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NONUNIFORM SMALL-GAIN THEOREMS FOR SYSTEMS WITH UNSTABLE INVARIANT SETS*

IVAN TYUKIN[†], ERIK STEUR[‡], HENK NIJMEIJER[‡], AND CEES VAN LEEUWEN[§]

Abstract. We consider the problem of asymptotic convergence to invariant sets in interconnected nonlinear dynamical systems. Standard approaches often require that the invariant sets be uniformly attracting, e.g., stable in the Lyapunov sense. This, however, is neither a necessary requirement nor is always useful. Systems may, for instance, be inherently unstable (e.g., intermittent, itinerant, meta-stable) or the problem statement may include requirements that cannot be satisfied with stable solutions. This is often the case in general optimization problems and in nonlinear parameter identification or adaptation. Conventional techniques for these cases either rely on detailed knowledge of the system's vector-fields or require boundedness of its states. The presently proposed method relies only on estimates of the input-output maps and steady-state characteristics. The method requires the possibility of representing the system as an interconnection of a stable and contracting part with an unstable and exploratory part. We illustrate with examples how the method can be applied to problems of analyzing the asymptotic behavior of locally unstable systems as well as to problems of parameter identification and adaptation in the presence of nonlinear parametrizations. The relation of our results to conventional small-gain theorems is discussed.

Key words. nonuniform convergence, weakly attracting sets, small-gain theorems, input-output stability

AMS subject classifications. 40A99, 34D05, 34D45, 93D25, 93B03, 93B30

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1. Notation. Throughout the paper we use the following notational conventions. The symbol \mathbb{R} denotes the field of real numbers; symbol \mathbb{R}_+ stands for the following subset of \mathbb{R} : $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$; and \mathbb{N} and \mathbb{Z} denote the set of natural numbers and its extension to the negative domain, respectively.

Let Ω be a set; by the symbol $\mathcal{S}\{\Omega\}$ we denote the set of all subsets of Ω . The symbol \mathcal{C}^k denotes the space of functions that are at least k times differentiable; \mathcal{K} denotes the class of all strictly increasing functions $\kappa : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\kappa(0) = 0$. If, in addition, $\lim_{s \rightarrow \infty} \kappa(s) = \infty$, we say that $\kappa \in \mathcal{K}_\infty$. Further, \mathcal{K}_e (or $\mathcal{K}_{e,\infty}$) denotes the class of functions of which the restriction to the interval $[0, \infty)$ belongs to \mathcal{K} (or \mathcal{K}_∞). The symbol \mathcal{KL} denotes the class of functions $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\beta(\cdot, s) \in \mathcal{K}$ and $\beta(r, \cdot)$ is monotonically decreasing for each $s, r \in \mathbb{R}_+$.

Let $\mathbf{x} \in \mathbb{R}^n$, and \mathbf{x} can be partitioned into two vectors $\mathbf{x}_1 \in \mathbb{R}^q$, $\mathbf{x}_1 = (x_{11}, \dots, x_{1q})^T$, $\mathbf{x}_2 \in \mathbb{R}^p$, $\mathbf{x}_2 = (x_{21}, \dots, x_{2p})^T$ with $q + p = n$; then \oplus denotes their concatenation: $\mathbf{x} = \mathbf{x}_1 \oplus \mathbf{x}_2$.

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The symbol $\|\mathbf{x}\|$ denotes the Euclidian norm in $\mathbf{x} \in \mathbb{R}^n$. By $L_\infty^n[t_0, T]$ we denote the space of all functions $\mathbf{f} : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ such that $\|\mathbf{f}\|_{\infty, [t_0, T]} = \sup\{\|\mathbf{f}(t)\|, t \in [t_0, T]\} < \infty$, and $\|\mathbf{f}\|_{\infty, [t_0, T]}$ stands for the $L_\infty^n[t_0, T]$ norm of $\mathbf{f}(t)$. Let \mathcal{A} be a set in \mathbb{R}^n and $\|\cdot\|$ be the usual Euclidian norm in \mathbb{R}^n . By the symbol $\|\cdot\|_{\mathcal{A}}$ we denote the following induced norm:

$$\|\mathbf{x}\|_{\mathcal{A}} = \inf_{\mathbf{q} \in \mathcal{A}} \{\|\mathbf{x} - \mathbf{q}\|\}.$$

Let $\Delta \in \mathbb{R}_+$; then the notation $\|\mathbf{x}\|_{\mathcal{A}_\Delta}$ stands for the following equality:

$$\|\mathbf{x}\|_{\mathcal{A}_\Delta} = \begin{cases} \|\mathbf{x}\|_{\mathcal{A}} - \Delta, & \|\mathbf{x}\|_{\mathcal{A}} > \Delta, \\ 0, & \|\mathbf{x}\|_{\mathcal{A}} \leq \Delta. \end{cases}$$

The symbol $\|\cdot\|_{\mathcal{A}_\infty, [t_0, t]}$ is defined as follows:

$$\|\mathbf{x}(\tau)\|_{\mathcal{A}_\infty, [t_0, t]} = \sup_{\tau \in [t_0, t]} \|\mathbf{x}(\tau)\|_{\mathcal{A}}.$$

2. Introduction. In many fields of science, such as systems and control theory, physics, chemistry, and biology, it is of fundamental importance to analyze the asymptotic behavior of dynamical systems. Most of these analyses are based around the concept of Lyapunov stability [15], [33], [32], i.e., continuity of the flow $\mathbf{x}(t, \mathbf{x}_0) : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow L_\infty^n[t_0, \infty]$ with respect to \mathbf{x}_0 [18], in combination with the standard notion of an *attracting set* [9], defined as follows.

DEFINITION 1. A set \mathcal{A} is an attracting set iff it is

- (i) closed, invariant, and
- (ii) for some neighborhood \mathcal{V} of \mathcal{A} and for all $\mathbf{x}_0 \in \mathcal{V}$ the following conditions hold:

$$(1) \quad \mathbf{x}(t, \mathbf{x}_0) \in \mathcal{V} \quad \forall t \geq 0;$$

$$(2) \quad \lim_{t \rightarrow \infty} \|\mathbf{x}(t, \mathbf{x}_0)\|_{\mathcal{A}} = 0.$$

Condition (1) in Definition 1 stipulates the existence of a trapping region \mathcal{V} which is a neighborhood of \mathcal{A} . Condition (2) assures convergence to \mathcal{A} . Due to condition (1), convergence to \mathcal{A} is uniform with respect to \mathbf{x}_0 in the neighborhood of \mathcal{A} ; i.e., every trajectory which starts in \mathcal{V} remains in \mathcal{V} for $t \geq 0$ and converges to \mathcal{A} at $t \rightarrow \infty$.

Although the conventional concepts of attracting set and Lyapunov stability are powerful in tandem in various applications, some problems cannot be solved within this framework. Condition (1), for example, could be violated in systems with intermittent, itinerant, or meta-stable dynamics. In general the condition does not hold when the system dynamics, loosely speaking, is exploring rather than contracting. Such systems appear naturally in the context of global optimization. For instance, in [22] finding the global minimum of a differentiable cost function $Q : \mathbb{R}^n \rightarrow \mathbb{R}_+$ in a bounded subset $\Omega_x \subset \mathbb{R}^n$ is achieved by splitting the search procedure into a locally attracting gradient \mathcal{S}_a and a wandering part \mathcal{S}_w :

$$(3) \quad \begin{aligned} \mathcal{S}_a : \dot{\mathbf{x}} &= -\mu_x \frac{\partial Q(\mathbf{x})}{\partial \mathbf{x}} + \mu_t T(t), \quad \mu_x, \mu_t \in \mathbb{R}_+, \\ \mathcal{S}_w : T(t) &= h\{t, \mathbf{x}(t)\}, \quad h : \mathbb{R}_+ \times L_\infty^n[t_0, t] \rightarrow L_\infty^n[t_0, t]. \end{aligned}$$

The trace function, $T(t)$, in (3) is supposed to cover (i.e., be dense in) the whole searching domain Ω_x . Even though the results in [22] are purely simulation studies,

they illustrate the superior performance of algorithms (3) in a variety of benchmark problems compared to standard local minimizers and classical methods of global optimization. Abandoning Lyapunov stability is likewise advantageous in problems of identification and adaptation in the presence of general nonlinear parametrization [28], in maneuvering and path searching [26], and in decision making in intelligent systems [30], [31]. Systems with attracting, yet unstable, invariant sets are relevant for modeling complex behavior in biological and physical systems [2]. Last but not least, Lyapunov-unstable attracting sets are relevant in problems of synchronization [5], [19], [27].¹

Even when it is appropriate to consider a system as stable, we may be limited in our success in meeting the requirement to identify a proper Lyapunov function. This is the case, for instance, when the system's dynamics is only partially known. Trading stability requirements for the sake of convergence might be a possible remedy. Known results in this direction can be found in [11], [21].²

In all the cases that are problematic under condition (1) of Definition 1, condition (2)—convergence of $\mathbf{x}(t, \mathbf{x}_0)$ to an invariant set \mathcal{A} —is still a requirement that has to be met. In order to treat these cases analytically we shall, first of all, move from the standard concept of attracting sets in Definition 1 to one that does not assume that the basin of attraction is necessarily a neighborhood of the invariant set \mathcal{A} . In other words we shall allow convergence which is not uniform in initial conditions. This requirement is captured by the concept of weak, or Milnor, attraction [17], defined as follows.

DEFINITION 2. *A set \mathcal{A} is weakly attracting, or Milnor attracting, iff*

- (i) *it is closed, invariant, and*
- (ii) *for some set \mathcal{V} (not necessarily a neighborhood of \mathcal{A}) with strictly positive measure and for all $\mathbf{x}_0 \in \mathcal{V}$ limiting relation (2) holds.*

Conventional methods such as La Salle's invariance principle [14] or center manifold theory [7] can, in principle, address the issue of convergence to weak equilibria. They do so, however, at the expense of requiring detailed knowledge of the vector-fields of the ordinary differential equations of the model. When such information is not available the system can be thought of as a mere interconnection of input-output maps. Small-gain theorems [34], [12] are usually efficient in this case. These results, however, apply only under the assumption of stability of each component in the interconnection.

In the present study we aim to find a proper balance between the generality of input-output approaches [34], [12] in the analysis of convergence and the specificity of the fundamental notions of limit sets and invariance that play a central role in [14], [7]. The object of our study is a class of systems that can be decomposed into an attracting, or stable, component \mathcal{S}_a and an exploratory, generally unstable, part \mathcal{S}_w . Typical systems of this class are nonlinear systems in cascaded form

$$(4) \quad \begin{aligned} \mathcal{S}_a : \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{z}), \\ \mathcal{S}_w : \dot{\mathbf{z}} &= \mathbf{q}(\mathbf{z}, \mathbf{x}), \end{aligned}$$

where the zero solution of the \mathbf{x} -subsystem is asymptotically stable in the absence of

¹See also [20], where the striking difference between stable and "almost stable" synchronization in terms of the coupling strengths for a pair of the Lorenz oscillators is demonstrated analytically.

²In Examples 1 and 2 in section 6, we demonstrate how explorative dynamics can solve the problem of simultaneous state and parameter observation for a system which cannot be transformed into a canonical adaptive observer form [3].

input \mathbf{z} , and the state of the \mathbf{z} -subsystem consists of functions of the type $\int_{t_0}^t \|\mathbf{x}(\tau)\| d\tau$. Even when both subsystems in (4) are stable and the \mathbf{x} -subsystem does not depend on state \mathbf{z} , the cascade can still be unstable [1]. We show, however, that for unstable interconnections (4), under certain conditions that involve only input-to-state properties of \mathcal{S}_a and \mathcal{S}_w , there is a set \mathcal{V} in the system state space such that trajectories starting in \mathcal{V} remain bounded. The result is formally stated in Theorem 3. In the case when an additional measure of invariance is defined for \mathcal{S}_a (in our case a steady-state characteristic), a weak, Milnor attracting set emerges. Its location is completely determined by the zeros of the steady-state response of system \mathcal{S}_a .

We demonstrate how this basic result can be used in problems of design and analysis of control systems and identification/adaptation algorithms. In particular, we present an adaptive observer of state and parameter values for uncertain systems which cannot be transformed into a canonic adaptive observer form [3]. In Examples 1 and 2 in section 6 we present an application of this result to the problem of reconstructing a dynamic model of neuronal cell activity.

The paper is organized as follows. In section 3 we formally state the problem and provide specific assumptions for the class of systems under consideration. Section 4 contains the main results of our present study. In section 5 we provide several corollaries of the main result that apply to specific problems. Section 6 contains examples, and section 7 concludes the paper. Proofs of all lemmas, theorems, and corollaries are provided in the appendix.

3. Problem formulation. Consider a system that can be decomposed into two interconnected subsystems, \mathcal{S}_a and \mathcal{S}_w :

$$(5) \quad \begin{aligned} \mathcal{S}_a &: (u_a, \mathbf{x}_0) \mapsto \mathbf{x}(t), \\ \mathcal{S}_w &: (u_w, \mathbf{z}_0) \mapsto \mathbf{z}(t), \end{aligned}$$

where $u_a \in \mathcal{U}_a \subseteq L_\infty[t_0, \infty]$, $u_w \in \mathcal{U}_w \subseteq L_\infty[t_0, \infty]$ are the spaces of inputs to \mathcal{S}_a and \mathcal{S}_w , respectively, $\mathbf{x}_0 \in \mathbb{R}^n$, $\mathbf{z}_0 \in \mathbb{R}^m$ represent initial conditions, and $\mathbf{x}(t) \in \mathcal{X} \subseteq L_\infty^n[t_0, \infty]$, $\mathbf{z}(t) \in \mathcal{Z} \subseteq L_\infty^m[t_0, \infty]$ are the system states.

System \mathcal{S}_a represents the contracting dynamics. More precisely, we require that \mathcal{S}_a is input-to-state stable³ [23] with respect to a compact set \mathcal{A} .

ASSUMPTION 1 (contracting dynamics).

$$(6) \quad \mathcal{S}_a : \|\mathbf{x}(t)\|_{\mathcal{A}} \leq \beta(\|\mathbf{x}(t_0)\|_{\mathcal{A}}, t - t_0) + c\|u_a(t)\|_{\infty, [t_0, t]} \quad \forall t_0 \in \mathbb{R}_+, t \geq t_0,$$

where the function $\beta(\cdot, \cdot) \in \mathcal{KL}$, and $c > 0$ is some positive constant.

The function $\beta(\cdot, \cdot)$ in (6) specifies the contraction property of the unperturbed dynamics of \mathcal{S}_a . In other words it models the rate with which the system forgets its initial conditions \mathbf{x}_0 , if left unperturbed. Propagation of the input to output is estimated in terms of a continuous mapping, $c\|u_a(t)\|_{\infty, [t_0, t]}$, which, in our case, is chosen for simplicity to be linear. Notice that this mapping should not necessarily be contracting. In what follows we will assume that the function $\beta(\cdot, \cdot)$ and constant c are known or can be estimated a priori.

For systems \mathcal{S}_a , of which a model is given by a system of ordinary differential equations

$$(7) \quad \dot{\mathbf{x}} = \mathbf{f}_x(\mathbf{x}, u_a), \quad \mathbf{f}_x(\cdot, \cdot) \in \mathcal{C}^1,$$

³In general, as will be demonstrated with examples, our analysis can be carried out for (integral) input-to-output/input-to-state stable systems as well.

