langle \sigma_{\mathsf{rel}}^{u+s}(x) \rangle \xmapsto{s} \langle p \rangle \tag{622}$$

$$\langle \sigma_{\mathsf{rel}}^{s}(\perp) \rangle \not\xmapsto{s} \tag{623}$$

The Transition 622 proves that a transition corresponding to Transition 613 holds for $\sigma_{\mathsf{rel}}^{u+s}(x)$.
The pair (p, p) is in $R \cup \mathcal{I}$

iii. Cases $t > s$

Reasoning is Similar to above case.

(c) Rule P1-23
Then the following must hold:

$$\langle \sigma_{\mathsf{rel}}^{s}(\perp) \rangle \xmapsto{t} \langle p \rangle \tag{624}$$

$$\langle \sigma_{\mathsf{rel}}^{u+s}(x) \rangle \not\xmapsto{t} \tag{625}$$

$$\langle \texttt{consistent } \sigma_{\mathsf{rel}}^{u+s}(x) \rangle \tag{626}$$

We distinguish between three cases for different values of t.

i. Case $t < s$

Let $s = t + t_1$.
For $s > t$, the Predicate 625 cannot be derived.

ii. Case $s = t$
Transition 624 cannot be derived.
If $u > 0$, then Predicate 625 can also not be derived.

iii. Case $t > s$

Transition 624 cannot be derived.

Hence, we conclude that Rule P1-23 cannot be used to derive Transition 613.

4.
$$\langle \sigma_{\mathsf{rel}}^{u+s}(x) \rangle \xmapsto{t} \langle p \rangle \implies \begin{array}{l} \exists p' \in P : \langle \sigma_{\mathsf{rel}}^{u+s}(x) + \sigma_{\mathsf{rel}}^{s}(\perp) \rangle \xmapsto{t} \langle p' \rangle \\ \text{and } (p', p) \in R \cup \mathcal{I} \end{array}$$

Suppose,

$$\langle \sigma_{\mathsf{rel}}^{u+s}(x) \rangle \xmapsto{t} \langle p \rangle \tag{627}$$

We distinguish between three cases for different values of t.

(a) Case $t < s$

Let $s = t + t_1$, for $t_1 > 0$.
Then Transition 627 must be derived from Rule P1-9 and $p = \sigma_{\mathsf{rel}}^{u+t_1}(x)$.
Rewriting Transition 627:

$$\langle \sigma_{\mathsf{rel}}^{u+t+t_1}(x) \rangle \overset{t}{\longmapsto} \langle \sigma_{\mathsf{rel}}^{u+t_1}(x) \rangle \tag{628}$$

By Rule P1-9, the following can be derived:

$$\langle \sigma_{\mathsf{rel}}^{t+t_1}(\bot) \rangle \overset{t}{\longmapsto} \langle \sigma_{\mathsf{rel}}^{t_1}(\bot) \rangle \tag{629}$$

Apply Rule P1-21 on the above transitions:

$$\langle \sigma_{\mathsf{rel}}^{u+t+t_1}(x) + \sigma_{\mathsf{rel}}^{t+t_1}(\bot) \rangle \overset{t}{\longmapsto} \langle \sigma_{\mathsf{rel}}^{u+t_1}(x) + \sigma_{\mathsf{rel}}^{t_1}(\bot) \rangle \tag{630}$$

Consider target process terms in Transition 628 and 630. For $t_1 > 0$, the pair $(\sigma_{\mathsf{rel}}^{u+t_1}(x) + \sigma_{\mathsf{rel}}^{t_1}(\bot), \sigma_{\mathsf{rel}}^{u+t_1}(x))$ is in R.

(b) Case $t = s$

Rewriting Transition 627:

$$\langle \sigma_{\mathsf{rel}}^{u+s}(x) \rangle \overset{s}{\longmapsto} \langle p \rangle \tag{631}$$

When $s = t$, the following predicate holds:

$$\langle \sigma_{\mathsf{rel}}^{s}(\bot) \rangle \overset{s}{\not\longmapsto} \tag{632}$$

And also:

$$\langle \texttt{consistent } \sigma_{\mathsf{rel}}^{s}(\bot) \rangle \tag{633}$$

Apply Rule P1-22 on the above transitions:

$$\langle \sigma_{\mathsf{rel}}^{u+s}(x) + \sigma_{\mathsf{rel}}^{s}(\bot) \rangle \overset{s}{\longmapsto} \langle p \rangle \tag{634}$$

Consider target process terms in Transition 627 and 634. The pair (p, p) is in $\mathcal{I}$.

(c) Case $t > s$

Let $t = s + s_1$, for some $s_1 > 0$
Rewriting Transition 627:

$$\langle \sigma_{\mathsf{rel}}^{u+s}(x) \rangle \overset{s+s_1}{\longmapsto} \langle p \rangle \tag{635}$$

Let us consider $\sigma_{\mathsf{rel}}^{s}(\bot)$. The Transition

$$\langle \sigma_{\mathsf{rel}}^{s}(\bot) \rangle \overset{s+s_1}{\longmapsto} \langle p' \rangle$$

cannot be derived for any process term p'.
Hence, the following predicate holds:

$$\langle \sigma^{s}_{\mathsf{rel}}(\perp) \rangle \not\xmapsto{s+s_1} \tag{636}$$

Also we know:

$$\langle \texttt{consistent } \sigma^{s}_{\mathsf{rel}}(\perp) \rangle$$

Apply Rule P1-22:

$$\langle \sigma^{u+s}_{\mathsf{rel}}(x) + \sigma^{s}_{\mathsf{rel}}(\perp) \rangle \xmapsto{s+s_1} \langle p \rangle \tag{637}$$

Consider the target process terms in Transitions 627 and 637. The pair (p, p) is in $\mathcal{I}$.

5.
$$\langle \sigma^{u+s}_{\mathsf{rel}}(x) + \sigma^{s}_{\mathsf{rel}}(\perp) \rangle \xrightarrow{a} \surd \iff \langle \sigma^{u+s}_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \surd$$

Trivial.

6.
$$\langle \texttt{consistent } \sigma^{u+s}_{\mathsf{rel}}(x) + \sigma^{s}_{\mathsf{rel}}(\perp) \rangle \iff \langle \texttt{consistent } \sigma^{u+s}_{\mathsf{rel}}(x) \rangle$$

Trivial.

G.17 Axiom SRT4

$$\sigma^{u}_{\mathsf{rel}}(x) \cdot y = \sigma^{u}_{\mathsf{rel}}(x \cdot y)$$

where $u \geq 0$.

We need to prove, $\sigma^{u}_{\mathsf{rel}}(x) \cdot y \leftrightarrows \sigma^{u}_{\mathsf{rel}}(x \cdot y)$.

We do the proof in two steps.

Case $u = 0$

Proof Trivial using Axiom SRT1 and the fact that bisimulation is a congruence.

Case $u > 0$

Let R be a binary relation on process terms defined as follows:

$$R = \quad \{ \quad (\sigma^{t}_{\mathsf{rel}}(x) \cdot y, \sigma^{t}_{\mathsf{rel}}(x \cdot y)) \mid x, y \in P, 0 < t \leq u\}$$

For all $x, y, p \in P$, $r > 0$, $a \in A$, the following holds:

1.
$$\langle \sigma^{t}_{\mathsf{rel}}(x) \cdot y \rangle \xrightarrow{a} \langle p \rangle \implies \exists z \in P : \langle \sigma^{t}_{\mathsf{rel}}(x \cdot y) \rangle \xrightarrow{a} \langle z \rangle \text{ and } (p, z) \in R \cup \mathcal{I}.$$

Suppose,

$$\langle \sigma^t_{\mathsf{rel}}(x) \cdot y \rangle \xrightarrow{a} \langle p \rangle \tag{638}$$

The above action step can only be derived from Rule P1-13 or 14. We discuss the two cases one by one:

(a) Rule P1-13

If Transition 638 is derived from this rule, then for some process term p', $p = p' \cdot y$. And from the premise of the rule, the following must be derivable,

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \langle p' \rangle \tag{639}$$

An action step for operator σ^t_{rel} with $t > 0$ cannot be derived from any rules. Hence we conclude that Rule P1-13 cannot be used to derive Transition 638.

(b) Rule P1-14

If Transition 638 is derived from this rule, then, $p = y$. And from the premise of the rule, the following must be derivable,

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \surd \tag{640}$$

A termination step for operator σ^t_{rel} with $t > 0$ cannot be derived from any rules. Hence we conclude that Rule P1-14 cannot be used to derive Transition 638.

Transition 638 cannot be derived from any rules. Since the left hand side of the implication does not hold, therefore the implication holds.

2.

$$\langle \sigma^t_{\mathsf{rel}}(x \cdot y) \rangle \xrightarrow{a} \langle p \rangle \implies \exists z \in P : \langle \sigma^t_{\mathsf{rel}}(x) \cdot y \rangle \xrightarrow{a} \langle z \rangle \text{ and } (z, p) \in R \cup \mathcal{I}.$$

Suppose,

$$\langle \sigma^t_{\mathsf{rel}}(x) \cdot y \rangle \xrightarrow{a} \langle p \rangle \tag{641}$$

An action step for operator σ^t_{rel} with $t > 0$ cannot be derived from any rules. Hence our supposition is wrong.

3.

$$\langle \sigma^t_{\mathsf{rel}}(x) \cdot y \rangle \xmapsto{r} \langle p \rangle \implies \exists z \in P : \langle \sigma^t_{\mathsf{rel}}(x \cdot y) \rangle \xmapsto{r} \langle z \rangle \text{ and } (p, z) \in R \cup \mathcal{I}.$$

Suppose,

$$\langle \sigma^t_{\mathsf{rel}}(x) \cdot y \rangle \xmapsto{r} \langle p \rangle \tag{642}$$

The above time step can only be derived from Rule P1-15. Then for some process term p', $p = p' \cdot y$. Rewriting Transition 642:

$$\langle \sigma^t_{\mathsf{rel}}(x) \cdot y \rangle \xrightarrow{r}\!\!\!\!\!\mapsto \langle p' \cdot y \rangle \tag{643}$$

From the premise of the rule the following holds:

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xrightarrow{r}\!\!\!\!\!\mapsto \langle p' \rangle \tag{644}$$

We distinguish between three cases for different values of r:

(a) <u>Case $r < t$</u>

Let $t = r + r_1$, for some $r_1 > 0$.
Then Transition 644 can only be derived from Rule P1-9. From the rule, we have $p' = \sigma^{r_1}_{\mathsf{rel}}(x)$. Rewriting Transitions 643 and 644:

$$\langle \sigma^{r+r_1}_{\mathsf{rel}}(x) \cdot y \rangle \xrightarrow{r}\!\!\!\!\!\mapsto \langle \sigma^{r_1}_{\mathsf{rel}}(x) \cdot y \rangle \tag{645}$$

$$\langle \sigma^{r+r_1}_{\mathsf{rel}}(x) \rangle \xrightarrow{r}\!\!\!\!\!\mapsto \langle \sigma^{r_1}_{\mathsf{rel}}(x) \rangle \tag{646}$$

From Rule P1-9, the following can be

$$\langle \sigma^{r+r_1}_{\mathsf{rel}}(x \cdot y) \rangle \xrightarrow{r}\!\!\!\!\!\mapsto \langle \sigma^{r_1}_{\mathsf{rel}}(x \cdot y) \rangle \tag{647}$$

Consider the target process terms in Transitions 645 and 647. For $0 < r_1 < t$, the pair $(\sigma^{r_1}_{\mathsf{rel}}(x) \cdot y, \sigma^{r_1}_{\mathsf{rel}}(x \cdot y))$ is in R.

(b) <u>Case $r = t$</u>
Then Transition 644 can only be derived from Rule P1-10. From the rule, we have $p' = x$. Rewriting Transitions 643 and 644:

$$\langle \sigma^t_{\mathsf{rel}}(x) \cdot y \rangle \xrightarrow{t}\!\!\!\!\!\mapsto \langle x \cdot y \rangle \tag{648}$$

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xrightarrow{t}\!\!\!\!\!\mapsto \langle x \rangle \tag{649}$$

From the premise of Rule P1-10, the following must hold:

$$\langle \texttt{consistent}\ x \rangle$$

Apply Rule P1-12 on the above predicate, we get:

$$\langle \texttt{consistent}\ x \cdot y \rangle \tag{650}$$

Apply Rule P1-10 on process term $\sigma^t_{\mathsf{rel}}(x \cdot y)$, we get:

$$\langle \sigma^t_{\mathsf{rel}}(x \cdot y) \rangle \xrightarrow{t}\!\!\!\!\!\mapsto \langle x \cdot y \rangle \tag{651}$$

Consider the target process terms in Transitions 648 and 651. The pair $(x \cdot y, x \cdot y)$ is in $\mathcal{I}$.

(c) Case $r > t$

Let $r = t + v$, for some $v > 0$.
Rewriting Transitions 643 and 644:

$$\langle \sigma^t_{\mathsf{rel}}(x) \cdot y \rangle \xmapsto{t+v} \langle p' \cdot y \rangle \tag{652}$$

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xmapsto{t+v} \langle p' \rangle \tag{653}$$

Transition 653 can only be derived from Rule P1-11. Then from the premise of the rule the following holds:

$$\langle x \rangle \xmapsto{v} \langle p' \rangle \tag{654}$$

Apply Rule P1-15 on the above transition, we get:

$$\langle x \cdot y \rangle \xmapsto{v} \langle p' \cdot y \rangle \tag{655}$$

Apply Rule P1-11 on the above transition, we get:

$$\langle \sigma^t_{\mathsf{rel}}(x \cdot y) \rangle \xmapsto{t+v} \langle p' \cdot y \rangle \tag{656}$$

Consider the target process terms in Transitions 652 and 656. The pair $(p' \cdot y, p' \cdot y)$ is in $\mathcal{I}$.

4.

$$\langle \sigma^t_{\mathsf{rel}}(x \cdot y) \rangle \xmapsto{r} \langle p \rangle \implies \begin{array}{l} \exists z \in P : \langle \sigma^t_{\mathsf{rel}}(x) \cdot y \rangle \xmapsto{r} \langle z \rangle \\ \text{and } (z, p) \in R \cup \mathcal{I}. \end{array}$$

Suppose,

$$\langle \sigma^t_{\mathsf{rel}}(x \cdot y) \rangle \xmapsto{r} \langle p \rangle \tag{657}$$

We distinguish between three cases for different values of r.

(a) Case $r < t$

Let $t = r + r_1$, for some $r_1 < t$.
Then Transition 657 can only be derived from Rule P1-9. From the rule, we have $p = \sigma^{r_1}_{\mathsf{rel}}(x \cdot y)$. Rewriting Transition 657:

$$\langle \sigma^{r+r_1}_{\mathsf{rel}}(x \cdot y) \rangle \xmapsto{r} \langle \sigma^{r_1}_{\mathsf{rel}}(x \cdot y) \rangle \tag{658}$$

From Rule P1-9, the following can be derived:

$$\langle \sigma^{r+r_1}_{\mathsf{rel}}(x) \rangle \xmapsto{r} \langle \sigma^{r_1}_{\mathsf{rel}}(x) \rangle$$

Apply Rule P1-15 on the above transition. We get:

$$\langle \sigma^{r+r_1}_{\mathsf{rel}}(x) \cdot y \rangle \xmapsto{r} \langle \sigma^{r_1}_{\mathsf{rel}}(x) \cdot y \rangle \tag{659}$$

Consider the target process terms in Transitions 658 and 659. The pair $(\sigma^{r_1}_{\mathsf{rel}}(x) \cdot y, \sigma^{r_1}_{\mathsf{rel}}(x \cdot y))$ is in R.

(b) Case $r = t$
Then Transition 657 can only be derived from Rule P1-10. From the rule, we have $p = x \cdot y$. Rewriting Transition 657:

$$\langle \sigma^t_{\mathsf{rel}}(x \cdot y) \rangle \xmapsto{t} \langle x \cdot y \rangle \tag{660}$$

The above time step can only be derived from Rule P1-10. From the premise of the rule,

$$\langle \texttt{consistent}\ x \cdot y \rangle$$

which can only be derived from Rule P1-12. Then the following must hold:

$$\langle \texttt{consistent}\ x \rangle$$

Apply Rule P1-10 on the above predicate, we get:

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xmapsto{t} \langle x \rangle \tag{661}$$

Apply Rule P1-15, we get:

$$\langle \sigma^t_{\mathsf{rel}}(x) \cdot y \rangle \xmapsto{t} \langle x \cdot y \rangle \tag{662}$$

Consider the target process terms in Transitions 651 and 662. The pair $(x \cdot y, x \cdot y)$ is in R.

(c) Case $r > t$
Let $r = v + t$, for some $v > 0$.
Then Transition 657 can only be derived from Rule P1-11. Rewriting Transition 657:

$$\langle \sigma^t_{\mathsf{rel}}(x \cdot y) \rangle \xmapsto{t+v} \langle p \rangle \tag{663}$$

From the premise of the rule,

$$\langle x \cdot y \rangle \xmapsto{v} \langle p \rangle \tag{664}$$

The above transition can only be derived from Rule P1-15. Then for some process term p', $p = p' \cdot y$. Rewriting Transitions 663 and 664, we get:

$$\langle \sigma^t_{\mathsf{rel}}(x \cdot y) \rangle \xmapsto{t+v} \langle p' \cdot y \rangle \tag{665}$$
$$\langle x \cdot y \rangle \xmapsto{v} \langle p' \cdot y \rangle \tag{666}$$

From the the premise of the rule,

$$\langle x \rangle \xmapsto{v} \langle p' \rangle \tag{667}$$

Apply Rule P1-11 on the above transition, we get:

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xmapsto{t+v} \langle p' \rangle \tag{668}$$

Apply Rule P1-15 on the above transition, we get:

$$\langle \sigma^t_{\mathsf{rel}}(x) \cdot y \rangle \xmapsto{t+v} \langle p' \cdot y \rangle \tag{669}$$

Consider the target process terms in Transitions 663 and 669. The pair $(p' \cdot y, p' \cdot y)$ is in R.

5.
$$\langle \sigma^t_{\mathsf{rel}}(x \cdot y) \rangle \xrightarrow{a} \surd \iff \langle \sigma^t_{\mathsf{rel}}(x) \cdot y \rangle \xrightarrow{a} \surd$$

6.
$$\langle \texttt{consistent } \sigma^t_{\mathsf{rel}}(x \cdot y) \rangle \iff \langle \texttt{consistent } \sigma^t_{\mathsf{rel}}(x) \cdot y \rangle$$

From Rule P1-8,
$$\langle \texttt{consistent } \sigma^t_{\mathsf{rel}}(x \cdot y) \rangle$$

From Rule P1-12 and Rule P1-8, it can be derived that:

$$\langle \texttt{consistent } \sigma^t_{\mathsf{rel}}(x) \cdot y \rangle$$

G.18 Axiom SRU1

$$\nu_{\mathsf{rel}}(\tilde{\tilde{a}}) = \tilde{\tilde{a}}$$

We need to prove, $\nu_{\mathsf{rel}}(\tilde{\tilde{a}}) \leftrightarrows \tilde{\tilde{a}}$.
Let R be a binary relation on process terms defined as follows:

$$R = \{(\nu_{\mathsf{rel}}(\tilde{\tilde{a}}), \tilde{\tilde{a}}) \mid a \in A\}$$

The relation $R \cup \mathcal{I}$ is a bisimulation relation.

G.19 Axiom SRU2

$$\nu_{\mathsf{rel}}(\sigma^u_{\mathsf{rel}}(x)) = \tilde{\tilde{\delta}}$$

We need to prove, $\nu_{\mathsf{rel}}(\sigma^u_{\mathsf{rel}}(x)) \leftrightarrows \tilde{\tilde{\delta}}$.
Let R be a binary relation on process terms defined as follows:

$$R = \{(\nu_{\mathsf{rel}}(\sigma^u_{\mathsf{rel}}(x)) \mid x \in P, u > 0\}$$

The relation $R \cup \mathcal{I}$ is a bisimulation relation.

G.20 Axiom SRU3

$$\nu_{\mathsf{rel}}(x + y) = \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y)$$

We need to prove, $\nu_{\mathsf{rel}}(x + y) \leftrightarrows \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y)$.
Let R be a binary relation on process terms defined as follows:

$$R = \{(\nu_{\mathsf{rel}}(x + y), \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y)) \mid x, y \in P\}$$

We prove that the relation $R \cup \mathcal{I}$ satisfies all conditions of bisimulation.
For all $a \in A, r > 0, x, y, p \in P$, the following holds:

1. $$\langle \nu_{\mathsf{rel}}(x + y) \rangle \xrightarrow{a} \langle p \rangle \implies \exists z \in P : \langle \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y) \rangle \xrightarrow{a} \langle z \rangle \text{ and } (p, z) \in R.$$

   Suppose,
   $$\langle \nu_{\mathsf{rel}}(x + y) \rangle \xrightarrow{a} \langle p \rangle \tag{670}$$

   The above transition can only be derived from Rule P1-26. Then from the premise the following holds:

   $$\langle x + y \rangle \xrightarrow{a} \langle p \rangle \tag{671}$$

   The above action step can be derived from two rules:

   (a) <u>Rule P1-19</u>
   If Transition 671 is derived from this rule, then from the premise of the rule, we have:

   $$\langle x \rangle \xrightarrow{a} \langle p \rangle \tag{672}$$
   $$\langle \texttt{consistent}\ \ y \rangle \tag{673}$$

   Apply Rule P1-26 on Transition 672, we get:

   $$\langle \nu_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \langle p \rangle \tag{674}$$

   Apply Rule P1-24 on Predicate 673, we get:

   $$\langle \texttt{consistent}\ \ \nu_{\mathsf{rel}}(y) \rangle \tag{675}$$

   Apply Rule P1-19 on Transition 674 and Predicate 675. We get:

   $$\langle \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y) \rangle \xrightarrow{a} \langle p \rangle \tag{676}$$

   Consider the target process terms in Transitions 671 and 676. The pair (p, p) is in R.

(b) Rule P1-20

If Transition 671 is derived from this rule, then from the premise of the rule, we have:

$$\langle y \rangle \xrightarrow{a} \langle p \rangle \tag{677}$$

$$\langle \texttt{consistent}\ x \rangle \tag{678}$$

Reasoning similar to that of Rule P1-19 applies here.

2.

$$\langle \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y) \rangle \xrightarrow{a} \langle p \rangle \implies \begin{array}{l} \exists z \in P : \langle \nu_{\mathsf{rel}}(x + y) \rangle \xrightarrow{a} \langle z \rangle \\ \text{and } (z, p) \in R. \end{array}$$

Suppose,

$$\langle \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y) \rangle \xrightarrow{a} \langle p \rangle \tag{679}$$

The above transition can be derived from Rule P1-19 or Rule P1-20. We discuss them one by one:

(a) Rule P1-19
If Transition 679 is derived from this rule, then from the premise of the rule, we have:

$$\langle \nu_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \langle p \rangle \tag{680}$$

$$\langle \texttt{consistent}\ \nu_{\mathsf{rel}}(y) \rangle \tag{681}$$

Transition 680 can only be derived from Rule P1-26. Predicate 681 can only be derived from Rule P1-24. From their premises, the following holds:

$$\langle x \rangle \xrightarrow{a} \langle p \rangle \tag{682}$$

$$\langle \texttt{consistent}\ y \rangle \tag{683}$$

Apply Rule P1-19 on the above transition and predicate, we get:

$$\langle x + y \rangle \xrightarrow{a} \langle p \rangle \tag{684}$$

Apply Rule P1-26 on the above transition, we get:

$$\langle \nu_{\mathsf{rel}}(x + y) \rangle \xrightarrow{a} \langle p \rangle \tag{685}$$

Consider the target process terms in Transitions 679 and 685. The pair (p, p) is in R.

(b) Rule P1-20
If Transition 679 is derived from this rule, then from the premise of the rule, we have:

$$\langle \nu_{\mathsf{rel}}(y) \rangle \xrightarrow{a} \langle p \rangle \tag{686}$$

$$\langle \texttt{consistent}\ \nu_{\mathsf{rel}}(x) \rangle \tag{687}$$

Reasoning similar to that given for Rule P1-19 applies here.

3.
$$\langle \nu_{\mathsf{rel}}(x+y)\rangle \xmapsto{r} \langle p\rangle \implies \begin{array}{l}\exists z \in P : \langle \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y)\rangle \xmapsto{r} \langle z\rangle \\ \text{and } (p,z) \in R.\end{array}$$

Suppose,
$$\langle \nu_{\mathsf{rel}}(x+y)\rangle \xmapsto{r} \langle p\rangle$$

A time step for now operator can not be derived from any rules. Hence our supposition cannot hold and the implication is trivially satisfied.

4.
$$\langle \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y)\rangle \xmapsto{r} \langle p\rangle \implies \begin{array}{l}\exists z \in P : \langle \nu_{\mathsf{rel}}(x+y)\rangle \xmapsto{r} \langle z\rangle \\ \text{and } (z,p) \in R.\end{array}$$

Suppose,
$$\langle \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y)\rangle \xmapsto{r} \langle p\rangle \tag{688}$$

A time step for an alternative composition can be derived from Rule P1-21, Rule P1-22 or Rule P1-23. We discuss them one by one:

(a) Rule P1-21
If transition 688 is derived from this rule, Then for some process term x_1, y_1, $p = x_1 + y_1$ and from the premise of the rule, the following holds:
$$\langle \nu_{\mathsf{rel}}(x)\rangle \xmapsto{r} \langle x_1\rangle \tag{689}$$
$$\langle \nu_{\mathsf{rel}}(y)\rangle \xmapsto{r} \langle y_1\rangle \tag{690}$$
A time step for now operator can not be derived from any rules. Hence Transitions 689 and 690 are not derivable. We conclude that Rule P1-21 cannot be used to derive Transition 688.

(b) Rule P1-22
If transition 688 is derived from this rule, Then the premise of the rule, the following holds:
$$\langle \nu_{\mathsf{rel}}(x)\rangle \xmapsto{r} \langle p\rangle \tag{691}$$
$$\langle \texttt{consistent}\ \nu_{\mathsf{rel}}(y)\rangle \tag{692}$$
$$\langle \nu_{\mathsf{rel}}(y)\rangle \not\xmapsto{r} \tag{693}$$
A time step for now operator can not be derived from any rules. Hence Transition 691 is not derivable. We conclude that Rule P1-22 cannot be used to derive Transition 688.

(c) Rule P1-23
Similarly, if transition 688 is derived from this rule, Then the premise of the rule, the following holds:
$$\langle \nu_{\mathsf{rel}}(y)\rangle \xmapsto{r} \langle p\rangle \tag{694}$$
$$\langle \texttt{consistent}\ \nu_{\mathsf{rel}}(x)\rangle \tag{695}$$
$$\langle \nu_{\mathsf{rel}}(x)\rangle \not\xmapsto{r} \tag{696}$$

A time step for now operator can not be derived from any rules. Hence Transition 694 is not derivable. We conclude that Rule P1-23 cannot be used to derive Transition 688.

No rules allow derivation of Transition 688. Hence our supposition cannot hold and the implication is trivially satisfied.

5.
$$\langle \nu_{\mathsf{rel}}(x+y) \rangle \xrightarrow{a} \surd \iff \langle \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y) \rangle \xrightarrow{a} \surd$$

<u>Left Implication</u>
Suppose,
$$\langle \nu_{\mathsf{rel}}(x+y) \rangle \xrightarrow{a} \surd \tag{697}$$

The above predicate can only be derived from Rule P1-25. Then from the premise the following holds:

$$\langle x+y \rangle \xrightarrow{a} \surd \tag{698}$$

The above action step can be derived from two rules:

(a) <u>Rule P1-17</u>
If Predicate 698 is derived from this rule, then from the premise of the rule, we have:

$$\langle x \rangle \xrightarrow{a} \surd \tag{699}$$
$$\langle \texttt{consistent}\ y \rangle \tag{700}$$

Apply Rule P1-25 on Predicate 699, we get:

$$\langle \nu_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \surd \tag{701}$$

Apply Rule P1-24 on Predicate 700, we get:

$$\langle \texttt{consistent}\ \nu_{\mathsf{rel}}(y) \rangle \tag{702}$$

Apply Rule P1-17 on Predicate 701 and Predicate 702. We get:

$$\langle \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y) \rangle \xrightarrow{a} \surd \tag{703}$$

(b) <u>Rule P1-18</u>

If Predicate 698 is derived from this rule, then from the premise of the rule, we have:

$$\langle y \rangle \xrightarrow{a} \surd \tag{704}$$
$$\langle \texttt{consistent}\ x \rangle \tag{705}$$

Reasoning similar to that of Rule P1-17 applies here.

Right Implication

Suppose,

$$\langle \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y) \rangle \xrightarrow{a} \surd \tag{706}$$

The above predicate can be derived from Rule P1-17 or Rule P1-18. We discuss them one by one:

(a) Rule P1-17
If Predicate 706 is derived from this rule, then from the premise of the rule, we have:

$$\langle \nu_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \surd \tag{707}$$

$$\langle \texttt{consistent } \nu_{\mathsf{rel}}(y) \rangle \tag{708}$$

Predicate 707 can only be derived from Rule P1-25. Predicate 708 can only be derived from Rule P1-24. From their premises, the following holds:

$$\langle x \rangle \xrightarrow{a} \surd \tag{709}$$

$$\langle \texttt{consistent } y \rangle \tag{710}$$

Apply Rule P1-17 on the above predicates, we get:

$$\langle x + y \rangle \xrightarrow{a} \surd \tag{711}$$

Apply Rule P1-25 on the above predicate, we get:

$$\langle \nu_{\mathsf{rel}}(x + y) \rangle \xrightarrow{a} \surd \tag{712}$$

(b) Rule P1-18
If Predicate 706 is derived from this rule, then from the premise of the rule, we have:

$$\langle \nu_{\mathsf{rel}}(y) \rangle \xrightarrow{a} \surd \tag{713}$$

$$\langle \texttt{consistent } \nu_{\mathsf{rel}}(x) \rangle \tag{714}$$

Reasoning similar to that given for Rule P1-17 applies here.

6.

$$\langle \texttt{consistent } \nu_{\mathsf{rel}}(x + y) \rangle \iff \langle \texttt{consistent } \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y) \rangle$$

Left Implication

Suppose

$$\langle \texttt{consistent } \nu_{\mathsf{rel}}(x + y) \rangle$$

The above predicate is only derivable from Rule P1-24. Then from premise of the rule, the following holds:

$$\langle \texttt{consistent} \ \ x + y \rangle \tag{715}$$

which is only derivable from Rule P1-16. Then from the premise the following holds:

$$\langle \texttt{consistent} \ \ x \rangle \tag{716}$$
$$\langle \texttt{consistent} \ \ y \rangle \tag{717}$$

Apply Rule P1-24 on the above predicates, we get:

$$\langle \texttt{consistent} \ \ \nu_{\mathsf{rel}}(x) \rangle \tag{718}$$
$$\langle \texttt{consistent} \ \ \nu_{\mathsf{rel}}(y) \rangle \tag{719}$$

Apply Rule P1-16 on the above predicates, we get the desired predicate:

$$\langle \texttt{consistent} \ \ \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y) \rangle$$

Right Implication

Suppose,

$$\langle \texttt{consistent} \ \ \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y) \rangle \tag{720}$$

which is only derivable from Rule P1-16. Then from the premise the following holds:

$$\langle \texttt{consistent} \ \ \nu_{\mathsf{rel}}(x) \rangle \tag{721}$$
$$\langle \texttt{consistent} \ \ \nu_{\mathsf{rel}}(y) \rangle \tag{722}$$

The above predicates are only derivable from Rule P1-24. Then from the premise the following must hold:

$$\langle \texttt{consistent} \ \ x \rangle \tag{723}$$
$$\langle \texttt{consistent} \ \ y \rangle \tag{724}$$

Apply Rule P1-16 on the above predicates, we get:

$$\langle \texttt{consistent} \ \ x + y \rangle \tag{725}$$

Apply Rule P1-24 on the above predicate, we get:

$$\langle \texttt{consistent} \ \ \nu_{\mathsf{rel}}(x + y) \rangle \tag{726}$$

G.21 Axiom SRU4

$$\nu_{\mathsf{rel}}(x \cdot y) = \nu_{\mathsf{rel}}(x) \cdot y$$

We need to prove, $\nu_{\mathsf{rel}}(x \cdot y) \leftrightarrows \nu_{\mathsf{rel}}(x) \cdot y$.
Let R be a binary relation on process terms defined as follows:

$$R = \{(\nu_{\mathsf{rel}}(x \cdot y), \nu_{\mathsf{rel}}(x) \cdot y) \mid x, y \in P\}$$

We prove that the relation $R \cup \mathcal{I}$ satisfies all conditions of bisimulation.
For all $a \in A, r > 0, x, y, p \in P$, the following holds:

1. $$\langle \nu_{\mathsf{rel}}(x \cdot y) \rangle \xrightarrow{a} \langle p \rangle \implies \begin{array}{l} \exists z \in P : \langle \nu_{\mathsf{rel}}(x) \cdot y \rangle \xrightarrow{a} \langle z \rangle \\ \text{and } (p, z) \in R. \end{array}$$

   Suppose,
   $$\langle \nu_{\mathsf{rel}}(x \cdot y) \rangle \xrightarrow{a} \langle p \rangle \tag{727}$$

   The above transition can only be derived from Rule P1-26. Then from the premise the following holds:

   $$\langle x \cdot y \rangle \xrightarrow{a} \langle p \rangle \tag{728}$$

   The above action step can be derived from two rules:

   (a) Rule P1-13
   If Transition 728 is derived from this rule, then for some process term p', $p = p' \cdot y$. Rewriting Transition 728:

   $$\langle x \cdot y \rangle \xrightarrow{a} \langle p' \cdot y \rangle \tag{729}$$

   From the premise of the rule, we have:

   $$\langle x \rangle \xrightarrow{a} \langle p' \rangle \tag{730}$$

   Apply Rule P1-26 on Transition 730, we get:

   $$\langle \nu_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \langle p' \rangle \tag{731}$$

   Apply Rule P1-13 on the above transition. We get:

   $$\langle \nu_{\mathsf{rel}}(x) \cdot y \rangle \xrightarrow{a} \langle p' \cdot y \rangle \tag{732}$$

   Consider the target process terms in Transitions 729 and 732. The pair $(p' \cdot y, p' \cdot y)$ is in $\mathcal{I}$.

