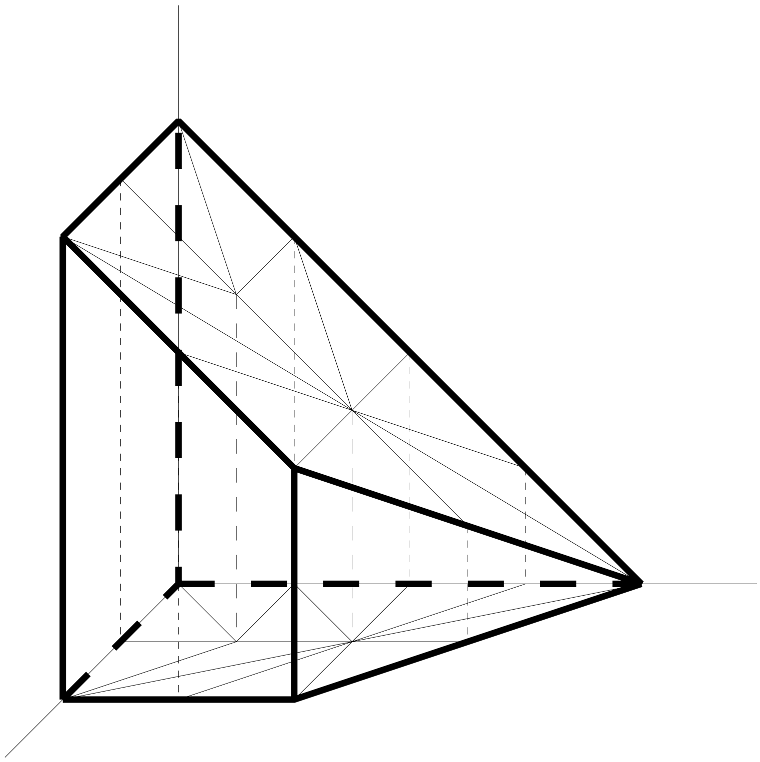
Lattice Polytopes and Triangulations

With Applications to Toric Geometry

Christian Haase



Lattice Polytopes and Triangulations

With Applications to Toric Geometry

vorgelegt von
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Thanks!

In 1996, I was about to finish being an undergraduate. What should I become as an adult? I could do real–world work and get rich, or I could be a mathematician and have fun — only where, what, how? At the time, there were no university posts for graduates, at least not in Berlin. In order to come to an end, I planned not to take part in any courses anymore, but to "dedicate my full working power"[1] to my Diplomarbeit[2] in algebraic topology. Nice plan, but as I do not like big decisions, I pushed my deadline further and further. Part of this strategy, I could not avoid the lecture entitled *Discrete structures and their topology*, given by a certain professor **Günter Ziegler**. This was my first contact with discrete mathematics. In a discussion about the perspectives of graduates, this Günter Ziegler mentioned that a graduate school could be an alternative to the usual university positions. Then he presented a number of challenging mathematical problems to me that could be the subject of a dissertation.

So there was my decision, and there was my deadline: **Bettina Felsner**, the secretary of the graduate school **Algorithmische diskrete Mathematik**[3], wanted a copy of my Diplomarbeit by January 1997. Suddenly, I really had to finish quickly. **Marion** and **Volkmar Scholz** supplied me with the necessary copying power and I was admitted. I started my discrete career in a very stimulating environment: the Berlin discrete community assembled in the graduate school and at the TU–Berlin, and remainders of the topological family around my former advisor professor **Elmar Vogt** helped me in many tea time discussions and built up this great atmosphere. One cannot overestimate the contribution of ice cream devouring Bettina Felsner in this context.

Günter Ziegler taught me how to research, how to give talks and much more. His door was always open for me, even when he had a huge pile of work on his desk. He introduced me to **Dimitrios Dais**, a very enthusiastic algebraic geometer, who reported lots of known and unknown algebraic geometry to me, prevented me from writing algebraic non–sense, and who co–authored Chapters III and IV. Once written down, this thesis had to pass my referees **Carsten Lange**, **Günter Paul Leiterer**, **Mark de Longueville**, **Frank Lutz**, **Eva–Nuria Müller** and **Carsten Schultz**.

So much about my academic support. Personally, I was supported by my two women: My mother **Heide Haase** did everything one can imagine that a mother can do — and a little bit more. In particular, I want to mention her help concerning my friends from Y–town[4]. My girlfriend **Kerstin Theurer** got me down to earth from time to time. I completely rely on the backing and love she gives to me.

Thank you all, .

[1] From: *Richtlinien für Stipendiaten* [GK, p. 2]

[2] Diploma thesis.

[3] Algorithmic discrete mathematics, DFG grant GRK 219/3.

[4] German Bundeswehr.

Contents

Chapter I. Introduction 9
1. What, why, who? 9
2. Notions from discrete and convex geometry 12
3. Notions from algebraic and toric geometry 18
4. From the dictionary 20

Chapter II. Lattice width of empty simplices 23
1. Introduction 23
2. Adding one dimension 26
3. Computer search in dimension **4** 31

Chapter III. Crepant resolutions of toric l.c.i.–singularities 39
1. Introduction 39
2. Proof of the Main Theorem 42
3. Applications 45

Chapter IV. Stringy Hodge numbers of hypersurfaces in toric varieties 51
1. Introduction 51
2. Symmetric Fano polytopes 58
3. Pyramids over Fano polytopes 67

Bibliography 79

CHAPTER I

Introduction

1. What, why, who?

This is a commercial. I want you to become friends with lattice polytopes. A lattice polytope is the convex hull of finitely many points from a given grid or lattice (usually $\mathbb{Z}^d$). The interaction between the combinatorial, geometric and algebraic information encoded makes them rewarding objects of study: Many algebraic or combinatorial results have a proof that uses lattice polytopes, and methods from the whole mathematical landscape are used to deal with lattice polytope problems.

Triangulations provide an important device in our tool box. In many circumstances it is useful to split a given lattice polytope into smaller pieces of simpler structure. The easiest (but by far not easy) species to handle are the simplices: the convex hull of affinely independent points. Usually there are several ways to triangulate a lattice polytope, i.e., to subdivide it into simplices. So it is natural to ask for 'good' triangulations for the specific problem. An example of a criterion for a 'good' triangulation is unimodularity: a (full dimensional) simplex is unimodular if, by integral affine combinations of its vertices one can reach the whole lattice. The problem about unimodular triangulations is that they do not always (only rarely ?) exist.

One of the many applications of lattice polytopes lies in the field of toric geometry. The discrete geometric object "lattice polytope" has an algebro geometric brother, a toric variety. It is amazing how discrete properties find their algebraic counterparts, how seemingly combinatorial results have algebraic proofs (and vice versa).

There are many other fields where lattice polytopes show up. Whether you come from discrete optimization, algebraic geometry, commutative algebra or geometry of numbers you have already seen them in action. On the other hand, methods from combinatorics, algebra and analysis are assembled in order to tackle lattice polytope problems. But let us get concrete.

For example, in discrete optimization and in geometry of numbers, lattice polytopes arise as the convex hull of the lattice points in a given convex body B. Fundamental problems in the area are the feasibility problem "$B \cap \mathbb{Z}^d = \varnothing$?", and, more general, the counting problem "$\mathrm{card}(B \cap \mathbb{Z}^d) = $?". The answer in dimension 2, Pick's famous formula

$$\mathrm{Vol}(P) = \mathrm{card}(P \cap \mathbb{Z}^2) - \frac{\mathrm{card}(\partial P \cap \mathbb{Z}^2)}{2} - 1 \quad , \quad P \subset \mathbb{R}^2 \text{ a lattice polygon,}$$

follows from the fact that polygons always admit unimodular triangulations. This does no longer hold in dimensions ≥ 3. Even worse, both above problems prove to be NP-hard when the dimension varies. Though, if the dimension is not considered as part of the input, Lenstra [Len83] used the concept of lattice width and an algorithmic version of the Flatness Theorem (cf. Chapter II) to provide a polynomial time algorithm for the feasibility problem. (The best known flatness–bound has an analytical proof [BLPS98].) Barvinok [Bar94] applied 'weighted' unimodular triangulations of cones to count lattice points in fixed dimension in polynomial time.

A unimodular triangulation, if it exists, tells you a lot about the polytope. E.g., the numbers of the simplices of the various dimensions completely determines (and is determined by) the Ehrhart polynomial, which counts lattice points in dilations of the polytope.

$$\mathrm{card}(k \cdot P \cap \mathbb{Z}^d) = \mathbf{Ehr}(P, k) = \sum_{i=0}^{d} a_i(P) k^i \quad \text{for} \quad k \geq 0.$$

Unfortunately, the typical lattice polytope does not admit such nice triangulations. One can try to weaken the triangulation property by several dissection or covering properties [BGT97, FZ99, Seb90]. But even a unimodular cover does not always exist. The situation changes if the polytope of concern is enlarged. In [KKMSD73], Knudsen and Mumford proved that each polytope can be multiplied by a factor to obtain a polytope with a unimodular triangulation. Bruns, Gubeladze and Trung [BGT97] show that a unimodular cover exists for all large enough factors.

In commutative algebra one associates a graded affine semigroup with a lattice polytope. The above covering/triangulation properties translate to algebraic properties of the semigroup ring and its Gröbner bases [BG99, BGH$^+$99, BGT97, Stu96].

Interestingly, the meanest examples with respect to these properties are the (non–unimodular) empty lattice simplices, simplices that do not contain any lattice points other than the vertices. They correspond to so called terminal singularities in toric geometry. White's Theorem which bounds the lattice width of empty lattice tetrahedra by 1 is the key ingredient in the classification of 3–dimensional terminal Abelian quotient singularities, i.e., of empty lattice tetrahedra [MS84].

This thesis takes you to three sites in the lattice polytope landscape with a focus on the algebraic geometry borderline. We visit the family of empty lattice simplices (Chapter II), which are flat by Khinchine's Flatness Theorem. But they are not as flat as one could expect. We present examples in dimension 4, and some constructions how to obtain thick d–simplices from thick $(d-1)$–dimensional ones. These results together disprove Bárány's pancake conjecture: thick empty simplices may have huge volume. The results of this chapter have appeared as [HZ00]. This is joint work with Günter Ziegler.

At our next stop (Chapter III), we look at those lattice polytopes whose toric brothers are local complete intersections (l.c.i.). We will see that they are not typical, in the sense that they do admit unimodular triangulations. So their brothers have nice resolutions. This supports the conjecture that any l.c.i. (not necessarily toric) admits such resolutions. This joint work with Dimitrios Dais and Günter Ziegler will appear as [DHZ00] (cf. also [DHZ98a]).

The last two locations (Chapter IV) present incidences of the above counting problem. This time there is no conjecture behind — well, there is one, far behind. Related to the mirror symmetry conjecture from theoretical physics, Batyrev and Dais [BD96] constructed certain invariants of varieties, the string theoretic Hodge numbers. We open a small zoo of examples for which these invariants can actually be computed. Therefore we have to evaluate Ehrhart polynomials and some other data. This is joint work with Dimitrios Dais [DHa].

This will finish our little journey. It will hopefully convince you of the beauty of the whole area and motivate you to explore the many unknown spots. But before we can really start, we have to work ourselves through a jungle of definitions and notation (the next three sections of this chapter.) I have tried to make it short and painless. Have a nice trip. . .

2. Notions from discrete and convex geometry

2.1. General notation. The *convex hull* and the *affine hull* of a set $S \subset \mathbb{R}^d$ are denoted by $\mathrm{conv}(S)$ and $\mathrm{aff}(S)$, respectively. The *dimension* $\dim(S)$ of S is the dimension of $\mathrm{aff}(S)$; the relative interior $\mathrm{relint}(S)$ is the interior with respect to $\mathrm{aff}(S)$. A *polyhedron* Q is a finite intersection of closed halfspaces in $\mathbb{R}^d$. The subset $F \subseteq Q$ that minimizes some linear functional on Q is a *face* of Q; we write $F \preceq Q$, and $F \prec Q$ if we want to exclude equality (F is a *proper face*). Zero–dimensional faces are called *vertices*, 1–dimensional bounded faces are *edges*, unbounded ones *rays* and $(\dim Q - 1)$–dimensional faces are *facets*. The faces of a polyhedron form a partially ordered set (poset) with respect to inclusion. Denote by $f_k(Q)$ the number of k–dimensional faces of Q.

In this thesis we will meet two types of polyhedra: a bounded polyhedron is a *polytope*, and a polyhedron which forms an additive semigroup with 0 is a *(polyhedral) cone*. For $S \subset \mathbb{R}^d$, let $\mathrm{pos}(S)$ denote the set of all real, non–negative linear combinations of elements of S. A set $\sigma \subset \mathbb{R}^d$ is a cone if and only if it equals $\mathrm{pos}(S)$ for some finite S. If σ has the vertex $\mathbf{0}$ or, equivalently, $\sigma \cap (-\sigma) = \{\mathbf{0}\}$, we will say that σ is *pointed*. We can pass from polytopes to pointed cones. If $P \subset \mathbb{R}^{d-1}$ is a polytope, then $\sigma(P) := \mathrm{pos}(P \times \{1\}) \subset \mathbb{R}^d$ is a pointed cone which will be referred to as the cone *spanned* by P. If we introduce the empty set $\varnothing$ as a (-1)–dimensional face of P, then there is an isomorphism between the face posets of P and $\sigma(P)$ that shifts the dimension by 1.

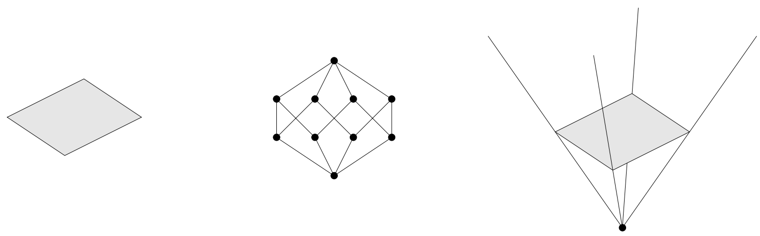


FIGURE 1. P, $\sigma(P)$ and their face poset.

For an arbitrary cone σ, the set $\sigma^\vee = \{\mathbf{x} \in (\mathbb{R}^d)^\vee : \langle \mathbf{x}, \sigma \rangle \geq 0\}$ is a cone, the *dual cone* of σ. There is an inclusion reversing bijection between the face posets of σ and of $\sigma^\vee$. By abuse of notation, for $\tau \preceq \sigma$, we write $\tau^\vee$ for the corresponding face of $\sigma^\vee$ under this bijection. If $\mathbf{0}$ is an interior point of P, then the cone $\sigma(P)^\vee$ is spanned by the polytope $P^\vee = \{\mathbf{x} \in (\mathbb{R}^{d-1})^\vee : \langle \mathbf{x}, P \rangle \geq -1\}$, the *dual polytope* of P. Abuse strikes again: $F^\vee$ will denote the *dual face* of F (cf. Figure 2).

In the following we refer to $\mathbb{Z}^d$ as *the lattice*; $(\mathbb{Z}^d)^\vee$ is the *dual lattice* of integral linear forms. A cone σ is *rational* if it is generated by lattice points; a polytope is a *lattice polytope* if all its vertices are lattice points. We will identify two polytopes (cones) Q, Q', if they are *affinely (linearly) lattice equivalent*, i.e., if they can

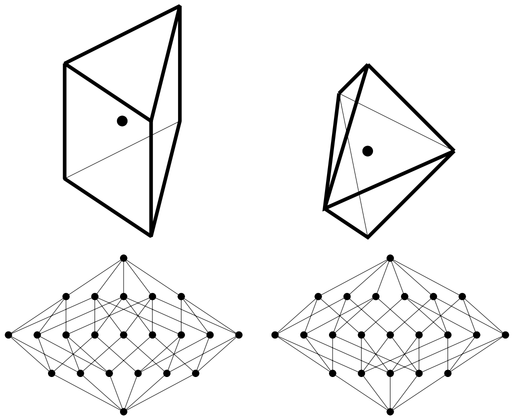


FIGURE 2. P, $P^\vee$ and their face posets.

be related by an affine (linear) map $\operatorname{aff}(Q) \to \operatorname{aff}(Q')$ that maps $\mathbb{Z}^d \cap \operatorname{aff}(Q)$ bijectively onto $\mathbb{Z}^{d'} \cap \operatorname{aff}(Q')$ and which maps Q to Q'. Such a map is a *lattice equivalence*.

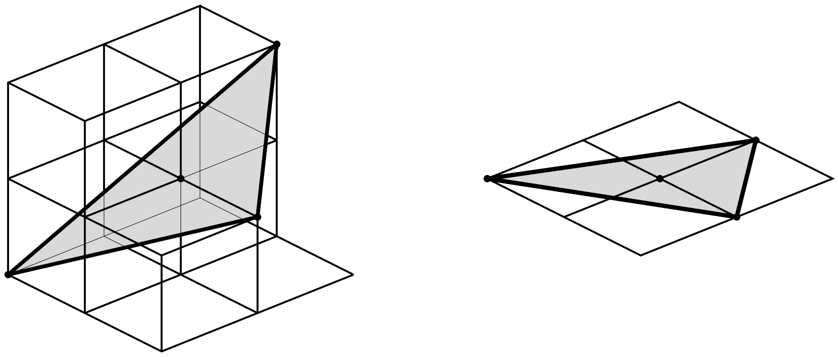


FIGURE 3. Lattice equivalent polytopes.

A cone is *simplicial* if it is generated by an $\mathbb{R}$–linearly independent set. A simplicial cone is *unimodular* (or *basic*) if it is lattice equivalent to the cone generated by the standard basis in $\mathbb{R}^{\dim \sigma}$. A polytope $\mathfrak{s}$ is a *simplex* if $\sigma(\mathfrak{s})$ is simplicial, and *unimodular* if $\sigma(\mathfrak{s})$ is, or equivalently, if $\mathfrak{s}$ is lattice equivalent to the *standard simplex* $\triangle^d$, the convex hull of $\mathbf{0}$ together with the standard unit vectors $\mathbf{e}_i$ $(1 \le i \le d)$ in $\mathbb{R}^d$. (Sometimes, $\triangle^d$ will mean the equivalent polytope $\operatorname{conv}\{\mathbf{e}_i : 1 \le i \le d+1\} \subset \mathbb{R}^{d+1}$, but this will be clear from the context.)

The *determinant* (also *normalized volume* or *index*) of a d–dimensional lattice simplex $\mathfrak{s} = \operatorname{conv}\{\mathbf{a}_0, \dots, \mathbf{a}_d\} \subset \mathbb{R}^d$ is $\det(\mathfrak{s}) := |\det(\mathbf{a}_1 - \mathbf{a}_0, \dots, \mathbf{a}_d - \mathbf{a}_0)|$. The relation to the Euclidian volume reads $\operatorname{Vol}(\mathfrak{s}) = \frac{1}{d!}\det(\mathfrak{s})$. The *normalized volume* of an arbitrary full–dimensional lattice polytope P is the integer(!) $(\dim P)! \cdot \operatorname{Vol}(P)$. In general, the determinant of a (possibly not full dimensional) simplex $\mathfrak{s} = \operatorname{conv}\{\mathbf{0}, \mathbf{a}_1, \ldots, \mathbf{a}_k\} \subset \mathbb{R}^d$ is the gcd of all $(k \times k)$–minors of the matrix A with columns $\mathbf{a}_i$: This gcd is not changed if one applies an integral linear transformation which differs from the identity only by one off–diagonal entry. These matrices, together with the permutations, generate the group of linear lattice equivalences. These can be used to transform A into the form $\binom{A'}{\mathbf{0}}$, where A' is a $(k \times k)$–matrix with $\det(\mathfrak{s}) = \det(A')$.

A lattice polytope is *empty* (also called *elementary* or *lattice point free*) if its vertices are the only lattice points it contains. Every unimodular simplex is empty, but the converse is not true in dimensions $d \geq 3$ (cf. the proof of Proposition II.2.1). A lattice polytope, which contains the unique interior lattice point $\mathbf{0}$ and whose dual has again integral vertices, is called *reflexive*.

2.2. Complexes and subdivisions. A *polyhedral complex* Σ is a finite collection of polyhedra such that the faces of any member belong to Σ and such that the intersection of any two members is a face of each of them. The *support* $\operatorname{supp}(\Sigma)$ of such a Σ is the union of its members. A single polyhedron together with all of its faces forms a polyhedral complex that we denote by the same symbol as the single object. For an element $\sigma \in \Sigma$ its *star* consists of all $\tau \in \Sigma$ with $\sigma \prec \tau$ and all their faces.

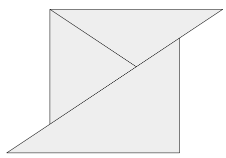


FIGURE 4. Not a complex.

A polyhedral complex that consists of rational pointed cones is a *fan* whereas a polyhedral complex all of whose members are lattice polytopes will be called a *complex of lattice polytopes*. A fan is *simplicial* (resp. *unimodular*) if all of its members are. We can construct fans from polytopes: Let F be a face of the polytope P. The normal cone of F is the set of all functionals that take their minimum over P on (all of) F. These cones fit together to form the *normal fan* $\mathcal{N}(P)$ of P. If P contains $\mathbf{0}$ in its interior, then the elements of $\mathcal{N}(P)$ are exactly the cones generated by the faces of $P^\vee$. A polytope is *simplicial* if all its proper faces are simplices and *simple* if its normalfan is simplicial. P is simplicial if and only if $P^\vee$ is simple.

A polyhedral complex $\mathcal{S}$ is a *subdivision* of Σ if every polyhedron in $\mathcal{S}$ is contained in a polyhedron of Σ and $\operatorname{supp}\mathcal{S} = \operatorname{supp}\Sigma$. A complex of lattice polytopes $\mathcal{T}$ that subdivides Σ into simplices is a (*lattice*) *triangulation*. A lattice triangulation is *unimodular* if its members are.

Let $\mathcal{S}$ be a subdivision of Σ. An *integral $\mathcal{S}$–linear (convex) support function* is a continuous function $\omega : \operatorname{supp}\Sigma \longrightarrow \mathbb{R}$, with $\omega(\operatorname{supp}\Sigma \cap \mathbb{Z}^d) \subseteq \mathbb{Z}$, which is

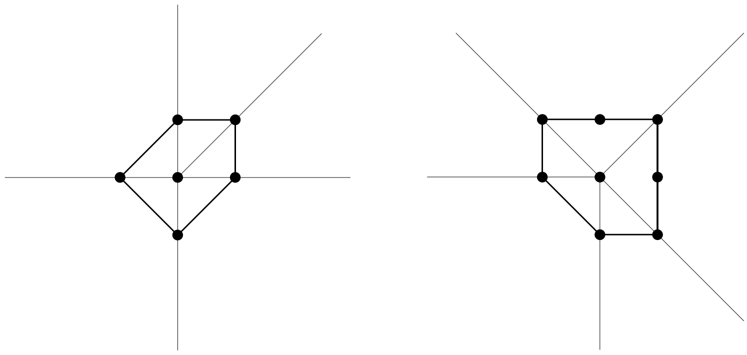


FIGURE 5. P, $\mathcal{N}(P^\vee)$ and $P^\vee$, $\mathcal{N}(P)$.

affine on each $\tau \in \mathcal{S}$ (and convex on every polyhedron $\sigma \in \Sigma$). If the domains of linearity of such a convex ω are exactly the maximal polyhedra of $\mathcal{S}$, then ω is said to be *strictly convex*. For a polyhedral complex Σ, equipped with an integral $\mathcal{S}$–linear strictly convex support function ω, we write $\mathcal{S} = \mathcal{S}_\omega$, and call $\mathcal{S}$ *coherent* (as a subdivision of Σ). This property is transitive, i.e., a coherent subdivision of a coherent subdivision is coherent. If $\mathrm{supp}(\Sigma)$ is a polyhedron itself, then Σ is coherent if it is coherent as a refinement of the trivial subdivision of $\mathrm{supp}(\Sigma)$.

One way to coherently refine a coherent subdivision $\mathcal{S} = \mathcal{S}_\omega$ of a polytope P is by *pulling* a (lattice) point $\mathbf{m} \in P$ (cf. [Lee97]). The refinement $\mathrm{pull}_{\mathbf{m}}(\mathcal{S})$ is defined as follows

- $\mathrm{pull}_{\mathbf{m}}(\mathcal{S})$ contains all $Q \in \mathcal{S}$ for which $\mathbf{m} \notin Q$, and
- if $\mathbf{m} \in Q \in \mathcal{S}$, then $\mathrm{pull}_{\mathbf{m}}(\mathcal{S})$ contains all the polytopes having the form $\mathrm{conv}(F \cup \mathbf{m})$, with F a facet of Q such that $\mathbf{m} \notin F$.

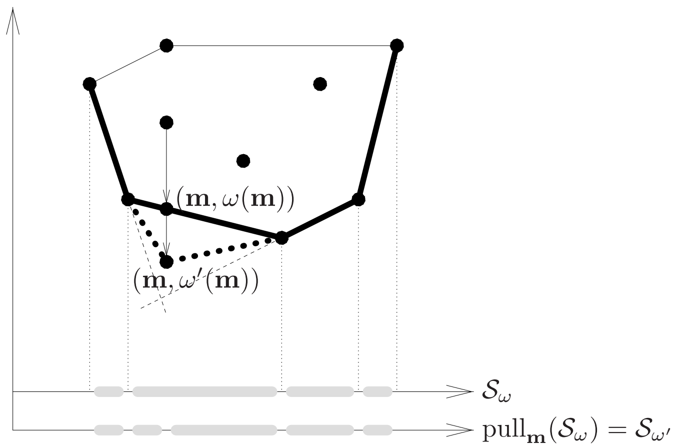


FIGURE 6. $\mathrm{pull}_{\mathbf{m}}(\mathcal{S})$ is coherent.

