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Large-time behaviour of Hele-Shaw flow with injection or suction for perturbed balls in $\mathbb{R}^N$

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Abstract

Large-time behaviour of Hele-Shaw flow with surface tension and with injection or suction in one point is discussed. We consider domains which are initially small perturbations of balls. Radially symmetric solutions are stationary after a suitable time-dependent rescaling. The evolution of perturbations can be described by a non-local nonlinear parabolic evolution equation. Global existence results and decay properties are derived using energy estimates in Sobolev spaces.

AMS subject classifications: 35R35, 35K55, 76D27

Key words: moving-boundary problem, non-local parabolic equation

1 Introduction

Hele-Shaw flow with surface tension and with injection or suction at a single point is described by a family of domains $t \mapsto \Omega(t)$ in $\mathbb{R}^N$, parameterised by time t, and two functions $v(\cdot, t) : \Omega(t) \to \mathbb{R}^N$ and $p(\cdot, t) : \Omega(t) \to \mathbb{R}$ that satisfy the following moving-boundary problem:

$$\operatorname{div} v = \mu\delta \qquad \text{in } \Omega(t), \tag{1}$$

$$v = -\nabla p \qquad \text{in } \Omega(t), \tag{2}$$

$$p = -\gamma\kappa \qquad \text{on } \Gamma(t) := \partial\Omega(t). \tag{3}$$

Here, $\kappa(\cdot, t) : \Gamma(t) \to \mathbb{R}$ is the mean curvature of the moving boundary $t \mapsto \Gamma(t)$ of the domain (taken negative if $\Omega(t)$ is convex), μ is the injection rate if $\mu > 0$ or the suction rate if $\mu < 0$, γ is a positive constant, and δ is the Dirac delta distribution. The evolution of the boundary $\Gamma(t)$ is specified by the requirement that its normal velocity v_n is given by

$$v_n = v \cdot n. \tag{4}$$

The equations (1), (2), and (3) can be combined to give

$$\Delta p = -\mu\delta \quad \text{in } \Omega(t), \tag{5}$$
$$p = -\gamma\kappa \quad \text{on } \Gamma(t) = \partial\Omega(t). \tag{6}$$

Hence, p solves a Dirichlet problem for any time t and on $\Gamma(t)$ we get from (4)

$$v_n = -\frac{\partial p}{\partial n}.$$

Besides liquid flow in a Hele-Shaw cell [2], the model and variations of it describe the growth of tumours [1] and porous media flow [3, 4]. Short-time existence of solutions for a closely related problem has been proved by Escher and Simonett [5]. For the three-dimensional injection problem, global existence of classical solutions and large-time behaviour have been found in [10] for domains that are initially small perturbations of balls. The three-dimensional suction case requires some extra conditions in order to get global existence results (see [10] as well). The ratio of suction rate to γ needs to be small and the position of the geometric centre of the initial domain has to be at the origin, where the suction point is located. In this paper we prove global existence results of classical solutions for $N = 2$ and $N \geq 4$.

Let σ_N be the area of the unit sphere $\mathbb{S}^{N-1}$. It is easy to check that if the initial domain has the same volume as the unit ball $\frac{\sigma_N}{N}$, then the volume V of the domain as a function of time is given by

$$V(t) = \frac{\sigma_N}{N} + \mu t. \tag{7}$$

The suction problem only makes sense for $t \leq T := -\frac{\sigma_N}{\mu N}$.

If $\Omega(0) = \mathbb{B}^N := \{x \in \mathbb{R}^N : |x| < 1\}$, then the moving domain will be a ball around the origin with radius $\alpha(t)$ given by

$$\alpha(t) = \sqrt[N]{\frac{\mu N t}{\sigma_N} + 1}.$$

In order to prove stability of this solution we rescale the domain by a factor $\alpha(t)$ such that $\mathbb{B}^N$ becomes a stationary solution. We introduce a function $r(\cdot, t) : \mathbb{S}^{N-1} \to \mathbb{R}$ that describes a small star-shaped perturbation of this stationary solution. In [10] an evolution equation for r in Hölder spaces is derived and linearised. This equation is autonomous only for $N = 3$. Therefore we cannot use the techniques that are used in [10] in other dimensions. Instead, we consider the problem in Sobolev spaces and work with energy estimates.

In Section 2 we introduce the same evolution equation on Sobolev spaces. In Section 3 we use the linearisation of the evolution operator to prove global existence and decay properties for perturbations. For the injection problems, we find existence for all $t > 0$. In the two-dimensional suction problem, we need the extra condition that the geometric centre of the initial domain and the suction point coincide, in order to prove that all liquid can be removed. The domain vanishes "as a round point". In contrast to the three-dimensional case, there is no restriction on the suction rate in the two-dimensional case. For $N \geq 4$, we do not know whether or not a global existence result for the suction problem can be found.

dimension	$\mu > 0$	$\mu < 0$
2	global existence and decay (see Theorem 3.3)	• global existence and decay for correct geometric centre (see Theorem 3.5) • almost global existence (see Theorem 4.1)
3	global existence and decay (see [10])	• global existence and decay for correct geometric centre when $\frac{\lvert\mu\rvert}{\gamma} < \frac{32\pi}{5}$ (see [10]) • almost global existence (see [10])
≥ 4	global existence and decay (see Theorem 3.6)	almost global existence (see Theorem 4.1)

Table 1: Large-time behaviour of Hele-Shaw flow with surface tension and injection or suction in one point

Both for $N = 2$ and for $N \geq 4$, $\mu < 0$, any portion of liquid, smaller than the entire domain, can be removed if the initial domain is close enough to a ball. We shall call this result almost global existence. It will be proved in Section 4.

2 The evolution equation and its linearisation

In [10], an evolution equation for the motion of the domain $\Omega(t)$ in Hölder spaces is derived and linearised. In this section we introduce the same equation in a Sobolev space setting.

Let $\mathfrak{H}_k^N$ be the vector space of harmonic homogeneous polynomials of degree k in N variables. Define the spherical harmonics as restrictions of these polynomials to the unit sphere $\mathbb{S}^{N-1}$:

$$\mathfrak{S}_k^N = \left\{ q \mid_{\mathbb{S}^{N-1}} : q \in \mathfrak{H}_k^N \right\}.$$

For each $k \in \mathbb{N}_0$, choose a basis $(s_{k,j})_{j=1}^{\nu(N,k)}$ of $\mathfrak{S}_k^N$ that is orthonormal in $\mathbb{L}_2(\mathbb{S}^{N-1})$:

$$\mathfrak{S}_k^N = \left\langle s_{k,1}, s_{k,2}, \ldots, s_{k,\nu(N,k)} \right\rangle,$$

where $\nu(N, k)$ is the dimension of $\mathfrak{S}_k^N$. From [7] Lemma 2, we know that the spherical harmonics

$$\bigcup_{k=0}^{\infty} \left\{ s_{k,1}, s_{k,2}, \ldots, s_{k,\nu(N,k)} \right\},$$

form an orthonormal basis for $\mathbb{L}_2(\mathbb{S}^{N-1})$. Let $(\cdot, \cdot)_0$ be the usual inner product on $\mathbb{L}_2(\mathbb{S}^{N-1})$. For each $r \in \mathbb{L}_2(\mathbb{S}^{N-1})$ define $r_{k,j}$ by

$$r_{k,j} := (r, s_{k,j})_0.$$

For all $s > 0$, equip the Sobolev space $\mathbb{H}^s(\mathbb{S}^{N-1})$ with the inner product

$$(r, \tilde{r})_s = \sum_{k,j} (k^2+1)^s r_{k,j} \tilde{r}_{k,j}.$$

In the sequel we will use the Sobolev embedding theorem: If $k \in \mathbb{N}_0$, $\beta \in (0,1)$, and $s > \frac{N-1}{2} + k + \beta$, then

$$\mathbb{H}^s(\mathbb{S}^{N-1}) \hookrightarrow \mathcal{C}^{k,\beta}(\mathbb{S}^{N-1})$$

and

$$\mathbb{H}^{s+\frac{1}{2}}(\mathbb{B}^N) \hookrightarrow \mathcal{C}^{k,\beta}(\overline{\mathbb{B}^N}).$$

We will also use the fact that for $s > \frac{N-1}{2}$, $\mathbb{H}^{s+\frac{1}{2}}(\mathbb{B}^N)$ and $\mathbb{H}^s(\mathbb{S}^{N-1})$ are Banach algebras.

We restrict ourselves to domain evolutions $t \mapsto \Omega(t)$ that can be described by a continuous function $R(\cdot, t) : \mathbb{S}^{N-1} \to (-1, \infty)$ satisfying

$$\Omega(t) = \Omega_{R(\cdot,t)} = \left\{ x \in \mathbb{R}^N \setminus \{0\} : |x| < 1 + R\left(\frac{x}{|x|}\right) \right\} \cup \{0\}.$$

Besides $R(\cdot, t)$ we introduce $r(\cdot, t)$ such that

$$\Omega_{r(\cdot,t)} = \alpha(t)^{-1} \Omega_{R(\cdot,t)},$$

which is equivalent to

$$1 + r(t) = \frac{1 + R(t)}{\alpha(t)}. \tag{8}$$

We will often write $r(t)$ instead of $r(\cdot, t)$ and if we consider a fixed domain, then the argument t will be suppressed. From now on we assume that $r \in \mathbb{H}^s(\mathbb{S}^{N-1})$ where

$$s > \frac{N+7}{2}. \tag{9}$$

Define $\Gamma_r := \partial \Omega_r$. Introduce

- $\tilde{z}(r, \cdot) : \mathbb{S}^{N-1} \to \Gamma_r$ by
$$\tilde{z}(r, \xi) = (1 + r(\xi))\, \xi,$$
- $n(r, \cdot)$ as the function that maps an element $\xi \in \mathbb{S}^{N-1}$ to the exterior unit normal vector on Γ_r at the point $\tilde{z}(r, \xi)$,
- $\kappa(r, \cdot)$ as the function that maps an element $\xi \in \mathbb{S}^{N-1}$ to the mean curvature of Γ_r at $\tilde{z}(r, \xi)$.

We will often use the notations $\tilde{z}(r)$, $n(r)$, and $\kappa(r)$ instead of $\tilde{z}(r, \cdot)$, $n(r, \cdot)$, and $\kappa(r, \cdot)$. In [8] Chapter 3 Lemma 16, it is proved that on a neighbourhood $\mathcal{U}$ of zero in $\mathbb{H}^s(\mathbb{S}^{N-1})$ the operators $n : \mathcal{U} \to \left(\mathbb{H}^{s-1}(\mathbb{S}^{N-1})\right)^N$ and $\kappa : \mathcal{U} \to \mathbb{H}^{s-2}(\mathbb{S}^{N-1})$ are analytic. By [9] Theorem 6.108, there exists an extension operator $E \in \mathcal{L}(\mathbb{H}^s(\mathbb{S}^{N-1}), \mathbb{H}^{s+\frac{1}{2}}(\mathbb{B}^N))$, such that

$$Er|_{\mathbb{S}^{N-1}} = r. \tag{10}$$

Define $z : \mathbb{H}^s(\mathbb{S}^{N-1}) \to \left(\mathbb{H}^{s+\frac{1}{2}}(\mathbb{B}^N)\right)^N$ by

$$z(r,x) := (1 + (Er)(x))\, x,$$

where $z(r,\cdot) = z(r)$.

