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THE DYNAMICS OF SYSTEMS OF DEFORMABLE BODIES

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Preface

This research was conducted at the section of Fundamentals of Mechanical Engineering, faculty of Mechanical Engineering, Eindhoven University of Technology. It was supervised by Dr. Fons Sauren and professor Dick van Campen.

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Deurne, November 1988 Willy Koppens

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Abstract

In this thesis, a mathematical description is presented of the dynamic behaviour of systems of interconnected deformable bodies.

The displacement field of a body is resolved into a displacement field due to a rigid body motion and a displacement field due to deformation. In order to get an unambiguous resolution, the displacement field due to deformation is required to be such that it cannot represent rigid body motions. This is achieved by prescribing either displacements due to deformation of selected particles or mean displacements due to deformation.

Starting from the equations of motion of a particle of the body, a variational formulation of the equations of motion of the free body is derived. These equations are simpler in case the mean displacements due to deformation are equal to zero. Approximate equations of motion are obtained by approximating the displacement field due to deformation by a linear combination of a set of assumed displacement fields. Three methods are described for generating assumed displacement fields, namely the assumed-modes method, the finite element method, and the modal synthesis method.

For formulating the equations of motion of a body which forms part of a system of bodies, the interconnections with other bodies must be accounted for. Energetic and active connections can be taken into account by adding the forces they generate to the applied forces on the free body. Kinematic connections constrain the relative motion of interconnected bodies. This can be accounted for with constraint equations, that can be used for partitioning the variables that describe the kinematics of the system of bodies into dependent and independent variables. For formulating constraint equations it is convenient to introduce variables that describe the relative motion of the interconnected bodies.

The simplification of the equations of motion in case the mean displacements due to deformation are chosen equal to zero, leads to a computation time reduction of a few decades of per cents in the most favourable case. For the systems investigated in this thesis the dynamic behaviour is approximated better in case displacement fields due to deformation are approximated by assumed displacement fields with mean displacements equal to zero. Caution must be taken in preventing rigid body motions of the displacement field due to deformation by prescribing displacements of selected particles of the body, since this may result in an incorrect

solution of the dynamic behaviour.

The assumed-modes method is only feasible for regularly shaped bodies. The finite element method and the modal synthesis method can be used for bodies with arbitrary shape. The finite element method leads often to a model with many degrees of freedom. The solution of such a model requires much computation time. The modal synthesis method can then be used with success to reduce the number of degrees of freedom such that the required computation time is cut down. The effectiveness of the modal synthesis method depends to a great extent on a proper choice of the assumed displacement fields. Such a choice can generally be made in advance on the basis of the load on the body. The lumped mass approximation, which is frequently used in literature, is feasible for determining time-independent mass coefficients from displacement fields which have been determined with a standard finite element program. One should bear in mind that a finer subdivision into elements may be required than would be necessary for determining the displacement fields sufficiently accurate.

A method is proposed to improve approximations for describing the dynamic behaviour of a body for a specific set of assumed displacement fields. This method has been used successfully for reducing the required computation time by lowering irrelevant high frequencies.

chapter 1

Introduction

The increase in capacity of computers has opened the possibility to simulate the dynamic behaviour of complex mechanical systems, such as spacecraft and vehicles, already in the design phase. This may save expensive modifications of prototypes which will be necessary in case the dynamic behaviour is inadequate. Increasing demands on the dynamic behaviour and more flexible system parts due to a more economical use of materials require that deformation of the parts is taken into account in determining the dynamic behaviour.

Mechanical systems differ in various ways, such as the number of bodies, the types of connections joining the bodies, and the topology. Many papers with the main objective to present a formalism to develop equations of motion for general mechanical systems have been published. Often the presented theory is restricted to a specific class of mechanical systems, such as systems with rigid bodies or two-dimensional systems. A survey will be given of three important topics related to the description of the dynamic behaviour of mechanical systems, namely the description of the kinematics of mechanical systems, mechanical principles for deriving the equations of motion, and deformability of bodies.

Two ways to describe the *kinematics* of mechanical systems are in common use, namely the global description and the relative description. In the global description, the positions of all bodies are described relative to an inertial space. In deriving the equations of motion, the kinematic connections between the bodies are taken into account separately by means of constraint equations. The resulting equations of motion are a set of mixed differential-algebraic equations having a simple form. Special techniques are required for solving these equations. The global description is used by, among others, Orlandea et al. (1977a, b), Wehage and Haug (1982), Haug et al. (1986), and Changizi et al. (1986).

The finite element formulation presented by Van der Werff and Jonker (1984) may be regarded as a variant of the global description. They describe the position and orientation of nodes relative to an inertial space. Both bodies and connections are considered as finite elements. This allows to obtain the equations of motion of a mechanical system by a standard assembly process. The relative motion of nodes of an element are described with deformation mode coordinates which are nonlinear functions of the nodal displacements. When a relative motion is constrained, such as may be the case for rigid bodies, the corresponding deformation coordinate equals zero which leads to a constraint equation.

In the relative description, the position of one arbitrary body is described relative to an inertial space; the positions of the other bodies are described relative to a body whose position has already been described, in terms of variables characterizing the relative motion. For systems without kinematically closed chains, these variables are independent. The resulting equations of motion are a set of differential equations of minimal dimension. In systems with kinematically closed chains, the closed chains are first opened by cutting the chains imaginarily. Then, the kinematics can be described in terms of the variables which characterize the relative motion of the bodies. Accounting for the cuts renders these variables dependent. For some mechanical systems with a simple geometric configuration only, this dependency can be eliminated. However, in general the resulting equations will be too involved. Therefore, the dependency is usually taken into account separately by means of constraint equations just as with the global description. As compared with the global description, the number of constraint equations going with the relative description is small; however, they involve the kinematic variables of all the bodies in a closed chain whereas in the global description only the kinematic variables of pairs of interconnected bodies are involved. The resulting equations of motion are a set of mixed differential-algebraic equations like with the global description. The relative description is used by, among others, Wittenburg (1977), Huston and Passerello (1979), Sol (1983), Schiehlen (1984), and Singh et al. (1985).

A combination of the global description and the relative description has been presented by Haug and McCullough (1986). They derived the equations of motion for recurring subsystems with a particular kinematic structure using the relative description. Special purpose modules are used to evaluate these equations of motion. The result is added to the equations of motion of the remaining part of the system whose kinematics is described using the global description. They observed a vastly improved computational efficiency as compared to a program based on the global description (McCullough and Haug, 1986).

The second important item is the *mechanical principle* used for deriving the equations of motion. Several papers (e.g. Schiehlen, 1981; Kane and Levinson, 1983; and Koplik and Leu, 1986) deal with the question: Which mechanical principle yields equations of motion in the least tedious way and having the simplest form? However, this is only an item of argument when the relative description is used for describing the kinematics of the mechanical system, because some principles, for example Lagrange's equations of motion, take the kinematics of the system into account from the start. When the global description is used, all mechanical principles yield without any trouble the equations of motion. In case the relative description is used, a variational formulation is most suited, such as Lagrange's form of d'Alembert's principle (Wittenburg, 1977), Kane's method of generalized speed (Kane, 1968), and Jourdain's principle (Schiehlen, 1986).

The third important item is the influence of *deformability* of bodies. Many papers deal with the dynamic analysis of mechanical systems that contain deformable bodies. Most papers use the same description of the kinematics of deformable bodies: the displacements of particles of a deformable body are resolved into displacements due to a rigid body motion of the body and displacements due to deformation of the body. This resolution is done in such a way that the strain-displacement relations may be linearized in case the strains are small. Further, most papers use Galerkin's method for obtaining an approximate solution of the equations of motion in the space domain. This involves an expansion of the displacements due to deformation of the body in a linear combination of linearly independent displacement fields. The papers differ in the way these displacement fields are generated: this is most often done by either the finite element method (Song and Haug, 1980; Thompson and Sung, 1984; Turcic and Midha, 1984a, b; Van der Weeën, 1985) or the modal method (Sunada and Dubowsky, 1981, 1983; Yoo and Haug, 1986a, b, c; Agrawal and Shabana, 1985). The papers differ further in the degree to which the coupling between rigid body motion and displacements due to deformation is included. The most simple analysis method considers only the quasi-static deflection caused by the inertia forces due to the motion which follows from a kinematic analysis of a corresponding rigid body model. Erdman and Sandor (1972) refer to such an analysis as elastodynamic analysis. In a more refined analysis, the inertia contribution corresponding to the displacements caused by deformation are also taken into account (Thompson and Sung, 1984; Turcic and Midha, 1984a, b). The most refined analysis departs from unknown rigid body motions and includes all coupling terms (Song and Haug, 1980; Sunada and Dubowsky, 1981, 1983; Yoo and Haug, 1986a, b, c; Agrawal and Shabana, 1985; Van der Weeën, 1985; Lilov and Wittenburg, 1986; Koppens et al., 1988).

The subjects of difference and resemblance of the numerous papers on the dynamics of systems of deformable bodies do not become clear from the literature. It is the purpose of this thesis to give a unified description of the dynamics of systems of deformable bodies. From this description the various descriptions that can be found in the literature can be derived. It is expected that this will increase the insight into the various descriptions.

The dynamics of an individual deformable body is presented in chapter 2. The displacements of the body are resolved into displacements due to deformation and displacements due to a rigid body motion. The order in which the displacements of the body are resolved is opposite to the usual order. This provides that the rotation tensor that describes the rigid body motion can be readily factored out of the deformation tensor, and that there is no need to introduce time differentiation of the displacements due to deformation relative to a rotating frame. However, the order of this resolution is immaterial for the ultimate equations of motion. The resolution of displacements of the body is ambiguous. Conditions are imposed on the displacement field due to deformation in order to get a unique resolution. Two types of conditions are described, namely conditions on displacements of selected material points of the body and conditions on the mean displacements of the body. Starting from the equations of motion of a particle of the body, a variational formulation for the equations of motion of the body is derived. It is shown that the equations of motion become considerably simpler when the displacements due to deformation satisfy the conditions on the mean displacements of the body. The equations of motion contain partial derivatives with respect to material coordinates. In general, such equations admit no closed-form solution. In view of this, approximate equations of motion are derived using Galerkin's method. This involves approximating the displacements due to deformation as a linear combination of assumed displacement fields. The above description of the kinematics of the body and the derivation of the equations of motion are done in terms of vectors and tensors in their symbolic form. Finally, the ultimate equations of motion are written in terms of the components of vectors and tensors relative to an orthonormal right-handed inertial base. It is shown that it is preferable to write the equations of motion in terms of the components of the angular velocity vector above the more used first and second time derivatives of angular orientation variables.

The most simple material behaviour is used namely isotropic linear elastic material behaviour, because the emphasis of this thesis is on the description of the dynamics of deformable bodies. However, anisotropic, nonlinear elastic, or visco-elastic material behaviour can be introduced without insurmountable difficulties by introducing the proper constitutive relation.

In chapter 3, three procedures for generating assumed displacement fields for approximating the displacements due to deformation are reviewed. The *assumed-modes method* can be used for regularly shaped bodies. The assumed displacement fields are analytic functions of the material coordinates. It is limited in scope because regularly shaped bodies are rare in practice. The *finite element method* is a more versatile method. It consists of subdividing the body into regularly shaped volumes. The displacements within such a volume can be easily approximated by analytic functions. These are chosen such that compatibility of displacements of neighbouring volumes can be easily ensured. However, the finite element method generally leads to a model with many degrees of freedom. These can be reduced by using a reduced set of linear combinations of finite element displacement fields. This approach is known as the *modal synthesis method*. It combines the versatility of the finite element method and the efficiency of the assumed-modes method.

The equations of motion of a system of bodies are considered in chapter 4. Bodies may be interconnected by energetic, active, and kinematic connections. The contribution of energetic and active connections can be readily introduced into the equations of motion of the single bodies. Kinematic connections render the variables that describe the motion of the individual bodies dependent. From examples for pairs of interconnected bodies it is shown how equations can be obtained that describe this dependency. It appears that the essential difference between the global description and the relative description is that for the latter approach extra variables are introduced to define the relative motion of the pair of bodies. This allows to write the rigid body motion of one body explicitly in terms of the remaining variables that describe the motion of the two bodies and the variables that describe their relative motion. From these equations it is possible to partition the variables into dependent variables and independent variables. The variational form of the equations of motion of the system of bodies is given. Using the partitioning of variables into dependent and independent variables, the equations of motion of the system of bodies can be written in terms of the independent variables. It is shown that these equations can be generated the same way for both the global description and the relative description. However, for the relative description use can be made of the fact that the rigid body motion can be solved from the equations that describe the dependency of variables due to kinematic connections.

In chapter 5, an assessment is given of descriptions and approximations. In section 5.2, potential savings of computation time from using the mean displacement conditions for the assumed displacement fields are evaluated. The finite element method and the modal synthesis method are considered in section 5.3 and section 5.4, respectively. Special attention is paid to preventing rigid body motions in the displacement field due to deformation. The effect of the frequently used lumped mass approximation is considered in section 5.4. The displacements due to deformation have been approximated by a linear combination of assumed displacement fields. Due to this approximation, some effects, such as for example the stiffening of a rotary wing due to centrifugal forces, are not present. This is discussed in section 5.5. In section 5.6 a procedure is presented for correcting eigenfrequencies going with a specific set of assumed displacement fields that do not agree with the actual eigenfrequencies. This procedure is used for alliviating the integration time step reducing effect of high frequencies. The numerical experiments presented in this chapter are done with the version of DADS for three-dimensional problems (CADSI, 1988). This general purpose multibody program is based on the global description. The subroutines that evaluate the equations of motion of a deformable body are replaced by subroutines based on the equations of motion presented in chapter 2. The solution algorithm used by DADS is described by Park and Haug (1985, 1986).

In this thesis, vectors and tensors are used in their symbolic form. Advantages of using the symbolic form over the component form are the notational convenience and the absence of the need to specify vector bases. Once the equations of interest are derived, they must be written in component form to allow for their numerical evaluation. In section A.2 some definitions and properties related to vectors and tensors are given. For a more detailed treatment the reader is referred to Malvern (1969) or Chadwick (1976).

chapter 2

The equations of motion of a deformable body

2.1 Introduction

The equations of motion of a deformable body constitute a building block for the equations of motion of a system of deformable bodies. They relate the acceleration of the body and the forces acting on the body. The motion of the body can be obtained by integration of the equations of motion.

In section 2.2 a description is given of the kinematics of a deformable body. Starting from the equations of motion of a particle of the body, the weak form of the equations of motion is derived in section 2.3. Since, in general, a closed-form solution to these equations does not exist, approximate equations of motion based on Galerkin's method are presented in section 2.4. The component form of the resulting equations of motion is presented in section 2.5.

2.2 The kinematics of a deformable body

A body consists of solid matter that occupies a region of the three-dimensional space. Following the customary simplifying concept of matter in continuum mechanics, bodies are assumed to be continuous, i.e. the atomic structure of matter is disregarded. An element of a body is called a particle. The region of a Euclidean point space occupied by the particles of a body is referred to as the current configuration of the body. A particle is identified by the position vector of the corresponding point of the Euclidean point space.

In solid mechanics it is customary to compare the current configuration of a body with a configuration of which all relevant quantities are known, the reference configuration. Usually, the unstressed state of the body is chosen as reference configuration. There is a continuous one-to-one mapping which maps the reference configuration onto the current configuration.

At first sight it is natural to describe the displacement field of a body with the displacement vectors of the particles relative to their position in the reference configuration. However, this has two drawbacks. Firstly, due to large rotations it is necessary to use nonlinear strain-displacement relations even when the strains are small. Secondly, discretization of a continuous body (which is necessary in order to be able to analyze the behaviour of the body with a computer) involves expressing the motion in terms of a (by preference) linear combination of independent displacement fields; this is only possible for some special bodies, such as a bar and a triangular plate with in-plane deformations. At first sight this is not a serious restriction since bodies can be built up from such special bodies, i.e. finite elements. However, this will result generally in a model with many degrees of freedom. Time integration of the equations of motion going with such a model is impractical. One might consider reducing the number of degrees of freedom by a linear transformation mapping the finite element nodal displacements onto generalized body degrees of freedom. Motions going with these body degrees of freedom may be for instance normal modes of free vibration. In order to be able to represent all possible motions of the body as close as possible, it is necessary to include motions that describe rigid body motions. However, in general it is not possible to describe large rigid body rotations as a linear combination of the finite element nodal displacements. These drawbacks are not present when the displacements of the body are resolved into displacements due to rigid body motion and displacements due to deformation.

Consider a body $\mathscr{B}$ with mass m in its configuration G_t at time t (see fig. 2.1). Let $\vec{r}$ be the position vector of an arbitrary particle P of the body, measured from an inertial point O. Let G be a time-independent reference configuration of which all relevant quantities are known. Let $\vec{x}$ be the position vector relative to O of the point of G corresponding to P. A continuous one-to-one mapping exists which maps $\vec{x}$ onto $\vec{r}$, i.e. $\vec{r} = \vec{r}(\vec{x},t)$.

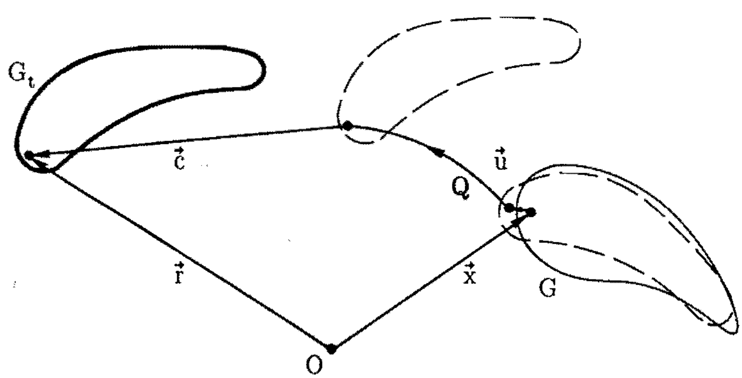

Fig. 2.1 Mapping of $\mathscr{B}$ from reference configuration onto current configuration

The body in G_t can be considered as the result of a deformation of the body in G with displacement field $\vec{u}(\vec{x},t)$, followed by in succession a rigid body rotation about O defined by the proper orthogonal tensor $\mathbf{Q}(t)$, and a rigid body translation defined by the vector $\vec{c}(t)$

$$\vec{r}(\vec{x},t) = \vec{c}(t) + \mathbf{Q}(t)\cdot\{\vec{x} + \vec{u}(\vec{x},t)\}. \tag{2.1}$$

The vector $d\vec{r}$ between two neighbouring points in G_t and the vector $d\vec{x}$ between the corresponding points in G are related by

$$d\vec{r} = \mathbf{Q}(t)\cdot\{d\vec{x} + \vec{u}(\vec{x}+d\vec{x},t) - \vec{u}(\vec{x},t)\}. \tag{2.2}$$

This equation can be rewritten as

$$d\vec{r} = \mathbf{F}\cdot d\vec{x}, \tag{2.3}$$

where, for infinitesimally small $d\vec{x}$, $\mathbf{F}$ is the deformation tensor given by

$$\mathbf{F} = \mathbf{Q}\cdot\{\mathbf{I} + (\vec{\nabla}\vec{u})^C\}. \tag{2.4}$$

Here, $\vec{\nabla}$ is the gradient operator referred to G.

The acceleration and a virtual displacement of a particle are obtained by differentiating (1) twice with respect to time and by taking the variation of (1), respectively. This yields

$$\ddot{\vec{r}} = \ddot{\vec{c}} + \mathbf{Q}\cdot\{\vec{\omega}\times[\vec{\omega}\times(\vec{x}+\vec{u})] + \dot{\vec{\omega}}\times(\vec{x}+\vec{u}) + 2\vec{\omega}\times\dot{\vec{u}} + \ddot{\vec{u}}\}, \tag{2.5}$$

$$\delta\vec{r} = \delta\vec{c} + \mathbf{Q}\cdot\{\delta\vec{\pi}\times(\vec{x}+\vec{u}) + \delta\vec{u}\}, \tag{2.6}$$

where $\vec{\omega}$ and $\delta\vec{\pi}$ are the axial vectors of the skew-symmetric tensors $\mathbf{Q}^C\cdot\dot{\mathbf{Q}}$ and $\mathbf{Q}^C\cdot\delta\mathbf{Q}$, respectively.

The angular velocity vector $\vec{\omega}$ differs from the usual angular velocity vector, which is defined as the axial vector of $\dot{\mathbf{Q}}\cdot\mathbf{Q}^C$. The reason for introducing this alternative angular velocity vector is that then the rotation tensor $\mathbf{Q}$ can be factored out in (5) which is advantageous in the derivation of the equations of motion. The same applies for the virtual rotation vector $\delta\vec{\pi}$.

The displacement field of the body has been resolved into a displacement field due to deformation defined by $\vec{u}$ and a displacement field due to a rigid body motion defined by $\vec{c}$ and $\mathbf{Q}$. In order to get a unique resolution, the displacement field due to deformation is not allowed to represent a rigid body motion. Two kinds of conditions for preventing rigid body motions can be found in literature, namely, conditions for displacements of selected particles of the body and conditions for mean displacements of the body (Koppens et al., 1988). The first kind of conditions is used by, among others, Sunada and Dubowsky (1981), Singh et al. (1985), Agrawal and Shabana (1985), and Haug et al. (1986). The second kind of conditions is used by Agrawal and Shabana (1985), McDonough (1976), and others.

Conditions for displacements of selected particles. This kind of conditions is also used in finite element analyses of structures (Przemieniecki, 1968). It comes to prescribing displacements due to deformation of a number of selected particles to prevent rigid body motion. For example, the displacements due to deformation of one particle P_c are required to be zero:

$$\vec{u}(P_c) = \vec{0}. \tag{2.7}$$

Now, the body can still perform rigid body rotations around P_c. Hence, in addition, rigid body rotations have to be prevented. This may be achieved by constraining the rotation of a body-fixed frame (cf. Singh et al., 1985), or by prescribing displacement components of other particles. In the latter case rigid body motions can be prevented by prescribing altogether six suitably chosen displacement components. When additional displacement components are prescribed, only a restricted class of all possible displacement fields can be described.

Conditions for mean displacements of the body. A rigid body translation involves a displacement of the centre of mass. Consequently, a displacement field $\vec{u}$ cannot represent a rigid body translation when it does not cause a displacement of the centre of mass. This can be expressed mathematically as

$$\int_\Omega \rho\, \vec{u}\; d\Omega = \vec{0}, \tag{2.8}$$

where ρ is the mass density of G and Ω is the reference volume. The translation vector $\vec{c}$ will represent the translation of the centre of mass of the body when this condition is used.

The displacement field due to an infinitesimally small rigid body rotation around O can be represented by the vector field

$$\vec{u}_{\text{rotation}} = \chi\, \vec{e} \times \vec{x}, \tag{2.9}$$

where χ is the rotation angle and $\vec{e}$ is a unit vector parallel to the rotation axis. It can be seen that the displacements due to this rigid body rotation are perpendicular to $\vec{x}$. Consequently, a displacement field $\vec{u}$ cannot represent a rigid body rotation when it is on a mean parallel to $\vec{x}$. This can be expressed mathematically as

$$\int_{\Omega} \rho\, \vec{x} \times \vec{u}\, d\Omega = \vec{0}. \tag{2.10}$$

(ρ has been used as a weighting factor in order to cancel some terms in the equations of motion.) Examples of displacement fields satisfying (8) and (10) are modes of free vibration (Ashley, 1967).

When in addition to condition (8), the centre of mass in the reference configuration is chosen to coincide with O, some more terms in the equations of motion will cancel.

2.3 The equations of motion of a deformable body

The equations representing local balance of linear momentum at an interior point of the body referring to the reference configuration are given by (cf. Malvern, 1969)

$$\vec{\nabla}\cdot(\mathrm{T}\cdot\mathbf{F}^{\mathrm{C}}) + \rho\vec{b} = \rho\ddot{\vec{r}}, \tag{2.11}$$

where T is the second Piola–Kirchhoff stress tensor and $\vec{b}$ is a specific body load vector. It is preferable starting from the equations of motion in this form to starting from the more well-known Cauchy's equations of motion, because the latter would require a transformation of variables referring to the current configuration onto variables referring to the reference configuration. This transformation has already been carried out for the equations of motion in the form (11).

The body is assumed to be stress-free in the reference configuration. Then, for isotropic linear elastic material behaviour, T is related to the strain by (cf. Gurtin, 1981)

$$\mathrm{T} = 2\,\mu\,\mathbf{E} + \lambda\,\mathrm{tr}(\mathbf{E})\,\mathbf{I}, \tag{2.12}$$

where μ and λ are the Lamé elastic constants, I is the identity tensor, and E is the Green–Lagrange strain tensor, defined by

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{F}^{\mathrm{C}}\cdot\mathbf{F} - \mathbf{I}). \tag{2.13}$$

Comparing this expression with the expression for the deformation tensor (4) reveals that the Green–Lagrange strain tensor does not depend on the rigid body motion. (13) may be linearized in case the gradients of the displacements due to deformation are small. In general, this would not be allowable in case the displacements had not been resolved into displacements due to a rigid body motion and displacements due to deformation.

The equations of motion of the body can be obtained by scalar multiplication of (11) with arbitrary test functions $\vec{\Psi}$. The resulting product is identically zero because (11) is identically zero. Consequently, integration of this product over the volume of the body yields

$$\int_\Omega \{\vec{\nabla}\cdot(\mathbf{T}\cdot\mathbf{F}^c) + \rho\vec{b} - \rho\ddot{\vec{r}}\}\cdot\vec{\Psi}\, d\Omega = 0. \tag{2.14}$$

The test functions will be restricted to functions for which this integral exists. Following the customary procedure in solid mechanics, the test functions are chosen from the space of variations of the displacement field of the body. Such a variation is denoted by $\delta\vec{r}$.