By pulling successively all the lattice points within a given lattice polytope P one obtains a triangulation into empty simplices. If $\mathcal{S} = \mathcal{S}_\omega$ is coherent, then $\mathrm{pull}_{\mathbf{m}}(\mathcal{S})$ is obtained by "pulling $\mathbf{m}$ from below". This means that, defining $\omega'(\mathbf{m}) = \omega(\mathbf{m}) - \varepsilon$ for ε small enough, and $\omega' = \omega$ on the remaining lattice points, and extending ω' by the maximal convex function ω'' whose values at the lattice points are not greater than the given ones, we get $\mathrm{pull}_{\mathbf{m}}(\mathcal{S}) = \mathcal{S}_{\omega''}$. Hence, if $\mathcal{S}$ is coherent, then so is $\mathrm{pull}_{\mathbf{m}}(\mathcal{S})$.

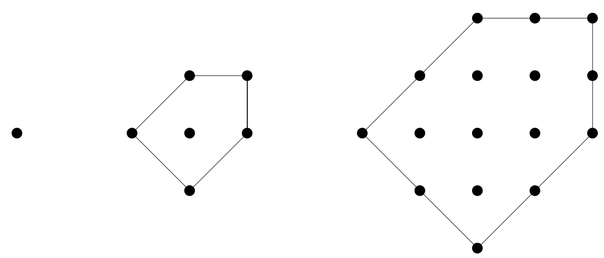
2.3. Counting lattice points. Let $P \subset \mathbb{R}^d$ be a lattice d–polytope. Then for $k \in \mathbb{Z}_{\geq 0}$, the number of lattice points in $k \cdot P$ is given by $\mathbf{Ehr}(P, k) \in \mathbb{Q}[k]$, the so–called *Ehrhart polynomial*:

$$\mathrm{card}(k \cdot P \cap \mathbb{Z}^d) = \mathbf{Ehr}(P, k) = \sum_{i=0}^{d} a_i(P) k^i \quad \text{for} \quad k \geq 0.$$

We have $a_d(P) = \mathrm{Vol}(P)$ and $a_0 = 1$. The corresponding *Ehrhart series*

$$\sum_{k \geq 0} \mathbf{Ehr}(P, k) t^k = \frac{\sum_{j=0}^{\dim P} \psi_j(P) t^j}{(1-t)^{\dim P + 1}},$$

gives rise to the Ψ*–polynomial* $\Psi(P, t) = \sum_{j=0}^{\dim P} \psi_j(P) t^j$ of P.



$0 \cdot P$ $1 \cdot P$ $2 \cdot P$

FIGURE 7. $\mathbf{Ehr}(P, k) = \frac{5}{2}k^2 + \frac{5}{2}k + 1$, $\Psi(P, t) = t^2 + 3t + 1$.

The coefficients are non–negative integers [Sta97]. They can be written as $\mathbb{Z}$–linear combinations of the first values of $\mathbf{Ehr}(P, k)$:

$$\psi_j(P) = \sum_{i=0}^{j} (-1)^i \binom{\dim P + 1}{i} \mathbf{Ehr}(P, j - i). \tag{2.1}$$

Another polynomial, $\mathbb{E}\mathrm{hr}$, counts lattice points in the interior of $k \cdot P$.

$$\mathrm{card}(k \cdot \mathrm{relint}(P) \cap \mathbb{Z}^d) = \mathbb{E}\mathrm{hr}(P, k) \quad \text{for} \quad k \geq 0,$$

with interior Ehrhart series

$$\sum_{k\geq 0} \mathbb{E}\mathrm{hr}(P,k)t^k = \frac{\sum_{j=1}^{\dim P+1} \varphi_j(P)t^j}{(1-t)^{\dim P+1}},$$

giving rise to the Φ–*polynomial*. The following well known reciprocity law holds [Ehr77, Sta97].

$$\mathbb{E}\mathrm{hr}(P,k) = (-1)^{\dim P}\mathbf{Ehr}(P,-k). \tag{2.2}$$

This implies:

2.1. PROPOSITION. *For any lattice polytope P, $\Phi(P,t) = t^{\dim P+1}\Psi(P,t^{-1})$ or, in other "words", $\phi_j(P) = \psi_{\dim P-j+1}(P)$.*

PROOF. The statement is true for the standard simplex. This follows from $\mathbf{Ehr}(\triangle^d,k) = \binom{k+d}{d}$, $\mathbb{E}\mathrm{hr}(\triangle^d,k) = \binom{k-1}{d}$ and $\Psi(\triangle^d,t) = 1$, $\Phi(\triangle^d,t) = t^{d+1}$. The Ehrhart polynomials of these simplices form a basis of $\mathbb{Q}[k]$: any (Ehrhart) polynomial can be written (uniquely) in the form $\sum_{i=0}^{\dim P}\lambda_i\binom{k+i}{i}$. Then, by (2.2), the interior Ehrhart polynomial is $\sum_{i=0}^{\dim P}(-1)^{d+i}\lambda_i\binom{k-1}{i}$. Hence, $\Psi = \sum_i \lambda_i(1-t)^{d-i}$ and $\Phi = \sum_i(-1)^{d+i}\lambda_i(1-t)^{d-i}t^{i+1}$. $\square$

If P is reflexive, then there are no lattice points between $k\cdot P$ and $(k+1)\cdot P$.

2.2. LEMMA [Hib92]. *The lattice polytope P (with $\mathbf{0}$ in its interior) is reflexive if and only if for every $k \in \mathbb{Z}_{\geq 0}$*

$$\mathbb{E}\mathrm{hr}(P,k+1) = \mathbf{Ehr}(P,k) \text{ for } k \geq 0, \tag{2.3}$$

or, equivalently,

$$\psi_j(P) = \psi_{d-j}(P).$$

PROOF. Equation (2.3) is equivalent to the fact that there is no lattice point $\mathbf{m}$ in the difference set $\mathrm{relint}((k+1)\cdot P) \setminus (k\cdot P)$. As

$$k\cdot P = \{\mathbf{y}\in\mathbb{R}^d : \langle \mathbf{v}_i,\mathbf{y}\rangle \geq -k \quad (\mathbf{v}_i \text{ vertex of } P^\vee)\},$$

the existence of such an $\mathbf{m}$ would imply that $k < \langle \mathbf{v}_i,\mathbf{m}\rangle < k+1$ for some i, contradicting the integrality of $\mathbf{v}_i$.

Conversely, assume that some $\mathbf{v}_i \notin \mathbb{Z}^d$. In this case, choose some lattice vector $\mathbf{n}_1$ from the interior of the cone $\mathrm{pos}(F_i)$, which is generated by the facet $F_i = \{\mathbf{y}\in P : \langle\mathbf{v}_i,\mathbf{y}\rangle = -1\}$. If already $\langle \mathbf{v}_i,\mathbf{n}_1\rangle \notin \mathbb{Z}$, we are done. Otherwise take any $\mathbf{n}_2 \in \mathbb{Z}^d$ with $\langle\mathbf{v}_i,\mathbf{n}_2\rangle \notin \mathbb{Z}$ (e.g., a suitable coordinate vector). Then for large $N \in \mathbb{Z}_{\geq 0}$ the lattice point $\mathbf{m} = N\mathbf{n}_1 + \mathbf{n}_2$ lies in $\mathrm{pos}(F_i)$ and satisfies $\langle\mathbf{v}_i,\mathbf{m}\rangle \notin \mathbb{Z}$. $\square$

3. Notions from algebraic and toric geometry

3.1. Properties of (local) rings. Let R denote a local Noetherian ring with maximal ideal $\mathfrak{m}$. R is *regular* if the following equality of Krull dimensions holds; $\dim(R) = \dim(\mathfrak{m}/\mathfrak{m}^2)$. R is *normal* if it is integrally closed. R is said to be a *complete intersection* if there exists a regular local ring R', such that $R \cong R'/\langle f_1, \dots, f_q\rangle$ for a finite set $\{f_1, \dots, f_q\} \subset R'$ whose cardinality equals $q = \dim(R') - \dim(R)$. R is called *Cohen–Macaulay* if $\operatorname{depth}(R) = \dim(R)$, where its depth is the maximum of the lengths of all regular sequences whose members belong to $\mathfrak{m}$. Such an R is *Gorenstein* if $\operatorname{Ext}_R^{\dim(R)}(R/\mathfrak{m}, R) \cong R/\mathfrak{m}$. The hierarchy of the above types of R's reads:

$$\text{regular} \Longrightarrow \text{complete intersection} \Longrightarrow \text{Gorenstein} \Longrightarrow \text{Cohen–Macaulay}.$$

An arbitrary Noetherian ring R and its associated affine scheme $\operatorname{Spec}(R)$ are called regular, Cohen–Macaulay, Gorenstein, or normal respectively, if all the localizations $R_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of R are of this type. In particular, if all $R_{\mathfrak{m}}$'s are complete intersections, then one says that R is a *local complete intersection* (l.c.i.).

3.2. Complex varieties and desingularizations. Throughout the thesis we consider only complex varieties $(X, \mathcal{O}_X)$, i.e., integral separated schemes of finite type over $\mathbb{C}$, and work within the analytic category (cf. the GAGA correspondence [Ser56]). The algebraic properties of 3.1 can be defined for the whole X via its affine coverings, and pointwise via the stalks $\mathcal{O}_{X,x}$ of the structure sheaf at $x \in X$. By $\operatorname{Sing}(X) = \{x \in X : \mathcal{O}_{X,x} \text{ non–regular}\}$ we denote the *singular* and by $\operatorname{Reg}(X) = X \smallsetminus \operatorname{Sing}(X)$ the *regular locus* of X. A *partial desingularization* $f : \widehat{X} \longrightarrow X$ of X is a proper holomorphic morphism of complex varieties with $\widehat{X}$ normal, such that there is a nowhere dense analytic set $S \subset X$, with $S \cap \operatorname{Sing}(X) \neq \varnothing$, whose inverse image $f^{-1}(S) \subset \widehat{X}$ is nowhere dense and such that the restriction of f to $\widehat{X} \smallsetminus f^{-1}(S)$ is biholomorphic. The map $f : \widehat{X} \longrightarrow X$ is called a *full desingularization* of X (or *full resolution of singularities* of X) if $\operatorname{Sing}(X) \subseteq S$ and $\operatorname{Sing}(\widehat{X}) = \varnothing$.

3.3. Divisors. A Weil divisor K_X of a normal complex variety X is *canonical* if its sheaf $\mathcal{O}_X(K_X)$ of fractional ideals is isomorphic to the sheaf of the (regular in codimension 1) Zariski differentials or, equivalently, if $\mathcal{O}_{\operatorname{Reg}(X)}(K_X)$ is isomorphic to the sheaf $\Omega^{\dim X}_{\operatorname{Reg}(X)}$ of the highest regular differential forms on $\operatorname{Reg}(X)$. As it is known, X is Gorenstein if and only if K_X is Cartier, i.e., if and only if $\mathcal{O}_X(K_X)$ is invertible. A birational morphism $f : X' \longrightarrow X$ between normal Gorenstein complex varieties is called *non–discrepant* or simply *crepant*, if the (up to rational equivalence uniquely determined) difference $K_{X'} - f^*(K_X)$ vanishes. Furthermore, $f : X' \to X$ is *projective* if X' admits an f–ample Cartier divisor.

Let X be a $\mathbb{Q}$–Gorenstein variety, i.e., rK_X is Cartier for some $r \in \mathbb{Z}_{\geq 1}$. Then X has *canonical singularities* if for every resolution $\phi : Y \to X$ with exceptional prime divisors $\{E_i\}$ one has $rK_Y - \phi^*(r\ K_X) = \sum \lambda_i E_i$ with non–negative coefficients λ_i. If always all λ_i are stricly positive, then X has *terminal* singularities (cf. [Rei87]).

3.4. Toric varieties. Let $\sigma \subset \mathbb{R}^d$ be a pointed rational cone and $\sigma^\vee$ its dual. Then the semigroup $\sigma^\vee \cap (\mathbb{Z}^d)^\vee$ is finitely generated — a generating set is called *Hilbert basis*. The semigroup ring $\mathbb{C}[\sigma^\vee \cap (\mathbb{Z}^d)^\vee]$ defines an affine complex variety $U_\sigma := \operatorname{Spec}(\mathbb{C}[\sigma^\vee \cap (\mathbb{Z}^d)^\vee])$. A general *toric variety* X_Σ associated with a fan Σ is the identification space $X_\Sigma := (\bigsqcup_{\sigma\in\Sigma} U_\sigma)/\sim$ over the equivalence relation defined by "$U_{\sigma_1} \ni u_1 \sim u_2 \in U_{\sigma_2}$ if and only if there is a face τ of both σ_1, σ_2, and $u_1 = u_2$ within U_τ." X_Σ is always normal and Cohen–Macaulay, and has at most rational singularities. Moreover, X_Σ admits a canonical group action which extends the multiplication of the algebraic torus $U_{\{\mathbf{0}\}} \cong (\mathbb{C}^*)^d$. The notion of *equivariance* will always be used with respect to this action. It partitions X_Σ into orbits that are in one–to–one correspondence with the cones of Σ. The orbit that corresponds to $\sigma \in \Sigma$ is a $(d - \dim\sigma)$–dimensional algebraic torus T_σ. Its closure D_σ is itself a toric variety. These are exactly the torus invariant subvarieties of X_Σ. Let σ be full–dimensional. Then the smallest orbit closure D_σ in U_σ is a singular point unless σ is unimodular (cf. [Oda88, Thm. 1.10]).

4. From the dictionary

In this section, we review some entries of the dictionary, that translates between convex and toric geometry. We refer to to the text books [Ewa96, Ful93, KKMSD73, Oda88] for further reading.

4.1. Toric morphisms. Let Σ, Σ' be fans in $\mathbb{R}^d$ respectively $\mathbb{R}^{d'}$. A *map of fans* between Σ and Σ' is a linear map $\phi : \mathbb{R}^d \to \mathbb{R}^{d'}$ with the property that for every $\sigma \in \Sigma$ there is some $\sigma' \in \Sigma'$ such that $\phi(\sigma \cap \mathbb{Z}^d) \subseteq \sigma' \cap \mathbb{Z}^{d'}$. Such a ϕ induces an equivariant holomorphic map $X_\Sigma \to X_{\Sigma'}$ (also denoted by ϕ), which is proper if and only if $\phi^{-1}(\operatorname{supp} \Sigma') = \operatorname{supp} \Sigma$. In particular, X_Σ is compact if and only if $\operatorname{supp} \Sigma = \mathbb{R}^d$. An example of a proper equivariant morphism is the map $\phi_{\mathcal{S}}$ induced by a subdivision $\mathcal{S}$ of a fan Σ.

4.2. Divisors and projective toric varieties. Invariant prime divisors of X_Σ are $(d-1)$–dimensional invariant subvarieties. These are exactly the orbit closures D_ϱ corresponding to the rays ϱ of Σ. So the torus invariant Weil divisors are formal $\mathbb{Z}$–linear combinations of the D_ϱ. Such a divisor $\sum \lambda_\varrho D_\varrho$ is Cartier if and only if for every maximal cone $\sigma \in \Sigma$ there is an element $\ell_\sigma \in (\mathbb{Z}^d)^\vee$ such that $\lambda_\varrho = \langle \ell_\sigma, \mathbf{p}(\varrho)\rangle$, for every one–dimensional subcone ϱ of σ, where $\mathbf{p}(\varrho)$ is the primitive lattice vector that generates ϱ.

If $\operatorname{supp}(\Sigma)$ is convex, we can consider Σ as a subdivision of $\operatorname{supp}(\Sigma)$. Then the restrictions $\ell_\sigma|_\sigma$ fit together to a (not necessarily convex) integral Σ–linear support function ω_ℓ. The Cartier divisor on X_Σ that is defined by ω_ℓ is ample if and only if ω_ℓ is strictly convex on $\operatorname{supp}(\Sigma)$ in the sense of 2.2. Hence X_Σ is quasi projective if and only if Σ admits a strictly convex support function ω, and the morphism $\phi_{\mathcal{S}}$ induced by a subdivision $\mathcal{S}$ is projective if and only if $\mathcal{S}$ is coherent.

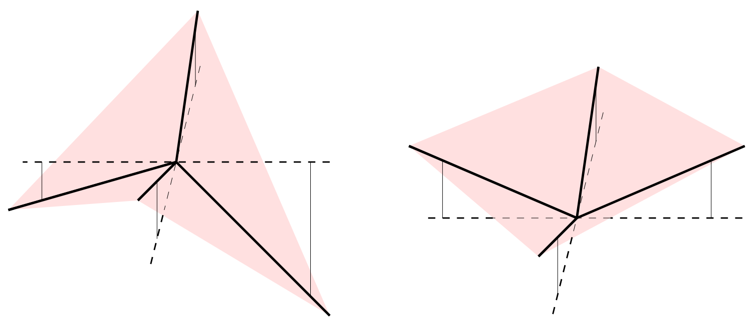


FIGURE 8. A ‘divisor’ and an ‘ample divisor’.

Let P be a lattice polytope with $\mathbf{0}$ in its interior. Then $X_{\mathcal{N}(P)}$ is projective, as witnessed by the support function $\omega(\mathbf{x}) = \min\{t \geq 0 : \mathbf{x} \in t \cdot P^\vee\}$. The polytope P even yields a projective embedding in the following way. The semigroup algebra $R = \mathbb{C}[\sigma(P) \cap \mathbb{Z}^{d+1}]$ has a natural grading by the last coordinate. The homogeneous components are $R_k = \mathbb{C}[\sigma(P) \cap (\mathbb{Z}^d \times \{k\})]$. So R is the quotient of a polynomial ring by a homogeneously generated ideal, and we have $X_{\mathcal{N}(P)} \cong \mathrm{Proj}(R)$. The dimension of R_k is just $\mathbf{Ehr}(P, k)$, and thus the degree of this embedding is $(\dim P)!$ times the leading coefficient of $\mathbf{Ehr}(P, k)$, which is the normalized volume of P.

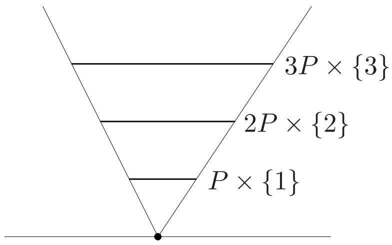


FIGURE 9. The grading of $\mathbb{C}[\sigma(P) \cap \mathbb{Z}^{d+1}]$.

CHAPTER II

Lattice width of empty simplices

1. Introduction

Geometric intuition may suggest that an empty lattice simplex must be "flat" in at least one direction, and if it is not "very flat," then its volume must be bounded. The concepts of "flat" and "very flat" are made precise using the notion of lattice width reviewed below. A "flat" simplex will be one whose lattice width is bounded by a certain constant $w(d)$ that depends only on the dimension, and it will be called "not very flat" if its lattice width is greater than another constant $\overline{w}(d-1)$. Using these concepts, we discuss in this chapter the (partial) validity of the geometric intuition.

Let $K \subseteq \mathbb{R}^d$ be any full–dimensional lattice polytope (or even a general full–dimensional convex body). For a linear form $\ell \in (\mathbb{R}^d)^\vee$ define the *width of K with respect to* ℓ as

$$\mathrm{width}_\ell(K) := \max \ell(K) - \min \ell(K).$$

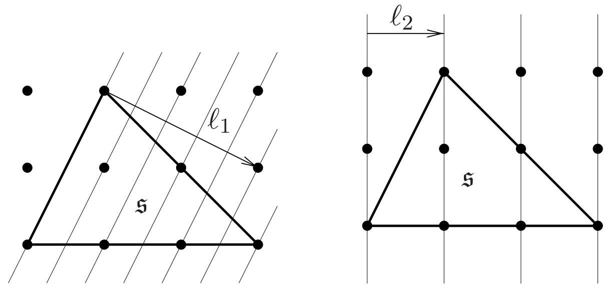


FIGURE 1. $\mathrm{width}_{\ell_1}(K) = 6$ $\qquad$ $\mathrm{width}_{\ell_2}(K) = 3$.

Given K, the assignment $\ell \mapsto \mathrm{width}_\ell(K)$ defines a norm on $(\mathbb{R}^d)^\vee$. The *(lattice) width* of K is

$$\mathrm{width}(K) := \min\{\mathrm{width}_\ell(K) \,:\, \ell \in (\mathbb{Z}^d)^\vee \setminus \{\mathbf{0}\}\}.$$

If K is not full–dimensional, we have to exclude not only $\ell = \mathbf{0}$, but all ℓ that are constant on the affine hull $\mathrm{aff}(K)$.

$$\mathrm{width}(K) = \min\{\mathrm{width}_\ell(K) \,:\, \ell \in (\mathbb{Z}^d)^\vee, \ell \ \text{ not constant on } \ \mathrm{aff}(K)\}.$$

This way the width becomes an invariant of lattice equivalence classes.

Another notion of width — more geometric and less number theoretic — is obtained if one minimizes over $\ell \in S^{d-1} \subset (\mathbb{R}^d)^\vee$ instead. This geometric width is a lower bound for our lattice width. It is not invariant under lattice equivalence and it will not be addressed any further.

By Khinchine's Flatness Theorem [Khi48], the lattice width of a d–dimensional empty lattice simplex is bounded by a constant which only depends on d. For each d the best bound is encoded in the *maximal width function*:

$$\begin{array}{rcl} w : \mathbb{Z}_{\geq 0} & \longrightarrow & \mathbb{Z}_{\geq 0} \\ d & \longmapsto & \max\{\mathrm{width}(\mathfrak{s}) : \mathfrak{s} \text{ is a } d\text{–dimensional empty lattice simplex}\}. \end{array}$$

Here are the main facts that are known about this function:

- $w(2) = 1$ (trivial),
- $w(3) = 1$ (This is White's Theorem [Whi64, MS84, Sca85, Seb98].),
- $w(4) \geq 4$ (We will see that the simplex spanned by $(6, 14, 17, 65)^t$ together with the four unit vectors in $\mathbb{R}^4$ is the smallest example of width 4; it seems to be the only one, up to lattice equivalence. In particular, we believe that $w(4) = 4$.),
- $w(d) \leq M d \log d$ for some M (Banaszczyk, Litvak, Pajor & Szarek [BLPS98]),
- $w(d) \geq d - 2$ for all $d \geq 1$, and $w(d) \geq d - 1$ for even d. (This was proved by Sebő and Bárány [Seb98], who gave explicit examples, cf. Proposition 1.1 below).

1.1. Proposition [Seb98]. *The d–dimensional simplex*

$$S_d(k) := \mathrm{conv}\{\mathbf{a}_1, \dots, \mathbf{a}_{d+1}\} \subset \mathbb{R}^{d+1},$$

where $\mathbf{a}_i = k\mathbf{e}_i + \mathbf{e}_{i+1}$ (indices are considered modulo $d+1$) is empty if $k < d$, and its width is k respectively $k-1$ according to whether d is even or odd.

Proof. (Sebő and Bárány [Seb98])
Emptiness: Suppose $\mathbf{m} \in S_d(k) \cap \mathbb{Z}^d$ and let $\lambda_1, \dots, \lambda_{d+1}$ be its barycentric coordinates, i.e., $m_{i+1} = k\lambda_{i+1} + \lambda_i$. If some $\lambda_i > 0$, $\lambda_{i+1} = 0$, then $\lambda_i = m_{i+1} \in \mathbb{Z}$ must be 1, so that $\mathbf{m} = \mathbf{a}_i$ is a vertex. Otherwise all $\lambda_i > 0$ and thus all $m_i > 0$. Hence $\sum m_i \geq d + 1$, but as element of $S_d(k)$ we should have $\sum m_i = k + 1$.
Width: The simplex $S_d(k)$ is not full–dimensional. Its affine hull is given by $V = \mathrm{aff}(S_d(k)) = \{\mathbf{x} \in \mathbb{R}^{d+1} : \sum x_i = k + 1\}$. A functional is constant on V if and only if all its coordinates are equal. So consider some ℓ which is not constant on V. Let i be any index such that l_i is maximal and j any index such that l_j is minimal. Then $l_i > l_j$ because of our assumption.

$$\begin{aligned} \ell(\mathbf{a}_i) - \ell(\mathbf{a}_j) &= kl_i + l_{i+1} - kl_j - l_{j+1} \\ &\geq kl_i + l_j - kl_j - l_i = (k-1)(l_i - l_j) \geq k - 1. \end{aligned}$$

Equality can only hold if $l_i - l_j = 1$ and for every maximal l_i, l_{i+1} is minimal and vice versa, i.e., if $\pm\ell = (l, l+1, l, l+1, \ldots, l, l+1)$ for some l. But this can only happen if $d+1$ is even. $\square$

Imre Bárány (personal communication 1997) related the volume of an empty lattice simplex to its width. He conjectured that the only way an empty simplex can have arbitrarily high volume is that it is flat like a pancake. More precisely, he claimed that in every fixed dimension the volume of empty lattice simplices of width ≥ 2 is bounded, or equivalently, there are only finitely many equivalence classes of such simplices. We will see that this is false in dimensions $d \geq 4$. Even the weaker conjecture that there are only finitely many different empty lattice simplices of width larger than $w(d-1)$ is not true for $d \geq 4$. However, we offer a new guess for a finiteness result in Conjecture 2.7.

