

Multilevel clustering of extremes

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S.Y.Novak
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Multilevel Clustering of Extremes

S. Y. Novak

*Eurandom, Technical University of Eindhoven,
PO Box 513, Eindhoven 5600 MB, Netherlands.*

Abstract

We investigate asymptotic properties of two–dimensional empirical point processes of exceedances (EPPE). We give a new description of a class of possible limit laws. Necessary and sufficient conditions for the weak convergence of one–dimensional EPPEs and complete convergence of two–dimensional EPPEs are established.

Keywords: *extreme values, point process, complete convergence.*

1 Introduction

The two–dimensional empirical point process of exceedances (EPPE) is a key tool for approximating probabilities of exceedances of distinct levels during different periods of time (cf. [15, 25, 8]). For instance, a stationary sequence $\{X_i\}$ of (dependent) random variables can represent claims to an insurance company. Let $N(t_i, I_i)$ denote the number of claims exceeding a level t_i in the time interval $I_i = [T_i^-, T_i^+]$. It can be of interest to approximate the probability $\mathbb{P}(N(t_1, I_1) = n_1, \dots, N(t_k, I_k) = n_k)$. This question can be easily addressed if the distribution of a two–dimensional EPPE has been approximated.

In the i.i.d. case, the limiting distribution of an EPPE is necessarily Poisson (see [24, 25, 15]). In the presence of dependence, extremes may appear in clusters. As a result, the limiting distribution of a one–dimensional EPPE counting locations of extremes is necessarily compound Poisson (see [12] and Theorem A below).

One–dimensional EPPEs describing heights of extremes have not been well studied in literature before (some limit properties of such processes can be obtained if a complete convergence theorem for a two–dimensional EPPE is proved). We fill that gap in Section 4.1. We describe the class \mathcal{P}' of possible limit laws and establish necessary and sufficient conditions for the weak convergence of an EPPE to a given element of \mathcal{P}' . We show also that corresponding jump processes have stochastically continuous trajectories.

Two–dimensional EPPEs count both locations and heights of extremes. The class \mathcal{P} of possible weak limits of two–dimensional EPPEs has been described by Mori [18] as a class of infinitely divisible point processes that are invariant under certain transformations. Hsing [10] represented a point process $P \in \mathcal{P}$ in terms of the points of a two–dimensional Poisson point process and a one–dimensional point process (see Theorem B below).

In this paper we suggest a different way of defining a two–dimensional EPPE. A feature of this approach is that one has to specify a minimal level u_n such that exceedances of u_n

are considered extreme. An advantage is that a weak limit of a two-dimensional EPPE appears a natural generalisation of a compound Poisson point process. Necessary and sufficient conditions for the complete convergence of a two-dimensional EPPE to a given limit are established.

2 Background

2.1 Number of exceedances and one-dimensional EPPEs

Let X, X_1, X_2, \dots be a strictly stationary sequence of (dependent) random variables (r.v.s). Denote

$$M_n = \max_{1 \leq i \leq n} X_i, \quad N_n(u) = \sum_{1 \leq i \leq n} \mathbb{I}\{X_i > u\}.$$

The random variable $N_n(u)$ is the number of exceedances over the level u by the random variables X_1, \dots, X_n . Let

$$X_{n,n} \leq \dots \leq X_{1,n} = M_n$$

be the sample order statistics. Evidently, $\{X_{k,n} \leq u\} = \{N_n(u) < k\}$.

We study asymptotic properties of the distribution of $N_n(u)$. It is clear that the sequence of threshold levels $\{u = u_n\}$ must satisfy an appropriate condition which guarantees that $\{N_n(u_n)\}$ has a non-degenerate limiting distribution. A common approach is to assume that there exists the limit

$$\lim_{n \rightarrow \infty} \mathbb{P}(M_n \leq u_n) := e^{-\lambda} \quad (0 < \lambda < \infty). \quad (2.1)$$

Let $\mathbf{\Pi}(\lambda, \zeta) \equiv \mathbf{\Pi}(\lambda, \mathcal{L}(\zeta))$ denote the compound Poisson distribution with intensity λ and multiplicity distribution $\mathcal{L}(\zeta)$, i.e.,

$$\mathbf{\Pi}(\lambda, \zeta) = \mathcal{L}\left(\sum_{i=0}^{\pi(\lambda)} \zeta_i\right),$$

where $\zeta_0 = 0$, $\zeta_i \stackrel{d}{=} \zeta$ ($i \geq 1$), the random variables $\pi(\lambda), \zeta, \zeta_1, \zeta_2, \dots$ are independent and $\pi(\lambda)$ has the Poisson $\mathbf{\Pi}(\lambda)$ distribution.

Denote by \mathcal{R} the class of sequences $\{r = r_n\}$ of natural numbers such that

$$r_n \rightarrow \infty, \quad r_n = o(n).$$

The random variable ζ is called the *limiting cluster size* if (2.1) holds and

$$\mathcal{L}(N_r(u_n) \mid N_r(u_n) > 0) \Rightarrow \mathcal{L}(\zeta) \quad (2.2)$$

for some sequence $\{r\} \in \mathcal{R}$. Under mild conditions, the only possible limit law for $N_n(u_n)$ is compound Poisson:

$$N_n(u_n) \Rightarrow \sum_{i=0}^{\pi(\lambda)} \zeta_i. \quad (2.3)$$

In Section 3 we present a multilevel generalisation of this limit theorem. The only possible limit law for a vector of numbers of exceedances of distinct levels is compound Poisson.

The important particular case is a pure Poisson limit. The problem of evaluating accuracy of Poisson approximation for a sum of dependent 0's and 1's attracted significant attention (see [2, 6, 5] and references therein). The accuracy of compound Poisson approximation for $\mathcal{L}(N_n(u_n))$ has been evaluated in [3, 19, 26, 28] (see also references in [2, 19]).

The next level of generality is to consider the one-level point process of exceedances $N_n[\cdot]$, where

$$N_n[B] \equiv N_n[B, u_n] = \sum_{1 \leq i \leq n} \mathbb{I}\{i/n \in B, X_i > u_n\} \quad (2.4)$$

for any Borel set $B \subset (0; 1]$.

The process (2.4) naturally appears when one wants to approximate the joint distribution of the numbers of exceedances *of the same level* during different periods of time.

The following result has been proved by Hsing et al. [12] under a mild mixing condition.

Theorem A [12]. *If (2.1) and (2.2) are in force then*

$$N_n[\cdot] \Rightarrow N[\cdot], \quad (2.5)$$

where $N[\cdot]$ is a compound Poisson point process with intensity rate λ and multiplicity distribution $\mathcal{L}(\zeta)$.