Lemma 2.1. *There exists a $\delta > 0$ such that if $\|r\|_s < \delta$ then $z(r) : \overline{\mathbb{B}^N} \to \overline{\Omega}_r$ is bijective and $z(r)^{-1} \in \left(\mathcal{C}^2(\overline{\Omega_r})\right)^N$.*

Proof. From the Sobolev embedding theorem we get $\mathbb{H}^s(\mathbb{S}^{N-1}) \hookrightarrow \mathcal{C}^4(\mathbb{S}^{N-1})$. The bijectivity follows from [11] Lemma 2.2. Using [11] Lemma 2.3, we can prove the other statement as well. □

On a neighbourhood $\mathcal{U}$ of zero in $\mathbb{H}^s(\mathbb{S}^{N-1})$ we define the following operators:

- $\mathcal{A} : \mathcal{U} \to \mathcal{L}(\mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^N), \mathbb{H}^{s-\frac{7}{2}}(\mathbb{B}^N))$ and $\mathcal{Q} : \mathcal{U} \to \mathcal{L}\left(\mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^N), \left(\mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N)\right)^N\right)$ by

$$\mathcal{A}(r)u := \left(\Delta\left(u \circ z(r)^{-1}\right)\right) \circ z(r) = \sum_{i,k,l} j^{i,l}(r) \frac{\partial}{\partial x_i}\left(j^{k,l}(r)\frac{\partial u}{\partial x_k}\right)$$

  and

$$\mathcal{Q}(r)u := \left(\nabla\left(u \circ z(r)^{-1}\right)\right) \circ z(r) = \sum_{i,k} j^{k,i}(r) \frac{\partial u}{\partial x_k} e_i,$$

  where e_i is the i-th unit vector in $\mathbb{R}^N$ and $j^{k,i}(r)$ are the coefficients of the inverse of the matrix

$$\mathcal{J}(r) = \frac{\partial z(r)}{\partial x} \in \left(\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)\right)^{N \times N}.$$

  The elements $j^{k,i}(r)$ are in $\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)$ for $\|r\|_s$ small because of continuity of inversion near the identity $\mathcal{J}(0)$ in Banach algebras. Because of (9) the space $\mathbb{H}^{s-\frac{7}{2}}(\mathbb{B}^N)$ is a Banach algebra. Therefore the operators $\mathcal{A}$ and $\mathcal{Q}$ are well-defined.

- $\mathcal{S} : \mathcal{U} \to \mathcal{L}(\mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^N), \mathbb{H}^{s-\frac{7}{2}}(\mathbb{B}^N) \times \mathbb{H}^{s-2}(\mathbb{S}^{N-1}))$ by

$$\mathcal{S}(r)u := \begin{pmatrix} \mathcal{A}(r)u \\ \mathrm{Tr}u \end{pmatrix}. \tag{11}$$

- $\varphi : \mathcal{U} \to \mathbb{H}^s(\mathbb{S}^{N-1})$ by

$$\varphi(r,x) := \Phi((1 + r(x))x), \tag{12}$$

  where $\Phi : \mathbb{R}^N \to \mathbb{R}$ is the fundamental solution of the Laplacian:

$$\Phi(x) := \begin{cases} -\dfrac{1}{2\pi}\ln|x| & N = 2, \\ \dfrac{1}{(N-2)\sigma_N |x|^{N-2}} - \dfrac{1}{(N-2)\sigma_N} & N \geq 3. \end{cases} \tag{13}$$

  We often write $\phi(r)$ instead of $\phi(r,\cdot)$.

Lemma 2.2. *For $\|r\|_s$ small, $\mathcal{S}(r)$ is an isomorphism between $\mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^N)$ and $\mathbb{H}^{s-\frac{7}{2}}(\mathbb{B}^N)\times \mathbb{H}^{s-2}(\mathbb{S}^{N-1})$.*

Proof. $\mathcal{S}(0)$ is an isomorphism between $\mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^N)$ and $\mathbb{H}^{s-\frac{7}{2}}(\mathbb{B}^N)\times \mathbb{H}^{s-2}(\mathbb{S}^{N-1})$. Because $\mathcal{S}$ is smooth ([11] Lemma 2.6), $\mathcal{S}(r)$ is an isomorphism for small r. □

Suppose that (9) holds. By [10] we have for r in a suitable neighbourhood $\mathcal{U}$ of zero in $\mathbb{H}^s(\mathbb{S}^{N-1})$

$$\frac{\partial r}{\partial t} = \frac{1}{\alpha(t)^3}\mathcal{F}(r,t), \tag{14}$$

where

$$\mathcal{F}(r,t) = \gamma\mathcal{F}_1(r) + \mu\alpha(t)^{3-N}\mathcal{F}_2(r), \tag{15}$$

for a third order operator $\mathcal{F}_1 : \mathcal{U} \to \mathbb{H}^{s-3}(\mathbb{S}^{N-1})$ and a first order operator $\mathcal{F}_2 : \mathcal{U} \to \mathbb{H}^{s-1}(\mathbb{S}^{N-1})$ given by

$$\mathcal{F}_1(r) = \mathcal{E}(r)\kappa(r),$$
$$\mathcal{F}_2(r) = \mathcal{E}(r)\varphi(r) + l(r),$$

where $\mathcal{E} : \mathcal{U} \to \mathcal{L}(\mathbb{H}^{s-2}(\mathbb{S}^{N-1}), \mathbb{H}^{s-3}(\mathbb{S}^{N-1}))$ and $l : \mathcal{U} \to \mathbb{H}^s(\mathbb{S}^{N-1})$ are defined by

$$\mathcal{E}(r)\psi := \xi \mapsto \frac{\mathrm{Tr}\left(\mathcal{Q}(r)\left[\mathcal{S}(r)^{-1}\left[\begin{array}{c} 0 \\ \psi \end{array}\right]\right]\right)(\xi)\cdot n(r,\xi)}{n(r,\xi)\cdot\xi}$$

and

$$l(r) := \frac{1}{\sigma_N(1+r)^{N-1}} - \frac{1+r}{\sigma_N}.$$

In fact, $\mathcal{F}_2$ also maps a neighbourhood of zero in $\mathbb{H}^{s-2}(\mathbb{S}^{N-1})$ to $\mathbb{H}^{s-3}(\mathbb{S}^{N-1})$.

Introduce a new time variable $\tau = \tau(t)$ such that $\tau(0) = 0$ and

$$\frac{d\tau}{dt} = \frac{1}{\alpha(t)^3}. \tag{16}$$

For $N \neq 3$ this implies

$$\tau(t) = \frac{\sigma_N}{\mu\,(N-3)}\left(\left(\frac{\mu N t}{\sigma_N}+1\right)^{1-\frac{3}{N}} - 1\right). \tag{17}$$

The original time interval was infinite for the injection problems and finite for the suction problems. Via (17), this does not change for the new time variable τ in the case $N \geq 4$. For $N = 2$ however, the new injection problem is defined on a finite time interval $(0, \tau_{\max})$ while the suction problem is defined on $(0, \infty)$. We have

$$\begin{aligned}
&\lim_{t\to\infty}\tau(t) = \frac{2\pi}{\mu}, && \text{for N} = 2,\ \mu > 0,\\
&\lim_{t\to T}\tau(t) = \infty, && \text{for N} = 2,\ \mu < 0,\\
&\lim_{t\to\infty}\tau(t) = \infty, && \text{for N} \geq 4,\ \mu > 0,\\
&\lim_{t\to T}\tau(t) = \frac{\sigma_N}{|\mu|(N-3)}, && \text{for N} \geq 4,\ \mu < 0.
\end{aligned}$$

We shall denote these limit values for τ by $\tau_{\max}$. We get

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r, t(\tau)) = \gamma \mathcal{F}_1(r) + \mu \alpha(t(\tau))^{3-N} \mathcal{F}_2(r).$$

Here $t(\tau)$ is the value of t that corresponds to τ. To simplify notation, we will write from now on $\alpha(\tau)$ instead of $\alpha(t(\tau))$ and $\mathcal{F}(r, \tau)$ instead of $\mathcal{F}(r, t(\tau))$, so

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r, \tau) = \gamma \mathcal{F}_1(r) + \mu \alpha(\tau)^{3-N} \mathcal{F}_2(r). \tag{18}$$

Note that the evolution operator $\mathcal{F}$ is independent of τ in the three dimensional case. In all other dimensions, time dependence only occurs in front of $\mathcal{F}_2$.

Lemma 2.3. *(Analyticity of the evolution operator)*
Suppose that $s > \frac{N+7}{2}$.

- *The mapping* $\mathcal{F}_1$ *is analytic from a neighbourhood* $\mathcal{U}$ *of zero in* $\mathbb{H}^s(\mathbb{S}^{N-1})$ *to* $\mathbb{H}^{s-3}(\mathbb{S}^{N-1})$.
- *The mapping* $\mathcal{F}_2$ *is analytic from a neighbourhood* $\mathcal{U}$ *of zero in* $\mathbb{H}^{s-2}(\mathbb{S}^{N-1})$ *to* $\mathbb{H}^{s-3}(\mathbb{S}^{N-1})$.

Proof. These statements can be proved in a similar way as was done in [8] Chapter 3 for Hele-Shaw flow without injection or suction and in [11] and [10] for $\mathcal{F}_1$ and $\mathcal{F}_2$ in Hölder spaces. □

The linearisations of $\mathcal{F}_1$ and $\mathcal{F}_2$ around zero are polynomials of the Dirichlet-to-Neumann mapping $\phi \mapsto \mathcal{N}\phi := \left(\mathrm{Tr} \nabla \mathcal{S}(0)^{-1} \begin{pmatrix} 0 \\ \phi \end{pmatrix} \right) \cdot n(0)$. This is a mapping of order 1 that satisfies

$$\mathcal{N} s = k s,$$

for all $s \in \mathfrak{S}_k^N$. From [10] we get

$$\mathcal{F}_1'(0) r = -p_1(\mathcal{N}) r \tag{19}$$

and

$$\mathcal{F}_2'(0) r = -p_2(\mathcal{N}) r \tag{20}$$

where p_1 and p_2 are the polynomials

$$p_1(k) = k^3 + (N-2)k^2 - (N-1)k$$

and

$$p_2(k) = \frac{1}{\sigma_N} k + \frac{N}{\sigma_N}.$$

3 Energy estimates and existence results

In this section we find estimates for $(r, \mathcal{F}(r,\tau))_s$, in order to prove stability of the stationary solution $r \equiv 0$. We use estimates for $(r, \mathcal{F}_1'(0)r)_s$ and $(r, \mathcal{F}_2'(0)r)_s$ that are easily obtained from (19) and (20). For the injection problems we prove that for small $r(0)$ there exists a global solution $r(t)$ of (14) that converges to zero if t goes to infinity. For the two dimensional suction problem we need to restrict ourselves to domains with certain geometric properties, in order to get an existence result on the time interval $[0, T]$.