2. Adding one dimension

The examples in Section 3 of empty 4–simplices of width greater than 1 together with the following proposition show that Bárány's pancake conjecture does not hold in dimension $d \geq 5$.

2.1. PROPOSITION. *Let $d \geq 3$. Every empty $(d-1)$–dimensional lattice simplex $\mathfrak{s} \subset \mathbb{R}^d$ is a facet of infinitely many pairwise non–equivalent empty d–dimensional lattice simplices $\tilde{\mathfrak{s}} \subset \mathbb{R}^d$ that have at least the same width,* $\operatorname{width}(\tilde{\mathfrak{s}}) \geq \operatorname{width}(\mathfrak{s})$.

2.2. COROLLARY. *The maximal width function is monotone: $w(d) \geq w(d-1)$.* □

2.3. COROLLARY. *For all $d \geq 3$, there are infinitely many equivalence classes of d–dimensional empty lattice simplices of width $\geq w(d-1)$.* □

PROOF OF PROPOSITION 2.1. Generalize Reeve's construction [Ree57] of arbitrarily large empty tetrahedra: $R(r) = \operatorname{conv}\{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + r\mathbf{e}_3\} \subset \mathbb{R}^3$.

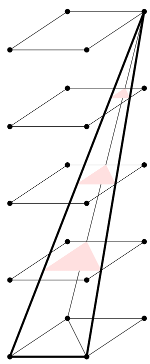


FIGURE 2. Reeve's tetrahedron $R(4)$.

Suppose that $\mathfrak{s} = \operatorname{conv}\{\mathbf{a}_0, \ldots, \mathbf{a}_{d-1}\} \subset \mathbb{R}^{d-1}$ is an empty simplex with $\mathbf{a}_0 = \mathbf{0}$. Every point $\mathbf{x}$ in the cone $\sigma = \operatorname{pos} \mathfrak{s}$ has a unique representation of the form $\mathbf{x} = \sum_{i=1}^{d-1} \lambda_i \mathbf{a}_i$ with $\lambda_i \geq 0$; define the *height* of $\mathbf{x}$ in σ as $\operatorname{ht}(\mathbf{x}) := \sum_{i=1}^{d-1} \lambda_i$.

Let $\mathbf{a}_d \in \operatorname{relint}(\sigma) \cap \mathbb{Z}^d$ be an integer point in the interior of σ with minimal height, that is, such that $\lambda_i > 0$ for all i and such that $\operatorname{ht}(\mathbf{a}_d) > 1$ is minimal. Then $\operatorname{conv}\{\mathbf{a}_0, \ldots, \mathbf{a}_d\} \subseteq \mathbb{R}^{d-1}$, a bipyramid over the facet $\operatorname{conv}\{\mathbf{a}_1, \ldots, \mathbf{a}_{d-1}\}$ of $\mathfrak{s}$, is empty (use $d-1 > 1$).

The d–dimensional simplex $\tilde{\mathfrak{s}}(h) = \operatorname{conv}\{\binom{\mathbf{a}_0}{0}, \ldots, \binom{\mathbf{a}_{d-1}}{0}, \binom{\mathbf{a}_d}{h}\} \subseteq \mathbb{R}^d$ derived from it "by lifting $\mathbf{a}_d$ to a new dimension" will satisfy our conditions if h is large enough. To see this, let $\mathbf{m} \in \tilde{\mathfrak{s}}(h) \cap \mathbb{Z}^d$. The projection of $\mathbf{m}$ to $\mathbb{R}^{d-1}$ is an integral point which lies in the bipyramid, so it must be one of the points $\mathbf{a}_0, \dots, \mathbf{a}_d$. But the only points of $\tilde{\mathfrak{s}}(h)$ with such a projection are the vertices.

Any functional $\ell \in (\mathbb{Z}^d)^\vee$ has the same values on the first d vertices of $\tilde{\mathfrak{s}}(h)$ as its restriction ℓ' to $\mathbb{R}^{d-1}$ takes on the vertices of $\mathfrak{s}$. Hence, we have $\mathrm{width}_\ell(\tilde{\mathfrak{s}}) \geq \mathrm{width}_{\ell'}(\mathfrak{s})$. This shows that $\mathrm{width}_\ell(\tilde{\mathfrak{s}})(h) \geq \mathrm{width}(\mathfrak{s})$, unless ℓ' is zero. In this last case ℓ has to be an integer multiple of the d–th coordinate function, which takes the values 0 and h on the vertices of $\tilde{\mathfrak{s}}(h)$. This establishes that $\mathrm{width}(\tilde{\mathfrak{s}})(h) \geq \min\{h, \mathrm{width}(\mathfrak{s})\}$. □

For a sharper analysis of the situation we introduce the following concept.

2.4. Definition. A lattice simplex without interior lattice points and with at least one empty facet is called *almost empty*. Let $\overline{w}(d)$ be the maximal width function for almost empty simplices.

The following Proposition in particular shows that $\overline{w}(d) \leq w(d+1)$ is finite.

2.5. Proposition. *For any $(d-1)$–dimensional almost empty simplex there are infinitely many pairwise non–equivalent d–dimensional empty simplices of at least the same width.*

As a special case of Proposition 2.5, we get the following infinite family of 4–dimensional empty lattice simplices of width $2 > w(3)$, which disproves the pancake conjecture in dimension 4. For this, write

$$\mathfrak{s}[\mathbf{m}] := \mathrm{conv}\big\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d, \mathbf{m}\big\},$$

to denote the convex hull of the standard unit vectors together with one additional vector $\mathbf{m} \in \mathbb{Z}^d$. We always assume that $\sum_{i=1}^d m_i =: D + 1 > 1$. ($D$ is the determinant of $\mathfrak{s}[\mathbf{m}]$.)

2.6. Proposition. *For every determinant $D \geq 8$, the 4–simplex $\mathfrak{s}[(2,2,3,D-6)^t]$, which is the convex hull of the columns of*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & D-6 \end{pmatrix},$$

has width 2. It is empty if and only if $\gcd(D, 6) = 1$. □

Proof of Proposition 2.5. Suppose that $\mathfrak{s} = \mathrm{conv}\{\mathbf{a}_0, \dots, \mathbf{a}_{d-1}\} \subseteq \mathbb{R}^{d-1}$ with $\mathbf{a}_0 = \mathbf{0}$ is an almost empty simplex with empty facet $\mathrm{conv}\{\mathbf{a}_1, \dots, \mathbf{a}_{d-1}\}$. Choose $\mathbf{a}_d$ as in the previous proof. Then $\mathfrak{s}' := \mathrm{conv}\{\mathbf{a}_1, \dots, \mathbf{a}_{d-1}, \mathbf{a}_d\}$ is an empty $(d-1)$–simplex by construction, and $\mathrm{conv}\{\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_{d-1}, \mathbf{a}_d\} = \mathfrak{s} \cup \mathfrak{s}'$ is a bipyramid with apexes $\mathbf{a}_0 = \mathbf{0}$ and $\mathbf{a}_d$, such that all the integer points in $\mathfrak{s} \cup \mathfrak{s}'$, except for the point $\mathbf{a}_d$, are contained in the facets of $\mathfrak{s}$ that contain the apex $\mathbf{a}_0 = \mathbf{0}$. Now we "lift $\mathbf{a}_0$ to the next dimension" to obtain

$$\tilde{\mathfrak{s}}(h) := \operatorname{conv}\left\{ \begin{pmatrix} \mathbf{a}_0 \\ h \end{pmatrix}, \begin{pmatrix} \mathbf{a}_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} \mathbf{a}_d \\ 0 \end{pmatrix} \right\}.$$

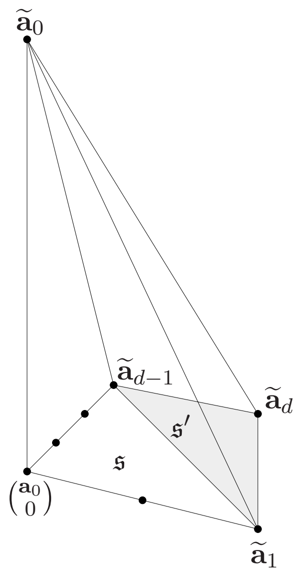


FIGURE 3. Lift $\mathbf{a}_0$ to the next dimension.

The following two claims establish that

(1) the simplex $\tilde{\mathfrak{s}}(h)$ has width $\operatorname{width}(\tilde{\mathfrak{s}}(h)) \geq \operatorname{width}(\mathfrak{s})$ if h is large enough, and

(2) it is empty for infinitely many h.

CLAIM (1) *There is a constant $H = H(\mathfrak{s}, \mathbf{a}_d)$ so that* $\operatorname{width}(\tilde{\mathfrak{s}}(h)) \geq \operatorname{width}(\mathfrak{s})$ *for every $h > H$.*

For $\ell \in (\mathbb{R}^d)^\vee$ let ℓ' denote its restriction to $\mathbb{R}^{d-1}$ and let l_d be the remaining component. Then $\operatorname{width}_\ell(\tilde{\mathfrak{s}}(h)) \geq \operatorname{width}_{\ell'}(\mathfrak{s}')$. Now $\mathfrak{s}'$ is a full–dimensional lattice simplex in $\mathbb{R}^{d-1}$. Hence the set $\{\ell' \in (\mathbb{R}^{d-1})^\vee : \operatorname{width}_{\ell'}(\mathfrak{s}') \leq \operatorname{width}(\mathfrak{s})\}$ is bounded. In particular, there is some M such that $\max|\ell'(\mathfrak{s}')| \leq M$ for every ℓ' from this set. From now on, we will only consider functionals ℓ that satisfy $\operatorname{width}_{\ell'}(\mathfrak{s}') < \operatorname{width}(\mathfrak{s})$ — otherwise the claim is clear anyway.

If $l_d = 0$ (and $\ell \neq \mathbf{0}$), then $\operatorname{width}_\ell(\tilde{\mathfrak{s}}(h)) = \operatorname{width}_{\ell'}(\mathfrak{s} \cup \mathfrak{s}') \geq \operatorname{width}(\mathfrak{s})$. But if $l_d \neq 0$ and $h > H := M + \operatorname{width}(\mathfrak{s})$, then

$$|\ell(\begin{pmatrix} \mathbf{a}_0 \\ h \end{pmatrix} - \begin{pmatrix} \mathbf{a}_i \\ 0 \end{pmatrix})| = |l_d h - \ell'(\mathbf{a}_i)| \geq h - \max|\ell'(\mathfrak{s}')| > \operatorname{width}(\mathfrak{s}).$$

This implies that $\operatorname{width}_\ell(\tilde{\mathfrak{s}}(h)) \geq \operatorname{width}(\mathfrak{s})$.

Claim (2) *Let $D = \det[\mathbf{a}_1, \dots, \mathbf{a}_{d-1}]$. If* $\gcd(D, h) = 1$*, then* $\tilde{\mathfrak{s}}(h)$ *is empty.*

Let once more $\mathrm{ht} \in (\mathbb{R}^{d-1})^\vee$ be the linear form on $\mathbb{R}^{d-1}$ that takes the value 1 on $\mathbf{a}_1, \dots, \mathbf{a}_{d-1}$ (and 0 on $\mathbf{a}_0 = \mathbf{0}$).

Let $\widetilde{\mathbf{m}} \in \tilde{\mathfrak{s}}(h)$ be an integer point. Its projection $\mathbf{m} \in \mathfrak{s} \cup \mathfrak{s}'$ to $\mathbb{R}^{d-1}$ has integral coordinates. If $\mathbf{m} = \mathbf{a}_d$, then $\widetilde{\mathbf{m}} = \binom{\mathbf{a}_d}{0}$ is a vertex of $\tilde{\mathfrak{s}}(h)$. Otherwise decompose

$$\widetilde{\mathbf{m}} = \begin{pmatrix} \mathbf{m} \\ m_d \end{pmatrix} = \frac{m_d}{h} \begin{pmatrix} \mathbf{0} \\ h \end{pmatrix} + \left(1 - \frac{m_d}{h}\right) \begin{pmatrix} \mathbf{x} \\ 0 \end{pmatrix},$$

for some $\mathbf{x} \in \mathfrak{s}'$. This yields $\mathbf{m} = \left(1 - \frac{m_d}{h}\right) \mathbf{x}$. If $\mathbf{m} = \mathbf{0}$, then $\widetilde{\mathbf{m}} = \binom{\mathbf{a}_0}{h}$ is a vertex of $\tilde{\mathfrak{s}}(h)$. Otherwise $m_d < h$, and $\mathbf{m}$ lies in a facet of $\mathfrak{s}$ (and thus of $\mathfrak{s} \cup \mathfrak{s}'$) that contains the vertex $\mathbf{a}_0 = \mathbf{0}$, while some multiple of $\mathbf{m} = (1 - \frac{m_d}{h})\mathbf{x}$, namely $\mathbf{x}$, lies in $\mathfrak{s}'$. Thus the geometry of the bipyramid $\mathfrak{s} \cup \mathfrak{s}'$ implies that $\mathbf{x} \in \mathfrak{s} \cap \mathfrak{s}'$, and thus $\mathrm{ht}(\mathbf{x}) = 1$. Hence

$$\mathrm{ht}(\mathbf{m}) = 1 - \frac{m_d}{h}.$$

On the other hand, by Cramer's rule, there are unique coefficients $\lambda_i \in \mathbb{Z}$, such that $\mathbf{m}$ has the representation $D \cdot \mathbf{m} = \sum_{i=1}^{d-1} \lambda_i \mathbf{a}_i$. This implies

$$D \cdot \mathrm{ht}(\mathbf{m}) \in \mathbb{Z}.$$

Conclude that

$$D \frac{m_d}{h} = D - D \cdot \mathrm{ht}(\mathbf{m}) \in \mathbb{Z}.$$

Hence, if $\gcd(D, h) = 1$, then $m_d = 0$, and thus $\widetilde{\mathbf{m}} \in \mathfrak{s}' \cap \mathbb{Z}^{d-1}$ is one of the vertices of $\tilde{\mathfrak{s}}(h)$. $\square$

Conversely, if $\mathfrak{s}$ does have interior lattice points, then $\tilde{\mathfrak{s}}(h)$ is an empty simplex for only finitely many values of h. To see this, consider the intersections of $\tilde{\mathfrak{s}}(h)$ with the hyperplanes $\{\widetilde{\mathbf{x}} \in \mathbb{R}^d : x_d = k\}$ for integers k. The projection $Z(h)$ of their union to $\mathbb{R}^{d-1}$ is a "forbidden zone" for integer points: if it contains an integer point, then $\tilde{\mathfrak{s}}(h)$ is not empty. Figure 4 illustrates this for dimension $d = 2 + 1$ and for the heights 4 and 8.

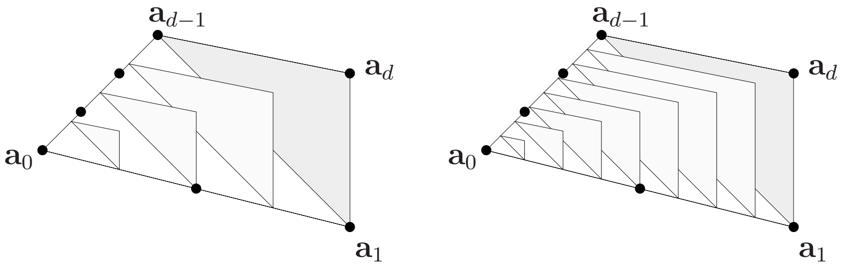


Figure 4. $Z(4)$ and $Z(8)$.

You can see that $Z(h)$ contains an inner parallel body of $\mathfrak{s}$ that grows with h and which completely fills the interior of $\mathfrak{s}$ for $h \to \infty$. So any fixed interior point of $\mathfrak{s}$ lies in $Z(h)$ if h is large enough.

As promised, we conclude this section with a new modified finiteness conjecture. Given some huge wide empty simplex $\tilde{\mathfrak{s}} \subset \mathbb{R}^d$, we can always suppose that the smallest facet $\mathfrak{s}$ lies in $\mathbb{R}^{d-1}$. Then our simplex is of the form $\tilde{\mathfrak{s}} = \tilde{\mathfrak{s}}(h)$ for some huge $h = \det(\tilde{\mathfrak{s}})/\det(\mathfrak{s})$. One is tempted to believe that h is large enough to (1): exclude interior lattice points from the projection of $\tilde{\mathfrak{s}}$ to $\mathbb{R}^{d-1}$, and (2): assure that any functional that realizes the width must live in $\mathbb{R}^{d-1}$.

2.7. CONJECTURE. *For every $d \geq 2$, there are only finitely many equivalence classes of empty d–simplices whose width is greater than $\overline{w}(d-1)$, the greatest width that can be achieved in dimension $d-1$ by almost empty simplices.*

This includes the conjecture that the maximal width of the bipyramids involved cannot be greater than $\overline{w}$.

3. Computer search in dimension 4

As already mentioned, 3–dimensional empty simplices always have width 1. The first examples of width 2 simplices in dimensions 4 and 5 were given by Uwe Wessels [Wes89]. In this section we describe the search for wide empty 4–simplices. The strategy is the following:

1. enumerate all equivalence classes of (possibly) empty simplices,
2. check whether or not the simplex is in fact empty, and
3. if so, calculate the width.

This relies on the two known results 3.1 and 3.3.

3.1. Theorem [Wes89]. *Every empty 4–simplex has at least two unimodular facets. In particular, every such simplex is equivalent to a simplex of the form* $\mathfrak{s}[\mathbf{m}]$. □

(The analogous statement is false in higher dimensions. For example, the simplex in $\mathbb{R}^5$ given by the columns of

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{pmatrix},$$

is empty, without any unimodular facet [Wes89, p. 21].)

We can further restrict our search for empty simplices, as follows.

3.2. Lemma. *The simplex* $\mathfrak{s}[\mathbf{m}]$ *is lattice equivalent to* $\mathfrak{s}[\mathbf{m}+D\varepsilon]$, *where* $\varepsilon \in \mathbb{Z}^d$ *is any vector with vanishing coordinate sum. In particular,* $\mathfrak{s}[\mathbf{m}]$ *is equivalent to some* $\mathfrak{s}[\mathbf{m}']$ *with* $\|\mathbf{m}'\|_\infty \leq D$. *Furthermore, any orientation preserving lattice equivalence which fixes the vertices* $\mathbf{e}_i$, *is of this type.* □

Given a determinant D, enumerate sorted 4–tuples $\mathbf{m}$ with $\sum m_i = D+1$ in the range given by Lemma 3.2. Then test emptiness by the following criterion, which is (up to the range restriction $k \leq D/2$) due to Herb Scarf.

3.3. Theorem [Sca85]. *The lattice simplex* $\mathfrak{s}[\mathbf{m}] = \operatorname{conv}\{\mathbf{e}_1, \ldots, \mathbf{e}_d, \mathbf{m}\}$ *of determinant* $D := \sum_{i=1}^d m_i - 1 > 0$ *is empty if and only if*

$$\sum_{i=1}^{d} \left\lceil \frac{km_i}{D} \right\rceil > k+1 \tag{3.1}$$

holds for all integers k *in the range* $1 \leq k \leq D/2$.

PROOF. Observe that

$$\sum_{i=1}^{d} \left\lceil \frac{km_i}{D} \right\rceil \geq k+1$$

always holds, since $\sum_{i=1}^{d} \lceil \frac{km_i}{D} \rceil \geq \sum_{i=1}^{d} \frac{km_i}{D} = \frac{k}{D} \sum_{i=1}^{d} m_i = k\frac{D+1}{D} > k$.

Also it is readily checked that $\sum_{i=1}^{d} x_i \geq 1$ together with the inequalities

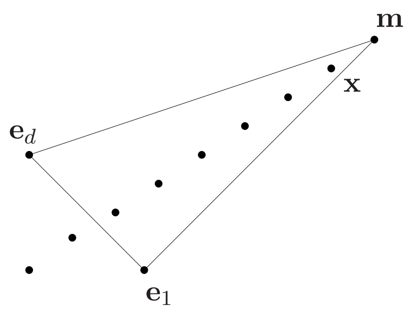
$$\frac{\sum_{i=1}^{d} x_i - 1}{D} m_j \leq x_j < \frac{\sum_{i=1}^{d} x_i - 1}{D} m_j + 1 \tag{3.2}$$

describes the set $\mathfrak{s} \smallsetminus \{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$: the weak inequalities describe $\mathfrak{s}$ and the strict ones cut off the vertices $\mathbf{e}_i$.

If there is a lattice point $\mathbf{x}$ in $\mathfrak{s} \smallsetminus \{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$, set $k = k(\mathbf{x}) := \sum_{i=1}^{d} x_i - 1$. Then by (3.2) $\mathbf{x}$ must have the coordinates

$$x_j = \left\lceil \frac{km_j}{D} \right\rceil. \tag{3.3}$$

Thus $\mathbf{x}$ violates (3.1), but it need not satisfy $k \leq D/2$. However, if $k > D/2$, then we get other lattice points $\mathbf{x}' := \mathbf{m} + n(\mathbf{x} - \mathbf{m})$ with $k' := k(\mathbf{x}') = D + n(k - D)$.



Choose $n = \lceil \frac{k}{D-k} \rceil$. Then

$$0 = D + (\frac{k}{D-k} + 1)(k - D) < k' \leq D + \frac{k}{D-k}(k - D) = D - k < D/2.$$

Thus from $k' \geq 0$ deduce that $\mathbf{x}'$ is another lattice point in $\mathfrak{s}[\mathbf{m}]$, which is not a vertex because of $k' > 0$, but whose $k' = \sum_{i=1}^{d} x_i' - 1$ is smaller than $D/2$, and which violates the condition (3.1). This finishes the *only if* part of the proof.

On the other hand, if for some k

$$\sum_{i=1}^{d} \left\lceil \frac{km_i}{D} \right\rceil = k+1,$$

then the vector $\mathbf{x}$ given by (3.3) satisfies (3.2), and thus provides a lattice point in $\mathfrak{s}[\mathbf{m}]$ which is not a vertex. $\square$

In principle, the width of a lattice simplex can be found by solving an integer program, as demonstrated by the following lemma.

3.4. Lemma. *Let W be an upper bound for the width of the simplex $\mathfrak{s}[\mathbf{m}]$. Then* $\mathrm{width}(\mathfrak{s}[\mathbf{m}])$ *is the optimal value of the following minimization problem:*

$$\begin{array}{rcccll} \multicolumn{6}{l}{\textit{minimize} \quad w \quad \textit{subject to}} \\ w_0 & \leq & l_i & \leq & w_0 + w & \textit{for} \ \ 1 \leq i \leq d, \\ w_0 & \leq & \sum_{i=1}^d l_i m_i & \leq & w_0 + w, & \\ & & \sum_{i=1}^d l_i W^{i-1} & \geq & 1, & \end{array} \tag{3.4}$$

with integer variables w, w_0, and l_i. (The values of the l_i–variables in an optimal solution yield a linear functional that realizes the width.)

Proof. The width is defined to be the minimal solution to the first constraints, excluding the zero solution ($\ell = \mathbf{0}$, $w = w_0 = 0$). This solution is cut off by the last constraint. We have to see that some $\ell \neq \mathbf{0}$ that realizes the width of $\mathfrak{s}[\mathbf{m}]$ satisfies this last constraint. By replacing the l_i by their negatives, we can assure the left hand side of this constraint to be non–negative. If it were zero, the first non–zero l_i would have to be a multiple of W and some other l_j would have the opposite sign, with the effect that $|\ell(\mathbf{e}_i) - \ell(\mathbf{e}_j)| = |l_i - l_j| > |l_i| \geq W$. $\square$

An integer programming formulation as in Lemma 3.4 also shows that the width of a general lattice simplex can be computed in polynomial time if the dimension is fixed. (A different IP formulation was provided by Sebő [Seb98, § 5].) Somewhat surprisingly, our computational tests using CPLEX* showed that the integer programs of Lemma 3.4 can indeed be solved fast and stably, provided that W is not too large. We used this for an enumeration of 4–dimensional empty lattice simplices up to determinant $D = 350$, and also for tests in dimension 5.

However, for larger determinants a less sophisticated criterion proved to be faster. Namely, a simplex $\mathfrak{s}[\mathbf{m}]$ has lattice width greater than w if and only if there is no solution to

$$\begin{array}{rcccl} 0 & \leq & l'_i & \leq & w \quad \text{for } 1 \leq i \leq d, \\ & & \sum_{i=1}^d l'_i m_i \pmod D & \leq & w, \\ & & (l'_1, \ldots, l'_d) & \neq & (0, \ldots, 0). \end{array}$$

This is easily derived from the system (3.4) using the substitutions $l'_i := l_i - w_0$. Thus, to test e.g. whether a 4–dimensional simplex has width greater than 2,

*CPLEX Linear Optimizer 4.0.8 with Mixed Integer & Barrier Solvers; ©CPLEX Optimization, Inc., 1989-1995 one simply has to see whether there is one of the $3^4 - 1 = 80$ different 4–tuples $(l'_1, \ldots, l'_4) \in \{0,1,2\}^4 \setminus \{\mathbf{0}\}$ that satisfies the modular inequality

$$\sum_{i=1}^{d} l'_i m_i \pmod{D} \;\leq\; 2.$$

This method yields a complete list of empty 4–simplices $\mathfrak{s}[\mathbf{m}]$ that have a given determinant and width ≥ 3. In order to exclude multiple representatives of the same equivalence class, we have to develop an equivalence test. A lattice equivalence that maps $\mathfrak{s}[\mathbf{m}]$ to some $\mathfrak{s}[\mathbf{m}']$ can either map $\mathbf{m}$ to $\mathbf{m}'$ (then it is a transformation as in Lemma 3.2 followed by a permutation of coordinates) or it maps $\mathbf{m}$ to some $\mathbf{e}_i$. In the latter case, we can — again, up to coordinate permutations — apply the following Lemma 3.5, to get a transformation that fixes the $\mathbf{e}_i$.