The continuity requirements for $\vec{r}$ can be lowered by integrating the first term of (14) by parts. This leads to more severe requirements on the continuity of the test functions but when the test functions are chosen from the space of variations of the displacement field of the body, these requirements will be satisfied. As stated by Zienkiewicz (1977), the solution to the resulting equation, the so-called weak form of (11), is often more realistic physically than the solution to the original problem (11). Application of the divergence theorem to the first term yields

$$-\int_\Omega \mathbf{T}:\delta\mathbf{E}\, d\Omega + \int_\Omega \rho\,\vec{b}\cdot\delta\vec{r}\, d\Omega - \int_\Omega \rho\,\ddot{\vec{r}}\cdot\delta\vec{r}\, d\Omega + \int_\Gamma (\mathbf{F}\cdot\mathbf{T}\cdot\vec{n})\cdot\delta\vec{r}\, d\Gamma = 0, \tag{2.15}$$

where Γ is the surface of G and $\vec{n}$ is the unit outward normal vector to Γ. The surface integral vanishes for that part of the surface where the displacements are prescribed since there $\delta\vec{r} \equiv \vec{0}$. On the remaining part of the surface, $\bar{\Gamma}$, a surface load of $\vec{p}$ per unit of undeformed area is prescribed and $\mathbf{T}$ has to satisfy

$$\vec{p} = \mathbf{F}\cdot\mathbf{T}\cdot\vec{n}. \tag{2.16}$$

Substituting (5), (6), (12) and (16) into (15) yields

$$-\int_\Omega \{2\,\mu\,\mathbf{E} + \lambda\,\mathrm{tr}(\mathbf{E})\,\mathbf{I}\}:\delta\mathbf{E}\, d\Omega$$

$$+\ \delta\vec{c}\cdot\left\{\vec{F} - m\ddot{\vec{c}} - \mathbf{Q}\cdot\left[\vec{\omega}\times\{\vec{\omega}\times(\vec{x}_0+\vec{u}_0)\} + \dot{\vec{\omega}}\times(\vec{x}_0+\vec{u}_0) + 2\vec{\omega}\times\dot{\vec{u}}_0 + \ddot{\vec{u}}_0\right]\right\}$$

$$+\ \delta\vec{\pi}\cdot\Big\{\vec{M} - (\vec{x}_0+\vec{u}_0)\times(\mathbf{Q}^c\cdot\ddot{\vec{c}}) - \vec{\omega}\times\{t(\vec{x}+\vec{u},\vec{x}+\vec{u})\cdot\vec{\omega}\} - t(\vec{x}+\vec{u},\vec{x}+\vec{u})\cdot\dot{\vec{\omega}}$$

$$- 2t(\dot{\vec{u}},\vec{x}+\vec{u})\cdot\vec{\omega} - \vec{v}(\vec{x}+\vec{u},\ddot{\vec{u}})\Big\}$$

$$-\ \delta\vec{u}_0\cdot\{\mathbf{Q}^c\cdot\ddot{\vec{c}}\} + \vec{\omega}\cdot\{t(\delta\vec{u},\vec{x}+\vec{u})\cdot\vec{\omega}\} + \dot{\vec{\omega}}\cdot\vec{v}(\delta\vec{u},\vec{x}+\vec{u}) + 2\vec{\omega}\cdot\vec{v}(\delta\vec{u},\dot{\vec{u}})$$

$$-\int_\Omega \rho\, \ddot{\vec{u}}\cdot\delta\vec{u}\, d\Omega + \int_\Omega \rho\, \delta\vec{u}\cdot(\mathbf{Q}^c\cdot\vec{b})\, d\Omega + \int_{\bar{\Gamma}} \delta\vec{u}\cdot(\mathbf{Q}^c\cdot\vec{p})\, d\bar{\Gamma} = 0, \tag{2.17}$$

where

$$\vec{F} = \int_\Omega \rho\, \vec{b}\, d\Omega + \int_{\bar{\Gamma}} \vec{p}\, d\bar{\Gamma}, \tag{2.18}$$

$$\vec{M} = \int_\Omega \rho\, \{(\vec{x}+\vec{u}) \times (\mathbf{Q}^c\cdot\vec{b})\}\, d\Omega + \int_{\bar{\Gamma}} \{(\vec{x}+\vec{u}) \times (\mathbf{Q}^c\cdot\vec{p})\}\, d\bar{\Gamma}, \tag{2.19}$$

$$\vec{x}_0 = \int_\Omega \rho\, \vec{x}\, d\Omega, \tag{2.20}$$

$$\vec{u}_0 = \int_\Omega \rho\, \vec{u}\, d\Omega, \tag{2.21}$$

$$t(\vec{a},\vec{b}) = \int_\Omega \rho\, \{(\vec{a}\cdot\vec{b})\mathrm{I} - \vec{a}\vec{b}\}\, d\Omega \qquad \forall\, \vec{a},\vec{b}, \tag{2.22}$$

$$\vec{v}(\vec{a},\vec{b}) = \int_\Omega \rho\, (\vec{a} \times \vec{b})\, d\Omega \qquad \forall\, \vec{a},\vec{b}. \tag{2.23}$$

The first term of (17) represents the variation of the strain energy δU of the body due to a virtual displacement $\delta\vec{r}$. In general, this expression is too complicated to evaluate; consequently approximations are used instead. For example the expression for the Green–Lagrange strain tensor (13) is often linearized. Also the body may be approximated by a two- (one-) dimensional body in case one (two) dimension(s) of the body is (are) considerably smaller compared with the other two (one) dimensions using an assumption from which the displacements of an arbitrary material point of the body can be written in terms of the displacement of a plane (line). With such a two- (one-) dimensional body goes an approximate expression for its strain energy. An example of such a body is a plate (beam).

As has been mentioned already at the end of section 2.2, some terms will cancel in the equations of motion when the mean displacement conditions (8) and (10) are used for eliminating rigid body motions. Substitution of successively (8) into (21), and (10) into (23) yields

$$\vec{u}_0 \equiv \vec{0}. \tag{2.24}$$

$$\vec{v}(\vec{x},\vec{u}) \equiv \vec{0}. \tag{2.25}$$

Consequently, also the time derivatives and variations of $\vec{u}_0$ and $\vec{v}(\vec{x},\vec{u})$ vanish.

When the centre of mass of the reference confuguration is chosen to coincide with O, also

$$\vec{x}_0 \equiv 0. \tag{2.26}$$

Substitution of (24)-(26) into (17) yields

$$\begin{aligned}
&-\int_\Omega \{2\,\mu\,\mathrm{E} + \lambda\,\mathrm{tr}(\mathrm{E})\,\mathrm{I}\}:\delta\mathrm{E}\,\mathrm{d}\Omega \\
&+ \delta\vec{c}\cdot\left\{\vec{F} - m\ddot{\vec{c}}\right\} \\
&+ \delta\vec{\pi}\cdot\left\{\vec{M} - \vec{\omega}\times\{\mathrm{t}(\vec{x}+\vec{u},\vec{x}+\vec{u})\cdot\vec{\omega}\} - \mathrm{t}(\vec{x}+\vec{u},\vec{x}+\vec{u})\cdot\dot{\vec{\omega}} - 2\mathrm{t}(\dot{\vec{u}},\vec{x}+\vec{u})\cdot\vec{\omega} - \vec{v}(\vec{u},\ddot{\vec{u}})\right\} \\
&+ \vec{\omega}\cdot\{\mathrm{t}(\delta\vec{u},\vec{x}+\vec{u})\cdot\vec{\omega}\} + \dot{\vec{\omega}}\cdot\vec{v}(\delta\vec{u},\vec{u}) + 2\vec{\omega}\cdot\vec{v}(\delta\vec{u},\dot{\vec{u}}) \\
&- \int_\Omega \rho\,\ddot{\vec{u}}\cdot\delta\vec{u}\,\mathrm{d}\Omega + \int_\Omega \rho\,\delta\vec{u}\cdot(\mathbf{Q}^{\mathrm{c}}\cdot\vec{b})\,\mathrm{d}\Omega + \int_{\bar{\Gamma}} \delta\vec{u}\cdot(\mathbf{Q}^{\mathrm{c}}\cdot\vec{p})\,\mathrm{d}\bar{\Gamma} = 0.
\end{aligned} \tag{2.27}$$

From a comparison of the coefficients of $\delta\vec{c}$ in (17) and (27), it is observed that in (27) the rotation and the displacement due to deformation are not coupled with the translational motion. Comparing the coefficients of $\delta\vec{\pi}$ reveals that the coupling between the rotational and the translational motion has vanished and that the coupling between the rotational motion and the displacement due to deformation is reduced. To conclude, also coupling due to terms that involve $\delta\vec{u}$ is reduced. All this may be advantageous in the numerical evaluation of the equations of motion since firstly, less terms have to be evaluated and secondly, the mass matrix has become more sparse. This is investigated more closely in section 5.2.

2.4 Approximate equations of motion: Galerkin's method

The weak form of the equations of motion of a single body have been presented as equations (17) in the preceding section. The contribution of the displacement due to deformation $\vec{u}$ makes that in general a closed-form solution to this equation does not exist or is not feasible. That is why one resorts to an approximate solution for $\vec{u}$. This solution is sought in a certain N-dimensional vector space of vector-valued functions defined on Ω. Then it can be represented as a linear combination of N functions that constitute a base of this vector space. In general (17) will not be satisfied by this approximate solution for any variation. In solid mechanics one usually only requires that (17) is satisfied for variations that can be written as a linear combination of the N base functions. From this condition the N unknown coefficients in the approximate solution can be determined. This procedure is known as Galerkin's method (Zienkiewicz, 1977). It leads to a system of ordinary differential equations which can be solved with numerical integration routines.

The above-mentioned vector space must be chosen such that its elements satisfy the same kinematic conditions as $\vec{u}$. Further, its elements must be continuous and once piecewise continuously differentiable such that (17) can be evaluated. In case the body has been approximated by a one- or two-dimensional body, the accompanying approximate expression for the strain energy may contain second order derivatives of the displacement field $\vec{u}$. Then the elements of the vector space and their first derivatives must be continuous, and their second derivatives must be piecewise continuous.

Let $\underset{\sim}{\vec{\Phi}}(\vec{x})$ be a column matrix of N vector-valued functions $\vec{\Phi}_i(\vec{x})$, $i = 1, 2,\dots, N$, that constitute a base of the N-dimensional vector space of vector-valued functions. Then, following Galerkin's method, both $\vec{u}$ and $\delta\vec{u}$ are approximated by a linear combination of these base functions:

$$\vec{u}(\vec{x},t) \approx \sum_{i=1}^{N} \alpha_i(t)\, \vec{\Phi}_i(\vec{x}) = \underset{\sim}{\alpha}^T(t)\, \underset{\sim}{\vec{\Phi}}(\vec{x}), \tag{2.28}$$

$$\delta\vec{u}(\vec{x}) \approx \sum_{i=1}^{N} \delta\alpha_i\, \vec{\Phi}_i(\vec{x}) = \delta\underset{\sim}{\alpha}^T\, \underset{\sim}{\vec{\Phi}}(\vec{x}), \tag{2.29}$$

where $\underset{\sim}{\alpha}(t)$ is a column matrix of generalized displacements $\alpha_i(t)$, $i = 1, 2,\dots, N$, and $\delta\underset{\sim}{\alpha}$ is a column matrix of the arbitrary constants $\delta\alpha_i(t)$, $i = 1, 2,\dots, N$. Substitution of these equations into (17) yields the variational form of the equations of motion for the approximated displacement fields (28) and (29):

$$\begin{aligned}
&+ \delta\vec{c}\cdot\left\{\vec{F} - m\ddot{\vec{c}} - \mathbf{Q}\cdot\left[\vec{\omega}\times(\vec{\omega}\times\vec{D}_1) + \dot{\vec{\omega}}\times\vec{D}_1 + 2\vec{\omega}\times(\dot{\underset{\sim}{\alpha}}^T\underset{\sim}{\vec{C}}_2) + \ddot{\underset{\sim}{\alpha}}^T\underset{\sim}{\vec{C}}_2\right]\right\} \\
&+ \delta\vec{\pi}\cdot\left\{\vec{M} - \vec{D}_1\times(\mathbf{Q}^C\cdot\ddot{\vec{c}}) - \vec{\omega}\times(\mathbf{J}\cdot\vec{\omega}) - \mathbf{J}\cdot\dot{\vec{\omega}} - 2(\dot{\underset{\sim}{\alpha}}^T\underset{\sim}{\mathbf{D}}_2)\cdot\vec{\omega} - \ddot{\underset{\sim}{\alpha}}^T\underset{\sim}{\vec{D}}_3\right\} \\
&+ \delta\underset{\sim}{\alpha}^T\left\{\underset{\sim}{f} - \ddot{\vec{c}}\cdot(\mathbf{Q}\cdot\underset{\sim}{\vec{C}}_2) + \vec{\omega}\cdot(\underset{\sim}{\mathbf{D}}_2\cdot\vec{\omega}) - \dot{\vec{\omega}}\cdot\underset{\sim}{\vec{D}}_3 + 2\vec{\omega}\cdot(\underline{\vec{C}}_7\dot{\underset{\sim}{\alpha}}) - \underline{C}_8\ddot{\underset{\sim}{\alpha}}\right\} = \delta U,
\end{aligned} \tag{2.30}$$

where

$$\vec{F} = \int_\Omega \rho\,\vec{b}\,d\Omega + \int_{\bar{\Gamma}} \vec{p}\,d\bar{\Gamma}, \tag{2.31}$$

$$\vec{M} = \int_{\Omega} \rho \, \{\vec{x} \times (\mathbf{Q}^{C} \cdot \vec{b})\} \, d\Omega + \int_{\bar{\Gamma}} \{\vec{x} \times (\mathbf{Q}^{C} \cdot \vec{p})\} \, d\bar{\Gamma}$$

$$+ \underset{\sim}{\alpha}^{T} \int_{\Omega} \rho \, \{\vec{\underset{\sim}{\Phi}} \times (\mathbf{Q}^{C} \cdot \vec{b})\} \, d\Omega + \underset{\sim}{\alpha}^{T} \int_{\bar{\Gamma}} \{\vec{\underset{\sim}{\Phi}} \times (\mathbf{Q}^{C} \cdot \vec{p})\} \, d\bar{\Gamma}, \tag{2.32}$$

$$\underset{\sim}{f} = \int_{\Omega} \rho \, \{\vec{\underset{\sim}{\Phi}} \cdot (\mathbf{Q}^{C} \cdot \vec{b})\} \, d\Omega + \int_{\bar{\Gamma}} \{\vec{\underset{\sim}{\Phi}} \cdot (\mathbf{Q}^{C} \cdot \vec{p})\} \, d\bar{\Gamma}, \tag{2.33}$$

$$\vec{C}_1 = \int_{\Omega} \rho \, \vec{x} \, d\Omega, \tag{2.34}$$

$$\vec{\underset{\sim}{C}}_2 = \int_{\Omega} \rho \, \vec{\underset{\sim}{\Phi}} \, d\Omega, \tag{2.35}$$

$$\mathbf{C}_3 = \int_{\Omega} \rho \, \{(\vec{x} \cdot \vec{x})\mathbf{I} - \vec{x}\vec{x}\} \, d\Omega, \tag{2.36}$$

$$\underset{\sim}{\mathbf{C}}_4 = \int_{\Omega} \rho \, \{(\vec{\underset{\sim}{\Phi}} \cdot \vec{x})\mathbf{I} - \vec{\underset{\sim}{\Phi}}\vec{x}\} \, d\Omega, \tag{2.37}$$

$$\underline{\mathbf{C}}_5 = \int_{\Omega} \rho \, \{(\vec{\underset{\sim}{\Phi}} \cdot \vec{\underset{\sim}{\Phi}}^{T})\mathbf{I} - \vec{\underset{\sim}{\Phi}}\vec{\underset{\sim}{\Phi}}^{T}\} \, d\Omega, \tag{2.38}$$

$$\vec{\underset{\sim}{C}}_6 = \int_{\Omega} \rho \, \{\vec{x} \times \vec{\underset{\sim}{\Phi}}\} \, d\Omega, \tag{2.39}$$

$$\vec{\underline{C}}_7 = \int_{\Omega} \rho \, \{\vec{\underset{\sim}{\Phi}} \times \vec{\underset{\sim}{\Phi}}^{T}\} \, d\Omega, \tag{2.40}$$

$$\underline{C}_8 = \int_{\Omega} \rho \, \{\vec{\underset{\sim}{\Phi}} \cdot \vec{\underset{\sim}{\Phi}}^{T}\} \, d\Omega, \tag{2.41}$$

$$\mathbf{J} = \mathbf{C}_3 + \underset{\sim}{\alpha}^{T} \, (\underset{\sim}{\mathbf{C}}_4 + \underset{\sim}{\mathbf{C}}_4^{C}) + \underset{\sim}{\alpha}^{T} \, \underline{\mathbf{C}}_5 \, \underset{\sim}{\alpha}, \tag{2.42}$$

$$\vec{D}_1 = \vec{C}_1 + \underset{\sim}{\alpha}^{T} \, \vec{\underset{\sim}{C}}_2, \tag{2.43}$$

$$\underset{\sim}{\mathbf{D}}_2 = \underset{\sim}{\mathbf{C}}_4 + \underline{\mathbf{C}}_5 \, \underset{\sim}{\alpha}, \tag{2.44}$$

$$\vec{\underset{\sim}{D}}_3 = \vec{\underset{\sim}{C}}_6 - \vec{\underline{C}}_7 \, \underset{\sim}{\alpha} \tag{2.45}$$

An underscore and a wavy underscore in these expressions denote an N × N matrix and an N × 1 column matrix, respectively. The quantities (34)-(41) are time-independent and consequently they have to be evaluated only once at the start of a numerical simulation. They can be determined once the base functions $\underset{\sim}{\vec{\Phi}}(\vec{x})$ have been chosen.

2.5 Approximate equations of motion in component form

The equations of motion as presented in the preceding section are in symbolic vector/tensor form. For computational purposes these equations must be rewritten in terms of the components of the vectors and tensors relative to some base. All vectors and tensors can easily be written in terms of their components relative to an inertial base thanks to the fact that the reference configuration is inertial. Consequently, there is no need to specify the base relative to which a certain vector or tensor is written. This is in contrast to the usual description found in the literature where both an inertial base and a body-fixed base is are introduced (Casey, 1983; Sol, 1983; McInnis and Liu, 1986).

Let the vectors and tensors be written in terms of their components relative to a right-handed orthonormal inertial base $\underset{\sim}{\vec{e}}$. The components of vectors and tensors will be stored in, respectively 3×1 column matrices and 3×3 matrices. Vectors in cross-product terms are replaced by the matrix representation of the corresponding skew-symmetric tensors which will be denoted by a wavy superscript (cf. A.1). Elements of the matrices defined in the preceding section are indicated by their row and column indices in order to obtain equations of motion in a form suitable for computer implementation. Using this notation and making use of the fact that $\underset{\sim}{\vec{e}} \cdot \underset{\sim}{\vec{e}}^{T} = \underline{I}$, the third-order unit matrix, the equations of motion (30) become in component form

$$
\begin{aligned}
&\delta \underset{\sim}{c}^{T}\Big\{\underset{\sim}{F} - m\ddot{\underset{\sim}{c}} - \underline{Q}\Big[\underline{\tilde{\omega}}\,\underline{\tilde{\omega}}\,\underset{\sim}{D}_1 - \underline{\tilde{D}}_1\dot{\underset{\sim}{\omega}} + 2\underline{\tilde{\omega}}\sum_j \dot{\alpha}(j)\underset{\sim}{C}_2(j) + \sum_j \ddot{\alpha}(j)\underset{\sim}{C}_2(j)\Big]\Big\} \\
&+ \delta\underset{\sim}{\pi}^{T}\Big\{\underset{\sim}{M} - \underline{\tilde{D}}_1\,\underline{Q}^{T}\ddot{\underset{\sim}{c}} - \underline{\tilde{\omega}}\,\underline{J}\,\underset{\sim}{\omega} - \underline{J}\dot{\underset{\sim}{\omega}} - 2\{\sum_j \dot{\alpha}(j)\underline{D}_2(j)\}\underset{\sim}{\omega} - \sum_j \ddot{\alpha}(j)\underset{\sim}{D}_3(j)\Big\} \\
&+ \sum_i \delta\alpha(i)\Big\{f(i) - \underset{\sim}{C}_2^{T}(i)\underline{Q}^{T}\ddot{\underset{\sim}{c}} + \underset{\sim}{\omega}^{T}\underline{D}_2(i)\underset{\sim}{\omega} - \underset{\sim}{D}_3^{T}(i)\dot{\underset{\sim}{\omega}} + 2\underset{\sim}{\omega}^{T}\sum_j \dot{\alpha}(j)\underset{\sim}{C}_7(i,j) \\
&\qquad\qquad\qquad\qquad - \sum_j \ddot{\alpha}(j)C_8(i,j)\Big\} = \delta U. \qquad (2.46)
\end{aligned}
$$

These equations cannot be integrated because the components of the angular velocity vector, $\underset{\sim}{\omega}$, cannot be integrated to obtain angular displacements, since they are non-integrable combinations of the first time derivatives of angular displacements. These angular displacements are required for evaluating the rotation matrix $\underline{Q}$. For this reason, differential equations must be added from which the angular displacements can be obtained.

Various kinds of angular displacements are in use, such as Euler angles, Bryant angles and Euler parameters (Wittenburg, 1977). The rotation matrix $\underline{Q}$ written in terms of Euler angles or Bryant angles contains the sine and cosine of these angles. Evaluation of these goniometric functions is laborious. These goniometric functions cause also a swell of terms when the first or second time derivative of $\underline{Q}$ is required. In addition, these angular displacements may suffer from singularities. Due to these drawbacks usually Euler parameters are preferable. A disadvantage of Euler parameters is that they are dependent.

The required differential equations for Euler parameters, which have been derived in appendix B, are

$$\dot{\underset{\sim}{q}} = \tfrac{1}{2}\,\underline{G}^{T}\underset{\sim}{\omega}, \tag{2.47}$$

where

$$\underset{\sim}{q} = [\, q_0 \;\; q_1 \;\; q_2 \;\; q_3 \,]^{T} \tag{2.48}$$

is a column matrix with Euler parameters, and

$$\underline{G} = \begin{bmatrix} -q_1 & q_0 & q_3 & -q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ -q_3 & q_2 & -q_1 & q_0 \end{bmatrix}. \tag{2.49}$$

In literature, $\underset{\sim}{\omega}$ and $\dot{\underset{\sim}{\omega}}$ are often written in terms of the angular displacements and their first and second time derivatives. This leads to more extended equations of motion. Moreover, when Euler parameters are used as angular displacements, an extra equation of motion is obtained. From this and in view of the results described by Nikravesh et al. (1985) for rigid bodies, it is discouraged to write the equations of motion in terms of the angular displacements and their first and second time derivatives.

However, the equations of motion on which the computer program DADS is based are written in terms of Euler parameters and their first and second time derivatives. Adapting the program to the above given preference would involve rewriting the program entirely. Because of the lack of the required source code and because of the large amount of work involved, the equations of motion are written partially in terms of Euler parameters, such that only a small part of the program has to be rewritten. From this the extra equation of motion that has been mentioned above is introduced. This makes the program less efficient. Consequently the computation times reported in chapter 5 are longer that those which would have been obtained with a rewritten program. However, the conclusions regarding computation time presented in chapter 5 are not affected.

From appendix B, $\delta \underset{\sim}{\pi}$ and $\dot{\underset{\sim}{\omega}}$ can be written in terms of the Euler parameters as

$$\delta \underset{\sim}{\pi} = 2\, \underline{G}\, \delta \underset{\sim}{q}, \tag{2.50}$$

$$\dot{\underset{\sim}{\omega}} = 2\, \underline{G}\, \ddot{\underset{\sim}{q}}. \tag{2.51}$$

Substitution into (46) yields

$$\begin{aligned}
&\delta \underset{\sim}{c}^T\Big\{\underset{\sim}{F} - m\ddot{\underset{\sim}{c}} - \underline{Q}\Big[\tilde{\underline{\omega}}\,\tilde{\underline{\omega}}\,\underset{\sim}{D}_1 - 2\tilde{\underline{D}}_1\underline{G}\ddot{\underset{\sim}{q}} + 2\tilde{\underline{\omega}}\sum_j \dot{\alpha}(j)\underset{\sim}{C}_2(j) + \sum_j \ddot{\alpha}(j)\underset{\sim}{C}_2(j)\Big]\Big\} \\
&+ 2\delta \underset{\sim}{q}^T\underline{G}^T\Big\{\underset{\sim}{M} - \tilde{\underline{D}}_1\,\underline{Q}^T\ddot{\underset{\sim}{c}} - \tilde{\underline{\omega}}\,\underline{J}\,\underset{\sim}{\omega} - 2\underline{J}\underline{G}\ddot{\underset{\sim}{q}} - 2\{\sum_j \dot{\alpha}(j)\underline{D}_2(j)\}\underset{\sim}{\omega} - \sum_j \ddot{\alpha}(j)\underset{\sim}{D}_3(j)\Big\} \\
&+ \sum_i \delta\alpha(i)\Big\{f(i) - \underset{\sim}{C}_2^T(i)\underline{Q}^T\ddot{\underset{\sim}{c}} + \underset{\sim}{\omega}^T\underline{D}_2(i)\underset{\sim}{\omega} - 2\underset{\sim}{D}_3^T(i)\underline{G}\ddot{\underset{\sim}{q}} + 2\underset{\sim}{\omega}^T\sum_j \dot{\alpha}(j)\underset{\sim}{C}_7(i,j) \\
&\qquad - \sum_j \ddot{\alpha}(j)C_8(i,j)\Big\} = \delta U.
\end{aligned} \tag{2.53}$$

These equations are somewhat less extended than those obtained by Yoo and Haug (1986a, b) as a result of the fact that terms involving $\underset{\sim}{\omega}$ are not replaced by their counterpart in terms of Euler parameters and as a result of the fact that some terms in their equations of motion would have cancelled when they had taken into account that $\dot{\underset{\sim}{q}}^T\underset{\sim}{q}$ is zero in their expression for the kinetic energy. As a consequence they have an additional time-independent term as compared to the time-independent terms given above.

These equations have been implemented in the computer program DADS-3D, replacing the routines based on the equations of motion of Yoo and Haug (1986a, b). Two additional versions have been created in order to investigate the feasibility of the mean displacement conditions for eliminating rigid body motions: in one version only the multiplications involving $\underset{\sim}{C}_1$, $\underset{\sim}{C}_2(i)$, and $\underset{\sim}{C}_6(i)$ are skipped in order to study the advantage of having simpler equations of motion separately; in the other version also the increased sparseness of the mass matrix is taken into account in solving the equations of motion.

chapter 3

Generating base functions

3.1 Introduction

In the preceding chapter the displacement field of a body has been resolved into a displacement field due to a rigid body motion and a displacement field $\vec{u}$ due to deformation. The instantaneous displacement field $\vec{u}$ is an element of a vector space of vector-valued functions defined on Ω which represent displacement fields that do not contain a rigid body motion. In section 2.4 this vector space has been replaced by an N-dimensional vector space. Any element of this vector space can be written in the form of a linear combination of base functions of this vector space. In case this vector space has been chosen properly the linear combination will be a good approximation of the actual solution.

In this chapter three methods for generating base functions will be discussed, namely the assumed-modes method, the finite element method and the modal synthesis method. From examples it will be illustrated how the time-independent inertia coefficients (2.34)-(2.41) and the stiffness terms originating from the variation of the strain energy can be derived for these base functions.

3.2 The assumed-modes method

In the assumed-modes method analytic base functions are used that are defined on the entire volume of the body. These can only be generated for regularly shaped bodies: these are bodies with a geometry that can be described analytically. Consequently, the assumed-modes method is restricted to such regularly shaped bodies. Advantage can be taken of knowledge of the behaviour of $\vec{u}$ by chosing a vector space that resembles the actual solution well. Then a good approximation can be obtained with only a few base functions. A more accurate solution will be obtained when the number of base functions is increased. However this is at the expense of an increase of the required computation time. Depending on the nature of the base functions, and the mass and stiffness distributions, the time-independent inertia and stiffness terms can be evaluated analytically or they must be determined numerically.

Example: uniform beam

A uniform beam made of homogeneous material has been selected to illustrate the assumed-modes method because of its simple geometry and since parts of mechanical systems can often be modelled as a uniform beam. Consider the uniform beam of length ℓ and mass m shown in fig. 3.1. We choose a reference configuration G with straight elastic axis and with its centre of mass coinciding with an inertial point O. Introduce an orthonormal right-handed vector base $\underset{\sim}{\vec{e}}$, such that $\vec{e}_1$ is parallel to the elastic axis of the beam. Consider in the first instance only displacements due to deformation in the plane spanned by $\vec{e}_1$ and $\vec{e}_2$. Using the Bernoulli-Euler beam theory, only the displacements of the elastic axis need to be considered. The elastic axis is assumed to be inextensible.

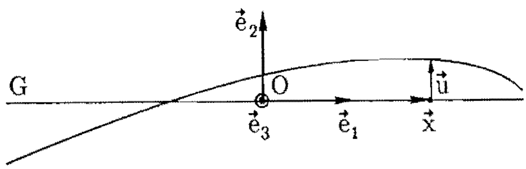

Fig. 3.1 Deformed beam and its reference configuration

The position vector of an arbitrary particle on the elastic axis of the beam in its reference configuration and its displacement vector due to deformation are resolved into their components in the base $\underset{\sim}{\vec{e}}$. This yields

$$\vec{x} = \tfrac{1}{2}\, \xi\, \ell\, \vec{e}_1, \tag{3.1}$$

$$\vec{u} = v(\xi,t)\, \vec{e}_2, \tag{3.2}$$

where ξ is the dimensionless distance in the reference configuration of an arbitrary particle on the elastic axis measured from the centre of mass and made dimensionless with $\ell/2$, and $v(\xi,t)$ is the transverse deflection of points on the elastic axis.

The rotary inertia of the cross-section of the beam will be neglected and the expression for the strain energy of the beam which will be used is

$$U = (4EI_3/\ell^3) \int_{-1}^{1} \{\partial^2 v/\partial\xi^2\}^2 d\xi, \tag{3.3}$$

where EI_3 is the rigidity of the beam for bending in a plane perpendicular to $\vec{e}_3$.

What choice will be appropriate for the vector space of functions for approximating the transverse deflections depends on the deflections which are to be expected. These deflections depend on the load on the beam which consists of load due to the acceleration of the beam, applied load, and load due to connections with other bodies. The acceleration due to rigid body motion varies linearly along the axis of the beam as can be easily veryfied from (2.5). The static deflection going with the corresponding inertia forces varies along the axis of the beam as a quintic polynomial. In table 3.1 the error of the eigenfrequencies, obtained with the assumed-modes method using quintic polynomials, of a uniform beam for various boundary conditions and made dimensionless with the corresponding analytic eigenfrequencies are presented. It can be concluded that quintic polynomials give a good approximation for the lowest eigenfrequencies. Applied concentrated loads and load due to connections at the ends of the beam cause a deflection which varies along the axis of the beam as a cubic polynomial. Consequently the vector space of quintic polynomials is capable of approximating the transverse deflections of uniform beams in many situations.