Assumption 1 is equivalent, for instance, to the combination of the following properties:⁴

1. Let $u_a(t) \equiv 0$ for all t ; then the set \mathcal{A} is Lyapunov stable and globally attracting for (7).
2. For all $u_a \in \mathcal{U}_a$ and $\mathbf{x}_0 \in \mathbb{R}^n$ there exists a nondecreasing function $\kappa : \mathbb{R}_+ \rightarrow \mathbb{R}_+ : \kappa(0) = 0$ such that

$$\inf_{t \in [0, \infty)} \|\mathbf{x}(t)\|_{\mathcal{A}} \leq \kappa(\|u_a(t)\|_{\infty, [t_0, \infty)}).$$

The system \mathcal{S}_w stands for the searching or wandering dynamics. We will consider \mathcal{S}_w subject to the following conditions.

ASSUMPTION 2 (wandering dynamics). *The system \mathcal{S}_w is forward-complete:*

$$u_w(t) \in \mathcal{U}_w \Rightarrow \mathbf{z}(t) \in \mathcal{Z} \quad \forall t \geq t_0, t_0 \in \mathbb{R}_+,$$

and there exists an “output” function $h : \mathbb{R}^m \rightarrow \mathbb{R}$, and two “bounding” functions $\gamma_0 \in \mathcal{K}_{\infty, e}$, $\gamma \in \mathcal{K}_{\infty, e}$ such that the following integral inequality holds:

$$(8) \quad \mathcal{S}_w : \int_{t_0}^t \gamma_1(u_w(\tau))d\tau \leq h(\mathbf{z}(t_0)) - h(\mathbf{z}(t)) \leq \int_{t_0}^t \gamma_0(u_w(\tau))d\tau$$

$$\forall t \geq t_0, t_0 \in \mathbb{R}_+.$$

In the case when system \mathcal{S}_w is specified in terms of vector-fields

$$(9) \quad \dot{\mathbf{z}} = \mathbf{f}_z(\mathbf{z}, u_w), \mathbf{f}_z(\cdot, \cdot) \in \mathcal{C}^1,$$

Assumption 2 can be viewed, for example, as postulating the existence of a function $h : \mathbb{R}^m \rightarrow \mathbb{R}_+$ of which the evolution in time is a mere integration of the input $u_w(t)$. In general, for $u_w : u_w(t) \geq 0$ for all $t \in \mathbb{R}_+$, inequality (8) implies *monotonicity* of function $h(\mathbf{z}(t))$ in t . Regarding the function $\gamma_0(\cdot)$ in (8), we assume that for any $M \in \mathbb{R}_+$ there exists a function $\gamma_{0,1} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and a *nondecreasing* function $\gamma_{0,2} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$(10) \quad \gamma_0(a \cdot b) \leq \gamma_{0,1}(a) \cdot \gamma_{0,2}(b) \quad \forall a, b \in [0, M].$$

Requirement (10) is a technical assumption which will be used in the formulation and proof of the main results of the paper. Yet, it is not too restrictive; it holds, for instance, for a wide class of locally Lipschitz functions $\gamma_0(\cdot) : \gamma_0(a \cdot b) \leq L_0(M) \cdot (a \cdot b)$, $L_0(M) \in \mathbb{R}_+$. Another example for which the assumption holds is the class of polynomial functions $\gamma_0(\cdot) : \gamma_0(a \cdot b) = (a \cdot b)^p = a^p \cdot b^p$, $p > 0$. No further restrictions will be imposed a priori on $\mathcal{S}_a, \mathcal{S}_w$.

Now consider the interconnection of (6), (8) with coupling $u_a(t) = h(\mathbf{z}(t))$ and $u_s(t) = \|\mathbf{x}(t)\|_{\mathcal{A}}$. Equations for the combined system can be written as

$$(11) \quad \begin{aligned} \|\mathbf{x}(t)\|_{\mathcal{A}} &\leq \beta(\|\mathbf{x}(t_0)\|_{\mathcal{A}}, t - t_0) + c\|h(\mathbf{z}(t))\|_{\infty, [t_0, t]}, \\ \int_{t_0}^t \gamma_1(\|\mathbf{x}(\tau)\|_{\mathcal{A}})d\tau &\leq h(\mathbf{z}(t_0)) - h(\mathbf{z}(t)) \leq \int_{t_0}^t \gamma_0(\|\mathbf{x}(\tau)\|_{\mathcal{A}})d\tau. \end{aligned}$$

⁴For a comprehensive characterization of the input-to-state stability and detailed mathematical arguments we refer to the paper by Sontag and Wang [24].

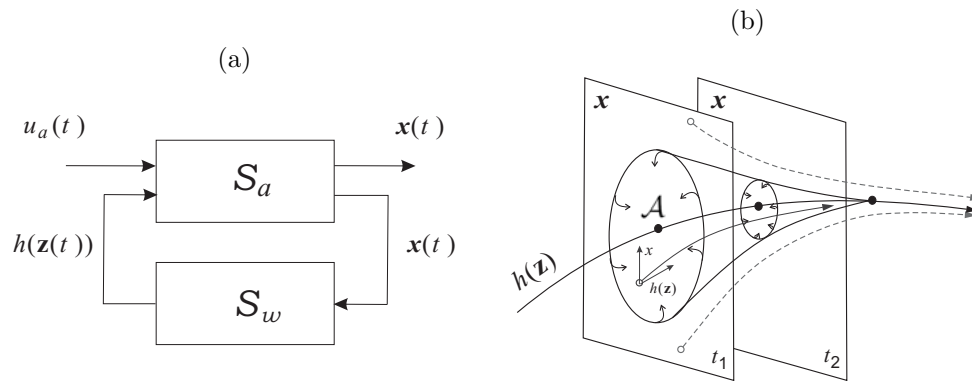


FIG. 1. The class of interconnected systems S_a and S_w (diagram (a)). System S_a , the “contracting system,” has an attracting invariant set \mathcal{A} in its state space. System S_w does not necessarily have an attracting set. This system represents the “wandering” dynamics. A typical example of such behavior is the dynamics of the flow in a neighborhood of a saddle point in three-dimensional space (diagram (b)).

A diagram illustrating the general structure of the entire system (11) is given in Figure 1.

Equations (11) capture the relevant interplay between contracting, S_a , and wandering, S_w , dynamics inherent in a variety of searching strategies in the realm of optimization, (3), and interconnections, (4), in general systems theory. In addition, this kind of interconnection describes the behavior of systems which undergo transcritical or saddle-node bifurcations. Consider, for instance, the following system:

$$(12) \quad \begin{aligned} \dot{x}_1 &= -x_1 + x_2, \\ \dot{x}_2 &= \varepsilon + \gamma x_1^2, \gamma > 0, \end{aligned}$$

where the parameter ε varies from negative to positive values. At $\varepsilon = 0$ stable and unstable equilibria collide, leading to the cascade satisfying (11). An alternative bifurcation scenario could be represented by the system

$$(13) \quad \begin{aligned} \dot{x}_1 &= -x_1 + x_2, \\ \dot{x}_2 &= \varepsilon + \gamma x_2^2, \gamma > 0. \end{aligned}$$

In this case, however, the dynamics of the variable x_2 is *independent* of x_1 , and analysis of the asymptotic behavior of (13) reduces to the analysis of each equation separately. Thus systems such as (13) are easier to deal with than (12). This constitutes an additional motivation for the present approach.

When analyzing the asymptotic behavior of interconnection (11) we will address the following question: Is there a set (a weak trapping set in the system state space) such that the trajectories which start in this set are bounded? It is natural to expect that the existence of such a set depends on the specific functions $\gamma_0(\cdot)$, $\gamma_1(\cdot)$ in (11), on properties of $\beta(\cdot, \cdot)$, and on values of c . In the case when such a set exists and could be defined, the next questions are, therefore, where will the trajectories converge and how can these domains be characterized?

4. Main results. In this section we provide a formal statement of the main results of our present study. In section 4.1, we formulate conditions ensuring that

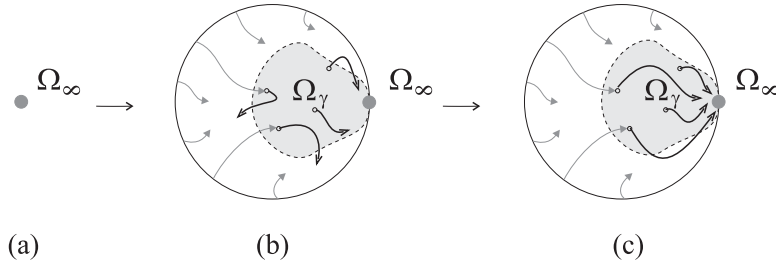


FIG. 2. Emergence of a weak (Milnor) attracting set Ω_∞ . Panel (a) depicts the target invariant set Ω_∞ as a filled circle. First (see Theorem 3), we investigate whether a domain $\Omega_\gamma \subset \mathbb{R}^n \times \mathbb{R}^m$ exists such that $\|\mathbf{x}(t)\|_{\mathcal{A}}, h(\mathbf{z}(t))$ are bounded for all $\mathbf{x}_0 \oplus \mathbf{z}_0 \in \Omega_\gamma$. In the text we refer to this set as a weak trapping region or simply a trapping region. The trapping region is shown as a grey domain in panel (b). In principle, the system's states can eventually leave the domain Ω_γ . They must, however, satisfy (14), ensuring boundedness of $\|\mathbf{x}(t)\|_{\mathcal{A}}, h(\mathbf{z}(t))$. As a result they will dwell within the region shown as a circle in panel (b). Notice that neither this domain nor the previous need be neighborhoods of Ω_∞ . Second (see Lemmas 6 and 7 and Corollary 8), we provide conditions which lead to the emergence of a weak attracting set in the trapping region Ω_γ . This is illustrated in panel (c).

there exists a point $\mathbf{x}_0 \oplus \mathbf{z}_0$ such that the ω -limit set of $\mathbf{x}_0 \oplus \mathbf{z}_0$ ⁵ is bounded in the following sense:

$$(14) \quad \|\omega_{\mathbf{x}}(\mathbf{x}_0 \oplus \mathbf{z}_0)\|_{\mathcal{A}} < \infty, \quad |h(\omega_{\mathbf{z}}(\mathbf{x}_0 \oplus \mathbf{z}_0))| < \infty.$$

These conditions and a specification of the set Ω_γ of points $\mathbf{x}' \oplus \mathbf{z}'$ for which the ω -limit set satisfies property (14) are provided in Theorem 3.

In order to verify whether an attracting set exists in $\omega(\Omega_\gamma)$ that is a subset of $\omega(\Omega_\gamma)$ we use an additional characterization of the contracting system \mathcal{S}_a . In particular, we introduce the intuitively clear notion of the input-to-state *steady-state characteristics*⁶ of a system. It is possible to show that in the case when system \mathcal{S}_a has a steady-state characteristic, there exists an attracting set in $\omega(\Omega_\gamma)$, and this set is uniquely defined by the zeros of the steady-state characteristics of \mathcal{S}_a . A diagram illustrating the steps of our analysis is provided in Figure 2, along with the sequence of conditions leading to the emergence of the attracting set in (11).

4.1. Emergence of the trapping region. Small-gain conditions. Before we formulate the main results of this section let us first comment briefly on the machinery of our analysis. First, we introduce three sequences

$$\begin{aligned} \mathcal{S} &= \{\sigma_i\}_{i=0}^\infty, \quad \sigma_i \in \mathbb{R}_+, \\ \Xi &= \{\xi_i\}_{i=0}^\infty, \quad \xi_i \in \mathbb{R}_+, \\ \mathcal{T} &= \{\tau_i\}_{i=0}^\infty, \quad \tau_i \in \mathbb{R}_+. \end{aligned}$$

The first sequence, \mathcal{S} , partitions the interval $[0, h(\mathbf{z}_0)]$, $h(\mathbf{z}_0) > 0$, into the union of shrinking subintervals H_i :

$$(15) \quad [0, h(\mathbf{z}_0)] = \cup_{i=0}^\infty H_i, \quad H_i = [\sigma_{i+1}h(\mathbf{z}_0), \sigma_i h(\mathbf{z}_0)].$$

⁵Recall that in our current notation a point $\mathbf{p} \in \mathbb{R}^{m+n}$ is an ω -limit point of $\mathbf{x}' \oplus \mathbf{z}'$ if there exists a sequence $\{t_i\}$, $i = 1, 2, \dots$, such that $\lim_{i \rightarrow \infty} t_i = \infty$ and $\lim_{i \rightarrow \infty} \mathbf{x}(t_i, \mathbf{x}' \oplus \mathbf{z}') \oplus \mathbf{z}(t_i, \mathbf{x}' \oplus \mathbf{z}') = \mathbf{p}$, where $\mathbf{x}(t, \mathbf{x}' \oplus \mathbf{z}') \oplus \mathbf{z}(t, \mathbf{x}' \oplus \mathbf{z}')$ denotes the flow of interconnection (11). A set of all ω -limit points of $\mathbf{x}' \oplus \mathbf{z}'$ is an ω -limit set of $\mathbf{x}' \oplus \mathbf{z}'$.

⁶A more precise definition of the steady-state characteristics is given in section 4.2.

For the sake of clarity, let us define this property formally in the form of Property 1 as follows.

PROPERTY 1 (partition of \mathbf{z}_0). *The sequence \mathcal{S} is strictly monotone and converging:*

$$(16) \quad \{\sigma_n\}_{n=0}^\infty : \lim_{n \rightarrow \infty} \sigma_n = 0, \sigma_0 = 1.$$

Sequences Ξ and \mathcal{T} will specify the desired rates $\xi_i \in \Xi$ of the contracting dynamics (6) in terms of function $\beta(\cdot, \cdot)$ and $\tau_i \in \mathcal{T}$. Let us, therefore, impose the following constraint on the choice of Ξ, \mathcal{T} .

PROPERTY 2 (rate of contraction, part 1). *Sequences Ξ and \mathcal{T} are such that for the given function $\beta(\cdot, \cdot) \in \mathcal{K}\mathcal{L}$ in (6) the following inequality holds:*

$$(17) \quad \beta(\cdot, T) \leq \xi_i \beta(\cdot, 0) \quad \forall T \geq \tau_i.$$

Property 2 states that for the given, yet arbitrary, factor ξ_i and time instant t_0 , time τ_i is needed for the state \mathbf{x} in order to reach the domain:

$$\|\mathbf{x}\|_{\mathcal{A}} \leq \xi_i \beta(\|\mathbf{x}(t_0)\|_{\mathcal{A}}, 0).$$

In order to specify the desired convergence rates ξ_i , it will be necessary to define another measure in addition to (17). This is a measure of the propagation of initial conditions \mathbf{x}_0 and input $h(\mathbf{z}_0)$ to the state $\mathbf{x}(t)$ of the contracting dynamics (6) when the system travels in $h(\mathbf{z}(t)) \in [0, h(\mathbf{z}_0)]$. For this reason we introduce two systems of functions, Φ and Υ :

$$(18) \quad \Phi : \begin{aligned} \phi_j(s) &= \phi_{j-1} \circ \rho_{\phi,j}(\xi_{i-j} \cdot \beta(s, 0)), \quad j = 1, \dots, i, \\ \phi_0(s) &= \beta(s, 0), \end{aligned}$$

$$(19) \quad \Upsilon : \begin{aligned} v_j(s) &= \phi_{j-1} \circ \rho_{v,j}(s), \quad j = 1, \dots, i, \\ v_0(s) &= \beta(s, 0), \end{aligned}$$

where the functions $\rho_{\phi,j}, \rho_{v,j} \in \mathcal{K}$ satisfy the following inequality:

$$(20) \quad \phi_{j-1}(a + b) \leq \phi_{j-1} \circ \rho_{\phi,j}(a) + \phi_{j-1} \circ \rho_{v,j}(b).$$

Notice that in the case when $\beta(\cdot, 0) \in \mathcal{K}_\infty$, the functions $\rho_{\phi,j}(\cdot), \rho_{v,j}(\cdot)$ will always exist [12]. The properties of sequence Ξ which ensure the desired propagation rate of the influence of initial condition \mathbf{x}_0 and input $h(\mathbf{z}_0)$ to the state $\mathbf{x}(t)$ are specified in Property 3.