If $N_n[\cdot]$ converges weakly to some point process $N[\cdot]$ then $N[\cdot]$ is a compound Poisson point process on $(0; 1]$ with some multiplicity distribution $\mathcal{L}(\zeta)$ and intensity rate λ that obeys (2.1) and, whenever $\lambda > 0$, (2.2) holds.

Let $\{u_n(\cdot), n \geq 1\}$ be a sequence of functions such that $u_n(\cdot)$ is strictly decreasing for all large enough n , $u_n(0) = \infty$ and

$$\lim_{n \rightarrow \infty} \mathbb{P}(M_n \leq u_n(t)) = e^{-t} \quad (t > 0). \quad (2.6)$$

If $\{M_n\}$ has a limiting distribution with normalising sequences $\{a_n\}$ and $\{b_n\}$, i.e., if

$$\lim_{n \rightarrow \infty} \mathbb{P}((M_n - b_n)/a_n \leq x) = G(x) \quad (x \in \mathbb{R}), \quad (2.7)$$

where G is necessarily a distribution function (d.f.) from one of the three extreme value types of d.f.s (see [9, 15]) then one can put $u_n(t) = a_n G^{-1}(e^{-t}) + b_n$.

The process (2.4) counts *location* points (along the horizontal axis) where exceedances of the level u_n occur. Equivalently, it can be viewed as a jump process $\{N_n[0; s], s \in (0; 1]\}$.

A natural question is to approximate the distribution of the jump process

$$\{N_n(u_n(t)), t \in [0; T]\} \quad (2.8)$$

which describes *heights* of extremes. Necessary and sufficient conditions for the weak convergence of the process (2.8) to a compound Poisson process are given in [19]. In Section 4.1 below we describe the class \mathcal{P}' of limiting distributions of the process (2.8) and present necessary and sufficient conditions for the weak convergence of the process (2.8) to a given $P' \in \mathcal{P}'$.

2.2 Two-dimensional EPPE

The two-dimensional point process of exceedances N_n^* can be defined by

$$N_n^*(A) := \sum_{i \geq 1} \mathbb{I}\{(i/n, u_n^{-1}(X_i)) \in A\} \quad (2.9)$$

for any Borel set $A \subset (0; \infty) \times [0; \infty)$. If $A_t = (0; 1] \times [0; t)$ then

$$N_n^*(A_t) = N_n(u_n(t)).$$

The weak convergence of N_n^* to a limiting point process is often called the *complete convergence*. Many results of Extreme Value Theory can be drawn as consequences if a complete convergence theorem is established (cf. [25, 19]).

If the r.v.s $\{X_i\}$ are independent (or dependent but without asymptotic clustering of extremes) then N^* converges weakly to a Poisson point process on $(0; \infty) \times [0; \infty)$ (see [1, 15, 24, 25]).

In the case of a stationary α -mixing sequence of random variables, the class \mathcal{P} of possible limit laws for N_n^* was described by Mori [18] as a class of "infinitely divisible point processes invariant under certain transformations". Hsing [10] represented elements of \mathcal{P} in terms of the points of a two-dimensional Poisson point process and a one-dimensional point process.

Theorem B [18, 10]. *Suppose that (2.6) holds and*

$$\lim_{s \rightarrow 0, t \rightarrow \infty} \mathbb{P}(u_n(t) < X < u_n(s)) = 1$$

for all large enough n . If N_n^ converges weakly to a point process N^* then N^* is an infinitely divisible point process with the following properties:*

- (1) $N^* \circ g_t \stackrel{d}{=} N^*$ ($t > 0$), where $g_t(x, y) = (x + t, y)$,*
- (2) $N^* \circ h_s \stackrel{d}{=} N^*$ ($s > 0$), where $h_s(x, y) = (x/s, ys)$,*
- (3) $\mathbb{P}(N^*((0; \varepsilon) \times (0; 1))) \rightarrow 0$ as $\varepsilon \rightarrow 0$,*
- (4) N^* has independent increments along the horizontal axis.*

The process N^ admits the representation*

$$N^*(\cdot) = \sum_{i \geq 1} \sum_{j=1}^{K_i} \mathbb{I}\{(Y_i, Z_i \kappa_{ij}) \in \cdot\}, \quad (2.10)$$

where $\{(Y_i, Z_i), i \geq 1\}$ are the points of a two-dimensional homogeneous Poisson point process η with the Lebesgue intensity measure, $\{\kappa_{ij}, 1 \leq j \leq K_i\}$ are the points of a point process κ_i on $[1; \infty)$, $\kappa_i \stackrel{d}{=} \kappa$ ($i \geq 1$), the process κ has an atom at 1, the processes $\eta, \kappa_1, \kappa_2, \dots$ are mutually independent.

Necessary and sufficient conditions for the convergence $N_n^* \Rightarrow P \in \mathcal{P}$ are given in [23] in the assumption that the sequence $\{X_i, i \geq 1\}$ possesses an *extremal index*. The idea of those conditions is that the distribution of the vector $\{N_r(u_n(t_1)), \dots, N_r(u_n(t_k))\}$ is assumed to be “close” to that of the limiting process for any choice of $t_1 < \dots < t_k, k \in \mathbb{N}$.

An important particular case among those possible limit laws is the compound Poisson. Necessary and sufficient conditions for the weak convergence of N_n^* to a compound Poisson point process (in terms of exceedances of two levels only) have been given in [19].

The Mori–Hsing characterisation (2.10) of the process N^* can be regarded as implicit. In Section 4 we suggest a different way of defining a two-dimensional EPPE and describe the class \mathcal{P}_* of limiting point processes. We represent an element of \mathcal{P}_* in terms of one-dimensional processes only (in particular, our representation immediately implies that the only possible limit law for the one-level process (2.4) is compound Poisson).

A weak limit of an EPPE is given in a form that seems to be a natural generalisation of a compound Poisson point process. Corresponding necessary and sufficient conditions for the complete convergence of a two-dimensional EPPE to a given $P_* \in \mathcal{P}_*$ are established.

Unless otherwise specified, limits are assumed as $n \rightarrow \infty$; a sum over \emptyset equals zero. Below, $\{\pi_\lambda(s), s \geq 0\}$ is a Poisson process with intensity rate λ and $\pi(\cdot) \equiv \pi_1(\cdot)$.

3 Exceedances of multiple levels

3.1 Conditions

Remind that the functions $u_n(\cdot)$ are strictly decreasing for all large enough n , $u_n(0) = \infty$ and (2.6) holds. In the rest of the paper we assume that

$$\limsup n\mathbb{P}(X_n > u_n(t)) < \infty \quad (0 < t < \infty). \quad (3.1)$$

Note that (2.6) does not imply (3.1) — for instance, consider the case $X_i \equiv X$. Denzel and O’Brien [7] give an example of an α -mixing sequence such that (2.6) holds though $n\mathbb{P}(X_n > u_n(t)) \rightarrow \infty$. On the other hand, (3.1) follows from (2.6) under a stronger mixing condition (cf. O’Brien [21], Lemma 3).