Table 3.1 Dimensionless error of approximated eigenfrequencies

	1	2	3	4
C-C	.003	.020		
C-P	.001	.015	.738	
C-G	.000	.011	.078	
C-F	.000	.006	.027	1.329
P-P	.000	.004	.484	.748
P-G	.000	.003	.079	.640
P-F	.002	.025	.574	1.102
G-G	.000	.038	.095	
G-F	.000	.017	.071	1.186
F-F	.009	.030	.847	1.280

C = clamped
P = pinned
G = guided
F = free

Consider the arbitrary quintic polynomial

$$\vec{\Phi}(\xi) = \{a_0 + a_1\,\xi + a_2\,\xi^2 + a_3\,\xi^3 + a_4\,\xi^4 + a_5\,\xi^5\}\,\vec{e}_2. \tag{3.4}$$

Deflections approximated with such a polynomial include also rigid body motions, which are not allowed in the displacement field $\vec{u}$ as has been discussed in the preceding chapter. When base functions are selected that do not contain rigid body motions, also the linear combinations (2.28) and (2.29) will be free of rigid body motions. For the present example the mean displacement conditions will be used to eliminate rigid body motions since these conditions yield the most simple equations of motion. Condition (2.8) applied to the quintic polynomial (4) leads to

$$a_0 + (1/3)a_2 + (1/5)a_4 = 0. \tag{3.5}$$

Condition (2.10) leads to

$$(1/3)a_1 + (1/5)a_3 + (1/7)a_5 = 0. \tag{3.6}$$

Displacement fields of quintic polynomials (4) that satisfy (5) and (6) are free of rigid body motions.

A base of the space of quintic polynomials that satisfies these conditions can be chosen in many ways. In order to minimize the number of nonzero terms in the mass matrix the base functions will be chosen orthogonal. Consequently, the off-diagonal terms of the modal mass matrix (2.41) will vanish. Base functions will be chosen to be either odd or even in order to be able to take advantage of possible symmetry and in order to make subsequent derivations easier. The base that has been chosen on account of these conditions is

$$\vec{\Phi}_1(\xi) = (1/2)(-1 + 3\xi^2)\,\vec{e}_2, \tag{3.7}$$

$$\vec{\Phi}_2(\xi) = (1/2)(-3\xi + 5\xi^3)\,\vec{e}_2, \tag{3.8}$$

$$\vec{\Phi}_3(\xi) = (1/8)(3 - 30\xi^2 + 35\xi^4)\,\vec{e}_2, \tag{3.9}$$

$$\vec{\Phi}_4(\xi) = (1/8)(15\xi - 70\xi^3 + 63\xi^5)\,\vec{e}_2. \tag{3.10}$$

These polynomials are normalized such that they equal 1 for $\xi = 1$. These functions are plotted in fig. 3.2.

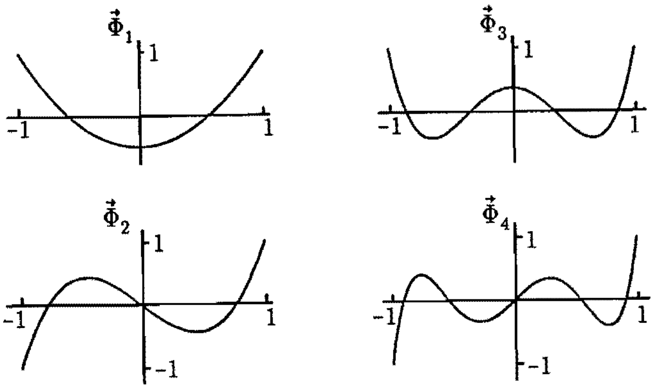

Fig. 3.2 Quintic polynomial base functions

The transverse deflections of the beam due to deformation, parallel to $\vec{e}_3$ can be approximated by the same base functions with $\vec{e}_2$ replaced by $\vec{e}_3$. Using this base for approximating the displacements due to deformation, the non-zero time-independent quantities (2.34)-(2.41) are

$$C_3 = (\vec{e}_2\vec{e}_2+\vec{e}_3\vec{e}_3)\ m\ell^2/12, \tag{3.11}$$

$$\underline{C}_5 = \begin{bmatrix} \underline{M} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underline{M} \end{bmatrix} \underset{\sim}{\vec{e}}^{T} \underset{\sim}{\vec{e}} - \begin{bmatrix} \underline{M}\vec{e}_2\vec{e}_2 & \underline{M}\vec{e}_2\vec{e}_3 \\ \underline{M}\vec{e}_3\vec{e}_2 & \underline{M}\vec{e}_3\vec{e}_3 \end{bmatrix}, \tag{3.12}$$

$$\underline{\vec{C}}_7 = \begin{bmatrix} \underset{\sim}{0} & \underline{M} \\ -\underline{M} & \underset{\sim}{0} \end{bmatrix} \vec{e}_1, \tag{3.13}$$

$$\underline{C}_8 = \begin{bmatrix} \underline{M} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underline{M} \end{bmatrix}, \tag{3.14}$$

where

$$\underline{M} = m \begin{bmatrix} 1/5 & 0 & 0 & 0 \\ 0 & 1/7 & 0 & 0 \\ 0 & 0 & 1/9 & 0 \\ 0 & 0 & 0 & 1/11 \end{bmatrix}. \tag{3.15}$$

Using equation (3) to evaluate the strain energy, the expression for the variation of the strain energy becomes

$$\delta U = \delta \underset{\sim}{\alpha}^{T} \begin{bmatrix} \underline{K}_3 & \underset{\sim}{0} \\ \underset{\sim}{0} & \underline{K}_2 \end{bmatrix} \underset{\sim}{\alpha}, \tag{3.16}$$

where

$$\underline{K}_i = \frac{EI_i}{\ell^3} \begin{bmatrix} 144 & 0 & 480 & 0 \\ 0 & 1200 & 0 & 3360 \\ 480 & 0 & 5520 & 0 \\ 0 & 3360 & 0 & 18480 \end{bmatrix}. \tag{3.17}$$

3.3 The finite element method

The finite element method is extensively used for the determination of the dynamic behaviour of structures. Basically, the finite element method as described in this section for approximating the displacement field due to deformation is the same as the regular finite element method. However, because of the subdivision of displacements, extra inertia properties of the finite elements are required (Shabana, 1986). In this section an overview of the general procedure of the finite element method is given. For a more detailed treatment the reader is referred to the literature which is plentiful available, e.g. Przemieniecki (1968), Ziekiewicz (1977), and Rao (1982). The derivation of the element properties is illustrated for a truss element with linear and quadratic shape functions.

The assumed-modes method as presented in the preceding section is inadequate for most practical problems since most bodies encountered in practice are not regularly shaped. The finite element method provides a way of generating base functions for arbitrarily shaped bodies. The basic idea is to subdivide bodies into small polyhedral parts called finite elements. For such elements it is possible to generate base functions. In general it is not possible to built up the volume of a body exactly with such elements due to the shape of the body. This error will decrease when the number of elements is increased.

On each finite element a number of points is selected, the nodes, usually situated on the boundary of the element. Nodes on the common boundary of neighbouring elements must coincide. In order to achieve this, the nodes on the boundaries are chosen in a systematic way, for instance at vertices. In order to satisfy the continuity requirements in a systematic way the base functions are chosen such that they are equal to unity at one node and zero at all the other nodes; they are only nonzero for the elements to which the node belongs except for the boundaries that do not contain the node. Since base functions extend only over elements with common nodes, base functions defined on elements that have no common nodes are orthogonal which renders the inertia and stiffness matrices of the finite element model of the body sparse. Because base functions are such that they are equal to unity at just one node and zero at all the other nodes, the coefficients of the base functions in the approximation for the displacement field $\vec{u}$ (2.28) represent nodal displacements. Consequently, the components of the nodal displacements relative to a common base can be used as the unknown coefficients in the linear combination (2.28). The functions defined on an element are called the element shape functions. A base function corresponding to a node is the junction of the element shape functions which equal unity at that node.

In general, a more accurate solution will be obtained when the number of finite elements is increased. In order to ensure convergence, the shape functions must be such that displacement fields can be described that correspond to a rigid body motion of the element, and displacement fields that correspond to a constant strain condition (Zienkiewicz, 1977), next to the continuity requirements. One often prefers to use polynomials as shape functions because inertia and stiffness properties can then be evaluated in closed form. The required minimum degree of these polynomials is determined by the convergence requirements on the shape functions. In general, for a given desired accuracy of the solution the total number of unknowns in a problem can be reduced when the degree of the polynomial is increased especially when the gradient of the displacement field varies sharply. However this leads to less sparse matrices and the effort required for formulating and evaluating the element inertia and stiffness properties increases. Consequently numerical experiments are necessary to determine whether it is advantageous to use polynomials with a higher degree than required. The number of nodes and the degree of the polynomial are linked in such a way that the total number of nodal displacements equals the number of coefficients in the polynomial.

The exact expression for the variation of the strain energy as given by the first term of (2.17) and its linearized counterpart contain only first order spatial derivatives of the displacement field due to deformation. Consequently the base functions must be continuous and piecewise continuously differentiable. Linear polynomials are the lowest degree polynomials that meet these requirements. An example of a class of elements that use linear polynomials are the simplex elements shown in fig. 3.3.

Their shape functions are the same for all displacement components. As a result

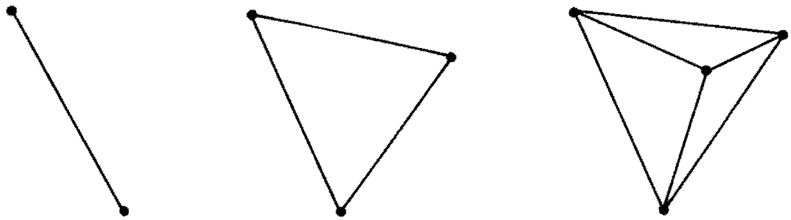

Fig. 3.3 Simplex elements

the displacement field within the element can be easily written in terms of the nodal displacement components relative to a base common for all elements. The shape functions can be used for interpolating the position vector of an arbitrary particle of the element from the position vectors of the nodes of the element in the reference configuration. When both the displacement field and the position vector are written in terms of their components relative to a common base then there will be no need to transform the element properties to a common base. This applies to all elements with shape functions that are independent of the orientation of the element.

Since the shape functions of these elements can describe large rigid body motions of the element, they can be used to describe the displacement field of a body without resolving the displacements into displacements due to a rigid body motion and displacements due to deformation. In fact this is done in literature for analyzing mechanical systems, e.g. the truss element considered by Jonker (1988). However, many regular finite elements are not capable of describing large rigid body rotations. Consequently special elements must be derived in case the displacement field of a body is not resolved into displacements due to a rigid body motion and displacements due to deformation. Jonker (1988) presented a spatial beam element for this purpose. The advantage of such an element description is that the contribution of the inertia of a body is taken into account by the assembled mass matrix, and the equations of motion of a system of bodies are ordinary differential equations which can be obtained by the regular assembly process. However, this has the two already mentioned drawbacks: firstly, nonlinear strain-displacement relations have to be used even when the strains are small and secondly, it is not possible to reduce the number of degrees of freedom using the modal synthesis method to be described in the next section. This way of description, i.e. without resolving the displacement field of the body, is not further considered in this thesis.

One may prefer to use different shape functions because the behaviour of one displacement component is different from the behaviour of other components. This occurs especially with one- or two-dimensional elements in a space of higher dimension. Particularly the behaviour of tangential displacements and transverse displacements generally differ: a transverse displacement field that causes bending must satisfy more severe continuity requirements because the expression for the variation of the strain energy due to bending contains second order derivatives, whereas its counterpart for tangential displacements contains first order derivatives; in case no bending is allowed the displacement field must be such that the element remains straight, i.e. the transverse displacement must vary linearly whereas the tangential displacement may be described by a higher degree polynomial. The shape functions are such that only infinitesimally small rigid rotations of the element can be described. When different shape functions are used it is necessary to transform quantities from an element base to a base common to all elements.

For all finite elements inertia properties have to be evaluated such that the time-independent inertia coefficients (2.34)-(2.41) can be determined. Only $\underline{C}_8$ is required for regular finite elements; the other coefficients are required as a result of the resolution of displacements into displacements due to rigid body motion and displacements due to deformation. The inertia and stiffness properties of an entire body can be obtained by adding the contribution of all elements. This process is identical to the assembly process of the standard finite element method. Rigid body motions of the assembled finite element model can be prevented in the same way as is customary for the standard finite element method.

The derivation of the element inertia and stiffness properties will be illustrated with two examples each from one of the two categories of elements discussed above, namely a uniform pin-jointed truss element with respectively a linearly and a quadratically varying displacement field.

Example 1: linear pin-jointed truss element

Consider the uniform pin-jointed truss element of length ℓ and mass m shown in fig. 3.4. For a truss element at least two nodes have to be introduced in order to be able to describe displacement fields that correspond to a rigid body motion of the element and displacement fields that correspond to a constant strain condition: with two nodes correspond six unknown displacements; five are required to describe a rigid body motion (a truss element has only five rigid degrees of freedom since a rotation about its axis is immaterial), consequently one degree of freedom is left for describing a constant strain condition. Introduce two nodes situated at the

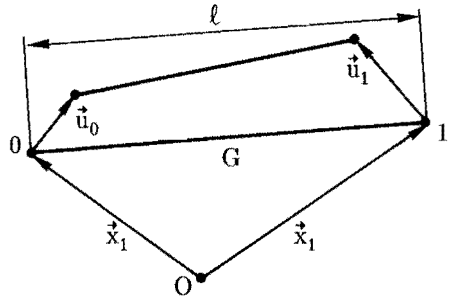

Fig. 3.4 Linear truss element

endpoints of the element. The constant strain condition corresponds to a linearly varying displacement along the axis of the element. From this the displacement of a particle P of the element can be written in terms of its nodal displacements

$$\vec{u}(\xi,t) = (1-\xi)\,\vec{u}_0(t) + \xi\,\vec{u}_1(t), \tag{3.18}$$

where ξ is the distance between P and node 0 in the undeformed configuration and made dimensionless with ℓ. This equation can be rewritten in a form equivalent to (2.28)

$$\vec{u}(\xi,t) = \left[\, \underset{\sim}{u}_0^T(t) \quad \underset{\sim}{u}_1^T(t) \,\right] \begin{bmatrix} (1-\xi)\underline{I} \\ \xi\,\underline{I} \end{bmatrix} \vec{\underset{\sim}{e}} = {}_e\underset{\sim}{\alpha}^T(t)\; {}_e\vec{\underset{\sim}{\Phi}}(\xi), \tag{3.19}$$

where $\underset{\sim}{u}_0$ and $\underset{\sim}{u}_1$ are the matrix representation of $\vec{u}_0$ and $\vec{u}_1$ relative to an inertial base $\vec{\underset{\sim}{e}}$; the subscript $_e$ refers to the contribution of the element to the quantity concerned. The shape functions of this element are $(1-\xi)\,\vec{\underset{\sim}{e}}$ and $\xi\,\vec{\underset{\sim}{e}}$.

The position vector of an arbitrary particle in the reference configuration can be expressed in terms of the position vectors of the nodes in the reference configuration

$$\vec{x}(\xi) = \left[\, \underset{\sim}{x}_0^T \quad \underset{\sim}{x}_1^T \,\right] \begin{bmatrix} (1-\xi)\underline{I} \\ \xi\,\underline{I} \end{bmatrix} \vec{\underset{\sim}{e}} = {}_e\underset{\sim}{X}^T\; {}_e\vec{\underset{\sim}{\Phi}}(\xi), \tag{3.20}$$

where $\underset{\sim}{x}_0$ and $\underset{\sim}{x}_1$ are the matrix representation of the position vectors of the nodes in the reference configuration. From this description of the displacement field of the truss element, its inertia and stiffness properties can be evaluated. Substitution of (19) and (20) into the time-independent inertia coefficients (2.34)-(2.41) yields

$${}_e\vec{C}_1 = {}_e\underset{\sim}{X}^T\; {}_e\vec{\underset{\sim}{C}}_2, \tag{3.21}$$

$$ {}_e\vec{C}_2 = \tfrac{1}{2}\mathrm{m} \begin{bmatrix} \underline{I} \\ \underline{I} \end{bmatrix} \vec{e}, \tag{3.22} $$

$$ {}_e C_3 = {}_e X^T \, {}_e\underline{C}_5 \, {}_e X, \tag{3.23} $$

$$ {}_e C_4 = {}_e\underline{C}_5 \, {}_e X, \tag{3.24} $$

$$ {}_e\underline{C}_5 = \frac{m}{6} \left\{ \begin{bmatrix} 2\underline{I} & \underline{I} \\ \underline{I} & 2\underline{I} \end{bmatrix} \vec{e}^T\vec{e} - \begin{bmatrix} 2\vec{e}\vec{e}^T & \vec{e}\vec{e}^T \\ \vec{e}\vec{e}^T & 2\vec{e}\vec{e}^T \end{bmatrix} \right\}, \tag{3.25} $$

$$ {}_e\vec{C}_6 = - {}_e\underline{\vec{C}}_7 \, {}_e X, \tag{3.26} $$

$$ {}_e\underline{\vec{C}}_7 = -\frac{m}{6} \begin{bmatrix} 2\tilde{\vec{e}} & \tilde{\vec{e}} \\ \tilde{\vec{e}} & 2\tilde{\vec{e}} \end{bmatrix}, \tag{3.27} $$

$$ {}_e\underline{C}_8 = \frac{m}{6} \begin{bmatrix} 2\underline{I} & \underline{I} \\ \underline{I} & 2\underline{I} \end{bmatrix}, \tag{3.28} $$

The matrix ${}_e\underline{C}_8$ is the regular mass matrix of a truss element (Przemieniecki, 1968).

The linearized expression for the strain energy of a truss element is

$$ U = \frac{EA}{2\ell} \int_0^1 \{\partial(\vec{u}\cdot\vec{\epsilon})/\partial\xi\}^2 d\xi = {}_e\alpha^T \, {}_e\underline{K} \, {}_e\alpha, \tag{3.29} $$

where EA is the extensional rigidity of the truss element and $\vec{\epsilon}$ is a unit vector parallel to the element in the reference configuration

$$ \vec{\epsilon} = (\vec{x}_1 - \vec{x}_0)/\ell, \tag{3.30} $$

and

$$ {}_e\underline{K} = \frac{EA}{\ell} \begin{bmatrix} \underline{\Lambda} & -\underline{\Lambda} \\ -\underline{\Lambda} & \underline{\Lambda} \end{bmatrix}, \tag{3.31} $$

with

$$ \underline{\Lambda} = (\vec{\epsilon}\cdot\vec{e})(\vec{\epsilon}\cdot\vec{e}^T). \tag{3.32} $$

${}_e\underline{K}$ is the regular stiffness matrix of a bar element (Przemieniecki, 1968).

Example 2: quadratic pin-jointed truss element

When a truss element is loaded by a distributed axial load or when it is not uniform, it may be advantageous to approximate the displacement field with a quadratic or higher degree polynomial. Consider the case with a quadratically varying displacement field. In order to remain straight, the transverse displacement field must vary linearly along the axis of the element. Consequently, the axial and transverse displacement fields must be interpolated differently. The displacement of a particle P on the element can be written in the form (see fig. 3.5)

$$\vec{u}(\xi,t) = (1-\xi)\,\vec{u}_0(t) + \xi\,\vec{u}_1(t) + \gamma(t)\,4\xi(1-\xi)\vec{\epsilon}^*, \tag{3.33}$$

where γ is a generalized displacement and $\vec{\epsilon}^*$ is a unit vector which is parallel to the element in its deformed configuration. The polynomial in the last term has been chosen such that it equals zero in the endpoints of the element. An extra node with unknown axial displacement must be introduced in order to be able to rewrite this expression in terms of unknown nodal displacements. However, the element properties will become more involved when they are referred to unknown nodal displacements as compared to the unknowns introduced in (33). Therefore the element properties will be derived using (33).

Equation (33) is not of the form (2.28) because $\vec{\epsilon}^*$ depends on $\vec{u}_0$ and $\vec{u}_1$. When the rotation of the element caused by deformation is small, $\vec{\epsilon}^*$ equals approximately $\vec{\epsilon}$, i.e. the unit vector which is parallel to the element in its undeformed configuration. Therefore $\vec{\epsilon}^*$ is replaced by $\vec{\epsilon}$. Using this approximation, (33) can be written in the form

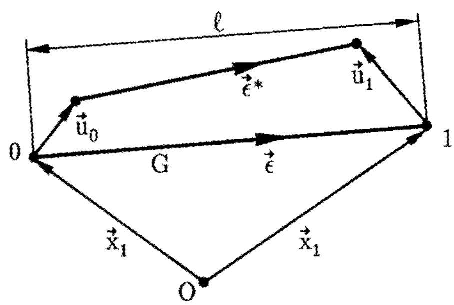

Fig. 3.5 Quadratic truss element

$$\vec{u} = \left[\, \underset{\sim}{u}_0^T \;\; \gamma \;\; \underset{\sim}{u}_1^T \,\right] \begin{bmatrix} (1-\xi)\underline{I} \\ 4\xi(1-\xi)\underset{\sim}{\epsilon}^T \\ \xi\,\underline{I} \end{bmatrix} \underset{\sim}{\vec{e}}, \tag{3.34}$$

where $\underset{\sim}{\epsilon}$ is a column matrix with the components of $\vec{\epsilon}$ relative to $\underset{\sim}{\vec{e}}$. This coefficient is the result of the transformation from an element base to a common base $\underset{\sim}{\vec{e}}$.

The position vector of an arbitrary material point can be expressed in terms of the position vectors of the end nodes using (20).

The inertia and stiffness properties of the quadratic truss element can be evaluated from this description of the displacement field. Substitution of (20) and (34) into the time-independent inertia coefficients (2.34)-(2.41) yields

$$_e\vec{C}_1 = \tfrac{1}{2} m \; {}_e\underset{\sim}{X}^T \begin{bmatrix} \underline{I} \\ \underline{I} \end{bmatrix} \underset{\sim}{\vec{e}}, \tag{3.35}$$

$$_e\underset{\sim}{\vec{C}}_2 = \frac{m}{6} \begin{bmatrix} 3\underline{I} \\ 4\underset{\sim}{\epsilon}^T \\ 3\underline{I} \end{bmatrix} \underset{\sim}{\vec{e}}, \tag{3.36}$$

$$_eC_3 = \frac{m}{6}\, {}_e\underset{\sim}{X}^T \left\{ \begin{bmatrix} 2\underline{I} & \underline{I} \\ \underline{I} & 2\underline{I} \end{bmatrix} \underset{\sim}{\vec{e}}^T \underset{\sim}{\vec{e}} - \begin{bmatrix} 2\underset{\sim}{\vec{e}}\underset{\sim}{\vec{e}}^T & \underset{\sim}{\vec{e}}\underset{\sim}{\vec{e}}^T \\ \underset{\sim}{\vec{e}}\underset{\sim}{\vec{e}}^T & 2\underset{\sim}{\vec{e}}\underset{\sim}{\vec{e}}^T \end{bmatrix} \right\} {}_e\underset{\sim}{X}, \tag{3.37}$$

$$_e\underset{\sim}{C}_4 = \frac{m}{6} \left\{ \begin{bmatrix} 2\underline{I} & \underline{I} \\ 2\underset{\sim}{\epsilon}^T & 2\underset{\sim}{\epsilon}^T \\ \underline{I} & 2\underline{I} \end{bmatrix} \underset{\sim}{\vec{e}}^T \underset{\sim}{\vec{e}} - \begin{bmatrix} 2\underset{\sim}{\vec{e}}\underset{\sim}{\vec{e}}^T & \underset{\sim}{\vec{e}}\underset{\sim}{\vec{e}}^T \\ 2\underset{\sim}{\epsilon}^T\underset{\sim}{\vec{e}}\underset{\sim}{\vec{e}}^T & 2\underset{\sim}{\epsilon}^T\underset{\sim}{\vec{e}}\underset{\sim}{\vec{e}}^T \\ \underset{\sim}{\vec{e}}\underset{\sim}{\vec{e}}^T & 2\underset{\sim}{\vec{e}}\underset{\sim}{\vec{e}}^T \end{bmatrix} \right\} {}_e\underset{\sim}{X}, \tag{3.38}$$

$$_e\underline{\underset{\sim}{C}}_5 = \frac{m}{30} \left\{ \begin{bmatrix} 10\underline{I} & 10\underset{\sim}{\epsilon} & 5\underline{I} \\ 10\underset{\sim}{\epsilon}^T & 16 & 10\underset{\sim}{\epsilon}^T \\ 5\underline{I} & 10\underset{\sim}{\epsilon} & 10\underline{I} \end{bmatrix} \underset{\sim}{\vec{e}}^T \underset{\sim}{\vec{e}} + \begin{bmatrix} 10\underset{\sim}{\vec{e}}\underset{\sim}{\vec{e}}^T & 10\underset{\sim}{\vec{e}}\underset{\sim}{\vec{e}}^T\underset{\sim}{\epsilon} & 5\underset{\sim}{\vec{e}}\underset{\sim}{\vec{e}}^T \\ 10\underset{\sim}{\epsilon}^T\underset{\sim}{\vec{e}}\underset{\sim}{\vec{e}}^T & 16\underset{\sim}{\vec{e}}^T\underline{\Lambda}\underset{\sim}{\vec{e}} & 10\underset{\sim}{\epsilon}^T\underset{\sim}{\vec{e}}\underset{\sim}{\vec{e}}^T \\ 5\underset{\sim}{\vec{e}}\underset{\sim}{\vec{e}}^T & 10\underset{\sim}{\vec{e}}\underset{\sim}{\vec{e}}^T\underset{\sim}{\epsilon} & 10\underset{\sim}{\vec{e}}\underset{\sim}{\vec{e}}^T \end{bmatrix} \right\} \tag{3.39}$$

$$_e\underset{\sim}{\vec{C}}_6 = -\frac{m}{6} \begin{bmatrix} 2\tilde{\underset{\sim}{\vec{e}}} & \tilde{\underset{\sim}{\vec{e}}} \\ 2\underset{\sim}{\epsilon}^T\tilde{\underset{\sim}{\vec{e}}} & 2\underset{\sim}{\epsilon}^T\tilde{\underset{\sim}{\vec{e}}} \\ \tilde{\underset{\sim}{\vec{e}}} & 2\tilde{\underset{\sim}{\vec{e}}} \end{bmatrix} {}_e\underset{\sim}{X}, \tag{3.40}$$

$$
{}_{e}\vec{\underline{C}}_7 = -\frac{m}{30}\begin{bmatrix} 10\tilde{\vec{\underline{e}}} & 10\tilde{\vec{\underline{e}}}\underset{\sim}{\epsilon} & 5\tilde{\vec{\underline{e}}} \\ 10\underset{\sim}{\epsilon}^T\tilde{\vec{\underline{e}}} & 0 & 10\underset{\sim}{\epsilon}^T\tilde{\vec{\underline{e}}} \\ 5\tilde{\vec{\underline{e}}} & 10\tilde{\vec{\underline{e}}}\underset{\sim}{\epsilon} & 10\tilde{\vec{\underline{e}}} \end{bmatrix}, \tag{3.41}
$$

$$
\underline{C}_8 = \frac{m}{30}\begin{bmatrix} 10\underline{I} & 10\underset{\sim}{\epsilon} & 5\underline{I} \\ 10\underset{\sim}{\epsilon}^T & 16 & 10\underset{\sim}{\epsilon}^T \\ 5\underline{I} & 10\underset{\sim}{\epsilon} & 10\underline{I} \end{bmatrix}. \tag{3.42}
$$

The linearized expression for the strain energy of this quadratic truss element is

$$
U = \frac{EA}{2\ell}\int_0^1 \{\partial(\vec{u}\cdot\vec{\epsilon})/\partial\xi\}^2 d\xi = \tfrac{1}{2}[\,\underset{\sim}{u}_0^T \;\; \gamma \;\; \underset{\sim}{u}_1^T\,]\; {}_{e}\underline{K} \begin{bmatrix} \underset{\sim}{u}_0^T \\ \gamma \\ \underset{\sim}{u}_1^T \end{bmatrix}, \tag{3.43}
$$

where the element stiffness matrix ${}_{e}\underline{K}$ is given by

$$
{}_{e}\underline{K} = \frac{EA}{3\ell}\begin{bmatrix} 3\underline{\Lambda} & \underset{\sim}{0} & -3\underline{\Lambda} \\ \underset{\sim}{0}^T & 16 & \underset{\sim}{0}^T \\ -3\underline{\Lambda} & \underset{\sim}{0} & 3\underline{\Lambda} \end{bmatrix}. \tag{3.44}
$$

$\underline{\Lambda}$ is defined by (32).