PROPERTY 3 (rate of contraction, part 2). *The sequences*

$$\sigma_n^{-1} \cdot \phi_n(\|\mathbf{x}_0\|_{\mathcal{A}}), \quad \sigma_n^{-1} \cdot \left(\sum_{i=0}^n v_i(c|h(\mathbf{z}_0)|\sigma_{n-i}) \right), \quad n = 0, \dots, \infty,$$

are bounded from above; e.g., there exist functions $B_1(\|\mathbf{x}_0\|), B_2(|h(\mathbf{z}_0)|, c)$ such that

$$(21) \quad \sigma_n^{-1} \cdot \phi_n(\|\mathbf{x}_0\|_{\mathcal{A}}) \leq B_1(\|\mathbf{x}_0\|_{\mathcal{A}}),$$

$$(22) \quad \sigma_n^{-1} \cdot \left(\sum_{i=0}^n v_i(c|h(\mathbf{z}_0)|\sigma_{n-i}) \right) \leq B_2(|h(\mathbf{z}_0)|, c)$$

for all $n = 0, 1, \dots, \infty$.

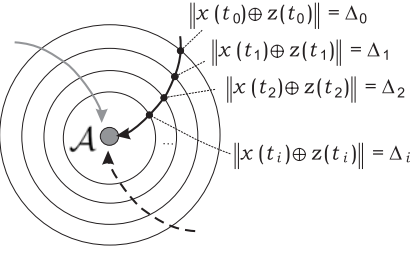
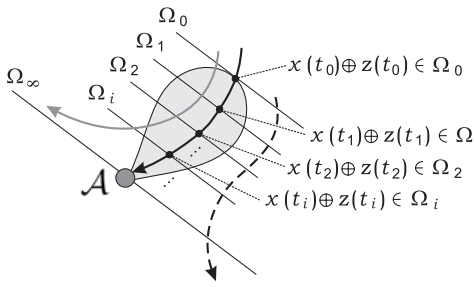
Standard	Proposed
<p>1) Domain of attraction is a neighborhood</p> <p>2) Implies Lyapunov stability</p> 	<p>1) Domain of attraction is a set of positive measure (not necessarily a neighborhood)</p> <p>2) Allows us to analyze convergence in Lyapunov-unstable systems</p> 
<p>Given: a sequence of diverging time instances t_i</p>	<p>Given: a sequence of sets Ω_i whose distance Δ_i to \mathcal{A} is converging to zero</p>
<p>Prove: convergence of norms $\ \mathbf{x}(t_i) \oplus \mathbf{z}(t_i)\ = \Delta_i$ to zero</p>	<p>Prove: divergence of $\{t_i\}$, where $t_i : \mathbf{x}(t_i) \oplus \mathbf{z}(t_i) \in \Omega_i$</p>

FIG. 3. Key differences between the conventional concept of convergence (left panel) and the concept of weak, nonuniform convergence (right panel). In the uniform case, trajectories which start in a neighborhood of \mathcal{A} remain in a neighborhood of \mathcal{A} (solid and dashed lines). In the nonuniform case, only a fraction of the initial conditions in a neighborhood of \mathcal{A} will produce trajectories which remain in a neighborhood of \mathcal{A} (solid black line). In the most general case a necessary condition for this to happen is that the sequence $\{t_i\}$ diverges. In our current problem statement divergence of $\{t_i\}$ implies boundedness of $\|\mathbf{x}(t)\|_{\mathcal{A}}$. To show state boundedness and convergence of $\mathbf{x}(t)$ to \mathcal{A} , additional information on the system dynamics will be required.

For a large class of functions $\beta(s, 0)$, for instance those that are Lipschitz in s , these conditions reduce to more transparent ones which can always be satisfied by an appropriate choice of sequences Ξ and \mathcal{S} . This case is considered in detail as a corollary of our main results in section 4.3.

The main differences between the standard and the presently proposed approaches for the analysis of asymptotic behavior of dynamical systems are illustrated in Figure 3. In order to prove the emergence of the trapping region we consider the following collection of volumes induced by the sequence \mathcal{S}_i and the corresponding partition (15) of the interval $[0, h(\mathbf{z}_0)]$:

$$(23) \quad \Omega_i = \{\mathbf{x} \in \mathcal{X}, \mathbf{z} \in \mathcal{Z} \mid h(\mathbf{z}(t)) \in H_i\}.$$

For the given initial conditions $\mathbf{x}_0 \in \mathcal{X}$, $\mathbf{z}_0 \in \mathcal{Z}$ two alternative possibilities exist. First, there exists an i such that the trajectory $\mathbf{x}(t, \mathbf{x}_0) \oplus \mathbf{z}(t, \mathbf{z}_0)$ enters Ω_i and stays there forever. Hence for $t \rightarrow \infty$ the state will converge into

$$(24) \quad \Omega_a = \{\mathbf{x} \in \mathcal{X}, \mathbf{z} \in \mathcal{Z} \mid \|\mathbf{x}\|_{\mathcal{A}} \leq c \cdot h(\mathbf{z}_0), \mathbf{z} : h(\mathbf{z}) \in [0, h(\mathbf{z}_0)]\}.$$

The second alternative is that for each $i = 0, 1, \dots$ the trajectory $\mathbf{x}(t, \mathbf{x}_0) \oplus \mathbf{z}(t, \mathbf{z}_0)$ enters Ω_i and leaves some time later. Let t_i be the time instances when it hits

the hypersurfaces $h(\mathbf{z}(t)) = h(\mathbf{z}_0)\sigma_i$. Then the state of the coupled system stays in $\cup_{i=0}^{\infty} \Omega_i$ only if the sequence $\{t_i\}_{i=0}^{\infty}$ diverges. Theorem 3 provides sufficient conditions specifying the latter case in terms of the properties of sequences \mathcal{S} , Ξ , \mathcal{T} and function $\gamma_0(\cdot)$ in (11). For a large class of interconnections (11) it is possible to formulate these conditions in terms of the input-output properties of systems \mathcal{S}_a and \mathcal{S}_w explicitly, i.e., in terms of functions $\beta(\cdot, \cdot)$ and $\gamma_0(\cdot)$ and the values of c . The results are presented as immediate corollaries of Theorem 3 in sections 4.3 and 5.1.

THEOREM 3 (nonuniform small-gain theorem). *Let systems \mathcal{S}_a , \mathcal{S}_w be given and satisfy Assumptions 1, 2. Consider their interconnection (11) and suppose there exist sequences \mathcal{S} , Ξ , and \mathcal{T} satisfying Properties 1–3. In addition, suppose that the following conditions hold:*

1. *There exists a positive number $\Delta_0 > 0$ such that*

$$(25) \quad \frac{1}{\tau_i} \frac{(\sigma_i - \sigma_{i+1})}{\gamma_{0,1}(\sigma_i)} \geq \Delta_0 \quad \forall i = 0, 1, \dots, \infty.$$

2. *The set Ω_γ of all points \mathbf{x}_0 , \mathbf{z}_0 satisfying the inequality*

$$(26) \quad \gamma_{0,2}(B_1(\|\mathbf{x}_0\|_{\mathcal{A}}) + B_2(|h(\mathbf{z}_0)|, c) + c|h(\mathbf{z}_0)|) \leq h(\mathbf{z}_0)\Delta_0$$

is not empty.

3. *Partial sums of elements from \mathcal{T} diverge:*

$$(27) \quad \sum_{i=0}^{\infty} \tau_i = \infty.$$

Then for all \mathbf{x}_0 , $\mathbf{z}_0 \in \Omega_\gamma$ the state $\mathbf{x}(t, \mathbf{z}_0) \oplus \mathbf{z}(t, \mathbf{z}_0)$ of system (11) converges into the set specified by (24):

$$\Omega_a = \{\mathbf{x} \in \mathcal{X}, \mathbf{z} \in \mathcal{Z} \mid \|\mathbf{x}\|_{\mathcal{A}} \leq c \cdot h(\mathbf{z}_0), \mathbf{z} : h(\mathbf{z}) \in [0, h(\mathbf{z}_0)]\}.$$

The proofs of Theorem 3 and subsequent results are provided in the appendix.

The major difference between the conditions of Theorem 3 and those of conventional small-gain theorems [34], [12] is that the latter involve only input-output or input-state mappings. Formulating conditions for state boundedness of the interconnection in terms of input-output or input-state mappings is possible in the traditional case because the interconnected systems are assumed to be input-to-state stable. Hence their internal dynamics can be neglected. In our case, however, the dynamics of \mathcal{S}_w is generally unstable in the Lyapunov sense. Hence, in order to ensure boundedness of $\mathbf{x}(t, \mathbf{x}_0)$ and $h(\mathbf{z}(t, \mathbf{z}_0))$, the rate/degree of stability of \mathcal{S}_a should be taken into account. Roughly speaking, system \mathcal{S}_a should ensure a sufficiently high degree of contraction in \mathbf{x}_0 while the input-output response of \mathcal{S}_w should be sufficiently small. The rate of contraction in \mathbf{x}_0 of \mathcal{S}_a , according to (6), is specified in terms of the function $\beta(\cdot, \cdot)$. Properties of this function that are relevant for convergence are explicitly accounted for in Property 3 and (27). The domain of admissible initial conditions, and actually the small-gain condition (input-state-output properties of \mathcal{S}_w and \mathcal{S}_a), are defined by (25), (26), respectively. Notice also that Ω_γ is not necessarily a neighborhood of Ω_a ; thus the convergence ensured by Theorem 3 is allowed to be nonuniform in \mathbf{x}_0 , \mathbf{z}_0 .

4.2. Characterization of the attracting set. Even for interconnections of Lyapunov-stable systems, small-gain conditions usually are effective merely for establishing boundedness of states or outputs. Yet, even in the setting of Theorem 3 it is still possible to derive estimates (such as, for instance, (24)) of the domains to which the state will converge. These estimates, however, are often too conservative. If a more precise characterization of these domains is required, additional information on the dynamics of systems \mathcal{S}_a and \mathcal{S}_w will be needed. The question, therefore, is how detailed this information should be. It appears that some additional knowledge of the steady-state characteristics of system \mathcal{S}_a is sufficient to improve the estimates (24) substantially.

Let us formally introduce the notion of steady-state characteristic as follows.

DEFINITION 4. We say that system (6) has steady-state characteristic $\chi : \mathbb{R} \rightarrow \mathcal{S}\{\mathbb{R}_+\}$ with respect to the norm $\|\mathbf{x}\|_{\mathcal{A}}$ iff for each constant \bar{u}_a the following holds:

$$(28) \quad \forall u_a(t) \in \mathcal{U}_a : \lim_{t \rightarrow \infty} u_a(t) = \bar{u}_a \Rightarrow \lim_{t \rightarrow \infty} \|\mathbf{x}(t)\|_{\mathcal{A}} \in \chi(\bar{u}_a).$$

The key property captured by Definition 4 is that there exists a limit of $\|\mathbf{x}(t)\|_{\mathcal{A}}$ as $t \rightarrow \infty$, provided that the limit for $u_a(t)$, $t \rightarrow \infty$, is defined and constant. Notice that the mapping χ is set-valued. This means that for each \bar{u}_a there is a set $\chi(\bar{u}_a) \subset \mathbb{R}_+$ such that $\|\mathbf{x}(t)\|_{\mathcal{A}}$ converges to an element of $\chi(\bar{u}_a)$ as $t \rightarrow \infty$. Therefore, our definition allows a fairly large amount of uncertainty for \mathcal{S}_a . It will be of essential importance, however, that such a characterization exists for the system \mathcal{S}_a .

Clearly, not every system obeys a steady-state characteristic $\chi(\cdot)$ of Definition 4. There are relatively simple systems of which the state does not converge even in the “norm” sense for constant converging inputs (condition (28)). In mechanics, physics, and biology such systems encompass the large class of nonlinear oscillators which can be excited by constant inputs. In order to take such systems into consideration, we introduce a weaker notion, that of a steady-state characteristic *on average*, defined as follows.

DEFINITION 5. We say that system (6) has steady-state characteristic on average $\chi_T : \mathbb{R} \rightarrow \mathcal{S}\{\mathbb{R}_+\}$ with respect to the norm $\|\mathbf{x}\|_{\mathcal{A}}$ iff for each constant \bar{u}_a and some $T > 0$ the following holds:

$$(29) \quad \forall u_a(t) \in \mathcal{U}_a : \lim_{t \rightarrow \infty} u_a(t) = \bar{u}_a \Rightarrow \lim_{t \rightarrow \infty} \int_t^{t+T} \|\mathbf{x}(\tau)\|_{\mathcal{A}} d\tau \in \chi_T(\bar{u}_a).$$

Steady-state characterizations of system \mathcal{S}_a allow us to further specify the asymptotic behavior of interconnection (11). These results are summarized in Lemmas 6 and 7 below.

LEMMA 6. Let system (11) be given and $h(\mathbf{z}(t, \mathbf{z}_0))$ be bounded for some $\mathbf{x}_0, \mathbf{z}_0$. Let, furthermore, system (6) have steady-state characteristic $\chi(\cdot) : \mathbb{R} \rightarrow \mathcal{S}\{\mathbb{R}_+\}$. Then the following limiting relations hold:⁷

$$(30) \quad \lim_{t \rightarrow \infty} \|\mathbf{x}(t, \mathbf{x}_0)\|_{\mathcal{A}} = 0, \quad \lim_{t \rightarrow \infty} h(\mathbf{z}(t, \mathbf{z}_0)) \in \chi^{-1}(0).$$

As follows from Lemma 6, in the case when the steady-state characteristic of \mathcal{S}_a is defined, the asymptotic behavior of interconnection (11) is characterized by the zeros

⁷The symbol $\chi^{-1}(0)$ in (30) denotes the set $\chi^{-1}(0) = \bigcup_{\bar{u}_a \in \mathbb{R}_+} \bar{u}_a : \chi(\bar{u}_a) \ni 0$.

of the steady-state mapping $\chi(\cdot)$. For the steady-state characteristics on average a slightly modified conclusion can be derived.

LEMMA 7. *Let system (11) be given, $h(\mathbf{z}(t, \mathbf{z}_0))$ be bounded for some $\mathbf{x}_0, \mathbf{z}_0$, $h(\mathbf{z}(t, \mathbf{z}_0)) \in [0, h(\mathbf{z}_0)]$, and system (6) have steady-state characteristic $\chi_T(\cdot) : \mathbb{R} \rightarrow \mathcal{S}\{\mathbb{R}_+\}$ on average. Furthermore, let there exist a positive constant $\bar{\gamma}$ such that the function $\gamma_1(\cdot)$ in (8) satisfies the following constraint:*

$$(31) \quad \gamma_1(s) \geq \bar{\gamma} \cdot s \quad \forall s \in [0, \bar{s}], \bar{s} \in \mathbb{R}_+ : \bar{s} > c \cdot h(\mathbf{z}_0).$$

In addition, suppose that $\chi_T(\cdot)$ has no zeros in the positive domain, i.e., $0 \notin \chi_T(\bar{u}_a)$ for all $\bar{u}_a > 0$. Then

$$(32) \quad \lim_{t \rightarrow \infty} \|\mathbf{x}(t, \mathbf{x}_0)\|_{\mathcal{A}} = 0, \quad \lim_{t \rightarrow \infty} h(\mathbf{z}(t, \mathbf{z}_0)) = 0.$$

An immediate outcome of Lemmas 6 and 7 is that in the case when the conditions of Theorem 3 are satisfied and system (6) has steady-state characteristic $\chi(\cdot)$ or $\chi_T(\cdot)$, the domain of convergence Ω_a becomes

$$(33) \quad \Omega_a = \{\mathbf{x} \in \mathcal{X}, \mathbf{z} \in \mathcal{Z} \mid \|\mathbf{x}\|_{\mathcal{A}} = 0, \mathbf{z} : h(\mathbf{z}) \in [0, h(\mathbf{z}_0)]\}.$$

It is possible, however, to improve estimate (33) further under additional hypotheses on system \mathcal{S}_a and \mathcal{S}_w dynamics. This result is formulated in the corollary below.