Denote $u_n(\bar{t}) = (u_n(t_1), \dots, u_n(t_k))$, and let $\mathcal{F}_{l,m} \equiv \mathcal{F}_{l,m}(u_n(\bar{t}))$ be the σ -field generated by the events $\{X_i > u_n(t_j)\}$, $l \leq i \leq m, 1 \leq j \leq k$. Put

$$\alpha(l, \{u_n(\bar{t})\}) = \sup |\mathbb{P}(AB) - \mathbb{P}(A)\mathbb{P}(B)|,$$

where the supremum is taken over $A \in \mathcal{F}_{1,m}, B \in \mathcal{F}_{m+l+1,n}, m \geq 1$ such that $\mathbb{P}(A) > 0$.

Condition $\Delta\{u_n(\bar{t})\}$ is said to hold if

$$\alpha(l_n, \{u_n(\bar{t})\}) \rightarrow 0$$

for some sequence $\{l_n\} \in \mathcal{R}$ (perhaps, dependent on $\{u_n(t_j)\}_{1 \leq j \leq k}$).

We say that *condition Δ holds* if $\Delta\{u_n(\bar{t})\}$ is in force for every choice of $0 < t_1 < \dots < t_k < \infty$, $k \in \mathbb{N}$.

Condition Δ^ is said to hold* if $\alpha([cn], \{u_n(t_j)\}_{1 \leq j \leq k}) \rightarrow 0$ for every $c \in (0; 1)$ and every choice of $0 < t_1 < \dots < t_k < \infty$, $k \in \mathbb{N}$ (thus, Δ^* implies Δ).

Denote by $\mathcal{R}(\bar{t}) \equiv \mathcal{R}(\{u_n(\bar{t})\})$ the class of sequences $\{r = r_n\} \in \mathcal{R}$ such that

$$n \gg r_n \gg l_n, nr_n^{-1} \alpha_n \rightarrow 0, \quad (3.2)$$

where $\alpha_n = \alpha(l_n, \{u_n(\bar{t})\})$. Evidently, $\mathcal{R}(\bar{t})$ is not empty: one can put

$$r_n = \max \left\{ [n\sqrt{\alpha_n}]; [\sqrt{nl_n}] \right\}. \quad (3.3)$$

If the sequence $\{X_i, i \geq 1\}$ is α -mixing then (3.2) holds with one and the same sequence $\{r_n\}$ for all $\bar{t} \in \mathbb{R}_k$ (r_n can be defined by (3.3)).

3.2 Exceedances of distinct levels

In this subsection we are interested in the joint limiting distribution of the vector

$$\{N_n(u_n(t_1)), \dots, N_n(u_n(t_k))\}$$

of the numbers of exceedances of several levels. Results on the limiting distribution of the vector $\{N_n(u_n(t_1)), \dots, N_n(u_n(t_k))\}$ imply, in particular, the corresponding ones on the joint limiting distribution of a finite number of upper order statistics.

The class of possible limit laws for the joint limiting distribution of the first and the second maxima has been described by Welsch [30] and Mori [17] (see also [20]). Welsch's result was generalised by Hsing [11]. Under the assumption that the sequence $\{X_i, i \geq 1\}$ is α -mixing and (2.6) holds, Hsing [11] shows that the probability

$$\mathbb{P}(X_{1,n} \leq u_n(s), X_{k,n} \leq u_n(t)) \equiv \mathbb{P}(N_n(u_n(s)) = 0, N_n(u_n(t)) < k) \quad (3.4)$$

converges for every $t > s > 0$ if and only if there exist functions $\rho_i(\cdot)$ and a sequence $\{r\} \in \mathcal{R}$ such that

$$\mathbb{P}(N_r(u_n(s)) = 0, N_r(u_n(t)) = i | N_r(u_n(t)) > 0) \rightarrow \rho_i \equiv \rho_i(s/t) \quad (i \geq 1)$$

for each $t > s > 0$ and $i \in \{1, \dots, k-1\}$. Notice that the expression suggested in [11] for the limit of probability (3.4) can be simplified to the form

$$\lim \mathbb{P}(N_n(u_n(s)) = 0, N_n(u_n(t)) < k) = \mathbb{P}\left(\sum_{i=1}^{\pi(t)} \zeta_i^* < k\right), \quad (3.5)$$

where the distribution of i.i.d.r.v.s $\{\zeta_i^*, i \geq 1\}$ depends on s/t : $\mathbb{P}(\zeta_1^* = i) = \rho_i(s/t)$.

Sufficient conditions for the weak convergence of the vector $\{N_n(u_n(s)), N_n(u_n(t))\}$ have been suggested in [19]. Notice that the weak convergence of M_n does not, in general, imply the weak convergence of the vector $\{N_n(u_n(s)), N_n(u_n(t))\}$. Mori [17] gives an example of a stationary sequence of one-dependent r.v.s such that (2.6) holds while $\{X_{1,n}, X_{2,n}\}$ does not converge.

It is convenient to write

$$N_n(a, t) = \sum_{1 \leq i \leq a} \mathbb{I}\{X_i > u_n(t)\} \quad (t \geq 0, a \geq 1). \quad (3.6)$$

Let $\bar{i} = (i_1, \dots, i_k)$, $\bar{t} = (t_1, \dots, t_k)$,

$$\mathbb{N}_k = \{\bar{i} \in \mathbf{Z}_+^k : i_1 + \dots + i_k > 0\}, \quad \mathbb{R}_k = \{\bar{t} \in \mathbb{R}^k : 0 < t_1 < \dots < t_k < \infty\}.$$

For any $m \in \mathbb{N}$, $\bar{t} \in \mathbb{R}_k$, put

$$N_n(m, \bar{t}) = \{N_n(m, t_1), \dots, N_n(m, t_k)\}.$$

Let $\zeta(\bar{t}, n) \equiv \{\zeta^1(\bar{t}, n), \dots, \zeta^k(\bar{t}, n)\}$ be a random vector with the distribution

$$\mathcal{L}(\zeta(\bar{t}, n)) = \mathcal{L}(N_n(r, \bar{t}) \mid N_n(r, t_k) > 0). \quad (3.7)$$

Proposition 1 *Assume condition $\Delta\{u_n(\bar{t})\}$. If*

$$\zeta(\bar{t}, n) \Rightarrow \exists \zeta(\bar{t}) \quad (3.8)$$

for some sequence $\{r\} \in \mathcal{R}(\bar{t})$ then

$$N_n(sn, \bar{t}) \Rightarrow \sum_{j=1}^{\pi(st_k)} \zeta_j(\bar{t}) \quad (\forall s > 0), \quad (3.9)$$

where $\{\zeta_j(\bar{t}), j \geq 1\}$ are independent copies of $\zeta(\bar{t})$.