(b) Rule P1-14

If Transition 728 is derived from this rule, then, $p = y$. Rewriting Transition 728:

$$\langle x \cdot y \rangle \xrightarrow{a} \langle y \rangle \tag{733}$$

From the premise of the rule, we have:

$$\langle x \rangle \xrightarrow{a} \surd \tag{734}$$

$$\langle \texttt{consistent}\ y \rangle \tag{735}$$

Apply Rule P1-25 on Transition 734, we get:

$$\langle \nu_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \surd \tag{736}$$

Apply Rule P1-14 on the above transition making use of predicate 735. We get:

$$\langle \nu_{\mathsf{rel}}(x) \cdot y \rangle \xrightarrow{a} \langle y \rangle \tag{737}$$

Consider the target process terms in Transitions 733 and 737. The pair (y, y) is in $\mathcal{I}$.

2.

$$\langle \nu_{\mathsf{rel}}(x) \cdot y \rangle \xrightarrow{a} \langle p \rangle \implies \exists z \in P : \langle \nu_{\mathsf{rel}}(x \cdot y) \rangle \xrightarrow{a} \langle z \rangle \text{ and } (z, p) \in R.$$

Suppose,

$$\langle \nu_{\mathsf{rel}}(x) \cdot y \rangle \xrightarrow{a} \langle p \rangle \tag{738}$$

The above action step can be derived from two rules:

(a) Rule P1-13

If Transition 738 is derived from this rule, then for some process term p', $p = p' \cdot y$. Rewriting Transition 738:

$$\langle \nu_{\mathsf{rel}}(x) \cdot y \rangle \xrightarrow{a} \langle p' \cdot y \rangle \tag{739}$$

And from the premise of the rule, the following holds:

$$\langle \nu_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \langle p' \rangle \tag{740}$$

The above transition can only be derived from Rule P1-26. Then from the premise the following holds:

$$\langle x \rangle \xrightarrow{a} \langle p' \rangle \tag{741}$$

Apply Rule P1-13 on the above transition, we get:

$$\langle x \cdot y \rangle \xrightarrow{a} \langle p' \cdot y \rangle \tag{742}$$

Again apply Rule P1-26 on Transition 742, we get:

$$\langle \nu_{\mathsf{rel}}(x \cdot y) \rangle \xrightarrow{a} \langle p' \cdot y \rangle \tag{743}$$

Consider the target process term in transitions 739 and 743. The pair $(p' \cdot y, p' \cdot y)$ is in $\mathcal{I}$.

(b) Rule P1-14
If Transition 738 is derived from this rule, then, $p = y$. Rewriting Transition 738:

$$\langle \nu_{\mathsf{rel}}(x) \cdot y \rangle \xrightarrow{a} \langle y \rangle \tag{744}$$

And from the premise of the rule, the following holds:

$$\langle \nu_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \surd \tag{745}$$

$$\langle \texttt{consistent}\ \ y \rangle \tag{746}$$

The Predicate 745 can only be derived from Rule P1-25. Then from the premise the following holds:

$$\langle x \rangle \xrightarrow{a} \surd \tag{747}$$

Apply Rule P1-14 on the above transition using Predicate 746:

$$\langle x \cdot y \rangle \xrightarrow{a} \langle y \rangle \tag{748}$$

Apply Rule P1-26 on the above transition, we get:

$$\langle \nu_{\mathsf{rel}}(x \cdot y) \rangle \xrightarrow{a} \langle y \rangle \tag{749}$$

Consider the target process term in transitions 739 and 749. The pair (y, y) is in $\mathcal{I}$.

3.
$$\langle \nu_{\mathsf{rel}}(x \cdot y) \rangle \overset{r}{\longmapsto} \langle p \rangle \implies \exists z \in P : \langle \nu_{\mathsf{rel}}(x) \cdot y \rangle \overset{r}{\longmapsto} \langle z \rangle \text{ and } (p, z) \in R.$$

Suppose,

$$\langle \nu_{\mathsf{rel}}(x \cdot y) \rangle \overset{r}{\longmapsto} \langle p \rangle$$

A time step for now operator can not be derived from any rules. Hence our supposition cannot hold and the implication is trivially satisfied.

4.
$$\langle \nu_{\mathsf{rel}}(x) \cdot y \rangle \overset{r}{\longmapsto} \langle p \rangle \implies \exists z \in P : \langle \nu_{\mathsf{rel}}(x \cdot y) \rangle \overset{r}{\longmapsto} \langle z \rangle \text{ and } (p, z) \in R.$$

Suppose,

$$\langle \nu_{\mathsf{rel}}(x) \cdot y \rangle \overset{r}{\longmapsto} \langle p \rangle \tag{750}$$

A time step for sequential composition can only be derived from Rule P1-15. The for some process term p', $p = p' \cdot y$. And from the premise the following holds:

$$\langle \nu_{\mathsf{rel}}(x) \rangle \overset{r}{\longmapsto} \langle p' \rangle \tag{751}$$

The above transition cannot be derived as a time step for now operator can not be derived from any rules. Hence the Transition 750 cannot hold and the implication is trivially proved.

5.
$$\langle \nu_{\mathsf{rel}}(x \cdot y)\rangle \xrightarrow{a} \surd \iff \langle \nu_{\mathsf{rel}}(x) \cdot y\rangle \xrightarrow{a} \surd$$

Left Implication

Suppose,
$$\langle \nu_{\mathsf{rel}}(x \cdot y)\rangle \xrightarrow{a} \surd \tag{752}$$

Predicate 752 can only be derived from Rule P1-25. Then from the premise of the rule, the following holds:

$$\langle x \cdot y\rangle \xrightarrow{a} \surd \tag{753}$$

A termination predicate for a sequential composition cannot be derived from any rules. Predicate 753 doesn't hold. hence our assumption predicate 752 doesn't hold.

Right Implication

Suppose,
$$\langle \nu_{\mathsf{rel}}(x) \cdot y\rangle \xrightarrow{a} \surd \tag{754}$$

A termination predicate for a sequential composition cannot be derived from any rules. Hence our assumption predicate 754 doesn't hold.

6.
$$\langle \texttt{consistent}\ \nu_{\mathsf{rel}}(x \cdot y)\rangle \iff \langle \texttt{consistent}\ \nu_{\mathsf{rel}}(x) \cdot y\rangle$$

Left Implication

Suppose

$$\langle \texttt{consistent}\ \nu_{\mathsf{rel}}(x \cdot y)\rangle$$

Only derivable from Rule P1-24. Then from premise of the rule, the following holds:

$$\langle \texttt{consistent}\ x \cdot y\rangle \tag{755}$$

which is only derivable from Rule P1-12. Then from the premise the following holds:

$$\langle \texttt{consistent}\ x\rangle \tag{756}$$

Apply Rule P1-24 on Predicate 756:

$$\langle \texttt{consistent}\ \nu_{\mathsf{rel}}(x)\rangle \tag{757}$$

Again apply Rule P1-12 on Predicate 757, we get the desired predicate:

$$\langle \texttt{consistent}\ \nu_{\mathsf{rel}}(x) \cdot y\rangle \tag{758}$$

Right Implication

Suppose,

$$\langle \texttt{consistent}\ \nu_{\mathsf{rel}}(x)\cdot y\rangle$$

which is only derivable from Rule P1-12. Then from the premise the following must hold:

$$\langle \texttt{consistent}\ \nu_{\mathsf{rel}}(x)\rangle \tag{759}$$

which is only derivable from Rule P1-24. Then from premise of the rule, the following holds:

$$\langle \texttt{consistent}\ x\rangle$$

Apply Rule P1-12 on the above predicate, we get:

$$\langle \texttt{consistent}\ x\cdot y\rangle \tag{760}$$

Apply Rule P1-24 on Predicate 760, we get the desired result:

$$\langle \texttt{consistent}\ \nu_{\mathsf{rel}}(x\cdot y)\rangle \tag{761}$$

G.22 Axiom NESRU

$$\nu_{\mathsf{rel}}(\bot)=\bot$$

We need to prove, $\nu_{\mathsf{rel}}(\bot) \leftrightarrows \bot$.

Let R be a binary relation on process terms defined as follows:

$$R=\{(\nu_{\mathsf{rel}}(\bot),\bot),(\bot,\nu_{\mathsf{rel}}(\bot))\}$$

The proof is trivial and therefore left.

H Soundness Proofs for Proposal 2

Let $\mathcal{I}$ be a binary relation on process terms defined as follows:

$$\mathcal{I} = \{(x, x) \mid x \in P\}$$

It is obvious that $\mathcal{I}$ is a bisimulation relation. We will use the relation $\mathcal{I}$ frequently in the proofs. We prove that the axioms given in Table 14 hold in the semantics given in Section 5.4.

The proofs of the soundness theorem use the following theorems.

H.1 Theorem : Sources of Transitions are consistent

Theorem 14 *For all closed terms p the following hold:*
For all $p', p'' \in P$, $a, b \in A$, $r, s > 0$:

$$(\langle p\rangle \xrightarrow{a} \langle p'\rangle) \vee (\langle p\rangle \overset{r}{\longmapsto} \langle p''\rangle) \vee (\langle p\rangle \xrightarrow{b} \surd) \vee (\langle p\rangle \overset{s}{\longmapsto}{}_{\perp}) \\ \Longrightarrow \langle \texttt{consistent}\ p\rangle$$

Proof We prove the above theorem by structural induction on a process term $p \in P$. The base case of the structural induction comprises of constant process terms, i.e. all undelayable actions in $\mathcal{A}$, the deadlock process term δ and the inconsistent process $\perp$.
Base Case

1. $p = \tilde{\tilde{a}}$.

   From Rule P2-2, $\langle \texttt{consistent}\ \tilde{\tilde{a}}\rangle$. Hence all conditions of the theorem are trivially satisfied.

2. $p = \tilde{\tilde{\delta}}$

   From Rule P2-1, $\langle \texttt{consistent}\ \tilde{\tilde{\tilde{\delta}}}\rangle$. Hence all conditions of the theorem are trivially satisfied.

3. $p = \perp$

   There are no rules for an inconsistent process $\perp$ in the semantics of $BPA^{srt}_{\perp}$. Hence all conditions of the theorem are trivially satisfied (as the left hand sides of the implications do not hold.)

By Induction Hypothesis

1. $p = \sigma^0_{\mathsf{rel}}(x)$, for a closed term x. We show that if p can perform an action or a time step or a termination or a future inconsistency predicate holds for p, then $\langle \texttt{consistent}\ p\rangle$ holds.

(a) *Action Step:*
Suppose,

$$\langle \sigma^0_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \langle p' \rangle$$

It can only be derived from Rule P2-4. From the premise of the rule,

$$\langle x \rangle \xrightarrow{a} \langle p' \rangle$$

By Induction on the above action step, we get:

$$\langle \texttt{consistent} \ \ x \rangle$$

Apply Rule P2-7. We get:

$$\langle \texttt{consistent} \ \ \sigma^0_{\mathsf{rel}}(x) \rangle$$

Hence proved.

(b) *Time Step:*
Suppose,

$$\langle \sigma^0_{\mathsf{rel}}(x) \rangle \overset{r}{\longmapsto} \langle p' \rangle$$

It can only be derived from Rule P2-6. From the premise of the rule,

$$\langle x \rangle \overset{r}{\longmapsto} \langle p' \rangle$$

By Induction on the above action step, we get:

$$\langle \texttt{consistent} \ \ x \rangle$$

Apply Rule P2-7. We get:

$$\langle \texttt{consistent} \ \ \sigma^0_{\mathsf{rel}}(x) \rangle$$

Hence proved.

(c) *Termination Predicate:*
Suppose,

$$\langle \sigma^0_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \surd$$

It can only be derived from Rule P2-5. From the premise of the rule,

$$\langle x \rangle \xrightarrow{a} \surd$$

By Induction on the above action step, we get:

$$\langle \texttt{consistent} \ \ x \rangle$$

Apply Rule P2-7. We get:

$$\langle \texttt{consistent} \ \ \sigma^0_{\mathsf{rel}}(x) \rangle$$

Hence proved.

(d) *A Future Inconsistency Predicate:*
Suppose,

$$\langle \sigma^0_{\mathsf{rel}}(x) \rangle \xmapsto{r} \perp$$

It can only be derived from Rule P2-8. From the premise of the rule,

$$\langle x \rangle \xmapsto{r} \perp$$

By Induction on the above action step, we get:

$$\langle \texttt{consistent}\ \ x \rangle$$

Apply Rule P2-7. We get:

$$\langle \texttt{consistent}\ \ \sigma^0_{\mathsf{rel}}(x) \rangle$$

Hence proved.

2. $p = \sigma^t_{\mathsf{rel}}(x) \qquad t > 0$
From Rule P2-12, for a process term $\sigma^t_{\mathsf{rel}}(x)$, with $t > 0$, the following holds:

$$\langle \texttt{consistent}\ \ \sigma^t_{\mathsf{rel}}(x) \rangle$$

Hence all conditions of the theorem are trivially proved.

3. $p = x \cdot y$.
We prove the four conditions of the theorem one by one.

(a) *Action Step:*
Suppose,

$$\langle x \cdot y \rangle \xrightarrow{a} \langle p' \rangle$$

It can only be derived from Rule P2-15 or Rule P2-16.

- *Rule P2-15*
Then for some process term p'', $p' = p'' \cdot y$. From the premise of the rule,

$$\langle x \rangle \xrightarrow{a} \langle p'' \rangle$$

By Induction on the above action step, we get:

$$\langle \texttt{consistent}\ \ x \rangle$$

Apply Rule P2-18. We get:

$$\langle \texttt{consistent}\ \ x \cdot y \rangle$$

Hence proved.

- *Rule P2-16*
  Then, $p' = y$. From the premise of the rule,

  $$\langle x \rangle \xrightarrow{a} \surd$$

  By Induction on the above predicate, we get:

  $$\langle \texttt{consistent} \ x \rangle$$

  Apply Rule P2-18. We get:

  $$\langle \texttt{consistent} \ x \cdot y \rangle$$

  Hence proved.

(b) *Time Step:*
Suppose,

$$\langle x \cdot y \rangle \overset{r}{\longmapsto} \langle p' \rangle$$

It can only be derived from Rule P2-17. From the premise of the rule,

$$\langle x \rangle \overset{r}{\longmapsto} \langle p' \rangle$$

By Induction on the above time step, we get:

$$\langle \texttt{consistent} \ x \rangle$$

Apply Rule P2-18. We get:

$$\langle \texttt{consistent} \ x \cdot y \rangle$$

Hence proved.

(c) *Termination Predicate:*
Suppose,

$$\langle x \cdot y \rangle \xrightarrow{a} \surd$$

There are no rules to derive a termination predicate for a sequential composition. Hence the left hand side of the implication does not hold and the implication is trivially satisfied.

(d) *A Future Inconsistency Predicate:*
Suppose,

$$\langle x \cdot y \rangle \overset{r}{\longmapsto}_{\perp}$$

It can only be derived from Rule P2-19. From the premise of the rule,

$$\langle x \rangle \overset{r}{\longmapsto}_{\perp}$$

By Induction on the above predicate, we get:

$$\langle \texttt{consistent}\ x \rangle$$

Apply Rule P2-18. We get:

$$\langle \texttt{consistent}\ x \cdot y \rangle$$

Hence proved.

4. $p = x + y$.
   We prove the four conditions of the theorem one by one.

   (a) *Action Step:*
   Suppose,

   $$\langle x + y \rangle \xrightarrow{a} \langle p' \rangle$$

   It can only be derived from Rule P2-20 or Rule P2-21.

   - *Rule P2-20*
     From the premise of the rule,

     $$\langle x \rangle \xrightarrow{a} \langle p' \rangle \tag{762}$$
     $$\langle \texttt{consistent}\ y \rangle \tag{763}$$

     By Induction on Transition 762, we get:

     $$\langle \texttt{consistent}\ x \rangle \tag{764}$$

     Apply Rule P2-27 on Predicates 763 and 764. We get:

     $$\langle \texttt{consistent}\ x + y \rangle$$

     Hence proved.
   - *Rule P2-21*
     From the premise of the rule,

     $$\langle y \rangle \xrightarrow{a} \langle p' \rangle \tag{765}$$
     $$\langle \texttt{consistent}\ x \rangle \tag{766}$$

     By Induction on Transition 765, we get:

     $$\langle \texttt{consistent}\ y \rangle \tag{767}$$

     Apply Rule P2-27 on Predicates 766 and 767. We get:

     $$\langle \texttt{consistent}\ x + y \rangle$$

     Hence proved.

(b) *Time Step:*
Suppose,

$$\langle x + y \rangle \overset{r}{\longmapsto} \langle p' \rangle$$

It can only be derived from Rule P2-24 or Rule P2-25 or Rule P2-26.

- *Rule P2-24*
Then for some process terms x_1, y_1, $p' = x_1 + y_1$. From the premise of the rule the following holds:

$$\langle x \rangle \overset{r}{\longmapsto} \langle x_1 \rangle \tag{768}$$
$$\langle y \rangle \overset{r}{\longmapsto} \langle y_1 \rangle \tag{769}$$

By Induction on the above time steps, we get:

$$\langle \texttt{consistent} \;\; x \rangle$$
$$\langle \texttt{consistent} \;\; y \rangle$$

Apply Rule P2-27 on the above Predicates. We get:

$$\langle \texttt{consistent} \;\; x + y \rangle$$

Hence proved.

- *Rule P2-25*
From the premise of the rule the following holds:

$$\langle x \rangle \overset{r}{\longmapsto} \langle p' \rangle \tag{770}$$
$$\langle \texttt{consistent} \;\; y \rangle \tag{771}$$
$$\langle y \rangle \overset{r}{\not\longmapsto} \tag{772}$$
$$\forall s \leq r, \;\; \langle y \rangle \overset{s}{\not\longmapsto}_{\perp} \tag{773}$$

By Induction on time step 770, we get:

$$\langle \texttt{consistent} \;\; x \rangle \tag{774}$$

Apply Rule P2-27 on Predicates 774 and 771. We get:

$$\langle \texttt{consistent} \;\; x + y \rangle$$

Hence proved.

- *Rule P2-26*
From the premise of the rule the following holds:

$$\langle y \rangle \overset{r}{\longmapsto} \langle p' \rangle \tag{775}$$
$$\langle \texttt{consistent} \;\; x \rangle \tag{776}$$
$$\langle x \rangle \overset{r}{\not\longmapsto} \tag{777}$$
$$\forall s \leq r, \;\; \langle x \rangle \overset{s}{\not\longmapsto}_{\perp} \tag{778}$$

By Induction on time step 775, we get:

$$\langle \texttt{consistent}\ y \rangle \tag{779}$$

Apply Rule P2-27 on Predicates 779 and 776. We get:

$$\langle \texttt{consistent}\ x + y \rangle$$

Hence proved.

(c) *Termination Predicate:*
Suppose,

$$\langle x + y \rangle \xrightarrow{a} \surd$$

It can only be derived from Rule P2-22 or Rule P2-23.

- *Rule P2-22*
  From the premise of the rule,

$$\langle x \rangle \xrightarrow{a} \surd \tag{780}$$
$$\langle \texttt{consistent}\ y \rangle \tag{781}$$

  By Induction on Predicate 780, we get:

$$\langle \texttt{consistent}\ x \rangle \tag{782}$$

  Apply Rule P2-27 on Predicates 781 and 782. We get:

$$\langle \texttt{consistent}\ x + y \rangle$$

  Hence proved.
- *Rule P2-23*
  From the premise of the rule,

$$\langle y \rangle \xrightarrow{a} \surd \tag{783}$$
$$\langle \texttt{consistent}\ x \rangle \tag{784}$$

  By Induction on Predicate 783, we get:

$$\langle \texttt{consistent}\ y \rangle \tag{785}$$

  Apply Rule P2-27 on Predicates 784 and 785. We get:

$$\langle \texttt{consistent}\ x + y \rangle$$

  Hence proved.

(d) *A Future Inconsistency Predicate:*
Suppose,

$$\langle x + y \rangle \overset{r}{\longmapsto}_{\perp}$$

It can only be derived from Rule P2-28 or Rule P2-29.

- *Rule P2-28*
  From the premise of the rule the following holds:

$$\langle x \rangle \xmapsto{r}{}_\perp \tag{786}$$
$$\langle \texttt{consistent}\ \ y \rangle \tag{787}$$
$$\forall s < r,\ \ \langle y \rangle \not\xmapsto{s}{}_\perp \tag{788}$$

  By Induction on predicate 786, we get:

$$\langle \texttt{consistent}\ \ x \rangle \tag{789}$$

  Apply Rule P2-27 on Predicates 789 and 787. We get:

$$\langle \texttt{consistent}\ \ x + y \rangle$$

  Hence proved.
- *Rule P2-29*
  From the premise of the rule the following holds:

$$\langle y \rangle \xmapsto{r}{}_\perp \tag{790}$$
$$\langle \texttt{consistent}\ \ x \rangle \tag{791}$$
$$\forall s < r,\ \ \langle x \rangle \not\xmapsto{s}{}_\perp \tag{792}$$

  By Induction on predicate 790, we get:

$$\langle \texttt{consistent}\ \ y \rangle \tag{793}$$

  Apply Rule P2-27 on Predicates 793 and 791. We get:

$$\langle \texttt{consistent}\ \ x + y \rangle$$

  Hence proved.

5. $p = \nu_{\mathsf{rel}}(x)$

   (a) *Action Step:*
   Suppose,

$$\langle \nu_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \langle p' \rangle$$

   It can only be derived from Rule P2-30. From the premise of the rule,

$$\langle x \rangle \xrightarrow{a} \langle p' \rangle$$

   By Induction on the above action step, we get:

$$\langle \texttt{consistent}\ \ x \rangle$$

   Apply Rule P2-32. We get:

$$\langle \texttt{consistent}\ \ \nu_{\mathsf{rel}}(x) \rangle$$

   Hence proved.

(b) *Time Step:*
No rule allows a derivation of a time step for the now operator.

(c) *Termination Predicate:*
Suppose,

$$\langle \nu_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \surd$$

It can only be derived from Rule P2-30. From the premise of the rule,

$$\langle x \rangle \xrightarrow{a} \surd$$

By Induction on the above predicate, we get:

$$\langle \texttt{consistent}\ x \rangle$$

Apply Rule P2-32. We get:

$$\langle \texttt{consistent}\ \nu_{\mathsf{rel}}(x) \rangle$$

Hence proved.

(d) *A future Inconsistency predicate:*
No rule allows a derivation of a future Inconsistency predicate for the now operator.

$\boxtimes$

H.2 Theorem : Future Inconsistency Predicates have Shortest Length

Theorem 15 *For all closed terms p, durations $r > 0$ the following holds:*

$$\langle p \rangle \overset{r}{\longmapsto}_{\perp} \implies \forall s < r,\ \langle p \rangle \overset{s}{\not\longmapsto}_{\perp}$$

<u>**Proof**</u> We prove the above theorem by structural induction on a process term $p \in P$. The base case of the structural induction comprises of constant process terms, i.e. all undelayable actions in $\mathcal{A}$, the deadlock process term δ and the inconsistent process $\perp$.

<u>Base Case</u>

1. $p = \tilde{\tilde{a}}$.

   There are no rules to derive a future Inconsistency predicate for an undelayable action. As the left hand side of the implication does not hold, therefore, the implication holds.

2. $p = \tilde{\tilde{\delta}}$

   There are no rules to derive a future Inconsistency predicate for an undelayable action.

3. $p = \bot$

   There are no rules for an inconsistent process $\bot$ in the semantics of $BPA_{\bot}^{srt}$.

By Induction Hypothesis

1. $p = \sigma_{\mathsf{rel}}^{0}(x)$, for a closed term x.
   Suppose,

$$\langle \sigma_{\mathsf{rel}}^{0}(x) \rangle \xmapsto{r}_{\bot}$$

   It can only be derived from Rule P2-8. From the premise of the rule,

$$\langle x \rangle \xmapsto{r}_{\bot}$$

   By Induction on the above predicate, we get:

$$\forall s < r, \;\; \langle x \rangle \not\xmapsto{s}_{\bot}$$

   Then for all durations $s < r$, the premise of Rule P2-8 is not satisfied. Since this is the only rule allowing a future Inconsistency predicate for the operator $\sigma_{\mathsf{rel}}^{0}$, therefore we conclude:

$$\forall s < r, \;\; \langle \sigma_{\mathsf{rel}}^{0}(x) \rangle \not\xmapsto{s}_{\bot}$$

   Proved.

2. $p = \sigma_{\mathsf{rel}}^{t}(x) \qquad t > 0$
   Suppose,

$$\langle \sigma_{\mathsf{rel}}^{t}(x) \rangle \xmapsto{r}_{\bot} \tag{794}$$

   We distinguish between three cases depending on the duration r.

   (a) Case $r < t$

   For a process term $\sigma_{\mathsf{rel}}^{t}(x)$, no future inconsistency predicate of length less than t can be derived.

   (b) Case $r = t$
   Rewriting Predicate 794:

$$\langle \sigma_{\mathsf{rel}}^{t}(x) \rangle \xmapsto{t}_{\bot}$$

From the case for $r < t$, the following always holds:

$$\forall s < t, \ \langle \sigma^t_{\mathsf{rel}}(x) \rangle \not\xmapsto{s} \bot \tag{795}$$

As Predicate 795 always holds, hence the theorem is proved for $\sigma^t_{\mathsf{rel}}(x)$.

(c) <u>Case $r > t$</u>
Let $r = u + t$, for $u > 0$.
Than a future Inconsistency predicate can only be derived from Rule P2-14. From the premise of the rule,

$$\langle x \rangle \xmapsto{u} \bot$$

By Induction on the above Predicate, we have:

$$\forall s < u, \ \langle x \rangle \not\xmapsto{s} \bot$$

Then Rule P2-14 cannot be applied for deriving a future Inconsistency predicate for $\sigma^t_{\mathsf{rel}}(x)$ with length $t + s$. As Rule P2-14 is the only such rule, therefore, we conclude that:

$$\forall s < u, \ \langle \sigma^t_{\mathsf{rel}}(x) \rangle \not\xmapsto{t+s} \bot$$

We can rewrite the above predicate as:

$$\forall s < (t + u), \ \langle \sigma^t_{\mathsf{rel}}(x) \rangle \not\xmapsto{s} \bot$$

Hence proved.

3. $p = x \cdot y$.
Suppose,

$$\langle x \cdot y \rangle \xmapsto{r} \bot$$

It can only be derived from Rule P2-19. From the premise of the rule,

$$\langle x \rangle \xmapsto{r} \bot$$

By Induction on the above predicate, we get:

$$\forall s < r, \ \langle x \rangle \not\xmapsto{s} \bot$$

Then Rule P2-19 cannot be applied for deriving a future Inconsistency predicate for $x \cdot y$ with length less than r. As Rule P2-19 is the only such rule, therefore, we conclude that:

$$\forall s < r, \ \langle x \cdot y \rangle \not\xmapsto{s} \bot$$

Hence proved.

4. $p = x + y$.

   Suppose,

$$\langle x + y \rangle \overset{r}{\longmapsto} \bot \tag{796}$$

   It can only be derived from Rule P2-28 or Rule P2-29.

   - *Rule P2-28* If Predicate 796 is derived from this rule, then from the premise of the rule the following holds:

$$\langle x \rangle \overset{r}{\longmapsto} \bot \tag{797}$$

$$\langle \texttt{consistent}\ \ y \rangle \tag{798}$$

$$\forall s < r,\ \ \langle y \rangle \overset{s}{\not\longmapsto} \bot \tag{799}$$

     By Induction on Predicate 797, we get:

$$\forall s < r,\ \ \langle x \rangle \overset{s}{\not\longmapsto} \bot \tag{800}$$

     From Predicate 799, Rule P2-29 cannot be applied to derive a future Inconsistency Predicate for $x + y$ for a duration less than r.

     Similarly, from Predicate 800, Rule P2-28 cannot be applied to derive a Future Inconsistency Predicate for $x + y$ for a duration less than r. Since rules 28 and 29 are the only such rules, hence we conclude:

$$\forall s < r,\ \ \langle x + y \rangle \overset{s}{\not\longmapsto} \bot$$

     Hence proved.

   - *Rule P2-29* If Predicate 796 is derived from this rule, then from the premise of the rule the following holds:

$$\langle y \rangle \overset{r}{\longmapsto} \bot \tag{801}$$

$$\langle \texttt{consistent}\ \ x \rangle \tag{802}$$

$$\forall s < r,\ \ \langle x \rangle \overset{s}{\not\longmapsto} \bot \tag{803}$$

     By Induction on Predicate 801, we get:

$$\forall s < r,\ \ \langle y \rangle \overset{s}{\not\longmapsto} \bot \tag{804}$$

     From Predicate 803, Rule P2-28 cannot be applied to derive a future Inconsistency Predicate for $x + y$ for a duration less than r.

     Similarly, from Predicate 804, Rule P2-29 cannot be applied to derive a Future Inconsistency Predicate for $x + y$ for a duration less than r. Since rules 28 and 29 are the only such rules, hence we conclude:

$$\forall s < r,\ \ \langle x + y \rangle \overset{s}{\not\longmapsto} \bot$$

     Hence proved.

5. $p = \nu_{\mathsf{rel}}(x)$

   There are no rules to derive a Future Inconsistency Predicate for the now operator. Hence the theorem trivially holds.

$\boxtimes$

H.3 Theorem : Time Determinism

Theorem 16 *For all closed terms p, durations $r > 0$ the following holds:*

$$\langle p \rangle \overset{r}{\longmapsto} \langle p_1 \rangle \wedge \langle p \rangle \overset{r}{\longmapsto} \langle p_2 \rangle \\ \implies p_1 \equiv p_2$$

Proof We prove the above theorem by structural induction on a process term $p \in P$. The base case of the structural induction comprises of constant process terms, i.e. all undelayable actions in $\mathcal{A}$, the deadlock process term δ and the inconsistent process $\perp$.

Base Case

1. $p = \tilde{\tilde{a}}$.

   There are no rules to derive a time step for an undelayable action.

2. $p = \tilde{\tilde{\delta}}$

   There are no rules to derive a future Inconsistency predicate for the deadlock constant.

3. $p = \perp$

   There are no rules for an inconsistent process $\perp$.

By Induction Hypothesis

1. $p = \sigma^0_{\mathsf{rel}}(x)$, for a closed term x.

   Suppose,

$$\langle \sigma^0_{\mathsf{rel}}(x) \rangle \overset{r}{\longmapsto} \langle p_1 \rangle \tag{805}$$

$$\langle \sigma^0_{\mathsf{rel}}(x) \rangle \overset{r}{\longmapsto} \langle p_2 \rangle \tag{806}$$

   Only Rule P2-6 allows derivation of a time step for the operator σ^0_{rel}. From the premise of the rule,

$$\langle x \rangle \overset{r}{\longmapsto} \langle p_1 \rangle \tag{807}$$

$$\langle x \rangle \overset{r}{\longmapsto} \langle p_2 \rangle \tag{808}$$

By Induction on the above predicate, we get:

$$p_1 \equiv p_2$$

Proved.

2. $p = \sigma^t_{\mathsf{rel}}(x) \qquad t > 0$
   Suppose,

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xmapsto{r} \langle p_1 \rangle \tag{809}$$

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xmapsto{r} \langle p_2 \rangle \tag{810}$$

   We distinguish between three cases depending on the duration r.

   (a) $\underline{\underline{\text{Case } r < t}}$

   Only Rule P2-9 can derive time steps 809 and 810. Then the target process terms in both time steps is $\sigma^{t-r}_{\mathsf{rel}}(x)$. I.e.,

$$p_1 = \sigma^{t-r}_{\mathsf{rel}}(x) \wedge p_2 = \sigma^{t-r}_{\mathsf{rel}}(x)$$

   Hence

$$p_1 \equiv p_2$$

   Proved.

   (b) $\underline{\underline{\text{Case } r = t}}$
   Rewriting time steps 809 and 810, we get:

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xmapsto{t} \langle p_1 \rangle \tag{811}$$

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xmapsto{t} \langle p_2 \rangle \tag{812}$$

   Only Rule P2-9 can derive time steps 811 and 812. Then the target process terms in both time steps is x. Hence,

$$p_1 \equiv p_2 \equiv x$$

   Proved.

   (c) $\underline{\underline{\text{Case } r > t}}$
   Let $r = u + t$, for $u > 0$.
   Rewriting time steps 809 and 810, we get:

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xmapsto{t+u} \langle p_1 \rangle \tag{813}$$

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xmapsto{t+u} \langle p_2 \rangle \tag{814}$$

   Only Rule P2-9 can derive time steps 813 and 814. From the premise of the rule, the following must hold:

$$\langle x \rangle \xmapsto{u} \langle p_1 \rangle \tag{815}$$

$$\langle x \rangle \xmapsto{u} \langle p_2 \rangle \tag{816}$$

By Induction,

$$p_1 \equiv p_2$$

Proved.

3. $p = x \cdot y$.

   Suppose,

$$\langle x \cdot y \rangle \xmapsto{r} \langle p_1 \rangle \tag{817}$$
$$\langle x \cdot y \rangle \xmapsto{r} \langle p_2 \rangle \tag{818}$$

   The above time steps can only be derived from Rule P2-17.
   Then for some process term p_1', $p_1 = p_1' \cdot y$.
   Rewriting Transition 817:

$$\langle x \cdot y \rangle \xmapsto{r} \langle p_1' \cdot y \rangle \tag{819}$$

   Also for some process term p_2', $p_1 = p_2' \cdot y$.
   Rewriting Transition 818:

$$\langle x \cdot y \rangle \xmapsto{r} \langle p_2' \cdot y \rangle \tag{820}$$

   From the premise of Rule P2-17 the following must hold:

$$\langle x \rangle \xmapsto{r} \langle p_1' \rangle \text{ and } \langle x \rangle \xmapsto{r} \langle p_2' \rangle \tag{821}$$

   By Induction

$$p_1' \equiv p_2'$$

   Hence,

$$p_1' \cdot y \equiv p_2' \cdot y \text{ I.e. } p_1 \equiv p_2$$

   Proved.