COROLLARY 8. *Let system (11) be given and satisfy the assumptions of Theorem 3. Let, in addition,*

(C1) *the flow $\mathbf{x}(t, \mathbf{x}_0) \oplus \mathbf{z}(t, \mathbf{z}_0)$ be generated by a system of autonomous differential equations with a locally Lipschitz right-hand side;*

(C2) *subsystem \mathcal{S}_w be practically integral-input-to-state stable:*

$$(34) \quad \|\mathbf{z}(\tau)\|_{\infty, [t_0, t]} \leq C_z + \int_0^t \gamma_1(u_w(\tau)) d\tau,$$

and let function $h(\cdot) \in \mathcal{C}^0$ in (8);

(C3) *system \mathcal{S}_a have steady-state characteristic $\chi(\cdot)$.*

Then for all $\mathbf{x}_0, \mathbf{z}_0 \in \Omega_\gamma$ the state of the interconnection converges to the set

$$(35) \quad \Omega_a = \{\mathbf{x} \in \mathcal{X}, \mathbf{z} \in \mathcal{Z} \mid \|\mathbf{x}\|_{\mathcal{A}} = 0, h(\mathbf{z}) \in \chi^{-1}(0)\}.$$

As follows from Corollary 8, zeros of the steady-state characteristic of system \mathcal{S}_a actually “control” the domains to which the state of interconnection (11) might potentially converge. This is illustrated in Figure 4. Notice also that in the case when condition C3 in Corollary 8 is replaced with the alternative,

(C3)' *system \mathcal{S}_a has a steady-state characteristic on average $\chi_T(\cdot)$,*

and then it is possible to show that the state converges to

$$(36) \quad \Omega_a = \{\mathbf{x} \in \mathcal{X}, \mathbf{z} \in \mathcal{Z} \mid \|\mathbf{x}\|_{\mathcal{A}} = 0, h(\mathbf{z}) = 0\}.$$

The proof follows straightforwardly from the proof of Corollary 8 and is therefore omitted.

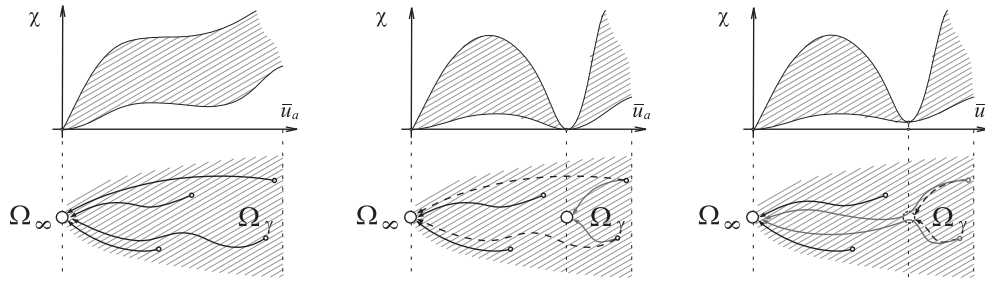


FIG. 4. Control of the attracting set by means of the system's steady-state characteristics.

4.3. Systems with contracting dynamics separable in space-time. In the previous sections we have presented convergence tests and estimates of the trapping region, and also characterized the attracting sets of interconnection (11) under assumptions of uniform asymptotic stability of \mathcal{S}_a and input-output properties (8), (34) of system \mathcal{S}_w . The conditions are given for rather general functions $\beta(\cdot, \cdot) \in \mathcal{KL}$ in (6) and $\gamma_0(\cdot), \gamma_1(\cdot)$ in (8). It appears, however, that these conditions can be substantially simplified if additional properties of $\beta(\cdot, \cdot)$ and $\gamma_0(\cdot)$ are available. This information is, in particular, the separability of function $\beta(\cdot, \cdot)$ or, equivalently, the possibility of factorization:

$$(37) \quad \beta(\|\mathbf{x}\|_{\mathcal{A}}, t) \leq \beta_x(\|\mathbf{x}\|_{\mathcal{A}}) \cdot \beta_t(t),$$

where $\beta_x(\cdot) \in \mathcal{K}$ and $\beta_t(\cdot) \in \mathcal{C}^0$ is strictly decreasing⁸ with

$$(38) \quad \lim_{t \rightarrow \infty} \beta_t(t) = 0.$$

In principle, as shown in [8], factorization (37) is achievable for a large class of uniformly asymptotically stable systems under an appropriate coordinate transformation. An immediate consequence of factorization (37) is that the elements of sequence Ξ in Property 2 are independent of $\|\mathbf{x}(t_i)\|_{\mathcal{A}}$. As a result, verification of Properties 2, 3 becomes easier. The most interesting case, however, occurs when the function $\beta_x(\cdot)$ in the factorization (37) is Lipschitz. For this class of functions the conditions of Theorem 3 reduce to a single and easily verifiable inequality. Let us consider this case in detail.

Without loss of generality, we assume that the state $\mathbf{x}(t)$ of system \mathcal{S}_a satisfies the equation

$$(39) \quad \|\mathbf{x}(t)\|_{\mathcal{A}} \leq \|\mathbf{x}(t_0)\|_{\mathcal{A}} \cdot \beta_t(t - t_0) + c \cdot \|h(\mathbf{z}(\tau, \mathbf{z}_0))\|_{\infty, [t_0, t]},$$

where $\beta_t(0)$ is greater than or equal to one. Given that $\beta_t(t)$ is strictly decreasing, the mapping $\beta_t : [0, \infty] \mapsto [0, \beta_t(0)]$ is injective. Moreover $\beta_t(t)$ is continuous, and then it is surjective and, therefore, bijective. In other words there is a (continuous) mapping $\beta_t^{-1} : [0, \beta_t(0)] \mapsto \mathbb{R}_+$:

$$(40) \quad \beta_t^{-1} \circ \beta_t(t) = t \quad \forall t > 0.$$

Conditions for emergence of the trapping region for interconnection (11) with dynamics of system \mathcal{S}_a governed by (39) are summarized below:

⁸If $\beta_t(\cdot)$ is not strictly monotone, it can always be majorized by a strictly decreasing function.

COROLLARY 9. *Let the interconnection (11) be given, system \mathcal{S}_a satisfy (39), and function $\gamma_0(\cdot)$ in (8) be Lipschitz:*

$$(41) \quad |\gamma_0(s)| \leq D_{\gamma,0} \cdot |s|.$$

Domain

$$(42) \quad \Omega_\gamma : D_{\gamma,0} \leq \left(\beta_t^{-1} \left(\frac{d}{\kappa} \right) \right)^{-1} \frac{\kappa - 1}{\kappa} \\ \times \frac{h(\mathbf{z}_0)}{\beta_t(0) \|\mathbf{x}_0\|_{\mathcal{A}} + \beta_t(0) \cdot c \cdot |h(\mathbf{z}_0)| \left(1 + \frac{\kappa}{1-d} \right) + c|h(\mathbf{z}_0)|}$$

is not empty for some $d < 1$, $\kappa > 1$. Then for all initial conditions $\mathbf{x}_0, \mathbf{z}_0 \in \Omega_\gamma$ the state $\mathbf{x}(t, \mathbf{x}_0) \oplus \mathbf{z}(t, \mathbf{z}_0)$ of interconnection (11) converges into the set Ω_a specified by (24). If, in addition, conditions (C1)–(C3) of Corollary 8 hold, then the domain of convergence is given by (33).

A practically important consequence of this corollary concerns systems \mathcal{S}_a which are exponentially stable:

$$(43) \quad \|\mathbf{x}(t)\|_{\mathcal{A}} \leq \|\mathbf{x}(t_0)\|_{\mathcal{A}} D_\beta \exp(-\lambda t) + c \cdot \|h(\mathbf{z}(t, \mathbf{z}_0))\|_{\infty, [t_0, t]}, \lambda > 0, D_\beta \geq 1.$$

In this case the domain (42) of initial conditions ensuring convergence into Ω_a is defined as

$$D_{\gamma,0} \leq \max_{\kappa > 1, d \in (0,1)} -\lambda \left(\ln \frac{d}{\kappa} \right)^{-1} \frac{\kappa - 1}{\kappa} \\ \times \frac{h(\mathbf{z}_0)}{D_\beta \|\mathbf{x}_0\|_{\mathcal{A}} + D_\beta \cdot c \cdot |h(\mathbf{z}_0)| \left(1 + \frac{\kappa}{1-d} \right) + c|h(\mathbf{z}_0)|}.$$

5. Discussion. In this section we discuss some practically relevant outcomes of the results of Theorem 3 and Corollaries 8, 9 and their potential applications to problems of analysis of asymptotic behavior in nonlinear dynamic systems.

First, in section 5.1 we specify conditions for existence of a trapping region of nonzero volume in $\mathbb{R}^n \oplus \mathbb{R}^m$ in terms of the parameters of system (11) without invoking dependence on $\mathbf{x}(t_0), \mathbf{z}(t_0)$, as was done in Theorem 3. The resulting criterion has a form similar to the standard small-gain conditions [34]. The differences and similarities between this new result and standard small-gain theorems are illustrated with an example.

Second, in section 5.2 we demonstrate how the results of our present contribution can be applied to address the problem of output nonlinear identification for systems which cannot be transformed into a canonic observer form and/or with nonlinear parametrization.

5.1. Relation to conventional small-gain theorems. Conditions specifying state boundedness formulated in Theorem 3 and Corollaries 8, 9 depend explicitly on initial conditions $\mathbf{x}(t_0), \mathbf{z}(t_0)$. Such dependence is inevitable when the convergence is allowed to be nonuniform. But if the mere existence of a trapping region is asked for, dependence on initial conditions may be removed from the statements of the results. The next corollary presents such modified conditions.

COROLLARY 10. *Consider interconnection (11), where the system \mathcal{S}_a satisfies inequality (39) and the function $\gamma_0(\cdot)$ obeys (41). Then there exists a set Ω_γ of initial*

conditions corresponding to the trajectories converging to Ω_a if the following condition is satisfied:

$$(44) \quad D_{\gamma,0} \cdot c \cdot \mathcal{G} < 1,$$

where

$$\mathcal{G} = \beta_t^{-1} \left(\frac{d}{\kappa} \right) \frac{k}{k-1} \left(\beta_t(0) \left(1 + \frac{\kappa}{1-d} \right) + 1 \right)$$

for some $d \in (0, 1)$, $\kappa \in (1, \infty)$. In particular, Ω_γ contains the following domain:

$$\|\mathbf{x}(t_0)\|_{\mathcal{A}} \leq \frac{h(\mathbf{z}(t_0))}{\beta_t(0)} \left[\frac{1}{D_{\gamma,0}} \left(\beta_t^{-1} \left(\frac{d}{\kappa} \right) \right)^{-1} \frac{k-1}{k} - c \left(\beta_t(0) \left(1 + \frac{\kappa}{1-d} \right) + 1 \right) \right].$$

In the case when the function $h(\mathbf{z})$ in (11) is continuous, the volume of the set Ω_γ is nonzero in $\mathbb{R}^n \oplus \mathbb{R}^m$.

Notice that in the case when the dynamics of the contracting subsystem \mathcal{S}_a is exponentially stable, i.e., it satisfies inequality (43), the term \mathcal{G} in condition (44) reduces to

$$(45) \quad \mathcal{G} = \frac{1}{\lambda} \cdot \ln \left(\frac{\kappa}{d} \right) \frac{k}{k-1} \left(D_\beta \left(1 + \frac{\kappa}{1-d} \right) + 1 \right).$$

For $D_\beta = 1$ the minimal value of \mathcal{G} in (45) can be estimated as

$$(46) \quad \mathcal{G}^* = \frac{1}{\lambda} \cdot \min_{d \in (0,1), \kappa \in (1,\infty)} \ln \left(\frac{\kappa}{d} \right) \frac{k}{k-1} \left(2 + \frac{\kappa}{1-d} \right) \approx \frac{15.6886}{\lambda} < \frac{16}{\lambda},$$

which leads to an even more simple formulation of (45):

$$D_{\gamma,0} \cdot \frac{c}{\lambda} \leq \frac{1}{16}.$$

Corollary 10 provides an explicit and easy-to-check condition for existence of a trapping region in the state space of a class of Lyapunov unstable systems. In addition, it allows us to specify explicitly points $\mathbf{x}(t_0)$, $\mathbf{z}(t_0)$ which belong to the emergent trapping region. Notice also that the existence condition, inequality (44), has the flavor of conventional small-gain constraints. Yet, it is substantially different from these classical results. This is because the input-output gain for the wandering subsystem, \mathcal{S}_w , may not be finite or need not even be defined.

To elucidate these differences as well as the similarities between conditions of conventional small-gain theorems and those formulated in Corollary 10 we provide an example. Consider the following systems:

$$(47a) \quad \begin{cases} \dot{x}_1 = -\lambda_1 x_1 + c_1 x_2, \\ \dot{x}_2 = -\lambda_2 x_2 - c_2 |x_1|, \end{cases}$$

$$(47b) \quad \begin{cases} \dot{x}_1 = -\lambda_1 x_1 + c_1 x_2, \\ \dot{x}_2 = -c_2 |x_1|. \end{cases}$$

System (47a) can be viewed as an interconnection of two input-to-state stable systems, x_1 and x_2 , with input-output L_∞ -gains c_1/λ_1 and c_2/λ_2 , respectively. Therefore, in order to prove state boundedness of (47a) we can, in principle, invoke the conventional small-gain theorem. The small-gain condition in this case is as follows:

$$(48a) \quad \frac{c_1}{\lambda_1} \cdot \frac{c_2}{\lambda_2} < 1.$$

The theorem, however, does not apply to system (47b) because the input-output gain of its second subsystem, x_2 , is infinite. Yet, by invoking Corollary 10 it is still possible to show existence of a weak attracting set in the state space of system (47b) and specify its basin of attraction. As follows from Corollary 10, condition

$$(48b) \quad \frac{c_1}{\lambda_1} \cdot \frac{c_2}{\lambda_1} < \frac{1}{16}$$

ensures existence of the trapping region, and the trapping region itself is given by

$$|x_1(t_0)| \leq \left[\frac{1}{c_2} \lambda_1 \left(\ln \frac{\kappa}{d} \right)^{-1} \frac{k-1}{k} - \frac{c_1}{\lambda_1} \left(2 + \frac{\kappa}{1-d} \right) \right] x_2(t_0).$$

5.2. Output nonlinear identification problem. In the literature on adaptive control, observation, and identification a few classes of systems are referred to as *canonic forms* because they guarantee existence of a solution to the problem and because a large variety of physical models can be transformed into this class. Among these, perhaps the most widely known is the *adaptive observer canonical form* [3]. Necessary and sufficient conditions for transformation of the original system into this canonical form can be found, for example, in [16]. These conditions, however, include restrictive requirements of linearization of uncertainty-independent dynamics by output injection, and they also require linear parametrization of the uncertainty. Alternative approaches [4] heavily rely on knowledge of the proper Lyapunov function for the uncertainty-independent part and still assume linear parametrization.