If the assumptions of Proposition 1 hold then (3.9) implies (3.5).

We say that the random vector Y has a compound Poisson distribution if

$$Y \stackrel{d}{=} X_1 + \dots + X_{\pi(t)},$$

where vectors $\{X_i, i \geq 1\}$ are independent, $X_i \stackrel{d}{=} X$ ($i \geq 1$) and $\pi(t)$ is a Poisson $\mathbb{I}(t)$ random variable.

The following result is a multilevel generalisation of a compound Poisson limit theorem for $N_n(u_n)$.

Theorem 2 Assume condition $\Delta\{u_n(\bar{t})\}$. If $N_n(n, \bar{t})$ converges weakly then (3.8) holds for any sequence $\{r\} \in \mathcal{R}(\bar{t})$. The distribution of the random vector $\zeta(\bar{t}) = \{\zeta^1(\bar{t}), \dots, \zeta^k(\bar{t})\}$ is scale-invariant:

$$\zeta(\bar{t}) \stackrel{d}{=} \zeta(a\bar{t}) \quad (\forall a > 0) \quad (3.10)$$

and does not depend on the choice of the sequence $\{r\} \in \mathcal{R}(\bar{t})$. The marginal distributions of $\zeta(\bar{t})$ obey formula (3.12) below. The weak limit of $N_n(n, \bar{t})$ is necessarily a compound Poisson random vector:

$$N_n(sn, \sigma\bar{t}) \Rightarrow \sum_{j=1}^{\pi(s\sigma t_k)} \zeta_j(\bar{t}) \quad (s > 0, \sigma > 0). \quad (3.11)$$

Notice that vectors $N_n(sn, \bar{t})$ and $N_n(n, s\bar{t})$ have the same limiting distribution.

Note also that Hsing's [11] description of the joint limiting distribution of the first and the k -th sample maxima is given in the assumption that probabilities (3.4) converge for all $t > s > 0$. The feature of our result is that we assume only the convergence of $N_n(n, \bar{t})$ for a fixed \bar{t} , and then show that this implies the weak convergence of $N_n(sn, \sigma\bar{t})$ for all $s > 0, \sigma > 0$.

Let ζ be a random variable with the limiting cluster size distribution (2.2), and let $\{\zeta_i, i \geq 1\}$ be independent copies of ζ . For any $a \in (0; 1]$, denote by $Z(a)$ the random variable with the distribution $\mathbb{P}(Z(a) = 0) = 1 - a$, $\mathbb{P}(Z(a) = i) = a\mathbb{P}(\zeta = i)$ ($i \geq 1$). Note that

$$Z(a) \stackrel{d}{=} \zeta\xi(a),$$

where $\xi(a)$ is independent of ζ and has Bernoulli $\mathbf{B}(a)$ distribution.

The property $\zeta(\bar{t}) \stackrel{d}{=} \zeta(a\bar{t})$ means that the marginal distributions of the vector $\zeta(\bar{t})$ are functionals of $\mathcal{L}(\zeta)$ and ratios t_l/t_k . We show in section 5 that

$$\zeta^l(\bar{t}) \stackrel{d}{=} Z(t_l/t_k). \quad (3.12)$$

In particular, this implies that the distribution of the limiting cluster size ζ in the limit theorem (2.3) does not depend on λ .

While the limiting cluster size ζ takes values in \mathbb{N} , the random variable $Z(a)$ does it in $\mathbf{Z}_+ = \mathbb{N} \cup \{0\}$. In other words, clusters at a level strictly above the "basic" one can be empty. This is a feature of the multilevel situation.

Proposition 1 and Theorem 2 hint that (2.3) is not the only way of formulating the limit theorem for $N_n(u_n)$. In fact, we have a variety of ways to define the limiting cluster size and formulate a limit theorem.

Indeed, let $\{\zeta_i(a), i \geq 1\}$ be independent copies of $Z(a)$, $a \in (0; 1]$. If Δ and (3.1) hold and $N_n(u_n(at))$ converges weakly then, according to (2.3),

$$N_n(u_n(at)) \Rightarrow \sum_{i=1}^{\pi(at)} \zeta_i. \quad (3.13)$$

If $\{Y_j\}$ are i.i.d. random vectors, $a \in [0; 1]$ and $\pi(at)$ is independent of $\{Y_j\}$ then it is easy to check that

$$\sum_{i=1}^{\pi(at)} Y_i \stackrel{d}{=} \sum_{i=1}^{\pi(t)} Y_i(a), \quad (3.14)$$

where $\{Y_i(a)\}$ are i.i.d. random vectors, $Y_i(a) \stackrel{d}{=} Y_i\xi(a)$ and $\pi(t)$ is independent of $\{Y_j(a)\}$. Therefore,

$$N_n(u_n(at)) \Rightarrow \sum_{i=1}^{\pi(t)} Z_i(a). \quad (3.13^*)$$

As one of possible applications of Theorem 2, consider the problem of approximating the distribution of the random vector

$$\nu_n(t_1, t_2) = \sum_{i=1}^n \mathbb{I}\{u_n(t_1) \geq X_i > u_n(t_2)\}.$$

This problem arises, for instance, when an insurance company is interested in approximating the distribution of a number of claims varying in a specified interval.

Assume conditions of Theorem 2. If $N_n(n, \bar{t})$ converges weakly then (3.11) entails

$$\nu_n(t_1, t_2) \Rightarrow \sum_{i=1}^{\pi(t_2)} Y_i, \quad (3.15)$$

where $\{Y_i\}$ are independent copies of $\zeta^2(\bar{t}) - \zeta^1(\bar{t})$ and $\bar{t} = (t_1, t_2)$.

4 Main results

In Theorems 3 – 5 and Corollary 6 below we assume condition Δ .

4.1 The process $\{N_n(u_n(t)), t \in [0; T]\}$

Recall that there is a one-to-one correspondence between one-dimensional point processes and jump processes (random step functions). In this subsection we find convenient to treat one-dimensional point processes of exceedances as jump processes.

For instance, the process (2.4) may alternatively be viewed as the jump process $\{N_{[sn]}(u_n), s \in (0; 1]\}$. It describes *locations* of exceedances of the level u_n .