Let ω be a bijection between $\{(l,m) \in \mathbb{N}^2 : 1 \leq l < m \leq N\}$ and $\{1, 2, \dots, \binom{N}{2}\}$. We define the following differential operators on functions on $\mathbb{S}^{N-1}$:

$$D_{\omega(l,m)} := x_l \frac{\partial}{\partial x_m} - x_m \frac{\partial}{\partial x_l}.$$

For each $i \in \{1, 2, \dots, \binom{N}{2}\}$, D_i is the infinitesimal generator of a semigroup of operators $h \mapsto R_h$:

$$R_h f = e^{hD_i} f = f \circ g_h, \qquad f \in \mathbb{L}_2(\mathbb{S}^{N-1}),$$

where $g_h : \mathbb{S}^{N-1} \to \mathbb{S}^{N-1}$ are rotations of the unit sphere.

Lemma 3.1. *Let $s > \frac{N+7}{2}$. For $r \in \mathbb{H}^{s+1}(\mathbb{S}^{N-1})$ with $\|r\|_s$ small, we have the generalised chain rule of differentiation:*

$$D_i \mathcal{F}_k(r) = \mathcal{F}_k'(r)[D_i r], \qquad k = 1, 2.$$

If in addition $r \in \mathbb{H}^{s+2}(\mathbb{S}^{N-1})$, then we have the second order generalised chain rule of differentiation:

$$D_i D_j \mathcal{F}_k(r) = \mathcal{F}_k'(r)[D_i D_j r] + \mathcal{F}_k''(r)[D_i r, D_j r], \qquad k = 1, 2. \tag{21}$$

Proof. Because $\mathcal{F}_k$ $(k = 1, 2)$ commutes with rotations, i.e.

$$\mathcal{F}_k(r) \circ g_h = \mathcal{F}_k(r \circ g_h),$$

we get

$$\begin{aligned} D_i \mathcal{F}_k(r) &= \lim_{h \to 0} \frac{1}{h}(R_h - I)\mathcal{F}_k(r) = \lim_{h \to 0} \frac{1}{h}\left(\mathcal{F}_k(r) \circ g_h - \mathcal{F}_k(r)\right) \\ &= \lim_{h \to 0} \frac{1}{h}\left(\mathcal{F}_k(r \circ g_h) - \mathcal{F}_k(r)\right) = \lim_{h \to 0} \frac{1}{h} \mathcal{F}_k'(r)[r \circ g_h - r] \\ &= \mathcal{F}_k'(r)\left[\lim_{h \to 0} \frac{r \circ g_h - r}{h}\right] = \mathcal{F}_k'(r)[D_i r], \end{aligned}$$

where I is the identity. From this we can derive the second order generalised chain rule as well (see also [6] Section 4 and [8] Chapter 5). □

Let for $\sigma > 0$, $\|\cdot\|_{\sigma-2,2}$ be the norm on $\mathbb{H}^\sigma(\mathbb{S}^{N-1})$ induced by the inner product

$$(r, \tilde{r})_{\sigma-2,2} := (r, \tilde{r})_{\sigma-2} + \sum_{i,j} (D_i D_j r, D_i D_j \tilde{r})_{\sigma-2}.$$

This norm is equivalent to the norm $\|\cdot\|_\sigma$ that we introduced earlier.

Lemma 3.2. *If $r \in \mathfrak{S}_k^N$ then $D_i r \in \mathfrak{S}_k^N$.*

Proof. The spaces $\mathfrak{S}_k^N$ are invariant under rotations. The lemma follows from this and the fact that D_i generates a semigroup of rotations. □

Theorem 3.3. *Let $N = 2$, $\mu > 0$, and $\lambda_0 \in \left(0, \frac{\mu}{2\pi}\right)$. Suppose that $s > 5$. There exists a $\delta > 0$ such that if $r_0 \in \mathbb{H}^s(\mathbb{S}^1)$ with $\|r_0\|_s < \delta$ then the problem*

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r,\tau), \tag{22}$$

with $r(0) = r_0$, has a solution $r \in \mathcal{C}([0,\tau_{\max}), \mathbb{H}^s(\mathbb{S}^1)) \cap \mathcal{C}^1([0,\tau_{\max}), \mathbb{H}^{s-3}(\mathbb{S}^1))$. Furthermore, $((\xi,\tau) \mapsto r(\tau)(\xi)) \in \mathcal{C}^\infty(\mathbb{S}^1 \times (0,\tau_{\max}))$. If we regard r as a function of the original time variable t, then

$$\|r(t)\|_{s-2,2} \le \left(\frac{\mu t}{\pi} + 1\right)^{-\frac{\pi\lambda_0}{\mu}} \|r_0\|_{s-2,2}, \quad t \in [0,\infty). \tag{23}$$

Proof. The theorem follows from the inequality

$$(r, \mathcal{F}(r,\tau))_{s-2,2} \le -\lambda_0 \alpha(\tau) \|r\|^2_{s-2,2}, \tag{24}$$

for all $r \in \mathbb{H}^{s+3}(\mathbb{S}^1)$ with $\|r\|_s$ small. First we find a similar estimate for the Fréchet derivatives $\mathcal{F}_1'(0)$ and $\mathcal{F}_2'(0)$. Perturbation arguments and the chain rule (21) lead to (24). Combining (16) and (24) we get algebraic decay of r as a function of t, given by (23).

Throughout the proof we will assume that $r \in \mathbb{H}^{s+3}(\mathbb{S}^1)$ with $\|r\|_s < \delta$, where δ is sufficiently small. The symbol C always denotes a sufficiently large constant, which is independent of r.

1. Take $\eta > 0$ such that $\lambda_0 < \frac{(1-\eta)\mu}{2\pi}$. Define

$$c_1 := \inf_{k \in \mathbb{N}_0, \tau \ge 0} \frac{\gamma p_1(k) + \eta\mu\alpha(\tau) p_2(k)}{(k^2+1)^{\frac{3}{2}}} = \inf_{k \in \mathbb{N}_0} \frac{\gamma p_1(k) + \eta\mu p_2(k)}{(k^2+1)^{\frac{3}{2}}} > 0,$$

and define $\varepsilon := \min\{c_1, \frac{(1-\eta)\mu}{2\pi} - \lambda_0\}$.

2. Let $\mathcal{F}'$ be the Frechet derivative of $\mathcal{F}$ with respect to the first argument. From (19) and (20) we have the following estimate for the linear part of $\mathcal{F}(r,\tau)$:

$$\begin{aligned}
&(r, \mathcal{F}'(0,\tau)r)_{s-2} \\
&= \gamma(r, \mathcal{F}_1'(0)r)_{s-2} + \mu\alpha(\tau)(r, \mathcal{F}_2'(0)r)_{s-2} \\
&= \gamma(r, \mathcal{F}_1'(0)r)_{s-2} + \eta\mu\alpha(\tau)(r, \mathcal{F}_2'(0)r)_{s-2} + (1-\eta)\mu\alpha(\tau)(r, \mathcal{F}_2'(0)r)_{s-2} \\
&= \sum_{k,j} (k^2+1)^{s-2+\frac{3}{2}} \frac{-\gamma p_1(k) - \eta\mu\alpha(\tau) p_2(k)}{(k^2+1)^{\frac{3}{2}}} r_{k,j}^2 \\
&\quad + (1-\eta)\alpha(\tau) \sum_{k,j} (k^2+1)^{s-2+\frac{1}{2}} \frac{-\mu p_2(k)}{(k^2+1)^{\frac{1}{2}}} r_{k,j}^2 \\
&\le -c_1 \|r\|^2_{s-\frac{1}{2}} - \frac{(1-\eta)\mu}{2\pi} \alpha(\tau) \|r\|^2_{s-\frac{3}{2}}.
\end{aligned} \tag{25}$$

In the last step we used $-\frac{p_2(k)}{\sqrt{k^2+1}} \leq -\frac{1}{2\pi}$.

3. Now we find an estimate for the remaining nonlinear part. From the analyticity of $\mathcal{F}_1$ and $\mathcal{F}_2$ near zero and the fact that $\mathcal{F}_1(0) = \mathcal{F}_2(0) = 0$, we have for r in a neighbourhood of zero in $\mathbb{H}^{s-\frac{1}{2}}(\mathbb{S}^1)$

$$\|\mathcal{F}_1(r) - \mathcal{F}_1'(0)r\|_{s-\frac{7}{2}} \leq C\|r\|^2_{s-\frac{1}{2}},$$

$$\|\mathcal{F}_2(r) - \mathcal{F}_2'(0)r\|_{s-\frac{5}{2}} \leq C\|r\|^2_{s-\frac{3}{2}}.$$

Here the demand $s > 5$ is crucial. Now we get

$$\begin{aligned} &\gamma(r, \mathcal{F}_1(r) - \mathcal{F}_1'(0)r)_{s-2} + \mu\alpha(\tau)(r, \mathcal{F}_2(r) - \mathcal{F}_2'(0)r)_{s-2} \\ \leq\; & C\left(\|r\|^3_{s-\frac{1}{2}} + \alpha(\tau)\|r\|^3_{s-\frac{3}{2}}\right). \end{aligned} \tag{26}$$

4. From the chain rule (21) we get

$$\begin{aligned} &(r, \mathcal{F}(r,\tau))_{s-2,2} \\ =\; &\gamma(r, \mathcal{F}_1(r))_{s-2} + \mu\alpha(\tau)(r, \mathcal{F}_2(r))_{s-2} \\ &+\gamma\sum_{i,j}(D_iD_jr, \mathcal{F}_1'(r)[D_iD_jr])_{s-2} + \mu\alpha(\tau)\sum_{i,j}(D_iD_jr, \mathcal{F}_2'(r)[D_iD_jr])_{s-2} \\ &+\gamma\sum_{i,j}(D_iD_jr, \mathcal{F}_1''(r)[D_ir, D_jr])_{s-2} + \mu\alpha(\tau)\sum_{i,j}(D_iD_jr, \mathcal{F}_2''(r)[D_ir, D_jr])_{s-2}. \end{aligned} \tag{27}$$

We divide the right-hand side into three parts and we estimate these parts separately.

5. Adding (25) and (26) we get for the first part of (27)

$$\begin{aligned} &\gamma(r, \mathcal{F}_1(r))_{s-2} + \mu\alpha(\tau)(r, \mathcal{F}_2(r))_{s-2} \\ \leq\; &-c_1\|r\|^2_{s-\frac{1}{2}} - \frac{(1-\eta)\mu}{2\pi}\alpha(\tau)\|r\|^2_{s-\frac{3}{2}} + C\left(\|r\|^3_{s-\frac{1}{2}} + \alpha(\tau)\|r\|^3_{s-\frac{3}{2}}\right) \\ \leq\; &-c_1\|r\|^2_{s-\frac{1}{2}} - \frac{(1-\eta)\mu}{2\pi}\alpha(\tau)\|r\|^2_{s-\frac{3}{2}} + C\delta\left(\|r\|^2_{s-\frac{1}{2}} + \alpha(\tau)\|r\|^2_{s-\frac{3}{2}}\right). \end{aligned} \tag{28}$$

6. For the second part of (27) we get from similar arguments

$$
\begin{aligned}
&\gamma(D_iD_jr, \mathcal{F}_1'(r)[D_iD_jr])_{s-2} + \mu\alpha(\tau)(D_iD_jr, \mathcal{F}_2'(r)[D_iD_jr])_{s-2} \\
&= \gamma(D_iD_jr, \mathcal{F}_1'(0)[D_iD_jr])_{s-2} + \mu\alpha(\tau)(D_iD_jr, \mathcal{F}_2'(0)[D_iD_jr])_{s-2} \\
&\quad +\gamma(D_iD_jr, \{\mathcal{F}_1'(r) - \mathcal{F}_1'(0)\}\,[D_iD_jr])_{s-2} \\
&\quad +\mu\alpha(\tau)(D_iD_jr, \{\mathcal{F}_2'(r) - \mathcal{F}_2'(0)\}\,[D_iD_jr])_{s-2} \\
&\le -c_1\|D_iD_jr\|^2_{s-\frac{1}{2}} - \frac{(1-\eta)\mu}{2\pi}\alpha(\tau)\|D_iD_jr\|^2_{s-\frac{3}{2}} \\
&\quad +C\delta\left(\|D_iD_jr\|^2_{s-\frac{1}{2}} + \alpha(\tau)\|D_iD_jr\|^2_{s-\frac{3}{2}}\right). \qquad (29)
\end{aligned}
$$

In the last step we used analyticity of $\mathcal{F}_1$ near zero in $\mathbb{H}^{s-\frac{1}{2}}(\mathbb{S}^{N-1})$ and analyticity of $\mathcal{F}_2$ near zero in $\mathbb{H}^{s-\frac{3}{2}}(\mathbb{S}^{N-1})$.