4. $p = x + y$.

   Suppose,

$$\langle x + y \rangle \xmapsto{r} \langle p_1 \rangle \tag{822}$$
$$\langle x + y \rangle \xmapsto{r} \langle p_2 \rangle \tag{823}$$

   Rule P2-24, Rule P2-25 or Rule P2-26 can be used to derive the above time steps. We discuss these rules one by one. We show both transitions are derived by the same rule and that only one rule is applicable at a time.

(a) *Rule P2-24*

Suppose Transition 822 is derived from this rule. Then for some process terms x_1, y_1,

$$p_1 = x_1 + y_1 \tag{824}$$

From the premise of the rule the following holds:

$$\langle x \rangle \xmapsto{r} \langle x_1 \rangle \tag{825}$$

$$\langle y \rangle \xmapsto{r} \langle y_1 \rangle \tag{826}$$

From Transition 825, $(\langle x \rangle \xmapsto{r} \langle x_1 \rangle)$, Rule P2-26 becomes inapplicable to derive a time step for $x + y$.

From Transition 826, $(\langle y \rangle \xmapsto{r} \langle y_1 \rangle)$, Rule P2-25 becomes inapplicable to derive a time step for $x + y$.

Therefore Transition 823 can also be only derived by Rule P2-24. From the premise of the rule, for some process terms x_2, y_2,

$$p_2 = x_2 + y_2 \tag{827}$$

and the following must hold:

$$\langle x \rangle \xmapsto{r} \langle x_2 \rangle \tag{828}$$

$$\langle y \rangle \xmapsto{r} \langle y_2 \rangle \tag{829}$$

Apply Induction Hypothesis on Transitions 825 and 828, and on Transitions 826 and 829. We get:

$$x_1 \equiv x_2$$
$$y_1 \equiv y_2$$

which implies

$$x_1 + y_1 \equiv x_2 + y_2$$

From Statements 824 and 827,

$$p_1 \equiv p_2$$

Proved.

(b) *Rule P2-25*

Suppose Transition 822 is derived from this rule. From the premise of the rule the following holds:

$$\langle x \rangle \xmapsto{r} \langle p_1 \rangle \tag{830}$$

$$\langle \texttt{consistent } y \rangle \tag{831}$$

$$\langle y \rangle \not\xmapsto{r} \tag{832}$$

$$\forall s \leq r, \ \langle y \rangle \not\xmapsto{s}_{\perp} \tag{833}$$

From Transition 830, ($\langle x\rangle \xmapsto{r} \langle p_1\rangle$), Rule P2-26 becomes inapplicable to derive a time step for $x + y$.

From Transition 826, ($\langle y\rangle \not\xmapsto{r}$), Rule P2-24 becomes inapplicable to derive a time step for $x + y$.

Hence Transition 823 can only be derived from Rule P2-25.

From the premise of the rule, in addition to Predicates 831, 832 and 833, the following holds:

$$\langle x\rangle \xmapsto{r} \langle p_2\rangle \tag{834}$$

Apply Induction Hypothesis on Transition 830 and Transition 834, we get:

$$p_1 \equiv p_2$$

Proved.

(c) *Rule P2-26*

Suppose Transition 822 is derived from this rule. From the premise of the rule the following holds:

$$\langle y\rangle \xmapsto{r} \langle p_1\rangle \tag{835}$$

$$\langle \texttt{consistent}\ x\rangle \tag{836}$$

$$\langle x\rangle \not\xmapsto{r} \tag{837}$$

$$\forall s \leq r,\ \ \langle x\rangle \not\xmapsto{s}_{\perp} \tag{838}$$

From Transition 835, ($\langle y\rangle \xmapsto{r} \langle p_1\rangle$), Rule P2-25 becomes inapplicable to derive a time step for $x + y$.

From Transition 838, ($\langle x\rangle \not\xmapsto{r}$), Rule P2-24 becomes inapplicable to derive a time step for $x + y$.

Hence Transition 823 can only be derived from Rule P2-26.

From the premise of the rule, in addition to Predicates 836, 837 and 838, the following holds:

$$\langle y\rangle \xmapsto{r} \langle p_2\rangle \tag{839}$$

Apply Induction Hypothesis on Transition 835 and Transition 839, we get:

$$p_1 \equiv p_2$$

Proved.

5. $p = \nu_{\mathsf{rel}}(x)$

There are no rules to derive a time step for the now operator. Hence the theorem trivially holds.

$\boxtimes$

H.4 Axiom A1 (Commutativity)

$x + y = y + x$ (Commutativity-A1)

We need to prove, $x + y \leftrightarrows y + x$.

Let R be a binary relation on process terms defined as follows:

$$R = \{(x + y, y + x) \mid x, y \in P\}$$

It is trivial to prove that $R \cup \mathcal{I}$ is a bisimulation relation.

H.5 Axiom A2 (Associativity of Choice)

$(x + y) + z = x + (y + z)$ (Associativity of Alternative Composition-A2).

We need to prove, $(x + y) + z \leftrightarrows x + (y + z)$.

Let R be a binary relation on process terms defined as follows:

$$R = \{ ((x + y) + z, x + (y + z)) \mid x, y, z \in P\}$$

We prove that the relation $R \cup \mathcal{I}$ is a bisimulation relation.

For all $a \in A, r > 0, x, y, z, p \in P$, the following holds:

1. $$\langle (x + y) + z\rangle \xrightarrow{a} \langle p\rangle \implies \begin{array}{l} \exists p' \in P : \langle x + (y + z)\rangle \xrightarrow{a} \langle p'\rangle \\ \text{and } (p, p') \in R \cup \mathcal{I} \end{array}$$

   Suppose,
   $$\langle (x + y) + z\rangle \xrightarrow{a} \langle p\rangle \tag{840}$$

   An action transition for an alternative composition can be derived only from rules P2 20 or P2 21. We discuss them one by one:

   (a) Rule P2 20

   If Transition 840 is derived from this rule, then from the premise the following must hold:

   $$\langle x + y\rangle \xrightarrow{a} \langle p\rangle \tag{841}$$
   $$\langle \texttt{consistent}\ z\rangle \tag{842}$$

   Again Transition 841 can be derived from Rule P2 20 or Rule P2-21.

   i. *Rule P2 20:*
   If Transition 841 is derived from this rule, then from the premise the following must hold:

   $$\langle x\rangle \xrightarrow{a} \langle p\rangle \tag{843}$$
   $$\langle \texttt{consistent}\ y\rangle \tag{844}$$

   Apply Rule P2-27 on predicates 842 and 844, we get:

   $$\langle \texttt{consistent}\ y + z\rangle \tag{845}$$

By applying Rule 20 on Transition 843, for any process term q with $\langle \texttt{consistent}\ \ q \rangle$, the following holds:

$$\langle x + q \rangle \xrightarrow{a} \langle p \rangle$$

The term q can be $y + z$. Hence we have,

$$\langle x + (y + z) \rangle \xrightarrow{a} \langle p \rangle \tag{846}$$

Consider the target process terms in Transition 840 and 846. The pair (p, p) is in $\mathcal{I}$.

ii. *Rule P2 21:*
If Transition 841 is derived from this rule, then from the premise the following must hold:

$$\langle y \rangle \xrightarrow{a} \langle p \rangle \tag{847}$$

$$\langle \texttt{consistent}\ \ x \rangle \tag{848}$$

By applying Rule 20 on Transition 847, using Predicate 842, we can derive the following transition:

$$\langle y + z \rangle \xrightarrow{a} \langle p \rangle \tag{849}$$

By applying Rule 21 on above transition, using Predicate 848, we can derive the following transition:

$$\langle x + (y + z) \rangle \xrightarrow{a} \langle p \rangle \tag{850}$$

Consider the target process terms in Transition 840 and 850. The pair (p, p) is in $\mathcal{I}$.

(b) Rule P2 21

If Transition 840 is derived from this rule, then from the premise the following must hold:

$$\langle z \rangle \xrightarrow{a} \langle p \rangle \tag{851}$$

$$\langle \texttt{consistent}\ \ x + y \rangle \tag{852}$$

Predicate 852 can only hold, if

$$\langle \texttt{consistent}\ \ x \rangle \tag{853}$$

$$\langle \texttt{consistent}\ \ y \rangle \tag{854}$$

By applying Rule 21 on Transition 851, using Predicate 854, we can derive the following transition:

$$\langle y + z \rangle \xrightarrow{a} \langle p \rangle \tag{855}$$

By applying Rule 21 on above transition, using Predicate 853, we can derive the following transition:

$$\langle x + (y + z)\rangle \xrightarrow{a} \langle p\rangle \tag{856}$$

Consider the target process terms in Transition 840 and 856. The pair (p, p) is in $\mathcal{I}$.

2.

$$\langle x + (y + z)\rangle \xrightarrow{a} \langle p\rangle \implies \begin{array}{l} \exists p' \in P : \langle (x + y) + z\rangle \xrightarrow{a} \langle p'\rangle \\ \text{and } (p', p) \in R \cup \mathcal{I} \end{array}$$

Suppose,

$$\langle x + (y + z)\rangle \xrightarrow{a} \langle p\rangle \tag{857}$$

An action transition for an alternative composition can be derived only from rules P2 20 or P2 21. We discuss them one by one:

(a) Rule P2-20

If Transition 857 is derived from this rule, then from the premise the following must hold:

$$\langle x\rangle \xrightarrow{a} \langle p\rangle \tag{858}$$

$$\langle \texttt{consistent}\ \ y + z\rangle \tag{859}$$

Predicate 859 can only hold, if

$$\langle \texttt{consistent}\ \ y\rangle \tag{860}$$

$$\langle \texttt{consistent}\ \ z\rangle \tag{861}$$

By applying Rule 20 on Transition 858, using Predicate 860, we can derive the following transition:

$$\langle x + y\rangle \xrightarrow{a} \langle p\rangle \tag{862}$$

By again applying Rule 20 on above transition, using Predicate 861, we can derive the following transition:

$$\langle (x + y) + z\rangle \xrightarrow{a} \langle p\rangle \tag{863}$$

Consider the target process terms in Transition 857 and 863. The pair (p, p) is in $\mathcal{I}$.

(b) Rule P2 21

If Transition 857 is derived from this rule, then from the premise the following must hold:

$$\langle y + z\rangle \xrightarrow{a} \langle p\rangle \tag{864}$$

$$\langle \texttt{consistent}\ \ x\rangle \tag{865}$$

Again Transition 864 can be derived from Rule P2 20 or Rule P2-21.

i. *Rule P2 20:*
If Transition 864 is derived from this rule, then from the premise the following must hold:

$$\langle y \rangle \xrightarrow{a} \langle p \rangle \tag{866}$$

$$\langle \texttt{consistent} \ z \rangle \tag{867}$$

By applying Rule 21 on Transition 866, using Predicate 865, we can derive the following transition:

$$\langle x + y \rangle \xrightarrow{a} \langle p \rangle \tag{868}$$

By applying Rule 20 on above transition, using Predicate 867, we can derive the following transition:

$$\langle (x + y) + z \rangle \xrightarrow{a} \langle p \rangle \tag{869}$$

Consider the target process terms in Transition 857 and 869. The pair (p, p) is in $\mathcal{I}$.

ii. *Rule P2 21:*
If Transition 864 is derived from this rule, then from the premise the following must hold:

$$\langle z \rangle \xrightarrow{a} \langle p \rangle \tag{870}$$

$$\langle \texttt{consistent} \ y \rangle \tag{871}$$

Apply Rule P2-27 on predicates 871 and 865, we get:

$$\langle \texttt{consistent} \ x + y \rangle \tag{872}$$

By applying Rule 21 on Transition 870, for any process term q with $\langle \texttt{consistent} \ q \rangle$, the following holds:

$$\langle q + z \rangle \xrightarrow{a} \langle p \rangle$$

The term q can be $x + y$. Hence we have,

$$\langle (x + y) + z \rangle \xrightarrow{a} \langle p \rangle \tag{873}$$

Consider the target process terms in Transition 857 and 873. The pair (p, p) is in $\mathcal{I}$.

3.

$$\langle (x + y) + z \rangle \xmapsto{r} \langle p \rangle \implies \begin{array}{l} \exists z' \in P : \langle x + (y + z) \rangle \xmapsto{r} \langle z' \rangle \\ \text{and } (p, z') \in R \cup \mathcal{I} \end{array}$$

Suppose,

$$\langle (x + y) + z \rangle \xmapsto{r} \langle p \rangle \tag{874}$$

Rules P2 24, Rule P2-25 or Rule P2-26 can be used to derive the above transition.

(a) Rule P2-24

If this rule is used to derive Transition 874, then for some process terms p_1, p_2, $p = p_1 + p_2$. Rewriting Transition 874:

$$\langle (x + y) + z \rangle \xmapsto{r} \langle p_1 + p_2 \rangle \tag{875}$$

From premise of the rule,

$$\langle x + y \rangle \xmapsto{r} \langle p_1 \rangle \tag{876}$$
$$\langle z \rangle \xmapsto{r} \langle p_2 \rangle \tag{877}$$

Again Transition 876 can be derived from one of the three rules: Rule P2 24, Rule P2-25 or Rule P2-26.

i. *Rule P2-24*
If Transition 876 is derived from this rule, then for some process terms x_1, y_1, $p_1 = x_1 + y_1$. Rewriting Transition 875 and Transition 876, we get:

$$\langle (x + y) + z \rangle \xmapsto{r} \langle (x_1 + y_1) + p_2 \rangle \tag{878}$$
$$\langle x + y \rangle \xmapsto{r} \langle x_1 + y_1 \rangle \tag{879}$$

From the premise of the rule,

$$\langle x \rangle \xmapsto{r} \langle x_1 \rangle \tag{880}$$
$$\langle y \rangle \xmapsto{r} \langle y_1 \rangle \tag{881}$$

Apply Rule P2-24 on Transition 877 and 881, we get:

$$\langle y + z \rangle \xmapsto{r} \langle y_1 + p_2 \rangle \tag{882}$$

Again apply Rule P2-24 on Transition 882 and 880, we get:

$$\langle x + (y + z) \rangle \xmapsto{r} \langle x_1 + (y_1 + p_2) \rangle \tag{883}$$

Consider the target process terms in transitions 878 and 883. The pair $((x_1 + y_1) + p_2, x_1 + (y_1 + p_2))$ is in R.

ii. *Rule P2-25*
If Transition 876 is derived from this rule, then from the premise of the rule,

$$\langle x \rangle \xmapsto{r} \langle p_1 \rangle \tag{884}$$
$$\langle \texttt{consistent}\ y \rangle \tag{885}$$
$$\langle y \rangle \not\xmapsto{r} \tag{886}$$
$$\forall s \leq r\ \ \langle y \rangle \not\xmapsto{s}_{\perp} \tag{887}$$

On Transitions (Predicates) 877, 885 , 886 and 887, apply Rule P2-26, we get:

$$\langle y+z\rangle \overset{r}{\longmapsto} \langle p_2\rangle \tag{888}$$

On Transitions 884 and 888, apply Rule P2-24, we get:

$$\langle x+(y+z)\rangle \overset{r}{\longmapsto} \langle p_1+p_2\rangle \tag{889}$$

Consider the target process terms in transitions 875 and 889. The pair (p_1+p_2, p_1+p_2) is in $\mathcal{I}$.

iii. *Rule P2-26*

If Transition 876 is derived from this rule, then from the premise of the rule,

$$\langle y\rangle \overset{r}{\longmapsto} \langle p_1\rangle \tag{890}$$

$$\langle \texttt{consistent}\ \ x\rangle \tag{891}$$

$$\langle x\rangle \overset{r}{\not\longmapsto} \tag{892}$$

$$\forall s \leq r\ \ \langle x\rangle \overset{s}{\not\longmapsto}_{\perp} \tag{893}$$

On Transitions 890 and 877, apply Rule P2-24, we get:

$$\langle y+z\rangle \overset{r}{\longmapsto} \langle p_1+p_2\rangle \tag{894}$$

On Transitions (Predicates) 894, 891, 892 and 893, apply Rule P2-26, we get:

$$\langle x+(y+z)\rangle \overset{r}{\longmapsto} \langle p_1+p_2\rangle \tag{895}$$

Consider the target process terms in transitions 875 and 895. The pair (p_1+p_2, p_1+p_2) is in $\mathcal{I}$.

(b) Rule P2-25

If Transition 874 is derived from this rule, then from the premise:

$$\langle x+y\rangle \overset{r}{\longmapsto} \langle p\rangle \tag{896}$$

$$\langle \texttt{consistent}\ \ z\rangle \tag{897}$$

$$\langle z\rangle \overset{r}{\not\longmapsto} \tag{898}$$

$$\forall s \leq r\ \ \langle z\rangle \overset{s}{\not\longmapsto}_{\perp} \tag{899}$$

Again Transition 896 can be derived from three rules. They are Rule P2-24, Rule P2-25 and Rule P2-26. We discuss them one by one.

i. *Rule P2-24*

If this rule is used to derive Transition 896, then for some process terms x_1, y_1, $p = x_1 + y_1$. Rewriting Transitions 874 and Transition 896, we get:

$$\langle (x+y)+z\rangle \overset{r}{\longmapsto} \langle x_1+y_1\rangle \tag{900}$$

$$\langle x+y\rangle \overset{r}{\longmapsto} \langle x_1+y_1\rangle \tag{901}$$

From the premise of the rule:

$$\langle x \rangle \xmapsto{r} \langle x_1 \rangle \tag{902}$$

$$\langle y \rangle \xmapsto{r} \langle y_1 \rangle \tag{903}$$

On Transitions (Predicates) 903, 897, 898 and 899, apply Rule P2-25, we get:

$$\langle y + z \rangle \xmapsto{r} \langle y_1 \rangle \tag{904}$$

On Transitions 904, 902, apply Rule P2-24, we get:

$$\langle x + (y + z) \rangle \xmapsto{r} \langle x_1 + y_1 \rangle \tag{905}$$

Consider target process terms in Transitions 900 and 905. The pair $(x_1 + y_1, x_1 + y_1)$ is in $\mathcal{I}$.

ii. *Rule P2-25*

If this rule is used to derive Transition 896, then from the premise of the rule, the following holds:

$$\langle x \rangle \xmapsto{r} \langle p \rangle \tag{906}$$

$$\langle \texttt{consistent}\ y \rangle \tag{907}$$

$$\langle y \rangle \not\xmapsto{r} \tag{908}$$

$$\forall s \leq r \quad \langle y \rangle \not\xmapsto{s}_{\perp} \tag{909}$$

On Predicates 897 and 907, apply Rule P2-27, we get:

$$\langle \texttt{consistent}\ y + z \rangle \tag{910}$$

A time transition for $y + z$ with duration r can either be derived from Rule P2-24, Rule P2-25 or Rule P2-26. From Predicate 908, Rules P2 24 and P2 25 cannot be applied. From Predicate 898, Rule P2 26 cannot be applied. Hence we can conclude,

$$\langle y + z \rangle \not\xmapsto{r} \tag{911}$$

A future inconsistency predicate for $y + z$ with duration $s \in (0, r]$ can either be derived from Rule P2-28, or Rule P2-29. From Predicate 909, Rule P2 28 cannot be applied to derive a future inconsistency predicate of length $s \in (0, r]$ for $y + z$. From Predicate 899, Rule P2 29 cannot be applied to derive a future inconsistency predicate of length $s \in (0, r]$ for $y + z$. Hence we can conclude,

$$\forall s \leq r \ \ \langle y + z \rangle \not\xmapsto{s}_{\perp} \tag{912}$$

On Transitions (Predicates) 910, 911, 912 and 906, apply Rule P2 25. We get:

$$\langle x + (y + z) \rangle \xmapsto{r} \langle p \rangle \tag{913}$$

Consider target process terms in Transitions 874 and 913. The pair (p, p) is in $\mathcal{I}$.

iii. *Rule P2-26*

If this rule is used to derive Transition 896, then from the premise of the rule, the following holds:

$$\langle y \rangle \xmapsto{r} \langle p \rangle \tag{914}$$

$$\langle \texttt{consistent}\ \ x \rangle \tag{915}$$

$$\langle x \rangle \not\xmapsto{r} \tag{916}$$

$$\forall s \leq r \ \ \langle x \rangle \not\xmapsto{s}_{\perp} \tag{917}$$

On Transitions (Predicates) 897, 898, 899 and 914, apply Rule P2-25, we get:

$$\langle y + z \rangle \xmapsto{r} \langle p \rangle \tag{918}$$

On Transitions (Predicates) 918, 915, 916 and 917, apply Rule P2-26, we get:

$$\langle x + (y + z) \rangle \xmapsto{r} \langle p \rangle \tag{919}$$

Consider target process terms in Transitions 874 and 919. The pair (p, p) is in $\mathcal{I}$.

(c) <u>Rule P2-26</u>

If Transition 874 is derived from this rule, then from the premise of the rule:

$$\langle z \rangle \xmapsto{r} \langle p \rangle \tag{920}$$

$$\langle \texttt{consistent}\ \ x + y \rangle \tag{921}$$

$$\langle x + y \rangle \not\xmapsto{r} \tag{922}$$

$$\forall s \leq r \ \ \langle x + y \rangle \not\xmapsto{s}_{\perp} \tag{923}$$

Predicate 921 can only be derived from Rule P2-27. Hence the premise of the rule must hold:

$$\langle \texttt{consistent}\ \ x \rangle \tag{924}$$

$$\langle \texttt{consistent}\ \ y \rangle \tag{925}$$

From Predicate 923, we want to prove that the following holds:

$$\forall s \leq r \ \ \langle x \rangle \not\xmapsto{s}_{\perp}$$

$$\forall s \leq r \ \ \langle y \rangle \not\xmapsto{s}_{\perp}$$

We prove the above predicates by contradiction.

Suppose,

$$\exists_{u,u' \leq r} : \ \ \langle x \rangle \xmapsto{u}_{\perp} \ \vee \langle y \rangle \xmapsto{u'}_{\perp}$$

The above statement is equivalent to the statement below:

$$\exists_{u \leq r} : \ \langle x \rangle \overset{u}{\longmapsto}_\perp \ \vee \langle y \rangle \overset{u}{\longmapsto}_\perp \tag{926}$$

We discuss different cases of the Disjunction Predicate 926 and show that all cases lead to a contradiction to Predicate 923.

i. *Case* $\langle x \rangle \overset{u}{\longmapsto}_\perp \wedge \langle y \rangle \overset{u}{\longmapsto}_\perp$
   Then by Theorem 15,

$$\forall t_1 < u \ \ \langle y \rangle \overset{t_1}{\not\longmapsto}_\perp \tag{927}$$

   From Predicate 925,

$$\langle \texttt{consistent} \ \ y \rangle \tag{928}$$

   Using Predicates 926, 928 and the assumption $\langle x \rangle \overset{u}{\longmapsto}_\perp$, apply Rule P2-28, we get:

$$\langle x + y \rangle \overset{u}{\longmapsto}_\perp \tag{929}$$

   which is a contradiction to Predicate 923.

ii. *Case* $\langle x \rangle \overset{u}{\longmapsto}_\perp \wedge \langle y \rangle \overset{u}{\not\longmapsto}_\perp$ For $v < u$, one of the two cases must hold:

   - *Case* $\langle y \rangle \overset{v}{\longmapsto}_\perp$
     Again from Theorem 15 using the assumption ($\langle x \rangle \overset{u}{\longmapsto}_\perp$), the following holds:

$$\forall t_1 < v \ \ \langle x \rangle \overset{t_1}{\not\longmapsto}_\perp \tag{930}$$

     Using Predicates 930, 924 and the assumption $\langle y \rangle \overset{v}{\longmapsto}_\perp$, apply Rule P2-28, we get:

$$\langle x + y \rangle \overset{v}{\longmapsto}_\perp \tag{931}$$

     which is a contradiction to Predicate 923.
   - *Case* $\forall v < u : \ \ \langle y \rangle \overset{v}{\not\longmapsto}_\perp$
     Using Predicate 925 and the assumptions $\langle x \rangle \overset{u}{\longmapsto}_\perp$ and $\forall v < u : \ \ \langle y \rangle \overset{v}{\not\longmapsto}_\perp$, apply Rule P2-28, we get:

$$\langle x + y \rangle \overset{u}{\longmapsto}_\perp \tag{932}$$

     which is again a contradiction to Predicate 923.

iii. *Case* $\langle x \rangle \overset{u}{\not\longmapsto}_\perp \wedge \langle y \rangle \overset{u}{\longmapsto}_\perp$
   Similar to as above.

Hence we conclude:

$$\forall s \leq r \ \ \langle x \rangle \overset{s}{\not\longmapsto}_\perp \tag{933}$$

$$\forall s \leq r \ \ \langle y \rangle \overset{s}{\not\longmapsto}_\perp \tag{934}$$

From Predicate 922, we conclude that none of the Rules P2 24, P2 25 and P2 26 are applicable. If Rule P2-24 is inapplicable, then x and y cannot delay together for r time units. Suppose one of x and y can delay. Suppose, for some x',

$$\langle x \rangle \overset{r}{\longmapsto} \langle x' \rangle$$
$$\langle y \rangle \overset{r}{\not\longmapsto}$$

Now from Transitions (Predicates) 934,925 and the time transition for x' and impossibility of delay for y given above, Rule P2-25 becomes applicable and we can derive,

$$\langle x + y \rangle \overset{r}{\longmapsto} \langle x' \rangle$$

which is a contradiction to Predicate 922. Similarly, if we suppose y can delay for r time units, then Rule P2 26 becomes applicable.

Hence we conclude that none of the process terms, x and y can delay.

$$\langle x \rangle \overset{r}{\not\longmapsto} \tag{935}$$
$$\langle y \rangle \overset{r}{\not\longmapsto} \tag{936}$$

On Transitions (Predicates) 920, 925, 934 and 936, apply Rule P2-26, we get:

$$\langle y + z \rangle \overset{r}{\longmapsto} \langle p \rangle \tag{937}$$

Again join Transitions (Predicates) 924, 933, 935 and 937 and apply Rule P2-26, we get:

$$\langle x + (y + z) \rangle \overset{r}{\longmapsto} \langle p \rangle \tag{938}$$

Consider target process terms in transitions 874 and 938. The pair (p, p) is in $\mathcal{I}$.

4.
$$\langle x + (y + z) \rangle \overset{r}{\longmapsto} \langle p \rangle \implies \begin{array}{l} \exists z' \in P : \langle (x + y) + z \rangle \overset{r}{\longmapsto} \langle z' \rangle \\ \text{and } (z', p) \in R \cup \mathcal{I} \end{array}$$

Suppose,

$$\langle x + (y + z) \rangle \overset{r}{\longmapsto} \langle p \rangle \tag{939}$$

Rules P2 24, Rule P2-25 or Rule P2-26 can be used to derive the above transition.

(a) Rule P2-24

If this rule is used to derive Transition 939, then for some process terms p_1, p_2, $p = p_1 + p_2$. Rewriting Transition 939:

$$\langle x + (y + z) \rangle \overset{r}{\longmapsto} \langle p_1 + p_2 \rangle \tag{940}$$

From premise of the rule,

$$\langle x \rangle \xmapsto{r} \langle p_1 \rangle \tag{941}$$

$$\langle y + z \rangle \xmapsto{r} \langle p_2 \rangle \tag{942}$$

Again Transition 942 can be derived from one of the three rules: Rule P2 24, Rule P2-25 or Rule P2-26.

i. *Rule P2-24*
If Transition 942 is derived from this rule, then for some process terms y_2, z_2, $p_2 = y_2 + z_2$. Rewriting Transition 940 and Transition 942, we get:

$$\langle x + (y + z) \rangle \xmapsto{r} \langle p_1 + (y_2 + z_2) \rangle \tag{943}$$

$$\langle y + z \rangle \xmapsto{r} \langle y_2 + z_2 \rangle \tag{944}$$

From the premise of the rule,

$$\langle y \rangle \xmapsto{r} \langle y_2 \rangle \tag{945}$$

$$\langle z \rangle \xmapsto{r} \langle z_2 \rangle \tag{946}$$

Apply Rule P2-24 on Transition 941 and 945, we get:

$$\langle x + y \rangle \xmapsto{r} \langle p_1 + y_2 \rangle \tag{947}$$

Again apply Rule P2-24 on Transition 947 and 946, we get:

$$\langle (x + y) + z) \rangle \xmapsto{r} \langle (p_1 + y_2) + z_2) \rangle \tag{948}$$

Consider the target process terms in transitions 943 and 948. The pair $(p_1 + (y_2 + z_2), (p_1 + y_2) + z_2))$ is in R.

ii. *Rule P2-25*
If Transition 942 is derived from this rule, then from the premise of the rule,

$$\langle y \rangle \xmapsto{r} \langle p_2 \rangle \tag{949}$$

$$\langle \texttt{consistent}\ \ z \rangle \tag{950}$$

$$\langle z \rangle \not\xmapsto{r} \tag{951}$$

$$\forall s \leq r \quad \langle z \rangle \not\xmapsto{s}_{\perp} \tag{952}$$

On Transitions 941 and 949, apply Rule P2-24, we get:

$$\langle x + y \rangle \xmapsto{r} \langle p_1 + p_2 \rangle \tag{953}$$

On Transitions (Predicates) 953 , 950, 951 and 952, apply Rule P2-25, we get:

$$\langle (x + y) + z \rangle \xmapsto{r} \langle p_1 + p_2 \rangle \tag{954}$$

Consider the target process terms in transitions 940 and 954. The pair $(p_1 + p_2, p_1 + p_2)$ is in $\mathcal{I}$.

iii. *Rule P2-26*

If Transition 942 is derived from this rule, then from the premise of the rule,

$$\langle z \rangle \xmapsto{r} \langle p_2 \rangle \tag{955}$$

$$\langle \texttt{consistent}\ \ y \rangle \tag{956}$$

$$\langle y \rangle \not\xmapsto{r} \tag{957}$$

$$\forall s \leq r \quad \langle y \rangle \not\xmapsto{s}_{\perp} \tag{958}$$

On Transitions (Predicates) 956 , 957, 958 and 941 and apply Rule P2-25, we get:

$$\langle x + y \rangle \xmapsto{r} \langle p_1 \rangle \tag{959}$$

On Transitions 955 and 959, and apply Rule P2-24, we get:

$$\langle x + (y + z) \rangle \xmapsto{r} \langle p_1 + p_2 \rangle \tag{960}$$

Consider the target process terms in transitions 940 and 960. The pair $(p_1 + p_2, p_1 + p_2)$ is in $\mathcal{I}$.

(b) <u>Rule P2-25</u>

If Transition 939 is derived from this rule, then from the premise of the rule:

$$\langle x \rangle \xmapsto{r} \langle p \rangle \tag{961}$$

$$\langle \texttt{consistent}\ \ y + z \rangle \tag{962}$$

$$\langle y + z \rangle \not\xmapsto{r} \tag{963}$$

$$\forall s \leq r \ \ \langle y + z \rangle \not\xmapsto{s}_{\perp} \tag{964}$$

Predicate 962 can only be derived from Rule P2-27. Hence the premise of the rule must hold:

$$\langle \texttt{consistent}\ \ y \rangle \tag{965}$$

$$\langle \texttt{consistent}\ \ z \rangle \tag{966}$$

From Predicates 964, 965 ,966 and Theorem 15, by employing the same reasoning as given before we conclude:

$$\forall s \leq r \ \ \langle y \rangle \not\xmapsto{s}_{\perp} \tag{967}$$

$$\forall s \leq r \ \ \langle z \rangle \not\xmapsto{s}_{\perp} \tag{968}$$

From Predicate 963, we conclude that none of the rules P2 24, P2 25 and P2 26 are applicable. Then y and z cannot both delay for r time units otherwise Rule P2-24 becomes applicable. Suppose one of y and z can delay. Suppose, for some y',

$$\langle y \rangle \xmapsto{r} \langle y' \rangle$$

$$\langle z \rangle \not\xmapsto{r}$$

Now from Transitions (Predicates) 968, 966, the time transition for y and the impossibility of delay predicate for z given above, Rule P2-25 becomes applicable and we can derive,

$$\langle y + z \rangle \xmapsto{r} \langle y' \rangle$$

which is a contradiction to Predicate 963. Similarly, if we suppose, z can delay then Rule P2 26 becomes applicable.
Hence we conclude that none of the process terms, y and z can delay.

$$\langle y \rangle \not\xmapsto{r} \tag{969}$$

$$\langle z \rangle \not\xmapsto{r} \tag{970}$$

On Transitions (Predicates) 961, 965, 967 and 969, apply Rule P2-25, we get:

$$\langle x + y \rangle \xmapsto{r} \langle p \rangle \tag{971}$$

Again join Transitions (Predicates) 966, 968, 970 and 971, apply Rule P2-25, we get:

$$\langle (x + y) + z \rangle \xmapsto{r} \langle p \rangle \tag{972}$$

Consider target process terms in transitions 939 and 972. The pair (p, p) is in $\mathcal{I}$.