3.5. Lemma. *The facet* $\operatorname{conv}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{m}\}$ *of* $\mathfrak{s}[\mathbf{m}]$ *is unimodular if and only if* $\gcd(D, m_4) = 1$. *In that case let* $p, q \in \mathbb{Z}$ *such that* $pm_4 - qD = 1$. *Then the affine map given by*

$$\mathbf{x} \longmapsto \begin{pmatrix} 1+qm_1 & qm_1 & qm_1 & (q-p)m_1 \\ qm_2 & 1+qm_2 & qm_2 & (q-p)m_2 \\ qm_3 & qm_2 & 1+qm_3 & (q-p)m_3 \\ km_4-q & km_4-q & km_4-q & p-q+k(m_4-D) \end{pmatrix} \mathbf{x} - \begin{pmatrix} qm_1 \\ qm_2 \\ qm_3 \\ km_4-q \end{pmatrix}$$

(with $k = q - p - 1$*) is lattice preserving and maps*

$$\begin{aligned} \mathbf{e}_1 &\longmapsto \mathbf{e}_1 \\ \mathbf{e}_2 &\longmapsto \mathbf{e}_2 \\ \mathbf{e}_3 &\longmapsto \mathbf{e}_3 \\ \mathbf{e}_4 &\longmapsto (-pm_1, -pm_2, -pm_3, p + (p-q+1)D)^t \\ \mathbf{m} &\longmapsto \mathbf{e}_4. \end{aligned}$$

□

The coordinate transformations can be ruled out by sorting the m_i and the Lemma 3.2 transformations by comparison of the moduli $m_i \pmod{D}$. The following theorem records our computational results, based on generation and test of all equivalence classes of empty lattice simplices of determinant $D \leq 1000$. It provides evidence for $w(4) = 4$ as well as for Conjecture 2.7.

3.6. Theorem. *Among the 4–dimensional empty lattice simplices of determinant $D \leq 1000$,*

- *there are no simplices of width 5 or larger,*
- *there is a unique equivalence class of simplices of width 4 which is represented by the simplex $\mathfrak{s}[(6, 14, 17, 65)^t]$, whose determinant is $D = 101$,*
- *all simplices of width 3 have determinant $D \leq 179$, where the (unique) smallest example, of determinant $D = 41$, is represented by $\mathfrak{s}[(-10, 4, 23, 25)^t]$, and the (unique) example of determinant $D = 179$ is represented by $\mathfrak{s}[(20, 36, 53, 71)^t]$.*

This result has been found independently by Fermigier and Kantor, and it was confirmed by Wahidi [Wah99].

If it were true that every empty 4–simplex of width 3 has determinant ≤ 179, it would follow that $\overline{w}(3) \leq 2$, the latter can be shown directly:

3.7. Proposition. *The lattice width of any almost empty tetrahedron is at most 2, i.e.,*

$$\overline{w}(3) = 2.$$

Proof. The almost empty tetrahedron $\mathfrak{s}[(2, 2, 3)^t]$ underlying Proposition 2.6 shows that $\overline{w}(3) \geq 2$.

Suppose that there is an almost empty tetrahedron $\mathfrak{s}$ of width ≥ 3. We first bring it into normal form: Up to a unimodular transformation, $\mathfrak{s}$ has vertices $\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, (x, y, z')^t$ with $z' \geq 1$ and $0 \leq x, y < z'$. If $z' \leq x + y - 2$, then $(1, 1, 1)^t$ would be an interior point. Hence $z' \geq x + y - 1$ and $\mathfrak{s}$ is equivalent to $\mathfrak{s}[(x, y, z)^t]$ with $z = z' - x - y + 1 \geq 0$.

Let us suppose $0 \leq x \leq y \leq z$. Because of width$(\mathfrak{s}) \geq 3$ we know that $x \geq 3$, $y - x \geq 2$, $z - y \geq 2$ and $|z - x - y| \geq 2$. From these inequalities we want to deduce that there is an interior lattice point. Therefore, observe that the lattice point given by (3.3) is interior if and only if both inequalities in (3.2) are strict. In other words, we have to produce some k such that

$$\left\lceil \frac{kx}{D} \right\rceil + \left\lceil \frac{ky}{D} \right\rceil + \left\lceil \frac{ky}{D} \right\rceil = k + 1,$$

and none of the three fractions is integral. There are two cases.

- $z - x - y \leq -2$: $(1, 1, 1)^t$ is an interior point ($k = 2$).
- $z - x - y \geq 2$: Abbreviate $p := \lfloor y/x \rfloor$. Then there is an integer q such that $(1, p, q)^t$ is an interior point ($k = p + q$).

We still have to prove the second claim. It is equivalent to

$$\left(\frac{(p+q)x}{D} < 1 \qquad \Longleftarrow \right) \qquad \frac{(p+q)y}{D} < p \qquad \text{and} \qquad \frac{(p+q)z}{D} < q.$$

So we are looking for an integer q such that

$$\frac{pz}{D-z} < q < \frac{p(D-y)}{y}. \tag{3.5}$$

Once more, we have several cases:

- $y \geq 2x$ $(p \geq 2)$,
- $y = 2x-1$, $z-x-y > 2$,
- $y = 2x-1$, $z-x-y = 2$ and
- $y < 2x-1$.

It is convenient to consider the most special case first. If $y = 2x-1$, $z-x-y = 2$, then (3.5) becomes

$$\frac{3x+1}{3x-2} < q < \frac{4x}{2x-1},$$

which is satisfied by $q = 2$.

In the other cases we will assure the existence of some q that satisfies (3.5) by showing that

$$\frac{p(D-y)}{y} - \frac{pz}{D-z} > 1,$$

or, equivalently,

$$z \cdot p(x-1) > (x+y-1) \cdot \big(y - p(x-1)\big).$$

- $y \geq 2x$: $p \geq 2$

$$\begin{aligned} y &\leq (p+1)x - 1 && (p = \lfloor y/x \rfloor) \\ &\leq (p+1)x + p - 3 && (p \geq 2) \\ &= (p+1)(x-1) + 2(p-1) \\ &\leq (p+1)(x-1) + (x-1)(p-1) && (x \geq 3) \\ &= 2p(x-1). \end{aligned}$$

  So $z > x+y-1$ and $p(x-1) \geq y - p(x-1)$.
- $y = 2x-1$, $z-x-y > 2$: $p = 1$ and $z \geq 3x+2$.

$$\begin{aligned} z \cdot 1(x-1) &\geq (3x+2)(x-1) \\ &> (3x+2)(x-1) - (x-2) \\ &= y - 1(x-1). \end{aligned}$$

- $y < 2x-1$:
  $x - 1 \geq y - (x-1)$ and $z > x+y-1$.

$\square$

All empty 4–simplices of width ≥ 3 and determinant ≤ 1000. There are 179 equivalence classes of empty 4–simplices whose width exceeds 2 and whose determinant is not larger than 1000. The following table lists exactly one 4–tuple $\mathbf{m}$ for each of them, such that $\mathfrak{s}[\mathbf{m}]$ represents this class. (Cf. [Wah99, p. 49f].)

det	m			
41	-10	4	23	25
43	3	5	11	25
	5	7	9	23
44	4	9	15	17
	5	9	12	19
47	3	7	12	26
48	4	7	9	29
49	4	9	15	22
	9	11	13	17
50	4	11	17	19
51	3	5	13	31
	5	7	9	31
52	5	8	11	29
53	3	10	18	23
	5	12	14	23
	6	8	11	29
	6	11	14	23
	7	13	16	18
	8	13	15	18
54	4	11	19	21
55	3	7	16	30
	3	8	14	31
56	8	13	17	19
57	3	5	13	37
58	3	11	20	25
	6	9	13	31
	7	11	16	25

det	m			
59	7	9	20	24
	7	10	12	31
	7	10	19	24
	8	10	15	27
	8	12	17	23
	8	15	17	20
	9	12	16	23
	9	14	17	20
	11	13	15	21
	11	13	16	20
60	11	13	16	21
61	-9	3	27	41
	3	5	13	41
	4	7	18	33
	5	7	18	32
	6	8	11	37
	6	9	13	34
	7	9	13	33
62	7	10	13	33
	7	13	16	27
	8	15	19	21
63	5	13	22	24
	8	10	13	33
64	4	13	23	25
	10	13	17	25
65	3	5	22	36
	6	14	17	29
	11	14	18	23
	12	14	17	23

det	m			
67	3	5	17	43
	3	5	23	37
	4	10	17	37
	7	15	17	29
	10	12	19	27
	13	15	17	23
68	5	12	21	31
	10	15	21	23
69	3	14	22	31
	10	14	19	27
71	5	12	26	29
	5	13	22	32
	7	11	16	38
	7	16	18	31
	8	11	15	38
	8	15	18	31
	9	15	22	26
	11	16	21	24
	11	17	20	24
73	3	8	14	49
	4	10	17	43
	6	8	11	49
	13	15	21	25
	13	16	20	25
74	4	15	27	29
	10	17	23	25
	12	15	19	29

det	**m**			
75	7	9	13	47
	7	17	19	33
	14	16	19	27
	14	17	19	26
76	8	11	17	41
	9	11	26	31
77	3	5	13	57
	8	18	23	29
	8	21	23	26
	12	17	23	26
	15	17	20	26
79	3	11	20	46
	4	14	25	37
	7	16	27	30
	9	12	16	43
	9	14	16	41
	9	14	20	37
	9	16	21	34
	10	16	23	31
	12	16	21	31
	13	16	20	31
	13	17	22	28
	14	16	23	27
81	6	14	25	37
82	7	13	30	33
83	6	14	17	47
	7	11	16	50
	8	11	15	50
	8	23	25	28
	9	11	14	50
	11	19	26	28
	11	21	24	28
	12	21	23	28
	13	20	23	28
	16	19	21	28

det	**m**			
84	8	13	19	45
85	3	5	32	46
	4	7	18	57
	6	8	11	61
	8	19	22	37
87	5	13	22	48
	17	19	22	30
89	4	7	18	61
	10	13	19	48
	11	18	26	35
91	3	5	23	61
	3	11	20	58
94	7	16	29	43
95	5	18	32	41
	11	24	29	32
	12	23	29	32
	13	21	30	32
	15	23	26	32
	17	21	26	32
97	3	5	23	67
	3	8	14	73
	3	13	28	54
	6	8	11	73
	9	15	22	52
	11	14	20	53
101	**6**	**14**	**17**	**65**
	9	21	34	38
	13	23	32	34
	16	21	31	34

det	**m**			
103	7	19	32	46
	8	11	26	59
	9	15	22	58
	10	16	23	55
	10	23	26	45
	11	13	21	59
	13	21	32	38
107	3	5	43	57
	8	18	33	49
	10	12	19	67
	15	23	34	36
	17	23	30	38
	17	25	27	39
	20	23	27	38
	21	23	28	36
109	10	22	29	49
	13	22	32	43
	16	22	29	43
	20	22	29	39
113	15	25	36	38
	20	25	31	38
119	15	27	38	40
	19	24	37	40
121	19	26	34	43
125	11	26	42	47
127	3	5	53	67
137	7	26	46	59
139	16	28	41	55
149	17	30	44	59
	23	32	42	53
169	19	34	50	67
179	20	36	53	71

In fact, all these simplices have exactly width 3, with the exception of the bold simplex $\mathfrak{s}[(6,14,17,65)^t]$, which has width 4.

CHAPTER III

Crepant resolutions of toric l.c.i.–singularities

1. Introduction

1.1. Motivation. In the past two decades *crepant* birational morphisms were mainly used in algebraic geometry to reduce the canonical singularities of algebraic d–folds, $d \geq 3$, to $\mathbb{Q}$–factorial terminal singularities, and to treat *minimal models* in high dimensions. From the late eighties onwards, crepant full desingularizations $\widehat{Y} \to Y$ of projective varieties Y with trivial dualizing sheaf and mild singularities (like quotient or toroidal singularities) play also a crucial role in producing *Calabi–Yau manifolds*, which serve as internal target spaces for non–linear super–symmetric sigma models in the framework of physical string–theory. This explains the recent mathematical interest in both local and global versions of the existence problem of *smooth* birational models of such Y's.

Locally, the high–dimensional *McKay correspondence* (cf. [IR94, Rei]) for the underlying spaces $\mathbb{C}^d/G$, $G \subset \mathrm{SL}(d, \mathbb{C})$, of Gorenstein quotient singularities was proved by Batyrev [Bat99, Theorem 8.4]. It states that the following two quantities are equal:

- the ranks of the non–trivial (=even) cohomology groups $H^{2k}(\widehat{X}, \mathbb{C})$ of the overlying spaces $\widehat{X}$ of crepant, full desingularizations $\widehat{X} \to X = \mathbb{C}^d/G$ on the one hand,
- the number of conjugacy classes of G having the weight (also called "age") k on the other.

Moreover, a one–to–one correspondence of McKay–type is also true for torus–equivariant, crepant, full desingularizations $\widehat{X} \to X = U_\sigma$ of the underlying spaces of Gorenstein toric singularities [BD96, §4]. Again, the non–trivial (even) cohomology groups of the $\widehat{X}$'s have the "expected" dimensions, which in this case are determined by the Ehrhart polynomials of the corresponding lattice polytopes (cf. Chapter IV). Thus in both situations the ranks of the cohomology groups of $\widehat{X}$'s turn out to be independent of the particular choice of a crepant resolution. Also in both situations, a crepant resolution always exists if $d \leq 3$, but not in general: for $d \geq 4$ there are, for instance, lots of *terminal* Gorenstein singularities in both classes, such as the toric quotient singularities $U_{\sigma(\mathfrak{s})}$ for empty simplices $\mathfrak{s}$ (cf. Chapter II).

We believe that a purely algebraic, sufficient condition for the existence of projective, crepant, full resolutions in *all* dimensions is to require from our singularities to be, in addition, *local complete intersections* (*l.c.i.'s*). In the toric category, this conjecture was verified for Abelian quotient singularities in [DHZ98b] via a Theorem by Kei–ichi Watanabe [Wat80]. (For non–Abelian groups acting on $\mathbb{C}^d$, it remains open.) Furthermore, Dais, Henk and Ziegler [DHZ98b, cf. §8(iii)] asked for geometric analogues of the joins and dilations occuring in their reduction theorem also for toric non–quotient l.c.i.–singularities. As we shall see below, such a characterization (in a somewhat different context) is indeed possible by making use of another beautiful classification theorem due to Haruhisa Nakajima [Nak86], which generalizes Watanabe's results to the entire class of toric l.c.i.'s. Based on this classification we prove the following:

1.1. Theorem. *The underlying spaces of all toric l.c.i.–singularities admit torus–equivariant, projective, crepant, full resolutions (i.e., smooth minimal models) in all dimensions.*

Families of Gorenstein non–l.c.i. toric singularities that have such special full resolutions seem to be very rare. This problem is discussed in [DHH98, DHb] for certain families of Abelian quotient non–l.c.i. singularities.

1.2. Crepant resolutions via triangulations. The divisor $K_\Sigma = -\sum D_\varrho$ on X_Σ is canonical. The affine piece U_σ is Gorenstein if and only if the canonical divisor is Cartier, i.e., there is an integral linear functional ℓ_σ that takes the value -1 on all primitive ray generating lattice vectors $\mathbf{p}(\varrho)$ (cf. I.4.2). Deduce:

1.2. Proposition. *X_Σ is Gorenstein if and only if every cone in Σ is equivalent to a cone spanned by some lattice polytope.*

Recall that a subdivision $\mathcal{S}$ of a fan Σ induces a proper birational morphism $\phi_\mathcal{S} : X_\mathcal{S} \to X_\Sigma$. In this situation, the affine pieces U_σ that cover the overlying space $X_\mathcal{S}$ are smooth if and only if the σ are unimodular:

1.3. Proposition. *The morphism $\phi_\mathcal{S} : X_\mathcal{S} \to X_\Sigma$ is a full desingularization if and only if $\mathcal{S}$ is a unimodular triangulation.*

Also, $\phi_\mathcal{S}$ is projective if and only if $\mathcal{S}$ is a coherent subdivision of Σ. Projective full desingularizations always exist for any X_Σ (see [KKMSD73, §I.2]). Nevertheless, we ask in this section about conditions under which such $\phi_\mathcal{S}$ are, in addition, crepant. This is a local question that only makes sense if both X_Σ and $X_\mathcal{S}$ are ($\mathbb{Q}$–)Gorenstein. So, locally, suppose that the cone $\sigma = \sigma(P) \subset \mathbb{R}^{d+1}$ is spanned by a lattice polytope $P \subset \mathbb{R}^d$. Then the canonical divisor K_{U_σ} (with $\mathcal{O}(K_{U_\sigma})$ trivial) is defined by the linear functional $\mathbf{x} \mapsto -x_{d+1}$. The pull back

$\phi^* K_{U_\sigma}$ as a Cartier divisor on $X_{\mathcal{S}}$ is defined by the pull back of the local equation: by the same functional. As a Weil divisor, it is $\phi^* K_{U_\sigma} = -\sum \mathbf{p}(\varrho)_{d+1} D_\varrho$, where the sum ranges over all rays ϱ of $\mathcal{S}$. On the other hand, the canonical divisor of $X_{\mathcal{S}}$ is $K_{X_{\mathcal{S}}} = -\sum D_\varrho$. Thus $\phi_{\mathcal{S}}$ is crepant if and only if $\mathbf{p}(\varrho)_{d+1} = 1$ for every ray $\varrho \in \mathcal{S}$.

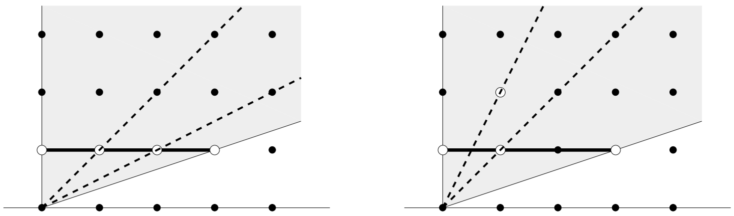


FIGURE 1. A 'crepant' and a 'non–crepant' subdivision.

This establishes the following criterion.

1.4. PROPOSITION. *Let σ be spanned by the lattice polytope P and let $\mathcal{S}$ be a subdivision of σ. The morphism $\phi_{\mathcal{S}} : X_{\mathcal{S}} \to U_{\sigma(P)}$ is crepant if and only if $\mathcal{S}$ is induced by a subdivision of P into lattice polytopes.*

2. Proof of the Main Theorem

2.1. Nakajima's classification. The question about the geometric description of the polytopes which span the cones defining affine toric l.c.i.–varieties was completely answered by Nakajima [Nak86]. The original definition reads as follows:

2.1. Definition. Associate to a sequence $\mathfrak{g} = (\mathbf{g}_1, \dots, \mathbf{g}_r)$ of nonzero forms in $(\mathbb{Z}^d)^\vee$ $(d > r)$, satisfying $g_{ij} = 0$ for $j > i$, the following sequence of polytopes:

$$P^{(1)} = \{(1, 0, \dots, 0)\} \subset \mathbb{R}^d,$$

$$P^{(i+1)} = \operatorname{conv}\left(P^{(i)} \cup \{(\mathbf{x}', \langle \mathbf{g}_i, \mathbf{x} \rangle, 0, \dots, 0) \in \mathbb{R}^d \,:\, \mathbf{x} = (\mathbf{x}', 0, 0, \dots, 0) \in P^{(i)}\}\right).$$

Call $\mathfrak{g}$ *admissible* if $\langle \mathbf{g}_i, \mathbf{x} \rangle \geq 0$ for all $\mathbf{x} \in P^{(i)}$. For $\mathfrak{g}$ admissible we call $P_\mathfrak{g} := P^{(r+1)}$ the *Nakajima polytope* or *lci polytope* associated to $\mathfrak{g}$.

2.2. Theorem [Nak86]. *An affine toric variety U_σ is a local complete intersection if and only if σ is lattice equivalent to a cone spanned by some Nakajima polytope.* □

For our purposes a recursive definition is more suitable (cf. Figure 2.)

2.3. Lemma. *A lattice polytope $P \subseteq \mathbb{R}^d$ is a Nakajima polytope if and only if it is a lattice point $P = \{\mathbf{e}_1\} \subset \mathbb{Z}^d$ or*

$$P = \{\mathbf{x} = (\mathbf{x}', x_d) \in F \times \mathbb{R} \,:\, 0 \leq x_d \leq \langle \mathbf{g}, \mathbf{x}' \rangle\}, \tag{2.1}$$

where the facet $F \subseteq \mathbb{R}^{d-1}$ is a Nakajima polytope, and $\mathbf{g} \in (\mathbb{Z}^{d-1})^\vee$ is a functional with non–negative values on $F \cap \mathbb{Z}^{d-1}$.

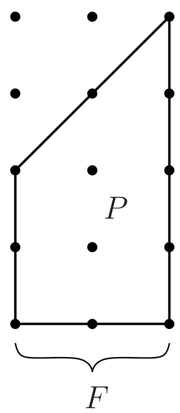


Figure 2. How to construct a Nakajima polytope.

PROOF. Let $\mathfrak{g}$ be an admissible sequence. Then the corresponding Nakajima polytope $P_{\mathfrak{g}}$ has the following description by inequalities:

$$P_{\mathfrak{g}} = \{\mathbf{x} \in \mathbb{R}^d \,:\, x_1 = 1 \text{ and } 0 \leq x_{i+1} \leq \langle \mathbf{g}_i, \mathbf{x} \rangle \text{ for } 1 \leq i \leq d-1\}, \tag{2.2}$$

so that $P_{\mathfrak{g}}$ can be reconstructed from the facet $F = P_{\mathfrak{g}'}$ and $\mathbf{g} = \mathbf{g}_r$, where $\mathfrak{g}'$ is the truncated sequence $(\mathbf{g}_1, \ldots, \mathbf{g}_{r-1})$. Conversely, given the situation (2.1), F is some $P_{\mathfrak{g}'}$. Then we can append $\mathbf{g}$ to $\mathfrak{g}'$ and obtain an admissible sequence for P. $\square$

2.2. The chimney lemma. Propositions 1.3 and 1.4 and Theorem 2.2 reduce the proof of Theorem 1.1 to the existence of coherent unimodular triangulations for all Nakajima polytopes of any dimension. We do not loose in generality if we henceforth assume that the considered Nakajima polytope $P \subset \mathbb{R}^d$ is full–dimensional. (Otherwise, $\sigma = \widetilde{\sigma} \oplus \{\mathbf{0}\}$ leads to the splitting $\mathrm{Sing}(U_P) = \mathrm{Sing}(X_{\widetilde{\sigma}}) \times (\mathbb{C}^*)^{d-\dim P}$, which does not cause any difficulties for the desingularization problem.)

Proceed by induction on the dimension of P. Zero– and one–dimensional polytopes always admit unique such triangulations. For the induction step proceed as follows: According to Lemma 2.3, you may assume that for a given $P_{\mathfrak{g}}$ the facet F (as in (2.1)) is already endowed with a coherent unimodular triangulation $\mathcal{T}_F$. This triangulation induces a coherent subdivision

$$\mathcal{S}_P = \{(\mathfrak{s} \times \mathbb{R}) \cap P_{\mathfrak{g}} \,:\, \mathfrak{s} \in \mathcal{T}_F\}$$

of $P_{\mathfrak{g}}$ into "chimneys" over the simplices of $\mathcal{T}_F$. The second step is to refine $\mathcal{S}_P$ coherently by pulling lattice points until we obtain a triangulation $\mathcal{T}_P$ into empty simplices (cf. Figure 3.)

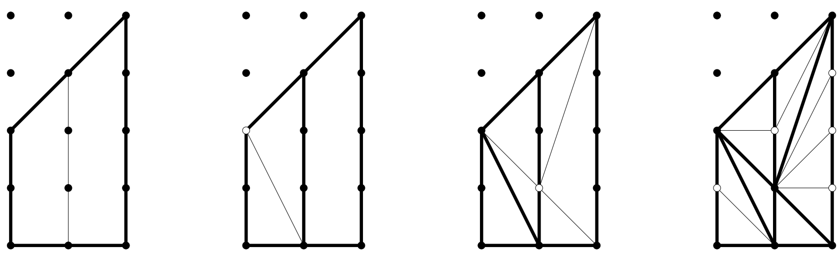


FIGURE 3. Refining the chimney subdivision.