The element properties referred to nodal displacements can be calculated from these element properties. However, this is not necessary when the extra node is not coupled to other elements. In general, elements will not be coupled via the extra node since that causes a jump in the gradient of the displacement field which cannot be described by (34). In order to show what transformation is required for replacing the generalized displacement γ by nodal displacements, γ must be written in terms of nodal displacements. Introduce an extra node at $\xi = \frac{1}{2}$. Let the axial displacement at this node be $\bar{u}$. This displacement must be equal to the axial displacement obtained from (34) for $\xi = \frac{1}{2}$. Hence

$$
\bar{u} = \tfrac{1}{2}\underset{\sim}{u}_0^T\,\underset{\sim}{\epsilon} + \tfrac{1}{2}\underset{\sim}{u}_1^T\,\underset{\sim}{\epsilon} + \gamma. \tag{3.45}
$$

Consequently, the column matrix with the generalized displacements and the column matrix of nodal degrees of freedom are related by

$$
[\,\underset{\sim}{u}_0^T \;\; \gamma \;\; \underset{\sim}{u}_1^T\,] = [\,\underset{\sim}{u}_0^T \;\; \bar{u} \;\; \underset{\sim}{u}_1^T\,]\underline{T}, \tag{3.46}
$$

where

$$\underline{T} = \begin{bmatrix} \underline{I} & -\frac{1}{2}\underset{\sim}{\epsilon} & \underline{0} \\ \underset{\sim}{0}^T & 1 & \underset{\sim}{0}^T \\ \underline{0} & -\frac{1}{2}\underset{\sim}{\epsilon} & \underline{I} \end{bmatrix}, \tag{3.47}$$

In order to transform the element properties to their counterpart with respect to nodal displacements, the column matrices ${}_e\vec{\underset{\sim}{C}}_2$, ${}_e\underset{\sim}{C}_4$ and ${}_e\vec{\underset{\sim}{C}}_6$ must be premultiplied with $\underline{T}$ and the matrices ${}_e\underline{C}_5$, ${}_e\vec{\underline{C}}_7$, ${}_e\underline{C}_8$ and ${}_e\underline{K}$ must be premultiplied with $\underline{T}$ and postmultiplied with $\underline{T}^T$.

3.4 Modal synthesis method

For bodies with a complex geometry, the finite element method generally leads to models with many degrees of freedom. This is undesirable from a computation costs point of view: in determining the transient response of a model many equations of motion must be integrated, and many degrees of freedom lead to a large variation of eigenfrequencies which reduces the integration time steps. Consequently, a reduction of degrees of freedom is desirable. The usual procedure to achieve this, which is called the modal synthesis method, is to use a set of linear combinations of the base functions generated with the finite element method. Such linear combinations of finite element base functions may be regarded as numerically generated base functions which are the counterpart of the analytic base functions considered in section 3.2. This approach combines the efficiency of the assumed-modes method and the versatility of the finite element method.

Let ${}_a\vec{\underset{\sim}{\Phi}}$ be an n × 1 column matrix of the assembled finite element shape functions of a body. Then a reduced set of base functions can be obtained with the transformation

$$\vec{\underset{\sim}{\Phi}} = \underline{\Sigma}^T {}_a\vec{\underset{\sim}{\Phi}}, \tag{3.48}$$

where $\vec{\underset{\sim}{\Phi}}$ is an N × 1 column matrix of base functions and $\underline{\Sigma}$ is an n × N matrix of constants. The columns of $\underline{\Sigma}$ contain nodal displacements. In general N will be much smaller than n. The time-independent coefficients for this new set of base functions can be obtained from the assembled time-independent coefficients obtained with the finite element method by premultiplying ${}_a\vec{\underset{\sim}{C}}_2$, ${}_a\underset{\sim}{C}_4$ and ${}_a\vec{\underset{\sim}{C}}_6$ with $\underline{\Sigma}^T$, and premultiplying ${}_a\underline{C}_5$, ${}_a\underline{\vec{C}}_7$, ${}_a\underline{C}_8$ and ${}_a\underline{K}$ with $\underline{\Sigma}^T$ and postmultiplying the result with $\underline{\Sigma}$.

An issue of primary concern of the modal synthesis method is the selection of the matrix $\underline{\Sigma}$. To take full advantage of the modal synthesis method one should select the matrix $\underline{\Sigma}$ such that only a small number of base functions can describe the displacement field due to deformation of the body satisfactorily. What choice will be most appropriate depends on the problem at hand. The most widely used choices originate from the reduction procedures used in structural dynamics.

The standard procedure to determine an approximate solution of the transient response of structures under time-dependent loads is to express the displacement of the structure as a linear combination of a reduced set of modes of free vibration (Przemieniecki, 1968). The corresponding set of equations of motion are uncoupled and its solution can be easily obtained. Complex structures are often regarded as an assembly of substructures. The deformation of each substructure is represented by a linear combination of deformation modes. These modes are chosen such that the equations of motion of the complete structure can be easily obtained from the equations of motion of the substructures (component mode synthesis method). Craig (1981) gives an overview of such modes.

The modes used in structural dynamics have distinct properties which facilitate solving the equations of motion. But these properties are not necessarily relevant in solving the equations of motion of a mechanical system. Modes of free vibration do not uncouple the equations of motion (2.30) because of the coupling of displacements due to deformation and displacements due to a rigid body motion. Moreover, the instantaneous modes of free vibration of a body in a system of bodies depend on the instantaneous configuration of the system, especially when it has a tree topology. Consequently there is no motive to prefer modes of free vibration above other displacement fields. Likewise the properties of the modes used in the component mode synthesis method are not relevant. For all that, these modes have been used by several investigators: Agrawal and Shabana (1985), and Singh et al. (1985) used modes of free vibration; Sunada and Dubowsky (1981, 1983) used component modes.

Yoo and Haug (1986a, 1986c) recognized that less modes are required when in the case of the presence of concentrated loads, the static deformation due to such loads is included. Also for structural dynamic problems a more accurate solution may be obtained when static deformation modes are included (Kline, 1986). Indeed those modes should be used which can represent the actual deformation of the body best. The selection of deformation modes will be discussed in section 5.4 with the help of some examples.

chapter 4

The equations of motion of a system of bodies

4.1 Introduction

The equations of motion of a single isolated deformable body have been derived in chapter 2. In the present chapter the equations of motion of a system of interconnected bodies will be considered. These can be obtained from the equations of motion of the single bodies by taking the forces caused by the connections into account.

Connections can be subdivided into energetic, active and kinematic connections (Sol, 1983). The forces caused by energetic and active connections can be obtained from constitutive equations. It will be assumed that forces caused by kinematic connections do not enter into the constitutive equations such as may be the case for Coulomb friction forces. The forces caused by energetic and active connections can be determined at the outset and consequently they can be taken into account by introducing them into the load integrals of (2.15).

Kinematic connections restrict relative motions of interconnected bodies. Consequently displacements of bodies are dependent. The equations relating displacements of kinematically connected bodies are called constraint equations. Only constraint equations of the form

$$\Theta(\underset{\sim}{z},t) = 0 \tag{4.1}$$

will be considered, where $\Theta(\underset{\sim}{z},t)$ is an algebraic function which is once continuously differentiable with respect to its arguments and $\underset{\sim}{z}$ is a column matrix of all scalar variables that are introduced to describe the kinematics of the system of bodies, named the generalized displacements of the system of bodies. Kinematic connections with constraint equations of this form or that can be reduced to this form are called holonomic. If a constraint equation does not contain time explicitly, it is called scleronomic; otherwise it is called rheonomic. Forces going with kinematic connections, constraint forces, cannot be determined at the outset. They may be introduced as additional unknowns. A practical way of doing this is by means of the Lagrange multiplier method (Meirovitch, 1970). Then variations of the generalized displacements of the system of bodies may be chosen arbitrarily. They may also be chosen such that the constraint equations are not violated.

Variations of the generalized displacements which are consistent with the constraint equations are termed virtual displacements. The work done by all constraint forces due to virtual displacements equals zero (Rosenberg, 1977) and consequently these forces do not appear in the assembled variational form of the equations of motion.

In section 4.2, energetic and active connections will be considered. The constraint equations for kinematic connections differ for the relative and the global description. This will be illustrated in section 4.3. Procedures for obtaining generalized virtual displacements will be given in section 4.4 for the relative and the global description. These are used for obtaining the equations of motion of a system of bodies.

4.2 Energetic and active connections

The forces caused by both energetic and active connections can be obtained from constitutive equations, which relate the forces, the relative displacements and velocities of the interconnected bodies and the time history of these quantities, and external parameters. The difference between energetic and active connections is that the former do not increase the total mechanical energy of the system of bodies, whereas the latter may increase the total mechanical energy. Examples of energetic connections are springs and viscous dampers; an example of an active element is an actuator. The derivation of the contribution of energetic and active connections to the equations of motion of a system of bodies is demonstrated with a linear spring, a linear viscous damper and an actuator.

Consider the bodies $\mathscr{B}^i$ and $\mathscr{B}^j$ interconnected by a linear spring, a linear viscous damper and an actuator as shown in fig. 4.1. Let the spring, the damper,

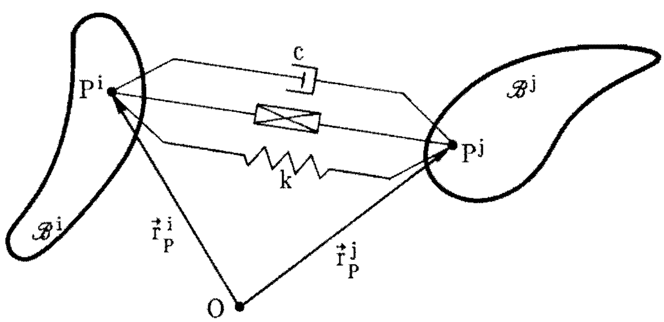

Fig. 4.1 Connection of spring, damper and actuator

and the actuator be attached to $\mathscr{B}^i$ at material point P^i and to $\mathscr{B}^j$ at material point P^j. The position vectors of these points are $\vec{r}_p^{\,i}$ and $\vec{r}_p^{\,j}$, respectively. The distance ℓ of the interconnected points is given by

$$\ell = \| \vec{r}_p^{\,i} - \vec{r}_p^{\,j} \| = \{(\vec{r}_p^{\,i} - \vec{r}_p^{\,j})\cdot(\vec{r}_p^{\,i} - \vec{r}_p^{\,j})\}^{\frac{1}{2}}. \tag{4.2}$$

The force caused by this connection can be determined with the constitutive equation

$$F_c = k(\ell - \ell_0) + c\,\dot{\ell} + F_a\,, \tag{4.3}$$

where k is the spring coefficient, ℓ_0 is the free spring length, c is the viscous damping coefficient, and F_a is the actuator force. The force on $\mathscr{B}^i$ is directed from P^i to P^j; the force on $\mathscr{B}^j$ is directed opposite:

$$\vec{F}_c^{\,i} = (F_c/\ell)\,(\vec{r}_p^{\,j} - \vec{r}_p^{\,i}), \tag{4.4}$$

$$\vec{F}_c^{\,j} = (F_c/\ell)\,(\vec{r}_p^{\,i} - \vec{r}_p^{\,j}). \tag{4.5}$$

These concentrated forces can be considered as volume or surface loads by representing them as spatial Dirac delta functions in their points of application. The contribution of these forces to the equations of motion can be obtained by substituting them into the expressions for the generalized forces (2.31)-(2.33). This yields for body i

$$\vec{F}^{\,i} = (F_c/\ell)\,(\vec{r}_p^{\,j} - \vec{r}_p^{\,i}), \tag{4.6}$$

$$\vec{M}^{\,i} = (F_c/\ell)\,(\vec{x}_p^{\,i} + \vec{u}_p^{\,i}) \times \{\mathbf{Q}^{iC}\cdot(\vec{r}_p^{\,j} - \vec{r}_p^{\,i})\}, \tag{4.7}$$

$$\underline{f}^{\,i} = (F_c/\ell)\,\vec{\underline{\Phi}}_p^{\,i}\cdot\{\mathbf{Q}^{iC}\cdot(\vec{r}_p^{\,j} - \vec{r}_p^{\,i})\}, \tag{4.8}$$

where $\vec{x}_p^{\,i}$ is the position vector of P^i in the reference configuration and $\vec{u}_p^{\,i}$ is the displacement of P^i due to deformation of $\mathscr{B}^i$. The expression for the contribution of the connection force to the generalized forces of body j are obtained from these equations by interchanging i and j.

4.3 Kinematic connections

Kinematic connections restrict relative motions of interconnected bodies. The relation between the generalized displacements of a pair of kinematically connected bodies are given by constraint equations. The expressions for the constraint equations of a specific kinematic connection differ for the global and the relative description. In the global description it is customary to relate the displacement fields $\vec{u}(\vec{x},t)$, the rotation tensors $\mathbf{Q}(t)$ and the translation vectors $\vec{c}(t)$ of the pair of interconnected bodies. In the relative description it is customary to write the rotation tensor $\mathbf{Q}(t)$ and the translation vector $\vec{c}(t)$ of one body in terms of $\mathbf{Q}(t)$ and $\vec{c}(t)$ of the other body, the displacement fields $\vec{u}(\vec{x},t)$ of both bodies, and some variables, equal in number to the number of relative degrees of freedom of the interconnected bodies, which can describe their relative motion. The description of a kinematic connection using the global and the relative description will be illustrated for a spherical joint.

Example: spherical joint

Consider the bodies $\mathscr{B}^i$ and $\mathscr{B}^j$ interconnected by a spherical joint shown in fig. 4.2. A spherical joint forces the two bodies to have two coinciding material points. Let this be P^i for body $\mathscr{B}^i$ and P^j for body $\mathscr{B}^j$. The constraint equation for a spherical joint using the global description can be written as

$$\vec{c}^{\,i}(t) + \mathbf{Q}^i(t)\cdot\{\vec{x}^{\,i}_P + \vec{u}^{\,i}(\vec{x}^{\,i}_P,t)\} = \vec{c}^{\,j}(t) + \mathbf{Q}^j(t)\cdot\{\vec{x}^{\,j}_P + \vec{u}^{\,j}(\vec{x}^{\,j}_P,t)\}. \tag{4.9}$$

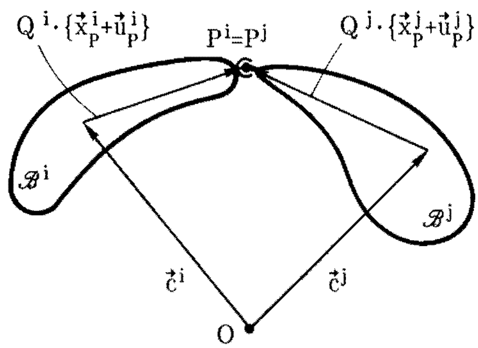

Fig. 4.2 Spherical joint

When the relative description is used, variables must be introduced with which the motion of $\mathscr{B}^j$ relative to $\mathscr{B}^i$ can be described. $\mathscr{B}^j$ is free to rotate relative to $\mathscr{B}^i$. Therefore a rotation tensor $\mathbf{Q}^{ji}(t)$ is introduced which defines the rigid body rotation of $\mathscr{B}^j$ relative to $\mathscr{B}^i$. Using this relative rotation tensor, $\mathbf{Q}^j(t)$ can be written in terms of $\mathbf{Q}^i(t)$ and $\mathbf{Q}^{ji}(t)$:

$$\mathbf{Q}^j(t) = \mathbf{Q}^{ji}(t)\cdot\mathbf{Q}^i(t). \tag{4.10}$$

Substitution of (10) into (9) yields for the translation vector of $\mathscr{B}^j$

$$\vec{c}^j(t) = \vec{c}^i(t) + \mathbf{Q}^i(t)\cdot\{\vec{x}_p^i + \vec{u}^i(\vec{x}_p^i,t)\} - \mathbf{Q}^{ji}(t)\cdot\mathbf{Q}^i(t)\cdot\{\vec{x}_p^j + \vec{u}^j(\vec{x}_p^j,t)\}. \qquad (4.11)$$

(10) and (11) constitute the relative description of a spherical joint.

The relative rotation tensor $\mathbf{Q}^{ji}(t)$ as used in the above description of a spherical joint does not represent the relative rotation within the joint. The way Sol (1983) and Singh et al. (1985) use the relative description, relative motion means relative motion within the joint. However, this leads to more involved expressions for a spherical joint. Only when the relative motion within the joint is of interest it may be advantageous to use the relative motion within the joint.

Also when the global description is used, it may be advantageous or even necessary to introduce variables which describe the relative motion of kinematically connected bodies. An example of such a connection is a translational joint (Li and Likins, 1987; Hwang and Haug, 1987).

Example: translational joint

Consider the bodies $\mathscr{B}^i$ and $\mathscr{B}^j$ interconnected by a translational joint shown in fig. 4.3. Material point P^j of $\mathscr{B}^j$ can slide along a guideway on $\mathscr{B}^i$. When $\mathscr{B}^i$ is in the reference configuration, it is assumed that the guideway is straight and $\mathscr{B}^j$ does not rotate around the guideway when it slides along it. The translational joint is modelled as a deformable line (the centreline of the guideway) and a muff which can only slide along that line; the muff rotates with the line as if it were rigidly

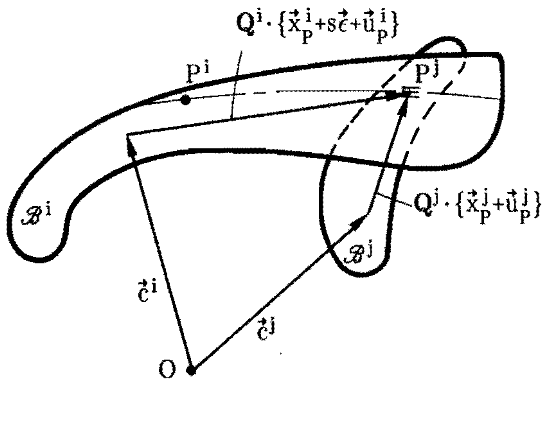

Fig. 4.3 Translational joint

attached to it. Let this line in the reference configuration be parallel to a unit vector $\vec{\epsilon}$. The relative motion can be described by one variable, namely the distance s between the muff and some reference point P^i on the line. Due to the translational joint, a material point of $\mathscr{B}^i$ must coincide with material point P^j. The position vector of this material point in the reference configuration is $\vec{x}_p^i + s\vec{\epsilon}$, where $\vec{x}_p^i$ is the position vector of P^i in the reference configuration. The constraint equation which describes that these material points must coincide is

$$\vec{c}^i(t) + \mathbf{Q}^i(t)\cdot\{\vec{x}_p^i + s\vec{\epsilon} + \vec{u}^i(\vec{x}_p^i{+}s\vec{\epsilon},t)\} = \vec{c}^j(t) + \mathbf{Q}^j(t)\cdot\{\vec{x}_p^j + \vec{u}^j(\vec{x}_p^j,t)\}. \quad (4.12)$$

Further, the points of contact cannot rotate relative to one another. It is assumed that the orientation of the bodies in the reference configuration has been chosen such that there is no relative rotation within the translational joint. Then the corresponding constraint equation is

$$\mathbf{Q}^i(t)\cdot\mathbf{R}^i(\vec{u}^i(\vec{x}_p^i{+}s\vec{\epsilon},t)) = \mathbf{Q}^j(t)\cdot\mathbf{R}^j(\vec{u}^j(x_p^j,t)), \quad (4.13)$$

where $\mathbf{R}^i(\vec{u}^i(\vec{x}_p^i{+}s\vec{\epsilon},t))$ and $\mathbf{R}^j(\vec{u}^j(x_p^j,t))$ account for the rotation due to deformation at the contact points. This equation can be rewritten as

$$\mathbf{R}^{j^C}(\vec{u}^j(x_p^j,t))\cdot\mathbf{Q}^{j^C}(t)\cdot\mathbf{Q}^i(t)\cdot\mathbf{R}^i(\vec{u}^i(\vec{x}_p^i{+}s\vec{\epsilon},t)) = \mathbf{I}. \quad (4.14)$$

These are nine equations; only three of them are independent due to properties of rotation tensors. In principle an arbitrary triplet may be chosen. However, because generally the rotation tensors due to deformation are linearized the equations on the diagonal of the matrix representation of this equation cannot be satisfied and corresponding off-diagonal terms are dependent. When this is taken into account in selecting constraint equations, the remaining independent constraint equations are the same as those presented by Hwang and Haug (1987).

In general it is not possible to eliminate the variable s from these constraint equations. Consequently, for deriving the constraint equations in the global description of a translational joint, it is necessary to introduce a variable which describes the relative motion.

The constraint equations for a translational joint in the relative description are obtained by solving $\vec{c}^j(t)$ and $\mathbf{Q}^j(t)$ from (12) and (13). This yields

$$\begin{aligned}\vec{c}^j(t) = \vec{c}^i(t) &+ \mathbf{Q}^i(t)\cdot\{\vec{x}_p^i + s\vec{\epsilon} + \vec{u}^i(\vec{x}_p^i{+}s\vec{\epsilon},t)\}\\ &- \mathbf{Q}^i(t)\cdot\mathbf{R}^i(\vec{u}^i(\vec{x}_p^i{+}s\vec{\epsilon},t))\cdot\mathbf{R}^{j^C}(\vec{u}^j(x_p^j,t))\cdot\{\vec{x}_p^j + \vec{u}^j(\vec{x}_p^j,t)\},\end{aligned} \quad (4.15)$$

$$\mathbf{Q}^j(t) = \mathbf{Q}^i(t)\cdot\mathbf{R}^i(\vec{u}^i(\vec{x}_p^i+s\vec{e},t))\cdot\mathbf{R}^{jc}(\vec{u}^j(x_p^j,t)). \tag{4.16}$$

In the relative description, the order of the bodies matters as will become clear in the next section. When the opposite order of bodies is selected the relative description of a translational joint becomes

$$\begin{aligned}\vec{c}^i(t) = \vec{c}^j(t) + \mathbf{Q}^j(t)\cdot\{\vec{x}_p^j + \vec{u}^j(\vec{x}_p^j,t)\} \\ - \mathbf{Q}^j(t)\cdot\mathbf{R}^j(\vec{u}^j(x_p^j,t))\cdot\mathbf{R}^{ic}(\vec{u}^i(\vec{x}_p^i+s\vec{e},t))\cdot\{\vec{x}_p^i + s\vec{e} + \vec{u}^i(\vec{x}_p^i+s\vec{e},t)\},\end{aligned} \tag{4.17}$$

$$\mathbf{Q}^i(t) = \mathbf{Q}^j(t)\cdot\mathbf{R}^j(\vec{u}^j(x_p^j,t))\cdot\mathbf{R}^{ic}(\vec{u}^i(\vec{x}_p^i+s\vec{e},t)). \tag{4.18}$$

4.4 The equations of motion of a system of bodies

The equations of motion in component form of a body, which is part of a system of bodies, can be written symbolically (cf. 2.46)

$$[\ \delta \underset{\sim}{c}^T\ \ \delta \underset{\sim}{\pi}^T\ \ \delta \underset{\sim}{\alpha}^T\ ]\left\{ \underline{M} \begin{bmatrix} \ddot{\underset{\sim}{c}} \\ \dot{\underset{\sim}{\omega}} \\ \ddot{\underset{\sim}{\alpha}} \end{bmatrix} + \underset{\sim}{g}(\underset{\sim}{c}, \dot{\underset{\sim}{c}}, \underset{\sim}{q}, \underset{\sim}{\omega}, \underset{\sim}{\alpha}, \dot{\underset{\sim}{\alpha}}, t) + \underset{\sim}{g}_c \right\} = 0. \tag{4.19}$$

where $\underset{\sim}{g}_c$ is a column matrix of the generalized constraint forces that act on the body. Using the generalized principle of d'Alembert (Meirovitch, 1970) the equations of motion for a system of NB bodies become

$$\sum_{i=1}^{NB} [\ \delta \underset{\sim}{c}^T\ \ \delta \underset{\sim}{\pi}^T\ \ \delta \underset{\sim}{\alpha}^T\ ]_i \left\{ \underline{M} \begin{bmatrix} \ddot{\underset{\sim}{c}} \\ \dot{\underset{\sim}{\omega}} \\ \ddot{\underset{\sim}{\alpha}} \end{bmatrix} + \underset{\sim}{g}(\underset{\sim}{c}, \dot{\underset{\sim}{c}}, \underset{\sim}{q}, \underset{\sim}{\omega}, \underset{\sim}{\alpha}, \dot{\underset{\sim}{\alpha}}, t) \right\}_i = 0. \tag{4.20}$$

The contribution of the constraint forces cancels only when the variations of the generalized displacements satisfy the variation of the constraint equations with time held fixed. This variation can be written in the form

$$\underline{H}\ \delta \underset{\sim}{z} = \underset{\sim}{0}, \tag{4.21}$$

where $\underline{H}$ is the m × n Jacobian matrix of the constraint equations of rank m, and $\delta \underset{\sim}{z}$ is an n × 1 column matrix which contains variations of all generalized displacements of the system of bodies; m is the number of constraint equations and n is the number of generalized displacements. The rank of the Jacobian will be less than m when the constraint equations are dependent or incompatible. In the former case, some of the constraint equations can be ignored such that the new Jacobian has full row rank; the latter case refers to an inconsistent set of constraint equations which allows no solution. Using (21), variations of m generalized displacements can be written in terms of variations of the remaining n − m generalized displacements. The latter are independent and may be chosen arbitrarily. The procedure for partitioning the virtual displacements into dependent and independent displacements differs for the global and the relative description. These two approaches will be considered more closely in the next two sections.

4.4.1 *The global description*

Wehage and Haug (1982) have presented an automatic procedure for partitioning displacements into dependent and independent displacements called the generalized coordinate partitioning method. They decompose the Jacobian $\underline{H}$ using full row and column pivoting into matrices $\underline{L}$, $\underline{U}^u$, and $\underline{U}^v$ such that

$$\underline{H} \sim \underline{L}[\,\underline{U}^u \;\; \underline{U}^v\,], \tag{4.22}$$

where $\underline{L}$ and $\underline{U}^u$ are nonsingular lower and upper triangular m × m matrices, respectively, and the symbol ~ denotes that the matrices on either side are equal apart from a possibly different order of columns due to the decompositioning of $\underline{H}$. The elements of $\delta\underset{\sim}{z}$ going with $\underline{U}^u$, $\delta\underset{\sim}{u}$, can be written in terms of the remaining elements $\delta\underset{\sim}{v}$ because $\underline{U}^u$ is nonsingular:

$$\delta\underset{\sim}{u} = -\,\underline{U}^{u^{-1}}\underline{U}^v\delta\underset{\sim}{v}. \tag{4.23}$$

Substitution of in succession n − m linearly independent combinations $\delta\underset{\sim}{v}$ and the corresponding variations $\delta\underset{\sim}{u}$ into (20) leads to n − m equations of motion. Usually, in succession one component of $\delta\underset{\sim}{v}$ is chosen equal to one and the remaining components are set equal to zero.

The resulting equations of motion contain still the dependent displacements and their time derivatives $\dot{\underset{\sim}{u}}$ and $\ddot{\underset{\sim}{u}}$. The dependent displacements can be solved from the constraint equations. Since these are generally nonlinear algebraic equations this must be done iteratively. $\dot{\underset{\sim}{u}}$ and $\ddot{\underset{\sim}{u}}$ can be written in terms of $\dot{\underset{\sim}{v}}$ and $\ddot{\underset{\sim}{v}}$ using, respectively, the first and second time derivative of the constraint equations. Using these, $\ddot{\underset{\sim}{v}}$ can be solved from the equations of motion. Then $\ddot{\underset{\sim}{v}}$ can be integrated using a standard numerical integration algorithm such as the variable order explicit predictor/implicit corrector algorithm DE (Shampine and Gordon, 1975).