The process $\{N_n(u_n(t)), t \in [0; T]\}$ describes *heights* of extremes. In this subsection we investigate asymptotic properties of the distribution of the process $\{N_n(u_n(\cdot))\}$.

Necessary and sufficient conditions for the weak convergence of $\{N_n(u_n(\cdot))\}$ to a compound Poisson process are given in [19]. According to Mori's [18] result, the family \mathcal{P}' of weak limits of $\{N_n(u_n(\cdot))\}$ is wider than the class of compound Poisson processes.

In this subsection we present necessary and sufficient conditions for the weak convergence of $\{N_n(u_n(\cdot))\}$ to a given process $P' \in \mathcal{P}'$. We show that every element $P' \in \mathcal{P}'$ is a sum of a Poisson number of jump processes.

More generally, we will study the limiting behaviour of the jump process

$$\{N_n(sn, t), t \in [0; T]\},$$

where $s > 0$ is a fixed number.

Note that $N_n(sn, t) = N_n^*((0; s] \times [0; t))$. According to Theorem B, if N_n^* converges to a point process N^* then

$$N_n(sn, \cdot) \Rightarrow N'_s(\cdot) \equiv N^*((0; s] \times [0; \cdot)), \quad (4.1)$$

where the process $\{N'_s(t), t > 0\}$ has the following properties:

(a) $N'_{as}(t) \stackrel{d}{=} N'_s(at)$

(b) $\mathbb{P}(N'_\varepsilon(1) > 0) = \mathbb{P}(N'_1(\varepsilon) > 0) \rightarrow 0$ as $\varepsilon \rightarrow 0$

Notice that (b) is a consequence of (2.6): $\mathbb{P}(N'_1(\varepsilon) = 0) = \lim \mathbb{P}(N_n(u_n(\varepsilon)) = 0) = e^{-\varepsilon} \rightarrow 1$ as $\varepsilon \rightarrow 0$.

In Theorems 3 and 4 below, s and τ are fixed positive numbers. Denote

$$\mathbb{R}_k^1 = \{\bar{t} \in \mathbb{R}_k : t_k = 1\}.$$

Theorem 3 *Suppose that there exists a jump process $\{\gamma(t), t \in [0; 1]\}$ with stochastically continuous trajectories such that for arbitrary $k \geq 1$ and $\bar{t} \in \mathbb{R}_k^1$,*

$$\zeta(\bar{t}, n) \Rightarrow (\gamma(t_1), \dots, \gamma(t_k)) \quad (4.2)$$

for some $\{r\} \in \mathcal{R}$. Then

$$\{N_n(sn, \tau t), t \in [0; 1]\} \Rightarrow \{N_\tau(s, t), t \in [0; 1]\}, \quad (4.3)$$

where

$$N_\tau(s, t) = \sum_{j=1}^{\pi_\tau(s)} \gamma_j(t), \quad (4.4)$$

$\{\gamma_j(\cdot)\}$ are independent copies of $\gamma(\cdot)$. The process (4.4) has the following property:

$$N_\tau(as, \cdot) \stackrel{d}{=} N_\tau(s, a\cdot) \quad (\forall a \in [0; 1]). \quad (4.5)$$

Evidently, (4.3) can be rewritten as follows:

$$\{N_n(sn, t), t \leq \tau\} \Rightarrow \left\{ \sum_{j=1}^{\pi_\tau(s)} \gamma_j(t/\tau), t \leq \tau \right\}. \quad (4.3^*)$$

The process $\left\{ \sum_{j=1}^{\pi(s)} \gamma_j(\cdot) \right\}$ can be called *Poisson cluster process* or *compound Poisson process of the second order* (regarding the standard compound Poisson process as a “compound Poisson process of the first order”). If $\{X_i\}$ are i.i.d.r.v.s then one can take $l = 0$ and $r = 1$ to show that $N_n(n, \cdot)$ converges to a pure Poisson process with intensity rate 1 (it admits the representation $\sum_{j=1}^{\pi(1)} \gamma_j(\cdot)$, where $\gamma(t) = \mathbb{I}\{\xi < t\}$ and ξ has a uniform $U[0; 1]$ distribution — cf. [27], ch. 1.).

While the random variable ζ represents the limiting cluster size, the process γ describes the variability of heights of cluster members.

Theorem 4 *If $\{N_n(n, t), t \in [0; 1]\}$ converges weakly to some jump process P then there exists a jump process $\{\gamma(t), t \in [0; 1]\}$ with stochastically continuous trajectories such that (4.2) holds and*

$$N_n(sn, \tau \cdot) \Rightarrow N_\tau(s, \cdot) \quad (4.6)$$

for every $s > 0, \tau > 0$. The marginal distributions of the process γ obey

$$\gamma(t) \stackrel{d}{=} Z(t) \quad (0 \leq t \leq 1). \quad (4.7)$$

Theorems 3 and 4 show that the class \mathcal{P}' of possible weak limits of the process (2.8) consists of processes $\sum_{j=1}^{\pi(T)} \gamma_j(\cdot/T)$, where $\gamma(\cdot)$ is a jump process on $[0; 1]$ with stochastically continuous trajectories such that $\mathbb{P}(\gamma(1) \geq 1) = 1$.

4.2 Complete convergence

Let T be a fixed positive number. We define the two-dimensional process $N_T^* \equiv N_T^*(\gamma)$ on $(0; 1] \times [0; 1)$ as a point process with the following properties (evidently, it suffices defining N_T^* on unions of rectangles):

(P1) N_T^* has independent increments along the horizontal axis,

(P2) $N_T^*((a; b] \times B) \stackrel{d}{=} N_T^*((0; b - a] \times B)$ for any Borel set $B \subset [0; 1)$,

(P3) $\{N_T^*((0; a] \times [0; t]), t \in [0; 1)\} \stackrel{d}{=} \{N_T(a, t), t \in [0; 1)\}$.

Otherwise, N_T^* can be viewed as a random measure

$$N_T^*(A) = \int_A N_T^*(dx \times dy),$$

where A is a Borel set in $(0; 1] \times [0; 1)$ and

$$N_T^*(dx \times dy) = \sum_{\pi_T(x) < j \leq \pi_T(x+dx)} (\gamma_j(y+dy) - \gamma_j(y)).$$

Note that the two-dimensional process N_T^* is constructed via one-dimensional processes.

Evidently, N_T^* has properties (1), (3) and (4) of Theorem B; property (2) follows from (4.5). Besides, it is easy to see that $N_T^*((0; a] \times [0; b]) \stackrel{d}{=} N_{sT}^*((0; a] \times [0; b])$.