We now demonstrate how these restrictions can be lifted by application of our result to the problem of state and parameter observation. Let us consider systems which can be transformed by means of static or dynamic feedback⁹ into the following form:

$$(49) \quad \dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}, t) + \mathbf{f}(\boldsymbol{\xi}(t), \boldsymbol{\theta}) - \mathbf{f}(\boldsymbol{\xi}(t), \hat{\boldsymbol{\theta}}) + \boldsymbol{\varepsilon}(t),$$

where

$$\boldsymbol{\varepsilon}(t) \in L_\infty^m[t_0, \infty], \quad \|\boldsymbol{\varepsilon}(\tau)\|_{\infty, [t_0, t]} \leq \Delta_\varepsilon$$

is an external perturbation with known Δ_ε , and $\mathbf{x} \in \mathbb{R}^n$. The function $\boldsymbol{\xi} : \mathbb{R}_+ \rightarrow \mathbb{R}^\xi$ is a function of time, which possibly includes available measurements of the state, and $\boldsymbol{\theta}, \hat{\boldsymbol{\theta}} \in \Omega_\theta \subset \mathbb{R}^d$ are the unknown and estimated parameters of the function $\mathbf{f}(\cdot)$, respectively, and the set Ω_θ is bounded. We assume that uniformly in $\boldsymbol{\xi}$, the function $\mathbf{f}(\boldsymbol{\xi}(t), \boldsymbol{\theta})$ is locally bounded in $\boldsymbol{\theta}$:

$$\|\mathbf{f}(\boldsymbol{\xi}(t), \boldsymbol{\theta}) - \mathbf{f}(\boldsymbol{\xi}(t), \hat{\boldsymbol{\theta}})\| \leq D_f \|\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\| + \Delta_f$$

⁹Notice that conventional observers in control theory could be viewed as dynamic feedbacks.

and the values of $D_f \in \mathbb{R}_+$, Δ_f are available. The function $\mathbf{f}_0(\cdot)$ in (49) is assumed to satisfy the following condition.

ASSUMPTION 3. *The system*

$$(50) \quad \dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}, t) + \mathbf{u}(t)$$

is forward-complete. Furthermore, for all $\mathbf{u}(t)$ such that

$$\|\mathbf{u}(t)\|_{\infty, [t_0, t]} \leq \Delta_u + \|\mathbf{u}_0(\tau)\|_{\infty, [t_0, t]}, \quad \Delta_u \in \mathbb{R}_+,$$

there exists a bounded set \mathcal{A} , $c > 0$ and a function $\Delta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying the following inequality:

$$\|\mathbf{x}(t)\|_{\mathcal{A}_{\Delta(\Delta_u)}} \leq \beta(t - t_0) \|\mathbf{x}(t_0)\|_{\mathcal{A}_{\Delta(\Delta_u)}} + c \|\mathbf{u}_0(\tau)\|_{\infty, [t_0, t]},$$

where $\beta(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\lim_{t \rightarrow \infty} \beta(t) = 0$ is a strictly decreasing function.

Consider the following auxiliary system:

$$(51) \quad \dot{\lambda} = S(\lambda), \quad \lambda(t_0) = \lambda_0 \in \Omega_\lambda \subset \mathbb{R}^\lambda,$$

where $\Omega_\lambda \subset \mathbb{R}^\lambda$ is a compact set, $\lambda(t, \lambda_0) \in \Omega_\lambda$ for all $t \geq t_0$, and $S(\lambda)$ is locally Lipschitz. Furthermore, suppose that the following assumption holds for system (51).

ASSUMPTION 4. *System (51) is Poisson stable in Ω_λ , that is,*

$$\forall \lambda' \in \Omega_\lambda, t' \in \mathbb{R}_+ \Rightarrow \exists t'' > t : \|\lambda(t'', \lambda') - \lambda'\| \leq \epsilon,$$

where ϵ is an arbitrary small positive constant. Moreover, the trajectory $\lambda(t, \lambda_0)$ is dense in Ω_λ :

$$\forall \lambda' \in \Omega_\lambda, \epsilon \in \mathbb{R}_{>0} \Rightarrow \exists t \in \mathbb{R}_+ : \|\lambda' - \lambda(t, \lambda_0)\| < \epsilon.$$

Now we are ready to formulate the following statement.

COROLLARY 11. *Consider system (49) and suppose that the following conditions hold:*

- (C4) *the vector-field $\mathbf{f}_0(\mathbf{x}, t)$ in (49) satisfies Assumption 3;*
- (C5) *there exists a (known) system (51) satisfying Assumption 4;*
- (C6) *there exists a locally Lipschitz $\eta : \mathbb{R}^\lambda \rightarrow \mathbb{R}^d$:*

$$\|\eta(\lambda') - \eta(\lambda'')\| \leq D_\eta \|\lambda' - \lambda''\|$$

such that the set $\eta(\Omega_\lambda)$ is dense in Ω_θ ;

(C7) *system (49) has a steady-state characteristic with respect to the norm*

$$\|\cdot\|_{\mathcal{A}_{\Delta(M)}}, \quad M = 2\Delta_f + \Delta_\epsilon + \delta,$$

and input $\hat{\theta}$, where δ is some positive (arbitrarily small) constant.

Consider the following interconnection of (49), (51):

$$(52) \quad \begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}_0(\mathbf{x}, t) + \mathbf{f}(\xi(t), \theta) - \mathbf{f}(\xi(t), \hat{\theta}) + \varepsilon(t), \\ \hat{\theta} &= \eta(\lambda), \\ \dot{\lambda} &= \gamma \|\mathbf{x}(t)\|_{\mathcal{A}_{\Delta(M)}} S(\lambda), \end{aligned}$$

where $\gamma > 0$ satisfies the following inequality:

$$(53) \quad \gamma \leq \left(\beta_t^{-1} \left(\frac{d}{\kappa} \right) \right)^{-1} \frac{\kappa - 1}{\kappa} \frac{1}{D_\lambda \left(\beta_t(0) \left(1 + \frac{\kappa}{1-d} \right) + 1 \right)},$$

$$D_\lambda = c \cdot D_f \cdot D_\eta \cdot \max_{\lambda \in \Omega_\lambda} \|S(\lambda)\|$$

for some $d \in (0, 1)$, $\kappa \in (1, \infty)$. Then, for $\lambda(t_0) = \lambda_0$, some $\theta' \in \Omega_\theta$, and all $\mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbb{R}^n$, the following holds:

$$(54) \quad \lim_{t \rightarrow \infty} \|\mathbf{x}(t)\|_{\mathcal{A}_{\Delta(M)}} = 0, \quad \lim_{t \rightarrow \infty} \hat{\theta}(t) = \theta' \in \Omega_\theta.$$

Notice that, as has been pointed out in the previous section, in the case when the dynamics of (50) is exponentially stable with a rate of convergence equal to ρ and $\beta(0) = D_\beta$, condition (53) will have the following form:

$$\gamma \leq -\rho \left(\ln \frac{d}{\kappa} \right)^{-1} \frac{\kappa - 1}{\kappa} \frac{1}{D_\lambda \left(D_\beta \left(1 + \frac{\kappa}{1-d} \right) + 1 \right)}.$$

According to Corollary 11, for the rather general class of systems (49) it is possible to design an estimator $\hat{\theta}(t)$ which guarantees not only that the “error” vector $\mathbf{x}(t)$ reaches a neighborhood of the origin, but also that the estimates $\hat{\theta}(t)$ converge to some θ' in Ω_θ . Both these facts, together with additional nonlinear persistent excitation conditions [6], [29]

$$\begin{aligned} &\exists T > 0, \rho \in \mathcal{K} : \forall \mathcal{T} = [t, t + T], t \in \mathbb{R}_+ \\ &\Rightarrow \exists \tau \in \mathcal{T} : |\mathbf{f}(\xi(\tau), \theta) - \mathbf{f}(\xi(\tau), \theta')| \geq \rho(\|\theta - \theta'\|), \end{aligned}$$

in principle allow us to estimate the domain of convergence for $\hat{\theta}(t)$.

Concluding this section we mention that statements of Theorem 3 and Corollaries 8–11 constitute additional theoretical tools for the analysis of asymptotic behavior of systems in cascaded form. In particular they are complementary to the results of [1], where *asymptotic stability* of systems of the following type:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}), \\ \dot{\mathbf{z}} &= \mathbf{q}(\mathbf{x}, \mathbf{z}), \quad \mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad \mathbf{q} : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m \end{aligned}$$

was considered under the assumption that the \mathbf{x} -subsystem is globally asymptotically stable and the \mathbf{z} -subsystem is integral input-to-state stable. In contrast to this, our results apply to establishing *asymptotic convergence* for systems with the following structure:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{z}), \\ \dot{\mathbf{z}} &= \mathbf{q}(\mathbf{x}, \mathbf{z}), \quad \mathbf{f} : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n, \end{aligned}$$

where the \mathbf{x} -subsystem is input-to-state stable, and the \mathbf{z} -subsystem could be practically integral input-to-state stable (see Corollary 8), although in general no stability assumptions are imposed on it.

6. Examples. In this section we provide two examples of parameter identification in nonlinearly parametrized systems that cannot be transformed into the canonical adaptive observer form.

The first example is merely an academical illustration of Corollary 11, where only one parameter is unknown and the system itself is a first-order differential equation. The second example illustrates a possible application of our results to the problem of identifying the dynamics in living cells.

Example 1. Consider the following system:

$$(55) \quad \dot{x} = -kx + \sin(x\theta + \theta) + u, \quad k > 0, \quad \theta \in [-a, a],$$

where θ is an unknown parameter and u is the control input. Without loss of generality we let $a = 1, k = 1$. The problem is to estimate the parameter θ from measurements of x and steer the system to the origin. Clearly, the choice $u = -\sin(x\hat{\theta} + \hat{\theta})$ transforms (55) into

$$(56) \quad \dot{x} = -kx + \sin(x\theta + \theta) - \sin(x\hat{\theta} + \hat{\theta}),$$

which satisfies Assumption 3. Moreover, the system

$$\begin{aligned} \dot{\lambda}_1 &= \lambda_1, \\ \dot{\lambda}_2 &= -\lambda_2, \quad \lambda_1^2(t_0) + \lambda_2^2(t_0) = 1 \end{aligned}$$

with mapping $\boldsymbol{\eta} = (1, 0)^T \boldsymbol{\lambda}$ satisfies Assumption 4 and therefore

$$(57) \quad \begin{aligned} \dot{\lambda}_1 &= \gamma|x|\lambda_1, \\ \dot{\lambda}_2 &= -\gamma|x|\lambda_2, \quad \lambda_1^2(t_0) + \lambda_2^2(t_0) = 1 \end{aligned}$$

would be a candidate for the control and parameter estimation algorithm. According to Corollary 11, the goal will be reached if the parameter γ in (57) obeys the following constraint:

$$\gamma \leq -\rho \left(\ln \frac{d}{\kappa} \right)^{-1} \frac{\kappa - 1}{\kappa} \frac{1}{D_\lambda \left(D_\beta \left(1 + \frac{\kappa}{1-d} \right) + 1 \right)}, \quad \rho = k = 1, \quad D_\beta = 1, \quad D_\lambda = 1$$

for some $d \in (0, 1), \kappa \in (1, \infty)$. Hence, choosing, for example, $d = 0.5, \kappa = 2$ we obtain that choice

$$0 < \gamma < -\ln \left(\frac{0.5}{2} \right)^{-1} \frac{1}{2} \cdot \frac{1}{6} = 0.0601$$

suffices to ensure that

$$\lim_{t \rightarrow \infty} x(t) = 0, \quad \lim_{t \rightarrow \infty} \hat{\theta}(t) = \theta.$$

We simulated system (56), (57) with $\theta = 0.3, \gamma = 0.05$ and initial conditions $x(t_0)$ randomly distributed in the interval $[-1, 1]$. Results of the simulation are illustrated in Figure 5, where the phase plots of system (56), (57) as well as the trajectories of $\hat{\theta}(t)$ are given.

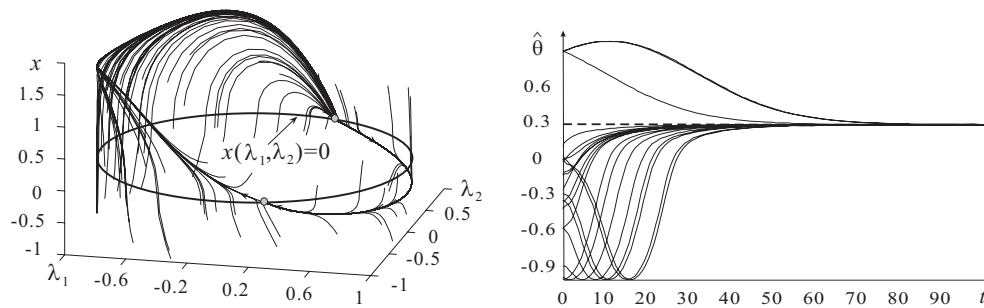


FIG. 5. Trajectories of system (56), (57) (left panel) and the family of estimates $\hat{\theta}(t)$ of parameter θ as functions of time t (right panel).

Example 2. Consider the problem of modeling electrical activity in biological cells from the input-output data in current clamp experiments. The simplest mathematical model, which captures a fairly large variety of phenomena such as periodic bursting in response to constant stimulation, is the classical Hindmarsh and Rose model neuron without adaptation currents [10]:

$$(58) \quad \begin{aligned} \dot{x}_1 &= -ax_1^3 + bx_1^2 + x_2 + \alpha u, \\ \dot{x}_2 &= c - \beta x_2 - dx_1^2, \end{aligned}$$

where variable x_1 is the membrane potential, x_2 stands for the ionic currents in the cell, u is the input current, and $a, b, c, d, \alpha, \beta \in \mathbb{R}$ are parameters. While the parameters of the first equation can, in principle, be identified experimentally by blocking the ionic channels in the cells and measuring the membrane conductance, identification of parameters β, d is a difficult problem, as information about ionic currents x_2 is rarely available.