(c) <u>Rule P2-26</u>
If Transition 939 is derived from this rule, then from the premise:

$$\langle y + z \rangle \xmapsto{r} \langle p \rangle \tag{973}$$

$$\langle \texttt{consistent } x \rangle \tag{974}$$

$$\langle x \rangle \not\xmapsto{r} \tag{975}$$

$$\forall s \leq r \ \ \langle x \rangle \not\xmapsto{s}_{\perp} \tag{976}$$

Again Transition 973 can be derived from three rules. They are Rule P2-24, Rule P2-25 and Rule P2-26. We discuss them one by one.

i. *Rule P2-24*
If this rule is used to derive Transition 973, then for some process terms y_1, z_1, $p = y_1 + z_1$. Rewriting Transitions 939 and Transition 973, we get:

$$\langle x + (y + z) \rangle \xmapsto{r} \langle y_1 + z_1 \rangle \tag{977}$$

$$\langle y + z \rangle \xmapsto{r} \langle y_1 + z_1 \rangle \tag{978}$$

From the premise of the rule:

$$\langle y \rangle \xmapsto{r} \langle y_1 \rangle \tag{979}$$

$$\langle z \rangle \xmapsto{r} \langle z_1 \rangle \tag{980}$$

On Transitions (Predicates) 979, 974, 975 and 976, apply Rule P2-26, we get:

$$\langle x + y \rangle \overset{r}{\longmapsto} \langle y_1 \rangle \tag{981}$$

On Transitions 981, 980, apply Rule P2-24, we get:

$$\langle (x + y) + z \rangle \overset{r}{\longmapsto} \langle y_1 + z_1 \rangle \tag{982}$$

Consider target process terms in Transitions 977 and 982. The pair $(y_1 + z_1, y_1 + z_1)$ is in $\mathcal{I}$.

ii. *Rule P2-25*
If this rule is used to derive Transition 973, then from the premise of the rule, the following holds:

$$\langle y \rangle \overset{r}{\longmapsto} \langle p \rangle \tag{983}$$
$$\langle \texttt{consistent}\ z \rangle \tag{984}$$
$$\langle z \rangle \overset{r}{\not\longmapsto} \tag{985}$$
$$\forall s \leq r \quad \langle z \rangle \overset{s}{\not\longmapsto}_{\perp} \tag{986}$$

On Transitions (Predicates) 974, 975, 976 and 983, apply Rule P2-26, we get:

$$\langle x + y \rangle \overset{r}{\longmapsto} \langle p \rangle \tag{987}$$

On Transitions (Predicates) 987, 984, 985 and 986, apply Rule P2-25, we get:

$$\langle (x + y) + z \rangle \overset{r}{\longmapsto} \langle p \rangle \tag{988}$$

Consider target process terms in Transitions 939 and 988. The pair (p, p) is in $\mathcal{I}$.

iii. *Rule P2-26*
If this rule is used to derive Transition 973, then from the premise of the rule, the following holds:

$$\langle z \rangle \overset{r}{\longmapsto} \langle p \rangle \tag{989}$$
$$\langle \texttt{consistent}\ y \rangle \tag{990}$$
$$\langle y \rangle \overset{r}{\not\longmapsto} \tag{991}$$
$$\forall s \leq r \, \langle y \rangle \overset{s}{\not\longmapsto}_{\perp} \tag{992}$$

On Predicates 974 and 990, apply Rule P2-27, we get:

$$\langle \texttt{consistent}\ x + y \rangle \tag{993}$$

A time transition for $x + y$ with duration r can either be derived from Rule P2-24, Rule P2-25 or Rule P2-26. From Predicate

975, Rules P2 24 and P2 25 cannot be applied. From Predicate 991, Rule P2 26 cannot be applied. Hence we can conclude,

$$\langle x + y \rangle \not\overset{r}{\mapsto} \tag{994}$$

A future inconsistency predicate for $x+y$ with duration $s \in (0, r]$ can either be derived from Rule P2-28, or Rule P2-29. From Predicate 976, Rule P2 28 cannot be applied to derive a future inconsistency predicate of length $s \in (0, r]$ for $x + y$. From Predicate 992, Rule P2 29 cannot be applied to derive a future inconsistency predicate of length $s \in (0, r]$ for $x + y$. Hence we can conclude,

$$\forall s \leq r \;\; \langle x + y \rangle \not\overset{r}{\mapsto}_{\perp} \tag{995}$$

On Transitions (Predicates) 989, 993, 994 and 995, apply Rule P2 26. We get:

$$\langle (x + y) + z \rangle \overset{r}{\longmapsto} \langle p \rangle \tag{996}$$

Consider target process terms in Transitions 939 and 996. The pair (p, p) is in $\mathcal{I}$.

5.
$$\langle (x + y) + z \rangle \xrightarrow{a} \surd \iff \quad \langle x + (y + z) \rangle \xrightarrow{a} \surd$$

Left Implication

Suppose,

$$\langle (x + y) + z \rangle \xrightarrow{a} \surd \tag{997}$$

A termination predicate for an alternative composition can be derived only from rules P2 22 or P2 23. We discuss them one by one:

(a) Rule P2 22

If Predicate 997 is derived from this rule, then from the premise the following must hold:

$$\langle x + y \rangle \xrightarrow{a} \surd \tag{998}$$
$$\langle \texttt{consistent } z \rangle \tag{999}$$

Again Predicate 998 can be derived from Rule P2 22 or Rule P2-23.

i. *Rule P2 22:*
If Predicate 998 is derived from this rule, then from the premise the following must hold:

$$\langle x \rangle \xrightarrow{a} \surd \tag{1000}$$
$$\langle \texttt{consistent } y \rangle \tag{1001}$$

Apply Rule P2-27 on predicates 999 and 1001, we get:

$$\langle \texttt{consistent}\ \ y + z\rangle \tag{1002}$$

By applying Rule 22 on Predicate 1000, for any process term q with $\langle \texttt{consistent}\ \ q\rangle$, the following holds:

$$\langle x + q\rangle \xrightarrow{a} \surd$$

The term q can be $y + z$. Hence we have,

$$\langle x + (y + z)\rangle \xrightarrow{a} \surd \tag{1003}$$

ii. *Rule P2 23:*
If Predicate 998 is derived from this rule, then from the premise the following must hold:

$$\langle y\rangle \xrightarrow{a} \surd \tag{1004}$$
$$\langle \texttt{consistent}\ \ x\rangle \tag{1005}$$

By applying Rule 22 on Predicate 1004, using Predicate 999, we can derive the following predicate:

$$\langle y + z\rangle \xrightarrow{a} \surd \tag{1006}$$

By applying Rule 23 on above predicate, using Predicate 1005, we can derive the following predicate:

$$\langle x + (y + z)\rangle \xrightarrow{a} \surd \tag{1007}$$

(b) Rule P2 23

If Predicate 997 is derived from this rule, then from the premise the following must hold:

$$\langle z\rangle \xrightarrow{a} \surd \tag{1008}$$
$$\langle \texttt{consistent}\ \ x + y\rangle \tag{1009}$$

Predicate 1009 can only hold, if

$$\langle \texttt{consistent}\ \ x\rangle \tag{1010}$$
$$\langle \texttt{consistent}\ \ y\rangle \tag{1011}$$

By applying Rule 23 on Predicate 1008, using Predicate 1011, we can derive the following predicate:

$$\langle y + z\rangle \xrightarrow{a} \surd \tag{1012}$$

By applying Rule 22 on above predicate, using Predicate 1010, we can derive the following predicate:

$$\langle x + (y + z)\rangle \xrightarrow{a} \surd \tag{1013}$$

Right Implication

Suppose,

$$\langle x + (y + z) \rangle \xrightarrow{a} \surd \tag{1014}$$

A termination predicate for an alternative composition can be derived only from rules P2 22 or P2 23. We discuss them one by one:

(a) Rule P2-22

If Predicate 1014 is derived from this rule, then from the premise the following must hold:

$$\langle x \rangle \xrightarrow{a} \surd \tag{1015}$$

$$\langle \texttt{consistent}\ \ y + z \rangle \tag{1016}$$

Predicate 1016 can only hold, if

$$\langle \texttt{consistent}\ \ y \rangle \tag{1017}$$

$$\langle \texttt{consistent}\ \ z \rangle \tag{1018}$$

By applying Rule 22 on Predicate 1015, using Predicate 1017, we can derive the following predicate:

$$\langle x + y \rangle \xrightarrow{a} \surd \tag{1019}$$

By again applying Rule 22 on above predicate, using Predicate 1018, we can derive the following predicate:

$$\langle (x + y) + z \rangle \xrightarrow{a} \surd \tag{1020}$$

(b) Rule P2 23

If Predicate 1014 is derived from this rule, then from the premise the following must hold:

$$\langle y + z \rangle \xrightarrow{a} \surd \tag{1021}$$

$$\langle \texttt{consistent}\ \ x \rangle \tag{1022}$$

Again Predicate 1021 can be derived from Rule P2 22 or Rule P2-23.

i. *Rule P2 22:*
If Predicate 1021 is derived from this rule, then from the premise the following must hold:

$$\langle y \rangle \xrightarrow{a} \surd \tag{1023}$$

$$\langle \texttt{consistent}\ \ z \rangle \tag{1024}$$

By applying Rule 23 on Predicate 1023, using Predicate 1022, we can derive the following predicate:

$$\langle x + y \rangle \xrightarrow{a} \surd \tag{1025}$$

By applying Rule 22 on above predicate, using Predicate 1024, we can derive the following predicate:

$$\langle (x + y) + z \rangle \xrightarrow{a} \surd \tag{1026}$$

ii. *Rule P2 23:*
If Predicate 1021 is derived from this rule, then from the premise the following must hold:

$$\langle z \rangle \xrightarrow{a} \surd \tag{1027}$$

$$\langle \texttt{consistent}\ \ y \rangle \tag{1028}$$

Apply Rule P2-27 on predicates 1028 and 1022, we get:

$$\langle \texttt{consistent}\ \ x + y \rangle \tag{1029}$$

By applying Rule 23 on Predicate 1027, for any process term q with $\langle \texttt{consistent}\ \ q \rangle$, the following holds:

$$\langle q + z \rangle \xrightarrow{a} \surd$$

The term q can be $x + y$. Hence we have,

$$\langle (x + y) + z \rangle \xrightarrow{a} \surd \tag{1030}$$

6.

$$\langle (x + y) + z \rangle \overset{r}{\longmapsto}_{\perp} \iff \langle x + (y + z) \rangle \overset{r}{\longmapsto}_{\perp}$$

Left Implication

Suppose,

$$\langle (x + y) + z \rangle \overset{r}{\longmapsto}_{\perp} \tag{1031}$$

Rule P2-28 or Rule P2-29 can be used to derive the above transition.

(a) Rule P2-28
If Predicate 1031 is derived from this rule, then from the premise:

$$\langle x + y \rangle \overset{r}{\longmapsto}_{\perp} \tag{1032}$$

$$\langle \texttt{consistent}\ \ z \rangle \tag{1033}$$

$$\forall s < r\, \langle z \rangle \overset{s}{\not\longmapsto}_{\perp} \tag{1034}$$

Again Predicate 1032 can be derived from two rules. They are Rule P2-28 and Rule P2-29. We discuss them one by one.

i. *Rule P2-28*
If this rule is used to derive Predicate 1032, then from the premise of the rule, the following holds:

$$\langle x \rangle \xrightarrow{r}_{\perp} \tag{1035}$$

$$\langle \texttt{consistent}\ \ y \rangle \tag{1036}$$

$$\forall s < r \quad \langle y \rangle \not\xrightarrow{s}_{\perp} \tag{1037}$$

On Predicates 1033 and 1036, apply Rule P2-27, we get:

$$\langle \texttt{consistent}\ \ y + z \rangle \tag{1038}$$

A future inconsistency predicate for $y + z$ with duration $s \in (0, r)$ can either be derived from Rule P2-28, or Rule P2-29. From Predicates 1037 and 1034, none of the rules can be applied. Hence we can conclude,

$$\forall s < r\, \langle y + z \rangle \not\xrightarrow{s}_{\perp} \tag{1039}$$

On Predicates 1038, 1039 and 1035, apply Rule P2 28. We get:

$$\langle x + (y + z) \rangle \xrightarrow{r}_{\perp} \tag{1040}$$

ii. *Rule P2-29*
If this rule is used to derive Predicate 1032, then from the premise of the rule, the following holds:

$$\langle y \rangle \xrightarrow{r}_{\perp} \tag{1041}$$

$$\langle \texttt{consistent}\ \ x \rangle \tag{1042}$$

$$\forall s < r \quad \langle x \rangle \not\xrightarrow{s}_{\perp} \tag{1043}$$

On Predicates 1033, 1034 and 1041, apply Rule P2-28, we get:

$$\langle y + z \rangle \xrightarrow{r}_{\perp} \tag{1044}$$

On Predicates 1044, 1042 and 1043, apply Rule P2-29, we get:

$$\langle x + (y + z) \rangle \xrightarrow{r}_{\perp} \tag{1045}$$

(b) <u>Rule P2-29</u>
If Predicate 1031 is derived from this rule, then from the premise of the rule:

$$\langle z \rangle \xrightarrow{r}_{\perp} \tag{1046}$$

$$\langle \texttt{consistent}\ \ x + y \rangle \tag{1047}$$

$$\forall s < r\, \langle x + y \rangle \not\xrightarrow{s}_{\perp} \tag{1048}$$

Predicate 1047 can only be derived from Rule P2-27. Hence the premise of the rule must hold:

$$\langle \texttt{consistent } x \rangle \tag{1049}$$

$$\langle \texttt{consistent } y \rangle \tag{1050}$$

From Predicates 1048, 1049 and 1050 and Theorem 15, we conclude:

$$\forall s < r \, \langle x \rangle \overset{s}{\not\mapsto}_{\perp} \tag{1051}$$

$$\forall s < r \, \langle y \rangle \overset{s}{\not\mapsto}_{\perp} \tag{1052}$$

On Predicates 1046, 1050 and 1052, apply Rule P2-29, we get:

$$\langle y + z \rangle \overset{r}{\mapsto}_{\perp} \tag{1053}$$

Again join Predicates 1049, 1051 and 1053 and apply Rule P2-29, we get:

$$\langle x + (y + z) \rangle \overset{r}{\mapsto}_{\perp} \tag{1054}$$

Right Implication

Suppose,

$$\langle x + (y + z) \rangle \overset{r}{\mapsto}_{\perp} \tag{1055}$$

Rule P2-28 or Rule P2-29 can be used to derive the above predicate.

(a) Rule P2-28
If Predicate 1055 is derived from this rule, then from the premise of the rule:

$$\langle x \rangle \overset{r}{\mapsto}_{\perp} \tag{1056}$$

$$\langle \texttt{consistent } y + z \rangle \tag{1057}$$

$$\forall s < r \, \langle y + z \rangle \overset{s}{\not\mapsto}_{\perp} \tag{1058}$$

Predicate 1057 can only be derived from Rule P2-27. Hence the premise of the rule must hold:

$$\langle \texttt{consistent } y \rangle \tag{1059}$$

$$\langle \texttt{consistent } z \rangle \tag{1060}$$

From Predicates 1058, 1059 and 1060 and Theorem 15, we conclude:

$$\forall s < r \, \langle y \rangle \overset{s}{\not\mapsto}_{\perp} \tag{1061}$$

$$\forall s < r \, \langle z \rangle \overset{s}{\not\mapsto}_{\perp} \tag{1062}$$

On Predicates 1056, 1059, 1061 ,

$$\langle x + y \rangle \overset{r}{\mapsto}_{\perp} \tag{1063}$$

Again join Predicates (Predicates) 1060, 1062, and 1063, apply Rule P2-28, we get:

$$\langle (x+y)+z\rangle \xmapsto{r}_{\perp}$$

(b) Rule P2-29

If Predicate 1055 is derived from this rule, then from the premise:

$$\langle y+z\rangle \xmapsto{r}_{\perp} \tag{1064}$$

$$\langle \texttt{consistent}\ \ x\rangle \tag{1065}$$

$$\forall s < r\, \langle x\rangle \not\xmapsto{s}_{\perp} \tag{1066}$$

Again Predicate 1064 can be derived from two rules. They are Rule P2-28 and Rule P2-29. We discuss them one by one.

i. *Rule P2-28*

If this rule is used to derive Predicate 1064, then from the premise of the rule, the following holds:

$$\langle y\rangle \xmapsto{r}_{\perp} \tag{1067}$$

$$\langle \texttt{consistent}\ \ z\rangle \tag{1068}$$

$$\forall s < r \quad \langle z\rangle \not\xmapsto{s}_{\perp} \tag{1069}$$

On Predicates 1065, 1066 and 1067, apply Rule P2-29, we get:

$$\langle x+y\rangle \xmapsto{r}_{\perp} \tag{1070}$$

On Predicates 1070, 1068 and 1069, apply Rule P2-28, we get:

$$\langle (x+y)+z\rangle \xmapsto{r}_{\perp} \tag{1071}$$

ii. *Rule P2-29*

If this rule is used to derive Predicate 1064, then from the premise of the rule, the following holds:

$$\langle z\rangle \xmapsto{r}_{\perp} \tag{1072}$$

$$\langle \texttt{consistent}\ \ y\rangle \tag{1073}$$

$$\forall s < r\, \langle y\rangle \not\xmapsto{s}_{\perp} \tag{1074}$$

On Predicates 1065 and 1073, apply Rule P2-27, we get:

$$\langle \texttt{consistent}\ \ x+y\rangle \tag{1075}$$

A future inconsistency predicate for $x+y$ with duration $s \in (0, r)$ can either be derived from Rule P2-28, or Rule P2-29. From Predicates 1066 and 1074 none of the rules can be applied. Hence we can conclude,

$$\forall s < r\, \langle x+y\rangle \not\xmapsto{r}_{\perp} \tag{1076}$$

On Predicates 1072, 1075 and 1076, apply Rule P2 29. We get:

$$\langle (x+y)+z\rangle \xmapsto{r}_{\perp} \tag{1077}$$

7.

$$\langle\texttt{consistent}\ (x+y)+z\rangle \iff \langle\texttt{consistent}\ x+(y+z)\rangle$$

Left Implication

Suppose,

$$\langle\texttt{consistent}\ (x+y)+z\rangle \tag{1078}$$

The above predicate can only be derived from Rule P2-27. From the premise of the rule, the following holds:

$$\langle\texttt{consistent}\ x+y\rangle \tag{1079}$$

$$\langle\texttt{consistent}\ z\rangle \tag{1080}$$

Again Predicate 1080 can only be derived from Rule P2-27. From the premise of the rule, the following holds:

$$\langle\texttt{consistent}\ x\rangle \tag{1081}$$

$$\langle\texttt{consistent}\ y\rangle \tag{1082}$$

Apply Rule P2-27 on Predicates 1082 and 1080, we get:

$$\langle\texttt{consistent}\ y+z\rangle \tag{1083}$$

Again apply Rule P2-27 on Predicates 1083 and 1081, we get:

$$\langle\texttt{consistent}\ x+(y+z)\rangle$$

Hence the left implication is proved.

Right Implication

Suppose,

$$\langle\texttt{consistent}\ x+(y+z)\rangle \tag{1084}$$

The above predicate can only be derived from Rule P2-27. From the premise of the rule, the following holds:

$$\langle\texttt{consistent}\ x\rangle \tag{1085}$$

$$\langle\texttt{consistent}\ y+z\rangle \tag{1086}$$

Again Predicate 1086 can only be derived from Rule P2-27. From the premise of the rule, the following holds:

$$\langle\texttt{consistent}\ y\rangle \tag{1087}$$

$$\langle\texttt{consistent}\ z\rangle \tag{1088}$$

Apply Rule P2-27 on Predicates 1087 and 1085, we get:

$$\langle\texttt{consistent}\ x+y\rangle \tag{1089}$$

Again apply Rule P2-27 on Predicates 1089 and 1088, we get:

$$\langle\texttt{consistent}\ (x+y)+z\rangle$$

Hence the right implication is proved.

H.6 Axiom A3 (Idempotency)

$x + x = x$ (Idempotency-A3)

We need to prove, $x + x \leftrightarrows x$.

Let R be a binary relation on process terms defined as follows:

$$R = \{(x + x, x) \mid x \in P\}$$

We prove that the relation $R \cup \mathcal{I}$ satisfies all conditions of a bisimulation relation. Below, we only prove that all pairs in R satisfy the conditions of bisimulation relation.

1. $$\langle x + x \rangle \xrightarrow{a} \langle p \rangle \implies \begin{array}{l} \exists z \in P : \langle x \rangle \xrightarrow{a} \langle z \rangle \\ \text{and } (p, z) \in R. \end{array}$$

   Suppose,

   $$\langle x + x \rangle \xrightarrow{a} \langle p \rangle \tag{1090}$$

   The above action step can be derived from either Rule P2-20 or Rule P2-21. The premise of each rule requires that the following holds:

   $$\langle x \rangle \xrightarrow{a} \langle p \rangle$$

   Consider the target process terms in the transition above and 1090. The pair (p, p) is in $\mathcal{I}$.

2. $$\langle x \rangle \xrightarrow{a} \langle p \rangle \implies \begin{array}{l} \exists z \in P : \langle x + x \rangle \xrightarrow{a} \langle z \rangle \\ \text{and } (z, p) \in R. \end{array}$$

   Suppose,

   $$\langle x \rangle \xrightarrow{a} \langle p \rangle \tag{1091}$$

   Then from Theorem 14, the following holds:

   $$\langle \texttt{consistent } x \rangle \tag{1092}$$

   Apply Rule P2-20 or Rule P2-21 on 1091 and 1092. We get:

   $$\langle x + x \rangle \xrightarrow{a} \langle p \rangle \tag{1093}$$

   Consider the target process terms in Transitions 1093 and 1091. The pair (p, p) is in $\mathcal{I}$.

3. $$\langle x + x \rangle \xmapsto{r} \langle p \rangle \implies \begin{array}{l} \exists z \in P : \langle x \rangle \xmapsto{r} \langle z \rangle \\ \text{and } (p, z) \in R. \end{array}$$

   Suppose,

   $$\langle x + x \rangle \xmapsto{r} \langle p \rangle \tag{1094}$$

The premises (namely $x \overset{r}{\not\mapsto}$) of Rule P2-25 and Rule P2-26 are not satisfied. Hence Transition 1094 can only be derived from Rule P2-24.

From Premise of Rule P2-24, for some process terms x_1 and y_1, $p = x_1 + y_1$. Rewriting Transition 1091, we get:

$$\langle x + x \rangle \overset{r}{\longmapsto} \langle x_1 + y_1 \rangle \tag{1095}$$

Also from the premise of Rule P2-24 the following holds:

$$\langle x \rangle \overset{r}{\longmapsto} \langle x_1 \rangle \tag{1096}$$

$$\langle x \rangle \overset{r}{\longmapsto} \langle y_1 \rangle \tag{1097}$$

By Theorem 16, $x_1 \equiv y_1$. Hence rewriting Transition 1095, we get:

$$\langle x + x \rangle \overset{r}{\longmapsto} \langle x_1 + x_1 \rangle \tag{1098}$$

Consider the target process terms in Transitions 1096 and 1098. The pair $(x_1 + x_1, x_1)$ is in R.

4.

$$\langle x \rangle \overset{r}{\longmapsto} \langle p \rangle \implies \begin{array}{l} \exists z \in P : \langle x + x \rangle \overset{r}{\longmapsto} \langle z \rangle \\ \text{and } (p, z) \in R. \end{array}$$

Suppose,

$$\langle x \rangle \overset{r}{\longmapsto} \langle p \rangle \tag{1099}$$

Then by Rule P2-24, the following holds:

$$\langle x + x \rangle \overset{r}{\longmapsto} \langle p + p \rangle \tag{1100}$$

Consider the target process terms in Transitions 1099 and 1100. The pair $(p + p, p)$ is in R.

5.

$$\langle x \rangle \xrightarrow{a} \surd \iff \langle x + x \rangle \xrightarrow{a} \surd$$

Left Implication

Suppose,

$$\langle x \rangle \xrightarrow{a} \surd \tag{1101}$$

Then from Theorem 14, the following holds:

$$\langle \texttt{consistent}\ \ x \rangle \tag{1102}$$

Apply Rule P2-22 or Rule P2-23 on 1101 and 1102. We get:

$$\langle x + x \rangle \xrightarrow{a} \surd$$

Right Implication

Suppose,

$$\langle x + x \rangle \xrightarrow{a} \surd$$

The above predicate can be derived from either Rule P2-22 or Rule P2-23. The premise of each rule requires that the following holds:

$$\langle x \rangle \xrightarrow{a} \surd$$

Proved.

6.

$$\langle x \rangle \overset{r}{\longmapsto}_{\perp} \iff \langle x + x \rangle \overset{r}{\longmapsto}_{\perp}$$

Left Implication

Suppose,

$$\langle x \rangle \overset{r}{\longmapsto}_{\perp} \tag{1103}$$

Then from Theorem 14, the following holds:

$$\langle \texttt{consistent} \;\; x \rangle \tag{1104}$$

Then from Theorem 15, the following holds:

$$\forall s < r, \langle x \rangle \overset{s}{\not\longmapsto}_{\perp} \tag{1105}$$

Apply one of Rule P2-28 or Rule P2-29 on Predicates 1103, 1104 and 1105. We get:

$$\langle x + x \rangle \overset{r}{\longmapsto}_{\perp}$$

Right Implication

Suppose,

$$\langle x + x \rangle \overset{r}{\longmapsto}_{\perp}$$

The above predicate can be derived from either Rule P2-28 or Rule P2-29. The premise of each rule requires that the following holds:

$$\langle x \rangle \overset{r}{\longmapsto}_{\perp}$$

Proved.

7.

$$\langle \texttt{consistent} \;\; x + x \rangle \iff \langle \texttt{consistent} \;\; x \rangle$$

Left Implication

Suppose,

$$\langle \texttt{consistent} \;\; x + x \rangle$$

This predicate can only be derived from Rule P2-27. From the premise of the rule,

$$\langle \texttt{consistent}\ x\rangle$$

Right Implication
Suppose,

$$\langle \texttt{consistent}\ x\rangle$$

Apply Rule P2-27. We get:

$$\langle \texttt{consistent}\ x+x\rangle$$

H.7 Axiom A4 (Right Distributivity)

$(x+y)\cdot z = x\cdot z + y\cdot z$ (Right Distributivity-A4).

We need to prove, $(x+y)\cdot z \leftrightarrows x\cdot z + y\cdot z$.

Let R be a binary relation on process terms defined as follows:

$$R \;=\; \{\;\; ((x+y)\cdot z, x\cdot z + y\cdot z) \mid x,y,z \in P\}$$

We show that the relation $R\cup\mathcal{I}$ is a bisimulation relation. Below we prove that all pairs in R satisfy the conditions of bisimulation.

For all $a\in A, r>0, x,y,z,p\in P$, the following holds:

1.
$$\langle (x+y)\cdot z\rangle \xrightarrow{a} \langle p\rangle \implies \begin{array}{l}\exists p'\in P : \langle x\cdot z + y\cdot z\rangle \xrightarrow{a} \langle p'\rangle \\ \text{and } (p,p')\in R\cup\mathcal{I}\end{array}$$

Suppose,

$$\langle (x+y)\cdot z\rangle \xrightarrow{a} \langle p\rangle \tag{1106}$$

An action transition for a sequential composition can be derived only from rules P2 15 or P2 16. We discuss them one by one:

(a) Rule P2 15

Then for some process term p', $p = p'\cdot z$. Rewriting Transition 1106, we get:

$$\langle (x+y)\cdot z\rangle \xrightarrow{a} \langle p'\cdot z\rangle \tag{1107}$$

From the premise of Rule P2 15, the following holds:

$$\langle x+y\rangle \xrightarrow{a} \langle p'\rangle \tag{1108}$$

The above transition can be derived from Rules P2 20 or P2 21.

i. *Rule P2 20*
If Transition 1108 is derived from this rule, then from the premise of the rule the following holds:

$$\langle x\rangle \xrightarrow{a} \langle p'\rangle \tag{1109}$$
$$\langle \texttt{consistent}\ \ y\rangle \tag{1110}$$

Apply Rule P2 15 on Transition 1109, we get:

$$\langle x\cdot z\rangle \xrightarrow{a} \langle p'\cdot z\rangle \tag{1111}$$

From Rule P2 20, for any term q with $\langle \texttt{consistent}\ \ q\rangle$, the following can be derived:

$$\langle x\cdot z+q\rangle \xrightarrow{a} \langle p'\cdot z\rangle \tag{1112}$$

From Predicate 1110, we can infer by using Rule P2-18, $\langle \texttt{consistent}\ \ y\cdot z\rangle$. Then q in Transition 1112 can be $y\cdot z$. Hence we get:

$$\langle x\cdot z+y\cdot z\rangle \xrightarrow{a} \langle p'\cdot z\rangle \tag{1113}$$

Consider the target process terms in Transitions 1107 and 1113. The pair $(p'\cdot z, p'\cdot z)$ is in $\mathcal{I}$.

ii. *Rule P2 21*
If Transition 1108 is derived from this rule, then from the premise of the rule, the following holds:

$$\langle y\rangle \xrightarrow{a} \langle p'\rangle \tag{1114}$$
$$\langle \texttt{consistent}\ \ x\rangle \tag{1115}$$

Similar reasoning as given above for Rule 20 applies here too.

(b) <u>Rule P2 16</u>
If this rule is used to derive transition 1106 Then, $p=z$. Rewriting Transition 1106, we get:

$$\langle (x+y)\cdot z\rangle \xrightarrow{a} \langle z\rangle \tag{1116}$$

And from the premise of Rule P2 16, the following holds:

$$\langle x+y\rangle \xrightarrow{a} \surd \tag{1117}$$
$$\langle \texttt{consistent}\ \ z\rangle \tag{1118}$$

The Transition 1117 can be derived from Rules P2 22 or P2 23.

i. *Rule P2 22*
If Transition 1117 is derived from this rule, then from the premise of the rule the following holds:

$$\langle x\rangle \xrightarrow{a} \surd \tag{1119}$$
$$\langle \texttt{consistent}\ \ y\rangle \tag{1120}$$

From Predicate 1118, we have $\langle \texttt{consistent}\ z \rangle$. Apply Rule P2 16 on Transition 1119 using process term z, we get:

$$\langle x \cdot z \rangle \xrightarrow{a} \langle z \rangle \tag{1121}$$

From Rule P2 20, for any term q with $\langle \texttt{consistent}\ q \rangle$, the following can be derived:

$$\langle x \cdot z + q \rangle \xrightarrow{a} \langle z \rangle \tag{1122}$$

From Predicate 1120, we infer $\langle \texttt{consistent}\ y \cdot z \rangle$. Then q in Transition 1122 can be $y \cdot z$. Hence we get:

$$\langle x \cdot z + y \cdot z \rangle \xrightarrow{a} \langle z \rangle \tag{1123}$$

Consider the target process terms in Transitions 1116 and 1123. The pair (z, z) is in $\mathcal{I}$.

ii. *Rule P2 23*
Similar reasoning as above applies.

2.
$$\langle x \cdot z + y \cdot z \rangle \xrightarrow{a} \langle p \rangle \implies \begin{array}{l} \exists p' \in P : \langle (x + y) \cdot z \rangle \xrightarrow{a} \langle p' \rangle \\ \text{and } (p', p) \in R \cup \mathcal{I} \end{array}$$

Suppose,

$$\langle x \cdot z + y \cdot z \rangle \xrightarrow{a} \langle p \rangle \tag{1124}$$

The above transition can be derived from Rules P2 20 or P2 21.

(a) Rule P2 20
If Transition 1124 is derived from this rule, then from the premise of the rule the following holds:

$$\langle x \cdot z \rangle \xrightarrow{a} \langle p \rangle \tag{1125}$$

$$\langle \texttt{consistent}\ y \cdot z \rangle \tag{1126}$$

Transition 1125 can be derived from Rule P2 15 or Rule P2 16. We discuss the two rules one by one.

i. *Rule P2 15*
Then for some process term p', $p = p' \cdot z$. Rewriting Transition 1124 and 1125, we get:

$$\langle x \cdot z + y \cdot z \rangle \xrightarrow{a} \langle p' \cdot z \rangle \tag{1127}$$

$$\langle x \cdot z \rangle \xrightarrow{a} \langle p' \cdot z \rangle \tag{1128}$$

From premise of the Rule P2 15, the following holds:

$$\langle x \rangle \xrightarrow{a} \langle p' \rangle$$

Apply Rule P2 20 on the above transition. Then for any term q, with $\langle \texttt{consistent}\ \ q\rangle$, the following holds:

$$\langle x + q\rangle \xrightarrow{a} \langle p'\rangle \tag{1129}$$

From Predicate 1126, it can be inferred that $\langle \texttt{consistent}\ \ y\rangle$. Hence q can be replaced by y in Transition 1129.

$$\langle x + y\rangle \xrightarrow{a} \langle p'\rangle \tag{1130}$$

Apply Rule P2 15 on the above Transition, we get:

$$\langle (x + y) \cdot z\rangle \xrightarrow{a} \langle p' \cdot z\rangle \tag{1131}$$

ii. *Rule P2 16*
If this rule is used to derive Transition 1125, then $p = z$. Rewriting Transitions 1124 and 1125:

$$\langle x \cdot z + y \cdot z\rangle \xrightarrow{a} \langle z\rangle \tag{1132}$$

$$\langle x \cdot z\rangle \xrightarrow{a} \langle z\rangle \tag{1133}$$

From Premise of Rule P2 16, the following holds:

$$\langle x\rangle \xrightarrow{a} \surd \tag{1134}$$

$$\langle \texttt{consistent}\ \ z\rangle \tag{1135}$$

Let q b a term with $\langle \texttt{consistent}\ \ q\rangle$. Apply Rule P2 22 on Transition 1134, we get:

$$\langle x + q\rangle \xrightarrow{a} \surd \tag{1136}$$

Consider Predicate 1126, $\langle \texttt{consistent}\ \ y{\cdot}z\rangle$. From the predicate, it can be inferred that:

$$\langle \texttt{consistent}\ \ y\rangle$$

Replace q by y in Transition 1136:

$$\langle x + y\rangle \xrightarrow{a} \surd \tag{1137}$$

From Predicate 1135, $\langle \texttt{consistent}\ \ z\rangle$. Using term z, apply Rule 16 on Transition 1137, we get:

$$\langle (x + y) \cdot z\rangle \xrightarrow{a} \langle z\rangle \tag{1138}$$

Consider the target process terms in Transitions 1132 and 1138. The pair (z, z) is in $\mathcal{I}$.

(b) Rule P2 21
Similar reasoning as given above applies.