7. Because of Lemma 2.3, there exists a $C > 0$ such that for r near the origin in $\mathbb{H}^{s-\frac{1}{2}}(\mathbb{S}^1)$ we have $\|\mathcal{F}_1''(r)\|_{X_1} \le C$, for $X_1 = \mathcal{L}^2(\mathbb{H}^{s-\frac{1}{2}}(\mathbb{S}^1) \times \mathbb{H}^{s-\frac{1}{2}}(\mathbb{S}^1), \mathbb{H}^{s-\frac{7}{2}}(\mathbb{S}^1))$ and $\|\mathcal{F}_2''(r)\|_{X_2} \le C$, for $X_2 = \mathcal{L}^2(\mathbb{H}^{s-\frac{3}{2}}(\mathbb{S}^1) \times \mathbb{H}^{s-\frac{3}{2}}(\mathbb{S}^1), \mathbb{H}^{s-\frac{5}{2}}(\mathbb{S}^1))$. Therefore, the third part of (27) can be estimated as follows:

$$
\begin{aligned}
&\gamma(D_iD_jr, \mathcal{F}_1''(r)[D_ir, D_jr])_{s-2} + \mu\alpha(\tau)(D_iD_jr, \mathcal{F}_2''(r)[D_ir, D_jr])_{s-2} \\
&\le C\left(\|r\|_{s+\frac{3}{2}}\|r\|^2_{s+\frac{1}{2}} + \alpha(\tau)\|r\|_{s+\frac{1}{2}}\|r\|^2_{s-\frac{1}{2}}\right) \\
&\le C\left(\|r\|_{s-\frac{1}{2}}\|r\|^2_{s+\frac{3}{2}} + \alpha(\tau)\|r\|_{s-\frac{3}{2}}\|r\|^2_{s+\frac{1}{2}}\right) \\
&\le C\delta\left(\|r\|^2_{s+\frac{3}{2}} + \alpha(\tau)\|r\|^2_{s+\frac{1}{2}}\right). \qquad (30)
\end{aligned}
$$

Here we used the following interpolation inequalities:

$$
\begin{aligned}
\|r\|^2_{s+\frac{1}{2}} &\le C\|r\|_{s-\frac{1}{2}}\|r\|_{s+\frac{3}{2}}, \\
\|r\|^2_{s-\frac{1}{2}} &\le C\|r\|_{s-\frac{3}{2}}\|r\|_{s+\frac{1}{2}}.
\end{aligned}
$$

8. Adding the results of (28), (29), and (30) and using equivalence of the norms $\|\cdot\|_\sigma$ and $\|\cdot\|_{\sigma-2,2}$ we get

$$
\begin{aligned}
(r, \mathcal{F}(r,\tau))_{s-2,2} &\le -c_1\|r\|^2_{s-\frac{1}{2},2} - \frac{(1-\eta)\mu}{2\pi}\alpha(\tau)\|r\|^2_{s-\frac{3}{2},2} \\
&\quad +C\delta\left(\|r\|^2_{s-\frac{1}{2},2} + \alpha(\tau)\|r\|^2_{s-\frac{3}{2},2}\right).
\end{aligned}
$$

If we choose $\delta < \frac{\varepsilon}{C}$, then we get

$$
\begin{aligned}
(r, \mathcal{F}(r,\tau))_{s-2,2} &\le -(c_1 - \varepsilon)\|r\|^2_{s-\frac{1}{2},2} - \left(\frac{(1-\eta)\mu}{2\pi} - \varepsilon\right)\alpha(\tau)\|r\|^2_{s-\frac{3}{2},2} \\
&\le -\lambda_0\alpha(\tau)\|r\|^2_{s-\frac{3}{2},2} \le -\lambda_0\alpha(\tau)\|r\|^2_{s-2,2}. \qquad (31)
\end{aligned}
$$

9. In [8] Chapter 6 Proposition 9 and 10, local existence results for Stokes flow with injection or suction are proved. Hele-Shaw flow can be treated in a similar way. Combining this local existence result and (31) we get global existence of a solution $r \in \mathcal{C}([0,\tau_{\max}),\mathbb{H}^s(\mathbb{S}^1)) \cap \mathcal{C}^1([0,\tau_{\max}),\mathbb{H}^{s-3}(\mathbb{S}^1))$ such that $r \in \mathcal{C}^1((0,\tau_{\max}),\mathcal{C}^\infty(\mathbb{S}^1))$. From this it follows that $((\xi,\tau) \mapsto r(\tau)(\xi)) \in \mathcal{C}^\infty(\mathbb{S}^1 \times (0,\tau_{\max}))$. Furthermore, for small $\vartheta > 0$ and $\tau > \vartheta$, we have $\|r(\tau)\|^2_{s-2,2} \le y(\tau)$ where $y : [\vartheta,\tau_{\max}) \to \mathbb{R}$ satisfies

$$\frac{dy}{d\tau} = -2\lambda_0 \alpha(\tau) y,$$

with $y(\vartheta) = \|r(\vartheta)\|^2_{s-2,2}$. Solving this ODE we get for $\tau > \vartheta$

$$y(\tau) = e^{-2\lambda_0 \int_\vartheta^\tau \alpha(\tilde{\tau})d\tilde{\tau}} \|r(\vartheta)\|^2_{s-2,2}.$$

After reintroducing the original time variable by (16) we get

$$\begin{aligned} y &= e^{-2\lambda_0 \int_{t(\vartheta)}^t \frac{1}{(\alpha(\tilde{t}))^2} d\tilde{t}} \|r(\vartheta)\|^2_{s-2,2} \\ &= \left(\frac{\mu(t - t(\vartheta))}{\pi} + 1 \right)^{-\frac{2\pi\lambda_0}{\mu}} \|r(\vartheta)\|^2_{s-2,2}. \end{aligned}$$

Here, $t(\vartheta)$ denotes the value of the original time variable t for τ equal to ϑ. Letting ϑ go to zero we find (23).

□

Define

$$\mathfrak{M}_1^N = \left\{ r \in \mathcal{C}^0(\mathbb{S}^{N-1}) : \int_{\Omega_r} dx = \frac{\sigma_N}{N}, \int_{\Omega_r} x dx = 0 \right\}.$$

Note that $\mathfrak{M}_1^N$ is the set of continuous functions on $\mathbb{S}^{N-1}$ for which the corresponding domains Ω_r have the volume of the unit ball and a geometric centre that coincides with the origin. It is proved in [10] Lemma 3.7, that if r is a solution of (18) with $r(0) \in \mathfrak{M}_1^N$, then $r(t) \in \mathfrak{M}_1^N$ for all t. Introduce the Hilbert spaces $\mathbb{H}_1^\sigma(\mathbb{S}^{N-1})$ as

$$\mathbb{H}_1^\sigma(\mathbb{S}^{N-1}) = \{ r \in \mathbb{H}^\sigma(\mathbb{S}^{N-1}) : (r,s)_0 = 0, \quad \forall s \in \mathfrak{S}_0^N \oplus \mathfrak{S}_1^N \}.$$

Define $f_1 : \mathbb{H}^s(\mathbb{S}^{N-1}) \to \mathbb{R} \times \mathbb{R}^N$ by

$$f_1(r) := \begin{pmatrix} \int_{\Omega_r} dx - \frac{\sigma_N}{N} \\ \int_{\Omega_r} x dx \end{pmatrix}.$$

Let $\mathcal{P}_1 : \mathbb{H}^s(\mathbb{S}^{N-1}) \to \mathbb{H}_1^s(\mathbb{S}^{N-1})$ be the orthogonal projection onto $\mathbb{H}_1^s(\mathbb{S}^{N-1})$ with respect to the $\mathbb{L}_2(\mathbb{S}^{N-1})$-inner product and let $\phi : \mathbb{H}^s(\mathbb{S}^{N-1}) \to \mathbb{R} \times \mathbb{R}^N \times \mathbb{H}_1^s(\mathbb{S}^{N-1})$ be defined by

$$\phi(r) := \begin{pmatrix} f_1(r) \\ \mathcal{P}_1 r \end{pmatrix}.$$

By the Implicit Function Theorem ϕ is an analytic bijection between a neighbourhood of zero in $\mathbb{H}^s(\mathbb{S}^{N-1})$ and a neighbourhood of zero in $\mathbb{R} \times \mathbb{R}^N \times \mathbb{H}_1^s(\mathbb{S}^{N-1})$. This is proved for different function spaces in [11]. On a suitable neighbourhood $\mathcal{U}$ of zero in $\mathbb{H}_1^s(\mathbb{S}^{N-1})$ we define $\psi : \mathcal{U} \to \mathfrak{M}_1^N$ by

$$\psi(\tilde{r}) := \phi^{-1}(0,\tilde{r}).$$

Lemma 3.4. *Let $s > \frac{N+7}{2}$. For $\tilde{r} \in \mathbb{H}_1^{s+2}(\mathbb{S}^{N-1})$ with $\|\tilde{r}\|_s$ small, we have*

$$D_i D_j \psi(\tilde{r}) = \psi'(\tilde{r})[D_i D_j \tilde{r}] + \psi''(\tilde{r})[D_i \tilde{r}, D_j \tilde{r}]. \tag{32}$$

Proof. It is sufficient to show that ψ commutes with rotations. If $\tilde{r} \in \mathbb{H}_1^{s+2}(\mathbb{S}^{N-1})$, then $\psi(\tilde{r}) \in \mathfrak{M}_1^N$ and for a rotation $g : \mathbb{R}^N \to \mathbb{R}^N$ we have $\tilde{r} \circ g \in \mathbb{H}_1^{s+2}(\mathbb{S}^{N-1})$ and $\psi(\tilde{r}) \circ g \in \mathfrak{M}_1^N$. Because rotations and $\mathcal{P}_1$ commute we have $\mathcal{P}_1(\psi(\tilde{r}) \circ g) = (\mathcal{P}_1 \psi(\tilde{r})) \circ g = \tilde{r} \circ g$. Therefore

$$\psi(\tilde{r}) \circ g = \phi^{-1}(0, \mathcal{P}_1(\psi(\tilde{r}) \circ g)) = \phi^{-1}(0, \tilde{r} \circ g) = \psi(\tilde{r} \circ g).$$

This proves the lemma. □

Now we derive a global existence result for the suction case for $N = 2$. Like in the proof of Theorem 3.3, we get this result from energy estimates. The suction case is more complicated then the injection case, first of all because we need to restrict ourselves to the subset $\mathfrak{M}_1^2$, which is not a vector space. This problem is solved by considering an equivalent problem on $\mathbb{H}_1^s(\mathbb{S}^1)$ given by equation (35) and using the bijection ψ between $\mathbb{H}_1^s(\mathbb{S}^1)$ and $\mathfrak{M}_1^2$. Once we have found existence of solutions $\tilde{r}$ of this equation, we have existence of corresponding solutions $r = \psi(\tilde{r})$ of equation (33). The second complication here is that we need to split up the time interval $[0, \infty)$ in two parts, $[0, \hat{T}]$ and $[\hat{T}, \infty)$. On the first interval, the norm of solutions $\tilde{r}$ that we find might grow up to a value δ'. On the second interval we need an energy estimate, that is sharper than the one that we found on the first interval, in order to obtain exponential decay for solutions of (35). For any ratio of $|\mu|$ to γ, a suitable $\hat{T}$ exists, because for large time surface tension dominates suction. For the three dimensional problem (see [10]) this is not the case, because eigenvalues of the linearisations of the evolution operators do not change in time.