Conventional techniques [3] cannot be applied directly to this problem as the model (58) is not in canonical adaptive observer form. Let us illustrate how our results can be used to derive the unknown parameters of (58) such that the reconstructed model fits the observed data. Assume, first, that parameters a, b, c, α in the first equation of (58) are known, whereas parameters β, d in the second equation are unknown. This corresponds to the realistic case, where the time constant of current x_2 and coupling between x_1 and x_2 are uncertain. In our example we assumed that

$$\beta \in \Omega_\beta = [0.3, 0.7], \quad d \in \Omega_d = [2, 3], \quad a = 1, \quad b = 3, \quad \alpha = 0.7, \quad c = 0.5.$$

As a candidate for the observer we select the following system:

$$(59) \quad \dot{\hat{x}} = \rho(x_1 - \hat{x}) - a\hat{x}_1^3 + b\hat{x}_1^2 + \alpha u + f(\hat{\beta}, \hat{d}, t), \quad \rho \in \mathbb{R}_{>0},$$

where $\hat{\beta}, \hat{d}$ are parameters to be adjusted and the function $f(\hat{\beta}, \hat{d}, t)$ is specified as

$$f(\hat{\beta}, \hat{d}, t) = \int_{t_0}^t e^{-\hat{\beta}(t-\tau)} (\hat{d}\hat{x}_1^2(\tau) + c) d\tau.$$

Then the dynamics of $\tilde{x}(t) = x(t) - \hat{x}(t)$ satisfies the following differential equation:

$$\dot{\tilde{x}} = -\rho\tilde{x} + f(\beta, d, t) - f(\hat{\beta}, \hat{d}, t).$$

The function $f(\beta, d, t)$ satisfies the following inequality:

$$\begin{aligned} |f(\beta, d, t) - f(\hat{\beta}, \hat{d}, t)| &\leq |f(\beta, d, t) - f(\hat{\beta}, d, t)| + |f(\hat{\beta}, d, t) - f(\hat{\beta}, \hat{d}, t)| \\ &\leq D_{f,\beta}|\beta - \hat{\beta}| + D_{f,d}|d - \hat{d}| + \epsilon(t), \end{aligned}$$

where $\epsilon(t)$ is an exponentially decaying term, and

$$(60) \quad D_{f,\beta} = \max_{\hat{\beta}, \beta \in \Omega_\beta, d \in \Omega_d} \left\{ \frac{1}{\hat{\beta}} (d \|x_1(\tau)\|_{\infty, [t_0, \infty]} + c) \right\}, \quad D_{f,d} = \max_{\hat{\beta} \in \Omega_\beta} \left\{ \frac{1}{\hat{\beta}} \|x_1(\tau)\|_{\infty, [t_0, \infty]} \right\}.$$

Furthermore, Assumption 3 is satisfied for system

$$(61) \quad \dot{\tilde{x}} = -\rho \tilde{x} + v(t),$$

with

$$\Delta(\Delta_u) = \frac{\Delta_u}{\rho}.$$

In particular, for all $v(t) : \|v(\tau)\|_{\infty, [t_0, t]} \leq \Delta_u + \|v_0(\tau)\|_{\infty, [t_0, t]}$ the following inequality holds:

$$(62) \quad \|\tilde{x}(t)\|_{\Delta(\Delta_u)} \leq e^{-\rho(t-t_0)} \|\tilde{x}(t_0)\|_{\Delta(\Delta_u)} + \frac{1}{\rho} \|v_0(\tau)\|_{\infty, [t_0, t]}.$$

To see this consider the general solution of (61):

$$\tilde{x}(t) = e^{-\rho(t-t_0)} \tilde{x}(t_0) + e^{-\rho t} \int_{t_0}^t e^{\rho\tau} v(\tau) d\tau$$

and derive an estimate of $|\tilde{x}(t)|$. This estimate has the following form:

$$\begin{aligned} |\tilde{x}(t)| &\leq e^{-\rho(t-t_0)} |\tilde{x}(t_0)| + \frac{1}{\rho} (1 - e^{-\rho(t-t_0)}) \|v(\tau)\|_{\infty, [t_0, t]} \\ &\leq e^{-\rho(t-t_0)} \left(|\tilde{x}(t_0)| - \frac{1}{\rho} \Delta_u \right) + \frac{1}{\rho} (\|v_0(\tau)\|_{\infty, [t_0, t]} + \Delta_u) \\ &\leq e^{-\rho(t-t_0)} \|\tilde{x}(t_0)\|_{\Delta(\Delta_u)} + \frac{1}{\rho} (\|v_0(\tau)\|_{\infty, [t_0, t]} + \Delta_u). \end{aligned}$$

Hence

$$|\tilde{x}(t)| - \frac{1}{\rho} \Delta_u \leq e^{-\rho(t-t_0)} \|\tilde{x}(t_0)\|_{\Delta(\Delta_u)} + \frac{1}{\rho} \|v_0(\tau)\|_{\infty, [t_0, t]},$$

which automatically implies (62).

Let us define subsystem (51). Consider the following system of differential equations:

$$(63) \quad \begin{aligned} \dot{\lambda}_1 &= \lambda_2, \\ \dot{\lambda}_2 &= -\omega_1^2 \lambda_1, \\ \dot{\lambda}_3 &= \lambda_4, \\ \dot{\lambda}_4 &= -\omega_2^2 \lambda_3, \quad \lambda_0 = (1, 0, 1, 0)^T, \end{aligned}$$

where Ω_λ is the ω -limit set of the point λ_0 , and $\omega_1, \omega_2 \in \mathbb{R}$. System (63), therefore, satisfies Assumption 4. Given that domains Ω_β, Ω_d are known, select

$$(64) \quad \boldsymbol{\eta} : \mathbb{R}^n \rightarrow \mathbb{R}^2, \quad \boldsymbol{\eta} = (\eta_1(\boldsymbol{\lambda}), \eta_2(\boldsymbol{\lambda})),$$

$$\hat{\beta} = \eta_1(\boldsymbol{\lambda}) = \frac{1}{2} \left(\frac{2 \arcsin(\lambda_1)}{\pi} + 1 \right) \cdot 0.4 + 0.3, \quad \hat{d} = \eta_2(\boldsymbol{\lambda}) = \frac{1}{2} \left(\frac{2 \arcsin(\lambda_3)}{\pi} + 1 \right) + 2.$$

Choosing

$$\frac{\omega_1}{\omega_2} = \pi$$

we ensure that $\boldsymbol{\eta}(\Omega_\lambda)$ is dense in $\Omega_\beta \times \Omega_d$. Given that $\hat{\beta}, \hat{d}$ are bounded and $\hat{\beta} \geq 0.3$, the values of $D_{f,\beta}, D_{f,d}$ are bounded since the signal $x_1(t)$ is always bounded for any $t \geq t_0$ for the given range of parameters. Hence, according to Corollary 11, interconnection of (59), (64), and

$$\begin{aligned} \dot{\lambda}_1 &= \gamma \|\tilde{x}(t)\|_{\Delta(\delta)} \cdot \lambda_2, \\ \dot{\lambda}_2 &= -\gamma \|\tilde{x}(t)\|_{\Delta(\delta)} \cdot \omega_1^2 \lambda_1, \\ \dot{\lambda}_3 &= \gamma \|\tilde{x}(t)\|_{\Delta(\delta)} \cdot \lambda_4, \\ \dot{\lambda}_4 &= -\gamma \|\tilde{x}(t)\|_{\Delta(\delta)} \cdot \omega_2^2 \lambda_3, \quad \boldsymbol{\lambda}_0 = (1, 0, 1, 0)^T, \end{aligned}$$

with arbitrary small $\delta > 0$ and properly chosen $\gamma > 0$, ensures that

$$\lim_{t \rightarrow \infty} \|\tilde{x}(t)\|_{\Delta(\delta)} = 0, \quad \lim_{t \rightarrow \infty} \hat{\beta}(t) = \beta' \in \Omega_\beta, \quad \lim_{t \rightarrow \infty} \hat{d}(t) = d' \in \Omega_d.$$

This in turn implies a successful fit of the model to the observations.

We simulated the system with $\rho = 10$ and $\gamma = 3 \cdot 10^{-4}$ for $\beta = 0.5, d = 2.5$. The results of the simulations are provided in Figure 6. It can be seen from this figure that the reconstruction is successful and that the parameters converge into a small neighborhood of the actual values. Further details explaining how this technique can be applied to model the dynamics of the evoked membrane potentials in real neural cells from input-output measurements in vitro are discussed in [25].

7. Conclusion. We proposed tools for the analysis of asymptotic behavior of a class of dynamical systems. In particular, we consider an interconnection of an input-to-state stable system with an unstable or integral input-to-state dynamics. Our results allow us to address a variety of problems in which convergence may not be uniform with respect to initial conditions. It is necessary to notice that the proposed method does not require complete knowledge of the dynamical systems in question. Only qualitative information such as, for instance, characterization of input-to-state stability is necessary for application of our results. We demonstrated how our analysis can be used in the problems of synthesis and design—in particular for problems of nonlinear regulation and parameter identification of nonlinear parametrized systems. The examples show the relevance of our approach in those domains where application of the standard techniques is either not possible or too complicated.

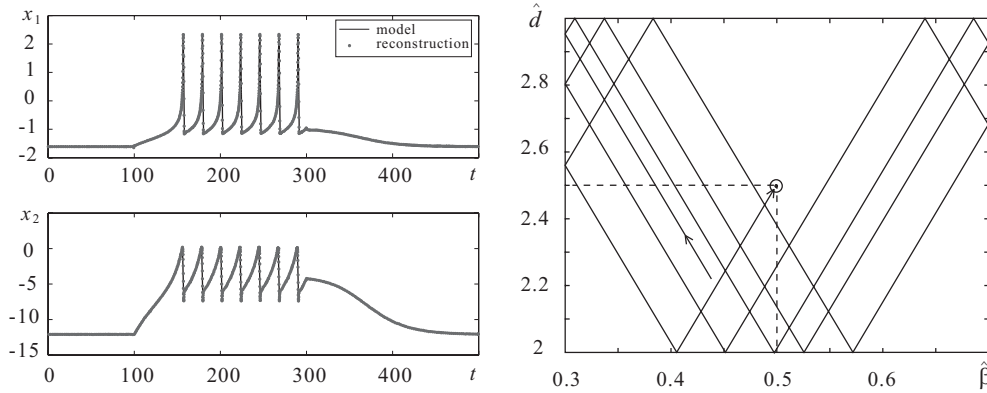


FIG. 6. Left panel: trajectories $x_1(t), x_2(t)$ of system (58) plotted for the nominal values of parameters $\beta = 0.5, d = 2.5$ (model), and for the values $\beta = \hat{\beta}(t_0 + T), d = \hat{d}(t_0 + T)$, where T is the total simulation time (reconstruction). Input $u(t)$ is a rectangular impulse with amplitude 0.7 starting at $t = 100$ and ending at $t = 300$. Right panel: searching dynamics in the bounded parameter space (a segment of the trajectory $\hat{\beta}(t), \hat{d}(t)$ towards the end of the simulation).

Appendix. Proofs of Theorem 3, lemmas, and corollaries.

A.1. Proof of Theorem 3. Let the conditions of the theorem be satisfied for given $t_0 \in \mathbb{R}_+$: $\mathbf{x}(t_0) = \mathbf{x}_0, \mathbf{z}(t_0) = \mathbf{z}_0$. Notice that in this case $h(\mathbf{z}_0) \geq 0$; otherwise requirement (26) will be violated. Consider the sequence (23) of volumes Ω_i induced by \mathcal{S} :

$$\Omega_i = \{\mathbf{x} \in \mathcal{X}, \mathbf{z} \in \mathcal{Z} \mid h(\mathbf{z}(t)) \in H_i\}.$$

To prove the theorem we show that $0 \leq h(\mathbf{z}(t)) \leq h(\mathbf{z}_0)$ for all $t \geq t_0$. For the given partition (23) we consider two alternatives.

First, in the degenerative case, the state $\mathbf{x}(t) \oplus \mathbf{z}(t)$ enters some $\Omega_j, j \geq 0$, and stays there afterward, which automatically guarantees that $0 \leq |h(\mathbf{z})| \leq h(\mathbf{z}_0)$. Then, according to (6) the trajectory $\mathbf{x}(t)$ satisfies the following inequality:

$$(65) \quad \|\mathbf{x}(t)\|_{\mathcal{A}} \leq \beta(\|\mathbf{x}_0\|_{\mathcal{A}}, t - t_0) + c\|h(\mathbf{z}(t))\|_{\infty, [t_0, t]} \leq \beta(\|\mathbf{x}_0\|_{\mathcal{A}}, t - t_0) + c|h(\mathbf{z}_0)|.$$

Taking into account that $\beta(\cdot, \cdot) \in \mathcal{KL}$ we can conclude that (65) implies that

$$(66) \quad \limsup_{t \rightarrow \infty} \|\mathbf{x}(t)\|_{\mathcal{A}} \leq c|h(\mathbf{z}_0)|.$$

Therefore the statements of the theorem hold.

Let us consider the second alternative, where the state $\mathbf{x}(t) \oplus \mathbf{z}(t)$ enters each Ω_j and leaves later. Given that $h(\mathbf{z}(t))$ is monotone and nonincreasing in t , this implies that there exists an ordered sequence of time instants t_j :

$$(67) \quad t_0 < t_1 < t_2 \cdots t_j < t_{j+1} \cdots$$

such that

$$(68) \quad h(\mathbf{z}(t_i)) = \sigma_i h(\mathbf{z}_0).$$

Hence in order to prove the theorem we must show that the sequence $\{t_i\}_{i=0}^{\infty}$ does not converge. In other words, the boundary $\sigma_{\infty} h(\mathbf{z}_0) = 0$ will not be reached in finite time.

In order to do this let us estimate the upper bounds for the following differences:

$$T_i = t_{i+1} - t_i.$$

Taking into account inequality (8) and the fact that $\gamma_0(\cdot) \in \mathcal{K}_e$, we can derive that

$$(69) \quad h(\mathbf{z}(t_i)) - h(\mathbf{z}(t_{i+1})) \leq T_i \max_{\tau \in [t_i, t_{i+1}]} \gamma_0(\|\mathbf{x}(\tau)\|_{\mathcal{A}}) \leq T_i \gamma_0(\|\mathbf{x}(\tau)\|_{\mathcal{A}_{\infty}, [t_i, t_{i+1}]}) .$$

According to the definition of t_i in (68) and noticing that the sequence \mathcal{S} is strictly decreasing, we have

$$h(\mathbf{z}(t_i)) - h(\mathbf{z}(t_{i+1})) = (\sigma_i - \sigma_{i+1})h(\mathbf{z}_0) > 0.$$

Hence $h(\mathbf{z}_0) > 0$ implies that $\gamma_0(\|\mathbf{x}(\tau)\|_{\mathcal{A}_{\infty}, [t_i, t_{i+1}]}) > 0$ and, therefore, (69) results in the following estimate of T_i :

$$(70) \quad T_i \geq \frac{h(\mathbf{z}(t_i)) - h(\mathbf{z}(t_{i+1}))}{\gamma_0(\|\mathbf{x}(\tau)\|_{\mathcal{A}_{\infty}, [t_i, t_{i+1}]})} = \frac{h(\mathbf{z}_0)(\sigma_i - \sigma_{i+1})}{\gamma_0(\|\mathbf{x}(\tau)\|_{\mathcal{A}_{\infty}, [t_i, t_{i+1}]})} .$$

Taking into account that $h(\mathbf{z}(t))$ is nonincreasing over $[t_i, t_{i+1}]$ and using (6) we can bound the norm $\|\mathbf{x}(\tau)\|_{\mathcal{A}_{\infty}, [t_i, t_{i+1}]}$ as follows:

$$(71) \quad \begin{aligned} \|\mathbf{x}(\tau)\|_{\mathcal{A}_{\infty}, [t_i, t_{i+1}]} &\leq \beta(\|\mathbf{x}(t_i)\|_{\mathcal{A}}, 0) + c\|h(\mathbf{z}(\tau))\|_{\infty, [t_i, t_{i+1}]} \\ &\leq \beta(\|\mathbf{x}(t_i)\|_{\mathcal{A}}, 0) + c \cdot \sigma_i h(\mathbf{z}_0). \end{aligned}$$

Hence, combining (70) and (71), we obtain that

$$T_i \geq \frac{h(\mathbf{z}_0)(\sigma_i - \sigma_{i+1})}{\gamma_0(\sigma_i(\sigma_i^{-1}\beta(\|\mathbf{x}(t_i)\|_{\mathcal{A}}, 0) + c \cdot h(\mathbf{z}_0)))} .$$

Then, using property (10) of function γ_0 we can derive that

$$(72) \quad T_i \geq \frac{h(\mathbf{z}_0)(\sigma_i - \sigma_{i+1})}{\gamma_{0,1}(\sigma_i)} \frac{1}{\gamma_{0,2}(\sigma_i^{-1}\beta(\|\mathbf{x}(t_i)\|_{\mathcal{A}}, 0) + c \cdot h(\mathbf{z}_0))} .$$

Taking into account condition (27) of the theorem, the theorem will be proved if we ensure that

$$(73) \quad T_i \geq \tau_i$$

for all $i = 0, 1, 2, \dots, \infty$. We prove this claim by induction with respect to the index $i = 0, 1, \dots, \infty$. We start with $i = 0$, and then show that for all $i > 0$ the following implication holds:

$$(74) \quad T_i \geq \tau_i \Rightarrow T_{i+1} \geq \tau_{i+1} .$$

Let us prove that (73) holds for $i = 0$. For this purpose consider the term $(\sigma_i - \sigma_{i+1})/\gamma_{0,1}(\sigma_i)$. As follows immediately from condition (25) of the theorem, we have that

$$(75) \quad \frac{\sigma_i - \sigma_{i+1}}{\gamma_{0,1}(\sigma_i)} \geq \tau_i \Delta_0 \quad \forall i \geq 0 .$$