Any such triangulation is automatically unimodular:

2.4. LEMMA. *Let $\pi : \mathbb{R}^d \longrightarrow \mathbb{R}^{d-1}$ be the deletion of the last coordinate. Let $\mathfrak{s} \subset \mathbb{R}^d$ be an empty lattice simplex whose projection $\mathfrak{s}' = \pi(\mathfrak{s})$ is unimodular. Then $\mathfrak{s}$ itself is unimodular.*

Proof. We can assume that $\mathfrak{s}$ is full–dimensional. After a lattice equivalence of $\mathbb{R}^{d-1}$ we can suppose that $\mathfrak{s}' = \triangle^{d-1}$ is the standard $(d-1)$–simplex. After a translation in the last coordinate, $\mathfrak{s}$ contains the origin. Now we can shift the lines $\pi^{-1}(\mathbf{e}_i)$ independently such that we finally obtain a simplex with non–negative last coordinate that contains $\triangle^{d-1} \times \{\mathbf{0}\}$. The fact that $\mathfrak{s}$ is empty implies that the additional vertex has last coordinate 1. □

Proof of Theorem 1.1. By construction, all simplices of $\mathcal{T}_P$ are empty because $P \cap \mathbb{Z}^d$ coincides with the set of vertices of $\mathcal{T}_P$ (we pulled them all.) Since their projections under π are the unimodular simplices of $\mathcal{T}_F$, they have themselves to be unimodular by the Chimney Lemma 2.4. □

3. Applications

3.1. Computing Betti numbers. This paragraph is, in a sense, a sneak preview of Chapter IV. We are interested in the rational cohomology groups of the crepant resolutions $X_{\mathcal{T}}$ as constructed in the previous section. They do in fact not depend on the particular choice of a triangulation, in odd dimensions they are trivial and the ranks in even dimensions are determined by the ψ–vectors of the faces of P (cf. Proposition IV.1.5.) The even cohomology ranks of the fiber over the distinguished point $D_{\sigma(P)} \in U_{\sigma(P)}$ are just the entries of the ψ–vector of P (cf. the remark after the proof of Proposition IV.1.5.)

From the description (2.2) of $P_{\mathfrak{g}}$ by inequalities, one deduces for its dilations:

$$k \cdot P_{\mathfrak{g}} = \{\mathbf{x} \in \mathbb{R}^d \ : \ x_1 = k \text{ and } 0 \leq x_{i+1} \leq \langle \mathbf{g}_i, \mathbf{x} \rangle \text{ for } 1 \leq i \leq d-1\}.$$

Thus, the Ehrhart polynomial and the ψ–vector (by (I.2.1)) of $P_{\mathfrak{g}}$ are given by

$$\mathbf{Ehr}(P_{\mathfrak{g}}, k) = \sum_{\nu_1=0}^{n g_{1,1}} \sum_{\nu_2=0}^{n g_{2,1}+\nu_1 g_{2,2}} \cdots \sum_{\nu_d=0}^{n g_{d,1}+\sum \nu_i g_{d-1,i}} 1 \ , \quad \text{respectively}$$

$$\psi_j(P_{\mathfrak{g}}) = \sum_{i=0}^{j} (-1)^i \binom{\dim P_{\mathfrak{g}}+1}{i} \sum_{\nu_1=0}^{(j-i)g_{1,1}} \sum_{\nu_2=0}^{(j-i)g_{2,1}+\nu_1 g_{2,2}} \cdots \sum_{\nu_d=0}^{(j-i)g_{d,1}+\sum \nu_i g_{d-1,i}} 1 \ .$$

For instance, the Ehrhart polynomial of the d–dimensional Nakajima polytope $P_{\mathfrak{g}}$ for $d \leq 3$ equals

$$\begin{aligned}
\mathbf{Ehr}(P_{\mathfrak{g}}, k) &= g_{1,1}k + 1, \text{ for } d = 1, \\
\mathbf{Ehr}(P_{\mathfrak{g}}, k) &= \left(\tfrac{1}{2} g_{2,2} g_{1,1}^2 + g_{2,1} g_{1,1}\right) k^2 + \left(g_{1,1} + \tfrac{1}{2} g_{2,2} g_{1,1} + g_{2,1}\right) k + 1, \text{ for } d = 2, \\
\mathbf{Ehr}(P_{\mathfrak{g}}, k) &= \big(g_{3,1} g_{2,1} g_{1,1} + \tfrac{1}{2} g_{3,2} g_{2,1} g_{1,1}^2 + \tfrac{1}{2} g_{3,3} g_{2,1}^2 g_{1,1} + \tfrac{1}{6} g_{3,3} g_{2,2}^2 g_{1,1}^3 \\
&\qquad + \tfrac{1}{2} g_{3,3} g_{2,2} g_{2,1} g_{1,1}^2 + \tfrac{1}{2} g_{3,1} g_{2,2} g_{1,1}^2 + \tfrac{1}{3} g_{3,2} g_{2,2} g_{1,1}^3\big)\, k^3 \\
&\quad + \big(g_{2,1} g_{1,1} + \tfrac{1}{2} g_{3,3} g_{2,1}^2 + \tfrac{1}{2} g_{3,1} g_{2,2} g_{1,1} + \tfrac{1}{4} g_{3,3} g_{2,2}^2 g_{1,1}^2 + g_{3,1} g_{2,1} \\
&\qquad + \tfrac{1}{2} g_{2,2} g_{1,1}^2 + \tfrac{1}{2} g_{3,2} g_{1,1}^2 + \tfrac{1}{2} g_{3,2} g_{2,2} g_{1,1}^2 + \tfrac{1}{4} g_{3,3} g_{2,2} g_{1,1}^2 \\
&\qquad + \tfrac{1}{2} g_{3,3} g_{2,1} g_{1,1} + g_{3,1} g_{1,1} + \tfrac{1}{2} g_{3,2} g_{2,1} g_{1,1} + \tfrac{1}{2} g_{3,3} g_{2,2} g_{2,1} g_{1,1}\big)\, k^2 \\
&\quad + \big(\tfrac{1}{2} g_{3,2} g_{1,1} + g_{2,1} + g_{1,1} + \tfrac{1}{2} g_{3,3} g_{2,1} + \tfrac{1}{2} g_{2,2} g_{1,1} + g_{3,1} + \tfrac{1}{12} g_{3,3} g_{2,2}^2 g_{1,1} \\
&\qquad + \tfrac{1}{6} g_{3,2} g_{2,2} g_{1,1} + \tfrac{1}{4} g_{3,3} g_{2,2} g_{1,1}\big)\, k \ + \ 1, \qquad \text{for } d = 3.
\end{aligned}$$

This is not a satisfactory formula (It is not *nice* in the sense of [BP98, § 1].) Nevertheless, in some concrete examples it is possible to compute the desired data (cf. § 3.3).

3.2. Nakajima polytopes are Koszul. A graded $\mathbb{C}$–algebra $R = \bigoplus_{i \geq 0} R_i$ is a *Koszul algebra* if the R–module $\mathbb{C} \cong R/\mathfrak{m}$ (for $\mathfrak{m}$ a maximal homogeneous ideal) has a linear free resolution, i.e., if there exists an exact sequence

$$\cdots \longrightarrow R^{n_{i+1}} \xrightarrow{\varphi_{i+1}} R^{n_i} \xrightarrow{\varphi_i} \cdots \xrightarrow{\varphi_2} R^{n_1} \xrightarrow{\varphi_1} R^{n_0} \longrightarrow R/\mathfrak{m} \longrightarrow 0$$

of graded free R–modules all of whose matrices (determined by the φ_i's) have entries which are forms of degree 1. Every Koszul algebra is generated by its component of degree 1 and is defined by relations of degree 2.

Recall the grading of the algebra $R = \mathbb{C}[\sigma(P) \cap \mathbb{Z}^{d+1}]$ associated with a lattice polytope $P \subset \mathbb{R}^d$ by $R_i = \mathbb{C}[\sigma(P) \cap (\mathbb{Z}^{d+1} \times \{i\})]$ (cf. I.4.2.) Call P *Koszul* if R is Koszul. Bruns, Gubeladze and Trung [BGT97] gave a sufficient condition for the Koszulness of P. In order to formulate it, we need the notion of a non–face of a lattice triangulation $\mathcal{T}$ of P. A subset $F \subset P \cap \mathbb{Z}^{d-1}$ is a *face* (of $\mathcal{T}$) if $\operatorname{conv}(F)$ is; otherwise F is said to be a *non–face*.

3.1. Proposition [BGT97, 2.1.3.]. *If the lattice polytope P has a coherent unimodular triangulation whose minimal non–faces (with respect to inclusion) consist of 2 points, then P is Koszul.* □

This property is satisfied for all triangulations obtained from a chimney subdivision as constructed in the previous section.

3.2. Corollary. *Nakajima polytopes are Koszul.*

Proof. Once more we proceed by induction. Let $P = P_{\mathfrak{g}} \subset \mathbb{R}^d$ be a Nakajima polytope and $F = P \cap \mathbb{R}^{d-1} \times \{0\}$ the Nakajima facet, both triangulated (by $\mathcal{T}_P$ respectively $\mathcal{T}_F$) as in Section 2. Let $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ denote the projection. By construction π maps faces of $\mathcal{T}_P$ to faces of $\mathcal{T}_F$. Choose a non–face $\mathfrak{n} \subseteq P \cap \mathbb{Z}^d$ of $\mathcal{T}_P$. We have to consider two cases:

1. The projection $\pi(\mathfrak{n})$ is a face of $\mathcal{T}_F$.
2. It is not.

In the first case we stay within the chimney over $\pi(\mathfrak{n}) \in \mathcal{T}_F$. Thus, we may assume that $\pi(P) = \pi(\mathfrak{n})$. For all interior points $\mathbf{x}$ of $\pi(P)$ the linear ordering of the maximal simplices of $\mathcal{T}_P$ by their intersections with the line $\pi^{-1}(\mathbf{x})$ is the same, say, $\mathfrak{s}_1, \ldots, \mathfrak{s}_r$. Assign to each lattice point $\mathbf{m} \in P \cap \mathbb{Z}^d$ two numbers (m, M): $M(\mathbf{m}) = \max\{i : \mathbf{m} \in \mathfrak{s}_i\}$ and $m(\mathbf{m}) = \min\{i : \mathbf{m} \in \mathfrak{s}_i\}$, as illustrated in Figure 4 for a 2–dimensional Nakajima polytope.

Two vertices $\mathbf{m}, \mathbf{m}'$ belong to the same maximal simplex $\mathfrak{s}_i$ of $\mathcal{T}_P$ if and only if $m(\mathbf{m}), m(\mathbf{m}') \le i \le M(\mathbf{m}), M(\mathbf{m}')$. Such an index i can be found if and only if $m(\mathbf{m}) \le M(\mathbf{m}')$ and $m(\mathbf{m}') \le M(\mathbf{m})$. Furthermore, if $\mathbf{m} \in \mathfrak{s}_i \cap \mathfrak{s}_j$, then also $\mathbf{m} \in \mathfrak{s}_k$ for all k between i and j.

Let $\mathbf{n}_\uparrow \in \mathfrak{n}$ be a vertex with maximal m and $\mathbf{n}_\downarrow \in \mathfrak{n}$ a vertex with minimal M. Since $\mathfrak{n}$ is a non–face, we have $m(\mathbf{n}_\uparrow) > M(\mathbf{n}_\downarrow)$. Hence $\mathbf{n}_\uparrow$ and $\mathbf{n}_\downarrow$ do not belong to a common maximal simplex, and $\{\mathbf{n}_\downarrow, \mathbf{n}_\uparrow\} \subseteq \mathfrak{n}$ is therefore a non–face of $\mathcal{T}_P$.

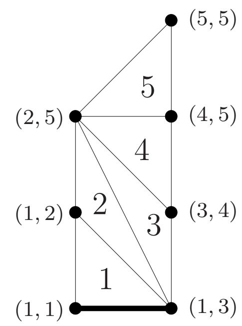


FIGURE 4. The linear ordering in a chimney.

In the second case, by induction $\pi(\mathfrak{n})$ contains a non–face of $\mathcal{T}_F$ of cardinality 2. Any two vertices in $\mathfrak{n}$ with this projection form a cardinality 2 non–face of $\mathcal{T}_P$. □

3.3. Examples. In this paragraph we apply our results to two classes of examples which are extreme in the sense that they achieve the lowest respectively highest possible numbers of faces of a Nakajima polytope. Moreover, we give the concrete binomial equations for the underlying spaces $U_{\sigma(P)}$ of the corresponding l.c.i.–singularities, and present closed formulae for the ψ–vectors.

Dilations of the standard simplex. Our first class of examples is the family of dilated standard simplices $k\cdot \triangle^{d-1}$. These polytopes have the Nakajima description $P_{\mathfrak{g}}$ by the admissible sequence $\mathfrak{g} = (k\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{d-1})$, and, being simplices, they achieve for all i the minimal number of i–faces any $(d-1)$–polytope can have. The cone spanned by this $P_{\mathfrak{g}}$ is $\sigma = \{\mathbf{x} \in \mathbb{R}^d : 0 \le x_d \le \ldots \le x_2 \le kx_1\}$, and its dual cone equals

$$\sigma^\vee = \operatorname{pos}\left(\mathbf{e}_d, \mathbf{e}_{d-1} - \mathbf{e}_d, \ldots, \mathbf{e}_2 - \mathbf{e}_3, k\mathbf{e}_1 - \mathbf{e}_2\right).$$

The semigroup $\sigma^\vee \cap \mathbb{Z}^d$ has the Hilbert basis $\mathcal{H} = \{\mathbf{f}_1, \ldots, \mathbf{f}_d, \mathbf{g}\}$, with elements $\mathbf{f}_1 = k\mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{f}_i = \mathbf{e}_i - \mathbf{e}_{i+1}$ for $i = 2, \ldots, d-1$, $\mathbf{f}_d = \mathbf{e}_d$ and $\mathbf{g} = \mathbf{e}_1$. We see that there is only one additive dependency $(k\mathbf{g}) - (\sum \mathbf{f}_i) = 0$. Introducing variables u_i and t_i to correspond to the evaluation of the torus characters at $\mathbf{f}_i$ and $\mathbf{g}$, respectively, this means that $U_\sigma = \{(t, u_1, \ldots, u_d) \in \mathbb{C}^{d+1} : t^k - \prod u_i = 0\}$ is a $(d; k)$–hypersurface (cf. [DHZ98b, 5.11]).

After computing the Erhart polynomial $\mathbf{Ehr}(k\cdot \triangle^{d-1}, n) = \binom{nk+d-1}{d-1}$, we obtain by (I.2.1)

$$\psi_j(k\cdot \triangle^{d-1}) = \sum_{i=0}^{j} (-1)^i \binom{d}{i} \binom{k(j-i)+d-1}{d-1}.$$

Among the triangulations of $k\cdot \triangle^{d-1}$ constructed by the inductive procedure of the previous section there is a coherent unimodular triangulation which is directly induced by an infinite hyperplane arrangement. The families of hyperplanes

$$H_{i,j}(k) = \{\mathbf{x} \in \mathbb{R}^{d-1} \,:\, x_i - x_j = k\} \quad (i,j \in \mathbb{Z}) \text{ and}$$
$$H_i(k) = \{\mathbf{x} \in \mathbb{R}^{d-1} \,:\, x_i = k\} \; (i \in \mathbb{Z})$$

form an arrangement $\mathfrak{H}$ that triangulates the entire ambient space $\mathbb{R}^{d-1}$ coherently and unimodularly. Since the facets of $k\cdot \Delta^{d-1}$ span hyperplanes of $\mathfrak{H}$, it is enough to consider the restriction $\mathcal{T}$ of this $\mathfrak{H}$–triangulation to $k\cdot \Delta^{d-1}$. The advantage of this new triangulation is the uniform nature of its vertex stars. Choose a triangulation–vertex, which is not a vertex of $k\cdot \Delta^{d-1}$, and translate it to the origin. Then the cones generated by the simplices that contain the given vertex determine a fan defining an exceptional prime divisor of $\phi_{\mathcal{T}} : X_{\mathcal{T}} \to U_\sigma$. In particular, the compactly supported exceptional prime divisors of $\phi_{\mathcal{T}}$ correspond to the vertices lying in the interior of $k\cdot \Delta^{d-1}$, and the star of each of them is nothing but the $\mathfrak{H}$–triangulation of a polytope which is lattice equivalent to the lattice zonotope

$$W^{(d-1)} = \operatorname{conv}\left([-1,0]^{d-1} \cup [0,1]^{d-1}\right).$$

This shows that each compactly supported exceptional prime divisor of $\phi_{\mathcal{T}}$ comes from the crepant full $\mathfrak{H}$–desingularization of a projective toric Fano variety, namely $X_{\mathcal{N}((W^{(d-1)})^\vee)}$. The lattice points in $(W^{(d-1)})^\vee$ are the origin together with the $d(d-1)$ points $\pm\mathbf{e}_i$ and $\mathbf{e}_i - \mathbf{e}_j$, for $i \neq j$. The corresponding points in the cone $\sigma((W^{(d-1)})^\vee)$ form a Hilbert basis, because $(W^{(d-1)})^\vee$ admits a unimodular cover (even a triangulation). Hence the target space of the embedding described in § I.4.2 is $\mathbb{P}^{d(d-1)}$. The degree of this embedding is $\binom{2(d-1)}{d-1}$, the normalized volume of $(W^{(d-1)})^\vee$: The intersection of $(W^{(d-1)})^\vee$ with the cube $[-1,0]^{d_-} \times [0,1]^{d_+}$ $(d_- + d_+ = d-1)$ is the product of a d_- and a d_+–dimensional unimodular simplex that has the normalized volume $\binom{d-1}{d_-}$. Coordinate permutations yield $\binom{d-1}{d_-}$ such intersections. Adding up, we obtain $\sum_{d_-} \binom{d-1}{d_-}\binom{d-1}{d_-} = \binom{2(d-1)}{d-1}$.

Products of intervals. Our second class of examples is the family of hyper–intervals $[\mathbf{0},\mathbf{a}] = [0,a_1]\times\cdots\times[0,a_d]$ for integers $a_i > 0$. These polytopes have the Nakajima description $P_{\mathfrak{g}}$ by the admissible sequence $\mathfrak{g} = (a_1\mathbf{e}_1,\ldots,a_d\mathbf{e}_1)$ and for all i achieve the maximal number of i–faces any d–dimensional Nakajima polytope can have. The cone spanned by this $P_{\mathfrak{g}}$ is $\sigma = \left\{\mathbf{x} \in \mathbb{R}^{d+1} \,:\, 0 \le x_i \le a_i x_{d+1}\right\}$,

$$\sigma^\vee = \operatorname{pos}\left(\mathbf{e}_1,\ldots,\mathbf{e}_d, a_1\mathbf{e}_{d+1} - \mathbf{e}_1,\ldots,a_d\mathbf{e}_{d+1} - \mathbf{e}_d\right),$$

and $\sigma^\vee \cap \mathbb{Z}^{d+1}$ is generated by $\mathcal{H} = \{\mathbf{f}_1,\ldots,\mathbf{f}_d,\mathbf{g}_1,\ldots,\mathbf{g}_d,\mathbf{h}\}$ with $\mathbf{f}_i = \mathbf{e}_i$ and $\mathbf{g}_i = a_i\mathbf{e}_{d+1} - \mathbf{e}_i$ for $i = 1,\ldots,d$, and $\mathbf{h} = \mathbf{e}_{d+1}$. The d linear relations $(\mathbf{f}_i + \mathbf{g}_i) - (a_i\mathbf{h}) = 0$ form a basis for the lattice of integral linear dependences of $\mathcal{H}$. Introducing one variable u_i for each $\mathbf{f}_i$, v_i for $\mathbf{g}_i$ and t for $\mathbf{h}$, these give rise to the equations $u_iv_i - t^{a_i} = 0$.

Using the multiplicative behavior of the Ehrhart polynomial for products of polytopes, one obtains the following formula for the ψ–vector:

$$\psi_j([\mathbf{0}, \mathbf{a}]) = \sum_{i=0}^{j} (-1)^j \binom{d+1}{j} \prod_{\nu=1}^{d} \big((j-i)a_\nu + 1\big).$$

Note that this example also admits the $\mathfrak{H}$–triangulation discussed before.

CHAPTER IV

Stringy Hodge numbers of hypersurfaces in toric varieties

1. Introduction

1.1. Hypersurfaces. The objects of study in this chapter are compactified generic hypersurfaces in projective toric varieties. To define these, we start with a lattice polytope $P \subset \mathbb{R}^d$. Every lattice point $\mathbf{m} \in \mathbb{Z}^d$ can be identified with a *Laurent monomial* $\mathbf{t}^{\mathbf{m}} = t_1^{m_1} \dots t_d^{m_d}$, which is a regular function on the d–dimensional algebraic torus $T = (\mathbb{C}^*)^d$. Consider the $\mathbf{Ehr}(P,1)$–dimensional vector space $\mathcal{L}(P)$ of $\mathbb{C}$–linear combinations $f = \sum_{\mathbf{m} \in P \cap \mathbb{Z}^d} \lambda_{\mathbf{m}} \mathbf{t}^{\mathbf{m}}$. The zero locus $Z_f \hookrightarrow T$ of such an f is a *hypersurface* in the torus. The projective toric variety $X_P := X_{\mathcal{N}(P)}$ contains T as an open and dense subset. The closure $\overline{Z}_f$ in the compactification X_P of T is a *compactified hypersurface*. The data f, Z_f and $\overline{Z}_f$ are called *generic* (or, more precisely P*–regular*) if for every face $F \preceq P$ the intersection $\overline{Z}_f \cap T_F$ with the $(\dim F)$–dimensional torus corresponding to F is either empty or a smooth hypersurface of T_F. The name is justified by the fact that the set of Laurent polynomials f, for which $\overline{Z}_f$ is generic, is open and dense in $\mathcal{L}(P)$ with respect to Zariski topology.

In the sequel we only write $\overline{Z} \hookrightarrow X_P$ and mean that the respective statement holds for every generic hypersurface. For a comprehensive treatment of these objects consult Batyrev [Bat94].

1.2. Hodge numbers. Let X be a compact smooth complex variety. Then there is a *pure Hodge decomposition* of its cohomology.

$$\mathbf{H}^k(X,\mathbb{Q}) \otimes \mathbb{C} = \bigoplus_{p+q=k} \mathbf{H}^{p,q}(X),$$

such that $\mathbf{H}^{p,q}(X) = \overline{\mathbf{H}^{q,p}(X)}$ (complex conjugation in $\mathbb{C}$). The dimensions

$$h^{p,q}(X) := \dim_{\mathbb{C}} \mathbf{H}^{p,q}(X,\mathbb{C})$$

are called the (*usual*) *Hodge numbers* of X (cf. [GH94]). They are symmetric, $h^{p,q} = h^{q,p}$, they satisfy Serre duality $h^{p,q} = h^{\dim X - p, \dim X - q}$, and they vanish unless $0 \le p, q \le \dim X$. For a general complex variety X Deligne [Del71, Del74] constructed a so–called *mixed Hodge structure*, giving rise to *Hodge–Deligne numbers* $h^{p,q}(\mathbf{H}^k(X,\mathbb{Q}))$, that agree with the usual ones in the smooth compact case: $h^{p,q}(\mathbf{H}^k(X,\mathbb{Q})) = \delta_{k,p+q} h^{p,q}(X)$. The same theory can be built on cohomology with compact support $\mathbf{H}^*_c(X,\mathbb{Q})$, which is the right theory for our computations.

1.1. Definition (Danilov and Khovanskiĭ [DK87]). Define the *E–polynomial* of a complex variety X by

$$e^{p,q}(X) := \sum_{k=0}^{2\dim_{\mathbb{C}} X} (-1)^k h^{p,q}(\mathbf{H}^k_c(X,\mathbb{Q})) ,$$

$$E(X;u,v) := \sum_{p,q=0}^{\dim_{\mathbb{C}} X} e^{p,q}(X) u^p v^q .$$

Observe that in the smooth compact case $e^{p,q}(X) = (-1)^{p+q} h^{p,q}(X)$, so the Hodge numbers can be read off from the E–polynomial. This polynomial has the following nice properties.

1.2. Theorem [DK87].

- $E(\{\text{point}\};u,v) = 1$.
- $E(X_1 \sqcup X_2;u,v) = E(X_1;u,v) + E(X_2;u,v)$ *for disjoint locally closed subvarieties* $X_1, X_2 \subset X$.
- $E(X_1 \times X_2;u,v) = E(X_1;u,v) \cdot E(X_2;u,v)$. □

So E–polynomials should be easy to compute for nicely stratified varieties.

1.3. Corollary [DK87]. *Let $\phi : \widehat{X} \longrightarrow X$ be a locally trivial fibering (in Zariski topology) with fiber F. Then*

$$E(\widehat{X};u,v) = E(X;u,v)\ E(F;u,v).$$

Proof. We use induction on the cardinality n of a trivializing open cover $X = \bigcup_{i=1}^n X_i$. For $n = 1$ the assertion is included in Theorem 1.2. For $n > 1$ decompose $X = (X' \setminus X_n) \sqcup (X' \cap X_n) \sqcup (X_n \setminus X')$ with $X' = \bigcup_{i=1}^{n-1} X_i$ and apply the induction hypothesis to the restrictions of ϕ. □

Theorem 1.2 enables us to calculate the first important E–polynomials.