We define the EPPE $N_{n,T}^*$ on $(0; 1] \times [0; 1)$ by the equation

$$N_{n,T}^*(A) = \sum_{i=1}^n \mathbb{I}\{(i/n, T^{-1}u_n^{-1}(X_i)) \in A\} \quad (4.8)$$

for any Borel set $A \subset (0; 1] \times [0; 1)$. In other words, we restrict our attention to the interval $X_i \in (u_n(T); \infty)$. The level $u_n(T)$ can be seen as a minimal threshold u such that X_i is considered “extreme” if it exceeds u .

Theorem 5 and Corollary 6 below show that processes N_T^* are the only possible weak limits for $N_{n,T}^*$.

Theorem 5 *Suppose that there exists a jump process $\{\gamma(t), t \in [0; 1]\}$ with stochastically continuous trajectories such that (4.2) holds. Then*

$$N_{n,T}^* \Rightarrow N_T^*(\gamma). \quad (4.9)$$

From Theorems 4 and 5 we deduce

Corollary 6 *If $N_{n,T}^*$ converges weakly to some point process then there exists a jump process $\{\gamma(t), t \in [0; 1]\}$ with stochastically continuous trajectories such that $N_{n,T}^* \Rightarrow N_T^*(\gamma)$.*

5 Proofs

The following well known fact (cf. [14, 22]) will be used in the proofs of Proposition 1 and Theorem 2.

Proposition 7 *Suppose that condition $\Delta\{u_n(t)\}$ holds for some $t > 0$. If (2.6) is in force then*

$$\mathbb{P}(N_r(u_n(t)) > 0) \sim tr/n \quad (5.1)$$

for any sequence $\{r\} \in \mathcal{R}(t)$. If (5.1) is valid for a sequence $\{r\} \in \mathcal{R}(t)$ then

$$\lim \mathbb{P}(M_{sn} \leq u_n(t)) = e^{-st} \quad (\forall s > 0). \quad (5.2)$$

Proof of Proposition 1. Let $\{r\} \in \mathcal{R}(t)$. Denote $m = [sn/r]$. Using Bernstein's blocks method (cf. Lemma 2.2 in [12]) and taking into account conditions $\Delta\{u_n(\bar{t})\}$ and (3.1), it is easy to show that

$$|\mathbb{E}e^{ivN_n(sn, \bar{t})} - \mathbb{E}^m e^{ivN_n(r, \bar{t})}| \rightarrow 0 \quad (5.3)$$

for any $v \in \mathbb{R}^k$. Note that

$$\mathbb{E}e^{ivN_n(r, \bar{t})} = \mathbb{P}(N_n(r, t_k) = 0) + \mathbb{E}\left\{e^{ivN_n(r, \bar{t})} \mid N_n(r, t_k) > 0\right\} \mathbb{P}(N_n(r, t_k) > 0).$$

Hence

$$\mathbb{E}e^{ivN_n(sn, \bar{t})} = \exp\left(m\mathbb{P}(N_n(r, t_k) > 0)\mathbb{E}\left\{e^{ivN_n(r, \bar{t})} - 1 \mid N_n(r, t_k) > 0\right\}\right) + o(1).$$

According to Proposition 7, $\mathbb{P}(N_r(u_n(t_k)) > 0) \sim t_k r/n$. Therefore,

$$\mathbb{E}e^{ivN_n(sn, \bar{t})} = \exp\left(st_k \left[\mathbb{E}e^{iv\zeta(\bar{t}, n)} - 1\right]\right) + o(1). \quad (5.4)$$

Relation (3.9) follows from (3.8) and (5.4). \square

Proof of Theorem 2. Assume that $N_n(sn, \bar{t}) \Rightarrow \exists N$ for some $s > 0$. This evidently implies

$$\lim \mathbb{E}e^{ivN_n(sn, \bar{t})} = \mathbb{E}e^{ivN} \quad (\forall v \in \mathbb{R}^k).$$

According to (5.4), there exists the limit $\lim \mathbb{E}e^{iv\zeta(\bar{t},n)} := \varphi_o(v)$. As a limit of a sequence of characteristic functions, it is a characteristic function itself. Hence (3.8) holds and

$$\mathbb{E}e^{ivN} = \exp(st_k[\varphi_o(v) - 1]),$$

i.e., N is a compound Poisson random vector with intensity st_k and multiplicity distribution $\mathcal{L}(\zeta)$ such that $\mathbb{E}e^{iv\zeta} = \varphi_o(v)$.

The distribution of the vector $\zeta(\bar{t})$ does not depend on the choice of a sequence $\{r\}$. Indeed, let $u_n = u_n(t)$, $t > 0$. If $\mathcal{L}(N_{r'}(u_n) \mid N_{r'}(u_n) > 0) \Rightarrow \mathcal{L}(\exists\zeta')$ for another sequence $\{r' = r'_n\} \in \mathcal{R}(\bar{t})$ then (3.9) entails $N_n(u_n) \Rightarrow \sum_{j=0}^{\pi(t)} \zeta_j$ and $N_n(u_n) \Rightarrow \sum_{j=0}^{\pi(t)} \zeta'_j$. Hence $\zeta \stackrel{d}{=} \zeta'$.

In order to show that the distribution of the vector $\zeta(\bar{t})$ is scale-invariant, denote $\bar{t}_* = (t_1/t_k, t_2/t_k, \dots, 1)$. By (3.9), $N_n(sn, \bar{t})$ converges for all $s > 0$. According to Lemma 8 below, $N_n(n, s\bar{t})$ converges for all $s > 0$ as well, and the limiting distributions of $N_n(sn, \bar{t})$ and $N_n(n, s\bar{t})$ coincide. Since

$$N_n(n, \bar{t}) \Rightarrow \sum_{j=1}^{\pi(t_k)} \zeta_j(\bar{t}), \quad N_n(t_k n, \bar{t}_*) \Rightarrow \sum_{j=1}^{\pi(t_k)} \zeta_j(\bar{t}_*)$$

according to (3.9), we have $\sum_{j=1}^{\pi(t_k)} \zeta_j(\bar{t}) \stackrel{d}{=} \sum_{j=1}^{\pi(t_k)} \zeta_j(\bar{t}_*)$. Hence $\zeta(\bar{t}) \stackrel{d}{=} \zeta(\bar{t}_*)$.

Formula (3.11) follows from (3.8) – (3.10).