In particular

$$\frac{\sigma_0 - \sigma_1}{\gamma_{0,1}(\sigma_0)} \geq \tau_0 \Delta_0.$$

Therefore, inequality (72) reduces to

$$(76) \quad T_0 \geq \tau_0 \Delta_0 \frac{h(\mathbf{z}_0)}{\gamma_{0,2}(\sigma_0^{-1} \beta(\|\mathbf{x}(t_0)\|_{\mathcal{A}}, 0) + c \cdot h(\mathbf{z}_0))}.$$

Moreover, taking into account Property 3 and (18), (19), we can derive the following estimate:

$$\begin{aligned} \sigma_0^{-1} \beta(\|\mathbf{x}(t_0)\|_{\mathcal{A}}, 0) &\leq \sigma_0^{-1} \phi_0(\|\mathbf{x}(t_0)\|_{\mathcal{A}}) + \sigma_0^{-1} v_0(c \cdot |h(\mathbf{z}_0)| \sigma_0) \\ &\leq B_1(\|\mathbf{x}_0\|_{\mathcal{A}}) + B_2(|h(\mathbf{z}_0)|, c). \end{aligned}$$

According to the theorem conditions, \mathbf{x}_0 and \mathbf{z}_0 satisfy inequality (26). This in turn implies that

$$(77) \quad \begin{aligned} &\gamma_{0,2}(\sigma_0^{-1} \beta(\|\mathbf{x}(t_0)\|_{\mathcal{A}}, 0) + c \cdot h(\mathbf{z}_0)) \\ &\leq \gamma_{0,2}(B_1(\|\mathbf{x}_0\|_{\mathcal{A}}) + B_2(|h(\mathbf{z}_0)|, c) + c \cdot h(\mathbf{z}_0)) \leq \Delta_0 \cdot h(\mathbf{z}_0). \end{aligned}$$

Combining (76) and (77), we obtain the desired inequality

$$T_0 \geq \tau_0 \Delta_0 \frac{h(\mathbf{z}_0)}{\gamma_{0,2}(\sigma_0^{-1} \beta(\|\mathbf{x}(t_0)\|_{\mathcal{A}}, 0) + c \cdot h(\mathbf{z}_0))} \geq \tau_0 \frac{\Delta_0 h(\mathbf{z}_0)}{\Delta_0 h(\mathbf{z}_0)} = \tau_0.$$

Thus the basis of induction is proved.

Let us assume that (73) holds for all $i = 0, \dots, n$, $n \geq 0$. We shall prove now that implication (74) holds for $i = n + 1$. Consider the term $\beta(\|\mathbf{x}(t_{n+1})\|_{\mathcal{A}}, 0)$:

$$\begin{aligned} \beta(\|\mathbf{x}(t_{n+1})\|_{\mathcal{A}}, 0) &\leq \beta(\beta(\|\mathbf{x}(t_n)\|_{\mathcal{A}}, T_n) + c \|h(\mathbf{z}(\tau))\|_{\infty, [t_n, t_{n+1}]}, 0) \\ &\leq \beta(\beta(\|\mathbf{x}(t_n)\|_{\mathcal{A}}, T_n) + c \cdot \sigma_n \cdot h(\mathbf{z}_0), 0). \end{aligned}$$

Taking into account Property 2 (specifically, inequality (17)) and (18)–(20) we can derive that

$$(78) \quad \begin{aligned} \beta(\|\mathbf{x}(t_{n+1})\|_{\mathcal{A}}, 0) &\leq \beta(\xi_n \cdot \beta(\|\mathbf{x}(t_n)\|_{\mathcal{A}}, 0) + c \cdot \sigma_n \cdot h(\mathbf{z}_0), 0) \\ &\leq \phi_1(\|\mathbf{x}(t_n)\|_{\mathcal{A}}) + v_1(c \cdot |h(\mathbf{z}_0)| \cdot \sigma_n). \end{aligned}$$

Notice that, according to the inductive hypothesis ($T_i \geq \tau_i$), the following holds:

$$(79) \quad \|\mathbf{x}(t_{i+1})\|_{\mathcal{A}} \leq \beta(\|\mathbf{x}(t_i)\|_{\mathcal{A}}, T_i) + c \cdot \sigma_i \cdot h(\mathbf{z}_0) \leq \xi_i \beta(\|\mathbf{x}(t_i)\|_{\mathcal{A}}, 0) + c \cdot \sigma_i \cdot h(\mathbf{z}_0)$$

for all $i = 0, \dots, n$. Then (78), (79), and (18)–(20) imply that

$$(80) \quad \begin{aligned} &\beta(\|\mathbf{x}(t_{n+1})\|_{\mathcal{A}}, 0) \leq \phi_1(\xi_n \beta(\|\mathbf{x}(t_{n-1})\|_{\mathcal{A}}, 0) + c \cdot \sigma_{n-1} \cdot h(\mathbf{z}_0)) \\ &\quad + v_1(c \cdot |h(\mathbf{z}_0)| \cdot \sigma_n) \leq \phi_2(\|\mathbf{x}(t_{n-1})\|_{\mathcal{A}}) + v_2(c \cdot |h(\mathbf{z}_0)| \cdot \sigma_{n-1}) \\ &\quad + v_1(c \cdot |h(\mathbf{z}_0)| \cdot \sigma_n) \leq \phi_{n+1}(\|\mathbf{x}_0\|_{\mathcal{A}}) + \sum_{i=1}^{n+1} v_i(c \cdot |h(\mathbf{z}_0)| \sigma_{n+1-i}) \\ &\leq \phi_{n+1}(\|\mathbf{x}_0\|_{\mathcal{A}}) + \sum_{i=0}^{n+1} v_i(c \cdot |h(\mathbf{z}_0)| \sigma_{n+1-i}). \end{aligned}$$

According to Property 3, the term

$$\sigma_{n+1}^{-1} \left(\phi_{n+1}(\|\mathbf{x}_0\|_{\mathcal{A}}) + \sum_{i=0}^{n+1} v_i(c \cdot |h(\mathbf{z}_0)|\sigma_{n+1-i}) \right)$$

is bounded from above by the sum

$$B_1(\|\mathbf{x}_0\|_{\mathcal{A}}) + B_2(|h(\mathbf{z}_0)|, c).$$

Therefore, monotonicity of $\gamma_{0,2}$, estimate (80), and inequality (26) lead to the following inequality:

$$\begin{aligned} \gamma_{0,2}(\sigma_{n+1}^{-1}\beta(\|\mathbf{x}(t_{n+1})\|_{\mathcal{A}}, 0) + c \cdot h(\mathbf{z}_0)) &\leq \gamma_{0,2}(B_1(\|\mathbf{x}_0\|_{\mathcal{A}}) + B_2(|h(\mathbf{z}_0)|, c) + c \cdot h(\mathbf{z}_0)) \\ &\leq h(\mathbf{z}_0)\Delta_0. \end{aligned}$$

Hence, according to (72) and (75) we have

$$\begin{aligned} T_{n+1} &\geq \frac{(\sigma_{n+1} - \sigma_{n+2})}{\gamma_{0,1}(\sigma_{n+1})} \frac{h(\mathbf{z}_0)}{\gamma_{0,2}(\sigma_{n+1}^{-1}\beta(\|\mathbf{x}(t_{n+1})\|_{\mathcal{A}}, 0) + c \cdot h(\mathbf{z}_0))} \\ &\geq \tau_{n+1} \frac{\Delta_0 h(\mathbf{z}_0)}{\Delta_0 h(\mathbf{z}_0)} = \tau_{n+1}. \end{aligned}$$

Thus implication (74) is proved. This implies that $h(\mathbf{z}(t)) \in [0, h(\mathbf{z}_0)]$ for all $t \geq t_0$ and, consequently, that (66) holds. \square

A.2. Proof of Lemma 6. As follows from the assumptions, $h(\mathbf{z}(t, \mathbf{z}_0))$ is bounded. Assume it belongs to the interval $[a, h(\mathbf{z}_0)]$, $a \leq h(\mathbf{z}_0)$. Taking into account that $h(\mathbf{z}(t, \mathbf{z}_0))$ is bounded and monotone in t (every subsequence of which is again monotone) and applying the Bolzano–Weierstrass theorem we can conclude that $h(\mathbf{z}(t, \mathbf{z}_0))$ converges in $[a, h(\mathbf{z}_0)]$. In particular, there exists $\bar{h} \in [a, h(\mathbf{z}_0)]$ such that

$$(81) \quad \lim_{t \rightarrow \infty} h(\mathbf{z}(t, \mathbf{z}_0)) = \bar{h}.$$

Therefore, as follows from (8) we can conclude that

$$\begin{aligned} (82) \quad 0 &\leq \lim_{t \rightarrow \infty} \int_{t_0}^t \gamma_1(\|\mathbf{x}(\tau, \mathbf{x}_0)\|_{\mathcal{A}}) d\tau \leq \lim_{t \rightarrow \infty} (h(\mathbf{z}_0) - h(\mathbf{z}(t, \mathbf{z}_0))) \\ &= h(\mathbf{z}_0) - \bar{h} \leq h(\mathbf{z}_0) - a < \infty. \end{aligned}$$

According to the lemma assumptions, system \mathcal{S}_a has steady-state characteristics. This means that there exists a constant $\bar{x} \in \mathbb{R}_+$ such that

$$(83) \quad \lim_{t \rightarrow \infty} \|\mathbf{x}(t, \mathbf{x}_0)\|_{\mathcal{A}} = \bar{x}.$$

Suppose that $\bar{x} > 0$. Then it follows from (83) that there exists time instant t_1 , $t_0 \leq t_1 < \infty$ and some constant $0 < \delta < \bar{x}$ such that

$$\|\mathbf{x}(t)\|_{\mathcal{A}} \geq \delta \quad \forall t \geq t_1.$$

Hence using (82) and noticing that $\gamma_1 \in \mathcal{K}_e$ we obtain

$$\infty > h(\mathbf{z}_0) - \bar{h} \geq \lim_{t \rightarrow \infty} \int_{t_0}^t \gamma_1(\|\mathbf{x}(\tau, \mathbf{x}_0)\|_{\mathcal{A}}) d\tau \geq \lim_{t \rightarrow \infty} \int_{t_1}^t \gamma_1(\delta) d\tau = \infty.$$

Thus we obtained a contradiction. Hence $\bar{x} = 0$ and, consequently,

$$\lim_{t \rightarrow \infty} \|\mathbf{x}(t)\|_{\mathcal{A}} = 0.$$

Then, according to the notion of steady-state characteristic in Definition 4, this is possible only if $\bar{h} \in \chi^{-1}(0)$. \square

A.3. Proof of Lemma 7. Analogously to the proof of Lemma 6 we notice that (82) holds. This, however, implies that for any constant and positive T , the limit

$$\lim_{t \rightarrow \infty} \int_t^{t+T} \gamma_1(\|\mathbf{x}(\tau)\|_{\mathcal{A}}) d\tau$$

exists and equals zero. Furthermore, $h(\mathbf{z}(t, \mathbf{z}_0)) \in [0, h(\mathbf{z}_0)]$ for all $t \geq t_0$. Hence there exists a time instant t' such that

$$\|\mathbf{x}(t)\|_{\mathcal{A}} \leq c \cdot h(\mathbf{z}_0) + \varepsilon \quad \forall t \geq t',$$

where $\varepsilon > 0$ is arbitrarily small. Then taking into account (31) we can conclude that

$$(84) \quad \lim_{t \rightarrow \infty} \int_t^{t+T} \gamma_1(\|\mathbf{x}(\tau)\|_{\mathcal{A}}) d\tau \geq \bar{\gamma} \int_t^{t+T} \|\mathbf{x}(\tau)\|_{\mathcal{A}} d\tau = 0.$$

Given that (81) holds, system (6) has the steady-state characteristic on average, and given that $\chi_T(\cdot)$ has no zeros in the positive domain, limiting relation (84) is possible only if $\bar{h} = 0$. Then, according to (6), $\lim_{t \rightarrow \infty} \|\mathbf{x}(t)\|_{\mathcal{A}} = 0$. \square

A.4. Proof of Corollary 8. As follows from Theorem 3, state $\mathbf{x}(t, \mathbf{x}_0) \oplus \mathbf{z}(t, \mathbf{z}_0)$ converges to the set Ω_a specified by (24). Hence $h(\mathbf{z}(t, \mathbf{z}_0))$ is bounded. Then, according to (8), estimate (82) holds. This, in combination with condition (34), implies that $\mathbf{z}(t, \mathbf{z}_0)$ is bounded. In other words,

$$\mathbf{x}(t, \mathbf{x}_0) \oplus \mathbf{z}(t, \mathbf{z}_0) \in \Omega' \quad \forall t \geq t_0,$$

where Ω' is a bounded subset in $\mathbb{R}^n \times \mathbb{R}^m$. Applying the Bolzano–Weierstrass theorem we can conclude that for every point $\mathbf{x}_0 \oplus \mathbf{z}_0 \in \Omega_\gamma$ there is an ω -limit set $\omega(\mathbf{x}_0 \oplus \mathbf{z}_0) \subseteq \Omega'$ (nonempty).