Theorem 3.5. *Let $N = 2$, $\mu < 0$, and take $\lambda_0 \in \left(0, \frac{6\gamma}{5\sqrt{5}}\right)$. Suppose that $s > 5$. There exists a $\delta > 0$ and a $M > 0$ such that if $r_0 \in \mathbb{H}^s(\mathbb{S}^1) \cap \mathfrak{M}_1^2$ with $\|r_0\|_s < \delta$ then the problem*

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r, \tau), \tag{33}$$

with $r(0) = r_0$, has a solution $r \in \mathcal{C}([0,\infty), \mathbb{H}^s(\mathbb{S}^1)) \cap \mathcal{C}^1([0,\infty), \mathbb{H}^{s-3}(\mathbb{S}^1))$ with $((\xi,\tau) \mapsto r(\tau)(\xi)) \in \mathcal{C}^\infty(\mathbb{S}^1 \times (0,\infty))$. We have

$$\|r(\tau)\|_{s-2,2} \leq M e^{-\lambda_0 \tau} \|r_0\|_{s-2,2}. \tag{34}$$

Proof. Again, the symbol C is used for a constant that may vary throughout the proof.

1. Note that $-\frac{p_1(k)}{(k^2+1)^{\frac{3}{2}}}$ decreases in k. Therefore we have for $k \geq 2$

$$-\frac{\gamma p_1(k)}{(k^2+1)^{\frac{3}{2}}} \leq -\frac{6\gamma}{5\sqrt{5}} < -\lambda_0.$$

Furthermore $\frac{p_2(k)}{(k^2+1)^{\frac{3}{2}}}$ is bounded and $\lim_{\tau\to\infty} \alpha(\tau) = 0$. Therefore there exists a $\hat{T}$ such that for $\tau \geq \hat{T}$ and $k \geq 2$

$$\frac{-\gamma p_1(k) + |\mu| \alpha(\tau) p_2(k)}{(k^2+1)^{\frac{3}{2}}} < -\lambda_0.$$

Choose $K \in \mathbb{N}$ such that for $k > K$ we have $-\gamma p_1(k) + |\mu| p_2(k) < 0$ and let $\mathcal{P}_K : \mathbb{L}_2(\mathbb{S}^1) \to \mathbb{L}_2(\mathbb{S}^1)$ be the orthogonal projection with respect to the $\mathbb{L}_2(\mathbb{S}^1)$-inner product onto the orthoplement of $\bigoplus_{k=0}^{K} \mathfrak{S}_k^2$. Define $c_1 > 0$ and $c_2 > 0$ by

$$c_1 := \inf_{k \geq 2, \tau \geq \hat{T}} \frac{\gamma p_1(k) - |\mu| \alpha(\tau) p_2(k)}{(k^2+1)^{\frac{3}{2}}} = \inf_{k \geq 2} \frac{\gamma p_1(k) - |\mu| \alpha(\hat{T}) p_2(k)}{(k^2+1)^{\frac{3}{2}}} > \lambda_0$$

and

$$c_2 := \inf_{k > K, \tau \geq 0} \frac{\gamma p_1(k) - |\mu| \alpha(\tau) p_2(k)}{(k^2+1)^{\frac{3}{2}}} = \inf_{k > K} \frac{\gamma p_1(k) - |\mu| p_2(k)}{(k^2+1)^{\frac{3}{2}}}.$$

The positivity of c_2 follows from the fact that $\frac{\gamma p_1(k) - |\mu| p_2(k)}{(k^2+1)^{\frac{3}{2}}}$ converges to γ if k goes to infinity. Define $\varepsilon := \min\{c_1 - \lambda_0, c_2\}$.

2. Assume for the moment that r satisfies (33). Then $\tilde{r} := \mathcal{P}_1 r$ satisfies

$$\frac{\partial \tilde{r}}{\partial \tau} = \mathcal{P}_1 \mathcal{F}(\psi(\tilde{r}), \tau). \tag{35}$$

First we will prove solvability of this equation, finding estimates for $(\tilde{r}, \mathcal{P}_1 \mathcal{F}(\psi(\tilde{r}), \tau))_{s-2,2}$ for $\tilde{r} \in \mathbb{H}_1^{s+3}(\mathbb{S}^1)$ and $\|\tilde{r}\|_s < \delta'$ with δ' small enough.

3. Introduce on a suitable neighbourhood $\mathcal{U}$ of zero in $\mathbb{H}_1^s(\mathbb{S}^1)$ the operators $\tilde{\mathcal{F}}_1 : \mathcal{U} \to \mathbb{H}_1^{s-3}(\mathbb{S}^1)$ and $\tilde{\mathcal{F}}_2 : \mathcal{U} \to \mathbb{H}_1^{s-1}(\mathbb{S}^1)$ as $\tilde{\mathcal{F}}_1 = \mathcal{P}_1 \circ \mathcal{F}_1 \circ \psi$ and $\tilde{\mathcal{F}}_2 = \mathcal{P}_1 \circ \mathcal{F}_2 \circ \psi$. These operators are compositions of analytic operators, so they are analytic themselves. From a simple calculation we see that $\psi'(0)$ is the identity on $\mathbb{H}_1^s(\mathbb{S}^1)$. Therefore,

$$\tilde{\mathcal{F}}_k'(0)\tilde{r} = \mathcal{F}_k'(0)\tilde{r},$$

for $k = 1, 2$ and we have the following estimate for the linear part of $\gamma \tilde{\mathcal{F}}_1 + \mu \alpha(\tau) \tilde{\mathcal{F}}_2$:

$$\begin{aligned}
&\gamma(\tilde{r}, \tilde{\mathcal{F}}_1'(0)\tilde{r})_{s-2} + \mu \alpha(\tau)(\tilde{r}, \tilde{\mathcal{F}}_2'(0)\tilde{r})_{s-2} \\
&= \sum_{k \leq K} (k^2+1)^{s-2+\frac{3}{2}} \frac{-\gamma p_1(k) + |\mu| \alpha(\tau) p_2(k)}{(k^2+1)^{\frac{3}{2}}} \tilde{r}_{k,j}^2 \\
&\quad + \sum_{k > K} (k^2+1)^{s-2+\frac{3}{2}} \frac{-\gamma p_1(k) + |\mu| \alpha(\tau) p_2(k)}{(k^2+1)^{\frac{3}{2}}} \tilde{r}_{k,j}^2 \\
&\leq C \|\tilde{r}\|_0^2 - c_2 \|\mathcal{P}_K \tilde{r}\|_{s-\frac{1}{2}}^2 \\
&= C \|\tilde{r}\|_0^2 + c_2 \|(\mathcal{I} - \mathcal{P}_K)\tilde{r}\|_{s-\frac{1}{2}}^2 - c_2 \|\tilde{r}\|_{s-\frac{1}{2}}^2 \\
&\leq C \|\tilde{r}\|_0^2 - c_2 \|\tilde{r}\|_{s-\frac{1}{2}}^2.
\end{aligned} \tag{36}$$

Here we used the fact that $\mathcal{I} - \mathcal{P}_K : \mathbb{L}_2(\mathbb{S}^1) \to \mathbb{H}^{s-\frac{1}{2}}(\mathbb{S}^1)$ is bounded.

4. For the suction problem we have $\alpha(\tau) \leq 1$. Therefore, the nonlinear parts can be estimated in the following way:

$$\gamma(\tilde{r}, \tilde{\mathcal{F}}_1(\tilde{r}) - \tilde{\mathcal{F}}_1'(0)\tilde{r})_{s-2} + \mu\alpha(\tau)(\tilde{r}, \tilde{\mathcal{F}}_2(\tilde{r}) - \tilde{\mathcal{F}}_2'(0)\tilde{r})_{s-2} \leq C\|\tilde{r}\|^3_{s-\frac{1}{2}}. \qquad (37)$$

Here we used the analyticity of $\tilde{\mathcal{F}}_1$ and $\tilde{\mathcal{F}}_2$, as we did for $\mathcal{F}_1$ and $\mathcal{F}_2$ in the proof of Theorem 3.3.

5. From the chain rule (21) we get

$$\begin{aligned}
&\gamma(\tilde{r}, \tilde{\mathcal{F}}_1(\tilde{r}))_{s-2,2} + \mu\alpha(\tau)(\tilde{r}, \tilde{\mathcal{F}}_2(\tilde{r}))_{s-2,2} \\
&= \gamma(\tilde{r}, \tilde{\mathcal{F}}_1(\tilde{r}))_{s-2} + \mu\alpha(\tau)(\tilde{r}, \tilde{\mathcal{F}}_2(\tilde{r}))_{s-2} \\
&\quad + \gamma\sum_{i,j}(D_iD_j\tilde{r}, \tilde{\mathcal{F}}_1'(\tilde{r})[D_iD_j\tilde{r}])_{s-2} + \mu\alpha(\tau)\sum_{i,j}(D_iD_j\tilde{r}, \tilde{\mathcal{F}}_2'(\tilde{r})[D_iD_j\tilde{r}])_{s-2} \\
&\quad + \gamma\sum_{i,j}(D_iD_j\tilde{r}, \tilde{\mathcal{F}}_1''(\tilde{r})[D_i\tilde{r}, D_j\tilde{r}])_{s-2} + \mu\alpha(\tau)\sum_{i,j}(D_iD_j\tilde{r}, \tilde{\mathcal{F}}_2''(\tilde{r})[D_i\tilde{r}, D_j\tilde{r}])_{s-2}.
\end{aligned}$$

The right-hand side consists of three parts that will be estimated separately on both intervals $[0, \hat{T}]$ and $[\hat{T}, \infty)$. We start with $[0, \hat{T}]$.