3.
$$\langle (x+y)\cdot z\rangle \overset{r}{\longmapsto} \langle p\rangle \implies \begin{array}{l}\exists z' \in P : \langle x\cdot z + y\cdot z\rangle \overset{r}{\longmapsto} \langle z'\rangle \\ \text{and } (p, z') \in R\cup\mathcal{I}\end{array}$$

Suppose,

$$\langle (x+y)\cdot z\rangle \overset{r}{\longmapsto} \langle p\rangle \tag{1139}$$

A time transition for a sequential composition can be derived only from rule P2 17. Then, for some $p' \in P$, p must be equal to $p'\cdot z$. Rewriting Transition 1139:

$$\langle (x+y)\cdot z\rangle \overset{r}{\longmapsto} \langle p'\cdot z\rangle \tag{1140}$$

And the following must hold from premise of Rule P2 17:

$$\langle x+y\rangle \overset{r}{\longmapsto} \langle p'\rangle \tag{1141}$$

The above transition can be derived from Rules P2 24, P2 25 or P2 26. We discuss them one by one:

(a) Rule P2 24

Then for some process term x_1, x_2, $p' = x_1 + x_2$. Re-writing Transitions 1140 and 1141:

$$\langle (x+y)\cdot z\rangle \overset{r}{\longmapsto} \langle (x_1+x_2)\cdot z\rangle \tag{1142}$$

$$\langle x+y\rangle \overset{r}{\longmapsto} \langle x_1+x_2\rangle \tag{1143}$$

From premise of Rule P2 24, the following must hold:

$$\begin{array}{c}\langle x\rangle \overset{r}{\longmapsto} \langle x_1\rangle \\ \langle y\rangle \overset{r}{\longmapsto} \langle x_2\rangle\end{array}$$

Apply Rule P2 17 on the above transitions, we get:

$$\langle x\cdot z\rangle \overset{r}{\longmapsto} \langle x_1\cdot z\rangle \tag{1144}$$

$$\langle y\cdot z\rangle \overset{r}{\longmapsto} \langle x_2\cdot z\rangle \tag{1145}$$

Apply Rule P2 24 on the above two transitions, we get:

$$\langle x\cdot x + y\cdot z\rangle \overset{r}{\longmapsto} \langle x_1\cdot z + x_2\cdot z\rangle \tag{1146}$$

Consider Transitions 1142 and 1146. The pair of their target process terms $((x_1+x_2)\cdot z, x_1\cdot z + x_2\cdot z)$ is in R.

(b) Rule P2 25

If Transition 1141 is derived from this rule, the from the premise of the rule, the following must hold:

$$\langle x\rangle \overset{r}{\longmapsto} \langle p'\rangle \tag{1147}$$

$$\langle \texttt{consistent}\ y\rangle \tag{1148}$$

$$\langle y\rangle \overset{r}{\not\longmapsto} \tag{1149}$$

$$\forall s \le r,\ \ \langle y\rangle \overset{s}{\not\longmapsto}_{\perp} \tag{1150}$$

Apply Rule P2 17 on Transition 1147.

$$\langle x \cdot z \rangle \xmapsto{r} \langle p' \cdot z \rangle \tag{1151}$$

We can infer from Predicate 1149 the following:

$$\langle y \cdot z \rangle \not\xmapsto{r} \tag{1152}$$

We can infer from Predicate 1148 by using Rule P2-18:

$$\langle \texttt{consistent}\ \ y \cdot z \rangle \tag{1153}$$

We can infer from predicate 1150

$$\forall s \leq r, \ \ \langle y \cdot z \rangle \not\xmapsto{s}_{\perp} \tag{1154}$$

Join Transitions (Predicates) 1151, 1152,1153 and 1154 and apply Rule P2-25. We get:

$$\langle x \cdot z + y \cdot z \rangle \xmapsto{r} \langle p' \cdot z \rangle \tag{1155}$$

Consider Transitions 1140 and 1155. The pair of their target process terms $(p' \cdot z, p' \cdot z)$ is in $\mathcal{I}$.

(c) Rule P2 26

If Transition 1141 is derived from this rule, then from the premise of the rule, the following must hold:

$$\langle y \rangle \xmapsto{r} \langle p' \rangle \tag{1156}$$

$$\langle \texttt{consistent}\ \ x \rangle \tag{1157}$$

$$\langle x \rangle \not\xmapsto{r} \tag{1158}$$

$$\forall s \leq r, \ \ \langle x \rangle \not\xmapsto{s}_{\perp} \tag{1159}$$

Similar reasoning as given above for Rule P2-25 should be applied here.

4.

$$\langle x \cdot z + y \cdot z \rangle \xmapsto{r} \langle p \rangle \implies \begin{array}{l} \exists z' \in P : \langle (x + y) \cdot z \rangle \xmapsto{r} \langle z' \rangle \\ \text{and } (z', p) \in R \cup \mathcal{I} \end{array}$$

Suppose,

$$\langle x \cdot z + y \cdot z \rangle \xmapsto{r} \langle p \rangle \tag{1160}$$

The above transition can be derived from Rule P2-24, Rule P2-25 or Rule P2-26. We discuss them one by one:

(a) Rule P2-24

Then for some process terms x', y', $p = x' + y'$. Rewriting Transition 1160,

$$\langle x \cdot z + y \cdot z \rangle \xmapsto{r} \langle x' + y' \rangle \tag{1161}$$

And from the premise of Rule P2-24, the following holds:

$$\langle x \cdot z \rangle \xmapsto{r} \langle x' \rangle \tag{1162}$$
$$\langle y \cdot z \rangle \xmapsto{r} \langle y' \rangle \tag{1163}$$

A time step for a sequential composition can only be derived from Rule P2-17. Then for some process terms x_1 and y_1, $x' = x_1 \cdot z$ and $y' = y_1 \cdot z$. Rewriting Transition 1161:

$$\langle x \cdot z + y \cdot z \rangle \xmapsto{r} \langle x_1 \cdot z + y_1 \cdot z \rangle \tag{1164}$$

From premise of Rule P2-17, the following must hold:

$$\langle x \rangle \xmapsto{r} \langle x_1 \rangle \tag{1165}$$
$$\langle y \rangle \xmapsto{r} \langle y_1 \rangle \tag{1166}$$

Apply Rule P2-24 on above transitions, we get:

$$\langle x + y \rangle \xmapsto{r} \langle x_1 + y_1 \rangle \tag{1167}$$

Apply Rule P2-17 on above transition. We get:

$$\langle (x + y) \cdot z \rangle \xmapsto{r} \langle (x_1 + y_1) \cdot z \rangle \tag{1168}$$

Consider Transitions 1164 and 1168. The pair of their target process terms $(x_1 \cdot z + y_1 \cdot z, (x_1 + y_1) \cdot z)$ is in R.

(b) Rule P2-25

If Transition 1160 is derived from this rule, the from the premise of the rule, the following must hold:

$$\langle x \cdot z \rangle \xmapsto{r} \langle p \rangle \tag{1169}$$
$$\langle \texttt{consistent}\ \ y \cdot z \rangle \tag{1170}$$
$$\langle y \cdot z \rangle \not\xmapsto{r} \tag{1171}$$
$$\forall s \leq r,\ \ \langle y \cdot z \rangle \not\xmapsto{s}_{\perp} \tag{1172}$$

Transition 1169 can only be derived from Rule P2 17. Then for some process term p', $p = p' \cdot z$. Rewriting Transition 1160 and Transition 1169, we get:

$$\langle x \cdot z + y \cdot z \rangle \xmapsto{r} \langle p' \cdot z \rangle \tag{1173}$$
$$\langle x \cdot z \rangle \xmapsto{r} \langle p' \cdot z \rangle \tag{1174}$$
$$\tag{1175}$$

From the premise of Rule P2-17, the following must hold:

$$\langle x \rangle \xmapsto{r} \langle p' \rangle \tag{1176}$$

Predicate 1171 can hold if Rule P2-17 cannot apply. Hence, its premise must not hold:

$$\langle y \rangle \not\xmapsto{r} \tag{1177}$$

Predicate 1170 can only be derived from Rule P2-18. Hence, its premise must hold:

$$\langle \texttt{consistent}\ \ y \rangle \tag{1178}$$

Predicate 1172 can hold if Rule P2-19 cannot apply. Hence, its premise must not hold:

$$\forall s \leq r,\ \ \langle y \rangle \not\xmapsto{s}_{\perp} \tag{1179}$$

Join Transitions (Predicates) 1176, 1177,1178 and 1179 and apply Rule P2-25. We get:

$$\langle x + y \rangle \xmapsto{r} \langle p' \rangle \tag{1180}$$

Apply Rule 17 on above transition, we get:

$$\langle (x + y) \cdot z \rangle \xmapsto{r} \langle p' \cdot z \rangle \tag{1181}$$

Consider Transitions 1173 and 1181. The pair of their target process terms $(p' \cdot z, p' \cdot z)$ is in $\mathcal{I}$.

(c) Rule P2-26

Same reasoning as given for Rule P2-25 applies here.

5.

$$\langle (x + y) \cdot z \rangle \xrightarrow{a} \surd \iff \langle x \cdot z + y \cdot z \rangle \xrightarrow{a} \surd$$

Left Implication

Suppose,

$$\langle (x + y) \cdot z \rangle \xrightarrow{a} \surd \tag{1182}$$

For a sequential composition, a termination predicate cannot be derived from any rules. Hence our supposition is wrong and the left implication trivially holds.

Right Implication

Suppose,

$$\langle x \cdot z + y \cdot z \rangle \xrightarrow{a} \surd \tag{1183}$$

The above predicate can be derived from Rule P2-22 or Rule P2-23.

(a) Rule P2-22

If Predicate 1183 is derived from this rule, then the following must hold:

$$\langle x \cdot z \rangle \xrightarrow{a} \surd \tag{1184}$$

$$\langle \texttt{consistent } y \cdot z \rangle \tag{1185}$$

Predicate 1184 cannot be derived from any rules. Hence Predicate 1183 cannot be derived from this rule.

(b) Rule P2-22

If Predicate 1183 is derived from this rule, then the following must hold:

$$\langle y \cdot z \rangle \xrightarrow{a} \surd \tag{1186}$$

$$\langle \texttt{consistent } x \cdot z \rangle \tag{1187}$$

Predicate 1186 cannot be derived from any rules. Hence Predicate 1183 cannot be derived from this rule.

The Predicate 1183 cannot be derived from any rules. Hence the right implication holds trivially.

6.

$$\langle (x + y) \cdot z \rangle \overset{r}{\longmapsto}_{\perp} \Longleftrightarrow \langle x \cdot z + y \cdot z \rangle \overset{r}{\longmapsto}_{\perp}$$

Left Implication

Suppose,

$$\langle (x + y) \cdot z \rangle \overset{r}{\longmapsto}_{\perp} \tag{1188}$$

A future Inconsistency predicate for a sequential composition can be derived only from rule P2 19. Then, the following must hold from premise of the rule:

$$\langle x + y \rangle \overset{r}{\longmapsto}_{\perp} \tag{1189}$$

The above transition can be derived from Rules P2 28 or P2 29.

(a) Rule P2 28

If Transition 1189 is derived from this rule, the from the premise of the rule, the following must hold:

$$\langle x \rangle \overset{r}{\longmapsto}_{\perp} \tag{1190}$$

$$\langle \texttt{consistent } y \rangle \tag{1191}$$

$$\forall s < r, \ \langle y \rangle \overset{s}{\not\longmapsto}_{\perp} \tag{1192}$$

Apply Rule P2 19 on Predicate 1190.

$$\langle x \cdot z \rangle \overset{r}{\longmapsto}_{\perp} \tag{1193}$$

We can infer from Predicate 1191 by using Rule P2-18:

$$\langle \texttt{consistent}\ y \cdot z \rangle \tag{1194}$$

From Predicate 1192, Rule P2 19 cannot be applied. It is the only rule by which a future Inconsistency predicate for a sequential composition can be derived. Hence we can conclude that:

$$\forall s < r,\ \ \langle y \cdot z \rangle \overset{s}{\not\longmapsto}_{\perp} \tag{1195}$$

Join Transitions (Predicates) 1193, 1194 and 1195 and apply Rule P2-28. We get:

$$\langle x \cdot z + y \cdot z \rangle \overset{r}{\longmapsto}_{\perp} \tag{1196}$$

(b) Rule P2 29
Similar reasoning as given above for Rule P2-28 is applied here.

Right Implication
Suppose,

$$\langle x \cdot z + y \cdot z \rangle \overset{r}{\longmapsto}_{\perp} \tag{1197}$$

A future inconsistency predicate for an alternative composition can only be derived from Rule P2-28 or Rule P2-29. We discuss them one by one:

(a) Rule P2-28

If Predicate 1197 is derived from this rule, then from the premise of the rule, the following must hold:

$$\langle x \cdot z \rangle \overset{r}{\longmapsto}_{\perp} \tag{1198}$$

$$\langle \texttt{consistent}\ y \cdot z \rangle \tag{1199}$$

$$\forall s < r,\ \ \langle y \cdot z \rangle \overset{s}{\not\longmapsto}_{\perp} \tag{1200}$$

Predicate 1198 can only be derived from Rule P2 19. Then from the premise of the rule, the following must hold:

$$\langle x \rangle \overset{r}{\longmapsto}_{\perp} \tag{1201}$$

Predicate 1199 can only be derived from Rule P2-18. Hence, its premise must hold:

$$\langle \texttt{consistent}\ y \rangle \tag{1202}$$

Predicate 1200 can only hold if Rule P2-19 cannot be applied. Hence, its premise may not hold, we conclude that:

$$\forall s < r,\ \ \langle y \rangle \overset{s}{\not\longmapsto}_{\perp} \tag{1203}$$

Join Transitions (Predicates) 1201, 1202 and 1203 and apply Rule P2-28. We get:

$$\langle x + y \rangle \overset{r}{\longmapsto} \perp \tag{1204}$$

Apply Rule 19 on above transition, we get:

$$\langle (x + y) \cdot z \rangle \overset{r}{\longmapsto} \perp \tag{1205}$$

Hence proved.

(b) Rule P2-29
Same reasoning as given for Rule P2-28 applies here.

7.

$$\langle \texttt{consistent}\ \ (x + y) \cdot z \rangle \iff \langle \texttt{consistent}\ \ x \cdot z + y \cdot z \rangle$$

Left Implication

Suppose,

$$\langle \texttt{consistent}\ \ (x + y) \cdot z \rangle \tag{1206}$$

This is only derivable from Rule P2 18. From the premise of the rule, the following must hold:

$$\langle \texttt{consistent}\ \ x + y \rangle \tag{1207}$$

This can only be derived from Rule P2 27. Then from the premise of the rules, the following hold:

$$\langle \texttt{consistent}\ \ x \rangle \tag{1208}$$

$$\langle \texttt{consistent}\ \ y \rangle \tag{1209}$$

Apply Rule P2 18 on Predicate 1208 with process term z. We get:

$$\langle \texttt{consistent}\ \ x \cdot z \rangle \tag{1210}$$

Apply Rule 18 on Predicate 1209 also with process term z. We get:

$$\langle \texttt{consistent}\ \ y \cdot z \rangle \tag{1211}$$

Apply Rule P2-27 on Predicates 1210 and 1211, we get:

$$\langle \texttt{consistent}\ \ x \cdot z + y \cdot z \rangle$$

Hence left implication is proved.

Right Implication
Suppose,

$$\langle \texttt{consistent}\ \ x \cdot z + y \cdot z \rangle$$

The above predicate can only be derived from Rule P2 27.

From the premise of the rule, the following must hold:

$$\langle \texttt{consistent}\ x \cdot z \rangle \tag{1212}$$

$$\langle \texttt{consistent}\ y \cdot z \rangle \tag{1213}$$

The above predicates can only be derived from Rule P2 18. Then the following must hold:

$$\langle \texttt{consistent}\ x \rangle \tag{1214}$$

$$\langle \texttt{consistent}\ y \rangle \tag{1215}$$

Apply Rule P2 27 on the above predicates, we get:

$$\langle \texttt{consistent}\ x + y \rangle$$

Apply Rule P2-18 on the above predicate, we get:

$$\langle \texttt{consistent}\ (x + y) \cdot z \rangle$$

Hence right implication is proved.

H.8 Axiom A5

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$ (Associativity of sequential composition-A5).

We need to prove, $(x \cdot y) \cdot z \leftrightarrows x \cdot (y \cdot z)$.

Let R be a binary relation on process terms defined as follows:

$$R = \{ \ ((x \cdot y) \cdot z, x \cdot (y \cdot z)) \mid x, y, z \in P\}$$

We prove that the relation $R \cup \mathcal{I}$ is a bisimulation relation. Below we show that all pairs in R satisfy the conditions of bisimulation.

For all $a \in A, r \in R^{>0}, x, y, z, p \in P$, the following holds:

1.
$$\langle (x \cdot y) \cdot z \rangle \xrightarrow{a} \langle p \rangle \implies \begin{array}{l} \exists z' \in P : \langle x \cdot (y \cdot z) \rangle \xrightarrow{a} \langle z' \rangle \\ \text{and } (p, z') \in R \cup \mathcal{I} \end{array}$$

Suppose,

$$\langle (x \cdot y) \cdot z \rangle \xrightarrow{a} \langle p \rangle \tag{1216}$$

An action transition for a sequential composition can be derived only from rules P2 15 or P2 16. We discuss them one by one:

(a) Rule P2 15

Then for some process term p', $p = p' \cdot z$. Rewriting Transition 1216, we get:

$$\langle (x \cdot y) \cdot z \rangle \xrightarrow{a} \langle p' \cdot z \rangle \tag{1217}$$

From the premise of Rule P2 15, the following holds:

$$\langle x \cdot y \rangle \xrightarrow{a} \langle p' \rangle \tag{1218}$$

The above transition can be derived from Rules P2 15 or P2 16.

i. *Rule P2 15*
If Transition 1218 is derived from this rule, then for some process term p'', $p' = p'' \cdot y$. Rewriting Transitions 1217 and 1218, we get:

$$\langle (x \cdot y) \cdot z \rangle \xrightarrow{a} \langle (p'' \cdot y) \cdot z \rangle \tag{1219}$$
$$\langle x \cdot y \rangle \xrightarrow{a} \langle p'' \cdot y \rangle \tag{1220}$$

From premise of Rule P2 15 the following holds:

$$\langle x \rangle \xrightarrow{a} \langle p'' \rangle \tag{1221}$$

Apply Rule P2 15 on Transition 1221. For any process term q we get:

$$\langle x \cdot q \rangle \xrightarrow{a} \langle p'' \cdot q \rangle$$

The term q can be $y \cdot z$.

$$\langle x \cdot (y \cdot z) \rangle \xrightarrow{a} \langle p'' \cdot (y \cdot z) \rangle \tag{1222}$$

Consider the target process terms in Transitions 1219 and 1222. The pair $((p'' \cdot y) \cdot z, p'' \cdot (y \cdot z))$ is in R.

ii. *Rule P2 16*
If Transition 1218 is derived from this rule, then for some process term, $p' = y$. Rewriting Transitions 1217 and 1218, we get:

$$\langle (x \cdot y) \cdot z \rangle \xrightarrow{a} \langle y \cdot z \rangle \tag{1223}$$
$$\langle x \cdot y \rangle \xrightarrow{a} \langle y \rangle \tag{1224}$$

From premise of Rule P2 16 the following holds:

$$\langle x \rangle \xrightarrow{a} \surd \tag{1225}$$
$$\langle \texttt{consistent}\ \ y \rangle \tag{1226}$$

From Predicate 1226 by Rule P2 18, the following holds:

$$\langle \texttt{consistent}\ \ y \cdot z \rangle$$

Apply Rule P2 16 on Transition 1225. For any process term q with $\langle \texttt{consistent}\ \ q \rangle$, have:

$$\langle x \cdot q \rangle \xrightarrow{a} \langle q \rangle$$

The process term q can be $y \cdot z$.

$$\langle x \cdot (y \cdot z) \rangle \xrightarrow{a} \langle y \cdot z \rangle \tag{1227}$$

Consider the target process terms in Transitions 1223 and 1227. The pair $(y \cdot z, y \cdot z)$ is in $\mathcal{I}$.

(b) Rule P2 16
If Transition 1216 is derived from this rule, then $p = z$. Rewriting Transition 1216, we get:

$$\langle (x \cdot y) \cdot z \rangle \xrightarrow{a} \langle z \rangle \tag{1228}$$

From premise of Rule P2 16 the following holds:

$$\langle x \cdot y \rangle \xrightarrow{a} \surd \tag{1229}$$

$$\langle \texttt{consistent}\ z \rangle \tag{1230}$$

The transition 1229 cannot be derived. (No termination transition for a sequential composition can be derived.)
Hence Rule P2 16 cannot be used to derive Transition 1216.

2.

$$\langle x \cdot (y \cdot z) \rangle \xrightarrow{a} \langle p \rangle \implies \begin{array}{l} \exists z' \in P : \langle (x \cdot y) \cdot z \rangle \xrightarrow{a} \langle z' \rangle \\ \text{and } (z', p) \in R \cup \mathcal{I} \end{array}$$

Suppose,

$$\langle x \cdot (y \cdot z) \rangle \xrightarrow{a} \langle p \rangle \tag{1231}$$

An action transition for a sequential composition can be derived only from rules P2 15 or P2 16. We discuss them one by one:

(a) Rule P2 15
If this rule is used to derive Transition 1231, then for some process term p', $p = p' \cdot (y \cdot z)$. Rewriting Transition 1231, we get:

$$\langle x \cdot (y \cdot z) \rangle \xrightarrow{a} \langle p' \cdot (y \cdot z) \rangle \tag{1232}$$

From premise of the rule, the following must hold:

$$\langle x \rangle \xrightarrow{a} \langle p' \rangle \tag{1233}$$

Apply Rule P2 15 on the above transitions twice. We get:

$$\langle (x \cdot y) \cdot z \rangle \xrightarrow{a} \langle (p' \cdot y) \cdot z \rangle \tag{1234}$$

Consider the target process terms in Transitions 1232 and 1234. The pair $(p' \cdot (y \cdot z), (p' \cdot y) \cdot z)$ is in R.

(b) Rule P2 16

If Transition 1231 is derived from this rule, then $p = y \cdot z$. Rewriting Transition 1231, we get:

$$\langle x \cdot (y \cdot z) \rangle \xrightarrow{a} \langle y \cdot z \rangle \tag{1235}$$

From premise of Rule P2 16 the following holds:

$$\langle x \rangle \xrightarrow{a} \surd \tag{1236}$$

$$\langle \texttt{consistent} \ \ y \cdot z \rangle \tag{1237}$$

A consistency predicate (Predicate 1237) for a sequential composition can only be derived from Rule P2-18. From the premise of the rule, the following must hold:

$$\langle \texttt{consistent} \ \ y \rangle \tag{1238}$$

Apply Rule P2 16 on Transition 1236 using Predicate 1238, we get:

$$\langle x \cdot y \rangle \xrightarrow{a} \langle y \rangle$$

Apply Rule P2 15 on the above transition. We get:

$$\langle (x \cdot y) \cdot z \rangle \xrightarrow{a} \langle y \cdot z \rangle \tag{1239}$$

Consider the target process terms in Transitions 1235 and 1239. The pair $(y \cdot z, y \cdot z)$ is in $\mathcal{I}$.

3.
$$\langle (x \cdot y) \cdot z \rangle \overset{r}{\longmapsto} \langle p \rangle \implies \begin{array}{c} \exists z' \in P : \langle x \cdot (y \cdot z) \rangle \overset{r}{\longmapsto} \langle z' \rangle \\ \text{and } (p, z') \in R \end{array}$$

Suppose,

$$\langle (x \cdot y) \cdot z \rangle \overset{r}{\longmapsto} \langle p \rangle \tag{1240}$$

A time step for a sequential composition can be derived only from rule P2 17. Then for some p', $p = p' \cdot z$. Rewriting Transition 1240:

$$\langle (x \cdot y) \cdot z \rangle \overset{r}{\longmapsto} \langle p' \cdot z \rangle \tag{1241}$$

From the premise of the rule, the following holds:

$$\langle x \cdot y \rangle \overset{r}{\longmapsto} \langle p' \rangle \tag{1242}$$

Again the above transition can only be derived from Rule P2 17. Then for some p'', $p' = p'' \cdot y$. Rewriting Transition 1241 and Transition 1242:

$$\langle (x \cdot y) \cdot z \rangle \overset{r}{\longmapsto} \langle (p'' \cdot y) \cdot z \rangle \tag{1243}$$

$$\langle x \cdot y \rangle \overset{r}{\longmapsto} \langle p'' \cdot y \rangle \tag{1244}$$

From the premise of the rule, the following holds:

$$\langle x \rangle \overset{r}{\longmapsto} \langle p'' \rangle \tag{1245}$$

Apply Rule P2 17 on Transition 1245. For any process term q we get:

$$\langle x \cdot q \rangle \xrightarrow{r} \langle p'' \cdot q \rangle$$

The term q can be $y \cdot z$.

$$\langle x \cdot (y \cdot z) \rangle \xrightarrow{r} \langle p'' \cdot (y \cdot z) \rangle \tag{1246}$$

Consider the target process terms in Transitions 1243 and 1246. The pair $((p'' \cdot y) \cdot z, p'' \cdot (y \cdot z))$ is in R.

4.
$$\langle x \cdot (y \cdot z) \rangle \xrightarrow{r} \langle p \rangle \implies \begin{array}{l} \exists z' \in P : \langle (x \cdot y) \cdot z \rangle \xrightarrow{r} \langle z' \rangle \\ \text{and } (z', p) \in R \end{array}$$

Suppose,
$$\langle x \cdot (y \cdot z) \rangle \xrightarrow{r} \langle p \rangle \tag{1247}$$

A time step for a sequential composition can be derived only from rule P2 17. Then, for some process term p', $p = p' \cdot (y \cdot z)$. Rewriting Transition 1247, we get:
$$\langle x \cdot (y \cdot z) \rangle \xrightarrow{r} \langle p' \cdot (y \cdot z) \rangle \tag{1248}$$

From the premise of the rule, the following must hold:

$$\langle x \rangle \xrightarrow{r} \langle p' \rangle \tag{1249}$$

Apply Rule P2 17 on the above time step with process term y , we get:

$$\langle x \cdot y \rangle \xrightarrow{r} \langle p' \cdot y \rangle$$

Again apply Rule P2 17 on the above time step with process term z , we get:
$$\langle (x \cdot y) \cdot z \rangle \xrightarrow{r} \langle (p' \cdot y) \cdot z \rangle \tag{1250}$$

Consider the target process terms in Transitions 1248 and 1250. The pair $(p' \cdot (y \cdot z), (p' \cdot y) \cdot z)$ is in R.

5.
$$\langle (x \cdot y) \cdot z \rangle \xrightarrow{r}_{\perp} \iff \langle x \cdot (y \cdot z) \rangle \xrightarrow{r}_{\perp}$$

Left Implication

Suppose,
$$\langle (x \cdot y) \cdot z \rangle \xrightarrow{r}_{\perp} \tag{1251}$$

A future inconsistency predicate for a sequential composition can be derived only from rules P2 19.

From the premise the rule, the following holds:

$$\langle x \cdot y \rangle \xrightarrow{r}_{\perp} \tag{1252}$$

Again the above transition can only be derived from Rule P2 19. From the premise of the rule, the following holds:

$$\langle x \rangle \overset{r}{\longmapsto}_{\perp} \tag{1253}$$

Apply Rule P2 19 on Transition 1253. For any process term q we get:

$$\langle x \cdot q \rangle \overset{r}{\longmapsto}_{\perp}$$

The term q can be $y \cdot z$.

$$\langle x \cdot (y \cdot z) \rangle \overset{r}{\longmapsto}_{\perp}$$

Right Implication

Suppose,

$$\langle x \cdot (y \cdot z) \rangle \overset{r}{\longmapsto}_{\perp} \tag{1254}$$

A future inconsistency predicate for a sequential composition can be derived only from rule P2 19. Then, from the premise of the rule, the following must hold:

$$\langle x \rangle \overset{r}{\longmapsto}_{\perp} \tag{1255}$$

Apply Rule P2 19 on the above predicate. We get:

$$\langle x \cdot y \rangle \overset{r}{\longmapsto}_{\perp}$$

By applying Rule P2 17 again on the above predicate, we get:

$$\langle (x \cdot y) \cdot z \rangle \overset{r}{\longmapsto}_{\perp} \tag{1256}$$

6.

$$\langle (x \cdot y) \cdot z \rangle \xrightarrow{a} \surd \iff \langle x \cdot (y \cdot z) \rangle \xrightarrow{a} \surd$$

Left Implication

Suppose,

$$\langle (x \cdot y) \cdot z \rangle \xrightarrow{a} \surd \tag{1257}$$

A termination predicate for a sequential composition cannot be derived from any rules. Hence the above predicate doesn't hold.

Right Implication

Suppose,

$$\langle x \cdot (y \cdot z) \rangle \xrightarrow{a} \surd \tag{1258}$$

A termination predicate for a sequential composition cannot be derived from any rules. Hence the above predicate doesn't hold.

7.
$$\langle \texttt{consistent}\ (x \cdot y) \cdot z \rangle \iff \langle \texttt{consistent}\ x \cdot (y \cdot z) \rangle$$

Left Implication

Suppose,

$$\langle \texttt{consistent}\ (x \cdot y) \cdot z \rangle$$

The above predicate can only be derived from Rule P2-18. Then from the premise of the rule,

$$\langle \texttt{consistent}\ x \cdot y \rangle$$

Again, the above predicate can only be derived from Rule P2-18. Hence,

$$\langle \texttt{consistent}\ x \rangle$$

Apply Rule P2-18 on the above predicate. For any process term q, the following holds:

$$\langle \texttt{consistent}\ x \cdot q \rangle$$

The process term q can be $y \cdot z$. Hence we have,

$$\langle \texttt{consistent}\ x \cdot (y \cdot z) \rangle$$

Right Implication

Suppose,

$$\langle \texttt{consistent}\ x \cdot (y \cdot z) \rangle$$

The above predicate can only be derived from Rule P2-18. Then from the premise of the rule,

$$\langle \texttt{consistent}\ x \rangle$$

Apply Rule P2-18 on the above predicate. For any process term q, the following holds:

$$\langle \texttt{consistent}\ x \cdot q \rangle$$

The process term q can be y. Hence we have,

$$\langle \texttt{consistent}\ x \cdot y \rangle$$

By repeating the same reasoning,

$$\langle \texttt{consistent}\ (x \cdot y) \cdot z \rangle$$

H.9 Axiom A6SR

$x + \tilde{\tilde{\delta}} = x$

We need to prove, $x + \tilde{\tilde{\delta}} \leftrightarrows x$.

Let R be a binary relation on process terms defined as follows:

$$R = \{(x + \tilde{\tilde{\delta}}, x) \mid x \in P\}$$

The relation $R \cup \mathcal{I}$ is a bisimulation relation.

H.10 Axiom A7SR

$\tilde{\tilde{\delta}} \cdot x = \tilde{\tilde{\delta}}$

We need to prove, $\tilde{\tilde{\delta}} \cdot x \leftrightarrows \tilde{\tilde{\delta}}$.

Let R be a binary relation on process terms defined as follows:

$$R = \{(\tilde{\tilde{\delta}} \cdot x, \tilde{\tilde{\delta}}) \mid x \in P\}$$

The relation $R \cup \mathcal{I}$ is a bisimulation relation.

H.11 Axiom NE1

$x + \bot = \bot$ NE1

We need to prove, $x + \bot \leftrightarrows \bot$.

Let R be a binary relation on process terms defined as follows:

$$R = \{(x + \bot, \bot) \mid x \in P\}$$

The relation $R \cup \mathcal{I}$ is a bisimulation relation.

H.12 Axiom NE2

$\bot \cdot x = \bot$ NE2

We need to prove, $\bot \cdot x \leftrightarrows \bot$.

Let R be a binary relation on process terms defined as follows:

$$R = \{(\bot \cdot x, \bot) \mid x \in P\}$$

The relation $R \cup \mathcal{I}$ is a bisimulation relation.

H.13 Axiom NE3SR

$\tilde{\tilde{a}} \cdot \bot = \tilde{\tilde{\delta}}$ NE3SR

We need to prove, $\tilde{\tilde{a}} \cdot \bot \leftrightarrows \tilde{\tilde{\delta}}$.

Let R be a binary relation on process terms defined as follows:

$$R = \{(\tilde{\tilde{a}} \cdot \bot, \tilde{\tilde{\delta}}), (\tilde{\tilde{\delta}}, \tilde{\tilde{a}} \cdot \bot) \mid a \in A\}$$

The relation $R \cup \mathcal{I}$ is a bisimulation relation.

H.14 Axiom SRT1

$\sigma^0_{\mathsf{rel}}(x) = x$. SRT1

We need to prove, $\sigma^0_{\mathsf{rel}}(x) \leftrightarrows x$.

Let R be a binary relation on process terms defined as follows:

$$R = \{(\sigma^0_{\mathsf{rel}}(x), x) \mid x \in P\}$$

Then $R \cup \mathcal{I}$ is a bisimulation relation that witnesses $\sigma^0_{\mathsf{rel}}(x) \leftrightarrows x$.

H.15 Axiom SRT2

$\sigma_{\mathsf{rel}}^{v}(\sigma_{\mathsf{rel}}^{u}(x)) = \sigma_{\mathsf{rel}}^{v+u}(x) \qquad v, u \geq 0$(SRT2)

We need to prove, $\sigma_{\mathsf{rel}}^{v}(\sigma_{\mathsf{rel}}^{u}(x)) \leftrightarrows \sigma_{\mathsf{rel}}^{v+u}(x)$.

We do the proof in four steps:

Case $u = 0, v = 0$

The proof is trivial using Axiom SRT1 and the fact that bisimulation is a congruence.

Case $u > 0, v = 0$

The proof is trivial using Axiom SRT1 and the fact that bisimulation is a congruence.

Case $u = 0, v > 0$

The proof is trivial using Axiom SRT1 and the fact that bisimulation is a congruence.