The efficiency of the procedure outlined above is limited by the requirements for iterative solution of the dependent generalized displacements from the constraint equations. This problem may be alleviated by using the relative description.

Another approach is to supplement the equations of motion with the second time derivative of the constraint equations and to integrate both the independent and the dependent displacements. This makes solving of the constraint equations and their first time derivative for the dependent displacements and their first time derivative superfluous. However, because of numerical errors, the constraint equations will be violated. Baumgarte (1972, 1978) proposed to replace the second time derivative of the constraint equations by the alternative equation

$$\ddot{\underset{\sim}{\Theta}}(\underset{\sim}{z},t) + 2\alpha\dot{\underset{\sim}{\Theta}}(\underset{\sim}{z},t) + \beta^2\underset{\sim}{\Theta}(\underset{\sim}{z},t) = \underset{\sim}{0}, \tag{4.24}$$

where $\underset{\sim}{\Theta}$ is a column matrix of the constraint functions, $\underset{\sim}{z}$ is a column matrix of the generalized displacements of the system of bodies, and α and β are positive constants. This approach is referred to as the constraint violation stabilization method. Usually α is chosen equal to β which corresponds to a critically damped system. Due to the extra terms, additional eigenfrequencies are introduced. A solution to this problem is proposed by Baumgarte (1978). A shortcoming of Baumgarte's approach is the issue of choosing proper values of α and β. Chang and Nikravesh (1985) proposed a modification of Baumgarte's method where the constants α and β are adjusted each integration time step. However, numerical errors accumulate due to lack of positive error control of the constraint violation stabilization method.

Park and Haug (1985, 1986) proposed a hybrid method that is the fusion of the generalized coordinate partitioning method and the constraint violation stabilization method. They partitioned the generalized displacements into dependent and independent displacements, and integrated both types of displacements using the extra terms in (24). Error control is imposed only on the independent displacements. The dependent generalized displacements are corrected by solving them from the constraint equations if the violation of the constraint equations exceeds a specified error tolerance. Numerical experiments done by Park and Haug reveal that this is required for only a small number of times which makes their approach more efficient than the generalized coordinate partitioning method. Their method (and the generalized coordinate partitioning method) succeed in solving problems where the constraint violation method fails due to lack of positive error control, which makes the hybrid method more stable than the constraint violation stabilization method.

4.4.2 *The relative description*

When the relative description is used, the relative motion of a pair of kinematically connected bodies must be described such that the rigid body motion of one body can be expressed explicitly in terms of the rigid body motion of the other body, the displacement fields due to deformation of both bodies, and variables that describe the motion of the bodies relative to one another (cf. (10)-(11), (15)-(16), and (17)-(18)). When the system of bodies does not have kinematically closed chains, the motion of any body of the system can be written explicitly in terms of independent generalized displacements thus avoiding the need to solve the constraint equations iteratively.

Consider a system of bodies without kinematically closed chains. Assume the system has a tree topology, i.e. one can go from one body to another along a unique alternating sequence of bodies and kinematic connections. In case such a sequence does not exist, the system may be thought of to consist of several trees or, kinematic connections with six degrees of freedom may be introduced in order to create such a sequence. The position of one body of a tree, the reference body, is described relative to an inertial space. In case the tree contains a body whose position is prescribed, that body should be selected as the reference body. The position of the other bodies are described in succession relative to a body whose position has been described already and with which it is joined by a kinematic connection. Starting from the reference body, the motion of a particular body of the system can be written in terms of the translation vector and the rotation tensor of the reference body, the displacement fields due to deformation, and the variables that describe the relative motion of the bodies on the path to the body at hand. All these variables are independent and may therefore be varied arbitrarily. Linearly independent virtual displacements can be obtained by letting in succession the variation of one of these variables equal to one and the remaining equal to zero. The accompanying dependent virtual displacements can best be computed recursively, i.e. the virtual displacements of a particular body are calculated from the virtual displacement of the contiguous body, starting from the reference body, and the variation of the variables that describe the relative motion of the bodies on the path to the body at hand.

Substitution of in succession a set of linearly independent virtual displacements into (20) leads to the equations of motion of the system of bodies. These still contain dependent displacements, namely the translation vectors and the rotation tensors of the bodies and their first and second time derivative. They can be evaluated recursively using the constraint equations of the kinematic connections and their first and second time derivative. The translation vector and the rotation tensor of a body are explicitly given by the constraint equations. Consequently, there is no need to solve constraint equations iteratively and the resulting equations of motion are a set of ordinary differential equations.

The partitioning of displacements as proposed by Wehage and Haug (1982) will lead to the same equations of motion as described above when the same description of the kinematic connections is used and the same independent displacements are selected. Therefore the relative description can be implemented without insurmountable difficulties in a general-purpose computer program based on the global description, thus leading to a general-purpose computer program based on the relative description.

The generalized displacements which have been introduced above for describing the kinematics of a system of deformable bodies, namely the translation vector and the rotation tensor of the reference body, the displacement fields due to deformation, and all variables that describe the relative motion of kinematically connected bodies, are not independent when the system has kinematically closed chains. The expression for the relationship of these generalized displacements can be obtained from the compatibility conditions at one fictitious cut for each kinematically closed chain. Either a kinematic connection (Wittenburg, 1977; Singh et al., 1986) or a body (Lilov and Chirikov, 1981; Samin and Willems, 1986; Wittenburg, 1987) may be cut imaginarilly. In the former case, the constraint equations are of the type as is used in the global description, i.e. there are no variables introduced that describe the relative motion in the cut joint. For each cut chain, the required number of constraint equations equals at most six minus the number of degrees of freedom of the cut connection; this number may be less when some of the compatibility conditions are already satisfied. In general, it is difficult to formulate these constraint equations in an automatic way.

The constraint equations can be obtained more easily when a body is cut, i.e. two translation vectors, $\vec{c}$ and $\vec{c}'$, and two rotation tensors, $\mathbf{Q}$ and $\mathbf{Q}'$, are introduced for the cut body; the displacement fields due to deformation are not affected by the cut. From the compatibility conditions, these translation vectors and rotation tensors satisfy

$$\vec{c} = \vec{c}', \tag{4.25}$$

$$\mathbf{Q} = \mathbf{Q}'. \tag{4.26}$$

By recursive use of the relative description of the kinematic connections in the kinematically closed chain, $\vec{c}'$ and $\mathbf{Q}'$ can be written in terms of $\vec{c}$, $\mathbf{Q}$, the displacement fields due to deformation, and the variables that describe the relative motion of the bodies. This leads to six constraint equations. This number exceeds the number of constraint equations needed when a kinematic connection is cut because in the latter case no variables have been introduced to describe the relative motion of the bodies joined by the cut connection.

Having the constraint equations available, the generalized displacements can be partitioned into independent and dependent displacements by for instance the procedure of Wehage and Haug (1982), which has already been outlined in section 4.4.1. Substitution of in succession a set of all independent combinations of a variation of the independent displacements and the corresponding variation of the dependent displacements and of the variables that describe the rigid body motion of all bodies except the reference body, into (20) leads to the equations of motion of the system of bodies. The resulting equations of motion contain still the dependent variables and their first and second time derivative. These can be eliminated by solving them from the constraint equations and their first and second time derivatives. Since the constraint equations are nonlinear this must be done iteratively. This may be overcome by using the hybrid method of Park and Haug (1985, 1986).
The variables that describe the rigid body motion of all bodies except the reference body, and their first and second time derivatives, can be eliminated by recursive use of the constraint equations and their first and second time derivatives.

chapter 5

Assessment of descriptions and approximations

5.1 Introduction

In this chapter miscellaneous topics related to the analysis of the dynamic behaviour of systems of deformable bodies are considered. In section 5.2 possible savings of computation time by using base functions for approximating the displacement field due to deformation that satisfy the mean displacement conditions (2.8) and (2.10) are investigated. The finite element method and the modal synthesis method are considered in section 5.3 and 5.4, respectively. Special attention is paid to preventing rigid body motions in the displacement field due to deformation. The effect of the frequently used lumped mass approximation is considered in section 5.4. In section 5.5 the influence of nonlinearities going with the displacement field due to deformation are considered. A procedure for correcting eigenfrequencies going with base functions that do not agree with the actual eigenfrequencies is presented in section 5.6. This procedure is used for alleviating the integration time step reducing effect of high frequencies.

The numerical examples presented in this chapter are carried out with the version of DADS (CADSI, 1988) for three-dimensional problems on a VAX 8530. The required CPU time for a specific problem depends on several input parameters. The reported CPU times apply for default values of the error tolerances, only binary output, no reaction forces, and a maximum integration time step size which is equal to the print interval.

5.2 Mean displacement conditions

In order to investigate possible merits of using base functions that satisfy the mean displacement conditions (2.8) and (2.10), the slider-crank mechanism shown in fig. 5.1 is analyzed. It consists of a rigid crank with length 0.15 m, a deformable connecting rod, and a rigid sliding block with mass equal to half the mass of the connecting rod. The connecting rod has a length $\ell = 0.3$ m and a circular cross section with diameter d = 0.006 m. It is made of steel with a modulus of elasticity $E = 0.2*10^{12}$ N/m^2 and a mass density $\rho = 7.87*10^3$ kg/m^3. Axial deformation of the connecting rod is neglected. The crank rotates with a constant angular velocity $\Omega = 150.0$ rad/s. In the initial configuration, the connecting rod is straight and it is

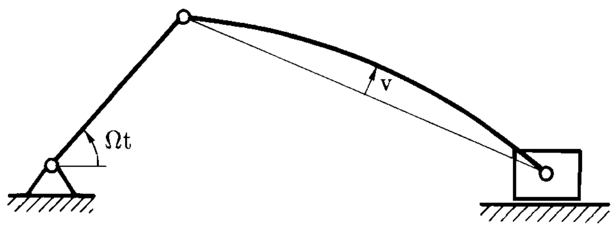

Fig. 5.1 Planar slider-crank mechanism

in a direct line with the crank. Its velocity field corresponds with that of a rigid connecting rod. Starting from this initial configuration, the deflection v of the middle of the connecting rod during 0.06 s is computed. Output is requested for every 0.0002 s. This problem has also been considered by Bakr and Shabana (1986) and Jonker (1988).

The reference configuration of the connecting rod has been chosen analogous to fig. 3.1. The displacements due to deformation of the connecting rod are approximated by the four quintic polynomials (3.7)-(3.10). Subsequently, in order to investigate the influence of the number of flexible bodies, the connecting rod is assumed to be built up of two identical rigidly connected deformable beams. In order to keep the total number of base functions constant, the displacements due to deformation of these beams are approximated by the two cubic polynomials (3.7)-(3.8). These two cases have been analyzed with the three versions of DADS mentioned at the end of section 2.5.

The version of DADS that does not take advantage of zero terms used 223 s and 238 s CPU time to compute the response for the quintic polynomial and the cubic polynomial approximation, respectively. The version that takes only advantage of the zero terms was somewhat faster: 0.97, respectively 0.96, times the corresponding CPU times given above. The version that takes also advantage of the increased sparseness of the mass matrix used 0.92, respectively 0.96, times the corresponding CPU times given above. The reduction of required CPU time are of the same order of magnitude as those obtained by Koppens et al. (1988) with the version of DADS for two-dimensional problems.

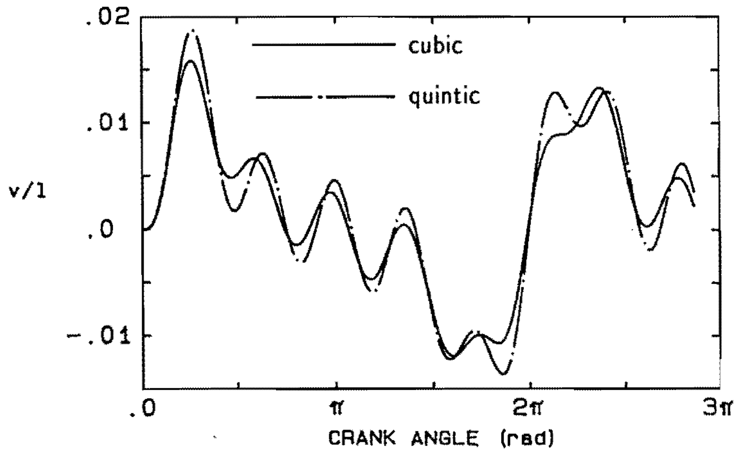

Fig. 5.2 Assumed modes solution using cubic and quintic polynomials

In fig. 5.2 the dimensionless deflection v/ℓ of the middle of the connecting rod is plotted versus the angle over which the crank has rotated. The response related to the quintic polynomial approximation deviates from the response corresponding to the cubic polynomial approximation. This can be attributed to nonlinearities which are taken into account for the cubic polynomial approximation because the connecting rod has been modelled as two rigidly connected beams as will be explained in section 5.5. The response corresponding to the cubic polynomial approximation is indistinguishable from the linear solution obtained by Jonker (1988).

The above slider-crank mechanism is actually a two-dimensional mechanism. A three-dimensional slider-crank mechanism is considered by Jonker (1988) (cf. fig. 5.3). It has the same dimensions and initial configuration as the above slider-crank mechanism.

The reference configuration of the connecting rod has been chosen analogous to fig. 3.1. The displacements due to deformation of the connecting rod are approximated by the four quintic polynomials (3.7)-(3.10) and the corresponding four quintic polynomials for displacements in the $\vec{e}_3$ direction. The connecting rod has also been divided into two identical rigidly connected deformable beams. The displacements due to deformation of these beams are approximated by the two cubic polynomials (3.7)-(3.8) and the corresponding two cubic polynomials for the displacements in the $\vec{e}_3$ direction. These two cases have been analyzed with the three versions of DADS mentioned at the end of section 2.5.

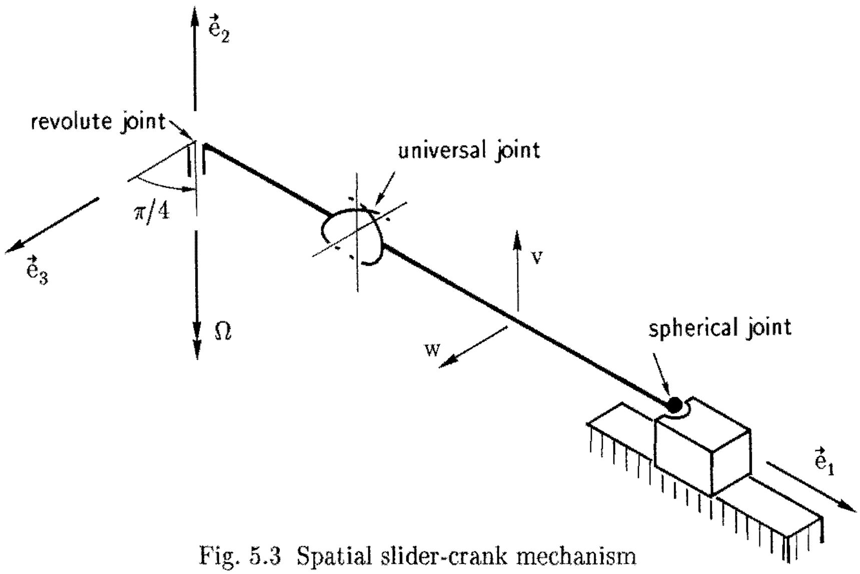

Fig. 5.3 Spatial slider-crank mechanism

The version of DADS that does not take advantage of zero terms required 1016 s and 1224 s CPU time to compute the response for the quintic polynomial and the cubic polynomial approximation, respectively. The version that takes only advantage of the zero terms was somewhat faster: 0.99, respectively 0.98, times the corresponding CPU times given above. The version that takes also advantage of the increased sparseness of the mass matrix used 0.83, respectively 0.82, times the corresponding CPU times given above.

Fig. 5.4 shows the components in $\vec{e}_2$ and $\vec{e}_3$ direction of the dimensionless deflection of the middle of the connecting rod versus the crank angle. The deviation of the responses going with the quintic polynomial approximation and the cubic polynomial approximation can be attributed to nonlinearities which are taken into account for the cubic polynomial approximation because then the connecting rod is modelled as two rigidly connected beams. The two-beams solution is close to the nonlinear solution obtained by Jonker (1988). The difference is of the same order of magnitude as the difference shown in fig. 5.4.

From the above results it can be concluded that only a small reduction of required CPU time is obtained when advantage is taken of zero terms. This reduction becomes larger when the number of assumed displacement fields is increased. For a small number of assumed displacement fields, the saving of computation time is obtained from evaluating the equations of motion; for a large number of assumed displacement fields the time-saving is obtained from solving the equations of motion.

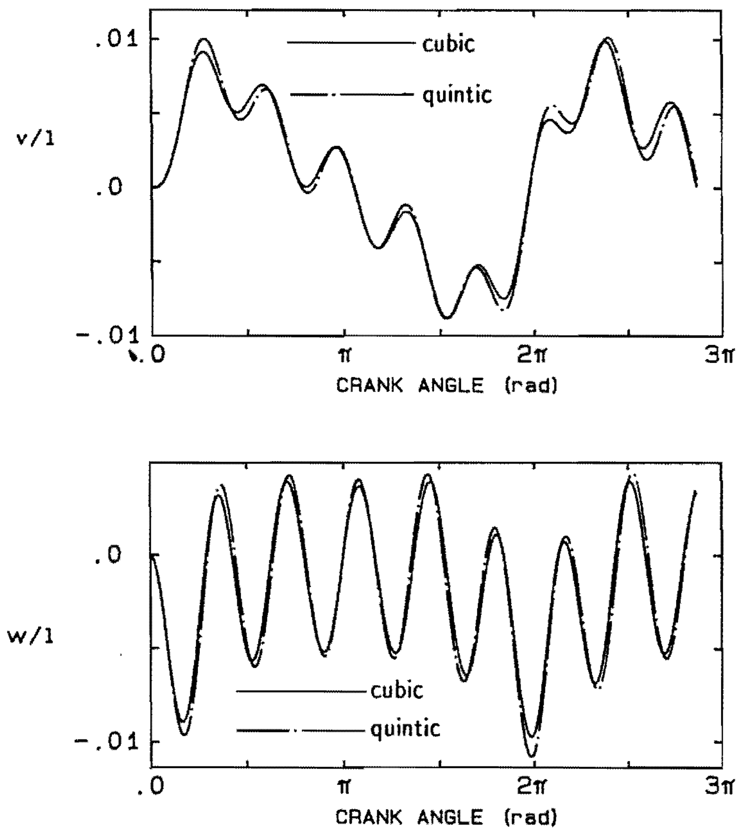

Fig. 5.4 Assumed modes solution using cubic and quintic polynomials

5.3 The finite element method

In the preceding section analytic functions have been used as base functions for approximating the displacement field due to deformation. However, as has been mentioned in chapter 3, this approach is only applicable for regularly shaped bodies. For complex shaped bodies, the finite element method is more appropriate for generating base functions. This method will be illustrated for the planar slider-crank mechanism considered in the preceding section. The finite element that has been used is described in appendix C using linearized strain-displacement relations. Axial deformation of the connecting rod is suppressed.

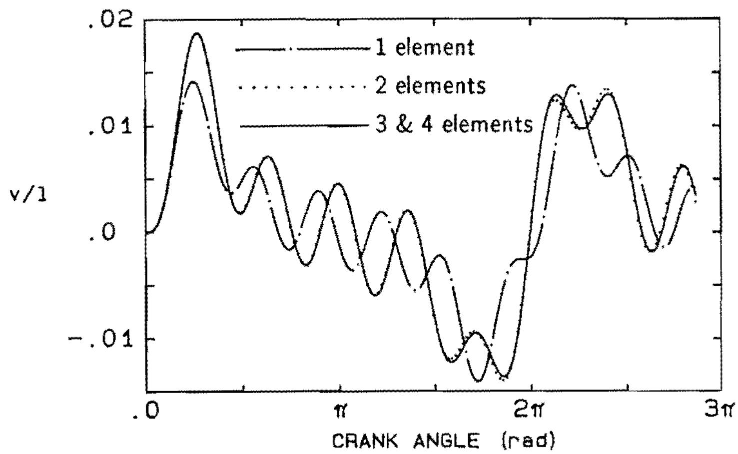

Fig. 5.5 Finite element solution: simply supported conditions

Fig. 5.5 shows the dimensionless deflection v/ℓ of the middle of the connecting rod for the case in which rigid body motions of the displacement field due to deformation are prevented by prescribing the deflection of the endpoints of the connecting rod. This case has been analyzed with the connecting rod divided into one, two, three, and four elements, respectively. The solutions for the three- and four-element subdivision are indistinguishable which indicates that a converged solution has been obtained for the three-element subdivision. This solution is indistinguishable from the quintic polynomial solution presented in the preceding section. The CPU times required for obtaining the solution for the one-, two-, three-, and four-element subdivision are respectively, 86 s, 283 s, 683 s, and 1455 s.

The planar slider-crank mechanism has also been analyzed for the case in which rigid body motions of the displacement field due to deformation are prevented by prescribing the deflection and the rotation of the endpoint of the connecting rod that is attached to the crank. (This may seem an odd choice. However, it simplifies formulating the constraint equations of two interconnected bodies because the displacements due to deformation of just one body are involved. In fact, this is used in formulating the equations of motion by the computer programs TREETOPS and CONTOPS (Singh et al. (1985, 1986)). This case has been analyzed with the connecting rod divided into one, two, three, and four elements, respectively. Fig. 5.6 shows the dimensionless deflection v/ℓ of the middle of the connecting rod. The solutions obtained for the three- and four-element subdivision are indistinguishable which indicates that a converged solution has been obtained for the three-element

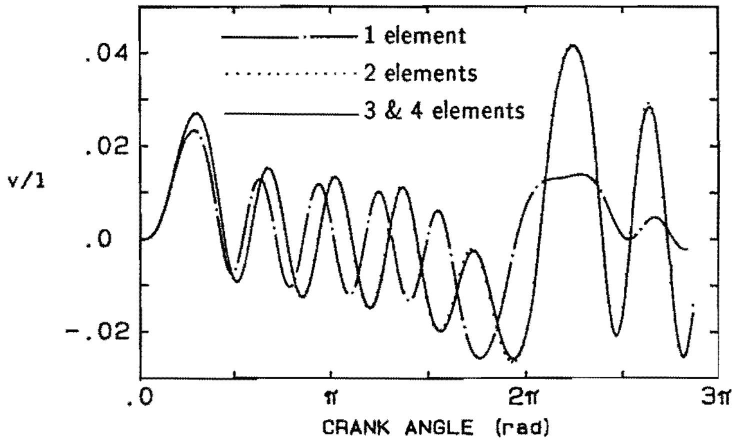

Fig. 5.6 Finite element solution: one-sided built in conditions

subdivision. The CPU time required for obtaining the solution for the one-, two-, three-, and four-element subdivision are respectively, 81 s, 244 s, 580 s, and 1294 s.

The results shown in fig. 5.5 and fig. 5.6 differ. The slider-crank mechanism is too complex to trace the cause of this discrepancy. For this reason the deformable beam shown in fig. 5.7 is considered. This beam and the connecting rod of the slider-crank mechanism are identical. In the initial configuration the beam is straight and its velocity is zero. The beam will bend under the action of gravity. The displacement field due to deformation is approximated with three finite elements using linearized strain-displacement relations. Rigid body motions of the displacement field due to deformation are prevented by prescribing either both endpoint deflections or the deflection and the rotation of the left endpoint. Three values of the axial force F are considered: F = 0 N, F = -200 N, and F = 200 N. (The order of magnitude of the axial force in the connecting rod of the slider-crank mechanism is 10^2 N.)

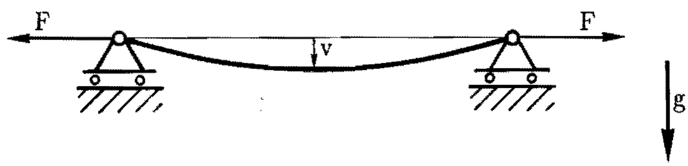

Fig. 5.7 Simply supported beam

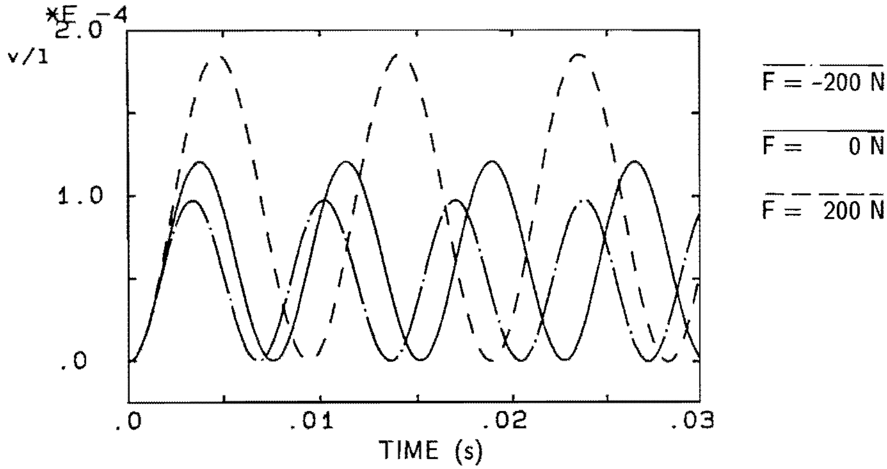

Fig. 5.8 Deflection of simply supported beam: one-sided built in conditions

Fig. 5.8 shows the dimensionless deflection v/ℓ of the middle of the beam for the above three values of the axial force F in case the deflection and the rotation of the left endpoint are prescribed. The mean value of the dimensionless deflection for $F = 0$ N agrees with the dimensionless static deflection of the beam: $(v_S/\ell) = (5/24)\rho\ell^3 g/Ed^2 = 60.3*10^{-6}$. The period of the response agrees with the analytic value of the period going with the lowest natural frequency of the beam: $T_a = (8/\pi)(\ell^2/d)\sqrt{\rho/E} = 7.58*10^{-3}$ s. The deflection and the period become smaller when a compressive load (F = -200 N) is applied, whereas the deflection and the period increase when a tensile load (F = 200 N) is applied! This is opposite to what happens in reality. This can be explained with the help of fig. 5.9. It shows the beam in a deflected configuration, and the reference configuration of the beam after a rigid body motion which corresponds to this deflected configuration. Forces normal to this reference configuration cause a deflection of the beam. The applied force F has a component normal to this reference configuration. For a positive value of F, this component will increase the deflection of the beam; for a negative value of F, this component will decrease the deflection of the beam. This agrees with the numerical results presented in fig. 5.8. For small deflections of the beam, the component of the applied force normal to the reference configuration is proportional to this deflection. Consequently it behaves as if it arises from a linear spring with a negative stiffness coefficient for a positive value of F, and a positive stiffness coefficient for a negative value of F. This explains the change in eigenfrequency due to the application of an axial force.

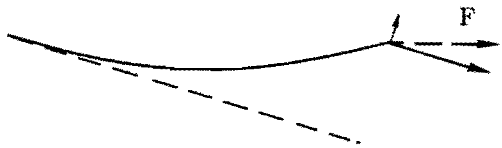

Fig. 5.9 Components of the axial load parallel and normal to the reference configuration

All this does not happen when the beam is analyzed with rigid body motions of the displacement field due to deformation eliminated by prescribing the deflection of the endpoints of the beam, because in that case the axial force is parallel to the reference configuration. This is confirmed by the results obtained for the case with prescribed endpoint deflections which are for all three values of F identical to the above result for F = 0 N.

From this result, it may be expected that the results for the slider-crank mechanism for both sets of conditions will agree better in case the axial force is smaller. This will be the case when the slider is lighter. In order to verify this anticipation, the slider-crank mechanism is analyzed with a slider mass equal to zero. The results for both sets of conditions are shown in fig. 5.10. It can be seen that the results agree much better, which supports the above explanation.