It remains to show that $\zeta^l(\bar{t}) \stackrel{d}{=} Z(t_l/t_k)$. Indeed, (3.9) entails $N_n(u_n(t_l)) \Rightarrow \sum_{j=0}^{\pi(t_k)} \zeta_j^l(\bar{t})$. According to (2.3), $N_n(u_n(t_l)) \Rightarrow \sum_{j=0}^{\pi(t_l)} \zeta_j$. Note that $Z(1) \stackrel{d}{=} \zeta$ and $\mathbf{\Pi}(as, \zeta) = \mathbf{\Pi}(s, Z(a))$ for any $s > 0$. Hence $\sum_{j=0}^{\pi(t_l)} \zeta_j \stackrel{d}{=} \sum_{j=0}^{\pi(t_k)} Z(t_l/t_k)$. Comparing the characteristic functions of $\sum_{j=0}^{\pi(t_k)} \zeta_j^l(\bar{t})$ and $\sum_{j=0}^{\pi(t_k)} Z(t_l/t_k)$, we get (3.12). \square

Let $\bar{t} \in \mathbb{R}_k$, and let I be an open interval in $(0; \infty)$. Denote

$$P_1(n, s) = \mathbb{P}(N_{sn}(u_n(t_1)) < i_1, \dots, N_{sn}(u_n(t_k)) < i_k), \quad (5.5)$$

$$P_2(n, s) = \mathbb{P}(N_n(u_n(st_1)) < i_1, \dots, N_n(u_n(st_k)) < i_k). \quad (5.6)$$

The following lemma follows the corresponding lines in Hsing [10, 11] but our mixing condition is weaker than the condition Δ^* assumed in [10] or the α -mixing condition assumed in [11].

Lemma 8 *Assume condition $\Delta\{u_n(\bar{t})\}$. If one of the probabilities (5.5) or (5.6) converges for each $s \in I$ then so does the other, and the limits coincide.*

Proof of Lemma 8. Let $s' > s > s''$ be the points from I . As was noticed in [11], $u_{[n/s']}(t) < u_n(s''t)$ for all sufficiently large n (this follows from (5.2)).

Suppose that the limit $g(s) \equiv g(s, \bar{t}) = \lim P_2(n, s)$ exists ($s \in I$). Then

$$\begin{aligned} \limsup P_1(n, s') &= \limsup P_1([n/s'], s') \\ &= \limsup \mathbb{P}(N_n(u_{[n/s']}(t_1)) < i_1, \dots, N_n(u_{[n/s']}(t_k)) < i_k) \\ &\leq \limsup \mathbb{P}(N_n(u_n(st_1)) < i_1, \dots, N_n(u_n(st_k)) < i_k) = \lim P_2(n, s). \end{aligned} \quad (5.7)$$

Similarly

$$\lim P_2(n, s) \leq \liminf P_1(n, s''). \quad (5.8)$$

Therefore,

$$\limsup P_1(n, s') \leq g(s_1) \leq g(s_2) \leq \liminf P_1(n, s'')$$

if $s' > s_1 > s_2 > s''$. Notice that

$$\begin{aligned} & \liminf P_1(n, s'') - \limsup P_1(n, s') \leq \limsup [P_1(n, s'') - P_1(n, s')] \\ & \leq \sum_{i=1}^k \lim \mathbb{P} \left(N_{[s''n]-[s'n]}(u_n(t_i)) > 0 \right) \leq \sum_{i=1}^k ([s''n] - [s'n]) \mathbb{P}(X > u_n(t_i)). \end{aligned} \quad (5.9)$$

This and (3.1) imply

$$0 \leq \liminf P_1(n, s'') - \limsup P_1(n, s') \rightarrow 0$$

as $s' - s'' \rightarrow 0$. Hence the function $g(s)$ is uniformly continuous in I .

If $s_1 > s > s_2$ are the points from I then (5.7) and (5.8) entail

$$g(s_1) = \lim P_2(n, s_1) \leq \liminf P_1(n, s) \leq \limsup P_1(n, s) \leq \lim P_2(n, s_2) = g(s_2).$$

Letting $s_1 \rightarrow s$ and $s_2 \rightarrow s$, we observe that the limit $\lim P_1(n, s)$ exists and equals $g(s)$.

Suppose that for every $s \in I$ there exists the limit $h(s) \equiv h(s, \bar{t}) = \lim P_1(n, s)$. Similarly to (5.7) and (5.8),

$$\lim P_1(n, s') \leq \liminf P_2(n, s) \leq \limsup P_2(n, s) \leq \lim P_1(n, s'').$$

From (5.9) we derive that $\lim P_1(n, s'') - \lim P_1(n, s') \rightarrow 0$ as $s' - s'' \rightarrow 0$. This entails the limit $\lim P_2(n, s)$ exists and equals $h(s)$. The proof is complete. \square

Proof of Theorem 3. Let $\bar{t} \in \mathbb{R}_k^1$, and denote $N_\tau(s, \bar{t}) = \{N_\tau(s, t_1), \dots, N_\tau(s, t_k)\}$. Relation (3.8) holds with $\zeta(\bar{t}) \stackrel{d}{=} (\gamma(t_1), \dots, \gamma(1))$. Proposition 1 and Theorem 2 entail

$$\{N_n(sn, \tau t_1), \dots, N_n(sn, \tau)\} \Rightarrow \{N_\tau(s, t_1), \dots, N_\tau(s, 1)\}$$

for every $\bar{t} \in \mathbb{R}_k^1$. Thus, finite-dimensional distributions of $\{N_n(sn, \tau t), t \in [0; 1]\}$ converge to those of $\{N_\tau(s, t), t \in [0; 1]\}$. In view of [13], ch. 4, this implies the weak convergence $N_n(sn, \tau \cdot) \Rightarrow N_\tau(s, \cdot)$.

In order to check (4.5), we must show that finite-dimensional distributions of the processes coincide:

$$\left\{ \sum_{j=1}^{\pi_\tau(as)} \gamma_j(t_1), \dots, \sum_{j=1}^{\pi_\tau(as)} \gamma_j(t_k) \right\} \stackrel{d}{=} \left\{ \sum_{j=1}^{\pi_\tau(s)} \gamma_j(at_1), \dots, \sum_{j=1}^{\pi_\tau(s)} \gamma_j(at_k) \right\}. \quad (5.10)$$

By (3.11), the left-hand side of (5.10) is the weak limit of $N_n(asn, \tau \bar{t})$. Let $\tilde{t} = \{at, 1\}$. Then $N_n(sn, \tau \tilde{t}) \Rightarrow N_\tau(s, \tilde{t})$ and hence $N_n(sn, \tau at) \Rightarrow N_\tau(s, at)$, the right-hand side

of (5.10). According to Theorem 2, the weak limits of $N_n(asn, \tau \bar{t})$ and $N_n(sn, \tau a \bar{t})$ coincide. This implies (5.10) and (4.5). \square

Proof of Theorem 4. Suppose that the process $N_n(n, \cdot)$ converges weakly to some jump process P . Let $k \in \mathbb{N}$, $\bar{t} \in \mathbb{R}_k^1$. Then

$$(N_n(n, t_1), \dots, N_n(n, 1)) \Rightarrow (P(t_1), \dots, P(1)). \quad (5.11)$$