6. From (36) and (37), we have for the first part

$$\begin{aligned}
&\gamma(\tilde{r}, \tilde{\mathcal{F}}_1(\tilde{r}))_{s-2} + \mu\alpha(\tau)(\tilde{r}, \tilde{\mathcal{F}}_2(\tilde{r}))_{s-2} \\
&\leq C\|\tilde{r}\|_0^2 - c_2\|\tilde{r}\|^2_{s-\frac{1}{2}} + C\delta'\|\tilde{r}\|^2_{s-\frac{1}{2}}.
\end{aligned} \qquad (38)$$

7. The second part can be treated in a way similar to the proof of Theorem 3.3. We use (36), the boundedness of $\alpha(\tau)$, and the analyticity of $\tilde{\mathcal{F}}_1$ and $\tilde{\mathcal{F}}_2$ to get

$$\begin{aligned}
&\gamma(D_iD_j\tilde{r}, \tilde{\mathcal{F}}_1'(\tilde{r})[D_iD_j\tilde{r}])_{s-2} + \mu\alpha(\tau)(D_iD_j\tilde{r}, \tilde{\mathcal{F}}_2'(\tilde{r})[D_iD_j\tilde{r}])_{s-2} \\
&= \gamma(D_iD_j\tilde{r}, \tilde{\mathcal{F}}_1'(0)[D_iD_j\tilde{r}])_{s-2} + \mu\alpha(\tau)(D_iD_j\tilde{r}, \tilde{\mathcal{F}}_2'(0)[D_iD_j\tilde{r}])_{s-2} \\
&\quad + \gamma(D_iD_j\tilde{r}, \{\tilde{\mathcal{F}}_1'(\tilde{r}) - \tilde{\mathcal{F}}_1'(0)\}[D_iD_j\tilde{r}])_{s-2} \\
&\quad + \mu\alpha(\tau)(D_iD_j\tilde{r}, \{\tilde{\mathcal{F}}_2'(\tilde{r}) - \tilde{\mathcal{F}}_2'(0)\}[D_iD_j\tilde{r}])_{s-2} \\
&\leq C\|D_iD_j\tilde{r}\|_0^2 - c_2\|D_iD_j\tilde{r}\|^2_{s-\frac{1}{2}} + C\delta'\|D_iD_j\tilde{r}\|^2_{s-\frac{1}{2}}.
\end{aligned} \qquad (39)$$

Here, we also used the fact that $D_iD_j\tilde{r} \in \mathbb{H}_1^s(\mathbb{S}^1)$ if $\tilde{r} \in \mathbb{H}_1^{s+3}(\mathbb{S}^1)$. This follows from Lemma 3.2.

8. For the third part, we refer to the proof of Theorem 3.3 as well and we use the boundedness of $\alpha(\tau)$ and the analyticity of $\tilde{\mathcal{F}}_1$ and $\tilde{\mathcal{F}}_2$. In this way we find

$$\gamma(D_iD_j\tilde{r}, \tilde{\mathcal{F}}_1''(\tilde{r})[D_i\tilde{r}, D_j\tilde{r}])_{s-2} + \mu\alpha(\tau)(D_iD_j\tilde{r}, \tilde{\mathcal{F}}_2''(\tilde{r})[D_i\tilde{r}, D_j\tilde{r}])_{s-2} \leq C\delta'\|\tilde{r}\|^2_{s+\frac{3}{2}}. \qquad (40)$$

9. Combining (38), (39), and (40) and using equivalence of the norms $\|\cdot\|_{s+\frac{3}{2}}$ and $\|\cdot\|_{s-\frac{1}{2},2}$ we get on the interval $[0,\hat{T}]$

$$\begin{aligned}&\gamma(\tilde{r},\tilde{\mathcal{F}}_1(\tilde{r}))_{s-2,2}+\mu\alpha(\tau)(\tilde{r},\tilde{\mathcal{F}}_2(\tilde{r}))_{s-2,2}\\ &\leq C\|\tilde{r}\|_{0,2}^2-c_2\|\tilde{r}\|_{s-\frac{1}{2},2}^2+C\delta'\|\tilde{r}\|_{s-\frac{1}{2},2}^2.\end{aligned}$$

If we take $\delta'<\frac{\varepsilon}{C}$, then we get

$$\begin{aligned}&\gamma(\tilde{r},\tilde{\mathcal{F}}_1(\tilde{r}))_{s-2,2}+\mu\alpha(\tau)(\tilde{r},\tilde{\mathcal{F}}_2(\tilde{r}))_{s-2,2}\\ &\leq C\|\tilde{r}\|_{0,2}^2\leq C\|\tilde{r}\|_{s-2,2}^2.\end{aligned}\tag{41}$$

Define $\tilde{r}_0:=\mathcal{P}_1 r_0$. Using local existence results as in Theorem 3.3 and diminishing δ' if necessary we find an $S>0$ such that if $\|\tilde{r}_0\|_{s-2,2}\leq\delta'$ then (35) has a solution $\tilde{r}$ on $[0,S]$, with $\tilde{r}(0)=\tilde{r}_0$. Take $\delta<e^{-C\hat{T}}\delta'$ and assume $\|\tilde{r}_0\|_{s-2,2}\leq\delta$. Then one can show by induction over $K\in\mathbb{N}_0$ that (35) has a solution on $[KS,(K+1)S]\cap[0,\hat{T}]$ that satisfies

$$\|\tilde{r}(\tau)\|_{s-2,2}\leq e^{C\tau}\|\tilde{r}_0\|_{s-2,2}$$

there. Therefore

$$\|\tilde{r}(\hat{T})\|_{s-2,2}\leq e^{C\hat{T}}\delta<\delta'.$$

10. Now we treat the interval $[\hat{T},\infty)$. Again we consider the chain rule and distinguish between three parts. Because of the boundedness of α we have for the first part

$$\begin{aligned}&\gamma(\tilde{r},\tilde{\mathcal{F}}_1(\tilde{r}))_{s-2}+\mu\alpha(\tau)(\tilde{r},\tilde{\mathcal{F}}_2(\tilde{r}))_{s-2}\\ &\leq\gamma(\tilde{r},\tilde{\mathcal{F}}_1'(0)\tilde{r})_{s-2}+\mu\alpha(\tau)(\tilde{r},\tilde{\mathcal{F}}_2'(0)\tilde{r})_{s-2}+C\|\tilde{r}\|_{s-\frac{1}{2}}^3\\ &=\sum_{k\geq 2}(k^2+1)^{s-2+\frac{3}{2}}\frac{-\gamma p_1(k)+|\mu|\alpha(\tau)p_2(k)}{(k^2+1)^{\frac{3}{2}}}\tilde{r}_{k,j}^2+C\|\tilde{r}\|_{s-\frac{1}{2}}^3\\ &\leq -c_1\|\tilde{r}\|_{s-\frac{1}{2}}^2+C\delta'\|\tilde{r}\|_{s-\frac{1}{2}}^2.\end{aligned}\tag{42}$$

Note that in the summation we start counting from $k=2$, because $\tilde{r}\in\mathbb{H}_1^s(\mathbb{S}^1)$.

11. For the second part we use the same strategy as for the first time interval, to obtain

$$\begin{aligned}&\gamma(D_iD_j\tilde{r},\tilde{\mathcal{F}}_1'(\tilde{r})[D_iD_j\tilde{r}])_{s-2}+\mu\alpha(\tau)(D_iD_j\tilde{r},\tilde{\mathcal{F}}_2'(\tilde{r})[D_iD_j\tilde{r}])_{s-2}\\ &\qquad\leq -c_1\|D_iD_j\tilde{r}\|_{s-\frac{1}{2}}^2+C\delta'\|D_iD_j\tilde{r}\|_{s-\frac{1}{2}}^2.\end{aligned}\tag{43}$$

12. For the third part, we get exactly the same result as for the first time interval, cf. (40).

13. Adding (42), (43), and (40) and using equivalence of the norms $\|\cdot\|_{s+\frac{3}{2}}$ and $\|\cdot\|_{s-\frac{1}{2},2}$ we get

$$\begin{aligned}&\gamma(\tilde{r},\tilde{\mathcal{F}}_1(\tilde{r}))_{s-2,2}+\mu\alpha(\tau)(\tilde{r},\tilde{\mathcal{F}}_2(\tilde{r}))_{s-2,2}\\ &\leq -c_1\|\tilde{r}\|_{s-\frac{1}{2},2}^2+C\delta'\|\tilde{r}\|_{s-\frac{1}{2},2}^2.\end{aligned}$$

Taking $\delta' < \frac{\varepsilon}{C}$ we find

$$\gamma(\tilde{r}, \tilde{\mathcal{F}}_1(\tilde{r}))_{s-2,2} + \mu\alpha(\tau)(\tilde{r}, \tilde{\mathcal{F}}_2(\tilde{r}))_{s-2,2} \leq -\lambda_0 \|\tilde{r}\|^2_{s-\frac{1}{2},2} \leq -\lambda_0 \|\tilde{r}\|^2_{s-2,2}.$$

Using [8] Chapter 6 Proposition 9 and 10 again, we can extend the solution $\tilde{r}$ that we found on $[0, \hat{T}]$ to $[\hat{T}, \infty)$, such that for $\tau \in [\hat{T}, \infty)$

$$\|\tilde{r}(\tau)\|_{s-2,2} \leq e^{-\lambda_0(\tau - \hat{T})} \|\tilde{r}(\hat{T})\|_{s-2,2}.$$

Combining the results on both intervals, we get existence of a $M' > 0$ independent of $\tilde{r}(0)$, such that for any $\tau \in [0, \infty)$

$$\|\tilde{r}(\tau)\|_{s-2,2} \leq M' e^{-\lambda_0 \tau} \|\tilde{r}(0)\|_{s-2,2}.$$

Define

$$r = \psi(\tilde{r}).$$

From the smoothness of ψ and the fact that $\psi(0) = 0$ we see that there exists a $M > 0$ such that if r_0 is small enough, then r is a solution to (33) that satisfies (34).

□

In contrast to the three-dimensional suction problem that is discussed in [10], we do not need any restriction on the ratio of suction rate to γ.

Now we derive a theorem for global existence for the higher dimensional case. For injection, we have to deal with the problem that eigenvalues of the linearisation corresponding to spherical harmonics of order zero and one go to zero for large time. The nonlinear part cannot be controlled anymore. In order to deal with this, we use the bijection ϕ near the origin between $\mathbb{H}^s(\mathbb{S}^{N-1})$ and $\mathbb{R} \times \mathbb{R}^N \times \mathbb{H}^s_1(\mathbb{S}^{N-1})$ and consider the evolution of $\mathcal{P}_1 r$ on $\mathbb{H}^s_1(\mathbb{S}^{N-1})$ and the evolution of $f_1(r)$, the zeroth and first Richardson moments, separately. We write down an equation for $\mathcal{P}_1 r$ and find an energy estimate for its evolution operator. This equation differs from the one that we found in the proof of Theorem 3.5 because the evolution may not be in $\mathfrak{M}^N_1$. We use the fact that the zeroth and first Richardson moments as function of time are known beforehand.

For the suction problem for $N \geq 4$ we do not get any global existence result because if t approaches T, more and more eigenvalues of the linearised evolution equation become positive. In other words: For large t, $\mathcal{F}_2$ dominates $\mathcal{F}_1$. Suction can no longer be controlled by surface tension.