1.4. Example. The projective line $\mathbb{P}^1$ is a smooth compact variety. As a real manifold it is a 2–sphere. The only non–trivial cohomology groups are $\mathbf{H}^0(\mathbb{P}^1;\mathbb{Q}) = \mathbf{H}^2(\mathbb{P}^1;\mathbb{Q}) = \mathbb{Q}$. This already determines its Hodge numbers. They have to be $h^{0,0}(\mathbb{P}^1) = h^{1,1}(\mathbb{P}^1) = 1$ and $h^{0,1}(\mathbb{P}^1) = h^{1,0}(\mathbb{P}^1) = 0$. So

$$\begin{array}{lcl} E(\mathbb{P}^1;u,v) & = & uv+1 , \\ E(\mathbb{C} = \mathbb{P}^1 \setminus \{\infty\};u,v) & = & uv , \\ E(\mathbb{C}^* = \mathbb{P}^1 \setminus \{0,\infty\};u,v) & = & uv-1 , \\ E((\mathbb{C}^*)^d;u,v) & = & (uv-1)^d . \end{array}$$

□

A *Calabi–Yau variety* X is a normal compact variety with at most Gorenstein canonical singularities, $K_X = 0$ and $\mathbf{H}^1(X, \mathcal{O}_X) = \ldots = \mathbf{H}^{\dim X - 1}(X, \mathcal{O}_X) = 0$. The *mirror duality conjecture* in theoretical physics suggests that every smooth d–dimensional Calabi–Yau variety X should have a family of "mirror partners" X°, whose Hodge numbers satisfy

$$h^{p,q}(X) = h^{\dim X - p,q}(X^\circ). \tag{1.1}$$

Among the many references concerning the mathematical aspects of mirror symmetry we recommend [CK99, Mor97].

In many known mirror constructions, mirror partners have singularities. In this case the mirror symmetry identity (1.1) is expected to hold for crepant resolutions instead — if such exist. One could try to replace smooth crepant desingularizations by maximal crepant partial desingularizations. Although these always exist, they are not unique and their cohomology groups and Hodge numbers are not determined by the underlying spaces. The way out is to consider new Hodge numbers $h^{p,q}_{Str}$ instead. They should satisfy three properties:

- $h^{p,q}_{Str}(X) = h^{p,q}(X)$ for smooth X,
- $h^{p,q}(\widehat{X}) = h^{p,q}_{Str}(X)$ for a crepant desingularization $\widehat{X} \longrightarrow X$,
- $h^{p,q}_{Str}(X) = h^{d-p,q}_{Str}(X^\circ)$.

Consider an affine Gorenstein toric variety $U_{\sigma(P)}$ that admits an invariant crepant desingularization $\phi : X_{\mathcal{T}} \to U_P$ induced by a unimodular triangulation $\mathcal{T}$ of P. We should have $h^{p,q}_{Str}(U_{\sigma(P)}) = h^{p,q}(X_{\mathcal{T}})$. So let us calculate the E–polynomial of $X_{\mathcal{T}}$.

1.5. Proposition (Batyrev and Dais [BD96]). *Let $\phi : X_{\mathcal{T}} \to U_{\sigma(P)}$ be an invariant crepant desingularization. Then*

$$E(X_{\mathcal{T}}; u, v) = \sum_{F \preceq P} E(T_F; u, v) \Psi(F; uv). \tag{1.2}$$

Cf. I.2.3 for the Ψ–polynomial of a lattice polytope.

Proof. The stratification $U_{\sigma(P)} = \bigsqcup_{F \preceq P} T_F$ into tori induces a stratification of $X_{\mathcal{T}} = \bigsqcup_{F \preceq P} \phi^{-1}(T_F)$. The stratum $\phi^{-1}(T_F)$ is itself stratified by tori $T_{\mathfrak{s}}$ of dimension $d - \dim \mathfrak{s}$ corresponding to simplices $\mathfrak{s} \in \mathcal{T}$ with $\operatorname{relint} \mathfrak{s} \subseteq \operatorname{relint} F$. (Denote by $f_k(\mathcal{T}_{\operatorname{relint} F})$ the number of such k–simplices.) Then

$$E(\phi^{-1}(T_F); u, v) = \sum_{k=0}^{\dim F} f_k(\mathcal{T}_{\operatorname{relint} F})(uv - 1)^{d-k}.$$

On the other hand we can count lattice points in dilations of F by inclusion/exclusion

$$\begin{aligned}\Psi(F;uv) &= (1-uv)^{\dim F+1}\sum_{n\geq 0}\mathbf{Ehr}(F,n)(uv)^n\\ &= (1-uv)^{\dim F+1}\sum_{n\geq 0}\sum_{k=0}^{\dim F}(-1)^{\dim F-k}f_k(\mathcal{T}_{\mathrm{relint}\,F})\binom{n+k}{k}(uv)^n\\ &= (1-uv)^{\dim F+1}\sum_{k=0}^{\dim F}(-1)^{\dim F-k}f_k(\mathcal{T}_{\mathrm{relint}\,F})\frac{1}{(1-uv)^{k+1}}\\ &= \sum_{k=0}^{\dim F}f_k(\mathcal{T}_{\mathrm{relint}\,F})(uv-1)^{\dim F-k}.\end{aligned}$$

Both expressions differ by a factor $(uv-1)^{d-\dim F}=E(T_F;u,v)$. Hence,

$$\begin{aligned}E(\phi^{-1}(T_F);u,v) &= E(T_F;u,v)\Psi(F;uv),\\ E(X_{\mathcal{T}};u,v) &= \sum_{F\preceq P}E(T_F;u,v)\Psi(F;uv).\end{aligned}$$

$\square$

The restriction $\phi:\phi^{-1}(T_F)\to T_F$ is T_F invariant. In general, given a group G and a G–equivariant map $\phi: X\twoheadrightarrow G$ onto G, the group operation $(g,x)\mapsto gx$ is an isomorphism $G\times\phi^{-1}(1_G)\to X$. Thus $\phi^{-1}(T_F)\cong\phi^{-1}(\text{point}\in T_F)\times T_F$, and we recover $\Psi(F;uv)$ as the E–polynomial of any fiber $\phi^{-1}(\text{point})$.

The right hand side of (1.2) can also be computed if $U_{\sigma(P)}$ does not admit any crepant desingularization. This motivates the following definition.

1.6. Definition [BD96]. Let $X=\bigsqcup X_i$ be a stratified variety with Gorenstein singularities such that (X,X_i) is locally isomorphic to $(\mathbb{C}^d\times U_{\sigma(P_i)},\mathbb{C}^d\times\mathbf{0})$. Then the polynomial

$$E_{Str}(X;u,v)=\sum e^{p,q}_{Str}(X)u^pv^q:=\sum E(X_i;u,v)\Psi(P_i;uv)$$

is the *string theoretic E–polynomial* and the numbers $h^{p,q}_{Str}(X):=(-1)^{p+q}e^{p,q}_{Str}(X)$ are the *string theoretic Hodge numbers*.

Though up to now these are mere formally defined numbers, they have many properties which one could expect from cohomology dimensions. They are non–negative and symmetric, they satisfy Serre duality and also our three conditions. It is expected that there is some *string theoretic cohomology theory* that produces these numbers [BB96, Remark 4.4]. Also, there already is a more or less concrete candidate, conjectured by Borisov [Bor, 9.23].

Stringy Hodge numbers are preserved by (maximal) crepant partial desingularizations.

1.7. Theorem [BD96, Theorem 6.12]. *Let $\widehat{X} \to X$ be a maximal crepant partial desingularization of a variety X with (at most) toroidal Gorenstein singularities. Then $h^{p,q}_{Str}(\widehat{X}) = h^{p,q}_{Str}(X)$.* □

If P is reflexive, then any generic hypersurface $\overline{Z} \hookrightarrow X_P$ is Calabi–Yau [Bat94, Theorem 4.1.9]. Good candidates for mirror partners are the generic hypersurfaces $\overline{Z}^\vee \hookrightarrow X_{P^\vee}$.

1.8. Theorem [BB96, Theorem 4.15]. *Any pair of compactified generic hypersurfaces $\overline{Z} \hookrightarrow X_P$ and $\overline{Z}^\vee \hookrightarrow X_{P^\vee}$ satisfies the (stringy) mirror symmetry identity $h^{p,q}_{Str}(\overline{Z}) = h^{d-p-1,q}_{Str}(\overline{Z}^\vee)$.* □

1.3. Danilov–Khovanskiĭ formulae. In this paragraph we will outline the principal ideas of Danilov and Khovanskiĭ [DK87], who gave formulae for the usual Hodge numbers of a generic hypersurface in a smooth projective toric variety in terms of the corresponding lattice polytope.

As our toric varieties X_P are stratified by tori — one $(\dim F)$–dimensional torus for the $(d - \dim F)$–dimensional normal cone corresponding to each face $F \preceq P$ — we have:

1.9. Proposition. *Let $P \subset \mathbb{R}^d$ be a d–dimensional lattice polytope and X_P the associated projective toric variety. Then*

$$E(X_P; u, v) = \sum_{F \preceq P} (uv - 1)^{\dim F}, \qquad \text{and thus}$$

$$e^{p,q}(X_P) = \delta_{p,q}(-1)^p \sum_{k=p}^{d} (-1)^k f_k(P) \binom{k}{p}.$$

For large p, q this result carries over to generic hypersurfaces $\overline{Z} \hookrightarrow X_P$ by the following Lefschetz–type theorem.

1.10. Theorem [DK87]. *Let $P \subset \mathbb{R}^d$ be a d–dimensional lattice polytope and $\overline{Z} \hookrightarrow X_P$ a compactified generic hypersurface in the associated projective toric variety. If $p + q > d - 1$, then $e^{p,q}(\overline{Z}) = e^{p+1,q+1}(X_P)$.* □

For the rest of the section we assume $\mathcal{N}(P)$ to be unimodular. Then both $\overline{Z}$ and X_P are smooth so that Poincaré duality holds:

$$e^{p,q}(\overline{Z}) = e^{d-p-1,d-q-1}(\overline{Z}).$$

This yields the $e^{p,q}(\overline{Z})$ for $p+q < d-1$. The only remaining $e^{p,q}$'s are the ones on the diagonal $p+q = d-1$ in the Hodge diamond Figure 1.

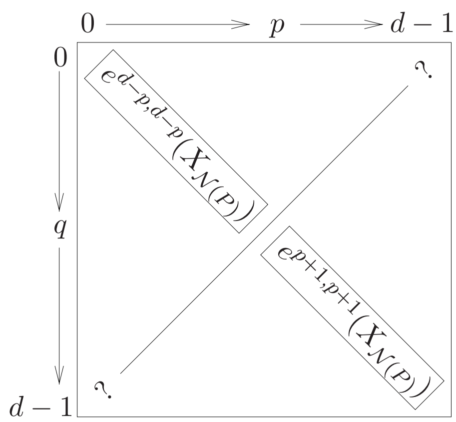


FIGURE 1. The $e^{p,q}$–diamond.

In order to evaluate these, it is sufficient to know the sums $\sum_q e^{p,q}(\overline{Z})$ over the columns, which for each p have at most two non–zero summands.

1.11. PROPOSITION [DK87]. *Let $P \subset \mathbb{R}^d$ be a d–dimensional lattice polytope and let $f \in \mathcal{L}(P)$ be a P–regular Laurent polynomial. Then*

$$(-1)^{d-1} \sum_q e^{p,q}(Z_f \hookrightarrow (\mathbb{C}^*)^d) = (-1)^p \binom{d}{p+1} + \psi_{p+1}(P).$$

□

The left–hand side equals the Euler characteristic of a certain sheaf on $\overline{Z}$ that Danilov and Khovanskiĭ compute "in a fairly standard way." [DK87, p. 290]

1.12. COROLLARY [DK87]. *Let $P \subset \mathbb{R}^d$ be a d–dimensional lattice polytope and $\overline{Z} \hookrightarrow X_P$ a compactified generic hypersurface in the associated projective toric variety. Then*

$$\sum_q e^{p,q}(\overline{Z}) = (-1)^p \sum_{F \preceq P} (-1)^{\dim F+1} \binom{\dim F}{p+1} + \sum_{F \preceq P} (-1)^{\dim F+1} \psi_{p+1}(F).$$

□

Observe that the first summand is just the contribution $e^{p+1,p+1}(X_P)$.

The following theorem reduces the computation of Hodge numbers to a purely combinatorial task. It is the main tool for the calculation of string theoretic Hodge numbers in the next section.

1.13. THEOREM [DK87, § 5.5]. *Let P be a d–polytope with unimodular normal fan, and let $\overline{Z} \hookrightarrow X_P$ be any generic compactified hypersurface. Then the Hodge numbers $h^{p,q}(\overline{Z})$ vanish unless $p = q$ or $p + q = d - 1$.*
For $2p > d - 1$,

$$h^{p,p}(\overline{Z}) = (-1)^p \sum_{k=p+1}^{d} (-1)^{k+1} \binom{k}{p+1} f_k(P),$$

$$h^{p,d-p-1}(\overline{Z}) = \sum_{F \preceq P} (-1)^{\dim F+1} \psi_{p+1}(F).$$

For $2p = d - 1$,

$$h^{p,p}(\overline{Z}) = (-1)^p \sum_{k=p+1}^{d} (-1)^{k+1} \binom{k}{p+1} f_k(P) + \sum_{F \preceq P} (-1)^{\dim F+1} \psi_{p+1}(F).$$

For $2p < d - 1$ the Hodge numbers may be obtained by duality:

$$h^{p,q}(\overline{Z}) = h^{d-p-1,d-q-1}(\overline{Z}).$$

□

2. Symmetric Fano polytopes

In this section we apply Theorem 1.13 in order to explicitly compute stringy Hodge numbers $h^{p,q}_{Str}(\overline{Z} \hookrightarrow X_P)$ for a whole family of reflexive polytopes P. At the same time we will show that the corresponding projective toric varieties admit projective crepant resolutions, that induce crepant resolutions of the generic $\overline{Z}$'s.

2.1. The classification. The polytopes for which we want to carry out the calculation are the pseudo–symmetric Fano polytopes. They constitute the only substantial, infinite class of reflexive polytopes for which a classification is available [Ewa96, Ewa88]. Before we do this, we have to fight our way through some terminology.

A polytope P is *centrally symmetric* if $P = -P$, *pseudo–symmetric* if it has two facets F, F' satisfying $F = -F'$, *Fano* if it is reflexive and all its proper faces are unimodular simplices. Let $\mathbb{1} = \sum \mathbf{e}_i$ denote the all–one–vector. For even $d \geq 2$ call

- $\mathrm{DP}^d = \mathrm{conv}(\pm \triangle^d, \pm\mathbb{1})$ the *del Pezzo polytope* and
- $\mathrm{preDP}^d = \mathrm{conv}(\pm \triangle^d, \mathbb{1})$ the *pre del Pezzo polytope*.

Let $P \subseteq \mathbb{R}^d$ and $P' \subseteq \mathbb{R}^{d'}$ be full–dimensional polytopes with $\mathbf{0}$ in their interior. We define

$$P \oplus P' = \mathrm{conv}(P \times \{\mathbf{0}\} \cup \{\mathbf{0}\} \times P') \quad \subseteq \mathbb{R}^{d+d'},$$

and say that $P \oplus P'$ *splits* into P and P'. In the sequel we will identify P with

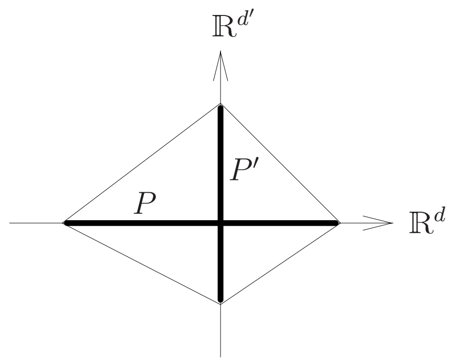


FIGURE 2. $P \oplus P'$.

$P \times \{\mathbf{0}\}$ and also P' with $\{\mathbf{0}\} \times P'$. The polar operation is the Cartesian product:

$$(P \oplus P')^\vee = P^\vee \times P'^\vee.$$

If P and P' are Fano polytopes, then so is $P \oplus P'$. One example for this construction is the d–dimensional *crosspolytope*

$$\Diamond^d := [-1, 1] \oplus \ldots \oplus [-1, 1] \qquad (d \text{ components}),$$

with its dual, the d–dimensional ± 1–*cube*, $\square^d := [-1,1]^d$.

Fight won; we can formulate:

2.1. THEOREM [Ewa96, Ewa88]. *Let $P \subseteq \mathbb{R}^d$ be a Fano polytope.*

- *If P is centrally symmetric, then it is equivalent to* $\Diamond^{d_0} \oplus \mathrm{DP}^{d_1} \oplus \ldots \oplus \mathrm{DP}^{d_r}$.
- *If P is pseudo–symmetric, then there is some centrally symmetric Fano polytope P', such that P is equivalent to* $P' \oplus \mathrm{preDP}^{d_1} \oplus \ldots \oplus \mathrm{preDP}^{d_r}$. □

We will achieve two things. We give give formulae for the $h^{p,q}_{\mathit{Str}}$ of a generic hypersurface within X_P, when P is a pseudo–symmetric Fano polytope, and we show that there is a crepant desingularization for them. Then our formulae compute the usual Hodge numbers of these smooth varieties.

Our strategy for the first goal is to compute instead $h^{d-p-1,q}_{\mathit{Str}}(\overline{Z}^\vee)$ of the mirror $(\overline{Z}^\vee \hookrightarrow X_{P^\vee})$, because $\mathcal{N}(P^\vee)$, whose cones are spanned by the faces of P, is unimodular so that we can apply Theorem 1.13. The input that we need are the ψ–vectors of all faces of $P^\vee$. So we have to analyze the components $((\mathrm{pre})\mathrm{DP}^d)^\vee$ and $[-1,1] = [-1,1]^\vee$, i.e., identify their faces and compute the Ehrhart polynomials on the one hand and investigate the behavior of the input data with respect to Cartesian products on the other.

By the results of § III.1.2, the existence of crepant desingularizations boils down to the existence of a unimodular triangulation of $\mathcal{N}(P)$ that is induced by triangulations of the polytopes that span the $\sigma \in \mathcal{N}(P)$, namely the faces of $P^\vee$. So, with both tasks we are left with questions about $P^\vee$.

2.2. The components. The triangulation part is an application of the up to now inappropriately unpublished Lemma 2.3, due to Francisco Santos (personal communication (1997)). The counting part is based on a nice geometric inclusion/exclusion argument.

Triangulations. The interval $[-1,1]$ admits the obvious coherent unimodular triangulation, whose unique minimal non–face is the non–edge $\{-1,1\}$. The fact that the polars of the (pre) del Pezzo's also admit such triangulations is embedded into a more general context. Once more we use the refinement of a given subdivision by pulling lattice points (cf. § I.2.2).

2.2. DEFINITION. Let P be a lattice polytope and let $\langle \mathbf{y}_i, \mathbf{x} \rangle \geq c_i$ be the facet defining inequalities with primitive integral $\mathbf{y}_i$. Then P has *facet width* $\max_i \mathrm{width}_{\mathbf{y}_i}(P)$. (Compare Section II.1 for the concept of width with respect to a given functional.)

In particular, P has facet width 1 if for every facet P lies between the hyperplane spanned by this facet and the next parallel lattice hyperplane.

2.3. Proposition (Paco's Lemma). *If the lattice polytope P has facet width 1, then every pulling triangulation of P is unimodular.*

Proof. By decreasing induction on the dimension one sees that every face of P has facet width 1. The restriction of a pulling triangulation to any face is a pulling triangulation itself and thus unimodular (by another induction). Hence, every maximal simplex in the triangulation of P is the join of a unimodular simplex in some facet with the first lattice point that was pulled. □

2.4. Proposition. *Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\} \subset \mathbb{Z}^d$ be a collection of vectors that span $\mathbb{R}^d$ and form a totally unimodular matrix, i.e., such that all $(d \times d)$–minors are either 0, 1 or -1. Then for any choice of integers c_i the polyhedron*

$$P = \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{v}_i, \mathbf{x} \rangle \geq c_i\}$$

admits a coherent unimodular triangulation.

Proof. The hyperplanes $H_i(k) = \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{v}_i, \mathbf{x} \rangle = k\}$ for integers k form an arrangement that subdivides P (and all of $\mathbb{R}^d$) coherently into polytopes. The vertices of these polytopes are lattice points by the determinant condition. Moreover, these polytopes have facet width 1. □

The vertices of the (pre) del Pezzo's satisfy the condition of Proposition 2.4. The polar polytope is then obtained by choosing all $c_i's$ to be -1. There is no further pulling necessary, because the hyperplane subdivision is already a triangulation. The minimal non–faces of such arrangement triangulations are easily controlled.

2.5. Proposition. *If the triangulation $\mathcal{T}$ of the polytope P is induced by a hyperplane arrangement, its minimal non–faces consist of 1 or 2 points. If the triangulation is unimodular, then all possible lattice points are used so that all minimal non–faces consist of 2 points.*

Proof. A subset $F \subset P \cap \mathbb{Z}^d$ spans a face of $\mathcal{T}$ if and only if every element of F is a vertex of $\mathcal{T}$ and all of F lies in the same halfspace with respect to each of the hyperplanes. So F is a non–face if and only if it contains a non–vertex or there is a hyperplane with respect to which two elements of F lie in different halfspaces. □

2.6. Corollary. *The polars of the (pre) del Pezzo's admit coherent unimodular triangulations whose minimal non–faces consist of 2 points.*

Ehrhart polynomials. As a warmup treat the interval $[-1,1]$. The Ehrhart polynomial is $\mathbf{Ehr}([-1,1],k)=2k+1$. Hence $\psi_0([-1,1])=\psi_1([-1,1])=1$. Fit as we are now, we can endure some more notation. Denote by

$$\binom{n}{k_1,\ldots,k_r}=\frac{n!}{k_1!\cdots k_r!},\qquad (k_1+\ldots+k_r=n),$$

the multinomial coefficient: the number of ordered partitions of an n–set into r sets of the respective sizes k_i. In this notation, our familiar binomial coefficient $\binom{n}{k}$ is also denoted by $\binom{n}{k,n-k}$ — so far the notational break.

The polar polytopes of the (pre) del Pezzos have the following description by inequalities:

$$(\mathrm{DP}^d)^\vee=\{\mathbf{x}\in[-1,+1]^d\,:\,\sum x_i\in[-1,+1]\}\quad\text{respectively}$$
$$(\mathrm{preDP}^d)^\vee=\{\mathbf{x}\in[-1,+1]^{d-1}\times(-\infty,1]\,:\,\sum x_i\in[-1,+1]\}.$$

They are lattice equivalent to

$$\begin{aligned}&\{\mathbf{x}\in[-1,+1]^{d+1}\,:\,\sum x_i=0\}\quad\text{respectively}\\ &\{\mathbf{x}\in[-1,+1]^{d}\times(-\infty,1]\,:\,\sum x_i=0\}.\end{aligned}\tag{2.1}$$

The (pre) del Pezzos are simplicial, such that the $((\mathrm{pre})\mathrm{DP}^d)^\vee$ are simple polytopes (d is assumed to be even). This means that every $(d-k)$–face of the latter is the intersection of k facets. Consider first the faces of $(\mathrm{DP}^d)^\vee$. The facet defining inequalities in (2.1) are either of the form $x_i\le 1$ or $x_i\ge -1$. A face F satisfies some of these inequalities with equality, say for $i\in I_+\subseteq\{1,\ldots,d+1\}$ the first one and for $i\in I_-$ the second one:

$$F=\{\mathbf{x}\in[-1,+1]^{d+1}\,:\,\sum x_i=0, x_i=1(i\in I_+), x_i=-1(i\in I_-)\}.$$

We say that such a face is *of type* $(d;s=\mathrm{card}(I_-),t=\mathrm{card}(I_+))$. It has dimension $d'=d-s-t$ and it is lattice equivalent to

$$F(d;s,t)=\{\mathbf{x}\in[-1,+1]^{d'+1}\,:\,\sum x_i=s-t\}.$$

If $s,t\le d/2$, then there are $\binom{d+1}{d'+1,s,t}$ such faces (otherwise $F(d;s,t)$ is empty).

In the face poset of $(\mathrm{preDP}^d)^\vee$ there are other faces showing up. The facet defining inequalities are the same as in the case of $(\mathrm{DP}^d)^\vee$, but the inequality $x_{d+1}\ge -1$ is missing. So, if $d+1\in I_+$, the considered face is of type $(d;s,t)$ and there are $\binom{d}{d'+1,s,t-1}$ such faces, provided $s,t\le d/2$. But if the $(d+1)$–st coordinate is not fixed, we get a new kind of faces. They are equivalent to

$$F'(d;s,t)=\{\mathbf{x}\in[-1,+1]^{d'}\times(-\infty,1]\,:\,\sum x_i=s-t\}.$$

If $s\le d/2$, then there are $\binom{d}{d',s,t}$ faces of that kind.

2.7. PROPOSITION. *The Ehrhart polynomials of $F(d;s,t)$ respectively $F'(d;s,t)$ are the following.*

$$\mathbf{Ehr}(F(d;s,t),k) = \sum_{r=0}^{d/2-t} (-1)^r \binom{d'+1}{r} \binom{(d-2t-2r+1)k+d'-r}{d'},$$

$$\mathbf{Ehr}(F'(d;s,t),k) = \sum_{r=0}^{d/2-t} (-1)^r \binom{d'}{r} \binom{(d-2t-2r+1)k+d'-r}{d'}.$$

PROOF. The polytope $F(d;s,t)+\mathbb{1}$ is a subset of the dilated standard simplex $(d-2t+1)\cdot \triangle^{d'}$, as illustrated in Figure 3 for $F(4;1,0)+\mathbb{1} \subset 5\cdot \triangle^3$.

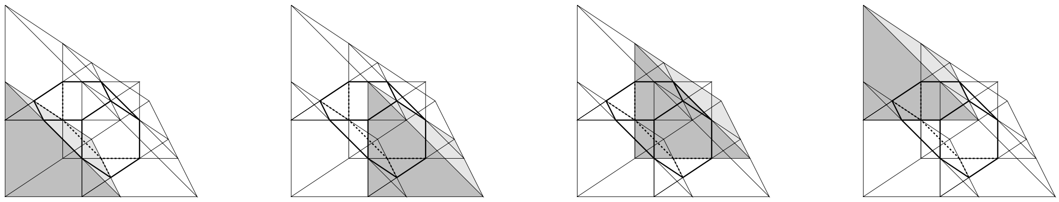


FIGURE 3. $F(d;s,t)+\mathbb{1}$ inscribed in $(d-2t+1)\,\triangle^{d'}$, and what juts out.