Theorem 2 and (5.11) imply (3.8). A comparison of (3.11) with (5.11) yields $P(\cdot) \stackrel{d}{=} N_1(1, \cdot)$. Moreover, (3.8) and (3.11) imply

$$(N_n(sn, \tau t_1), \dots, N_n(sn, \tau)) \Rightarrow (N_\tau(s, t_1), \dots, N_\tau(s, 1)) \quad (\forall s > 0, \tau > 0). \quad (5.12)$$

Since the distributions (3.7) are consistent, so are the distributions of $\zeta(\bar{t})$, $\bar{t} \in \mathbb{R}_k^1$, $k \geq 1$. By Kolmogorov's theorem, there exists a process $\gamma = \{\gamma(t), t \in [0; 1]\}$ such that $\{\mathcal{L}(\zeta(\bar{t})), \bar{t} \in \mathbb{R}_k^1\}_{k \geq 1}$ are the finite-dimensional distributions of γ . Evidently, γ is a jump process. The weak convergence (4.6) follows from (5.12) and Proposition 9 below.

In order to show that $\gamma(t) \stackrel{d}{=} Z(t)$ for any $t \in [0; 1]$, remind that

$$N_n(n, t) \Rightarrow \sum_{j=1}^{\pi(t)} \zeta_j \stackrel{d}{=} \sum_{j=1}^{\pi(1)} Z_j(t)$$

and $N_n(n, \bar{t}) \Rightarrow \sum_{j=1}^{\pi(1)} \bar{\zeta}_j$, where $\bar{t} = (t, 1)$ and $\bar{\zeta}_j = (\gamma_j(t), \gamma_j(1))$. Hence $N_n(n, t) \Rightarrow \sum_{j=1}^{\pi(1)} \gamma_j(t)$. Therefore, $\sum_{j=1}^{\pi(1)} Z_j(t) \stackrel{d}{=} \sum_{j=1}^{\pi(1)} \gamma_j(t)$. This entails (4.7). \square

Evidently, Theorem 4 remains valid if its assumption is replaced by the following one: "for some $T > 0$, the process $\{N_n(n, t), t \in [0; T]\}$ converges weakly to some jump process P_T ".

Proposition 9 *Let $\{r\} \in \mathcal{R}$. If, for arbitrary $k \geq 1$ and $\bar{t} \in \mathbb{R}_k^1$, (4.2) holds for some $\{r\} \in \mathcal{R}$ then trajectories of the process γ are stochastically continuous on $[0; 1]$.*

Define the random measure $Q\{\cdot\}$ by the equation

$$Q\{[s; t]\} = \gamma(t) - \gamma(s) \quad (0 \leq s < t \leq 1) \quad (5.13)$$

(Q is defined on intervals in $[0; 1]$ and thus on all Borel sets in $[0; 1]$). Note that (5.13) stipulates a one-to-one correspondence between $\gamma(\cdot)$ and the point process Q (if we had a point process Q on $[0; 1]$ then we could define a jump process $\{\gamma(t), t \in [0; 1]\}$ by the equation $\gamma(t) = Q\{[0; t]\}$). Proposition 9 states that

$$\mathbb{P}(Q\{t\} > 0) = 0 \quad (\forall t \in [0; 1]). \quad (5.14)$$

Proof of Proposition 9. Evidently, $\gamma(0) = 0$. The fact that $\mathbb{P}(Q\{0\} > 0) = 0$ (equivalently, $\mathbb{P}(\gamma(s) > 0) \rightarrow 0$ as $s \rightarrow 0$) follows from (4.7).

Let $t \in (0; 1]$. If $\mathbb{P}(Q\{t\} > 0) > 0$ then (3.15) implies

$$\begin{aligned} \mathbb{P}(\nu_n(\sigma t-, \sigma t+) > 0) &\rightarrow \mathbb{P}\left(\sum_{j=1}^{\pi(\sigma t)} Q_j\{t\} > 0\right) \\ &= 1 - \exp(-\sigma t \mathbb{P}(Q\{t\} > 0)) > 0 \end{aligned} \quad (5.15)$$

for every $\sigma \in (0; 1]$, where $\{Q_j, j \geq 1\}$ are independent copies of Q .

Denote by γ_t the weak limit of the process $\{N_n(n, \sigma t), \sigma \in (0; 1]\}$, and let Q_t be the corresponding point process. Relation (5.15) means that the set $\{\sigma : \mathbb{P}(Q_t\{\sigma\} > 0) > 0\}$ is uncountable. This contradicts to [16], Proposition 1.1.5. Hence (5.14) holds. \square

Proof of Theorem 5. Proposition 9 ensures that $\mathbb{P}(N_T^*((0; 1] \times \{b\}) > 0) = 0$ for any $b \in [0; 1)$. Because of (3.1),

$$\mathbb{P}(N_T^*({a} \times [0; 1)) > 0) = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \mathbb{P}\left(N_{[\varepsilon n]}(u_n(T)) > 0\right) = 0$$

for any $a \in (0; 1]$. Thus, $\mathbb{P}(N_T^*(\partial A) > 0) = 0$ if A is a rectangle on $(0; 1] \times [0; 1)$. Therefore (see [13], chapter 4), (4.9) follows if we show that

$$\{N_{n,T}^*(A_1), \dots, N_{n,T}^*(A_k)\} \Rightarrow \{N_T^*(A_1), \dots, N_T^*(A_k)\} \quad (5.16)$$

for any array $\{A_1, \dots, A_k\}$ of finite unions of rectangles.

Splitting rectangles in a proper way, we observe that it suffices to prove (5.16) in the case $A_i = (a_i; b_i] \times \bigcup_{j=1}^{m_i} [c_{ij}; d_{ij})$, where the intervals $(a_i; b_i]$ are disjoint and for each i , the intervals $[c_{ij}; d_{ij})$ are disjoint too.

Property (P1) implies the random variables $\{N_T^*(A_i)\}$ are independent. By standard arguments (cf. [10, 11, 19]), the random variables $\{N_{n,T}^*(A_i)\}$ are asymptotically independent as well. Thus, it remains to show that

$$N_{n,T}^*(A) \Rightarrow N_T^*(A) \quad (5.17)$$

for any set $A = (a; b] \times \bigcup_{j=1}^m [c_j; d_j) \subset (0; 1] \times [0; 1)$, where the intervals $[c_j; d_j)$ are disjoint.

Theorem 3 establishes (5.17) in the case $(a; b] = (0; 1]$. The arguments are evidently valid for an arbitrary interval $(a; b] \subset (0; 1]$. \square

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