Theorem 3.6. *Let $N \geq 4$ and $\mu > 0$. Suppose that $s > \frac{N+8}{2}$. There exists a $\delta > 0$ and a $M > 0$ such that if $r_0 \in \mathbb{H}^s(\mathbb{S}^{N-1})$ with $\|r_0\|_s < \delta$ then the problem*

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r, \tau), \tag{44}$$

with $r(0) = r_0$, has a solution $r \in \mathcal{C}([0,\infty), \mathbb{H}^s(\mathbb{S}^{N-1})) \cap \mathcal{C}^1([0,\infty), \mathbb{H}^{s-3}(\mathbb{S}^{N-1}))$. Furthermore, $((\xi,\tau) \mapsto r(\tau)(\xi)) \in \mathcal{C}^\infty(\mathbb{S}^{N-1} \times (0,\infty))$. If we regard r as a function of t conform (17), then

$$\|r(t)\|_s \leq \frac{M}{\frac{\mu N t}{\sigma_N} + 1} \|r_0\|_s.$$

Proof. 1. Introduce the number $c_1 > 0$ as

$$c_1 := \inf_{k \geq 2} \frac{\gamma p_1(k)}{(k^2+1)^{\frac{3}{2}}}.$$

Choose $\lambda_0 \in (0, \frac{c_1}{3})$ and define $\varepsilon := \frac{c_1}{3} - \lambda_0$.

2. It is easily checked that the geometric centre of a moving domain $\Omega_{R(t)}$ satisfying (1)-(4) does not change in time. From this and (7) it follows that solutions r of (44) satisfy

$$f_1(r(\tau)) = \begin{pmatrix} \frac{V_0}{\alpha(\tau)^N} \\ \frac{1}{\alpha(\tau)^{N+1}} m_0 \end{pmatrix} =: \begin{pmatrix} V_\tau \\ m_\tau \end{pmatrix},$$

where

$$\begin{pmatrix} V_0 \\ m_0 \end{pmatrix} := f_1(r_0).$$

For notational convenience we introduce $q_\tau := (V_\tau, m_\tau)^T$. Assume for the moment that r satisfies (44). Then $\tilde{r} := \mathcal{P}_1 r$ satisfies

$$\frac{\partial \tilde{r}}{\partial \tau} = \mathcal{P}_1 \mathcal{F}\left(\phi^{-1}\left(q_\tau, \tilde{r}\right), \tau\right). \tag{45}$$

First we prove solvability of this equation, finding estimates for $\left(\tilde{r}, \mathcal{P}_1 \mathcal{F}\left(\phi^{-1}\left(q_\tau, \tilde{r}\right), \tau\right)\right)_{s-2,2}$, for $|q_0|$ small, $\tilde{r} \in \mathbb{H}_1^{s+3}(\mathbb{S}^{N-1})$ and $\|\tilde{r}\|_s < \delta$, with δ small enough.

3. Because $\tilde{r} \in \mathbb{H}_1^s(\mathbb{S}^{N-1})$ we have

$$\begin{aligned} \gamma(\tilde{r}, \mathcal{F}_1'(0)\tilde{r})_{s-2} + \mu\alpha(\tau)^{3-N}(\tilde{r}, \mathcal{F}_2'(0)\tilde{r})_{s-2} &\leq \gamma(\tilde{r}, \mathcal{F}_1'(0)\tilde{r})_{s-2} \\ &\leq -c_1\|\tilde{r}\|^2_{s-\frac{1}{2}}. \end{aligned} \tag{46}$$

4. Because of Lipschitz continuity of $\mathcal{F}_k \circ \phi^{-1}$, for $k = 1, 2$, and because $\psi = \phi^{-1}(0, \cdot)$ we have

$$\begin{aligned} &\|\mathcal{P}_1\mathcal{F}_k\left(\phi^{-1}\left(q_\tau, \tilde{r}\right)\right) - \mathcal{P}_1\mathcal{F}_k(\psi(\tilde{r}))\|_{s-2} \\ &= \|\mathcal{P}_1\mathcal{F}_k\left(\phi^{-1}\left(q_\tau, \tilde{r}\right)\right) - \mathcal{P}_1\mathcal{F}_k(\phi^{-1}\left(0, \tilde{r}\right))\|_{s-2} \\ &\leq C|q_\tau|. \end{aligned} \tag{47}$$

From a simple calculation it follows that $\psi'(0)$ is the identity on $\mathbb{H}_1^{s-\frac{1}{2}}(\mathbb{S}^{N-1})$. Therefore the restriction of $\mathcal{F}_k'(0)$ to $\mathbb{H}_1^{s-\frac{1}{2}}(\mathbb{S}^{N-1})$ is the Fréchet derivative around zero of the analytic mapping $\mathcal{P}_1 \circ \mathcal{F}_k \circ \psi$ on $\mathbb{H}_1^{s-\frac{1}{2}}(\mathbb{S}^{N-1})$ and we have

$$\|\mathcal{P}_1\mathcal{F}_k(\psi(\tilde{r})) - \mathcal{F}_k'(0)\tilde{r}\|_{s-\frac{7}{2}} \leq C\|\tilde{r}\|^2_{s-\frac{1}{2}}. \tag{48}$$

Combining (47) and (48) we get the following estimate:

$$\begin{aligned}
&\gamma\left\{\left(\tilde r,\mathcal{P}_1\mathcal{F}_1\left(\phi^{-1}\left(q_\tau,\tilde r\right)\right)\right)_{s-2}-(\tilde r,\mathcal{F}_1'(0)\tilde r)_{s-2}\right\}\\
&+\mu\alpha(\tau)^{3-N}\left\{\left(\tilde r,\mathcal{P}_1\mathcal{F}_2\left(\phi^{-1}\left(q_\tau,\tilde r\right)\right)\right)_{s-2}-(\tilde r,\mathcal{F}_2'(0)\tilde r)_{s-2}\right\}\\
\le{}& C\Big(|q_\tau|\|\tilde r\|_{s-2}+\alpha(\tau)^{3-N}|q_\tau|\|\tilde r\|_{s-2}+\|\tilde r\|^3_{s-\frac12}+\alpha(\tau)^{3-N}\|\tilde r\|^3_{s-\frac12}\Big)\\
\le{}& C\Big(|q_\tau|\|\tilde r\|_{s-2}+\|\tilde r\|^3_{s-\frac12}\Big). \qquad (49)
\end{aligned}$$

Here we used the fact that $\alpha(\tau)^{3-N}\le 1$.

5. From the chain rule (21) we get

$$\begin{aligned}
&\left(\tilde r,\mathcal{P}_1\mathcal{F}\left(\phi^{-1}\left(q_\tau,\tilde r\right),\tau\right)\right)_{s-2,2}\\
&=\gamma\left(F_1+G_1+H_1\right)+\mu\alpha(\tau)^{3-N}\left(F_2+G_2+H_2\right), \qquad (50)
\end{aligned}$$

where for $k=1,2$

$$\begin{aligned}
F_k&=(\tilde r,\mathcal{P}_1\mathcal{F}_k\left(\phi^{-1}\left(q_\tau,\tilde r\right)\right))_{s-2},\\
G_k&=\sum_{i,j}(D_iD_j\tilde r,\mathcal{P}_1\mathcal{F}_k'\left(\phi^{-1}\left(q_\tau,\tilde r\right)\right)\left[D_iD_j\phi^{-1}(q_\tau,\tilde r)\right])_{s-2},\\
H_k&=\sum_{i,j}(D_iD_j\tilde r,\mathcal{P}_1\mathcal{F}_k''\left(\phi^{-1}\left(q_\tau,\tilde r\right)\right)\left[D_i\phi^{-1}(q_\tau,\tilde r),D_j\phi^{-1}(q_\tau,\tilde r)\right])_{s-2}.
\end{aligned}$$

We will estimate the terms containing F_k, G_k, and H_k separately.

6. Using (46) and (49), we have

$$\gamma F_1+\mu\alpha(\tau)^{3-N}F_2\le -c_1\|\tilde r\|^2_{s-\frac12}+C|q_\tau|\|\tilde r\|_{s-2}+C\|\tilde r\|^3_{s-\frac12}. \qquad (51)$$

7. Now we find an estimate for the terms of G_1. Because by definition $\psi(\tilde r)=\phi^{-1}(0,\tilde r)$, we have

$$G_1=\sum_{i,j}(D_iD_j\tilde r,I_{ij}+J_{ij}+K_{ij})_{s-2}, \qquad (52)$$

where

$$\begin{aligned}
I_{ij}&=\mathcal{P}_1\mathcal{F}_1'\left(\phi^{-1}\left(q_\tau,\tilde r\right)\right)\left[D_iD_j\phi^{-1}(q_\tau,\tilde r)\right]-\mathcal{P}_1\mathcal{F}_1'\left(\phi^{-1}\left(0,\tilde r\right)\right)\left[D_iD_j\phi^{-1}(0,\tilde r)\right],\\
J_{ij}&=\mathcal{P}_1\mathcal{F}_1'\left(\psi(\tilde r)\right)\left[\psi'(\tilde r)[D_iD_j\tilde r]\right],\\
K_{ij}&=\mathcal{P}_1\mathcal{F}_1'\left(\psi(\tilde r)\right)\left[\psi''(\tilde r)[D_i\tilde r,D_j\tilde r]\right].
\end{aligned}$$

Here we used Lemma 3.4. Because $\mathcal{P}_1\mathcal{F}_1'(\phi^{-1}(\cdot))[D_iD_j\phi^{-1}(\cdot)]$ is Lipschitz continuous from $\mathbb{R}\times\mathbb{R}^N\times\mathbb{H}_1^{s+3}(\mathbb{S}^{N-1})$ to $\mathbb{H}_1^{s-2}(\mathbb{S}^{N-1})$, we get

$$\|I_{ij}\|_{s-2} \le C|q_\tau|.$$

By the Cauchy-Schwarz inequality we have

$$(D_iD_j\tilde{r}, I_{ij})_{s-2} \le C|q_\tau|\|D_iD_j\tilde{r}\|_{s-2}. \tag{53}$$

Using (46), the analyticity of $\mathcal{F}_1\circ\psi$, the fact that $\psi'(0)$ is the identity and $\psi(0)=0$, we get

$$\begin{aligned}&(D_iD_j\tilde{r}, J_{ij})_{s-2}\\ &\le (D_iD_j\tilde{r}, \mathcal{P}_1\mathcal{F}_1'(0)\,[D_iD_j\tilde{r}])_{s-2} + C\|\tilde{r}\|_{s-\frac{1}{2}}\|D_iD_j\tilde{r}\|^2_{s-\frac{1}{2}}\\ &\le -c_1\|D_iD_j\tilde{r}\|^2_{s-\frac{1}{2}} + C\|\tilde{r}\|_{s-\frac{1}{2}}\|\tilde{r}\|^2_{s+\frac{3}{2}}.\end{aligned} \tag{54}$$

There exists a $C>0$, such that for $\tilde{r}$ near the origin in $\mathbb{H}_1^{s-\frac{1}{2}}(\mathbb{S}^{N-1})$ we have $\|\mathcal{P}_1\circ\mathcal{F}_1'(\psi(\tilde{r}))\circ\psi''(\tilde{r})\|_X \le C$ for $X=\mathcal{L}^2(\mathbb{H}_1^{s-\frac{1}{2}}(\mathbb{S}^{N-1})\times\mathbb{H}_1^{s-\frac{1}{2}}(\mathbb{S}^{N-1}),\mathbb{H}_1^{s-\frac{7}{2}}(\mathbb{S}^{N-1}))$. Therefore we have

$$\|K_{ij}\|_{s-\frac{7}{2}} \le C\|\tilde{r}\|^2_{s+\frac{1}{2}}.$$

By an interpolation inequality we have

$$(D_iD_j\tilde{r}, K_{ij})_{s-2} \le C\|\tilde{r}\|_{s+\frac{3}{2}}\|\tilde{r}\|^2_{s+\frac{1}{2}} \le C\|\tilde{r}\|^2_{s+\frac{3}{2}}\|\tilde{r}\|_{s-\frac{1}{2}}. \tag{55}$$