We want to count the integral points in $kF(d;s,t)+k\mathbb{1}$. The Ehrhart polynomial is determined by its values for large k. So suppose from now on $k > d/2-t$. If we denote by M_j the set of those points of $k\cdot(d-2t+1)\cdot \triangle^{d'}$ whose j–th coordinate exceeds $2k+1$:

$$M_j = \{\mathbf{x} \in \mathbb{Z}_{\geq 0}^{d'+1} \;:\; \sum x_i = k\cdot(d-2t+1), x_j \geq 2k+1\},$$

then the points we seek are the same as the integral points in

$$\mathbb{Z}^{d'+1} \cap k\cdot(d-2t+1)\cdot \triangle^{d'} \smallsetminus \bigcup_{j=1}^{d'+1} M_j.$$

Thus we have to count lattice points in $M_{j_1\dots j_r} = M_{j_1} \cap \ldots \cap M_{j_r}$. Up to a translation by $-(2k+1)\sum \mathbf{e}_{j_i}$ this is just a simplex (cf. Figure 4.):

$$M_{j_1\dots j_r} - (2k+1)\sum \mathbf{e}_{j_i} = \begin{cases} \big(k\cdot(d-2t-2r+1)-r\big)\cdot \triangle^{d'} & \text{if } r \leq d/2-t, \\ \varnothing & \text{if } r > d/2-t. \end{cases}$$

Hence, $M_{j_1\dots j_r}$ contains $\mathbf{Ehr}\big(\triangle^{d'}, k\cdot(d-2t-2r+1)-r\big) = \binom{d'-r+k(d-2t-2r+1)}{d'}$ lattice points (provided $r \leq d/2-t$), and the formula follows by inclusion/exclusion. The argument for $F'(d;s,t)$ is analogous. $\square$

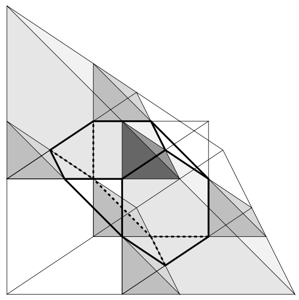


FIGURE 4. Intersecting prominent parts.

In order to deduce the Ehrhart series we have to compute the following kind of sums:

$$\sum_{k\geq 0}\binom{pk+q}{n}\tau^k = \frac{\sum_{i=0}^{n}\begin{bmatrix} n & i \\ p & q\end{bmatrix}\tau^i}{(1-\tau)^{n+1}}.$$

The coefficients

$$\begin{bmatrix} n & i \\ p & q\end{bmatrix} := \sum_{j=0}^{i}(-1)^j\binom{n+1}{j}\binom{pi-pj+q}{n}$$

are *non–central Eulerian numbers* [Cha82]. We obtain:

2.8. PROPOSITION.

$$\psi_k(F(d;s,t)) = \sum_{r=0}^{d/2-t}(-1)^r\binom{d'+1}{r}\begin{bmatrix} d' & k \\ d-2t-2r+1 & d'-r\end{bmatrix},$$

$$\psi_k(F'(d;s,t)) = \sum_{r=0}^{d/2-t}(-1)^r\binom{d'}{r}\begin{bmatrix} d' & k \\ d-2t-2r+1 & d'-r\end{bmatrix}.$$

□

2.3. Cartesian products. In Section 2.2 we finished the difficult part. It remains to multiply the components and to see what happens.

Triangulations. It is easy to define the product of two subdivisions. Unfortunately, the product of two triangulations is not a triangulation. It is a subdivision into products of simplices. Nevertheless:

2.9. Proposition. *Let P and P' be lattice polytopes. If both admit a (coherent) unimodular triangulation (whose minimal non–faces consist of 2 points), then so does $P \times P'$.*

Proof. Any triangulation that refines the obvious (coherent) subdivision of $P \times P'$ into products of unimodular simplices is unimodular, because any triangulation of $\triangle^{d_1} \times \triangle^{d_2}$ is. See e.g. [Stu96, p.72, Ex.(9)] or [Lee97, p. 282]. By the way, this product is a Nakajima polytope and it has facet width 1.

In order to preserve the non–face property, we have to be (a little) more careful. Order the lattice points $p_1, \dots, p_r$ in P and $p'_1, \dots, p'_s$ in P'. Then pull the lattice points (p_i, p'_j) in $P \times P'$ lexicographically.

Consider a non–face F. If one of its projections to P or P' is a non–face, we can lift a non–edge to F. Otherwise, F is a non–face in the staircase triangulation of a product of simplices (cf. [Lee97, p. 282]), i.e., there must be two points $(p, p'), (q, q') \in F$ that are not comparable in lexicographic order. $\square$

The Propositions 2.5 and 2.9 generalize a result of Bruns, Gubeladze and Trung [BGT97, Theorem 2.3.10]. Together they imply:

2.10. Corollary. *Let $P \subset \mathbb{R}^d$ be a pseudo–symmetric Fano polytope. Then its polar $P^\vee$ admits a coherent unimodular triangulation whose minimal non–faces consist of 2 points. In particular, $P^\vee$ is Koszul.* $\square$

Retranslated into algebraic geometry this reads

2.11. Theorem. *The toric Fano varieties X_P associated with pseudo–symmetric Fano polytopes P admit projective crepant resolutions.* $\square$

Ehrhart polynomials. The faces of $P \times P'$ are the sets of the form $F \times F'$ for faces F, F' of P, P'. Clearly

$$\mathbf{Ehr}((F \times F'), n) = \mathbf{Ehr}(F, n) \cdot \mathbf{Ehr}(F', n).$$

In order to calculate the ψ–vector of a general face

$$F = \square^d \times F(d_1; s_1, t_1) \times \cdots \times F(d_p; s_p, t_p) \times F'(d'_1; s'_1, t'_1) \times \cdots \times F'(d'_q; s'_q, t'_q),$$

one uses (I.2.1):

$$
\begin{aligned}
\psi_j(F) = \sum_{i=0}^{j} (-1)^i \binom{\dim F + 1}{i} \quad & (2(j-i)+1)^d \\
& \cdot \prod_{\nu=1}^{p} \mathbf{Ehr}(F(d_\nu; s_\nu, t_\nu), j-i) \\
& \cdot \prod_{\mu=1}^{q} \mathbf{Ehr}(F'(d'_\mu; s'_\mu, t'_\mu), j-i).
\end{aligned}
$$

This will get a little bit (too) complicated — especially if we plug it into 1.13. But it yields the complete answer. We spare you one full page of bird tracks. Let us rather look at specific examples instead.

2.4. Examples. Now that we have the ψ–vectors at hand, we would like to actually compute stringy Hodge numbers. Consider the del Pezzos, and let $\overline{Z}^\vee \hookrightarrow X_{(\mathrm{DP}^d)^\vee}$ be a generic hypersurface. Then for $p \geq d/2$,

$$
h^{p,p}(\overline{Z}) = h^{d-p-1,d-p-1}(\overline{Z}) = (-1)^{p+1} \sum_{k=p+1}^{d} 2^{d-k} \binom{d+1}{k+1} \binom{k}{p+1},
$$

and

$$
\begin{aligned}
h^{p,d-p-1}&(\overline{Z}) = h^{d-p-1,p}(\overline{Z}) \\
&= \sum_{s,t=0}^{d/2} \sum_{r=0}^{d/2-t} (-1)^{d'-r+1} \binom{d+1}{d'-r+1, s, t, r} \begin{bmatrix} d' & p+1 \\ d-2t-2r+1 & d'-r \end{bmatrix},
\end{aligned}
$$

where we set once more $d' = d - s - t$. Now we can apply mirror symmetry to obtain the stringy Hodge diamonds for a generic $\overline{Z} \hookrightarrow X_{\mathrm{DP}^d}$. They are depicted according to the pattern of Figure 1 for the first interesting values of d:

$$
d = 4: \qquad
\begin{array}{|cccc|}
\hline
1 & & & 1 \\
 & 6 & 46 & \\
 & 46 & 6 & \\
1 & & & 1 \\
\hline
\end{array}
$$

$$
d = 6: \qquad
\begin{array}{|cccccc|}
\hline
1 & & & & & 1 \\
 & 8 & & & 386 & \\
 & & 29 & 4187 & & \\
 & & 4187 & 29 & & \\
 & 386 & & & 8 & \\
1 & & & & & 1 \\
\hline
\end{array}
$$

$$
d = 8: \quad \boxed{\begin{array}{cccccccc}
1 & & & & & & & 1 \\
 & 10 & & & & & 3130 & \\
 & & 46 & & & 138070 & & \\
 & & & 130 & 743290 & & & \\
 & & & 743290 & 130 & & & \\
 & & 138070 & & & 46 & & \\
 & 3130 & & & & & 10 & \\
1 & & & & & & & 1
\end{array}}
$$

3. Pyramids over Fano polytopes

In the previous section we were able to compute stringy Hodge numbers by the shortcut via Theorem 1.13, because we treated a simple/simplicial pair $(P, P^\vee)$. In this section we want to do the computation for a pyramid over the crosspolytope $\diamond^d$. Given a d–polytope $P \subset \mathbb{R}^d$, define the *pyramid over* P (reflexive version) to be the $(d+1)$–dimensional polytope

$$\operatorname{pyr} P = \operatorname{conv}\big((P \times \{-1\}) \cup (\mathbf{0} \times \{1\})\big) \subset \mathbb{R}^{d+1}.$$

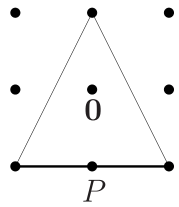


FIGURE 5. $\operatorname{pyr} P$.

Unless P is a simplex, this is neither a simple nor a simplicial polytope. So we cannot use Theorem 1.13.

Luckily, other people prepared the soil for us. In order to prove the stringy mirror duality for complete intersections in toric Fano varieties, Batyrev and Borisov [BB96] developed the following formula for the stringy E–polynomial of the generic hypersurfaces $\overline{Z} \hookrightarrow X_P$ associated with a reflexive polytope $P \subset \mathbb{R}^d$.

$$(3.1)\quad (-1)^{d+1} uv\, E_{Str}(\overline{Z}; u, v) = \sum_{(\mathbf{m},\mathbf{n}) \in \Lambda(\sigma,\sigma^\vee)} \left(\frac{u}{v}\right)^{\operatorname{ht}(\mathbf{m})} \left(\frac{1}{uv}\right)^{\operatorname{ht}(\mathbf{n})} (v-u)^{d(\mathbf{m})} (1-uv)^{d(\mathbf{n})} B(\mathcal{P}_{\mathbf{m},\mathbf{n}}; u, v).$$

Before we continue, we have to explain what the symbols mean. Recall the cones $\sigma = \sigma(P)$ and $\sigma^\vee = \sigma(P^\vee)$ used in order to projectively embed X_P and $X_{P^\vee}$ (§ I.4.2). The sum ranges over $\Lambda(\sigma, \sigma^\vee)$ — those pairs of lattice points $(\mathbf{m}, \mathbf{n}) \in \sigma \times \sigma^\vee$ for which $\langle \mathbf{m}, \mathbf{n} \rangle = 0$. This is an infinite sum; it will become clear in a moment that the expression is indeed a polynomial.

By $\operatorname{ht}(\mathbf{m}) := m_{d+1}$ and $\operatorname{ht}(\mathbf{n}) := n_{d+1}$ we denote the degrees of $\mathbf{m}$ and $\mathbf{n}$ in the respective graded semigroup(algebra)s. We write $d(\mathbf{m})$ and $d(\mathbf{n})$ for the dimensions $\dim \tau_{\mathbf{m}}$ and $\dim \tau_{\mathbf{n}}$ of the faces $\tau_{\mathbf{m}} \preceq \sigma$ and $\tau_{\mathbf{m}} \preceq \sigma^\vee$ that contain $\mathbf{m}$ respectively $\mathbf{n}$ in their relative interior.

$\mathcal{P}_{\mathbf{m},\mathbf{n}}$ is the part of the face poset of $\sigma^\vee$ that consists of all the faces of $\sigma^\vee$ that contain $\mathbf{n}$ and lie in the kernel of $\mathbf{m}$. The B–polynomial $B(\mathcal{P}; u, v)$ is a polynomial that depends on an Eulerian poset $\mathcal{P}$; this is the subject of § 3.1.

Before we take a closer look on Eulerian posets and B–polynomials, let us first reformulate Equation (3.1). The summand that corresponds to $(\mathbf{m}, \mathbf{n}) \in \Lambda(\sigma, \sigma^\vee)$

only depends on the cones $\tau_{\mathbf{m}}$ spanned by $F_{\mathbf{m}} \preceq P$ respectively $\tau_{\mathbf{n}}$ spanned by $F_{\mathbf{n}} \preceq P^\vee$ on the one hand and on the heights $m := \mathrm{ht}(\mathbf{m})$ and $n := \mathrm{ht}(\mathbf{n})$ on the other. In particular, the poset $\mathcal{P}_{\mathbf{m},\mathbf{n}}$ is an interval in the face poset of $\sigma^\vee$; $[\tau_{\mathbf{n}}, \tau_{\mathbf{m}}^\vee] := \{\tau \preceq \sigma^\vee : \tau_{\mathbf{n}} \preceq \tau \preceq \tau_{\mathbf{m}}^\vee\}$. There are $\mathbb{Ehr}(F_{\mathbf{m}}, m)$ lattice points with height m in the relative interior of $\sigma_{\mathbf{m}}$ and $\mathbb{Ehr}(F_{\mathbf{n}}, n)$ lattice points with height n in the relative interior of $\sigma_{\mathbf{n}}$.

$$
\begin{aligned}
&(-1)^{d+1} uv\ E_{Str}(\overline{Z}; u, v) \\
&\qquad = \sum_{\tau_{\mathbf{m}} \preceq \sigma} (v-u)^{\dim \tau_{\mathbf{m}}} \sum_{m \geq 0} \mathbb{Ehr}(F_{\mathbf{m}}, m) \left(\frac{u}{v}\right)^m \\
&\qquad\qquad \cdot \sum_{\tau_{\mathbf{n}} \prec \tau_{\mathbf{m}}^\vee} B([\tau_{\mathbf{n}}, \tau_{\mathbf{m}}^\vee]; u, v) \\
&\qquad\qquad \cdot (1-uv)^{\dim \tau_{\mathbf{n}}} \sum_{n \geq 0} \mathbb{Ehr}(F_{\mathbf{n}}, n) \left(\frac{1}{uv}\right)^n .
\end{aligned}
\tag{3.2}
$$

The remaining infinite sums are inner Ehrhart series (cf. § I.2.3) so that we obtain

$$
\begin{aligned}
&(-1)^{d+1} uv\ E_{Str}(\overline{Z}; u, v) \\
&\qquad = \sum_{F_{\mathbf{m}} \preceq P} \sum_{j=1}^{\dim F_{\mathbf{m}}+1} \varphi_j(F_{\mathbf{m}}) u^j v^{\dim F_{\mathbf{m}} - j + 1} \\
&\qquad\qquad \cdot \sum_{F_{\mathbf{n}} \preceq F_{\mathbf{m}}^\vee} B([F_{\mathbf{n}}, F_{\mathbf{m}}^\vee]; u, v) \\
&\qquad\qquad \cdot (-1)^{\dim F_{\mathbf{n}}+1} \sum_{j=1}^{\dim F_{\mathbf{n}}+1} \varphi_j(F_{\mathbf{n}}) (uv)^{\dim F_{\mathbf{n}} - j + 1}.
\end{aligned}
\tag{3.3}
$$

This is in fact a polynomial.

3.1. Eulerian posets. Let $\mathcal{P}$ be a finite poset with unique minimal element $\hat{0}$ and unique maximal element $\hat{1}$. An *interval* is a subposet of the form $[p, p'] = \{q \in \mathcal{P} : p \leq q \leq p'\}$. We will use the shorthand $[p, p]$ to denote the one point poset. Define recursively the *Möbius function* μ on such posets by $\mu[p, p] = 1$ and $\sum_{q \in [p,p']} \mu[p, q] = 0$ for $p < p'$. Call $\mathcal{P}$ *graded* if every maximal chain has the same length $\mathrm{rk}(\mathcal{P})$. If $\mathcal{P}$ is graded then so is every interval. Define the *rank function* $\mathrm{rk}(p)$ to be the length of a maximal chain in $[\hat{0}, p]$. A graded poset $\mathcal{P}$ is *Eulerian* [Sta97, § 3.14] if for any pair $p \leq p'$ in $\mathcal{P}$ we have $\mu[p, p'] = (-1)^{\mathrm{rk}(p') - \mathrm{rk}(p)}$.

The face poset of a pointed polyhedral cone σ is an example of an Eulerian poset with minimum $\{\mathbf{0}\}$, maximum σ, and rank function $\mathrm{rk}(\tau) = \dim \tau$.

3.1. Definition. Let $\mathcal{P}$ be an Eulerian poset of rank r. Define recursively polynomials $G(\mathcal{P};t)$ by

$$G(\mathcal{P};t) = 1 \qquad \text{if } r = 0,$$
$$\deg G(\mathcal{P};t) < r/2 \ \text{ and } \ \sum_{p\in\mathcal{P}} (t-1)^{\mathrm{rk}(p)} G([p,\hat{1}];t) \equiv 0 \mod t^{\lceil r/2\rceil} \ \text{ if } \ r \geq 1.$$

Use these polynomials to define the polynomial $B(\mathcal{P};u,v)$ by

$$B(\mathcal{P};u,v) = 1 \ \text{ if } \ r = 0,$$
$$\sum_{p\in\mathcal{P}} B([\hat{0},p];u,v) u^{r-\mathrm{rk}(p)} G([p,\hat{1}],\frac{v}{u}) = G(\mathcal{P};uv) \ \text{ if } \ r \geq 1.$$

One way of producing new posets from old ones is to take the Cartesian product with the component–wise partial order. If $\mathcal{P}$ and $\mathcal{P}'$ are Eulerian, then so is $\mathcal{P}\times\mathcal{P}'$. In the case of face posets, the product operation corresponds to the join of cones.

A map H that associates to every (isomorphism class of an) Eulerian poset $\mathcal{P}$ an element $H(\mathcal{P})$ of a fixed ring (e.g. $\mathbb{Z}[t_1 \dots t_r]$) is *multiplicative* if

$$H(\mathcal{P}\times\mathcal{P}') = H(\mathcal{P})H(\mathcal{P}')$$

for every pair of Eulerian posets. Observe that $H[p,p] = 1$ in an integral domain, unless H vanishes on all Eulerian posets.

3.2. Lemma. *Given H, H_1, H_2, with multiplicative H, such that*

$$H_1[p,p] = 1, \quad H_2[p,p] = 1, \quad \textit{and} \quad \sum_{p\in\mathcal{P}} H_1[\hat{0},p]H_2[p,\hat{1}] = H(\mathcal{P}).$$

Then H_1 is multiplicative if and only if H_2 is.

Proof. Use induction on the pairs $(\mathrm{rk}(\mathcal{P}), \mathrm{rk}(\mathcal{P}'))$. Suppose that H_1 is multiplicative, and use that intervals in product posets can be decomposed into products $[(p,q),(p',q')] = [p,p']\times[q,q']$.

$$\begin{aligned}
0 &= H(\mathcal{P}\times\mathcal{P}') - H(\mathcal{P})H(\mathcal{P}') \\
&= \sum_{(p,p')\in\mathcal{P}\times\mathcal{P}'} H_1([\hat{0},p]\times[\hat{0},p'])H_2([p,\hat{1}]\times[p',\hat{1}]) - H_1[\hat{0},p]\ H_1[\hat{0},p']\ H_2[p,\hat{1}]\ H_2[p',\hat{1}] \\
&= H_1([\hat{0},\hat{0}]\times[\hat{0},\hat{0}])H_2(\mathcal{P}\times\mathcal{P}') \ - \ H_1[\hat{0},\hat{0}]\ H_1[\hat{0},\hat{0}]\ H_2(\mathcal{P})H_2(\mathcal{P}') \\
&= H_2(\mathcal{P}\times\mathcal{P}') \ - \ H_2(\mathcal{P})H_2(\mathcal{P}').
\end{aligned}$$

The converse is shown the same way. □

3.3. Corollary. *The B–polynomial is multiplicative.*

PROOF. The proof of Lemma 3.2 applies to G in the following way. The polynomials $H = 0$ and $H_1(\mathcal{P}) = (t-1)^{\mathrm{rk}(\mathcal{P})}$ are multiplicative. The second equality holds only up to terms of degree $\geq \lceil\frac{\mathrm{rk}(\mathcal{P})+\mathrm{rk}(\mathcal{P}')}{2}\rceil$ for the first summand and $\geq \lceil\frac{\mathrm{rk}(\mathcal{P})}{2}\rceil + \lceil\frac{\mathrm{rk}(\mathcal{P}')}{2}\rceil$ for the second. Altogether this is a congruence modulo terms of degree $\geq \lceil\frac{\mathrm{rk}(\mathcal{P})+\mathrm{rk}(\mathcal{P}')}{2}\rceil$, which is sufficient as it dominates the degree of both $G(\mathcal{P}\times\mathcal{P}';t)$ and of $G(\mathcal{P};t)G(\mathcal{P}';t)$. Another application of Lemma 3.2 yields the multiplicativity of B. □

Now the B–polynomial of a *boolean algebra* $\mathcal{B}_r$ (the face poset of an r–dimensional simplicial cone) is easy to compute. As $\mathcal{B}_{r_1+r_2} = \mathcal{B}_{r_1}\times\mathcal{B}_{r_2}$, it is sufficient to consider the two element poset $\mathcal{B}_1$. One gets $G(\mathcal{B}_1;t) = 1$ and $B(\mathcal{B}_1;u,v) = 1-u$ and by multiplicativity:

3.4. LEMMA. $B(\mathcal{B}_r;u,v) = (1-u)^r$.

3.2. Pyramids. The face poset of the pyramid $\mathrm{pyr}\,P$ over a d–polytope is isomorphic to the product of the face poset of P with $\mathcal{B}_1$: The pair $(F,\hat{0})$ corresponds to the face $F\times\{-1\}$ and the pair $(F,\hat{1})$ corresponds to the face $\mathbf{e}_{d+1}\cdot(F\times\{-1\}) := \mathrm{conv}(\mathbf{e}_{d+1}\cup(F\times\{-1\}))$. If $\langle\mathbf{y},\mathbf{x}\rangle\geq -1$ is a facet defining inequality for P, then $2\langle\mathbf{y},\mathbf{x}\rangle - x_{d+1}\geq -1$ defines the corresponding apex–containing facet of $\mathrm{pyr}\,P$. The only non–apex–containing facet $P\times\{-1\}$ of $\mathrm{pyr}\,P$ is defined by $x_{d+1}\geq -1$. This shows that $(\mathrm{pyr}\,P)^\vee = \mathrm{pyr}\,(2P^\vee)$ and $\mathrm{pyr}\,P$ is in fact a reflexive polytope if P is.

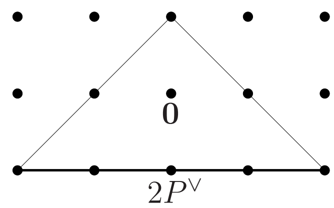


FIGURE 6. $(\mathrm{pyr}\,P)^\vee$.

The apex–containing faces $\mathbf{e}_{d+1}\cdot(F\times\{-1\})$ of $\mathrm{pyr}\,P$ are lattice equivalent to $\mathrm{conv}(F\cup\mathbf{0})\subset\mathbb{R}^d$. To see this, let $\langle\mathbf{y},\mathbf{x}\rangle\geq -1$ be a facet defining inequality that holds with equality on F. The lattice equivalence $(\mathbf{x},x_{d+1})\mapsto(\mathbf{x},x_{d+1}-2\langle\mathbf{y},\mathbf{x}\rangle)$ maps $\mathbf{0}\mapsto\mathbf{0}$ and $F\times\{-1\}\mapsto F\times\{0\}$. Similarly one can see that the apex–containing faces $\mathbf{e}_{d+1}\cdot(2F\times\{-1\})$ of $\mathrm{pyr}\,(2P^\vee)$ are lattice equivalent to $\mathrm{conv}(2F\cup\mathbf{0})\subset\mathbb{R}^d$.

If P admits a unimodular triangulation then so does $\mathrm{pyr}\,P$. The subdivision of the normal fan $\mathcal{N}\big((\mathrm{pyr}\,P)^\vee\big)$ induced by this triangulation is the join (in the sense of [Ewa96, III, 1.12-14]) of the fan Σ whose cones are spanned by the simplices in the faces of $P\times\{-1\}$ with the fan $\Sigma' = \big\{\varrho,-\varrho,\{\mathbf{0}\}\big\}$.

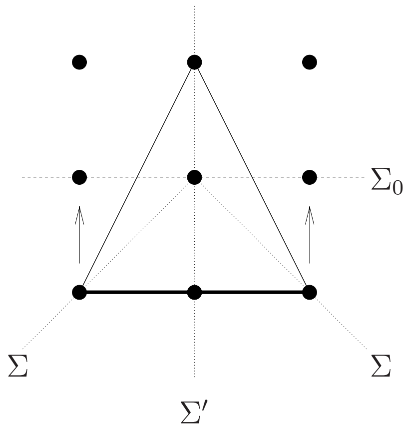


FIGURE 7. $\mathcal{N}\big((\operatorname{pyr} P)^\vee\big) = \Sigma \cdot \Sigma'$.