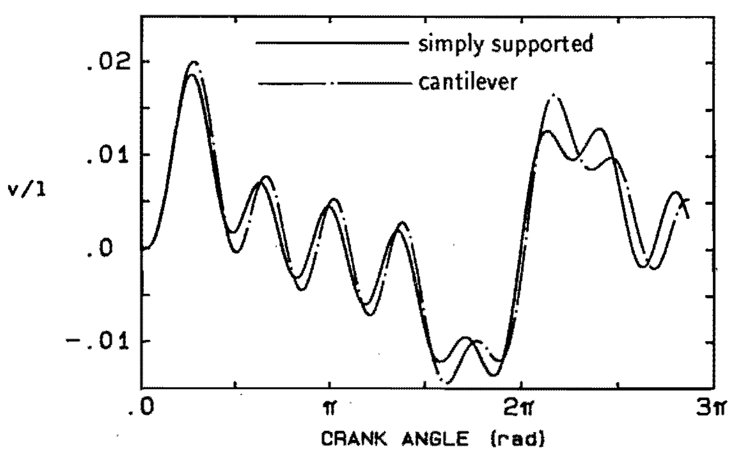

Fig. 5.10 Finite element solution: slider mass equal to zero

Another numerical experiment reveals an interesting phenomenon. The rigid body properties m, $\vec{C}_1$ and C_3 of the connecting rod are set equal to zero, and a rigid body with mass properties identical to those of the connecting rod is rigidly attached to the connecting rod at the endpoint with prescribed deflection and rotation. The results obtained for this problem are indistinguishable from those obtained with prescribed endpoint deflection and rotation, which indicates that these two problems are identical. The forces that are necessary for the motion of the extra body are reacted by a concentrated force and moment at the endpoint of the connecting rod. Since both problems are identical, this occurs also when rigid body motions are prevented by prescribing endpoint deflection and rotation of the connecting rod. Since in reality there is no concentrated moment at the end of the connecting rod, this may lead to incorrect results.

5.4 The modal synthesis method

From the CPU times reported in the preceding section it can be concluded that the CPU time required for a finite element analysis is about directly proportional to the square of the number of degrees of freedom. A regularly shaped body such as the connecting rod does not need to be divided into many elements. However, complex shaped bodies must be divided into many elements in order to be able to describe the shape of the body sufficiently accurate. The solution of the corresponding equations of motion requires much computation time. Therefore it is desirable to reduce the number of degrees of freedom in order to shorten the required computation time. The modal synthesis method can be used for this purpose. The effectiveness of the modal synthesis method depends to a great extent on a proper choice of base functions for approximating the displacement field due to deformation. Considerations that lead to a specific choice of base functions are given in subsection 5.4.1 for the planar slider-crank mechanism. In subsection 5.4.2 the conditions that are used for eliminating rigid body motions are considered. An approximate method for obtaining the time-independent inertia coefficients (2.34)-(2.41), the lumped mass approximation, is investigated in subsection 5.4.3.

5.4.1 *Selection of base functions*

In selecting base functions for the connecting rod, the forces that cause the deformation must be considered. In order to identify the importance of different forces causing the deformation, fig. 5.11 gives the total deflection of the middle of the connecting rod obtained with three finite elements (cf. fig. 5.5), and the quasi-

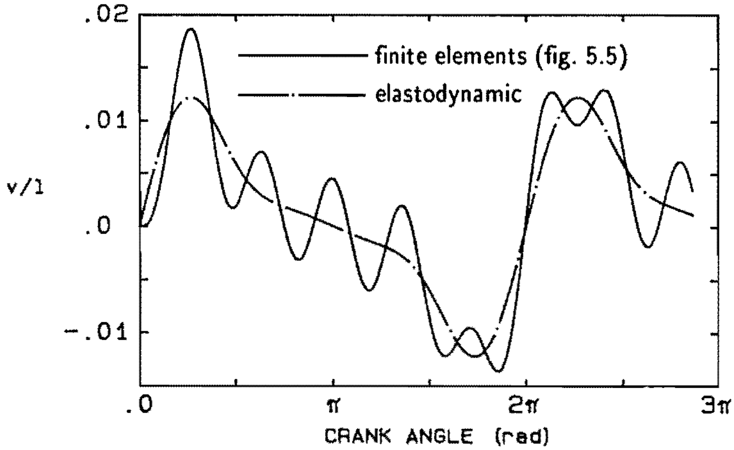

Fig. 5.11 Deflection of the middle of the connecting rod

static deflection resulting from the inertia load and the corresponding reaction forces at the endpoints of the connecting rod that go with the motion which follows from a kinematic analysis of the corresponding slider-crank mechanism with a rigid connecting rod (elastodynamic deflection, cf. appendix D). It can be seen that the main contribution can be attributed to the load arising from the rigid body motion of the connecting rod. This inertia load is a distributed load varying linearly along the axis of the connecting rod as has been mentioned already in section 3.2. It can be subdivided into a constant distributed load and a linearly distributed load that is antisymmetric relative to the middle of the connecting rod (fig. 5.12). The deflection of the middle of the connecting rod due to the antisymmetric load is zero. Consequently only the deflection resulting from the uniform load is of interest when one is concerned with the deflection of the middle of the connecting rod. The slider-crank mechanism has been analyzed using the deflection of the connecting rod

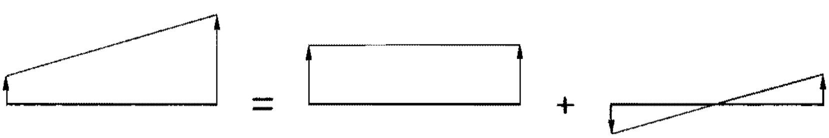

Fig. 5.12 Resolution of load into symmetric and antisymmetric component

resulting from a uniform load as base function for the connecting rod. This displacement field has been determined using a four-element subdivision. The response obtained with this displacement field is indistinguishable from the response obtained with the connecting rod divided into three finite elements. However, the required CPU time equals 59 s which is only 0.09 times the CPU time used for the finite element analysis with three elements. The reduction is even larger when the CPU time used for the modal synthesis analysis is compared with a finite element analysis without constraining the axial deformation of the connecting rod: the required CPU time for the modal analysis amounts then 0.02 times the required CPU time for the finite element analysis. It is more precise to sustain the latter reduction because modes of deformation cannot be readily eliminated from a finite element model. From this result it can be concluded that a substantial reduction of required CPU time can be obtained with the modal synthesis method.

A significant portion of the response of the connecting rod arises from its fundamental eigenfunction. The result obtained with just the static displacement field of the connecting rod resulting from a constant distributed load is so close to the total response because this displacement field is close to its fundamental eigenfunction. The magnitude of the contribution of the fundamental eigenfunction is determined by the exciting load, and the initial displacement field due to deformation and the corresponding velocity field of the connecting rod. The main exciting load consists of the inertia load and the corresponding reaction forces at the endpoints of the connecting rod that go with the rigid body motion of the connecting rod. The response due to this exciting force will be close to the quasi-static response, because the main frequency of the excitation ($\Omega = 150.0$ rad/s) is well below the fundamental frequency of the connecting rod ($\nu = (\pi^2/4)(\mathrm{d}/\ell^2)\sqrt{\mathrm{E}/\rho} = 829.2$ rad/s). For this reason the portion of the response of the connecting rod going with the fundamental eigenfunction is mainly due to the initial displacement field due to deformation and the corresponding velocity field of the connecting rod. The initial conditions used in the above analysis are not realistic. This follows also from experimental results presented by Sung et al. (1986). The influence of initial conditions will fade away due to damping and the response will become nearly periodic in process of time. Such a periodic response will be obtained immediately when the analysis is started using proper initial conditions. It may be expected that these initial conditions will be close to the conditions which follow from a quasi-static analysis. From a quasi-static analysis follows that the initial deflection of the middle of the connecting rod equals 0 m, and the initial velocity equals $(15/128)\rho\Omega^3\ell^5/\mathrm{Ed}^2 = 1.05$ m/s (cf. D.11). Fig. 5.13

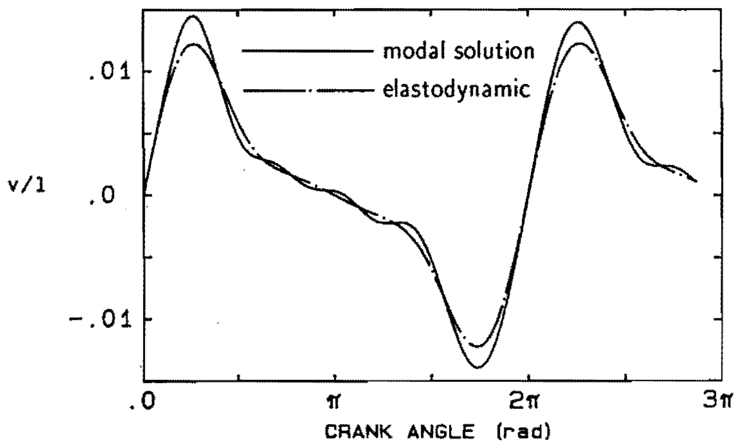

Fig. 5.13 Influence of initial condition on deflection

shows the dimensionless deflection of the middle of the connecting rod using these initial conditions for the displacement field due to deformation. The response is now almost periodic. It can be seen that the contribution of the fundamental eigenfunction has become much smaller, indicating that the response of the connecting rod is mainly due to the quasi-static response.

Some investigators (for instance Agrawal and Shabana, 1985; Singh et al., 1985) use eigenfunctions as base functions. These do not necessarily represent the actual displacement field due to deformation of a body well, especially when concentrated forces are applied. Moreover, the eigenfunctions of a specific body of a system of bodies may change when the configuration of the system changes. This can be illustrated with the two-link robot shown in fig. 5.14. The two links are identical

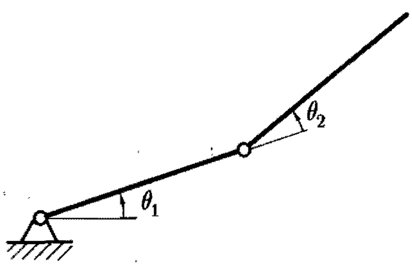

Fig. 5.14 Two link robot arm

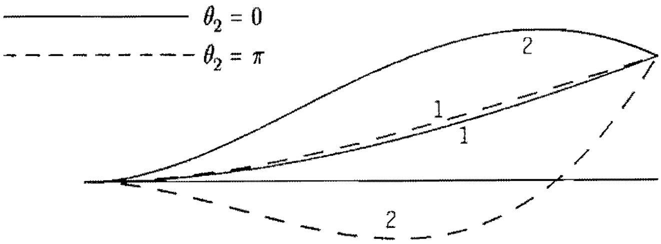

Fig. 5.15 Eigenfunctions of inner link

with the exception of the bending stiffness which is assumed to be infinite for the outer link. The angles θ_1 and θ_2 are kinematically driven. Consider the cases with θ_1 constant, and $\theta_2 = 0$, respectively $\theta_2 = \pi$.

Fig. 5.15 shows the first and second eigenfunction of the inner link for these two situations. It can be seen that especially the second eigenfunction depends on the configuration of the robot arm. This example illustrates that it may be difficult to decide which eigenfunctions should be used.

However, it is not difficult to decide which static displacement fields will describe the actual displacement field of the inner link probably well. The outer link applies a force and a moment to the inner link. Consequently the two static displacement fields resulting from either a concentrated force or a moment applied at the endpoint of the inner link will probably be a good approximation of the displacement field of the inner link. In any case they describe the quasi-static deflection due to the force and moment from the outer link exactly.

Fig. 5.16 shows the approximated eigenfunctions of the inner link for $\theta_2 = 0$ using the above suggested base functions. The fundamental eigenfunction is indistinguishable from the exact eigenfunction. The second eigenfunction is close to the exact eigenfunction. The approximated eigenfunctions for $\theta_2 = \pi$ are indistinguishable from the corresponding exact eigenfunctions. Consequently the above suggested base functions give also a good approximation of the dynamic behaviour of the inner link.

The load applied to a specific body should be considered in selecting base functions for approximating its displacement field. One should consider to use also eigenfunctions as base functions in case the frequency of the excitation is close to an eigenfrequency of the body.

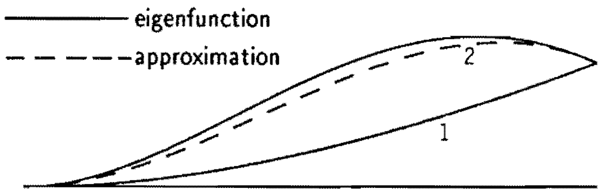

Fig. 5.16 Approximated eigenfunctions of the inner link for $\theta_2 = 0$

5.4.2 *Elimination of rigid body motions*

The base function used in the above analysis of the slider-crank mechanism does not contain a rigid body motion because the deflections at the endpoints of the connecting rod are equal to zero. The results obtained with the finite element method are incorrect in case rigid body motions are prevented by prescribing the endpoint deflection and rotation of the connecting rod as has been discussed in section 5.3. In order to investigate whether this is also the case when the modal synthesis method is used, the planar slider-crank mechanism is analyzed using eigenfunctions of the connecting rod going with zero deflection and rotation of the endpoint that is attached to the crank.

Fig. 5.17 shows the dimensionless deflection of the middle of the connecting rod in case two, respectively three eigenfunctions are used as base functions. These eigenfunctions have been determined with the connecting rod divided into four finite elements. The solid curve is the response obtained with the finite element method using a three-element approximation with rigid body motions eliminated by prescribing the endpoint deflection and rotation (cf. fig. 5.6). It can be concluded that the solution using the eigenfunctions of the cantilever connecting rod converges to this finite element solution when the number of eigenfunctions is increased. Consequently, the model using eigenfunctions suffers from the same problems as the finite element model.

Agrawal and Shabana (1985) obtained similar results in case they approximated the displacement field due to deformation with eigenfunctions of the cantilever connecting rod. They attributed these incorrect results to the fact that these eigenfunctions poorly represent the deflected shape of the connecting rod. In order to show that this explanation is incorrect, the conditions that are used for eliminating rigid body motions are changed. This is done by adding small rigid body motions to the original base functions. This transformation does not change

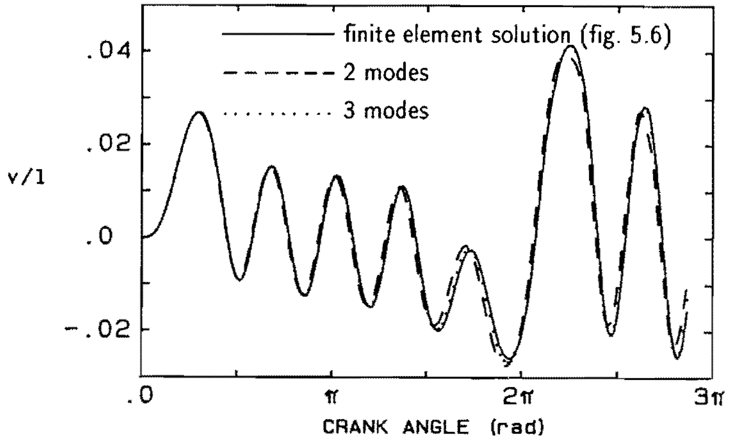

Fig. 5.17 Modal synthesis solution using eigenfunctions of cantilever beam

the shape of the base functions. Such a transformed set of base functions $\vec{\underset{\sim}{\Phi}}^*(\vec{x})$ can be written as

$$\vec{\underset{\sim}{\Phi}}^*(\vec{x}) = \vec{\underset{\sim}{\Phi}}(\vec{x}) + \vec{\underset{\sim}{\Phi}}_0 + \vec{\underset{\sim}{\vartheta}} \times \vec{x}, \tag{5.1}$$

where $\vec{\underset{\sim}{\Phi}}_0$ represents rigid body translations and $\vec{\underset{\sim}{\vartheta}} \times \vec{x}$ represents small rigid body rotations. (Actually, the base functions $\vec{\underset{\sim}{\Phi}}(\vec{x})$ are not necessarily small. This depends on how they have been normalized. However, the base functions multiplied by the corresponding generalized coordinates $\underset{\sim}{q}(t)$ are small. Consequently, the rigid body motions that are added to the original base functions are not necessarily small. But, when multiplied by the generalized coordinates $\underset{\sim}{q}(t)$, they are small and therefore the rigid body rotation that is added to the original base functions may be written as $\vec{\underset{\sim}{\vartheta}} \times \vec{x}$.) The unknown small rigid body translations and rotations follow from the conditions that are used for eliminating rigid body motions. Consider for instance the case for which these conditions are the mean displacement conditions (2.8) and (2.10). Substitution of (1) into (2.8) and (2.10) yields, respectively

$$\vec{\underset{\sim}{C}}_2 + m\,\vec{\underset{\sim}{\Phi}}_0 + \vec{\underset{\sim}{\vartheta}} \times \vec{\underset{\sim}{C}}_1 = \vec{0}, \tag{5.2}$$

$$\vec{\underset{\sim}{C}}_6 + \vec{\underset{\sim}{C}}_1 \times \vec{\underset{\sim}{\Phi}}_0 + \mathbf{C}_3 \cdot \vec{\underset{\sim}{\vartheta}} = \vec{0}. \tag{5.3}$$

$\vec{\underset{\sim}{\Phi}}_0$ and $\vec{\underset{\sim}{\vartheta}}$ can be solved from these equations because $\mathbf{C}_3$ is a positive-definite tensor

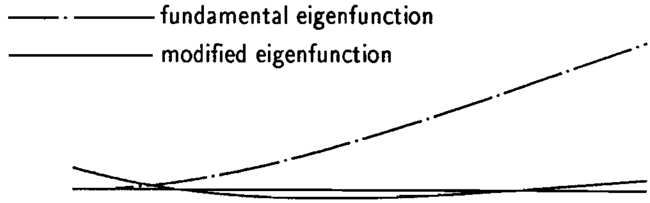

Fig. 5.18 Fundamental and modified eigenfunction of cantilever beam

(it is the inertia tensor of the body in the reference configuration). Substitution of the result into (1) yields the modified base functions. Fig. 5.18 shows the fundamental eigenfunction of the cantilever connecting rod and the base function that is obtained using the above transformation.

The slider-crank mechanism has been analyzed using eigenfunctions of the connecting rod going with zero deflection and rotation of the endpoint that is attached to the crank after the above described transformation. The deflection of the middle of the connecting rod obtained with two, respectively three of such base functions is given in fig. 5.19 . The solid curve is the response obtained with the finite element method using a three-element approximation with rigid body motions eliminated by prescribing the deflection of the endpoints of the connecting rod (cf.

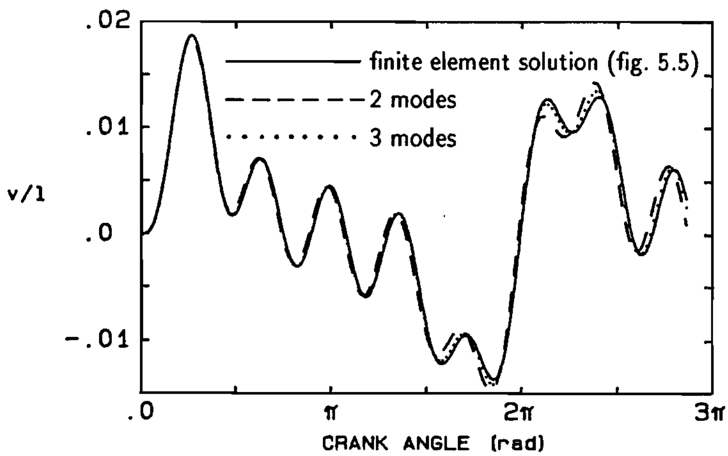

Fig. 5.19 Modal synthesis solution using modified eigenfunctions of cantilever beam

fig. 5.5). It can be seen that the solution obtained with modified eigenfunctions converges to this finite element solution.

This leads to the conclusion that the solution obtained with eigenfunctions of the cantilever connecting rod converges also to the correct solution provided suitable conditions are used for eliminating rigid body motions. However, when compared with the static deflection mode used in the beginning of this section, more base functions are needed because the eigenfunctions of the cantilever connecting rod do represent the deflected shape of the connecting rod worse.

5.4.3 *Lumped mass approximation*

In the preceding part of this section, the time-independent inertia coefficients (2.34)-(2.41) have been assembled from the corresponding coefficients of the beam element, which have been evaluated analytically. In literature a lumped mass approximation is frequently used (Sunada and Dubowsky, 1981, 1983; Yoo and Haug, 1986a, 1986b). The advantage of that approach is that the time-independent coefficients can be computed from nodal displacements and nodal masses which can be determined with a standard finite element computer program. It may be expected that a lumped mass approximation is less accurate than a consistent mass approximation. In order to evaluate the consequences of using a lumped mass approximation, the planar slider-crank mechanism of fig. 5.1 is considered. The displacement field due to deformation is approximated with the static displacement field resulting from a uniform load that has been used in subsection 5.4.1.

From subsection 5.4.1 it can be concluded that a four-element subdivision of the connecting rod yields a sufficiently accurate base function and corresponding time-independent inertia coefficients when a consistent mass approximation is used. Fig. 5.20 shows the dimensionless deflection of the middle of the connecting rod in case a lumped mass approximation is used. The results are obtained in case the base function is obtained with a four-element subdivision and an eight-element subdivision, respectively. The result going with the eight-element subdivision is indistinguishable from the result obtained with the three-element subdivision shown in fig. 5.5. The computation of the response going with a four-element subdivision and an eight-element subdivision took 59 s, respectively 62 s CPU time.

From these results it can be concluded that when compared with the consistent mass approximation, the lumped mass approximation may require a finer subdivision into elements for determining base functions in order to obtain the corresponding time-independent coefficients sufficiently accurate. This is not a severe dismerit because the bodies encountered in practice have generally a complex

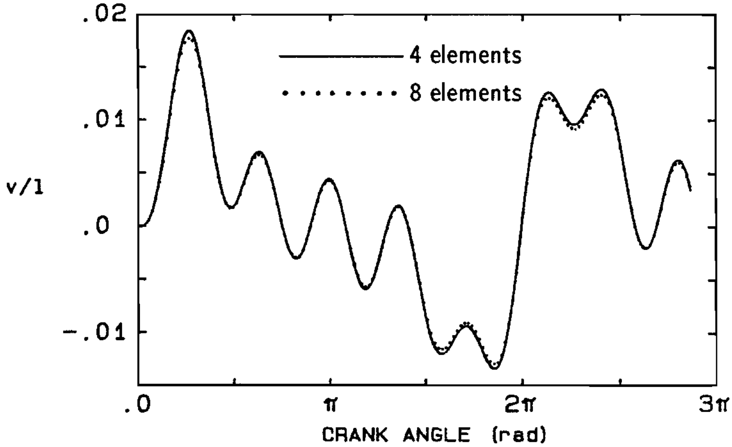

Fig. 5.20 Modal synthesis solution using lumped mass approximation

shape which requires already a fine subdivision into elements in order to describe their shape sufficiently accurate. Moreover, a finer subdivision requires only a small increase of required CPU time because the time-independent coefficients are evaluated only once. Observing this and in view of the fact that a lumped mass approximation is more versatile (a consistent mass approximation requires a library of time-independent coefficients of elements), a lumped mass approximation is preferable to a consistent mass approximation when the modal synthesis method is used. However, one should be aware that a finer subdivision into elements may be required than is needed for obtaining the base functions with sufficient accuracy.

5.5 Nonlinearities corresponding to displacements due to deformation

Because of the linearization of the strain-displacement relations and the approximation of the displacement field due to deformation by a linear combination of assumed displacement fields, there is no coupling between the transverse displacements and the axial displacements of the beam element used in the preceding sections. In reality, however, such a coupling may exist. For instance, an axial force changes the transverse stiffness of a beam, and transverse displacements give rise to axial displacements. These two items will be considered in this section.

5.5.1 *Nonlinear strain-displacement relations*

From the results presented by Bakr and Shabana (1986) it can be concluded that for the planar slider-crank mechanism considered in the preceding sections, nonlinear strain-displacement relations must be used in order to predict correctly the transverse deflection of the connecting rod. The axial force F is required in determining the geometric stiffness matrix going with the nonlinear strain-displacement relations (cf. appendix C). This force can be obtained from the axial deformation of the connecting rod. For this reason, the axial deformation will be included in the analysis.

The planar slider-crank mechanism considered in the preceding sections is analyzed using the finite element properties going with the nonlinear strain-displacement relations. The transverse displacement field is approximated by the static displacement field used in section 5.4. The axial displacement field is approximated by the static displacement field resulting from a constant axial force. Both displacement fields are obtained using a four-element approximation. Fig. 5.21 shows the deflection of the middle of the connecting rod obtained with the analyses using linearized and nonlinear strain-displacement relations. It can be seen that the nonlinear terms have a significant influence on the response and consequently linearization of the strain-displacement relations is not allowable for the present problem.

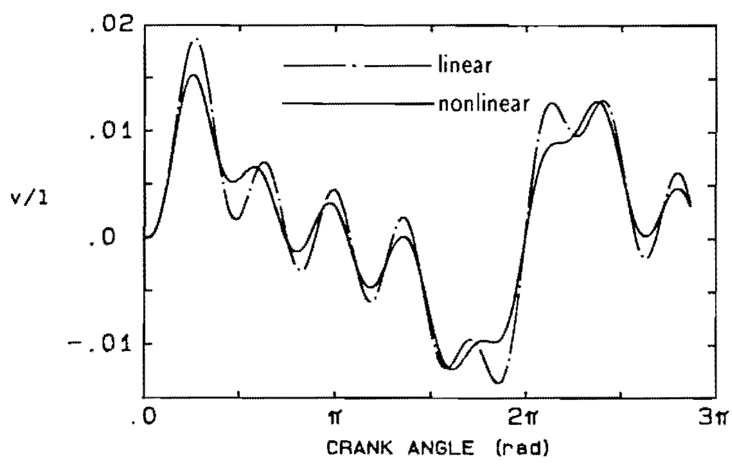

Fig. 5.21 Influence of nonlinearities

The linear solution obtained for the case in which the connecting rod is divided into two identical beams using a cubic polynomial approximation of the displacement field due to deformation (cf. fig. 5.2) is close to the nonlinear solution shown in fig. 5.21. This can be explained with the help of fig. 5.22. It shows a current configuration of the connecting rod (solid curve), and the reference configuration after the rigid body motion corresponding to this current configuration (dashed lines). The difference between these configurations represents displacements due to deformation. These are much smaller when the connecting rod is divided into two beams. Consequently, linearization of strain-displacement relations is allowable for much larger deflections of the connecting rod in case it is approximated with a number of beams. However, then it is not possible to reduce the number of degrees of freedom using the modal synthesis method. Because of the required computation time, an analysis based on the modal synthesis method combined with nonlinear strain-displacement relations is preferable to an analysis based on dividing the connecting rod into a number of beams combined with linearized strain-displacement relations.

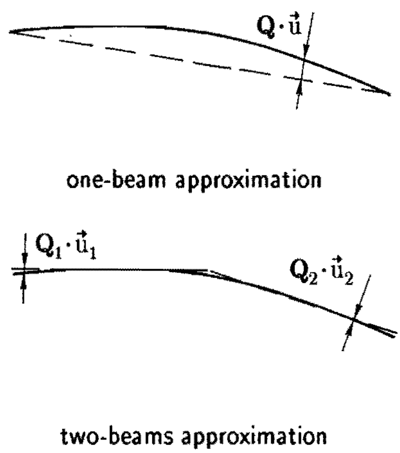

Fig. 5.22 Displacements due to deformation when the connecting rod is divided into several beams

5.5.2 *Nonlinear combinations of assumed displacement fields*

The displacement field due to deformation has been approximated by a linear combination of assumed displacement fields (cf. 2.28). However, it may be necessary to include also nonlinear terms in order to predict the dynamic behaviour of bodies correctly. This may be the case for instance for rotating beams.

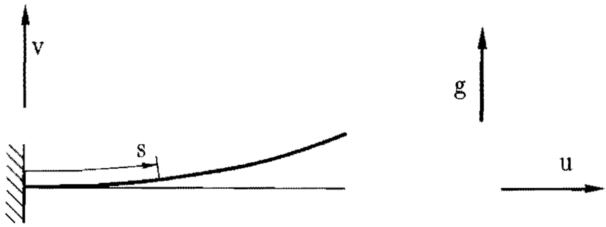

Fig. 5.23 Cantilever beam

Transverse displacements of a beam give rise to axial displacements. This can be illustrated with the cantilever beam shown in fig. 5.23. Assume that the elastic axis of the beam is inextensible. When higher order terms are neglected, the axial displacement field u due to the transverse displacement field v is given by

$$u = -\int_0^s \tfrac{1}{2}(\partial v/\partial s)^2 ds, \tag{5.4}$$

where s is the distance, measured along the elastic axis of the beam, between the built in end and an arbitrary point on the elastic axis.