Case $u > 0, v > 0$

Let R be a binary relation on process terms defined as follows:

$$R = \quad \{ \quad (\sigma_{\mathsf{rel}}^{t}(\sigma_{\mathsf{rel}}^{u}(x)), \sigma_{\mathsf{rel}}^{t+u}(x)), \mid x \in P, 0 < t \leq v\}$$

We prove that the relation $R \cup \mathcal{I}$ satisfies all conditions of bisimulation.

For all $a \in A, r > 0, x, y \in P$, the following holds:

1.
$$\langle \sigma_{\mathsf{rel}}^{t}(\sigma_{\mathsf{rel}}^{u}(x)) \rangle \xrightarrow{a} \langle y \rangle \implies \begin{array}{l} \exists z' \in P : \langle \sigma_{\mathsf{rel}}^{t+u}(x) \rangle \xrightarrow{a} \langle z' \rangle \\ \text{and } (p, z') \in R \cup \mathcal{I}. \end{array}$$

   Suppose,
$$\langle \sigma_{\mathsf{rel}}^{t}(\sigma_{\mathsf{rel}}^{u}(x)) \rangle \xrightarrow{a} \langle y \rangle$$

   A process term with relative delay operator with duration greater than 0 cannot perform an action step. Hence our supposition doesn't hold.

2.
$$\langle \sigma_{\mathsf{rel}}^{t+u}(x) \rangle \xrightarrow{a} \langle y \rangle \implies \begin{array}{l} \exists z' \in P : \langle \sigma_{\mathsf{rel}}^{t}(\sigma_{\mathsf{rel}}^{u}(x)) \rangle \xrightarrow{a} \langle z' \rangle \\ \text{and } (p, z') \in R \cup \mathcal{I}. \end{array}$$

   Suppose,
$$\langle \sigma_{\mathsf{rel}}^{t+u}(x) \rangle \xrightarrow{a} \langle y \rangle$$

   A process term with relative delay operator with duration greater than 0 cannot perform an action step. Hence our supposition doesn't hold.

3.
$$\langle\sigma^t_{\mathsf{rel}}(\sigma^u_{\mathsf{rel}}(x))\rangle \xmapsto{r} \langle y\rangle \implies \begin{array}{l}\exists z' \in P : \langle\sigma^{t+u}_{\mathsf{rel}}(x)\rangle \xmapsto{r} \langle z'\rangle \\ \text{and } (p, z') \in R \cup \mathcal{I}.\end{array}$$

Suppose,
$$\langle\sigma^t_{\mathsf{rel}}(\sigma^u_{\mathsf{rel}}(x))\rangle \xmapsto{r} \langle y\rangle \tag{1259}$$

We distinguish between three cases for different values of r.

(a) Case $r < t$

Let $t = r + r_1$ for some r_1 with $0 < r_1 < t$.
Then Transition 1259 is derived from Rule P2-9 and $y = \sigma^{r_1}_{\mathsf{rel}}(\sigma^u_{\mathsf{rel}}(x))$.
Rewriting Transition 1259:

$$\langle\sigma^{r+r_1}_{\mathsf{rel}}(\sigma^u_{\mathsf{rel}}(x))\rangle \xmapsto{r} \langle\sigma^{r_1}_{\mathsf{rel}}(\sigma^u_{\mathsf{rel}}(x))\rangle \tag{1260}$$

By Rule P2-9 the following can be derived:

$$\langle\sigma^{r+r_1+u}_{\mathsf{rel}}(x)\rangle \xmapsto{r} \langle\sigma^{r_1+u}_{\mathsf{rel}}(x)\rangle \tag{1261}$$

Consider the target process terms in Transitions 1260 and 1261. The pair $(\sigma^{r_1}_{\mathsf{rel}}(\sigma^u_{\mathsf{rel}}(x)), \sigma^{r_1+u}_{\mathsf{rel}}(x))$, where $0 < r_1 < t$ is in R.

(b) Case $r = t$

Then Transition 1259 is derived from Rule P2-10. Then $y = \sigma^u_{\mathsf{rel}}(x)$.
Rewriting Transition 1259:

$$\langle\sigma^t_{\mathsf{rel}}(\sigma^u_{\mathsf{rel}}(x))\rangle \xmapsto{t} \langle\sigma^u_{\mathsf{rel}}(x)\rangle \tag{1262}$$

By Rule P2-9 the following can be derived:

$$\langle\sigma^{t+u}_{\mathsf{rel}}(x)\rangle \xmapsto{t} \langle\sigma^u_{\mathsf{rel}}(x)\rangle \tag{1263}$$

Consider the target process terms in Transitions 1262 and 1263. The pair $(\sigma^u_{\mathsf{rel}}(x), \sigma^u_{\mathsf{rel}}(x))$ is in $\mathcal{I}$.

(c) Case $r > t$

Let $r = t + s$, for some $s > 0$. Rewriting Transition 1259,

$$\langle\sigma^t_{\mathsf{rel}}(\sigma^u_{\mathsf{rel}}(x))\rangle \xmapsto{t+s} \langle y\rangle \tag{1264}$$

The above transition can only be derived from Rule P2-11. From the premise of the rule, the following holds:

$$\langle\sigma^u_{\mathsf{rel}}(x)\rangle \xmapsto{s} \langle y\rangle \tag{1265}$$

We distinguish between three cases depending on different values of the duration s of the time step.

i. Case $s < u$

Let $u = s + s_1$, for some s_1 with $0 < s_1 < s$.
Then Transition 1265 can only be derived from Rule P2-9. Then $y = \sigma_{\mathsf{rel}}^{s_1}(x)$. Rewriting Transitions 1264 and 1265, we get:

$$\langle \sigma_{\mathsf{rel}}^{t}(\sigma_{\mathsf{rel}}^{u}(x)) \rangle \xmapsto{t+s} \langle \sigma_{\mathsf{rel}}^{s_1}(x) \rangle \tag{1266}$$
$$\langle \sigma_{\mathsf{rel}}^{u}(x) \rangle \xmapsto{s} \langle \sigma_{\mathsf{rel}}^{s_1}(x) \rangle \tag{1267}$$

From Rule P2-9, the following can be derived:

$$\langle \sigma_{\mathsf{rel}}^{t+u}(x) \rangle \xmapsto{t+s} \langle \sigma_{\mathsf{rel}}^{s_1}(x) \rangle \tag{1268}$$

Consider the target process terms in Transitions 1266 and 1268. The pair $(\sigma_{\mathsf{rel}}^{s_1}(x), \sigma_{\mathsf{rel}}^{s_1}(x))$ is in $\mathcal{I}$.

ii. Case $s = u$
Then Transition 1265 can only be derived from Rule P2-10. Then $y = x$. Rewriting Transitions 1264 and 1265, we get:

$$\langle \sigma_{\mathsf{rel}}^{t}(\sigma_{\mathsf{rel}}^{u}(x)) \rangle \xmapsto{t+u} \langle x \rangle \tag{1269}$$
$$\langle \sigma_{\mathsf{rel}}^{u}(x) \rangle \xmapsto{u} \langle x \rangle \tag{1270}$$

From the premise of Rule P2-10, the following must hold:

$$\langle \texttt{consistent}\ \ x \rangle$$

Applying Rule P2-10 on process term $\sigma_{\mathsf{rel}}^{t+u}(x)$, the following can be derived:

$$\langle \sigma_{\mathsf{rel}}^{t+u}(x) \rangle \xmapsto{t+u} \langle x \rangle \tag{1271}$$

Consider the target process terms in Transitions 1269 and 1271. The pair (x, x) is in $\mathcal{I}$.

iii. Case $s > u$
Let $s = u + t_1$, for some $t_1 > 0$. Rewriting Transitions 1264 and 1265, we get:

$$\langle \sigma_{\mathsf{rel}}^{t}(\sigma_{\mathsf{rel}}^{u}(x)) \rangle \xmapsto{t+u+t_1} \langle y \rangle \tag{1272}$$
$$\langle \sigma_{\mathsf{rel}}^{u}(x) \rangle \xmapsto{u+t_1} \langle y \rangle \tag{1273}$$

Transition 1273 can only be derived from Rule P2-11. Then from the premise of the rule the following must hold:

$$\langle x \rangle \xmapsto{t_1} \langle y \rangle \tag{1274}$$

Apply Rule P2-11 on the above transition. For any $m > 0$, the following is derivable:

$$\langle \sigma_{\mathsf{rel}}^{m}(x) \rangle \xmapsto{m+t_1} \langle y \rangle$$

In the above transition, m can be $t+u$. Hence, we get:

$$\langle \sigma_{\mathsf{rel}}^{t+u}(x) \rangle \xmapsto{t+u+t_1} \langle y \rangle \tag{1275}$$

Consider the target process terms in Transition 1272 and Transition 1275. The pair (y, y) is in $\mathcal{I}$.

4.
$$\langle \sigma_{\mathsf{rel}}^{t+u}(x) \rangle \xmapsto{r} \langle y \rangle \implies \begin{array}{l} \exists z' \in P : \langle \sigma_{\mathsf{rel}}^{t}(\sigma_{\mathsf{rel}}^{u}(x)) \rangle \xmapsto{r} \langle z' \rangle \\ \text{and } (p, z') \in R \cup \mathcal{I}. \end{array}$$

Suppose,

$$\langle \sigma_{\mathsf{rel}}^{t+u}(x) \rangle \xmapsto{r} \langle y \rangle \tag{1276}$$

We distinguish between three cases for different values of r.

(a) Case $r < (t+u)$

Again we distinguish between three cases:

i. *Case* $r < t$
Let $t = r + r_1$, for some r_1 such that, $0 < r_1 < t$.
Then Transition 1276 can only be derived from Rule P2-9. Then $y = \sigma_{\mathsf{rel}}^{r_1+u}(x)$. Rewriting Transition 1276, we get:

$$\langle \sigma_{\mathsf{rel}}^{r+r_1+u}(x) \rangle \xmapsto{r} \langle \sigma_{\mathsf{rel}}^{r_1+u}(x) \rangle \tag{1277}$$

Then from Rule P2-9, the following can be derived:

$$\langle \sigma_{\mathsf{rel}}^{r+r_1}(\sigma_{\mathsf{rel}}^{u}(x)) \rangle \xmapsto{r} \langle \sigma_{\mathsf{rel}}^{r_1}(\sigma_{\mathsf{rel}}^{u}(x)) \rangle \tag{1278}$$

Consider the target process terms in Transitions 1277 and 1278. For $0 < r_1 < t$, the pair $(\sigma_{\mathsf{rel}}^{r_1}(\sigma_{\mathsf{rel}}^{u}(x)), \sigma_{\mathsf{rel}}^{r_1+u}(x))$ is in R.

ii. *Case* $r = t$
Then Transition 1276 can only be derived from Rule P2-9. Then $y = \sigma_{\mathsf{rel}}^{u}(x)$. Rewriting Transition 1276, we get:

$$\langle \sigma_{\mathsf{rel}}^{t+u}(x) \rangle \xmapsto{t} \langle \sigma_{\mathsf{rel}}^{u}(x) \rangle \tag{1279}$$

From Rule P2-10, the following can be derived:

$$\langle \sigma_{\mathsf{rel}}^{t}(\sigma_{\mathsf{rel}}^{u}(x)) \rangle \xmapsto{t} \langle \sigma_{\mathsf{rel}}^{u}(x) \rangle \tag{1280}$$

Consider the target process terms in Transitions 1279 and 1280. The pair $(\sigma_{\mathsf{rel}}^{u}(x), \sigma_{\mathsf{rel}}^{u}(x))$ is in $\mathcal{I}$.

iii. *Case* $r > t$
Let $r = t + s$ for some $s > 0$.
Note that $s < u$ because of our assumption that $r < (t+u)$. Let $u = s + s_1$ for some s_1 such that $0 < s_1 < u$.

Rewriting Transition 1276, we get:

$$\langle \sigma_{\mathsf{rel}}^{t+s+s_1}(x) \rangle \xmapsto{t+s} \langle \sigma_{\mathsf{rel}}^{s_1}(x) \rangle \tag{1281}$$

By Rule P2-9, the following can be derived:

$$\langle \sigma_{\mathsf{rel}}^{s+s_1}(x) \rangle \xmapsto{s} \langle \sigma_{\mathsf{rel}}^{s_1}(x) \rangle \tag{1282}$$

Apply Rule P2-11 on the above transition. We get:

$$\langle \sigma_{\mathsf{rel}}^{t}(\sigma_{\mathsf{rel}}^{s+s_1}(x)) \rangle \xmapsto{t+s} \langle \sigma_{\mathsf{rel}}^{s_1}(x) \rangle \tag{1283}$$

Consider the target process terms in Transitions 1281 and 1283. The pair $(\sigma_{\mathsf{rel}}^{s_1}(x), \sigma_{\mathsf{rel}}^{s_1}(x))$ is in $\mathcal{I}$.

(b) Case $r = (t + u)$

Then Transition 1276 can only be derived from Rule P2-10 and $y = x$. Rewriting Transition 1276, we get:

$$\langle \sigma_{\mathsf{rel}}^{t+u}(x) \rangle \xmapsto{t+u} \langle x \rangle \tag{1284}$$

From the premise of the rule, the following holds:

$$\langle \texttt{consistent } x \rangle$$

Apply Rule P2-10 on the above predicate, we get:

$$\langle \sigma_{\mathsf{rel}}^{u}(x) \rangle \xmapsto{u} \langle x \rangle \tag{1285}$$

Apply Rule P2-11 on the above transition. We get:

$$\langle \sigma_{\mathsf{rel}}^{t}(\sigma_{\mathsf{rel}}^{u}(x)) \rangle \xmapsto{t+u} \langle x \rangle \tag{1286}$$

Consider the target process terms in Transitions 1284 and 1286. The pair (x, x) is in $\mathcal{I}$.

(c) Case $r > (t + u)$

Let $r = t + u + t_1$, for some $t_1 > 0$. Rewriting Transition 1276, we get:

$$\langle \sigma_{\mathsf{rel}}^{t+u}(x) \rangle \xmapsto{t+u+t_1} \langle y \rangle \tag{1287}$$

From the premise of the rule the following must hold:

$$\langle x \rangle \xmapsto{t_1} \langle y \rangle \tag{1288}$$

Apply Rule P2-11 on the above transition. We get:

$$\langle \sigma_{\mathsf{rel}}^{u}(x) \rangle \xmapsto{u+t_1} \langle y \rangle \tag{1289}$$

Again apply Rule P2-11 on the above transition. We get:

$$\langle \sigma_{\mathsf{rel}}^{t}(\sigma_{\mathsf{rel}}^{u}(x)) \rangle \xmapsto{t+u+t_1} \langle y \rangle \tag{1290}$$

Consider the target process terms in Transitions 1287 and 1290. The pair (y, y) is in $\mathcal{I}$.

5.
$$\langle \sigma^t_{\mathsf{rel}}(\sigma^u_{\mathsf{rel}}(x))\rangle \overset{r}{\longmapsto}_{\perp} \Longleftrightarrow \langle \sigma^{t+u}_{\mathsf{rel}}(x)\rangle \overset{r}{\longmapsto}_{\perp}$$

Left Implication

Suppose
$$\langle \sigma^t_{\mathsf{rel}}(\sigma^u_{\mathsf{rel}}(x))\rangle \overset{r}{\longmapsto}_{\perp} \tag{1291}$$

The above predicate can only be derived from Rule P2-13 or Rule P2-14. We discuss them one by one.

(a) Rule P2-13

If Predicate 1291 is derived from this rule, then $r = t$. Rewriting Predicate 1291:
$$\langle \sigma^t_{\mathsf{rel}}(\sigma^u_{\mathsf{rel}}(x))\rangle \overset{t}{\longmapsto}_{\perp} \tag{1292}$$

From the premise of the rule, $\sigma^u_{\mathsf{rel}}(x)$ must not be consistent. But from Rule P2-12, a consistency predicate for process term $\sigma^u_{\mathsf{rel}}(x)$, with $u > 0$, always holds. We are discussing the case with $u > 0$. Hence Predicate 1292 cannot be derived.

(b) Rule P2-14

Then the length r of future inconsistency predicate 1291 is greater than t. Let $r = t + s$, for some $s > 0$. Rewriting Predicate 1291, we get:
$$\langle \sigma^t_{\mathsf{rel}}(\sigma^u_{\mathsf{rel}}(x))\rangle \overset{t+s}{\longmapsto}_{\perp} \tag{1293}$$

From the premise of Rule P2-14, the following holds:
$$\langle \sigma^u_{\mathsf{rel}}(x)\rangle \overset{s}{\longmapsto}_{\perp} \tag{1294}$$

The above predicate can again only be derived from Rule P2-13 or Rule P2-14. We discuss them one by one.

i. Rule P2-13

If Predicate 1294 is derived from this rule, then $s = u$. Rewriting Predicates 1293 and 1294, we get:
$$\langle \sigma^t_{\mathsf{rel}}(\sigma^u_{\mathsf{rel}}(x))\rangle \overset{t+u}{\longmapsto}_{\perp} \tag{1295}$$
$$\langle \sigma^u_{\mathsf{rel}}(x)\rangle \overset{u}{\longmapsto}_{\perp} \tag{1296}$$

From the premise of the rule, the following holds:
$$\neg\langle \texttt{consistent}\ x\rangle \tag{1297}$$

Apply Rule P2-13 on the above predicate. For any $m > 0$, the following is derivable:

$$\langle \sigma_{\mathsf{rel}}^{m}(x) \rangle \xmapsto{m} \perp$$

Then m can be $t + u$ and hence the following is derivable:

$$\langle \sigma_{\mathsf{rel}}^{t+u}(x) \rangle \xmapsto{t+u} \perp \tag{1298}$$

Consider Predicates 1295 and 1298. The left implication is proved.

ii. Rule P2-14

If Predicate 1294 is derived from this rule, then $s > u$. Let $s = u + t_1$, for some $t_1 > 0$. Rewriting Predicates 1293 and 1294, we get:

$$\langle \sigma_{\mathsf{rel}}^{t}(\sigma_{\mathsf{rel}}^{u}(x)) \rangle \xmapsto{t+u+t_1} \perp \tag{1299}$$

$$\langle \sigma_{\mathsf{rel}}^{u}(x) \rangle \xmapsto{u+t_1} \perp \tag{1300}$$

And from the premise of the rule, the following holds:

$$\langle x \rangle \xmapsto{t_1} \perp \tag{1301}$$

Apply Rule P2-14 on the above predicate. For any $m > 0$, the following is derivable:

$$\langle \sigma_{\mathsf{rel}}^{m}(x) \rangle \xmapsto{m+t_1} \perp \tag{1302}$$

Then m can be $t + u$ and the following holds:

$$\langle \sigma_{\mathsf{rel}}^{t+u}(x) \rangle \xmapsto{t+u+t_1} \perp \tag{1303}$$

Consider Predicates 1299 and 1303. The left implication is proved. Hence the left implication is proved.

Right Implication

Suppose

$$\langle \sigma_{\mathsf{rel}}^{t+u}(x) \rangle \xmapsto{r} \perp \tag{1304}$$

The above predicate can only be derived from Rule P2-13 or Rule P2-14. We discuss them one by one.

(a) Rule P2-13

If Predicate 1304 is derived from this rule, then $r = t + u$. Rewriting Predicate 1304:

$$\langle \sigma_{\mathsf{rel}}^{t+u}(x) \rangle \xmapsto{t+u} \perp \tag{1305}$$

From the premise of Rule P2-13, the following holds:

$$\neg\langle \texttt{consistent}\ x\rangle$$

Using Rule P2-13 on the above predicate, the following can be derived:

$$\langle \sigma^{u}_{\mathsf{rel}}(x)\rangle \xmapsto{u} \bot \tag{1306}$$

Using Rule P2-14 on the above predicate, the following can be derived:

$$\langle \sigma^{t}_{\mathsf{rel}}(\sigma^{u}_{\mathsf{rel}}(x))\rangle \xmapsto{t+u} \bot \tag{1307}$$

Consider Predicates 1304 and 1307. The right implication is proved.

(b) Rule P2-14
If Predicate 1304 is derived from this rule, then $r > t + u$.
Let $r = t + u + t_1$, for some $t_1 > 0$. Rewriting Predicate 1304, we get:

$$\langle \sigma^{t+u}_{\mathsf{rel}}(x)\rangle \xmapsto{t+u+t_1} \bot \tag{1308}$$

From the premise of Rule P2-14, the following holds:

$$\langle x\rangle \xmapsto{t_1} \bot \tag{1309}$$

Apply Rule P2-14 on the above predicate, the following can be derived:

$$\langle \sigma^{u}_{\mathsf{rel}}(x)\rangle \xmapsto{u+t_1} \bot \tag{1310}$$

Again apply Rule P2-14 on the above predicate, the following can be derived:

$$\langle \sigma^{t}_{\mathsf{rel}}(\sigma^{u}_{\mathsf{rel}}(x))\rangle \xmapsto{t+u+t_1} \bot \tag{1311}$$

Hence the right implication is proved.

6.

$$\langle \sigma^{t}_{\mathsf{rel}}(\sigma^{u}_{\mathsf{rel}}(x))\rangle \xrightarrow{a} \surd \iff \langle \sigma^{t+u}_{\mathsf{rel}}(x)\rangle \xrightarrow{a} \surd$$

Trivial. Both process terms cannot perform an action.

7.

$$\langle \texttt{consistent}\ \sigma^{t}_{\mathsf{rel}}(\sigma^{u}_{\mathsf{rel}}(x))\rangle \iff \langle \texttt{consistent}\ \sigma^{t+u}_{\mathsf{rel}}(x)\rangle$$

Trivial. Both are consistent.

H.16 Axiom SRT3 (Time Determinism)

$\sigma^u_{\mathsf{rel}}(x) + \sigma^u_{\mathsf{rel}}(y) = \sigma^u_{\mathsf{rel}}(x+y) \qquad u \geq 0$ (Time determinism-SRT3).

We prove the soundness of the axiom in two steps:

Case $u = 0$

The proof is trivial using Axiom SRT1.

Case $u > 0$

We need to prove, $\sigma^u_{\mathsf{rel}}(x) + \sigma^u_{\mathsf{rel}}(y) \leftrightarrows \sigma^u_{\mathsf{rel}}(x+y)$.

Let R be a binary relation on process terms defined as follows:

$$R = \quad \{ \quad (\sigma^t_{\mathsf{rel}}(x+y), \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y)), \mid x, y \in P, 0 < t \leq u\}$$

We prove that the relation $R \cup \mathcal{I}$ satisfies all conditions of bisimulation.

For all $a \in A, r > 0, x, y, z \in P$, the following holds:

1.
$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y) \rangle \xrightarrow{a} \langle z \rangle \implies \begin{array}{l} \exists z' \in P : \langle \langle \sigma^t_{\mathsf{rel}}(x+y) \rangle \rangle \xrightarrow{a} \langle z' \rangle \\ \text{and } (z, z') \in R \cup \mathcal{I} \end{array}$$

Suppose,

$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y) \rangle \xrightarrow{a} \langle z \rangle \tag{1312}$$

The above transition can be derived from Rules P2-20 or P2-21.

(a) *Rule P2-20*

From the premise of the rule, the following must hold:

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \langle z \rangle \tag{1313}$$

$$\langle \texttt{consistent } \sigma^t_{\mathsf{rel}}(y) \rangle \tag{1314}$$

Again there are no rules for $\sigma^t_{\mathsf{rel}}(x),$ with $t > 0$ to perform an action. Hence the transition 1312 cannot be derived from Rule P2-20.

(b) *Rule P2-21*

For similar reasons as given above for Rule P2-20, the transition 1312 cannot be derived from Rule P2-21.

2.
$$\langle \sigma^t_{\mathsf{rel}}(x+y) \rangle \xrightarrow{a} \langle z \rangle \implies \begin{array}{l} \exists z' \in P : \langle \sigma^t_{\mathsf{rel}}(x) + \sigma^u_{\mathsf{rel}}(y) \rangle \xrightarrow{a} \langle z' \rangle \\ \text{and } (z', z) \in R \cup \mathcal{I} \end{array}$$

Suppose,

$$\langle \sigma^t_{\mathsf{rel}}(x+y) \rangle \xrightarrow{a} \langle z \rangle$$

There are no rules allowing a process term $\sigma^r_{\mathsf{rel}}(x),$ with $r > 0$ to perform an action. Hence the above transition with an action step for $\sigma^t_{\mathsf{rel}}(x+y)$ does not exist. Since the left hand side of the implication is impossible, therefore we do not need to show that the right hand side holds.

3.
$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y)\rangle \xmapsto{r} \langle z\rangle \implies \begin{array}{l}\exists z' \in P : \langle \sigma^t_{\mathsf{rel}}(x+y)\rangle \xmapsto{r} \langle z'\rangle \\ \text{and } (z, z') \in R \cup \mathcal{I}\end{array}$$

Suppose,
$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y)\rangle \xmapsto{r} \langle z\rangle \tag{1315}$$

A time step for an alternative composition can be derived from one of the three rules P2 24, P2 25 or P2 26.

(a) Rule P2 24
Then for some process terms x' and y', the process term z in Transition 1315 is $x' + y'$.
Rewriting Transition 1315, we get:
$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y)\rangle \xmapsto{r} \langle x' + y'\rangle \tag{1316}$$
From the premise of Rule P2 24 the following must be derivable:
$$\langle \sigma^t_{\mathsf{rel}}(x)\rangle \xmapsto{r} \langle x'\rangle, \tag{1317}$$
$$\langle \sigma^t_{\mathsf{rel}}(y)\rangle \xmapsto{r} \langle y'\rangle \tag{1318}$$
We distinguish between three cases for the derivation of above time steps for different values of duration r.

i. Case $r < t$:

Let $t = r + r_1$, for some $0 < r_1 < t$.
Only Rule P2 9 allows to derive a time step of duration less than t for a process term $\sigma^t_{\mathsf{rel}}(x)$. According to the rule, the target process terms x' and y' in Transitions 1317 and 1318 are $\sigma^{r_1}_{\mathsf{rel}}(x)$ and $\sigma^{r_1}_{\mathsf{rel}}(y)$ respectively. Rewriting Transition 1316, we get:
$$\langle \sigma^{r+r_1}_{\mathsf{rel}}(x) + \sigma^{r+r_1}_{\mathsf{rel}}(y)\rangle \xmapsto{r} \langle \sigma^{r_1}_{\mathsf{rel}}(x) + \sigma^{r_1}_{\mathsf{rel}}(y)\rangle \tag{1319}$$
The following time step can also be derived from Rule P2 9:
$$\langle \sigma^{r+r_1}_{\mathsf{rel}}(x+y)\rangle \xmapsto{r} \langle \sigma^{r_1}_{\mathsf{rel}}(x+y)\rangle \tag{1320}$$
Consider the target process terms in Transitions 1319 and 1320. The pair $(\sigma^{r_1}_{\mathsf{rel}}(x) + \sigma^{r_1}_{\mathsf{rel}}(y), \sigma^{r_1}_{\mathsf{rel}}(x+y))$, for $0 < r_1 < t$ is in R.

ii. Case $r = t$:
Then Transitions 1317 and 1318 can only be derived from Rule P2 10. According to the rule,
$$x' = x \text{ and } y' = y$$
Rewriting Transition 1316, we get:
$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y)\rangle \xmapsto{r} \langle x + y\rangle \tag{1321}$$

From the premise of Rule P2 10, the following must hold:

$$\langle \texttt{consistent}\ x\rangle \text{ and } \langle \texttt{consistent}\ y\rangle$$

which implies by rule P2 27

$$\langle \texttt{consistent}\ x + y\rangle$$

Consequently, Rule P2 10 becomes applicable on $\sigma^t_{\mathsf{rel}}(x+y)$ to derive the following time step:

$$\langle \sigma^t_{\mathsf{rel}}(x+y)\rangle \xmapsto{r} \langle x+y\rangle \tag{1322}$$

Consider the target process terms in Transitions 1321 and 1322. The pair $(x+y, x+y)$ is in R.

iii. Case $r > t$:

Let $t = r + t_1$, for some t_1 with $0 < t_1 < t$.
Then Transitions 1317 and 1318 can only be derived from Rule P2 11. According to the premise of the rule, the following must be derivable:

$$\langle x\rangle \xmapsto{t_1} \langle x'\rangle, \tag{1323}$$

$$\langle y\rangle \xmapsto{t_1} \langle y'\rangle \tag{1324}$$

Joining the two transitions and applying Rule P2-24, we get:

$$\langle x+y\rangle \xmapsto{t_1} \langle x'+y'\rangle \tag{1325}$$

Apply Rule P2 11 on the above Transition. We get:

$$\langle \sigma^t_{\mathsf{rel}}(x+y)\rangle \xmapsto{r+t_1} \langle x'+y'\rangle \tag{1326}$$

Consider the target process terms in Transitions 1316 and 1326. The pair $(x'+y', x'+y')$ is in R.

(b) Rule P2 25

We now inspect the case when Transition 1315 has been derived from Rule P2 25.

$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y)\rangle \xmapsto{r} \langle z\rangle \qquad (1315)$$

If Rule P2 25 is used to derive Transition 1315, then according to the rule $\sigma^t_{\mathsf{rel}}(x)$ must do the time step of duration r, $\sigma^t_{\mathsf{rel}}(y)$ must be unable to delay for duration r and $\sigma^t_{\mathsf{rel}}(y)$ must remain consistent throughout the delay. Mathematically, the requirements can be written as:

$$\langle \sigma^t_{\mathsf{rel}}(x)\rangle \xmapsto{r} \langle z\rangle, \tag{1327}$$

$$\langle \texttt{consistent}\ \sigma^t_{\mathsf{rel}}(y)\rangle, \tag{1328}$$

$$\langle \sigma^t_{\mathsf{rel}}(y)\rangle \not\xmapsto{r}, \tag{1329}$$

$$(\forall s \le r,\ \ \langle \sigma^t_{\mathsf{rel}}(y)\rangle \not\xmapsto{s} \perp) \tag{1330}$$

Again we distinguish three cases for different values of r.

i. <u>Case $r < t$:</u>

Let $t = r + r_1$, for some $0 < r_1 < t$.
A time step $\langle\sigma_{\mathsf{rel}}^{r+r_1}(y)\rangle \overset{r}{\longmapsto} \langle\sigma_{\mathsf{rel}}^{r_1}(y)\rangle$ is always derivable (Rule P2 9). Hence Predicate 1329 does not hold for $r < t$.
We conclude that Transition 1315 cannot be derived from Rule P2 25 for $r < t$.

ii. <u>Case $r = t$:</u>

Rewriting the requirements for Rule P2 25 for $r = t$, we get:

$$\langle\sigma_{\mathsf{rel}}^{t}(x)\rangle \overset{t}{\longmapsto} \langle z\rangle, \tag{1331}$$
$$\langle\texttt{consistent}\ \sigma_{\mathsf{rel}}^{t}(y)\rangle, \tag{1332}$$
$$\langle\sigma_{\mathsf{rel}}^{t}(y)\rangle \overset{t}{\not\longmapsto}, \tag{1333}$$
$$(\forall s \leq t,\ \ \langle\sigma_{\mathsf{rel}}^{t}(y)\rangle \overset{s}{\not\longmapsto}_{\perp}) \tag{1334}$$

The Predicate 1333 indicates that Rule P2 10 is not applicable. Therefore y must be inconsistent. I.e.,

$$\neg\langle\texttt{consistent}\ \ y\rangle$$

If that is the case, then Rule P2 13 becomes applicable and the following can be derived:

$$\langle\sigma_{\mathsf{rel}}^{t}(y)\rangle \overset{t}{\longmapsto}_{\perp} \tag{1335}$$

which contradicts predicate 1334.
We conclude that Transition 1315 cannot be derived from Rule P2 25 for $r = t$.

iii. <u>Case $r > t$:</u>

Let $r = t + v$, for some $v > 0$.
If Rule P2 25 is used to derive Transition 1315, then according to the rule the following must hold:

$$\langle\sigma_{\mathsf{rel}}^{t}(x)\rangle \overset{t+v}{\longmapsto} \langle z\rangle, \tag{1336}$$
$$\langle\texttt{consistent}\ \sigma_{\mathsf{rel}}^{t}(y)\rangle, \tag{1337}$$
$$\langle\sigma_{\mathsf{rel}}^{t}(y)\rangle \overset{t+v}{\not\longmapsto}, \tag{1338}$$
$$(\forall s \leq (t+v),\ \ \langle\sigma_{\mathsf{rel}}^{t}(y)\rangle \overset{s}{\not\longmapsto}_{\perp}) \tag{1339}$$

Transition 1336 can only be derived from Rule P2- 11. Then from the premise of the rule the following must be derivable:

$$\langle x\rangle \overset{v}{\longmapsto} \langle z\rangle \tag{1340}$$

From Predicate 1339 we can infer,

$$\langle \sigma^t_{\mathsf{rel}}(y) \rangle \overset{t}{\not\longmapsto}_{\perp} \tag{1341}$$

Hence Rule P2 13 must not be applicable. Therefore y must be consistent. I.e.,

$$\langle \texttt{consistent}\ y \rangle \tag{1342}$$

Predicate 1338 indicates that Rule P2 11 cannot be applied to process term $\sigma^t_{\mathsf{rel}}(y)$. Hence the premise of the rule doesn't hold. I.e.,

$$\langle y \rangle \overset{v}{\not\longmapsto} \tag{1343}$$

Consider Predicate 1339. If we weaken the predicate, we have,

$$\forall s : t < s \leq (t+v),\ \ \langle \sigma^t_{\mathsf{rel}}(y) \rangle \overset{s}{\not\longmapsto}_{\perp} \tag{1344}$$

(Note we are considering a future inconsistency predicate over a reduced range of s).

The above statement indicates that process term $\sigma^t_{\mathsf{rel}}(y)$ does not have a future inconsistency predicate of length greater than t and less than or equal to $t+v$. This means that Rule P2- 14 is not applicable for any duration in interval $(t, t+v]$. Hence the premise of the rule doesn't hold in the duration $(0, v]$.

$$\forall s \leq v,\ \ \langle y \rangle \overset{s}{\not\longmapsto}_{\perp} \tag{1345}$$

Apply Rule P2-25 to Transitions (Predicates) 1340,1342,1343 and 1345, we get:

$$\langle x + y \rangle \overset{v}{\longmapsto} \langle z \rangle \tag{1346}$$

Apply Rule P2 11 to the above transition. We get:

$$\langle \sigma^t_{\mathsf{rel}}(x+y) \rangle \xrightarrow{u+v} \langle z \rangle \tag{1347}$$

Consider the target process terms in Transitions 1315 and 1347. The pair (z, z) is in $\mathcal{I}$.