In this decomposition, Σ has the fan Σ_0 spanned by the simplices in the faces of $P \times \{0\}$ as a projection fan (in the sense of [Ewa96, VI, 6.5]). Hence we see that the crepant resolution $X_{\Sigma\cdot\Sigma'}$ of $X_{(\operatorname{pyr} P)^\vee}$ is a fiber bundle over the crepant resolution X_{Σ_0} of $X_{P^\vee}$ with fiber $\mathbb{P}^1 = X_{\Sigma'}$.

The hypersurfaces we are looking at do not have such a fiber bundle structure. We have to get down to earth to evaluate (3.3) for them. Denote by ϱ the ray spanned by $\mathbf{e}_{d+1}$ (in both $\mathbb{R}^{d+1}$ and in $(\mathbb{R}^{d+1})^\vee$), by σ the cone spanned by $P \times \{-1\}$, and by $\operatorname{pyr}\sigma$ the cone spanned by $\operatorname{pyr} P$. In order to evaluate (3.3) we distinguish 9 cases.

1. $\tau_{\mathbf{m}} \prec \sigma, \quad \varrho \prec \tau_{\mathbf{n}}$.
2. $\tau_{\mathbf{m}} \prec \sigma, \quad \tau_{\mathbf{n}} = \varrho$.
3. $\tau_{\mathbf{m}} \prec \sigma, \quad \varrho \not\prec \tau_{\mathbf{n}} \neq \{\mathbf{0}\}$.
4. $\tau_{\mathbf{m}} \prec \sigma, \quad \tau_{\mathbf{n}} = \{\mathbf{0}\}$.
5. $\tau_{\mathbf{m}} = \sigma, \quad \tau_{\mathbf{n}} = \varrho$.
6. $\tau_{\mathbf{m}} = \sigma, \quad \tau_{\mathbf{n}} = \{\mathbf{0}\}$.
7. $\varrho \prec \tau_{\mathbf{m}} \neq \operatorname{pyr}\sigma, \quad \varrho \not\prec \tau_{\mathbf{n}} \neq \{\mathbf{0}\}$.
8. $\varrho \prec \tau_{\mathbf{m}} \neq \operatorname{pyr}\sigma, \quad \tau_{\mathbf{n}} = \{\mathbf{0}\}$.
9. $\tau_{\mathbf{m}} = \operatorname{pyr}\sigma, \quad \tau_{\mathbf{n}} = \{\mathbf{0}\}$.

The big sum decomposes into 9 sums; we will group them in the following way: 1,3,7 and 2,4,8 and 5,6,9. This makes sense because in each group the summation runs over sets of pairs $(\tau_{\mathbf{m}}, \tau_{\mathbf{n}})$ that allow a canonical bijection. A pair $(\tau_{\mathbf{m}}, \tau_{\mathbf{n}})$ in Case 3 corresponds to the pair $(\tau_{\mathbf{m}}, \varrho \cdot \tau_{\mathbf{n}})$ in Case 1 and to the pair $(\varrho \cdot \tau_{\mathbf{m}}, \tau_{\mathbf{n}})$ in Case 7 (and vice versa). Similarly one obtains pairs in Cases 2 and 8 from a pair in Case 4 respectively the pairs in Cases 5 and 9 from the pair in Case 6.

The sum in the Cases 1,3,7. In these three cases we sum over pairs $(\tau_{\mathbf{m}}, \tau_{\mathbf{n}})$, where $\tau_{\mathbf{m}} \prec \sigma$ and $\varrho \not\prec \tau_{\mathbf{n}} \neq \{\mathbf{0}\}$. Then $\tau_{\mathbf{m}}$ is equivalent to the cone spanned by some face $F_{\mathbf{m}} \prec P$, and $\tau_{\mathbf{n}}$ is equivalent to the cone spanned by some face $\varnothing \neq F_{\mathbf{n}} \preceq F_{\mathbf{m}}^{\vee} \preceq P^{\vee}$, where duality takes place in $\mathbb{R}^d$.

In Case 1 we then have to evaluate the summand for the pair $(F_{\mathbf{m}}, \mathbf{e}_{d+1} \cdot F_{\mathbf{n}})$, in Case 3 for $(F_{\mathbf{m}}, F_{\mathbf{n}})$, and in Case 7 for $(\mathbf{e}_{d+1} \cdot F_{\mathbf{m}}, F_{\mathbf{n}})$. Each summand is a product of three factors. We investigate how they are affected when $\mathbf{e}_{d+1}$ is joined to $F_{\mathbf{m}}$ respectively to $F_{\mathbf{n}}$.

- $\sum_{j=1}^{\dim F_{\mathbf{m}}+1} \varphi_j(F_{\mathbf{m}}) u^j v^{\dim F_{\mathbf{m}}-j+1}$:
  The cone $\varrho \cdot \tau_{\mathbf{m}}$ is spanned by (a polytope that is lattice equivalent to) $\mathbf{0} \cdot F_{\mathbf{m}}$. There are $\mathbb{E}\mathrm{hr}(\mathbf{0} \cdot F_{\mathbf{m}}, k) = \sum_{j=1}^{k-1} \mathbb{E}\mathrm{hr}(F_{\mathbf{m}}, j)$ lattice points interior to the dilations of this polytope. This affects the φ–vector as follows.

$$\begin{aligned} \sum_{k\geq 1} \sum_{j=1}^{k-1} \mathbb{E}\mathrm{hr}(F,j) t^k &= \sum_{j\geq 1} \mathbb{E}\mathrm{hr}(F,j) \sum_{k\geq j+1} t^k \\ &= \sum_{j\geq 1} \mathbb{E}\mathrm{hr}(F,j) \frac{t^{j+1}}{1-t} \\ &= \frac{t}{1-t} \sum_{j\geq 1} \mathbb{E}\mathrm{hr}(F,j) t^j. \end{aligned}$$

  Hence we get a factor u:

$$\sum_{j=1}^{\dim F_{\mathbf{m}}+2} \varphi_j(\mathbf{e}_{d+1} \cdot F_{\mathbf{m}}) u^j v^{\dim F_{\mathbf{m}}-j+2} = u \sum_{j=1}^{\dim F_{\mathbf{m}}} \varphi_j(F_{\mathbf{m}}) u^j v^{\dim F_{\mathbf{m}}-j+1}.$$

- $B([\tau_{\mathbf{n}}, \tau_{\mathbf{m}}^{\vee}]; u, v)$:
  ϱ is a face of $\tau_{\mathbf{m}}$ if and only if ϱ on the dual side is not a face of $\tau_{\mathbf{m}}^{\vee}$. Joining ϱ to $\tau_{\mathbf{m}}$ corresponds under the above identification of face posets to descending from the pair $(F_{\mathbf{m}}^{\vee}, \hat{1})$ to $(F_{\mathbf{m}}^{\vee}, \hat{0})$. In a similar way, joining ϱ to $\tau_{\mathbf{n}}$ corresponds to ascending from $(F_{\mathbf{n}}, \hat{0})$ to $(F_{\mathbf{n}}, \hat{1})$. We have the following isomorphisms.

$$[\tau_{\mathbf{n}}, \tau_{\mathbf{m}}^{\vee}] \cong \mathcal{B}_1 \times [\tau_{\mathbf{n}}, (\varrho \cdot \tau_{\mathbf{m}})^{\vee}] \cong \mathcal{B}_1 \times [\varrho \cdot \tau_{\mathbf{n}}, \tau_{\mathbf{m}}^{\vee}] \cong \mathcal{B}_1 \times [F_{\mathbf{n}}, F_{\mathbf{m}}^{\vee}],$$

  where $F_{\mathbf{m}}^{\vee}$ is once more the face of $P^{\vee}$ dual to $F_{\mathbf{m}}$. This yields the identities

$$\begin{aligned} B([\tau_{\mathbf{n}}, \tau_{\mathbf{m}}^{\vee}]; u, v) &= (1-u) B([\tau_{\mathbf{n}}, (\varrho \cdot \tau_{\mathbf{m}})^{\vee}]; u, v) \\ &= (1-u) B([\varrho \cdot \tau_{\mathbf{n}}, \tau_{\mathbf{m}}^{\vee}]; u, v) \\ &= (1-u) B([F_{\mathbf{n}}, F_{\mathbf{m}}^{\vee}]; u, v). \end{aligned}$$

- $(-1)^{\dim F_{\mathbf{n}}+1} \sum_{j=1}^{\dim F_{\mathbf{n}}+1} \varphi_j(F_{\mathbf{n}}) (uv)^{\dim F_{\mathbf{n}}-j+1}$:
  For the moment we cannot say anything about this factor but that the sign changes if the dimension of $\tau_{\mathbf{n}}$ rises by 1.

Altogether the contribution of the Cases 1,3,7 is

$$\sum_{F_{\mathbf{m}} \prec P} \sum_{\varnothing \neq F_{\mathbf{n}} \preceq F_{\mathbf{m}}^{\vee}} \sum_{j=1}^{\dim F_{\mathbf{m}}+1} \varphi_j(F_{\mathbf{m}}) u^j v^{\dim F_{\mathbf{m}}-j+1} (-1)^{\dim F_{\mathbf{n}}+1} B([F_{\mathbf{n}}, F_{\mathbf{m}}^{\vee}]; u, v)$$

$$\Bigg(- \sum_{j=1}^{\dim F_{\mathbf{n}}+2} \varphi_j(\mathbf{0} \cdot (2F_{\mathbf{n}}))(uv)^{\dim F_{\mathbf{n}}+2-j} + (1-u) \sum_{j=1}^{\dim F_{\mathbf{n}}+1} \varphi_j(F_{\mathbf{n}})(uv)^{\dim F_{\mathbf{n}}-j+1} + u \sum_{j=1}^{\dim F_{\mathbf{n}}} \varphi_j(F_{\mathbf{n}})(uv)^{\dim F_{\mathbf{n}}-j+1} \Bigg)$$

$$= \sum_{F_{\mathbf{m}} \prec P} \sum_{\varnothing \neq F_{\mathbf{n}} \preceq F_{\mathbf{m}}^{\vee}} \sum_{j=1}^{\dim F_{\mathbf{m}}+1} \varphi_j(F_{\mathbf{m}}) u^j v^{\dim F_{\mathbf{m}}-j+1} (-1)^{\dim F_{\mathbf{n}}+1} B([F_{\mathbf{n}}, F_{\mathbf{m}}^{\vee}]; u, v)$$

$$\cdot \sum_{j=0}^{\dim F_{\mathbf{n}}+1} \Big(\varphi_j(F_{\mathbf{n}}) - \varphi_{j+1}(\mathbf{0} \cdot (2F_{\mathbf{n}})) \Big) (uv)^{\dim F_{\mathbf{n}}-j+1}.$$

The sum in the Cases 2,4,8. In these three cases we only sum over $\tau_{\mathbf{m}} \prec \sigma$ respectively $F_{\mathbf{m}} \prec P$, and take $\tau_{\mathbf{n}} = \{\mathbf{0}\}$ respectively $F_{\mathbf{n}} = \varnothing$.

In Case 2 we then have to evaluate the summand for the pair $(F_{\mathbf{m}}, \{\mathbf{e}_{d+1}\})$, in Case 4 for $(F_{\mathbf{m}}, \varnothing)$, and in Case 8 for $(\mathbf{0} \cdot F_{\mathbf{m}}, \varnothing)$.

- $\sum_{j=1}^{\dim F_{\mathbf{m}}+1} \varphi_j(F_{\mathbf{m}}) u^j v^{\dim F_{\mathbf{m}}-j+1}$:
  As before joining $\tau_{\mathbf{m}}$ with ϱ gives a factor u.
- $B([\tau_{\mathbf{n}}, \tau_{\mathbf{m}}^{\vee}]; u, v)$:
  $B([F_{\mathbf{n}}, F_{\mathbf{m}}^{\vee}]; u, v)$ in Cases 2 and 8, $(1-u)B([F_{\mathbf{n}}, F_{\mathbf{m}}^{\vee}]; u, v)$ in Case 4.
- $(-1)^{\dim F_{\mathbf{n}}+1} \sum_{j=1}^{\dim F_{\mathbf{n}}+1} \varphi_j(F_{\mathbf{n}})(uv)^{\dim F_{\mathbf{n}}-j+1}$:
  Easy: 1 in Cases 4 and 8, -1 in Case 2.

So this contribution is no contribution.

$$\sum_{F_{\mathbf{m}} \prec P} \sum_{j=1}^{\dim F_{\mathbf{m}}+1} \varphi_j(F_{\mathbf{m}}) u^j v^{\dim F_{\mathbf{m}}-j+1} B([\varnothing, F_{\mathbf{m}}^{\vee}]; u, v) \big(-1 + (1-u) + u \big) = 0.$$

The sum in the Cases 5,6,9. These three cases contain one pair each; $(P, \mathbf{e}_{d+1})$ in Case 5, $(P, \varnothing)$ in Case 6, and $(\operatorname{pyr} P, \varnothing)$ in Case 9.

- $\sum_{j=1}^{\dim F_{\mathbf{m}}+1} \varphi_j(F_{\mathbf{m}}) u^j v^{\dim F_{\mathbf{m}}-j+1}$:
  This time we have to count in $\operatorname{pyr} P$. The intersection of $k \operatorname{pyr} P$ with the hyperplane $\{x_{d+1} = j\}$ for $j \in [-k+1, k-1]$ is equivalent to $\frac{k-j}{2} P$. Thus, if $k - j = 2i$ is even, then there are $\mathbb{E}\mathrm{hr}(P, i)$ interior lattice points, and, if

$k-j=2i-1$ is odd, then there are $\mathbb{E}\mathrm{hr}(P,i)$ of them (recall that there are no lattice points between iP and $(i+1)P$).

$$\mathbb{E}\mathrm{hr}(\operatorname{pyr} P, k) = \sum_{i=1}^{k} \mathbb{E}\mathrm{hr}(P,i) + \sum_{i=1}^{k-1} \mathbb{E}\mathrm{hr}(P,i).$$

We can once more exchange the summation.

$$\begin{aligned}\sum_{k\geq 1}\left(\sum_{i=1}^{k} \mathbb{E}\mathrm{hr}(P,i) + \sum_{i=1}^{k-1} \mathbb{E}\mathrm{hr}(P,i)\right) t^k &= \sum_{i\geq 1} \mathbb{E}\mathrm{hr}(P,i)\left(\frac{t^j}{1-t}+\frac{t^{j+1}}{1-t}\right)\\ &= \frac{1+t}{1-t}\sum_{i\geq 1}\mathbb{E}\mathrm{hr}(P,i)t^i.\end{aligned}$$

Thus we get a factor of $v+u$ in Case 9.

- $B([\tau_{\mathbf{n}}, \tau_{\mathbf{m}}^{\vee}]; u, v)$:
  Easy: 1 in Cases 5 and 9, $1-u$ in Case 6.
- $(-1)^{\dim F_{\mathbf{n}}+1}\sum_{j=1}^{\dim F_{\mathbf{n}}+1}\varphi_j(F_{\mathbf{n}})(uv)^{\dim F_{\mathbf{n}}-j+1}$:
  Once more 1 in Cases 6 and 9, -1 in Case 5.

So there is a contribution

$$\sum_{j=1}^{d+1}\varphi_j(P)u^jv^{d+1-j}\big(-1+(1-u)+(u+v)\big) = \sum_{j=1}^{d+1}\varphi_j(P)u^jv^{d+2-j}.$$

Let us summarize what we have got.

3.5. Theorem. *Let $P\subset\mathbb{R}^d$ be a reflexive d–polytope. The stringy E–polynomial $E_{Str}(\overline{Z};u,v)$ of a generic hypersurface $\overline{Z}\hookrightarrow X_{\operatorname{pyr} P}$ equals the following expression.*

$$\begin{aligned}\frac{(-1)^d}{uv}\Bigg(&\sum_{j=1}^{d+1}\varphi_j(P)u^jv^{d+2-j}\\ &+\sum_{F_{\mathbf{m}}\prec P}\ \sum_{\varnothing\neq F_{\mathbf{n}}\preceq F_{\mathbf{m}}^{\vee}}\ \sum_{j=1}^{\dim F_{\mathbf{m}}+1}\varphi_j(F_{\mathbf{m}})u^jv^{\dim F_{\mathbf{m}}-j+1}(-1)^{\dim F_{\mathbf{n}}+1}B([F_{\mathbf{n}},F_{\mathbf{m}}^{\vee}];u,v)\\ &\qquad\cdot\sum_{j=0}^{\dim F_{\mathbf{n}}+1}\Big(\varphi_j(F_{\mathbf{n}})-\varphi_{j+1}(\mathbf{0}\cdot 2F_{\mathbf{n}})\Big)(uv)^{\dim F_{\mathbf{n}}-j+1}\Bigg).\end{aligned}$$

3.6. COROLLARY. *Let $P \subset \mathbb{R}^d$ be a d–dimensional Fano polytope. The stringy Hodge numbers $h^{p,q}_{Str}(\overline{Z} \hookrightarrow X_{\mathrm{pyr}\,P})$ of a generic hypersurface vanish unless $p = q$ or $p + q = d$. In the latter cases they are given by*

$$h^{p,p}_{Str}(\overline{Z}) = (-1)^{d+1} \sum_{\varnothing \neq F \preceq P^\vee} (-1)^{\dim F} \big(\psi_{p+1}(2F) - \psi_{p+1}(\mathbf{0} \cdot 2F)\big),$$

$$h^{p,d-p}_{Str}(\overline{Z}) = \varphi_{p+1}(P),$$

if $2p \neq d$. If $2p = d$ then one has

$$h^{p,p}_{Str}(\overline{Z}) = \varphi_{p+1}(P) + (-1)^{d+1} \sum_{\varnothing \neq F \preceq P^\vee} (-1)^{\dim F} \big(\psi_{p+1}(2F) - \psi_{p+1}(\mathbf{0} \cdot 2F)\big).$$

PROOF. If P is Fano, then all its proper faces are unimodular, with the effect that $\varphi_j(F_{\mathbf{m}}) = \delta_{j,\dim F_{\mathbf{m}}+1}$ and all intervals $[F_{\mathbf{n}}, F^\vee_{\mathbf{m}}]$ with $F_{\mathbf{n}} \neq \varnothing$ are boolean, with the effect that $B([F_{\mathbf{n}}, F^\vee_{\mathbf{m}}]; u, v) = (1-u)^{d-\dim F_{\mathbf{m}}-\dim F_{\mathbf{n}}-1}$. Exchange the summation to obtain

$$\frac{(-1)^d}{uv}\Bigg(\sum_{j=1}^{d+1} \varphi_j(P) u^j v^{d+2-j} + \sum_{\varnothing \neq F_{\mathbf{n}} \preceq P^\vee} (-1)^{\dim F_{\mathbf{n}}+1} \sum_{j=0}^{\dim F_{\mathbf{n}}+1} \Big(\varphi_j(2F_{\mathbf{n}}) - \varphi_{j+1}(\mathbf{0} \cdot 2F_{\mathbf{n}})\Big)(uv)^{\dim F_{\mathbf{n}}-j+1} \cdot \sum_{F_{\mathbf{m}} \preceq F^\vee_{\mathbf{n}}} u^{\dim F_{\mathbf{m}}+1}(1-u)^{d-\dim F_{\mathbf{m}}-\dim F_{\mathbf{n}}-1}\Bigg).$$

The summands of the last sum only depend on $\dim F_{\mathbf{m}}$. Because $F_{\mathbf{n}} \neq \varnothing$, $F^\vee_{\mathbf{n}}$ is a simplex, which has $\binom{d-\dim F_{\mathbf{n}}}{d'+1}$ d'–faces. So this last sum evaluates to

$$\sum_{d'=-1}^{d-\dim F_{\mathbf{n}}-1} \binom{d-\dim F_{\mathbf{n}}}{d'+1} u^{d'+1}(1-u)^{d-\dim F_{\mathbf{n}}-d'-1} = 1.$$

It remains to extract the coefficients of $(uv)^p$ and of $u^p v^{d-p}$. Use the fact that $\varphi_{\dim F_{\mathbf{n}}-p}(2F_{\mathbf{n}}) = \psi_{p+1}(2F_{\mathbf{n}})$ and $\varphi_{\dim F_{\mathbf{n}}-p+1}(\mathbf{0} \cdot 2F_{\mathbf{n}}) = \psi_{p+1}(\mathbf{0} \cdot 2F_{\mathbf{n}})$. □

If $P = \Diamond^d$ is the crosspolytope, then two faces of $P^\vee$ of the same dimension are equivalent. Furthermore they are reflexive themselves so that we can compute the expression $\psi_{p+1}(2F) - \psi_{p+1}(\mathbf{0} \cdot 2F)$ as follows.

$$\mathbf{Ehr}(\mathbf{0} \cdot 2F, k) = \sum_{j=0}^{2k} \mathbf{Ehr}(F, j).$$

It is slightly more subtle to exchange the summation.

$$\begin{aligned}\Psi(\mathbf{0}\cdot 2F;t) &= \sum_{k\geq 0}\sum_{j=0}^{2k}\mathbf{Ehr}(F,j)t^k = \sum_{j\geq 0}\mathbf{Ehr}(F,j)\sum_{k\geq\lceil\frac{j}{2}\rceil}t^k\\ &= \frac{1}{1-t}\left(\sum_{i\geq 0}\mathbf{Ehr}(F,2i)t^i + \sum_{i\geq 1}\mathbf{Ehr}(F,2i-1)t^i\right)\\ &= \frac{1}{1-t}\left(\Psi(2F;t) + \sum_{i\geq 1}\mathbb{Ehr}(F,2i)t^i\right)\\ &= \frac{1}{1-t}\big(\Psi(2F;t)+\Phi(2F;t)\big),\end{aligned}$$

where we used $\mathbf{Ehr}(F,2i-1)=\mathbb{Ehr}(F,2i)$. It follows that

$$\psi_{p+1}(2F)-\psi_{p+1}(\mathbf{0}\cdot 2F) = -\varphi_{p+1}(2F).$$

3.7. Corollary. *Let $\overline{Z}\hookrightarrow X_{\mathrm{pyr}\,\diamondsuit^d}$ be a generic hypersurface. Then*

$$\begin{aligned}h^{p,p}_{Str}(\overline{Z}) &= \sum_{d'=p}^{d}(-1)^{d-d'}\binom{d}{d'}2^{d-d'}\varphi_{p+1}(2\square^{d'}),\\ h^{p,d-p}_{Str}(\overline{Z}) &= \binom{d}{p},\end{aligned}$$

if $2p\neq d$. If $2p=d$, then

$$h^{p,p}_{Str}(\overline{Z}) = \binom{d}{p} + \sum_{d'=p}^{d}(-1)^{d-d'}\binom{d}{d'}2^{d-d'}\varphi_{p+1}(2\square^{d'}).$$

Proof. After the above considerations, all we have to do is to convince ourselves that

$$\varphi_{p+1}(\diamondsuit^d) = \binom{d}{p}.$$

The crosspolytope has an obvious unimodular triangulation with $\binom{d}{d'}2^{d'}$ simplices of dimension d' in its interior. Hence

$$\begin{aligned}
\Phi(\Diamond^d;t) &= (1-t)^{d+1}\sum_{k\geq 1}\left(\sum_{d'=0}^{d}\binom{d}{d'}2^{d'}\binom{k-1}{d'}\right)t^k\\
&= (1-t)^{d+1}\sum_{d'=0}^{d}\binom{d}{d'}2^{d'}\sum_{k\geq 1}\binom{k-1}{d'}t^k\\
&= (1-t)^{d+1}\sum_{d'=0}^{d}\binom{d}{d'}2^{d'}\frac{t^{d'+1}}{(1-t)^{d'+1}}\\
&= t\sum_{d'=0}^{d}\binom{d}{d'}(2t)^{d'}(1-t)^{d-d'} = t(t+1)^d.
\end{aligned}$$

$\square$

If we plug in

$$\varphi_p(2\square^{d'}) = \sum_{i=0}^{j-1}(-1)^i\binom{d'+1}{i}\big(4(j-i)-1\big)^{d'},$$

we get the following Hodge diamonds in small dimensions:

$$d=3:\quad \boxed{\begin{array}{ccc} 1 & & 1 \\ & 20 & \\ 1 & & 1 \end{array}}$$

$$d=4:\quad \boxed{\begin{array}{cccc} 1 & & & 1 \\ & 115 & 3 & \\ & 3 & 115 & \\ 1 & & & 1 \end{array}}$$

$$d=5:\quad \boxed{\begin{array}{ccccc} 1 & & & & 1 \\ & 612 & & 4 & \\ & & 2508 & & \\ & 4 & & 612 & \\ 1 & & & & 1 \end{array}}$$

$$d = 6: \quad \boxed{\begin{array}{cccccc}
1 & & & & & 1 \\
 & 3109 & & & 5 & \\
 & & 34154 & 10 & & \\
 & & 10 & 34154 & & \\
 & 5 & & & 3109 & \\
1 & & & & & 1
\end{array}}$$

This is the end of our tour. I hope you had fun and this little commercial convinced you to come back one of these days . . .

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"Grundsätzlich glaube ich nicht, daß Wahnsinn auf Mathematiker beschränkt ist." *John F. Nash jun.*