When the transverse displacement field of the beam due to deformation is approximated by a linear combination of assumed displacement fields, the corresponding axial displacement field will be a quadratic function of the coefficients of the assumed displacement fields; the coefficients in this quadratic function depend on the assumed displacement fields. Such an axial displacement field cannot be taken into account when displacement fields due to deformation are approximated by a linear combination of assumed displacement fields. Generally, the nonlinear terms are negligible. However, this may be inadmissible for a beam that rotates about an axis which is perpendicular to its elastic axis. The nonlinear terms that account for the axial displacements due to the transverse displacements lead to terms in the equations of motion that are linear in the coefficients of the assumed displacement fields, namely $\vec{\omega}\cdot\{t(\delta\vec{u},\vec{x})\}\cdot\vec{\omega}$, and $\dot{\vec{\omega}}\cdot\vec{v}(\delta\vec{u},\delta\vec{x})$ (cf. 2.17). The first term accounts for the centrifugal stiffening of the beam. The influence of neglecting this term will be illustrated with a numerical example.

Consider the cantilever beam shown in fig. 5.23. Its properties are identical to those of the connecting rod used in previous examples. The beam rotates with a constant angular velocity about an axis which is perpendicular to the undeformed beam and which passes through the built in end of the beam. In order to excite the beam, it is subjected to the acceleration of gravity which is parallel to the axis of rotation. Only transverse deflections are considered. The displacement field due to deformation is approximated by the eigenfunctions corresponding to the cantilever beam. Analyses for several values of the angular velocity show no dependency of the response on the angular velocity which can be attributed to the missing of the centrifugal stiffening term.

This failure may be partially obviated by changing the conditions for eliminating rigid body motions in the displacement field due to deformation such that the rigid body motion going with a deflection of the beam cause displacements towards the axis of rotation. This may be achieved for instance by changing to the mean axis conditions using eqs. (5.1)-(5.3). Fig. 5.24 shows the displacement of the free end of the beam parallel to the axis of rotation for an angular velocity of 150.0 rad/s (dashed curve). It has been obtained with the two lowest eigenfunctions of the cantilever beam transformed according to eqs. (5.1)-(5.3). Increasing the number of assumed displacement fields does not change the response, which indicates that a converged solution is obtained for two assumed displacement fields. The dot and dash curve represents the response obtained in case the two lowest eigenfunctions of the cantilever beam are used. Comparison of the two curves shows that the eigenfrequency is higher and the deflection is smaller when the transformed eigenfunctions are used. This is what should happen when the beam rotates.

Linearized strain-displacement relations have been used for the above-described analysis. Consequently, the stiffening of the beam due to an axial tensile force is not taken into account. In order to study the influence of this force, the rotating beam has also been analyzed using nonlinear strain-displacement relations. The axial force, which is required for determining the geometric stiffness matrix (cf.

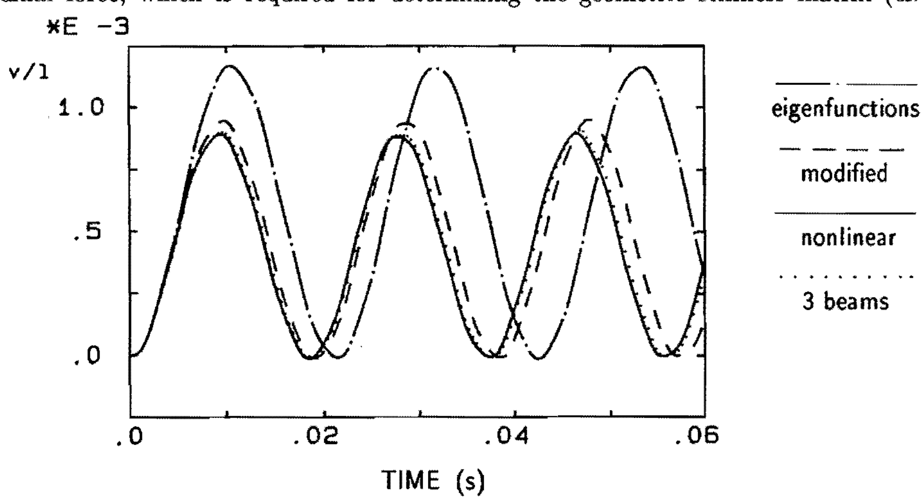

Fig. 5.24 Deflection of the endpoint of the rotating beam

appendix C), is obtained from the axial deformation of the beam. For this purpose, an extra assumed displacement field is used which represents the quasi-static displacement field of the beam resulting from the inertia forces which arise from a rotation of the beam in case it is undeformed. The response of the endpoint of the beam for this approximation is given by the solid curve in fig. 5.24. As can be seen it is not allowable to use linearized strain-displacement relations for this problem.

Another way to obviate the absence of quadratic terms in the displacement field due to deformation is to divide the beam into a number of parts. As has been mentioned in the preceding subsection, the displacements due to deformation are then much smaller. Hence, also the corresponding axial displacements due to the transverse displacements will be much smaller and are consequently negligible for much larger deflections. The rotating beam has been approximated by dividing it into three identical parts. The displacements due to deformation of these parts are approximated by the cubic polynomials (3.7)-(3.8). The response obtained for this approximation is given by the dotted curve of fig. 5.24. As can be seen the agreement between the three beam approximation and the approximation obtained with the transformed eigenfunctions using nonlinear strain-displacement relations is good.

Terms in the equations of motion that are linear in the displacements due to deformation are discarded due to the approximation of the displacement field due to deformation by a linear combination of a set of assumed displacement fields. For a rotating beam, the influence of these terms is taken into account when use is made of nonlinear strain-displacement relations and assumed displacement fields that satisfy the mean displacement conditions. More research is necessary to find out whether this holds more generally. When this is not the case, also nonlinear terms must be included for the approximation of displacement fields due to deformation.

5.6 Shifting of frequencies

5.6.1 *Tuning of frequencies*

One is often more concerned with the dynamic behaviour of the system of bodies as a whole rather than with the displacement field of the individual bodies. The displacements of the points of the bodies that are connected to other bodies should be approximated well in order to have a good approximation of the global motion of the system of bodies. Due to the connections, concentrated loads act on the bodies

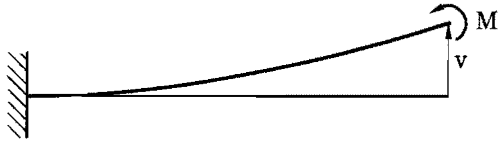

Fig. 5.25 Cantilever beam

at these points. The corresponding quasi-static displacement field due to deformation is a linear combination of the displacement fields resulting from concentrated loads applied at these points. However, these displacement fields do not necessarily approximate the dynamic behaviour well. A procedure is proposed in this section for improving the approximation of this dynamic behaviour.

Consider the cantilever beam shown in fig. 5.25. It has the same properties as the connecting rod used in previous examples. The beam is initially straight and has zero velocity. The applied moment equals 1.0 Nm. The quasi-static deflection is described without error when the displacement field is approximated with the static displacement resulting from a moment applied at the free end of the beam. However, the eigenfrequency corresponding to this displacement field (ν = 375.7 rad/s) does not agree with the fundamental frequency of the beam (ν = 295.4 rad/s), which leads to a discrepancy of the deflection of the endpoint of the beam. This is illustated by fig. 5.26. It shows the dimensionless deflection obtained with the finite element method using four elements (solid curve) and the response obtained with the above mentioned displacement field (dot and dash curve). It can be seen that the amplitude of the deflection obtained with this displacement field agrees well, but the corresponding frequency is too high.

A procedure to shift the approximate frequency to the fundamental frequency consists of multiplying the modal mass going with the static displacement field by the square of the ratio of the approximate frequency and the fundamental frequency: $(375.7/295.4)^2$. The result obtained with the static displacement field using this modified modal mass is given by the dotted curve shown in fig. 5.26. It can be seen that the modification of the modal mass leads to a much better approximation.

The approximation will also be better when the displacement field of the beam is approximated by a linear combination of displacement fields. Consider for instance the case in which, next to the above used static displacement field

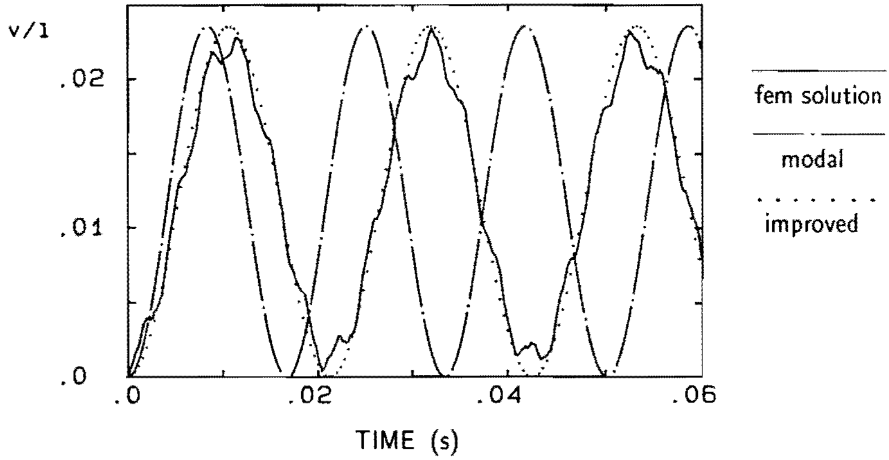

Fig. 5.26 Influence of modifying mass matrix: one mode solution

resulting from a moment applied at the free end of the beam, the static displacement resulting from a transverse force applied at the free end is used. Fig. 5.27 shows the response obtained with these two displacement fields (dot and dash curve) and the response obtained with the finite element method using four elements (solid curve). It can be seen that the approximation is now much better. This is primarily due to the fact that the lowest eigenfrequency corresponding to

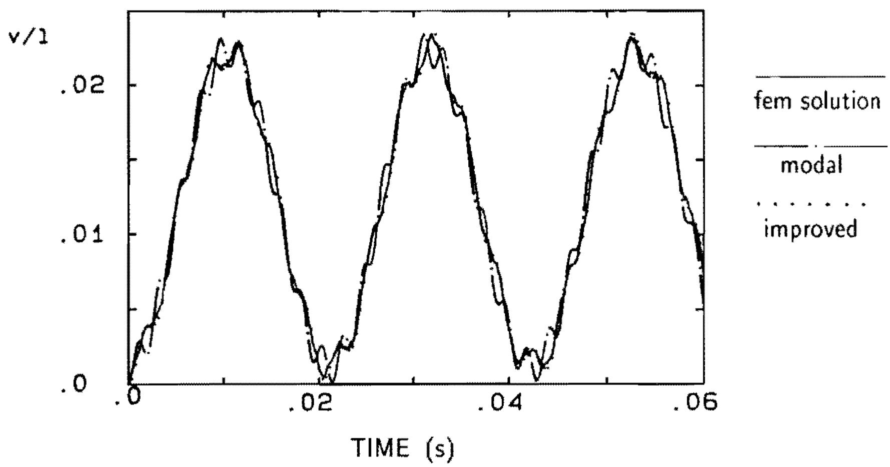

Fig. 5.27 Influence of modifying mass matrix: two mode solution

the two displacement fields approximation is close to the fundamental frequency of the beam.

The solution using two displacement fields can be improved with the above-described shifting procedure. The two displacement fields are orthogonalized before the procedure is applied. This is accomplished by using them as base functions of the space of eigenfunctions of the beam. The coefficients of the approximate eigenfunctions relative to this base follow from the eigenvalue problem going with the approximate equations of motion of the beam

$$\underline{C}_8\ddot{\underset{\sim}{\alpha}} + \underline{K}\underset{\sim}{\alpha} = \underset{\sim}{0}, \tag{5.4}$$

where $\underline{C}_8$ and $\underline{K}$ represent the modal mass matrix and the modal stiffness matrix, respectively, going with the two static displacement fields. The two approximate eigenfunctions which result from this analysis are used as base functions. From this eigenvalue problem follows also an approximation of the two lowest eigenfrequencies of the beam. The modal masses going with the approximate eigenfunctions are multiplied by the square of the ratio of the corresponding approximate eigenfrequency and the corresponding exact eigenfrequency.

This procedure is applied to the above two static displacement fields. The results obtained with the approximate eigenfunctions are identical to the result obtained with the two static displacement fields, because they span the same vector space of functions. The result obtained after modifying the modal masses is given by the dotted curve in fig. 5.27. It can be seen that the modification of the modal masses leads to a better approximation.

From the above it can be concluded that a more accurate discription of the dynamic behaviour of a body can be obtained by shifting the approximate eigenfrequencies of the body to its exact eigenfrequencies. However, more research is needed to fully understand the proposed procedure and to justify its application. It is felt that the procedure can also be used for improving the approximate solution of the dynamic behaviour of a body which is part of a system of bodies. An alternative application of the procedure is given in the next subsection.

5.6.2 *Lowering of frequencies*

The procedure that has been presented in the preceding subsection can also be used to lower high frequencies in a model of a system of bodies. This may be desirable because high frequencies slow down the integration process. The modal synthesis method is often used to reduce the number of degrees of freedom. The eigenfrequencies corresponding to the retained displacement fields are generally low. However, sometimes it may be beneficial to retain displacement fields with which correspond high frequencies. For example, an axial displacement field was included in determining the nonlinear response of the planar slider-crank mechanism in subsection 5.5.1 in order to determine the axial load in the connecting rod. The frequency related to this displacement field is high as compared with the frequency going with the transverse deflection of the connecting rod, which slows down the integration process. It is not important that the frequency corresponding to the axial displacement field is approximated correctly because the axial displacement field is of minor importance for the deflection of the connecting rod; only its mean value should be correct in order to obtain an accurate mean value of the axial load.

The procedure presented in the preceding section is applied to the modal mass corresponding to the axial displacement field of the connecting rod for the problem considered in subsection 5.5.1. The modal mass is multiplied by respectively 10.0, 25.0, 50.0, and 100.0. The CPU times required for the analyses are respectively 0.5, 0.4, 0.3, and 0.2 times the CPU time required for the analysis using a consistent modal mass. The deflection of the middle of the connecting rod is not influenced when the modal mass is multiplied by 10.0 or 25.0; the response is slightly different when the multiplier equals 50.0 or 100.0. This may be attributed to the fact that the frequency is shifted too close to the eigenfrequency related to the transverse deflection of the connecting rod.

It can be concluded that a high eigenfrequency can be lowered by multiplying the modal mass by a factor > 1.0. This factor should be such that the lowered frequency does not approach the eigenfrequencies corresponding to relevant displacement fields too close. The lowering of frequencies leads to CPU time savings.

chapter 6

Concluding remarks

6.1 Conclusions

In this thesis, a mathematical model is presented for determining the dynamic behaviour of a deformable body which forms part of a system of bodies. For describing the kinematics of the body, its displacement field is resolved into a displacement field due to a rigid body motion and a displacement field due to deformation. The displacement field due to deformation is chosen such that it does not contain rigid body motions. Two kinds of conditions are considered to achieve this, namely conditions for displacements of selected particles of the body and conditions for mean displacements of the body. The latter conditions lead to simpler equations of motion. The displacement field due to deformation is approximated by a linear combination of a number of assumed displacement fields. Three methods are considered for generating such displacement fields, namely the assumed-modes method, the finite element method, and the modal synthesis method.

The simplification of the equations of motion when the mean displacements going with the displacement field due to deformation are chosen equal to zero, leads to a computation time reduction of a few decades of per cents in the most favourable case. For the systems investigated in this thesis the dynamic behaviour is approximated better when the displacement field due to deformation is approximated by assumed displacement fields with mean displacements equal to zero. Caution must be taken in preventing rigid body motions of the displacement field due to deformation by prescribing displacements of selected particles of the body, since the solution of the dynamic behaviour may be incorrect.

The assumed-modes method is only feasible for regularly shaped bodies. The finite element method and the modal synthesis method can be used for bodies with arbitrary shape. The finite element method leads often to a model with many degrees of freedom. The solution of such a model requires much computation time. The modal synthesis method can then be used to reduce the number of degrees of freedom such that the required computation time is cut down. The effectiveness of the modal synthesis method depends to a great extent on a proper choice of the assumed displacement fields. Such a choice can generally be made in advance on the basis of the load on the body. The lumped mass approximation which is frequently used in literature is feasible for determining time-independent mass coefficients from displacement fields which have been determined with a standard finite element program. One should bear in mind that a finer subdivision into elements may be required than would be necessary for determining the displacement fields sufficiently accurate.

A method is proposed to improve approximations for describing the dynamic behaviour of a body for a specific set of assumed displacement fields. This method has been used successfully for reducing the required computation time by lowering irrelevant high frequencies.

6.2 Suggestions for further research

In section 2.5, the equations of motion of a body are written in terms of Euler parameters and their first and second time derivatives in order to allow their implementation in the computer program DADS. This leads to four equations that describe the rotational motion of the body whereas three equations suffice. Consequently, a constraint equation must be introduced which takes the dependency of these equations into account. The equations of motion are simpler when they are written in terms of the angular velocity vector and its first time derivative. Further, time derivatives of constraint equations are more easily formulated in terms of the angular velocity vector and its first time derivative. All this and the results obtained by Nikravesh et al. (1985) indicate that a computer program based on the equations of motion in terms of the angular velocity vector and its first time derivative will be more efficient. This may be a fruitful topic for further research.

Computer programs for the analysis of the dynamic behaviour of multibody systems are based on either the global description or the relative description. The description given by Haug and McCullough (1986) may be regarded as an impulse to arrive at a program which is based on a synthesis of the global description and the relative description. They derived the equations of motion of recurring subsystems in terms of variables that describe the relative motion of the interconnected bodies in the subsystem and implemented them into the multibody program DADS, which is based on the global description. This approach could be extended such that it is not necessary to derive the equations of motion of a specific subsystem in symbolic form and to implement them into a multibody program. This requires the addition of a module that automatically evaluates the equations of motion of any subsystem using the relative description. Perhaps it is even possible to give guidelines on which description is most efficient for a specific system. These can then be used by the program to select the most efficient description automatically.

The constraint equations for kinematic connections which are implemented in computer programs that are based on the global description are generally written in terms of the kinematic variables of the interconnected bodies. These equations cannot be solved explicitly for specific variables. However, they can be solved explicitly when variables that describe the relative motion of the interconnected bodies are introduced such as is done in the relative description. This can be used in solving the equations of motion of multibody systems with a tree structure, by circumventing the need to solve the constraint equations iteratively and to partition variables into dependent and independent variables numerically. Actually, such a description leads to the relative description within a program based on the global description. It can be implemented without difficulty in a multibody program such as DADS, since some kinematic connections are already described using the variables that describe the relative motion of the interconnected bodies. Therefore, research should be devoted to a solution procedure that takes advantage of the fact that the constraint equations can be solved explicitly.

A good choice of assumed displacement fields for a specific body can generally be made on the basis of the load on the body. This presents no difficulties when the type of load does not change much. However, the selection of assumed displacement fields is not straightforward when the type of load changes considerably. This may be the case when the point of application of the load changes, such as occurs for a body which is interconnected by a translational joint to another body. The best choice for a set of assumed displacement fields depends on the instantaneous location of the point of contact. Such a dependency cannot be accounted for with the description given in this thesis because the assumed displacement fields are taken invariable. Wang and Wei (1987) presented a mathematical model for a beam which takes a change of assumed displacement fields into account. They approximated the displacement field of the beam with instantaneous eigenfunctions. However, the instantaneous eigenfrequencies go to infinity when the point of contact approaches the end of the beam which slows down the integration process. Another option might be to waive the wish to use an instantaneous good set of assumed displacement fields and use displacement fields that do not change. This approach has been followed by Hwang and Haug (1987). A failure of this approach is that the displacement fields due to deformation of the body on both sides of the translational joint are coupled, which may lead to incorrect results. Further, it may be necessary to include a relatively large number of assumed displacement fields which leads to long computation times. The selection of assumed displacement fields for bodies with the above discribed loading requires further research.

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appendix A

Mathematical notation

A.1 Matrices

An m × n *matrix* is defined as a set of objects arranged in m rows and n columns. The objects may be for instance scalars, vectors, tensors or matrices. Matrices will be designated by an underscore; for legibility, column matrices will be designated by a wavy underscore. Matrices are sometimes denoted by the elements enclosed in brackets. A typical element of a matrix is denoted by the matrix symbol subscripted by its row and column number. The column number 1 for elements of a column matrix will be omitted. The name-giving of special matrices and operations on matrices agree with the normally used definitions (cf. e.g. Arnold; 1962).

Multiplication of a matrix with another object is evaluated by the common multiplication rule of matrix algebra with the understanding that the operation on the elements of the matrix and the other object is governed by the operator between the matrix and the object. For example, when the elements of the matrix and the object are vector quantities, then a · indicates that the resulting matrix elements are scalars obtained from scalar multiplication of the elements of the matrix and the objects.

The skew-symmetric 3 × 3 matrix that corresponds to a 3 × 1 column matrix $\utilde{a}$ in accordance with the definition

$$\underline{\tilde{a}} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{A.1}$$

is given a wavy superscript in order to discriminate it from other matrices. The elements of $\utilde{a}$ may be scalars, vectors or tensors.

A.2 Vectors and tensors

Vectors are entities defined in a Euclidean vector space that possess both magnitude and direction. They are often visualized by an arrow that is pointing in the direction associated with the vector and that has a length equal to the magnitude of the vector. Vectors will be designated by letters with an arrow as superscript.

Vectors as used in this thesis have the usual properties (cf. e.g. Newell, 1962).

The *scalar product* of an arbitrary pair of vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}\cdot\vec{b}$. It is a scalar with magnitude equal to the product of the magnitude of the two vectors and the cosine of the angle between the vectors.

The *vector product* of an arbitrary pair of vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a} \times \vec{b}$. It is a vector perpendicular to the two vectors with magnitude equal to the product of the magnitude of the two vectors and the sine of the angle between the two vectors; the sense of the vector is such that $\vec{a}$, $\vec{b}$ and $\vec{a} \times \vec{b}$ make a right-handed system.

The *tensor product* of an arbitrary pair of vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}\vec{b}$. It is a linear vector operator, called dyad, that can be used to form for instance the scalar product with an arbitrary vector $\vec{c}$ yielding a vector such that

$$(\vec{a}\vec{b})\cdot\vec{c} = (\vec{b}\cdot\vec{c})\vec{a}, \tag{A.2}$$

$$\vec{c}\cdot(\vec{a}\vec{b}) = (\vec{a}\cdot\vec{c})\vec{b}. \tag{A.3}$$

A second order *tensor* is a linear combination of dyads. In this thesis only second order tensors are used. They are identified by boldface letters. Such as for dyads, second order tensors can be used to form scalar products with an arbitrary vector yielding a vector. This vector follows from the definitions of the scalar product of a dyad and a vector.

The *product* of two second order tensors, **A** and **B**, is denoted by **A·B**. It is a second order tensor such that

$$\vec{a}\cdot(\mathbf{A}\cdot\mathbf{B})\cdot\vec{b} = (\vec{a}\cdot\mathbf{A})\cdot(\mathbf{B}\cdot\vec{b}) \tag{A.4}$$

for all vectors $\vec{a}$ and $\vec{b}$.

The *trace* of a tensor **A**, denoted by tr(**A**), is a scalar such that for $\vec{a}\cdot(\vec{b} \times \vec{c}) \neq 0$

$$(\mathbf{A}\cdot\vec{a})\cdot(\vec{b} \times \vec{c}) + \vec{a}\cdot\{(\mathbf{A}\cdot\vec{b}) \times \vec{c}\} + \vec{a}\cdot\{\vec{b} \times (\mathbf{A}\cdot\vec{c})\} = \mathrm{tr}(\mathbf{A})\ \vec{a}\cdot(\vec{b} \times \vec{c}) \tag{A.5}$$

This scalar does not depend on $\vec{a}$, $\vec{b}$ or $\vec{c}$.

The *scalar product* of two second order tensors, **A** and **B**, is denoted by **A:B**. It is a scalar such that

$$\mathbf{A}:\mathbf{B} = \mathrm{tr}(\mathbf{A}\cdot\mathbf{B}). \tag{A.6}$$

The *conjugate tensor* of an arbitrary tensor **A**, denoted by $\mathbf{A}^{\mathrm{C}}$, is a second order tensor such that

$$\vec{b}\cdot\mathbf{A}\cdot\vec{a} = \vec{a}\cdot\mathbf{A}^{C}\cdot\vec{b} \tag{A.7}$$

for all vectors $\vec{a}$ and $\vec{b}$.

The second order *unit tensor* **I** and the second order zero tensor **0** are defined by the requirements that

$$\mathbf{I}\cdot\vec{a} = \vec{a}\cdot\mathbf{I} = \vec{a}, \tag{A.8}$$

and

$$\mathbf{0}\cdot\vec{a} = \vec{a}\cdot\mathbf{0} = \vec{0} \tag{A.9}$$

for all vectors $\vec{a}$.

A tensor is called *symmetric* when it equals its conjugate. A tensor is called *skew-symmetric* when it equals the negative of its conjugate. A vector $\vec{a}$ is associated with a skew-symmetric tensor **A**, such that

$$\mathbf{A}\cdot\vec{b} = \vec{a}\times\vec{b} \tag{A.10}$$

for all vectors $\vec{b}$. $\vec{a}$ is termed the axial vector of **A**.