(c) Rule P2 26

Reasoning similar to above applies.

4.
$$\langle \sigma^t_{\mathsf{rel}}(x+y) \rangle \overset{r}{\longmapsto} \langle z \rangle \implies \begin{array}{l} \exists z' \in P : \langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y) \rangle \overset{r}{\longmapsto} \langle z' \rangle \\ \text{and } (z', z) \in R \cup \mathcal{I} \end{array}$$

Suppose,

$$\langle \sigma^t_{\mathsf{rel}}(x+y) \rangle \overset{r}{\longmapsto} \langle z \rangle \tag{1348}$$

We distinguish three cases depending on the length of duration r.

(a) Case $r < t$:

Let $t = r + r_1$, for some r_1 such that $0 < r_1 < t$.
When $r < t$, then Transition 1348 can only be derived from Rule P2-9. This rule has no premise. It can always be applied. Then, z must be $\sigma_{\mathsf{rel}}^{r_1}(x + y)$.

$$\langle \sigma_{\mathsf{rel}}^{r+r_1}(x + y)\rangle \xmapsto{r} \langle \sigma_{\mathsf{rel}}^{r_1}(x + y)\rangle \tag{1349}$$

The Rule P2-9 can be used to derive the following time steps:

$$\langle \sigma_{\mathsf{rel}}^{r+r_1}(x)\rangle \xmapsto{r} \langle \sigma_{\mathsf{rel}}^{r_1}(x)\rangle \tag{1350}$$

$$\langle \sigma_{\mathsf{rel}}^{r+r_1}(y)\rangle \xmapsto{r} \langle \sigma_{\mathsf{rel}}^{r_1}(y)\rangle \tag{1351}$$

Apply Rule P2 24 on the above transitions, we get:

$$\langle \sigma_{\mathsf{rel}}^{r+r_1}(x) + \sigma_{\mathsf{rel}}^{r+r_1}(y)\rangle \xmapsto{r} \langle \sigma_{\mathsf{rel}}^{r_1}(x) + \sigma_{\mathsf{rel}}^{r_1}(y)\rangle \tag{1352}$$

Consider the target process terms in transitions 1349 and 1352. For $r_1 < t$, the pair $(\sigma_{\mathsf{rel}}^{r_1}(x) + \sigma_{\mathsf{rel}}^{r_1}(y), \sigma_{\mathsf{rel}}^{r_1}(x + y))$ is in R.

(b) Case $r = t$:
When $r = t$, then Transition 1348 can only be derived from Rule P2 10. Then z must be of the form $x + y$. Rewriting Transition 1348

$$\langle \sigma_{\mathsf{rel}}^{t}(x + y)\rangle \xmapsto{t} \langle x + y\rangle \tag{1353}$$

And from the premise of the rule, the following must hold:

$$\langle \texttt{consistent}\ \ x + y\rangle \tag{1354}$$

The above predicate can only hold when both x and y are consistent. I.e.,

$$\langle \texttt{consistent}\ \ x\rangle \text{ and } \langle \texttt{consistent}\ \ y\rangle \tag{1355}$$

Then we can apply rule P2 10 to derive the following transitions for process terms $\sigma_{\mathsf{rel}}^{r}(x)$ and $\sigma_{\mathsf{rel}}^{r}(y)$:

$$\langle \sigma_{\mathsf{rel}}^{t}(x)\rangle \xmapsto{t} \langle x\rangle \tag{1356}$$

$$\langle \sigma_{\mathsf{rel}}^{t}(y)\rangle \xmapsto{t} \langle y\rangle \tag{1357}$$

Apply Rule P2 24 on the above transitions, we get:

$$\langle \sigma_{\mathsf{rel}}^{t}(x) + \sigma_{\mathsf{rel}}^{t}(y)\rangle \xmapsto{t} \langle x + y\rangle \tag{1358}$$

Consider the target process terms in transitions 1353 and 1358. The pair $(x + y, x + y)$ is in $\mathcal{I}$.

(c) Case $r > t$:
Let $r = t + v$, for some $v > 0$.
When $r > t$, then Transition 1348 can only be derived from Rule P2 11. From the premise of the rule, the following must hold:

$$\langle x + y \rangle \xmapsto{v} \langle z \rangle \tag{1359}$$

A time step for an alternative composition can be derived from rules P2 24, P2 25 and P2 26. We discuss each of the rules one by one:

i. *Rule P2 24:*
If this rule is used to derive Transition 1359, then both process terms x and y can do the time step. From the premise of the rule, for some process terms x', y', z must be of the form $x' + y'$. Rewriting Transitions 1348 and 1359:

$$\langle \sigma^t_{\mathsf{rel}}(x + y) \rangle \xmapsto{t+v} \langle x' + y' \rangle \tag{1360}$$
$$\langle x + y \rangle \xmapsto{v} \langle x' + y' \rangle \tag{1361}$$

And the following must hold:

$$\langle x \rangle \xmapsto{v} \langle x' \rangle \tag{1362}$$
$$\langle y \rangle \xmapsto{v} \langle y' \rangle \tag{1363}$$

On each of the above transitions, apply Rule P2 11, we get:

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xmapsto{t+v} \langle x' \rangle \tag{1364}$$
$$\langle \sigma^t_{\mathsf{rel}}(y) \rangle \xmapsto{t+v} \langle y' \rangle \tag{1365}$$

Apply Rule P2 24 on the above transitions, we get:

$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y) \rangle \xmapsto{t+v} \langle x' + y' \rangle \tag{1366}$$

Consider the target process terms in transitions 1360 and 1366. The pair $(x' + y', x' + y')$ is in $\mathcal{I}$.

ii. *Rule P2 25:*
If this rule is used to derive Transition 1359, then according to the rule x must do the time step of duration v, y must be unable to delay for duration v and y must remain consistent throughout the delay. Mathematically, the requirements can be written as:

$$\langle x \rangle \xmapsto{v} \langle z \rangle, \tag{1367}$$
$$\langle \texttt{consistent } y \rangle, \tag{1368}$$
$$\langle y \rangle \not\xmapsto{v}, \tag{1369}$$
$$(\forall s \leq v,\ \ \langle y \rangle \not\xmapsto{s}_{\perp}) \tag{1370}$$

Apply Rule P2 11 on Transition 1367. We get:

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xmapsto{t+v} \langle z \rangle \tag{1371}$$

A process term $\sigma^t_{\mathsf{rel}}(y)$ may have future inconsistency predicates with durations greater than or equal to t. For a process term $\sigma^t_{\mathsf{rel}}(y)$, there are no rules allowing a predicate of future inconsistency with a duration s which is strictly less than t. Hence, the following predicate holds:

$$\forall s < t,\ \ \langle \sigma^t_{\mathsf{rel}}(y) \rangle \not\xmapsto{s}_{\perp} \tag{1372}$$

For $\sigma^t_{\mathsf{rel}}(y)$ to have a future inconsistency with duration t, Rule P2 13 must be applicable. The rule has a premise that the predicate of consistency does not hold for y. But from Predicate 1368, we have: $\langle \texttt{consistent}\ \ y \rangle$. Hence Rule P2 13 cannot be applied and statement 1372 can be made stronger:

$$\forall s \le t,\ \ \langle \sigma^t_{\mathsf{rel}}(y) \rangle \not\xmapsto{s}_{\perp} \tag{1373}$$

Rule P2 14 is the only rule by which we can derive a future inconsistency predicate of length $t + s$, (where $s > 0$) for the process term $\sigma^t_{\mathsf{rel}}(y)$. The rule requires that a predicate of future inconsistency with length s must hold for y, i.e. the predicate,

$$\langle y \rangle \xmapsto{s}_{\perp} \tag{1374}$$

must hold.
If there does not hold such a predicate for y, then a future inconsistency predicate for $\sigma^t_{\mathsf{rel}}(y)$ with length $t + s$ cannot hold. Hence from Predicate 1370, $(\forall s \le v,\ \ \langle y \rangle \not\xmapsto{s}_{\perp})$
We have,

$$\forall s \le v,\ \ \langle \sigma^t_{\mathsf{rel}}(y) \rangle \not\xmapsto{t+s}_{\perp} \tag{1375}$$

Combining Predicates 1373 and 1375, we get:

$$\forall s \le (t + v),\ \ \langle \sigma^t_{\mathsf{rel}}(y) \rangle \not\xmapsto{s}_{\perp} \tag{1376}$$

From Rule P2-12 the following holds:

$$\langle \texttt{consistent}\ \ \sigma^t_{\mathsf{rel}}(y) \rangle \tag{1377}$$

Consider a process term $\sigma^t_{\mathsf{rel}}(z)$. Rule P2 11 is the only rule that allows $\sigma^t_{\mathsf{rel}}(z)$ to delay for a time duration $t + v$. The rule has a premise that z must be delayable for v time units. Hence from Predicate 1369 $(\langle y \rangle \not\xmapsto{v})$, we can infer the following:

$$\langle \sigma^t_{\mathsf{rel}}(y) \rangle \not\xmapsto{t+v} \tag{1378}$$

Apply Rule P2-25 to Transitions (Predicates) 1371,1376, 1377 and 1378 we get:

$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y) \rangle \xmapsto{t+v} \langle z \rangle \tag{1379}$$

Consider the target process terms in Transitions 1348 and 1379. The pair (z, z) is in R.

iii. *Rule P2 26:*
If this rule is used to derive Transition 1359, then according to the rule y must do the time step of duration r, x must be unable to delay for duration r and x must remain consistent throughout the delay.
Reasoning similar to the Rule P2 25 applies.

5. $$\langle \sigma^t_{\mathsf{rel}}(x + y) \rangle \xrightarrow{a} \surd \iff \langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y) \rangle \xrightarrow{a} \surd$$

Left Implication

Suppose,

$$\langle \sigma^t_{\mathsf{rel}}(x + y) \rangle \xrightarrow{a} \surd$$

There are no rules allowing a process term $\sigma^r_{\mathsf{rel}}(x),$ with $r > 0$ to perform an action. Hence the above transition with an action step for $\sigma^t_{\mathsf{rel}}(x + y)$ does not exist.

Right Implication

Suppose,

$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y) \rangle \xrightarrow{a} \surd \tag{1380}$$

The above transition can be derived from Rules P2-22 or P2-23.

(a) *Rule P2-22*
From the premise of the rule, the following must hold:

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \surd \tag{1381}$$

$$\langle \texttt{consistent}\ \sigma^t_{\mathsf{rel}}(y) \rangle \tag{1382}$$

Again there are no rules for $\sigma^t_{\mathsf{rel}}(x),$ with $t > 0$ to perform an action. Hence the transition 1380 cannot be derived from Rule P2-22.

(b) *Rule P2-23*
For similar reasons as given above for Rule P2-22, the transition 1380 cannot be derived from Rule P2-23.

6. $$\langle \sigma^t_{\mathsf{rel}}(x + y) \rangle \overset{r}{\mapsto}_{\perp} \iff \langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y) \rangle \overset{r}{\mapsto}_{\perp}$$

Left Implication

Suppose,

$$\langle \sigma^t_{\mathsf{rel}}(x + y)\rangle \overset{r}{\longmapsto}_{\perp} \tag{1383}$$

We distinguish three cases depending on the length of duration r.

(a) Case $r < t$:
There are no rules to derive a future inconsistency predicate for a process term $\sigma^t_{\mathsf{rel}}(z)$, with a length r which is less than u. Hence Predicate 1383 cannot be derived for $r < t$.

(b) Case $r = t$:

$$\langle \sigma^t_{\mathsf{rel}}(x + y)\rangle \overset{t}{\longmapsto}_{\perp} \tag{1384}$$

Only Rule P2 13 can be used to derive the above predicate. From the premise of the Rule, the following holds:

$$\neg\langle \texttt{consistent}\ x + y\rangle \tag{1385}$$

A consistency predicate for an alternative composition is derived from Rule P2-27. From Predicate 1385, the premise of the rule must not hold. Hence, at least one of the following holds:

$$\neg\langle \texttt{consistent}\ x\rangle \text{ and } \neg\langle \texttt{consistent}\ y\rangle \tag{1386}$$

i. Case $\neg\langle \texttt{consistent}\ x\rangle$
If $\neg\langle \texttt{consistent}\ x\rangle$, then Rule P2 13 can be applied to derive the following:

$$\langle \sigma^t_{\mathsf{rel}}(x)\rangle \overset{t}{\longmapsto}_{\perp} \tag{1387}$$

There are no rules to derive a predicate of Future Inconsistency for a process term $\sigma^t_{\mathsf{rel}}(y)$, with a length r which is less than u. Hence the following holds:

$$\forall s < t,\ \ \langle \sigma^t_{\mathsf{rel}}(y)\rangle \overset{s}{\not\longmapsto}_{\perp} \tag{1388}$$

Also, for $u > 0$, from Rule P2-12, the following holds:

$$\langle \texttt{consistent}\ \sigma^t_{\mathsf{rel}}(y)\rangle \tag{1389}$$

Apply Rule P2 28 on Predicates 1387, 1388 and 1389. We get:

$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^u_{\mathsf{rel}}(y)\rangle \overset{t}{\longmapsto}_{\perp} \tag{1390}$$

Consider Predicates 1384 and 1390. The Left implication is proved.

ii. Case $\neg\langle\texttt{consistent}\ y\rangle$

If $\neg\langle\texttt{consistent}\ y\rangle$, then by reasoning that is similar to above and by application of Rule P2 29, we get the following predicate:

$$\langle\sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y)\rangle \xmapsto{t} \bot$$

(c) Case $r > t$:

Let $r = t + v$, where $v > 0$.

When $r > t$, then Predicate 1383 can only be derived from Rule P2 14. Rewriting Predicate 1383:

$$\langle\sigma^t_{\mathsf{rel}}(x + y)\rangle \xmapsto{t+v} \bot \tag{1391}$$

From the premise of the rule, the following must hold:

$$\langle x + y\rangle \xmapsto{v} \bot \tag{1392}$$

A predicate of future inconsistency for an alternative composition can only be derived from rules P2 28 or P2 29. We discuss each of the rules one by one:

i. *Rule P2 28:*

From the premise of the rule the following must hold:

$$\langle x\rangle \xmapsto{v} \bot \tag{1393}$$

$$\langle\texttt{consistent}\ y\rangle \tag{1394}$$

$$\forall s < v,\ \ \langle y\rangle \not\xmapsto{s} \bot \tag{1395}$$

Apply Rule 14 on Predicate 1393. We get:

$$\langle\sigma^t_{\mathsf{rel}}(x)\rangle \xmapsto{t+v} \bot \tag{1396}$$

No rules for deriving a future inconsistency predicate of length less than r for a process term $\sigma^r_{\mathsf{rel}}(z)$. Hence the following predicate holds:

$$\forall s < t,\ \ \langle\sigma^t_{\mathsf{rel}}(y)\rangle \not\xmapsto{s} \bot \tag{1397}$$

From Predicate 1394, Rule P2 13 cannot be applied. Hence:

$$\langle\sigma^t_{\mathsf{rel}}(y)\rangle \not\xmapsto{t} \bot \tag{1398}$$

Combining Predicates 1397 and 1398, we get:

$$\forall s \le t,\ \ \langle\sigma^t_{\mathsf{rel}}(y)\rangle \not\xmapsto{s} \bot \tag{1399}$$

For process term $\sigma^t_{\mathsf{rel}}(y)$, a future inconsistency predicate of duration $t + s$, (where $s > 0$) can only be derived from Rule P2 14. From Predicate 1395, Rule P2 14 cannot be applied on process term $\sigma^t_{\mathsf{rel}}(y)$ for any duration in interval $(t, t+v)$. Hence the following predicate holds:

$$\forall s < v, \ \ \langle \sigma^t_{\mathsf{rel}}(y) \rangle \overset{t+s}{\not\longmapsto} \bot \tag{1400}$$

Combining Predicate 1399 and Predicate 1400, we get:

$$\forall s < t+v, \ \ \langle \sigma^t_{\mathsf{rel}}(y) \rangle \overset{s}{\not\mapsto} \bot \tag{1401}$$

Also the following holds:

$$\langle \texttt{consistent} \ \ \sigma^t_{\mathsf{rel}}(y) \rangle \tag{1402}$$

Apply Rule P2 28 on Transitions (Predicates) 1396, 1402 and 1401. We get:

$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y) \rangle \overset{t+v}{\longmapsto} \bot$$

Hence the left implication is proved.

ii. *Rule P2 29:*
Reasoning similar to Rule P2 28 applies.

Right Implication

Suppose

$$\langle \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y) \rangle \overset{r}{\mapsto} \bot \tag{1403}$$

Rules P2 28 or P2 29 can be applied to derive the above predicate.

(a) *Rule P2 28:*
If this rule is used to derive Predicate 1403, then according to the rule the following must hold:

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \overset{r}{\mapsto} \bot, \tag{1404}$$

$$\langle \texttt{consistent} \ \ \sigma^t_{\mathsf{rel}}(y) \rangle, \tag{1405}$$

$$(\forall s < r, \ \ \langle \sigma^t_{\mathsf{rel}}(y) \rangle \overset{s}{\not\mapsto} \bot) \tag{1406}$$

We distinguish three cases depending on different values of r:

i. *Case $r < u$*
Then it is not possible to derive Predicate 1404. As the premises of Rule P2 28 are not satisfied, therefore we conclude that Rule P2 28 cannot be used to derive Predicate 1403 for $r < t$.

ii. *Case $r = t$:*
Rewriting the requirements of Rule P2 28:

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \overset{t}{\mapsto} \bot, \tag{1407}$$

$$\langle \texttt{consistent} \ \ \sigma^t_{\mathsf{rel}}(y) \rangle, \tag{1408}$$

$$(\forall s < t, \ \ \langle \sigma^t_{\mathsf{rel}}(y) \rangle \overset{s}{\not\mapsto} \bot) \tag{1409}$$

Now Predicate 1407 can only be derived from Rule P2 13. Hence the premise of the rule must hold. I.e.,

$$\neg\langle \texttt{consistent}\ x\rangle$$

For an alternative composition, both alternatives must be consistent. Only then a consistency predicate for that alternative composition can be derived. Hence, from $\neg\langle \texttt{consistent}\ x\rangle$, the following holds:

$$\neg\langle \texttt{consistent}\ x+y\rangle$$

Apply Rule P2 13 on the above predicate:

$$\langle \sigma^{u}_{\mathsf{rel}}(x+y)\rangle \xmapsto{u} \perp \tag{1410}$$

Hence left implication is proved.

iii. *Case $r > t$:*
Let $r = t + v$, for some $v > 0$. Rewriting premises of Rule 28:

$$\langle \sigma^{t}_{\mathsf{rel}}(x)\rangle \xmapsto{t+v} \perp, \tag{1411}$$

$$\langle \texttt{consistent}\ \sigma^{t}_{\mathsf{rel}}(y)\rangle, \tag{1412}$$

$$(\forall s < (t+v),\ \ \langle \sigma^{t}_{\mathsf{rel}}(y)\rangle \not\xmapsto{s} \perp) \tag{1413}$$

Predicate 1411 can only be derived from Rule P2 14. Hence the following must hold:

$$\langle x\rangle \xmapsto{v} \perp \tag{1414}$$

Now from Predicate 1413, a future inconsistency predicate for process term $\sigma^{t}_{\mathsf{rel}}(y)$ with length t does not hold. I.e.,

$$\langle \sigma^{t}_{\mathsf{rel}}(y)\rangle \not\xmapsto{t} \perp$$

That means Rule P2 13 is not applicable. Hence its premise must not hold. I.e.,

$$\langle \texttt{consistent}\ y\rangle \tag{1415}$$

By weakening Predicate 1413, the following is inferred:

$$\forall s : t < s < (t+v),\ \ \langle \sigma^{t}_{\mathsf{rel}}(y)\rangle \not\xmapsto{s} \perp$$

Rewriting the above predicate,

$$\forall s : s < v,\ \ \langle \sigma^{t}_{\mathsf{rel}}(y)\rangle \not\xmapsto{t+v} \perp$$

The above Predicate states that Rule P2 14 is not applicable for process term, $\sigma^{t}_{\mathsf{rel}}(y)$ for any duration in interval $(t, t+v)$. Hence the premise of the rule must not hold:

$$\forall s < v,\ \ \langle y\rangle \not\xmapsto{s} \perp \tag{1416}$$

Apply Rule P2 14 on Predicates 1414, 1415 and 1416, we get:

$$\langle x + y \rangle \overset{v}{\longmapsto} \bot \tag{1417}$$

Apply Rule P2 14 on the above predicate, we get:

$$\langle \sigma^t_{\mathsf{rel}}(x + y) \rangle \overset{t+v}{\longmapsto} \bot$$

(b) *Rule P2 29:*
If this rule is used to derive Predicate 1403, then according to the rule the following must hold:

$$\langle \sigma^t_{\mathsf{rel}}(y) \rangle \overset{r}{\longmapsto} \bot, \tag{1418}$$
$$\langle \texttt{consistent}\ \ \sigma^t_{\mathsf{rel}}(x) \rangle, \tag{1419}$$
$$(\forall s < r,\ \ \langle \sigma^t_{\mathsf{rel}}(x) \rangle \overset{s}{\not\longmapsto} \bot) \tag{1420}$$

Reasoning similar to given above for Rule P2 28 applies.

7.
$$\langle \texttt{consistent}\ \ \sigma^t_{\mathsf{rel}}(x + y) \rangle \iff \langle \texttt{consistent}\ \ \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y) \rangle$$

Left Implication

Suppose,

$$\langle \texttt{consistent}\ \ \sigma^t_{\mathsf{rel}}(x + y) \rangle$$

Rule P2-12 indicates that a process term with a relative delay of $t > 0$ time units is always consistent. Hence, the following holds:

$$\langle \texttt{consistent}\ \ \sigma^t_{\mathsf{rel}}(x) \rangle$$
$$\langle \texttt{consistent}\ \ \sigma^t_{\mathsf{rel}}(y) \rangle$$

Apply Rule P2-27 on the above two predicates, we get:

$$\langle \texttt{consistent}\ \ \sigma^t_{\mathsf{rel}}(x) \rangle + \langle \texttt{consistent}\ \ \sigma^t_{\mathsf{rel}}(y) \rangle$$

Right Implication

Suppose,

$$\langle \texttt{consistent}\ \ \sigma^t_{\mathsf{rel}}(x) + \sigma^t_{\mathsf{rel}}(y) \rangle$$

From Rule P2-12, a process term with a relative delay of $t > 0$ time units is always consistent. Hence, the following holds:

$$\langle \texttt{consistent}\ \ \sigma^t_{\mathsf{rel}}(x + y) \rangle$$

H.17 Axiom SRT4

$\sigma^u_{\mathsf{rel}}(x) \cdot y = \sigma^u_{\mathsf{rel}}(x \cdot y) \qquad u \geq 0$ (SRT4)

We give the proof in two steps:

Case $u = 0$

The proof is trivial using Axiom SRT1.

Case $u > 0$

We need to prove, $\sigma^u_{\mathsf{rel}}(x) \cdot y \leftrightarrows \sigma^u_{\mathsf{rel}}(x \cdot y)$.

Let R be a binary relation on process terms defined as follows:

$$R = \{ \quad (\sigma^t_{\mathsf{rel}}(x) \cdot y, \sigma^t_{\mathsf{rel}}(x \cdot y)) \mid x, y \in P, 0 < t \leq u\}$$

For all $x, y, p \in P$, $r > 0$, $a \in A$, the following holds:

1.
$$\langle \sigma^t_{\mathsf{rel}}(x) \cdot y \rangle \xrightarrow{a} \langle p \rangle \implies \begin{array}{l} \exists z \in P : \langle \sigma^t_{\mathsf{rel}}(x \cdot y) \rangle \xrightarrow{a} \langle z \rangle \\ \text{and } (p, z) \in R \cup \mathcal{I}. \end{array}$$

Suppose,

$$\langle \sigma^t_{\mathsf{rel}}(x) \cdot y \rangle \xrightarrow{a} \langle p \rangle \tag{1421}$$

The above action step can only be derived from Rule P2-15 or 16. We discuss the two cases one by one:

(a) Rule P2-15

If Transition 1421 is derived from this rule, then for some process term p', $p = p' \cdot y$. And from the premise of the rule, the following must be derivable,

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \langle p' \rangle \tag{1422}$$

An action step for operator σ^t_{rel} with $t > 0$ cannot be derived from any rules. Hence we conclude that Rule P2-15 cannot be used to derive Transition 1421.

(b) Rule P2-16

If Transition 1421 is derived from this rule, then, $p = y$. And from the premise of the rule, the following must be derivable,

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \surd \tag{1423}$$

A termination step for operator σ^t_{rel} with $t > 0$ cannot be derived from any rules. Hence we conclude that Rule P2-16 cannot be used to derive Transition 1421.

Transition 1421 cannot be derived from any rules. Since the left hand side of the implication does not hold, therefore the implication holds.

2.
$$\langle \sigma^t_{\mathsf{rel}}(x \cdot y)\rangle \xrightarrow{a} \langle p\rangle \implies \begin{array}{l}\exists z \in P : \langle \sigma^t_{\mathsf{rel}}(x) \cdot y\rangle \xrightarrow{a} \langle z\rangle \\ \text{and } (z, p) \in R \cup \mathcal{I}.\end{array}$$

Suppose,

$$\langle \sigma^t_{\mathsf{rel}}(x) \cdot y\rangle \xrightarrow{a} \langle p\rangle \tag{1424}$$

An action step for operator σ^t_{rel} with $t > 0$ cannot be derived from any rules. Hence our supposition is wrong.

3.
$$\langle \sigma^t_{\mathsf{rel}}(x) \cdot y\rangle \overset{r}{\longmapsto} \langle p\rangle \implies \begin{array}{l}\exists z \in P : \langle \sigma^t_{\mathsf{rel}}(x \cdot y)\rangle \overset{r}{\longmapsto} \langle z\rangle \\ \text{and } (p, z) \in R \cup \mathcal{I}.\end{array}$$

Suppose,

$$\langle \sigma^t_{\mathsf{rel}}(x) \cdot y\rangle \overset{r}{\longmapsto} \langle p\rangle \tag{1425}$$

The above time step can only be derived from Rule P2-17. Then for some process term p', $p = p' \cdot y$. Rewriting Transition 1425:

$$\langle \sigma^t_{\mathsf{rel}}(x) \cdot y\rangle \overset{r}{\longmapsto} \langle p' \cdot y\rangle \tag{1426}$$

From the premise of the rule the following holds:

$$\langle \sigma^t_{\mathsf{rel}}(x)\rangle \overset{r}{\longmapsto} \langle p'\rangle \tag{1427}$$

We distinguish between three cases for different values of r:

(a) Case $r < t$

Let $t = r + r_1$, for some $r_1 > 0$.
Then Transition 1427 can only be derived from Rule P2-9. From the rule, we have $p' = \sigma^{r_1}_{\mathsf{rel}}(x)$. Rewriting Transitions 1426 and 1427:

$$\langle \sigma^{r+r_1}_{\mathsf{rel}}(x) \cdot y\rangle \overset{r}{\longmapsto} \langle \sigma^{r_1}_{\mathsf{rel}}(x) \cdot y\rangle \tag{1428}$$

$$\langle \sigma^{r+r_1}_{\mathsf{rel}}(x)\rangle \overset{r}{\longmapsto} \langle \sigma^{r_1}_{\mathsf{rel}}(x)\rangle \tag{1429}$$

From Rule P2-9, the following can be

$$\langle \sigma^{r+r_1}_{\mathsf{rel}}(x \cdot y)\rangle \overset{r}{\longmapsto} \langle \sigma^{r_1}_{\mathsf{rel}}(x \cdot y)\rangle \tag{1430}$$

Consider the target process terms in Transitions 1428 and 1430. For $0 < r_1 < t$, the pair $(\sigma^{r_1}_{\mathsf{rel}}(x) \cdot y, \sigma^{r_1}_{\mathsf{rel}}(x \cdot y))$ is in R.

(b) Case $r = t$
Then Transition 1427 can only be derived from Rule P2-10. From the rule, we have $p' = x$. Rewriting Transitions 1426 and 1427:

$$\langle \sigma^t_{\mathsf{rel}}(x) \cdot y \rangle \overset{t}{\longmapsto} \langle x \cdot y \rangle \tag{1431}$$

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \overset{t}{\longmapsto} \langle x \rangle \tag{1432}$$

From the premise of Rule P2-10, the following must hold:

$$\langle \texttt{consistent}\ \ x \rangle$$

Apply Rule P2-18 on the above predicate, we get:

$$\langle \texttt{consistent}\ \ x \cdot y \rangle \tag{1433}$$

Apply Rule P2-10 on process term $\sigma^t_{\mathsf{rel}}(x \cdot y)$, we get:

$$\langle \sigma^t_{\mathsf{rel}}(x \cdot y) \rangle \overset{t}{\longmapsto} \langle x \cdot y \rangle \tag{1434}$$

Consider the target process terms in Transitions 1431 and 1434. The pair $(x \cdot y, x \cdot y)$ is in $\mathcal{I}$.

(c) Case $r > t$
Let $r = t + v$, for some $v > 0$.
Rewriting Transitions 1426 and 1427:

$$\langle \sigma^t_{\mathsf{rel}}(x) \cdot y \rangle \overset{t+v}{\longmapsto} \langle p' \cdot y \rangle \tag{1435}$$

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \overset{t+v}{\longmapsto} \langle p' \rangle \tag{1436}$$

Transition 1436 can only be derived from Rule P2-11. Then from the premise of the rule the following holds:

$$\langle x \rangle \overset{v}{\longmapsto} \langle p' \rangle \tag{1437}$$

Apply Rule P2-17 on the above transition, we get:

$$\langle x \cdot y \rangle \overset{v}{\longmapsto} \langle p' \cdot y \rangle \tag{1438}$$

Apply Rule P2-11 on the above transition, we get:

$$\langle \sigma^t_{\mathsf{rel}}(x \cdot y) \rangle \overset{t+v}{\longmapsto} \langle p' \cdot y \rangle \tag{1439}$$

Consider the target process terms in Transitions 1435 and 1439. The pair $(p' \cdot y, p' \cdot y)$ is in $\mathcal{I}$.

4.
$$\langle \sigma^t_{\mathsf{rel}}(x \cdot y) \rangle \overset{r}{\longmapsto} \langle p \rangle \implies \begin{array}{l} \exists z \in P : \langle \sigma^t_{\mathsf{rel}}(x) \cdot y \rangle \overset{r}{\longmapsto} \langle z \rangle \\ \text{and } (z, p) \in R \cup \mathcal{I}. \end{array}$$

Suppose,

$$\langle \sigma^t_{\mathsf{rel}}(x \cdot y) \rangle \overset{r}{\longmapsto} \langle p \rangle \tag{1440}$$

We distinguish between three cases for different values of r.

(a) Case $r < t$

Let $t = r + r_1$, for some $r_1 < t$.
Then Transition 1440 can only be derived from Rule P2-9. From the rule, we have $p = \sigma_{\mathsf{rel}}^{r_1}(x \cdot y)$. Rewriting Transition 1440:

$$\langle \sigma_{\mathsf{rel}}^{r+r_1}(x \cdot y)\rangle \xmapsto{r} \langle \sigma_{\mathsf{rel}}^{r_1}(x \cdot y)\rangle \tag{1441}$$

From Rule P2-9, the following can be derived:

$$\langle \sigma_{\mathsf{rel}}^{r+r_1}(x)\rangle \xmapsto{r} \langle \sigma_{\mathsf{rel}}^{r_1}(x)\rangle$$

Apply Rule P2-17 on the above transition. We get:

$$\langle \sigma_{\mathsf{rel}}^{r+r_1}(x) \cdot y\rangle \xmapsto{r} \langle \sigma_{\mathsf{rel}}^{r_1}(x) \cdot y\rangle \tag{1442}$$

Consider the target process terms in Transitions 1441 and 1442. The pair $(\sigma_{\mathsf{rel}}^{r_1}(x) \cdot y), \sigma_{\mathsf{rel}}^{r_1}(x \cdot y))$ is in R.

(b) Case $r = t$
Then Transition 1440 can only be derived from Rule P2-10. From the rule, we have $p = x \cdot y$. Rewriting Transition 1440:

$$\langle \sigma_{\mathsf{rel}}^{t}(x \cdot y)\rangle \xmapsto{t} \langle x \cdot y\rangle \tag{1443}$$

The above time step can only be derived from Rule P2-10. From the premise of the rule,

$$\langle \texttt{consistent}\ \ x \cdot y\rangle$$

which can only be derived from Rule P2-18. Then the following must hold:

$$\langle \texttt{consistent}\ \ x\rangle$$

Apply Rule P2-10 on the above predicate, we get:

$$\langle \sigma_{\mathsf{rel}}^{t}(x)\rangle \xmapsto{t} \langle x\rangle \tag{1444}$$

Apply Rule P2-17, we get:

$$\langle \sigma_{\mathsf{rel}}^{t}(x) \cdot y\rangle \xmapsto{t} \langle x \cdot y\rangle \tag{1445}$$

Consider the target process terms in Transitions 1434 and 1445. The pair $(x \cdot y, x \cdot y)$ is in R.

