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Citation for published version (APA):

Florack, L. M. J. (2009). *Coarse-to-fine partitioning of signals*. (CASA-report; Vol. 0912). Technische Universiteit Eindhoven.

Document status and date:

Published: 01/01/2009

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

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Coarse-to-fine partitioning of signals

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An empirically acquired signal can be analyzed in a multi-scale framework. Its multi-scale structure induces a hierarchical partitioning of the signal domain into topologically meaningful segments. A method is proposed to operationalize this using elementary results from singularity theory for certain generic solutions of the one-dimensional heat equation.

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1 Introduction

We define the multi-scale extension $u(x, s)$ of a real-valued signal $f(x)$, and the associated signal $g(x, s)$, cf. Table 1, as

$$u(x, s) = \exp\left(s \frac{d^2}{dx^2}\right) f(x) \quad \text{resp.} \quad g(x, s) = \frac{1}{2} u_x^2(x, s).$$

The s -parameterized families $u(x, s)$ and $g(x, s)$ represent information contained in the raw signal $f(x)$ as a function of resolution (inverse “inner scale”), [1, 2]. The inner scale for resolving structure along the x -axis equals $\sigma \propto \sqrt{s}$. We write $u_0^{(k)}$ for a k -th order x -derivative at $(x, s) = (0, 0)$, assuming $u_0^{(1)} = 0$ henceforth.

a_{pq}	0	1	2
0	$u_0^{(1)} u_0^{(1)}$	$2u_0^{(1)} u_0^{(3)}$	$u_0^{(1)} u_0^{(5)} + u_0^{(3)} u_0^{(3)}$
1	$2u_0^{(1)} u_0^{(2)}$	$2u_0^{(1)} u_0^{(4)} + 2u_0^{(2)} u_0^{(3)}$	*
2	$u_0^{(1)} u_0^{(3)} + u_0^{(2)} u_0^{(2)}$	$u_0^{(1)} u_0^{(5)} + 2u_0^{(2)} u_0^{(4)} + u_0^{(3)} u_0^{(3)}$	*
3	$\frac{1}{3} u_0^{(1)} u_0^{(4)} + u_0^{(2)} u_0^{(3)}$	*	*
4	$\frac{1}{12} u_0^{(1)} u_0^{(5)} + \frac{1}{3} u_0^{(2)} u_0^{(4)} + \frac{1}{4} u_0^{(3)} u_0^{(3)}$	*	*

Table 1 Relevant coefficients of $2g(x, s) = \sum_{pq} a_{pq} x^p s^q$, $0 \leq p + 2q \leq 4$, with p as row index and q as column index.

We consider two partitioning methods, based on the spatial *critical paths* defined by $u_x(x, s) = 0$, respectively $g_x(x, s) = 0$.

1. $u_x(x, s) = 0$:

- $u_0^{(2)} \neq 0$ corresponds to a *regular critical path*.
- $u_0^{(2)} = 0$ indicates an *annihilation event*. In Thom’s “List of the Seven Elementary Catastrophes” [4] this represents a *fold catastrophe*, with control parameter s .
- *Inflection paths* defined by $u_{xx}(x, s) = 0$ provide *separatrices* in (x, s) -space, separating peaks (regions with a single maximum), dales (containing a single minimum), and void regions. They connect in a similar annihilation event.

2. $g_x(x, s) = 0$: This captures two types of critical points.

(a) Type I: $u_x(x, s) = 0$:

- $u_0^{(2)} \neq 0$ corresponds to a regular critical path.
- $u_0^{(2)} = 0$ corresponds to a “pitchfork”: 3 regular critical paths for $s < 0$ meet at the origin, leaving 1 for $s > 0$. In Thom’s list this represents a *fold catastrophe*, with 1 control parameter, viz. s .

(b) Type II: $u_{xx}(x, s) = 0$:

- $u_0^{(2)} = 0$ corresponds to a regular critical path.
- $u_0^{(2)} = u_0^{(3)} = 0$ indicates an *annihilation event*. The critical points involved are not critical points of u . In Thom’s list this represents a *cuspl catastrophe* with 2 control parameters, s and $u_0^{(1)}$.

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Under very mild conditions on a positive compactly supported signal there exists a scale $S > 0$ such that there is only one extremum for $s > S$, viz. a maximum. The relevant theorems are stated below. Proofs can be found in the literature [3].

Theorem 1.1 *Let $\int_{\mathbb{R}} f(x) dx > 0$, $\int_{\mathbb{R}} x^2 f(x) dx < \infty$, and $r > 0$. Then $\exists \delta > 0, s_0 > 0$ such that $\forall s > s_0$ the signal $u(\cdot, s)$ has a unique critical point $\xi(s) \in (-\delta\sqrt{s}, \delta\sqrt{s})$, viz. a maximum. Furthermore,*

$$\lim_{s \rightarrow \infty} \xi(s) = \frac{\int_{\mathbb{R}} x f(x) dx}{\int_{\mathbb{R}} f(x) dx}.$$

Theorem 1.2 *Let $f : \mathbb{R} \rightarrow \mathbb{R}_0^+$ be a nonnegative signal with support $[a, b] \subset \mathbb{R}$, then the critical points of its multi-scale signal, $u(x, s) = \exp(s\Delta)f(x)$, are spatially confined to $[a, b] \subset \mathbb{R}$.*

Theorem 1.3 *Let $f : \mathbb{R} \rightarrow \mathbb{R}_0^+$ be a nonnegative signal with support $[a, b] \subset \mathbb{R}$, then all singularities of its multi-scale signal, $u(x, s) = \exp(s\Delta)f(x)$, are contained in $[a, b] \times [0, S(f)] \subset \mathbb{R} \times \mathbb{R}_0^+$, where $S(f) = (b - a)^2/8$.*

Thus under the stated conditions all singular points are confined to an operationally meaningful region of (x, s) -space, and all critical paths can be tracked to the fiducial abscissa $s = 0$. (Note that critical paths cannot form closed loops in (x, s) -space.) Upon increasing scale, starting from an arbitrarily defined lowest scale, all regions as described previously (peaks, dales, and void) will merge into an encompassing region.

2 Summary

A one-dimensional, empirically acquired signal admits a coarse-to-fine hierarchical partitioning. It is most natural to use the singularity set and global morsification of the auxiliary signal $g(x, s)$ associated with the original multi-scale signal, $u(x, s)$, in order to obtain a coarse-to-fine partitioning of the signal, since this not only yields the part labels (viz. certain critical points uniquely attached to those parts) but also the part boundaries. This may help to establish a desired partitioning despite the presence of noise inherent in any empirically acquired signal, and depending on one's task. The hierarchies thus obtained are completely characterized by the scale catastrophe spectrum for generic scale transitions.

Acknowledgements The Netherlands Organisation for Scientific Research (NWO) is gratefully acknowledged for financial support.

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