Some useful identities are

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}, \tag{A.11}$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} = \{\vec{b}\vec{a} - (\vec{a}\cdot\vec{b})\mathbf{I}\}\cdot\vec{c}, \tag{A.12}$$

$$\vec{a}\times\{\vec{b}\times(\vec{a}\times\vec{b})\} = \vec{b}\times\{\vec{a}\times(\vec{a}\times\vec{b})\}. \tag{A.13}$$

The matrix representation of an arbitrary vector $\vec{c}$ relative to a base $\underset{\sim}{\vec{a}}$ is given by

$$\underset{\sim}{c}^{a} = \begin{bmatrix} \vec{a}_1\cdot\vec{c} \\ \vec{a}_2\cdot\vec{c} \\ \vec{a}_3\cdot\vec{c} \end{bmatrix} = \underset{\sim}{\vec{a}}\cdot\vec{c}. \tag{A.14}$$

The matrix representation $\underline{C}^{ab}$ of an arbitrary tensor **C** relative to the bases $\underset{\sim}{\vec{a}}$ and $\underset{\sim}{\vec{b}}$ is given by

$$\underline{C}^{ab} = \begin{bmatrix} \vec{a}_1 \cdot C \cdot \vec{b}_1 & \vec{a}_1 \cdot C \cdot \vec{b}_2 & \vec{a}_1 \cdot C \cdot \vec{b}_3 \\ \vec{a}_2 \cdot C \cdot \vec{b}_1 & \vec{a}_2 \cdot C \cdot \vec{b}_2 & \vec{a}_2 \cdot C \cdot \vec{b}_3 \\ \vec{a}_3 \cdot C \cdot \vec{b}_1 & \vec{a}_3 \cdot C \cdot \vec{b}_2 & \vec{a}_3 \cdot C \cdot \vec{b}_3 \end{bmatrix} = \underset{\sim}{\vec{a}} \cdot C \cdot \underset{\sim}{\vec{b}}^T. \tag{A.15}$$

In this thesis the matrix representation of all vectors and tensors are relative to the same base, namely an orthonormal right-handed inertial base $\underset{\sim}{\vec{e}}$. Consequently there is no need to specify the base that is used for determining the matrix representation.

appendix B

Description of rotational motion in terms of Euler parameters

From Euler's theorem, a rotation about a fixed point may be conceived as the result of a rotation about a unique axis passing through that point (Wittenburg, 1977). Let the direction of that axis be defined by the unit vector $\vec{e}$ and the angle of rotation by χ. Then from fig. B.1 the position vector of an arbitrary material point P in its reference configuration, $\vec{x}$, and the position vector of P after a rotation about an axis passing through an inertial point O, $\vec{r}$, are related by the equation

$$\vec{r} = \vec{x} + (1 - \cos\chi)\vec{w} + \sin\chi\,\vec{v}, \tag{B.1}$$

where $\vec{v}$ is perpendicular to the plane spanned by $\vec{x}$ and $\vec{e}$, and $\vec{w}$ is perpendicular to $\vec{e}$ and $\vec{v}$; the length of both $\vec{v}$ and $\vec{w}$ equal the radius of the circle traversed by P due to the rotation. From this $\vec{v}$ and $\vec{w}$ can be written in terms of $\vec{e}$ and $\vec{x}$. This yields

$$\vec{v} = \vec{e} \times \vec{x}, \tag{B.2}$$

$$\vec{w} = \vec{e} \times \vec{v} = \vec{e} \times (\vec{e} \times \vec{x}). \tag{B.3}$$

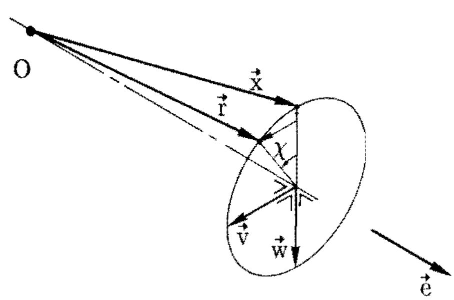

Fig. B.1 Rotation about $\vec{e}$

Substitution of (2) and (3) into (1) yields

$$\vec{r} = \vec{x} + 2\vec{q} \times (\vec{q} \times \vec{x}) + 2q_0\vec{q} \times \vec{x}, \tag{B.4}$$

where

$$q_0 = \cos(\chi/2), \tag{B.5}$$

$$\vec{q} = \vec{e}\,\sin(\chi/2). \tag{B.6}$$

q_0 and the components of $\vec{q}$ relative to an arbitrary orthonormal inertial base $\underset{\sim}{\vec{e}}$ are called Euler parameters. Since $\vec{e}$ is a unit vector, the Euler parameters are related by

$$q_0^2 + \vec{q}\cdot\vec{q} = 1. \tag{B.7}$$

For any vector $\vec{a}$ and $\vec{q}$ there exists a skew-symmetric tensor $\mathbf{R}$ such that

$$\vec{q} \times \vec{a} = \mathbf{R}\cdot\vec{a}. \tag{B.8}$$

From this equation it can be shown that $\mathbf{R}$ is related to the components of $\vec{q}$ relative to the base $\underset{\sim}{\vec{e}}$ by

$$\mathbf{R} = \underset{\sim}{\vec{e}}^{\,T}\begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix}\underset{\sim}{\vec{e}}. \tag{B.9}$$

Substitution of (8) into (4) yields

$$\vec{r} = \{\mathbf{I} + 2\mathbf{R}\cdot\mathbf{R} + 2q_0\mathbf{R}\}\cdot\vec{x} \tag{B.10}$$

Comparing this expression with the expression for a rigid body rotation

$$\vec{r} = \mathbf{Q}\cdot\vec{x} \tag{B.11}$$

reveals that

$$\mathbf{Q} = \mathbf{I} + 2\mathbf{R}\cdot\mathbf{R} + 2q_0\mathbf{R}. \tag{B.12}$$

Substitution of (9) into (12), using (7), yields for the components of $\mathbf{Q}$ relative to $\underset{\sim}{\vec{e}}$

$$\underline{Q} = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_1q_2 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_2q_3 + q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}. \tag{B.13}$$

The angular velocity vector $\vec{\omega}$ is the axial vector of $\mathbf{Q}^C \cdot \dot{\mathbf{Q}}$. After some algebraic manipulation, using (7) and its first time derivative, the following result can be obtained

$$\vec{\omega} = 2\, \vec{\underset{\sim}{e}}^T \underline{G}\, \dot{\underset{\sim}{q}}, \tag{B.14}$$

where

$$\underset{\sim}{q} = [\, q_0 \;\; q_1 \;\; q_2 \;\; q_3 \,]^T, \tag{B.15}$$

and

$$\underline{G} = \begin{bmatrix} -q_1 & q_0 & q_3 & -q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ -q_3 & q_2 & -q_1 & q_0 \end{bmatrix}. \tag{B.16}$$

Similarly, the virtual angular rotation equals

$$\delta\vec{\pi} = 2\, \vec{\underset{\sim}{e}}^T \underline{G}\, \delta\underset{\sim}{q}. \tag{B.17}$$

It may be verified that

$$\dot{\underline{G}}\, \dot{\underset{\sim}{q}} = \underset{\sim}{0}. \tag{B.18}$$

Consequently the expression for the angular acceleration becomes

$$\dot{\vec{\omega}} = 2\, \vec{\underset{\sim}{e}}^T \underline{G}\, \ddot{\underset{\sim}{q}}. \tag{B.19}$$

The component form of (14) relative to $\vec{\underset{\sim}{e}}$ is obtained from scalar premultiplication of (14) by $\vec{\underset{\sim}{e}}$. This yields

$$\underset{\sim}{\omega} = 2\, \underline{G}\, \dot{\underset{\sim}{q}}, \tag{B.20}$$

where $\underset{\sim}{\omega}$ is the column matrix of the components of $\vec{\omega}$ relative to $\vec{\underset{\sim}{e}}$. Solving (20) and the first time derivative of (7) for $\dot{\underset{\sim}{q}}$ yields

$$\dot{\underset{\sim}{q}} = \tfrac{1}{2}\, \underline{G}^T \underset{\sim}{\omega}. \tag{B.21}$$

Beam element

The objective of this appendix is to present a beam element with uniform cross section that can be used for modelling straight beams that deform in just one plane. The displacements due to deformation are assumed to satisfy the Bernoulli-Euler hypothesis.

Consider the beam element of length ℓ and mass m shown in fig. C.1. An inertial base $\underset{\sim}{\vec{e}}$ and an inertial point O are chosen such that $\vec{e}_1$ and the line connecting O with an arbitrary point of the elastic axis in the reference configuration are parallel to the elastic axis in the reference configuration. In each node three degrees of freedom are introduced in order to be able to satisfy the continuity requirements between elements. The position vector of an arbitrary point P of the elastic axis in the reference configuration can be expressed in terms of the nodal position vectors

$$\vec{x} = (1-\xi)\, x_0\vec{e}_1 + \xi\, x_1\vec{e}_1 = \begin{bmatrix} x_0 & x_1 \end{bmatrix} \begin{bmatrix} (1-\xi) \\ \xi \end{bmatrix} \vec{e}_1, \tag{C.1}$$

where ξ is the distance between node 0 and P which has been made dimensionless

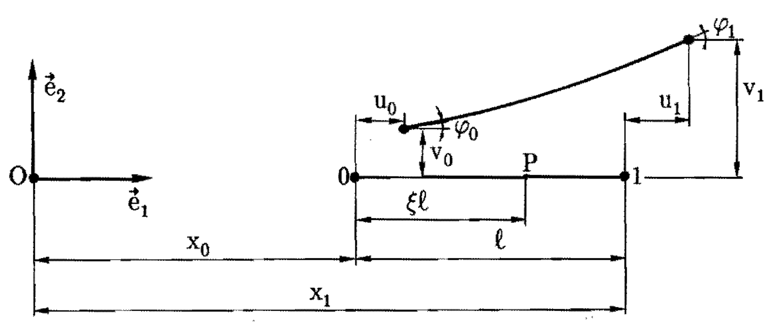

Fig. C.1 Beam element

with ℓ. The displacements between the nodes are interpolated with the usual Hermite polynomials

$$\vec{u}(\xi,t) = {}_e\underset{\sim}{\alpha}^T(t)\ {}_e\underset{\sim}{\vec{\Phi}}(\xi), \tag{C.2}$$

where

$${}_e\underset{\sim}{\alpha}(t) = [\ u_0(t) \quad v_0(t) \quad \varphi_0(t) \quad u_1(t) \quad v_1(t) \quad \varphi_1(t)\ ]^T, \tag{C.3}$$

$$\begin{aligned}{}_e\underset{\sim}{\vec{\Phi}}(\xi) = [\ (1-\xi)\vec{e}_1 \quad (1-3\xi^2+2\xi^3)\vec{e}_2 \quad \ell(\xi-2\xi^2+\xi^3)\vec{e}_2 \\ \xi\vec{e}_1 \quad (3\xi^2-2\xi^3)\vec{e}_2 \quad \ell(-\xi^2+\xi^3)\vec{e}_2\ ]^T.\end{aligned} \tag{C.4}$$

Using this approximation of the displacement field due to deformation, the time-independent inertia coefficients (2.34)-(2.41) become for this beam element

$${}_e\vec{C}_1 = \tfrac{1}{2}m[\ x_0 \quad x_1\ ]\begin{bmatrix}1\\1\end{bmatrix}\vec{e}_1, \tag{C.5}$$

$${}_e\underset{\sim}{\vec{C}}_2 = m[\ \tfrac{1}{2}\vec{e}_1 \quad \tfrac{1}{2}\vec{e}_2 \quad (\ell/12)\vec{e}_2 \quad \tfrac{1}{2}\vec{e}_1 \quad \tfrac{1}{2}\vec{e}_2 \quad -(\ell/12)\vec{e}_2\ ]^T, \tag{C.6}$$

$${}_e\mathbf{C}_3 = (m/6)[\ x_0 \quad x_1\ ]\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}x_0\\x_1\end{bmatrix}(\vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3), \tag{C.7}$$

$${}_e\underset{\sim}{\mathbf{C}}_4 = \{\underline{M}_1(\vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3) - \underline{M}_2\vec{e}_2\vec{e}_1\}\begin{bmatrix}x_0\\x_1\end{bmatrix}, \tag{C.8}$$

$${}_e\underline{\mathbf{C}}_5 = \underline{M}_3(\vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3) + \underline{M}_4(\vec{e}_1\vec{e}_1 + \vec{e}_3\vec{e}_3) - \underline{M}_5^T\vec{e}_1\vec{e}_2 - \underline{M}_5\vec{e}_2\vec{e}_1, \tag{C.9}$$

$${}_e\underset{\sim}{\vec{C}}_6 = \underline{M}_2\begin{bmatrix}x_0\\x_1\end{bmatrix}\vec{e}_3, \tag{C.10}$$

$${}_e\underline{\vec{C}}_7 = (\underline{M}_5^T - \underline{M}_5)\vec{e}_3, \tag{C.11}$$

$${}_e\underline{C}_8 = \underline{M}_3 + \underline{M}_4, \tag{C.12}$$

where

$$\underline{M}_1 = (m/6)\begin{bmatrix} 2 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{C.13}$$

$$\underline{M}_2 = (m/60)\begin{bmatrix} 0 & 0 \\ 21 & 9 \\ 3\ell & 2\ell \\ 0 & 0 \\ 9 & 21 \\ -2\ell & -3\ell \end{bmatrix}, \tag{C.14}$$

$$\underline{M}_3 = (m/6)\begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{C.15}$$

$$\underline{M}_4 = (m/420)\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 156 & 22\ell & 0 & 54 & -13\ell \\ 0 & 22\ell & 4\ell^2 & 0 & 13\ell & -3\ell^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 54 & 13\ell & 0 & 156 & -22\ell \\ 0 & -13\ell & -3\ell^2 & 0 & -22\ell & 4\ell^2 \end{bmatrix}, \tag{C.16}$$

$$\underline{M}_5 = (m/60)\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 21 & 0 & 0 & 9 & 0 & 0 \\ 3\ell & 0 & 0 & 2\ell & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 9 & 0 & 0 & 21 & 0 & 0 \\ -2\ell & 0 & 0 & -3\ell & 0 & 0 \end{bmatrix}. \tag{C.17}$$

The expression for the strain energy of a beam, using linearized strain-displacement relations, is given by

$$U_l = \tfrac{1}{2}(EA/\ell)\int_0^1 (\partial u/\partial\xi)^2 d\xi + \tfrac{1}{2}(EI/\ell^3)\int_0^1 (\partial^2 v/\partial\xi^2)^2 d\xi, \tag{C.18}$$

where u and v are respectively the axial and the transverse displacement field of the beam. Substitution of the displacement field (2) into this expression yields

$$U_l = \tfrac{1}{2}\,{}_e\underline{\alpha}^T\,{}_e\underline{K}\,{}_e\underline{\alpha}, \tag{C.19}$$

where ${}_e\underline{K}$ is the usual element stiffness matrix given by

$${}_e\underline{K} = (EI/\ell^3)\begin{bmatrix} \gamma & 0 & 0 & -\gamma & 0 & 0 \\ 0 & 12 & 6\ell & 0 & -12 & 6\ell \\ 0 & 6\ell & 4\ell^2 & 0 & -6\ell & 2\ell^2 \\ -\gamma & 0 & 0 & \gamma & 0 & 0 \\ 0 & -12 & -6\ell & 0 & 12 & -6\ell \\ 0 & 6\ell & 2\ell^2 & 0 & -6\ell & 4\ell^2 \end{bmatrix}, \tag{C.20}$$

with $\gamma = A\ell^2/I$.

An approximate expression for the strain energy, using nonlinear strain-displacement relations, is given by (Przemieniecki, 1968)

$$U_{nl} = U_l + \tfrac{1}{2}(EA/\ell^2)\int_0^1 (\partial u/\partial\xi)(\partial v/\partial\xi)^2 d\xi. \tag{C.21}$$

Substitution of the displacement field (2) into this expression yields

$$U_{nl} = \tfrac{1}{2}\,{}_e\underline{\alpha}^T\,({}_e\underline{K} + {}_e\underline{K}_G)\,{}_e\underline{\alpha}, \tag{C.22}$$

where ${}_e\underline{K}_G$ is the geometric stiffness matrix of the beam element given by

$${}_e\underline{K}_G = (F/30\ell)\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 36 & 3\ell & 0 & -36 & 3\ell \\ 0 & 3\ell & 4\ell^2 & 0 & -3\ell & -\ell^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -36 & -3\ell & 0 & 36 & -3\ell \\ 0 & 3\ell & -\ell^2 & 0 & -3\ell & 4\ell^2 \end{bmatrix}, \tag{C.23}$$

where $F = (EA/\ell)/(u_1 - u_0)$. F represents the axial force in the beam element.

appendix D

Elastodynamic analysis of a slider-crank mechanism

An elastodynamic analysis of a system of bodies consists of determining the motion of the system in case its bodies are rigid, followed by determining the quasi-static deformation of the bodies resulting from the load which goes with the rigid body motion. This can be done analytically for a planar slider-crank mechanism with a uniform deformable connecting rod because of its simple geometry.

Consider the planar slider-crank mechanism shown in fig. D.1. The length of the crank equals α times the length ℓ of the connecting rod with $0 < \alpha < 1$. From geometric considerations follows

$$\sin\theta = \alpha \sin\Omega t, \tag{D.1}$$

where Ω is the constant angular velocity of the crank. From this equation follows, using the relation $\sin^2\theta + \cos^2\theta = 1$

$$\cos\theta = \{1 - \alpha^2 \sin^2\Omega t\}^{\frac{1}{2}}. \tag{D.2}$$

The sign is correct for any value of t because $-\pi/2 < \theta < \pi/2$. Differentiating (1) once with respect to t yields

$$\dot\theta = \alpha\,\Omega \cos\Omega t \,/\, \cos\theta . \tag{D.3}$$

Differentiating (3) once with respect to t and using (1)-(3) yields

$$\ddot\theta = \alpha\,(\alpha^2-1)\Omega^2 \sin\Omega t \,/\cos^3\theta. \tag{D.4}$$

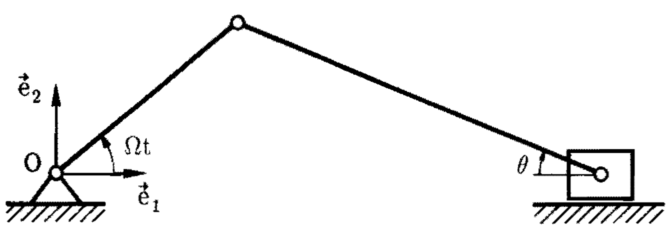

Fig. D.1 Planar slider-crank mechanism

The position vector of an arbitrary material point of the connecting rod is given by

$$\vec{r} = \{\alpha \ell \cos\Omega t + \xi \ell \cos\theta\}\vec{e}_1 + \{(1-\xi)\ell \sin\theta\}\vec{e}_2, \tag{D.5}$$

where $\xi\ell$ is the distance between the material point and the left end of the connecting rod. Differentiating (5) twice with respect to t yields

$$\ddot{\vec{r}} = \{-\alpha\Omega^2\ell \cos\Omega t - \xi\ell\,(\dot{\theta}^2\cos\theta + \ddot{\theta}\sin\theta)\}\vec{e}_1 +$$

$$\{(1-\xi)\ell\,(-\dot{\theta}^2\sin\theta + \ddot{\theta}\cos\theta)\}\vec{e}_2. \tag{D.6}$$

For determining the transverse deflection of the connecting rod, the component of the acceleration perpendicular to the connecting rod is required. It can be obtained from the scalar product of the acceleration and the unit vector $\vec{n}$ which is perpendicular to the connecting rod and is defined by

$$\vec{n} = \vec{e}_1\sin\theta + \vec{e}_2\cos\theta. \tag{D.7}$$

Taking the scalar product of $\ddot{\vec{r}}$ and $\vec{n}$ yields

$$a = \ddot{\vec{r}}\cdot\vec{n} = -\alpha\Omega^2\ell \cos\Omega t\,\sin\theta - \ell\dot{\theta}^2\sin\theta\,\cos\theta - \xi\ell\ddot{\theta} + \ell\ddot{\theta}\cos^2\theta. \tag{D.8}$$

The component of the acceleration perpendicular to the connecting rod varies linearly between the endpoints. It can be resolved into a symmetric- and an anti-symmetric contribution. The magnitude of the symmetric contribution equals the acceleration of the middle of the connecting rod. Substitution of $\xi = \frac{1}{2}$ into (8), and using (1)-(4) yields

$$\bar{a} = -\tfrac{1}{2}\alpha\Omega^2\ell \sin\Omega t\,\{2(\alpha\cos\Omega t + \cos\theta) + (\alpha^2-1)/\cos^3\theta\}. \tag{D.9}$$

The connecting rod may be considered to be simply supported. The load going with the antisymmetric contribution of the acceleration does not cause a deflection of the middle of the connecting rod. The deflection resulting from the symmetric contribution of the acceleration can be obtained from beam theory. This yields for a connecting rod with a bending rigidity EI

$$v = -(5/384)\bar{a}m\ell^3/EI. \tag{D.10}$$

Fig. D.2 shows the dimensionless elastodynamic deflection $v/(\Omega^2 m\ell^4/EI)$ of the middle of the connecting rod for $\alpha = 0.25$, 0.5, and 0.75.

The initial velocity of the displacement field due to deformation can be obtained

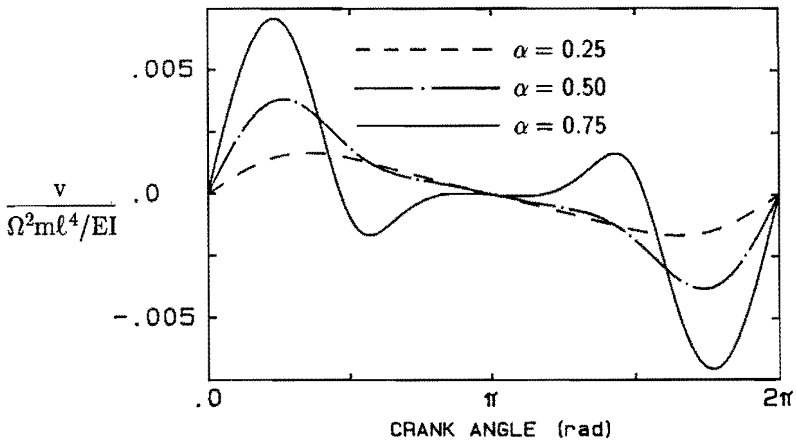

Fig. D.1 Elastodynamic deflection of the middle of the connecting rod

from differentiation of (10) once with respect to t and substituting t = 0 s. This yields

$$\dot{v} = -(5/384)\dot{\bar{a}}_0 m\ell^3/EI, \tag{D.11}$$

where $\dot{\bar{a}}_0$ is the time derivative of the mean acceleration for t = 0 s, given by

$$\dot{\bar{a}}_0 = -\tfrac{1}{2}\alpha(1+\alpha)^2\Omega^3\ell. \tag{D.12}$$

Samenvatting

In dit proefschrift wordt een wiskundige beschrijving gegeven van het dynamisch gedrag van stelsels van onderling verbonden vervormbare lichamen.

Het verplaatsingsveld van een lichaam wordt opgesplitst in een verplaatsingsveld als gevolg van een beweging als star lichaam en een verplaatsingsveld als gevolg van vervorming. Om deze opsplitsing eenduidig te maken, wordt van het verplaatsingsveld als gevolg van vervorming geëist dat het geen beweging als star lichaam representeert. Dit kan worden bereikt door òf verplaatsingen als gevolg van vervorming van een aantal materiële punten òf gemiddelde verplaatsingen als gevolg van vervorming, voor te schrijven.

Uitgaande van de bewegingsvergelijkingen voor een infinitesimaal volume-element van het lichaam wordt een variationele formulering voor de bewegingsvergelijkingen van het vrijgemaakte lichaam afgeleid. Deze vergelijkingen zijn eenvoudiger indien de gemiddelde verplaatsingen als gevolg van vervorming gelijk aan nul zijn. Een benaderingsoplossing voor de bewegingsvergelijkingen wordt verkregen door het verplaatsingsveld als gevolg van vervorming te benaderen met een lineaire combinatie van een aantal aangenomen verplaatsingsvelden. Drie methoden voor het genereren van dergelijke verplaatsingsvelden worden beschreven, namelijk de "assumed-modes" methode, de eindige elementen methode en de "modal synthesis" methode.

Voor het opstellen van de bewegingsvergelijkingen van het lichaam zoals het voorkomt in een stelsel van lichamen, moet rekening worden gehouden met verbindingen met andere lichamen. Energetische en actieve verbindingen kunnen in rekening worden gebracht door de krachten die zij veroorzaken toe te voegen aan de krachten op het vrijgemaakte lichaam. Kinematische verbindingen beperken de bewegingsvrijheid van de onderling verbonden lichamen. Dit kan in rekening worden gebracht met constraintvergelijkingen, waarmee de variabelen die de kinematica van het stelsel van lichamen beschrijven, kunnen worden onderverdeeld in afhankelijke en onafhankelijke variabelen. Het is handig om bij het formuleren van de constraintvergelijkingen, variabelen te introduceren die de relatieve beweging van de verbonden lichamen ten opzichte van elkaar beschrijven.

Het eenvoudiger zijn van de bewegingsvergelijkingen wanneer de gemiddelde verplaatsingen als gevolg van vervorming gelijk aan nul zijn gekozen, leidt in het gunstigste geval tot een rekentijdbesparing van enkele tientallen procenten. Van

de in dit proefschrift onderzochte systemen wordt het dynamisch gedrag beter benaderd indien verplaatsingsvelden als gevolg van vervorming worden benaderd met aangenomen verplaatsingsvelden waarvan de gemiddelde verplaatsingen als gevolg van vervorming gelijk aan nul zijn. Het voorkomen van bewegingen als star lichaam in het verplaatsingsveld als gevolg van vervorming door het voorschrijven van componenten van verplaatsingen van materiële punten moet behoedzaam gebeuren omdat de oplossing voor het dynamisch gedrag onjuist kan zijn.

De "assumed-modes" methode is alleen geschikt voor lichamen met een eenvoudige vorm. De eindige elementen methode en de "modal synthesis" methode kunnen worden gebruikt voor willekeurig gevormde lichamen. De eindige elementen methode blijkt al snel tot een model te leiden met veel vrijheidsgraden hetgeen lange rekentijden veroorzaakt. De "modal synthesis" methode kan dan met vrucht worden gebruikt om het aantal vrijheidsgraden te verkleinen en daarmee de benodigde rekentijd te reduceren. De effectiviteit van de "modal synthesis" methode hangt voor een belangrijk deel af van goed gekozen aangenomen verplaatsingsvelden. Een geschikte keuze kan in het algemeen worden gemaakt op grond van de op het lichaam werkende belasting. Een benadering met geconcentreerde massa's blijkt erg geschikt te zijn om, uitgaande van met een standaard eindig elementenpakket bepaalde verplaatsingsvelden, tijdsonafhankelijke massatraagheidstermen in de bewegingsvergelijkingen te bepalen. Bedacht moet worden dat dan een fijnere elementenverdeling noodzakelijk kan zijn dan nodig is voor het voldoende nauwkeurig bepalen van de verplaatsingsvelden.

Tot slot wordt een methode voorgesteld waarmee, bij gegeven verplaatsingsvelden, de benaderingsoplossing voor het dynamisch gedrag van een lichaam kan worden verbeterd. Deze methode is met succes gebruikt voor het verkorten van de benodigde rekentijd door het verlagen van niet-relevante hoge frequenties.

STELLINGEN

behorende bij het proefschrift

THE DYNAMICS OF SYSTEMS OF DEFORMABLE BODIES

1. Het is efficiënter om in bewegingsvergelijkingen van lichamen de rotatiebeweging te schrijven in termen van hoeksnelheden dan in termen van tijdsafgeleiden van variabelen die de rotatiebeweging beschrijven.

Dit proefschrift, hoofdstuk 2.

2. Eigentrillingsvormen en/of component modes worden vaak gebruikt voor het benaderen van verplaatsingen als gevolg van vervorming van vervormbare lichamen die deel uitmaken van mechanische systemen. Echter, zij zijn hiervoor vaak minder geschikt.

Dit proefschrift, hoofdstuk 5.

3. Het beschrijven van niet-kinematische verbindingen als kinematische verbindingen om een multibody-systeem met een boomstructuur te krijgen is vanuit een numeriek oogpunt een slechte werkwijze.

Wittenburg, J., 1977, *Dynamics of Systems of Rigid Bodies*, B. G. Teubner, Stuttgart.

4. Het opstellen van bewegingsvergelijkingen voor mechanische systemen met behulp van de vergelijking van Lagrange is vaak omslachtig in vergelijking met andere methoden uit de analytische mechanica en kan tot complexere bewegingsvergelijkingen leiden.

Schiehlen, W., 1984, "Dynamics of Complex Multibody Systems," *SM Archives,* Vol. 9, pp. 159-195.

5. De bewering van Wehage en Haug, en Nikravesh en Haug dat niet-standaard kinematische verbindingen moeilijk kunnen worden geïmplementeerd in een programma dat is gebaseerd op de relatieve beschrijvingswijze, is onjuist.

Nikravesh, P. E., and Haug, E. J., 1983, "Generalized Coordinate

Partitioning for Analysis of Mechanical Systems with Nonholonomic Constraints," *J. of Mechanisms, Transmissions, and Automation in Design*, Vol. 105, pp. 379-384.

Wehage, R. A., and Haug, E. J., 1982, "Generalized Coordinate Partitioning for Dimension Reduction in Analysis of Constrained Dynamic Systems," *J. of Mechanical Design*, Vol. 104, pp. 247-255.

6. De in de eindige elementenmethode gangbare benaming "gegeneraliseerde knooppuntskrachten" is misleidend voor elementen met inwendige knooppunten daar deze suggereert dat dit in de knooppunten aangrijpende geconcentreerde krachten zijn.

Przemieniecki, J. S., 1968, *Theory of Matrix Structural Analysis*, McGraw-Hill, New York.

7. Het introduceren van vectorbases alvorens de theorie op te stellen door mensen die de voorkeur geven aan het werken met vectoren en tensoren in symbolische vorm, is in strijd met hun voorkeur.

Sol, E. J., 1983, *Kinematics and Dynamics of Multibody Systems*, proefschrift Technische Universiteit Eindhoven, Eindhoven.

8. Door het toenemend gebruik van de computer voor het berekenen van het gedrag van constructies neemt het belang van het kunnen schatten van dat gedrag toe.

9. De toenemende individualisering in de maatschappij veroorzaakt een spanningsveld omdat mensen behoefte hebben aan waardering van anderen.

10. Ter bescherming van mensen die liever niet roken zouden de Nederlandse Spoorwegen het opschrift "bij vol balkon liever niet roken" moeten vervangen door "verboden te roken".

11. Het plaatsen van een vraagteken of uitroepteken aan het begin van een zin vergroot het leesgemak.