(c) Case $r > t$
Let $r = v + t$, for some $v > 0$.
Then Transition 1440 can only be derived from Rule P2-11. Rewriting Transition 1440:

$$\langle \sigma_{\mathsf{rel}}^{t}(x \cdot y)\rangle \xmapsto{t+v} \langle p\rangle \tag{1446}$$

From the premise of the rule,

$$\langle x \cdot y \rangle \xmapsto{v} \langle p \rangle \tag{1447}$$

The above transition can only be derived from Rule P2-17. Then for some process term p', $p = p' \cdot y$. Rewriting Transitions 1446 and 1447, we get:

$$\langle \sigma^t_{\mathsf{rel}}(x \cdot y) \rangle \xmapsto{t+v} \langle p' \cdot y \rangle \tag{1448}$$

$$\langle x \cdot y \rangle \xmapsto{v} \langle p' \cdot y \rangle \tag{1449}$$

From the premise of the rule,

$$\langle x \rangle \xmapsto{v} \langle p' \rangle \tag{1450}$$

Apply Rule P2-11 on the above transition, we get:

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xmapsto{t+v} \langle p' \rangle \tag{1451}$$

Apply Rule P2-17 on the above transition, we get:

$$\langle \sigma^t_{\mathsf{rel}}(x) \cdot y \rangle \xmapsto{t+v} \langle p' \cdot y \rangle \tag{1452}$$

Consider the target process terms in Transitions 1446 and 1452. The pair $(p' \cdot y, p' \cdot y)$ is in $\mathcal{I}$.

5.
$$\langle \sigma^t_{\mathsf{rel}}(x \cdot y) \rangle \xrightarrow{a} \surd \iff \langle \sigma^t_{\mathsf{rel}}(x) \cdot y \rangle \xrightarrow{a} \surd$$

Trivial therefore left.

6.
$$\langle \sigma^t_{\mathsf{rel}}(x) \cdot y \rangle \xmapsto{r}_{\perp} \iff \langle \sigma^t_{\mathsf{rel}}(x \cdot y) \rangle \xmapsto{r}_{\perp}$$

Left Implication

Suppose,

$$\langle \sigma^t_{\mathsf{rel}}(x) \cdot y \rangle \xmapsto{r}_{\perp} \tag{1453}$$

The above predicate can only be derived from Rule P2-19. From the premise of the rule, the following holds:

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xmapsto{r}_{\perp} \tag{1454}$$

We distinguish between three cases for different values of r.

(a) Case $r < t$

For $r < t$, Predicate 1454 cannot be derived. We conclude that Predicate 1453 cannot be derived for $r < t$.

(b) Case $r = t$

Then Predicate 1454 can only be derived from Rule P2-13. Rewriting Predicate 1454, we get:

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \overset{t}{\longmapsto} \bot \tag{1455}$$

From the premise of the rule, the following holds:

$$\neg\langle \texttt{consistent}\ \ x \rangle$$

Then the following also holds:

$$\neg\langle \texttt{consistent}\ \ x \cdot y \rangle$$

Apply Rule P2-13 on the process term $\sigma^t_{\mathsf{rel}}(x \cdot y)$, we get:

$$\langle \sigma^t_{\mathsf{rel}}(x \cdot y) \rangle \overset{t}{\longmapsto} \bot$$

(c) Case $r > t$

Let $r = t + v$, for some $v > 0$.
Rewriting Predicate 1454:

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \overset{t+v}{\longmapsto} \bot \tag{1456}$$

The above Predicate can only be derived from Rule P2-14. From the premise of the rule, the following holds:

$$\langle x \rangle \overset{v}{\longmapsto} \bot \tag{1457}$$

Apply Rule P2-19 on the above predicate, we get:

$$\langle x \cdot y \rangle \overset{v}{\longmapsto} \bot$$

Apply Rule P2-14 on the above predicate, we get:

$$\langle \sigma^t_{\mathsf{rel}}(x \cdot y) \rangle \overset{t+v}{\longmapsto} \bot$$

Right Implication

Suppose,

$$\langle \sigma^t_{\mathsf{rel}}(x \cdot y) \rangle \overset{r}{\longmapsto} \bot \tag{1458}$$

We distinguish between three cases for different values of r.

(a) Case $r < t$
A Future Inconsistency predicate for a process term with operator σ^t_{rel} of duration less than t cannot be derived. Hence, for $r < t$, Predicate 1458 cannot hold.

(b) Case $r = t$
Then Predicate 1458 can only be derived from Rule P2-13. From the premise of the rule, the following holds:

$$\neg\langle \texttt{consistent}\ \ x \cdot y \rangle$$

The above predicate can hold only if, $\neg\langle \texttt{consistent}\ \ x \rangle$ holds.

$$\neg\langle \texttt{consistent}\ \ x \rangle \tag{1459}$$

Apply Rule P2-13 on the above predicate. We get:

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xmapsto{t} \bot \tag{1460}$$

Apply Rule P2-19 on the above predicate, we get:

$$\langle \sigma^t_{\mathsf{rel}}(x) \cdot y \rangle \xmapsto{t} \bot$$

(c) Case $r > t$
Let $r = t + v$, for some $v > 0$.
Rewriting Predicate 1458, we get:

$$\langle \sigma^t_{\mathsf{rel}}(x \cdot y) \rangle \xmapsto{t+v} \bot \tag{1461}$$

Predicate 1461 can only be derived from Rule P2-14. Then from the premise of the rule, the following holds:

$$\langle x \cdot y \rangle \xmapsto{v} \bot \tag{1462}$$

Predicate 1462 can only be derived from Rule P2-19. Then from the premise of the rule, the following holds:

$$\langle x \rangle \xmapsto{v} \bot$$

Apply Rule P2-14 on the above predicate, we get:

$$\langle \sigma^t_{\mathsf{rel}}(x) \rangle \xmapsto{t+v} \bot \tag{1463}$$

Apply Rule P2-19 on the above predicate, we get:

$$\langle \sigma^t_{\mathsf{rel}}(x) \cdot y \rangle \xmapsto{t+v} \bot \tag{1464}$$

7.
$$\langle \texttt{consistent}\ \ \sigma^t_{\mathsf{rel}}(x \cdot y) \rangle \iff \langle \texttt{consistent}\ \ \sigma^t_{\mathsf{rel}}(x) \cdot y \rangle$$

From Rule P2-12,
$$\langle \texttt{consistent}\ \ \sigma^t_{\mathsf{rel}}(x \cdot y) \rangle$$

From Rule P2-18 and Rule P2-12, it can be derived that:

$$\langle \texttt{consistent}\ \ \sigma^t_{\mathsf{rel}}(x) \cdot y \rangle$$

H.18 Axiom SRU1

$\nu_{\mathsf{rel}}(\tilde{\tilde{a}}) = \tilde{\tilde{a}}$ (SRU1)

We need to prove, $\nu_{\mathsf{rel}}(\tilde{\tilde{a}}) \leftrightarrows \tilde{\tilde{a}}$.

Let R be a binary relation on process terms defined as follows:

$$R = \{(\nu_{\mathsf{rel}}(\tilde{\tilde{a}}), \tilde{\tilde{a}}) \mid a \in A\}$$

The proof that $R \cup \mathcal{I}$ is a bisimulation relation is trivial and therefore left out.

H.19 Axiom SRU2

$\nu_{\mathsf{rel}}(\sigma^r_{\mathsf{rel}}(x)) = \tilde{\tilde{\delta}}$ $r > 0$ (SRU2)

We need to prove, $\nu_{\mathsf{rel}}(\sigma^u_{\mathsf{rel}}(x)) \leftrightarrows \tilde{\tilde{\delta}}$.

Let R be a binary relation on process terms defined as follows:

$$R = \{(\nu_{\mathsf{rel}}(\sigma^u_{\mathsf{rel}}(x)) \mid x \in P, u > 0\}$$

The proof that $R \cup \mathcal{I}$ is a bisimulation relation is trivial and therefore left out.

H.20 Axiom SRU3

$\nu_{\mathsf{rel}}(x + y) = \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y).$ (SRU3)

We need to prove, $\nu_{\mathsf{rel}}(x + y) \leftrightarrows \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y)$.

Let R be a binary relation on process terms defined as follows:

$$R \quad = \quad \{ \quad (\nu_{\mathsf{rel}}(x + y), \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y)) \mid x, y \in P\}$$

We prove that the relation $R \cup \mathcal{I}$ satisfies all conditions of bisimulation.

For all $a \in A, r > 0, x, y, p \in P$, the following holds:

1. $$\langle \nu_{\mathsf{rel}}(x + y)\rangle \xrightarrow{a} \langle p\rangle \implies \exists z \in P : \langle \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y)\rangle \xrightarrow{a} \langle z\rangle \text{ and } (p, z) \in R.$$

   Suppose,
   $$\langle \nu_{\mathsf{rel}}(x + y)\rangle \xrightarrow{a} \langle p\rangle \tag{1465}$$

   The above transition can only be derived from Rule P2-30. Then from the premise the following holds:
   $$\langle x + y\rangle \xrightarrow{a} \langle p\rangle \tag{1466}$$

   The above action step can be derived from two rules:

   (a) Rule P2-20
   If Transition 1466 is derived from this rule, then from the premise of the rule, we have:
   $$\langle x\rangle \xrightarrow{a} \langle p\rangle \tag{1467}$$
   $$\langle \texttt{consistent}\ y\rangle \tag{1468}$$

Apply Rule P2-30 on Transition 1467, we get:

$$\langle \nu_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \langle p \rangle \tag{1469}$$

Apply Rule P2-32 on Predicate 1468, we get:

$$\langle \texttt{consistent}\ \nu_{\mathsf{rel}}(y) \rangle \tag{1470}$$

Apply Rule P2-20 on Transition 1469 and Predicate 1470. We get:

$$\langle \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y) \rangle \xrightarrow{a} \langle p \rangle \tag{1471}$$

Consider the target process terms in Transitions 1466 and 1471. The pair (p, p) is in $\mathcal{I}$.

(b) Rule P2-21

If Transition 1466 is derived from this rule, then from the premise of the rule, we have:

$$\langle y \rangle \xrightarrow{a} \langle p \rangle \tag{1472}$$

$$\langle \texttt{consistent}\ x \rangle \tag{1473}$$

Reasoning similar to that of Rule P2-20 applies here.

2.

$$\langle \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y) \rangle \xrightarrow{a} \langle p \rangle \implies \begin{array}{l} \exists z \in P : \langle \nu_{\mathsf{rel}}(x + y) \rangle \xrightarrow{a} \langle z \rangle \\ \text{and } (p, z) \in R. \end{array}$$

Suppose,

$$\langle \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y) \rangle \xrightarrow{a} \langle p \rangle \tag{1474}$$

The above transition can be derived from Rule P2-20 or Rule P2-21. We discuss them one by one:

(a) Rule P2-20

If Transition 1474 is derived from this rule, then from the premise of the rule, we have:

$$\langle \nu_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \langle p \rangle \tag{1475}$$

$$\langle \texttt{consistent}\ \nu_{\mathsf{rel}}(y) \rangle \tag{1476}$$

Transition 1475 can only be derived from Rule P2-30. Predicate 1476 can only be derived from Rule P2-32. From their premises, the following holds:

$$\langle x \rangle \xrightarrow{a} \langle p \rangle \tag{1477}$$

$$\langle \texttt{consistent}\ y \rangle \tag{1478}$$

Apply Rule P2-20 on the above transition and predicate, we get:

$$\langle x + y \rangle \xrightarrow{a} \langle p \rangle \tag{1479}$$

Apply Rule P2-30 on the above transition, we get:

$$\langle \nu_{\mathsf{rel}}(x + y) \rangle \xrightarrow{a} \langle p \rangle \tag{1480}$$

Consider the target process terms in Transitions 1474 and 1480. The pair (p, p) is in $\mathcal{I}$.

(b) Rule P2-21
If Transition 1474 is derived from this rule, then from the premise of the rule, we have:

$$\langle \nu_{\mathsf{rel}}(y) \rangle \xrightarrow{a} \langle p \rangle \tag{1481}$$

$$\langle \texttt{consistent } \nu_{\mathsf{rel}}(x) \rangle \tag{1482}$$

Reasoning similar to that given for Rule P2-20 applies here.

3.
$$\langle \nu_{\mathsf{rel}}(x + y) \rangle \overset{r}{\longmapsto} \langle p \rangle \implies \exists z \in P : \langle \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y) \rangle \overset{r}{\longmapsto} \langle z \rangle \text{ and } (p, z) \in R.$$

Suppose,

$$\langle \nu_{\mathsf{rel}}(x + y) \rangle \overset{r}{\longmapsto} \langle p \rangle$$

A time step for now operator can not be derived from any rules. Hence our supposition cannot hold and the implication is trivially satisfied.

4.
$$\langle \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y) \rangle \overset{r}{\longmapsto} \langle p \rangle \implies \exists z \in P : \langle \nu_{\mathsf{rel}}(x + y) \rangle \overset{r}{\longmapsto} \langle z \rangle \text{ and } (p, z) \in R.$$

Suppose,

$$\langle \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y) \rangle \overset{r}{\longmapsto} \langle p \rangle \tag{1483}$$

A time step for an alternative composition can be derived from Rule P2-24, Rule P2-25 or Rule P2-26. We discuss them one by one:

(a) Rule P2-24
If transition 1483 is derived from this rule, Then for some process term $x_1, y_1, p = x_1 + y_1$ and from the premise of the rule, the following holds:

$$\langle \nu_{\mathsf{rel}}(x) \rangle \overset{r}{\longmapsto} \langle x_1 \rangle \tag{1484}$$

$$\langle \nu_{\mathsf{rel}}(y) \rangle \overset{r}{\longmapsto} \langle y_1 \rangle \tag{1485}$$

A time step for now operator can not be derived from any rules. Hence Transitions 1484 and 1485 are not derivable. We conclude that Rule P2-24 cannot be used to derive Transition 1483.

(b) Rule P2-25

If transition 1483 is derived from this rule, Then the premise of the rule, the following holds:

$$\langle \nu_{\mathsf{rel}}(x) \rangle \xmapsto{r} \langle p \rangle \tag{1486}$$

$$\langle \texttt{consistent}\ \nu_{\mathsf{rel}}(y) \rangle \tag{1487}$$

$$\langle \nu_{\mathsf{rel}}(y) \rangle \not\xmapsto{r} \tag{1488}$$

$$\forall s \leq r\ \ \langle \nu_{\mathsf{rel}}(y) \rangle \not\xmapsto{s}_{\perp} \tag{1489}$$

A time step for now operator can not be derived from any rules. Hence Transition 1486 is not derivable. We conclude that Rule P2-25 cannot be used to derive Transition 1483.

(c) Rule P2-26

Similarly, if transition 1483 is derived from this rule, Then the premise of the rule, the following holds:

$$\langle \nu_{\mathsf{rel}}(y) \rangle \xmapsto{r} \langle p \rangle \tag{1490}$$

$$\langle \texttt{consistent}\ \nu_{\mathsf{rel}}(x) \rangle \tag{1491}$$

$$\langle \nu_{\mathsf{rel}}(x) \rangle \not\xmapsto{r} \tag{1492}$$

$$\forall s \leq r\ \ \langle \nu_{\mathsf{rel}}(x) \rangle \not\xmapsto{s}_{\perp} \tag{1493}$$

A time step for now operator can not be derived from any rules. Hence Transition 1490 is not derivable. We conclude that Rule P2-26 cannot be used to derive Transition 1483.

No rules allow derivation of Transition 1483. Hence our supposition cannot hold and the implication is trivially satisfied.

5.

$$\langle \nu_{\mathsf{rel}}(x+y) \rangle \xmapsto{r}_{\perp} \iff \langle \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y) \rangle \xmapsto{r}_{\perp}$$

Left Implication

Suppose,

$$\langle \nu_{\mathsf{rel}}(x+y) \rangle \xmapsto{r}_{\perp}$$

The above predicate can not be derived from any rules. Hence the left implication is trivially satisfied.

Right Implication

Suppose,

$$\langle \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y) \rangle \xmapsto{r}_{\perp} \tag{1494}$$

A future inconsistency predicate for an alternative composition can be derived from Rule P2-28 or Rule P2-29.

We discuss them one by one:

(a) Rule P2-28
If predicate 1494 is derived from this rule, Then from the premise of the rule, the following holds:

$$\langle \nu_{\mathsf{rel}}(x) \rangle \overset{r}{\longmapsto}\bot \tag{1495}$$

$$\langle \texttt{consistent}\ \nu_{\mathsf{rel}}(y) \rangle \tag{1496}$$

$$\forall s < r\ \ \langle \nu_{\mathsf{rel}}(y) \rangle \overset{s}{\not\longmapsto} \bot \tag{1497}$$

A future inconsistency predicate for a now operator can not be derived from any rules. Hence Predicate 1495 is not derivable. We conclude that Rule P2-28 cannot be used to derive Predicate 1494.

(b) Rule P2-29
If predicate 1494 is derived from this rule, Then the premise of the rule, the following holds:

$$\langle \nu_{\mathsf{rel}}(y) \rangle \overset{r}{\longmapsto}\bot \tag{1498}$$

$$\langle \texttt{consistent}\ \nu_{\mathsf{rel}}(x) \rangle \tag{1499}$$

$$\forall s < r\ \ \langle \nu_{\mathsf{rel}}(x) \rangle \overset{r}{\not\longmapsto}\bot \tag{1500}$$

A future inconsistency predicate for now operator can not be derived from any rules. Hence Predicate 1498 is not derivable. We conclude that Rule P2-29 cannot be used to derive Predicate 1494.

No rules allow derivation of Predicate 1494. Hence our supposition cannot hold and the implication is trivially satisfied.

6.
$$\langle \nu_{\mathsf{rel}}(x+y) \rangle \xrightarrow{a} \surd \iff \langle \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y) \rangle \xrightarrow{a} \surd$$

Left Implication
Suppose,

$$\langle \nu_{\mathsf{rel}}(x+y) \rangle \xrightarrow{a} \surd \tag{1501}$$

The above predicate can only be derived from Rule P2-31. Then from the premise the following holds:

$$\langle x + y \rangle \xrightarrow{a} \surd \tag{1502}$$

The above action step can be derived from two rules:

(a) Rule P2-22
If Predicate 1502 is derived from this rule, then from the premise of the rule, we have:

$$\langle x \rangle \xrightarrow{a} \surd \tag{1503}$$

$$\langle \texttt{consistent}\ y \rangle \tag{1504}$$

Apply Rule P2-31 on Predicate 1503, we get:

$$\langle \nu_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \surd \tag{1505}$$

Apply Rule P2-32 on Predicate 1504, we get:

$$\langle \mathtt{consistent}\ \nu_{\mathsf{rel}}(y) \rangle \tag{1506}$$

Apply Rule P2-22 on Predicate 1505 and Predicate 1506. We get:

$$\langle \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y) \rangle \xrightarrow{a} \surd \tag{1507}$$

(b) Rule P2-23

If Predicate 1502 is derived from this rule, then from the premise of the rule, we have:

$$\langle y \rangle \xrightarrow{a} \surd \tag{1508}$$

$$\langle \mathtt{consistent}\ x \rangle \tag{1509}$$

Reasoning similar to that of Rule P2-22 applies here.

Right Implication

Suppose,

$$\langle \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y) \rangle \xrightarrow{a} \surd \tag{1510}$$

The above predicate can be derived from Rule P2-22 or Rule P2-23. We discuss them one by one:

(a) Rule P2-22
If Predicate 1510 is derived from this rule, then from the premise of the rule, we have:

$$\langle \nu_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \surd \tag{1511}$$

$$\langle \mathtt{consistent}\ \nu_{\mathsf{rel}}(y) \rangle \tag{1512}$$

Predicate 1511 can only be derived from Rule P2-31. Predicate 1512 can only be derived from Rule P2-32. From their premises, the following holds:

$$\langle x \rangle \xrightarrow{a} \surd \tag{1513}$$

$$\langle \mathtt{consistent}\ y \rangle \tag{1514}$$

Apply Rule P2-22 on the above predicates, we get:

$$\langle x + y \rangle \xrightarrow{a} \surd \tag{1515}$$

Apply Rule P2-31 on the above predicate, we get:

$$\langle \nu_{\mathsf{rel}}(x + y) \rangle \xrightarrow{a} \surd \tag{1516}$$

(b) Rule P2-23

If Predicate 1510 is derived from this rule, then from the premise of the rule, we have:

$$\langle \nu_{\mathsf{rel}}(y) \rangle \xrightarrow{a} \surd \tag{1517}$$

$$\langle \mathtt{consistent}\ \ \nu_{\mathsf{rel}}(x) \rangle \tag{1518}$$

Reasoning similar to that given for Rule P2-22 applies here.

7.

$$\langle \mathtt{consistent}\ \ \nu_{\mathsf{rel}}(x+y) \rangle \iff \langle \mathtt{consistent}\ \ \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y) \rangle$$

Left Implication

Suppose

$$\langle \mathtt{consistent}\ \ \nu_{\mathsf{rel}}(x+y) \rangle$$

The above predicate is only derivable from Rule P2-32. Then from premise of the rule, the following holds:

$$\langle \mathtt{consistent}\ \ x+y \rangle \tag{1519}$$

which is only derivable from Rule P2-27. Then from the premise the following holds:

$$\langle \mathtt{consistent}\ \ x \rangle \tag{1520}$$

$$\langle \mathtt{consistent}\ \ y \rangle \tag{1521}$$

Apply Rule P2-32 on the above predicates, we get:

$$\langle \mathtt{consistent}\ \ \nu_{\mathsf{rel}}(x) \rangle \tag{1522}$$

$$\langle \mathtt{consistent}\ \ \nu_{\mathsf{rel}}(y) \rangle \tag{1523}$$

Apply Rule P2-27 on the above predicates, we get the desired predicate:

$$\langle \mathtt{consistent}\ \ \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y) \rangle$$

Right Implication

Suppose,

$$\langle \mathtt{consistent}\ \ \nu_{\mathsf{rel}}(x) + \nu_{\mathsf{rel}}(y) \rangle \tag{1524}$$

which is only derivable from Rule P2-27. Then from the premise the following holds:

$$\langle \mathtt{consistent}\ \ \nu_{\mathsf{rel}}(x) \rangle \tag{1525}$$

$$\langle \mathtt{consistent}\ \ \nu_{\mathsf{rel}}(y) \rangle \tag{1526}$$

The above predicates are only derivable from Rule P2-32. Then from the premise the following must hold:

$$\langle \texttt{consistent}\ x \rangle \tag{1527}$$

$$\langle \texttt{consistent}\ y \rangle \tag{1528}$$

Apply Rule P2-27 on the above predicates, we get:

$$\langle \texttt{consistent}\ x + y \rangle \tag{1529}$$

Apply Rule P2-32 on the above predicate, we get:

$$\langle \texttt{consistent}\ \nu_{\mathsf{rel}}(x + y) \rangle \tag{1530}$$

H.21 Axiom SRU4

$\nu_{\mathsf{rel}}(x \cdot y) = \nu_{\mathsf{rel}}(x) \cdot y.$ (SRU4)

We need to prove, $\nu_{\mathsf{rel}}(x \cdot y) \leftrightarrows \nu_{\mathsf{rel}}(x) \cdot y$.

Let R be a binary relation on process terms defined as follows:

$$R \quad = \quad \{ \quad (\nu_{\mathsf{rel}}(x \cdot y), \nu_{\mathsf{rel}}(x) \cdot y) \mid x, y \in P\}$$

We prove that the relation $R \cup \mathcal{I}$ satisfies all conditions of bisimulation.

For all $a \in A, r > 0, x, y, p \in P$, the following holds:

1.
$$\langle \nu_{\mathsf{rel}}(x \cdot y) \rangle \xrightarrow{a} \langle p \rangle \implies \quad \exists z \in P : \langle \nu_{\mathsf{rel}}(x) \cdot y \rangle \xrightarrow{a} \langle z \rangle \text{ and } (p, z) \in R.$$

Suppose,

$$\langle \nu_{\mathsf{rel}}(x \cdot y) \rangle \xrightarrow{a} \langle p \rangle \tag{1531}$$

The above transition can only be derived from Rule P2-30. Then from the premise the following holds:

$$\langle x \cdot y \rangle \xrightarrow{a} \langle p \rangle \tag{1532}$$

The above action step can be derived from two rules:

(a) <u>Rule P2-15</u>

If Transition 1532 is derived from this rule, then for some process term p', $p = p' \cdot y$. Rewriting Transition 1532:

$$\langle x \cdot y \rangle \xrightarrow{a} \langle p' \cdot y \rangle \tag{1533}$$

From the premise of the rule, we have:

$$\langle x \rangle \xrightarrow{a} \langle p' \rangle \tag{1534}$$

Apply Rule P2-30 on Transition 1534, we get:

$$\langle \nu_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \langle p' \rangle \tag{1535}$$

Apply Rule P2-15 on the above transition. We get:

$$\langle \nu_{\mathsf{rel}}(x) \cdot y \rangle \xrightarrow{a} \langle p' \cdot y \rangle \tag{1536}$$

Consider the target process terms in Transitions 1533 and 1536. The pair $(p' \cdot y, p' \cdot y)$ is in $\mathcal{I}$.

(b) Rule P2-16

If Transition 1532 is derived from this rule, then, $p = y$. Rewriting Transition 1532:

$$\langle x \cdot y \rangle \xrightarrow{a} \langle y \rangle \tag{1537}$$

From the premise of the rule, we have:

$$\langle x \rangle \xrightarrow{a} \surd \tag{1538}$$

$$\langle \texttt{consistent}\ \ y \rangle \tag{1539}$$

Apply Rule P2-31 on Transition 1538, we get:

$$\langle \nu_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \surd \tag{1540}$$

Apply Rule P2-16 on the above transition making use of predicate 1539. We get:

$$\langle \nu_{\mathsf{rel}}(x) \cdot y \rangle \xrightarrow{a} \langle y \rangle \tag{1541}$$

Consider the target process terms in Transitions 1537 and 1541. The pair (y, y) is in $\mathcal{I}$.

2.

$$\langle \nu_{\mathsf{rel}}(x) \cdot y \rangle \xrightarrow{a} \langle p \rangle \implies \exists z \in P : \langle \nu_{\mathsf{rel}}(x \cdot y) \rangle \xrightarrow{a} \langle z \rangle \text{ and } (p, z) \in R.$$

Suppose,

$$\langle \nu_{\mathsf{rel}}(x) \cdot y \rangle \xrightarrow{a} \langle p \rangle \tag{1542}$$

The above action step can be derived from two rules:

(a) Rule P2-15

If Transition 1542 is derived from this rule, then for some process term p', $p = p' \cdot y$. Rewriting Transition 1542:

$$\langle \nu_{\mathsf{rel}}(x) \cdot y \rangle \xrightarrow{a} \langle p' \cdot y \rangle \tag{1543}$$

And from the premise of the rule, the following holds:

$$\langle \nu_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \langle p' \rangle \tag{1544}$$

The above transition can only be derived from Rule P2-30. Then from the premise the following holds:

$$\langle x \rangle \xrightarrow{a} \langle p' \rangle \tag{1545}$$

Apply Rule P2-15 on the above transition, we get:

$$\langle x \cdot y \rangle \xrightarrow{a} \langle p' \cdot y \rangle \tag{1546}$$

Again apply Rule P2-30 on Transition 1546, we get:

$$\langle \nu_{\mathsf{rel}}(x \cdot y) \rangle \xrightarrow{a} \langle p' \cdot y \rangle \tag{1547}$$

Consider the target process term in transitions 1543 and 1547. The pair $(p' \cdot y, p' \cdot y)$ is in $\mathcal{I}$.

(b) Rule P2-16
If Transition 1542 is derived from this rule, then, $p = y$. Rewriting Transition 1542:

$$\langle \nu_{\mathsf{rel}}(x) \cdot y \rangle \xrightarrow{a} \langle y \rangle \tag{1548}$$

And from the premise of the rule, the following holds:

$$\langle \nu_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \surd \tag{1549}$$

$$\langle \texttt{consistent}\ \ y \rangle \tag{1550}$$

The Transition 1549 can only be derived from Rule P2-31. Then from the premise the following holds:

$$\langle \nu_{\mathsf{rel}}(x) \rangle \xrightarrow{a} \surd \tag{1551}$$

From Premise of Rule P2-31, the following holds:

$$\langle x \rangle \xrightarrow{a} \surd \tag{1552}$$

Apply Rule P2-16 on the above transition using Predicate 1550:

$$\langle x \cdot y \rangle \xrightarrow{a} \langle y \rangle \tag{1553}$$

Apply Rule P2-30 on the above transition, we get:

$$\langle \nu_{\mathsf{rel}}(x \cdot y) \rangle \xrightarrow{a} \langle y \rangle \tag{1554}$$

Consider the target process term in transitions 1543 and 1554. The pair (y, y) is in $\mathcal{I}$.

3.
$$\langle \nu_{\mathsf{rel}}(x \cdot y) \rangle \overset{r}{\longmapsto} \langle p \rangle \implies \begin{array}{l} \exists z \in P : \langle \nu_{\mathsf{rel}}(x) \cdot y \rangle \overset{r}{\longmapsto} \langle z \rangle \\ \text{and } (p, z) \in R. \end{array}$$

Suppose,

$$\langle \nu_{\mathsf{rel}}(x \cdot y) \rangle \overset{r}{\longmapsto} \langle p \rangle$$

A time step for now operator can not be derived from any rules. Hence our supposition cannot hold and the implication is trivially satisfied.

4.
$$\langle \nu_{\mathsf{rel}}(x) \cdot y \rangle \xmapsto{r} \langle p \rangle \implies \begin{array}{l}\exists z \in P : \langle \nu_{\mathsf{rel}}(x \cdot y) \rangle \xmapsto{r} \langle z \rangle \\ \text{and } (p, z) \in R.\end{array}$$

Suppose,
$$\langle \nu_{\mathsf{rel}}(x) \cdot y \rangle \xmapsto{r} \langle p \rangle \tag{1555}$$

A time step for sequential composition can only be derived from Rule P2-17. The for some process term p', $p = p' \cdot y$. And from the premise the following holds:
$$\langle \nu_{\mathsf{rel}}(x) \rangle \xmapsto{r} \langle p' \rangle \tag{1556}$$

The above transition cannot be derived as a time step for now operator can not be derived from any rules. Hence the Transition 1555 cannot hold and the implication is trivially proved.

5.
$$\langle \nu_{\mathsf{rel}}(x \cdot y) \rangle \xrightarrow{a} \surd \iff \langle \nu_{\mathsf{rel}}(x) \cdot y \rangle \xrightarrow{a} \surd$$

Left Implication

Suppose,
$$\langle \nu_{\mathsf{rel}}(x \cdot y) \rangle \xrightarrow{a} \surd \tag{1557}$$

Predicate 1557 can only be derived from Rule P2-31. Then from the premise of the rule, the following holds:

$$\langle x \cdot y \rangle \xrightarrow{a} \surd \tag{1558}$$

A termination predicate for a sequential composition cannot be derived from any rules. Predicate 1558 doesn't hold. hence our assumption predicate 1557 doesn't hold.

Right Implication

Suppose,
$$\langle \nu_{\mathsf{rel}}(x) \cdot y \rangle \xrightarrow{a} \surd \tag{1559}$$

A termination predicate for a sequential composition cannot be derived from any rules. Hence our assumption predicate 1559 doesn't hold.

6.
$$\langle \nu_{\mathsf{rel}}(x \cdot y) \rangle \xmapsto{r}_{\perp} \iff \langle \nu_{\mathsf{rel}}(x) \cdot y \rangle \xmapsto{r}_{\perp}$$

Left Implication

Suppose,
$$\langle \nu_{\mathsf{rel}}(x \cdot y) \rangle \xmapsto{r}_{\perp} \tag{1560}$$

The above predicate can not be derived from any rules. Hence the left implication is trivially satisfied.

Right Implication

Suppose,

$$\langle \nu_{\mathsf{rel}}(x) \cdot y \rangle \overset{r}{\longmapsto}_{\perp} \tag{1561}$$

A future inconsistency predicate for sequential composition can only be derived from Rule P2-19. Then from the premise the following holds:

$$\langle \nu_{\mathsf{rel}}(x) \rangle \overset{r}{\longmapsto}_{\perp} \tag{1562}$$

The above predicate cannot be derived as a future inconsistency predicate for now operator can not be derived from any rules. Hence the Transition 1561 cannot hold and the implication is trivially proved.

7.

$$\langle \texttt{consistent}\ \nu_{\mathsf{rel}}(x \cdot y) \rangle \iff \langle \texttt{consistent}\ \nu_{\mathsf{rel}}(x) \cdot y \rangle$$

Left Implication

Suppose

$$\langle \texttt{consistent}\ \nu_{\mathsf{rel}}(x \cdot y) \rangle$$

Only derivable from Rule P2-32. Then from premise of the rule, the following holds:

$$\langle \texttt{consistent}\ x \cdot y \rangle \tag{1563}$$

which is only derivable from Rule P2-18. Then from the premise the following holds:

$$\langle \texttt{consistent}\ x \rangle \tag{1564}$$

Apply Rule P2-32 on Predicate 1564:

$$\langle \texttt{consistent}\ \nu_{\mathsf{rel}}(x) \rangle \tag{1565}$$

Again apply Rule P2-18 on Predicate 1565, we get the desired predicate:

$$\langle \texttt{consistent}\ \nu_{\mathsf{rel}}(x) \cdot y \rangle \tag{1566}$$

Right Implication

Suppose,

$$\langle \texttt{consistent}\ \nu_{\mathsf{rel}}(x) \cdot y \rangle$$

which is only derivable from Rule P2-18. Then from the premise the following must hold:

$$\langle \texttt{consistent}\ \nu_{\mathsf{rel}}(x) \rangle \tag{1567}$$

which is only derivable from Rule P2-32. Then from premise of the rule, the following holds:

$$\langle \texttt{consistent } x \rangle$$

Apply Rule P2-18 on the above predicate, we get:

$$\langle \texttt{consistent } x \cdot y \rangle \tag{1568}$$

Apply Rule P2-32 on Predicate 1568, we get the desired result:

$$\langle \texttt{consistent } \nu_{\mathsf{rel}}(x \cdot y) \rangle \tag{1569}$$

H.22 Axiom NESRU

$\nu_{\mathsf{rel}}(\perp) = \perp \qquad \text{(NESRU)}$

We need to prove, $\nu_{\mathsf{rel}}(\perp) \leftrightarrows \perp$.

Let R be a binary relation on process terms defined as follows:

$$R = \{(\nu_{\mathsf{rel}}(\perp), \perp)\}$$

The proof is trivial and therefore left out.