Adding (53), (54), and (55) we get for (52)

$$\begin{aligned}&(D_iD_j\tilde{r}, I_{ij}+J_{ij}+K_{ij})_{s-2}\\ &\le -c_1\|D_iD_j\tilde{r}\|^2_{s-\frac{1}{2}} + C\Big(\|\tilde{r}\|^2_{s+\frac{3}{2}}\|\tilde{r}\|_{s-\frac{1}{2}} + |q_\tau|\|D_iD_j\tilde{r}\|_{s-2}\Big).\end{aligned}$$

For the terms of G_2 we get from similar arguments

$$\begin{aligned}&(D_iD_j\tilde{r}, \mathcal{P}_1\mathcal{F}_2'\left(\phi^{-1}\left(q_\tau,\tilde{r}\right)\right)\left[D_iD_j\phi^{-1}(q_\tau,\tilde{r})\right])_{s-2}\\ &\le C\Big(\|\tilde{r}\|^2_{s+\frac{1}{2}}\|\tilde{r}\|_{s-\frac{3}{2}} + |q_\tau|\|D_iD_j\tilde{r}\|_{s-2}\Big).\end{aligned}$$

Here, we used the estimate $(D_iD_j\tilde{r}, \mathcal{P}_1\mathcal{F}_2'(0)D_iD_j\tilde{r})_{s-2}\le 0$. This is a consequence of (20). Because $\alpha(\tau)^{3-N}\le 1$, we have

$$\begin{aligned}&\gamma G_1 + \mu\alpha(\tau)^{3-N}G_2\\ &\le -c_1\|D_iD_j\tilde{r}\|^2_{s-\frac{1}{2}} + C\Big(\|\tilde{r}\|^2_{s+\frac{3}{2}}\|\tilde{r}\|_{s-\frac{1}{2}} + |q_\tau|\|D_iD_j\tilde{r}\|_{s-2}\Big).\end{aligned} \tag{56}$$

8. From arguments that we used in Theorem 3.3 step 7 we obtain

$$\begin{aligned}\gamma H_1 + \mu\alpha(\tau)^{3-N} H_2 &\le C\|\tilde r\|_{s+\frac{3}{2}} \|\phi^{-1}(q_\tau, \tilde r)\|^2_{s+\frac{1}{2}} \\ &\le C\left(|q_\tau|^2 \|\tilde r\|_{s+\frac{3}{2}} + \|\tilde r\|^2_{s+\frac{1}{2}} \|\tilde r\|_{s+\frac{3}{2}}\right) \\ &\le C\left(|q_\tau|^2 \|\tilde r\|_{s+\frac{3}{2}} + \|\tilde r\|_{s-\frac{1}{2}} \|\tilde r\|^2_{s+\frac{3}{2}}\right). \qquad (57)\end{aligned}$$

Again we used an interpolation inequality.

9. Adding (51), (56), and (57) we get for (50)

$$\begin{aligned}&(\tilde r, \mathcal{P}_1 \mathcal{F}\left(\phi^{-1}\left(q_\tau, \tilde r\right), \tau\right))_{s-2,2} \\ &\le -c_1 \|\tilde r\|^2_{s-\frac{1}{2},2} + C\Big(|q_\tau| \|\tilde r\|_{s-2,2} + \delta \|\tilde r\|^2_{s-\frac{1}{2},2} + |q_\tau|^2 \|\tilde r\|_{s-\frac{1}{2},2}\Big) \\ &\le (-c_1 + C\delta) \|\tilde r\|^2_{s-\frac{1}{2},2} + C|q_\tau|^2 + \frac{c_1}{3} \|\tilde r\|^2_{s-2,2} + C|q_\tau|^4 + \frac{c_1}{3} \|\tilde r\|^2_{s-\frac{1}{2},2} \\ &\le (-\frac{c_1}{3} + C\delta) \|\tilde r\|^2_{s-\frac{1}{2},2} + C|q_\tau|^2.\end{aligned}$$

Here we used Cauchy's inequality and the fact that $|q_\tau|^4 \le |q_\tau|^2$ for small $|q_0|$. If we choose $\delta < \frac{\varepsilon}{C}$, then

$$\begin{aligned}(\tilde r, \mathcal{P}_1 \mathcal{F}\left(\phi^{-1}\left(q_\tau, \tilde r\right), \tau\right))_{s-2,2} &\le -\lambda_0 \|\tilde r\|^2_{s-\frac{1}{2},2} + C|q_\tau|^2 \\ &\le -\lambda_0 \|\tilde r\|^2_{s-\frac{1}{2},2} + C\frac{|q_0|^2}{\alpha(\tau)^{2N}}.\end{aligned}$$

10. Arguing as in the proof of Theorem 3.5 we get global existence of a solution $\tilde r$ of (45) for fixed q_0 and for $\tilde r(0) = \mathcal{P}_1 r_0$ small enough. Furthermore, we have $\|\tilde r(\tau)\|^2_s \le y(\tau)$ where $y : [0,\infty) \to \mathbb{R}$ satisfies

$$\frac{dy}{d\tau} = -2\lambda_0 y + C\frac{|q_0|^2}{\alpha(\tau)^{2N}},$$

with $y(0) = \|\mathcal{P}_1 r_0\|^2_s$. This ODE can be solved using the variation of constants formula:

$$y(\tau) = e^{-2\lambda_0 \tau} y(0) + C|q_0|^2 \int_0^\tau \frac{e^{2\lambda_0(\tilde\tau - \tau)}}{\alpha(\tilde\tau)^{2N}} d\tilde\tau.$$

We have

$$\begin{aligned}\int_0^\tau \frac{e^{2\lambda_0(\tilde\tau-\tau)}}{\alpha(\tilde\tau)^{2N}} d\tilde\tau &\le \int_0^{\frac{\tau}{2}} e^{2\lambda_0(\tilde\tau-\tau)} d\tilde\tau + \frac{1}{\alpha(\frac{\tau}{2})^{2N}} \int_{\frac{\tau}{2}}^\tau e^{2\lambda_0(\tilde\tau-\tau)} d\tilde\tau \\ &\le \frac{1}{2\lambda_0}\left(e^{-\lambda_0\tau} - e^{-2\lambda_0\tau} + \frac{1}{\alpha(\frac{\tau}{2})^{2N}}\right) \\ &\le \frac{C}{\alpha(\frac{\tau}{2})^{2N}} \le \frac{C}{\alpha(\tau)^{2N}}.\end{aligned}$$

We omitted the exponential terms because they are dominated by the algebraic terms. The result is

$$\|\tilde{r}(\tau)\|_s \leq Ce^{-\lambda_0\tau}\|\mathcal{P}_1 r_0\|_s + \frac{C}{\alpha(\tau)^N}|q_0|.$$

11. Now we construct a solution r of the original problem by setting

$$r(\tau) := \phi^{-1}(q_\tau, \tilde{r}(\tau)).$$

From the boundedness of ϕ^{-1} near the origin we get

$$\|r(\tau)\|_s \leq Ce^{-\lambda_0\tau}\|\mathcal{P}_1 r_0\|_s + \frac{C}{\alpha(\tau)^N}|q_0| \tag{58}$$

or

$$\begin{aligned}\|r(t)\|_s &\leq Ce^{-\lambda_0\tau(t)}\|\mathcal{P}_1 r_0\|_s + \frac{C}{\alpha(t)^N}|q_0| \\ &\leq \frac{C}{\alpha(t)^N}\left(|q_0| + \|\mathcal{P}_1 r_0\|_s\right) \\ &\leq \frac{C}{\alpha(t)^N}\|r_0\|_s.\end{aligned}$$

□

Note that because of (58), if we restrict ourselves to the case $r_0 \in \mathfrak{M}_1^{N-1}$, which means $q_0 = 0$, then we have faster convergence.

We do not treat the three dimensional problems here. Finding energy estimates for $N = 3$ is easier because the evolution operator does not depend on time. The results are similar to those that are found in [10] in a Hölder space setting.

4 Almost global existence results for the suction problems

In this section we find almost global existence results for the suction problems. Both the cases $N = 2$ and $N \geq 4$ will be treated. For almost global existence we do not need to restrict ourselves to the case $r(0) \in \mathfrak{M}_1^N$. Remember that

$$\tau_{\max} := \begin{cases} \infty & \text{for N = 2,} \\ \frac{\sigma_N}{|\mu|(N-3)} & \text{for N} \geq 4. \end{cases}$$

Theorem 4.1. *Let $N = 2$ or $N \geq 4$ and $\mu < 0$. Let $T_+ \in (0, \tau_{\max})$ and $s > \frac{N+8}{2}$. There exists a $\delta > 0$ such that if $r_0 \in \mathbb{H}^s(\mathbb{S}^{N-1})$ with $\|r_0\|_s < \delta$, then there exists a solution $r \in \mathcal{C}([0, T_+), \mathbb{H}^s(\mathbb{S}^{N-1})) \cap \mathcal{C}^1([0, T_+), \mathbb{H}^{s-3}(\mathbb{S}^{N-1}))$ of*

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r, \tau), \tag{59}$$

with $r(0) = r_0$. Furthermore, $((\xi, \tau) \mapsto r(\tau)(\xi)) \in \mathcal{C}^\infty(\mathbb{S}^{N-1} \times (0, T_+))$.

Proof. For the case $N = 2$, we argue as in the proof of Theorem 3.5. There, we split up the time interval in two parts. A different approach for the second time interval was necessary there, because we wanted to show global existence and exponential decay in τ assuming that $r \in \mathfrak{M}_1^2$. Here we only consider the first time interval, choosing $\hat{T} \geq T_+$. In the estimates in steps 3-8 of the proof of Theorem 3.5 we replace $\tilde{r}$ by r and $\tilde{\tilde{\mathcal{F}}}_k$ by $\mathcal{F}_k$. All estimates that are found for the evolution operators $\tilde{\mathcal{F}}_k$ $(k = 1, 2)$ on the first interval hold for the operators $\mathcal{F}_k$ as well, because up to equation (41) we did not use the fact that $r \in \mathfrak{M}_1^2$. In this way we derive that if $\|r\|_s < \delta'$, for δ' small, then we have

$$(r, \mathcal{F}(r, \tau))_{s-2,2} \leq C\|r\|_{s-2,2}^2.$$

We choose $\delta < \delta' e^{-CT_+}$ and use local existence results as before to prove the theorem for $N = 2$.

For $N \geq 4$, $\alpha(\tau)^{3-N}$ goes to infinity if τ approaches $\tau_{\max}$. However, on the time interval $[0, T_+)$, $\alpha(\tau)^{3-N}$ is bounded. Therefore we can use the same strategy as in the proof of Theorem 3.5 on the first of the two intervals to prove the theorem. □

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