As follows from (C3) and Lemma 6, the following holds:

$$\lim_{t \rightarrow \infty} h(\mathbf{z}(t, \mathbf{z}_0)) \in \chi^{-1}(0).$$

Therefore, given that $h(\cdot) \in \mathcal{C}^0$, we can obtain that

$$\lim_{t_i \rightarrow \infty} (\mathbf{z}(t_i, \mathbf{z}_0)) = h \left(\lim_{t_i \rightarrow \infty} \mathbf{z}(t_i, \mathbf{z}_0) \right) = h(\omega_z(\mathbf{x}_0 \oplus \mathbf{z}_0)) \in \chi^{-1}(0).$$

In other words,

$$\omega_z(\mathbf{x}_0 \oplus \mathbf{z}_0) \subseteq \Omega_h = \{\mathbf{x} \in \mathbb{R}^n, \mathbf{z} \in \mathbb{R}^m \mid h(\mathbf{z}) \in \chi^{-1}(0)\}.$$

Moreover,

$$\omega_x(\mathbf{x}_0 \oplus \mathbf{z}_0) \subseteq \Omega_a = \{\mathbf{x} \in \mathbb{R}^n, \mathbf{z} \in \mathbb{R}^m \mid \|\mathbf{x}\|_{\mathcal{A}} = 0\}.$$

According to assumption (C1), the flow $\mathbf{x}(t, \mathbf{x}_0) \oplus \mathbf{z}(t, \mathbf{z}_0)$ is generated by a system of autonomous differential equations with a locally Lipschitz right-hand side. Then, as follows from [13, Lemma 4.1, page 127],

$$\lim_{t \rightarrow \infty} \text{dist}(\mathbf{x}(t, \mathbf{x}_0) \oplus \mathbf{z}(t, \mathbf{z}_0), \omega(\mathbf{x}_0 \oplus \mathbf{z}_0)) = 0.$$

Noticing that

$$\text{dist}(\mathbf{x}(t, \mathbf{x}_0) \oplus \mathbf{z}(t, \mathbf{z}_0), \omega(\mathbf{x}_0 \oplus \mathbf{z}_0)) \geq \text{dist}(\mathbf{x}(t, \mathbf{x}_0), \Omega_a) + \text{dist}(\mathbf{z}(t, \mathbf{z}_0), \Omega_h),$$

we can finally obtain that

$$\lim_{t \rightarrow \infty} \text{dist}(\mathbf{x}(t, \mathbf{x}_0), \Omega_a) = 0, \quad \lim_{t \rightarrow \infty} \text{dist}(\mathbf{z}(t, \mathbf{z}_0), \Omega_h) = 0. \quad \square$$

A.5. Proof of Corollary 9. As follows from Theorem 3, the corollary will be proved if Properties 1–3 are satisfied and also if (25), (26), and (27) hold. In order to satisfy Property 1 we select the following sequence \mathcal{S} :

$$(85) \quad \mathcal{S} = \{\sigma_i\}_{i=0}^{\infty}, \quad \sigma_i = \frac{1}{\kappa^i}, \quad \kappa \in \mathbb{R}_+, \quad \kappa > 1.$$

Let us chose sequences \mathcal{T} and Ξ as follows:

$$(86) \quad \mathcal{T} = \{\tau_i\}_{i=0}^{\infty}, \quad \tau_i = \tau^*,$$

$$(87) \quad \Xi = \{\xi_i\}_{i=0}^{\infty}, \quad \xi_i = \xi^*,$$

where τ^* , ξ^* are positive constants yet to be defined. Notice that choosing \mathcal{T} as in (86) automatically fulfills condition (27) of Theorem 3. On the other hand, taking into account (17), (39), and that $\beta_t(t)$ is monotonically decreasing in t , this choice defines a constant ξ^* as follows:

$$(88) \quad \beta_t(\tau^*) \leq \xi^* \beta_t(0) < \beta_t(0), \quad 0 \leq \xi^* < 1.$$

Given that the inverse β_t^{-1} exists, (40), this choice is always possible. In particular, (88) will be satisfied for the following values of τ^* :

$$(89) \quad \tau^* \geq \beta_t^{-1}(\xi^* \beta_t(0)).$$

Let us now find the values for τ^* and ξ^* such that Property 3 is also satisfied. For this purpose consider systems of functions Φ , Υ specified by (18), (19). Notice that function $\beta(s, 0)$ in (18), (19) is linear for system (39),

$$\beta(s, 0) = s \cdot \beta_t(0),$$

and therefore the functions $\rho_{\phi, j}(\cdot)$, $\rho_{v, j}$ are identity maps. Hence Φ , Υ reduce to the following:

$$(90) \quad \Phi : \begin{aligned} \phi_j(s) &= \phi_{j-1} \cdot \xi^* \cdot \beta(s, 0) = \xi^* \cdot \beta_t(0) \cdot \phi_{j-1}(s), \quad j = 1, \dots, i, \\ \phi_0(s) &= \beta_t(0) \cdot s, \end{aligned}$$

$$(91) \quad \Upsilon : \begin{aligned} v_j(s) &= \phi_{j-1}(s), \quad j = 1, \dots, i, \\ v_0(s) &= \beta_t(0) \cdot s. \end{aligned}$$

Taking into account (85), (90), and (91), let us explicitly formulate requirements (21), (22) in Property 3. These conditions are equivalent to the boundedness of the following functions:

$$(92) \quad \|\mathbf{x}(t_0)\|_{\mathcal{A}} \cdot \beta_t(0) \cdot \kappa^n (\xi^* \cdot \beta_t(0))^n;$$

$$(93) \quad \begin{aligned} & \kappa^n \left(\beta_t(0) \frac{c|h(\mathbf{z}_0)|}{\kappa^n} + \frac{\beta_t(0)c|h(\mathbf{z}_0)|}{\kappa^{n-1}} + \beta_t(0) \sum_{i=2}^n c|h(\mathbf{z}_0)| \frac{1}{\kappa^{n-i}} (\xi^* \cdot \beta_t(0))^{i-1} \right) \\ & = \beta_t(0)c|h(\mathbf{z}_0)| + \beta_t(0)c|h(\mathbf{z}_0)|\kappa \left(1 + \sum_{i=2}^n \kappa^{i-1} (\xi^* \cdot \beta_t(0))^{i-1} \right). \end{aligned}$$

Boundedness of the functions $B_1(\|\mathbf{x}_0\|_{\mathcal{A}})$ and $B_2(|h(\mathbf{z}_0)|, c)$ is ensured if ξ^* satisfies the inequality

$$(94) \quad \xi^* \leq \frac{d}{\kappa \cdot \beta_t(0)}$$

for some $0 \leq d < 1$. Notice that $\kappa > 1$, $\beta_t(0) \geq 1$ imply that $\xi^* \leq 1$, and therefore constant τ^* satisfying (89) will always be defined. Hence according to (92) and (93), the functions $B_1(\|\mathbf{x}_0\|_{\mathcal{A}})$ and $B_2(|h(\mathbf{z}_0)|, c)$ satisfying Property 3 can be chosen as

$$(95) \quad B_1(\|\mathbf{x}_0\|_{\mathcal{A}}) = \beta_t(0) \|\mathbf{x}_0\|_{\mathcal{A}}; \quad B_2(|h(\mathbf{z}_0)|, c) = \beta_t(0) \cdot c \cdot |h(\mathbf{z}_0)| \left(1 + \frac{\kappa}{1-d} \right).$$

In order to apply Theorem 3 we have to check the remaining conditions (25) and (26). This requires the possibility of factorization (10) for the function $\gamma_0(\cdot)$. According to assumption (41) of the corollary the function $\gamma_0(\cdot)$ is Lipschitz:

$$|\gamma_0(s)| \leq D_{\gamma,0} \cdot |s|.$$

This allows us to choose functions $\gamma_{0,1}(\cdot)$ and $\gamma_{0,2}(\cdot)$ as follows:

$$(96) \quad \gamma_{0,1}(s) = s, \quad \gamma_{0,2}(s) = D_{\gamma,0} \cdot s.$$

Condition (25), therefore, is equivalent to solvability of the following inequality:

$$(97) \quad \left(\frac{1}{\kappa^i} - \frac{1}{\kappa^{i+1}} \right) \frac{\kappa^i}{\tau^*} \geq \Delta_0.$$

Taking into account inequalities (89) and (94), we can derive that solvability of

$$(98) \quad \Delta_0 = \left(\beta_t^{-1} \left(\frac{d}{\kappa} \right) \right)^{-1} \frac{\kappa - 1}{\kappa}$$

implies existence of $\Delta_0 > 0$ satisfying (97) and, consequently, condition (25) of Theorem 3. Given that $d < 1$, $\kappa > 1$, and $\beta_t(0) \geq 1$, a positive solution to (98) is always defined. Hence the proof will be complete and the claim is nonvacuous if the domain

$$(99) \quad \begin{aligned} & D_{\gamma,0} \leq \left(\beta_t^{-1} \left(\frac{d}{\kappa} \right) \right)^{-1} \frac{\kappa - 1}{\kappa} \\ & \times \frac{h(\mathbf{z}_0)}{\beta_t(0) \|\mathbf{x}_0\|_{\mathcal{A}} + \beta_t(0) \cdot c \cdot |h(\mathbf{z}_0)| \left(1 + \frac{\kappa}{1-d} \right) + c|h(\mathbf{z}_0)|} \end{aligned}$$

is not empty. \square

A.6. Proof of Corollary 10. It follows from Corollary 9 that the state of the interconnection converges into Ω_a for all initial conditions $\mathbf{x}_0, \mathbf{z}_0$ satisfying (99). In other words, the following inequality should hold:

$$(100) \quad \begin{aligned} & D_{\gamma,0} \left(\beta_t(0) \|\mathbf{x}_0\|_{\mathcal{A}} + \beta_t(0) \cdot c \cdot |h(\mathbf{z}_0)| \left(1 + \frac{\kappa}{1-d} \right) + c|h(\mathbf{z}_0)| \right) \\ & \leq \left(\beta_t^{-1} \left(\frac{d}{\kappa} \right) \right)^{-1} \frac{\kappa-1}{\kappa} \cdot h(\mathbf{z}_0). \end{aligned}$$

Hence assuming that $h(\mathbf{z}_0) > 0$, we can rewrite (100) in the following way:

$$(101) \quad \begin{aligned} & D_{\gamma,0} \cdot \beta_t(0) \|\mathbf{x}_0\|_{\mathcal{A}} \\ & \leq \left(\left(\beta_t^{-1} \left(\frac{d}{\kappa} \right) \right)^{-1} \frac{\kappa-1}{\kappa} - D_{\gamma,0} \cdot c \left(\beta_t(0) \cdot \left(1 + \frac{\kappa}{1-d} \right) + 1 \right) \right) h(\mathbf{z}_0). \end{aligned}$$

Solutions to (101) exist, however, if the inequality

$$\left(\beta_t^{-1} \left(\frac{d}{\kappa} \right) \right)^{-1} \frac{\kappa-1}{\kappa} \geq D_{\gamma,0} \cdot c \left(\beta_t(0) \cdot \left(1 + \frac{\kappa}{1-d} \right) + 1 \right)$$

or, equivalently,

$$(102) \quad D_{\gamma,0} \cdot c \cdot \left(\beta_t(0) \cdot \left(1 + \frac{\kappa}{1-d} \right) + 1 \right) \cdot \beta_t^{-1} \left(\frac{d}{\kappa} \right) \frac{\kappa}{\kappa-1} < 1$$

is satisfied. The estimate of the trapping region follows from (101).

Let us finally show that continuity of $h(\mathbf{z})$ implies that the volume of Ω_γ is nonzero in $\mathbb{R}^n \oplus \mathbb{R}^m$. For the sake of compactness we rewrite inequality (101) in the following form:

$$(103) \quad \|\mathbf{x}_0\|_{\mathcal{A}} \leq C_\gamma h(\mathbf{z}_0),$$

where C_γ is a constant depending on $d, \kappa, \beta_t(0)$, and $D_{\gamma,0}$. Given that (102) holds we can conclude that $C_\gamma > 0$. According to (103), domain Ω_γ contains the following set:

$$\{\mathbf{x}_0 \in \mathbb{R}^n, \mathbf{z}_0 \in \mathbb{R}^m \mid h(\mathbf{z}_0) > D_z \in \mathbb{R}_+, \|\mathbf{x}_0\|_{\mathcal{A}} \leq C_\gamma D_z\}.$$

Consider the following domain: $\Omega_{\mathbf{x},\gamma} = \{\mathbf{x}_0 \in \mathbb{R}^n \mid \|\mathbf{x}_0\|_{\mathcal{A}} \leq C_\gamma D_z\}$. Clearly, it contains a point $\mathbf{x}_{0,1} \in \mathbb{R}^n : \|\mathbf{x}_{0,1}\|_{\mathcal{A}} = \frac{C_\gamma D_z}{2}$. For the point $\mathbf{x}_{0,1}$ and for all $\boldsymbol{\varepsilon}_1 \in \mathbb{R}^n : \|\boldsymbol{\varepsilon}_1\| \leq \frac{C_\gamma D_z}{4}$ we have that $\|\mathbf{x}_{0,1} + \boldsymbol{\varepsilon}_1\|_{\mathcal{A}} = \inf_{\mathbf{q} \in \mathcal{A}} \|\mathbf{x}_{0,1} + \boldsymbol{\varepsilon}_1 - \mathbf{q}\| \leq \inf_{\mathbf{q} \in \mathcal{A}} \{\|\mathbf{x}_{0,1} - \mathbf{q}\| + \|\boldsymbol{\varepsilon}_1\|\} \leq \frac{3C_\gamma D_z}{4}$. On the other hand, $\|\mathbf{x}_{0,1} + \boldsymbol{\varepsilon}_1\|_{\mathcal{A}} = \inf_{\mathbf{q} \in \mathcal{A}} \|\mathbf{x}_{0,1} + \boldsymbol{\varepsilon}_1 - \mathbf{q}\| \geq \inf_{\mathbf{q} \in \mathcal{A}} \{\|\mathbf{x}_{0,1} - \mathbf{q}\| - \|\boldsymbol{\varepsilon}_1\|\} \geq \frac{C_\gamma D_z}{4}$. This implies that there exists a set of points $\mathbf{x}_{0,2} = \mathbf{x}_{0,1} + \boldsymbol{\varepsilon}_1 \in \mathbb{R}^n : \|\mathbf{x}_{0,1} - \mathbf{x}_{0,2}\| \leq \frac{C_\gamma D_z}{4}, \mathbf{x}_{0,2} \notin \mathcal{A}, \|\mathbf{x}_{0,2}\|_{\mathcal{A}} \leq C_\gamma D_z$.

Consider now the following domain: $\Omega_{\mathbf{z},\gamma} = \{\mathbf{z}_0 \in \mathbb{R}^m \mid h(\mathbf{z}_0) > D_z\}$. Let us pick $\mathbf{z}_{0,1} \in \Omega_{\mathbf{z},\gamma} : h(\mathbf{z}_{0,1}) = 2D_z$. Because $h(\cdot)$ is continuous we have that

$$\forall \varepsilon > 0, \exists \delta > 0 : \|\mathbf{z}_{0,1} - \mathbf{z}_{0,2}\| < \delta \Rightarrow |h(\mathbf{z}_{0,1}) - h(\mathbf{z}_{0,2})| < \varepsilon.$$

Let $\varepsilon = D_z$; then $-D_z < h(\mathbf{z}_{0,1}) - h(\mathbf{z}_{0,2}) < D_z$ and therefore $h(\mathbf{z}_{0,2}) > D_z$. Hence there exists a set of points $\mathbf{z}_{0,2} \in \mathbb{R}^m : \|\mathbf{z}_{0,1} - \mathbf{z}_{0,2}\| < \delta, \mathbf{z}_{0,2} \in \Omega_{\mathbf{z},\gamma}$.

Consider the following set:

$$\Omega_{\mathbf{xz},\gamma} = \left\{ \mathbf{x}' \in \mathbb{R}^n, \mathbf{z}' \in \mathbb{R}^m \mid \|\mathbf{x}_{0,1} - \mathbf{x}'\|^2 + \|\mathbf{z}_{0,1} - \mathbf{z}'\|^2 \leq r^2, r = \min \left\{ \delta, \frac{C_\gamma D_z}{4} \right\} \right\}.$$

For all $\mathbf{x}_0, \mathbf{z}_0 \in \Omega_{\mathbf{xz},\gamma}$ we have that $\mathbf{x}_0 \in \Omega_{\mathbf{x},\gamma}, \mathbf{z}_0 \in \Omega_{\mathbf{z},\gamma}$. Hence inequality (103) holds, and $\mathbf{x}_0 \oplus \mathbf{z}_0 \in \Omega_\gamma$. The volume of the set $\Omega_{\mathbf{xz},\gamma}$ is defined by the volume of the interior of a sphere in \mathbb{R}^{n+m} with nonzero radius. Thus the volume of $\Omega_\gamma \supset \Omega_{\mathbf{xz},\gamma}$ is also nonzero. \square

A.7. Proof of Corollary 11. Let $\lambda(\tau, \lambda_0)$ be a solution of system (51). Consider it as a function of variable τ . Let us pick some monotone, strictly increasing function σ such that the following holds:

$$\tau = \sigma(t), \sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+.$$

Given that $\eta(\Omega_\lambda)$ is dense in Ω_θ , for any $\theta \in \Omega_\theta$ there always exists a vector $\lambda_\theta \in \Omega_\lambda$ such that $\eta(\lambda_\theta) = \theta + \epsilon_\theta$, where $\|\epsilon_\theta\|$ is arbitrarily small. Furthermore, $\lambda(\tau)$ is dense in Ω_λ , and hence there is a point $\lambda^* = \lambda(\tau^*, \lambda_0)$ which is arbitrarily close to λ_θ . Consider the following difference:

$$\mathbf{f}(\xi(t), \theta) - \mathbf{f}(\xi(t), \hat{\theta}) = \mathbf{f}(\xi(t), \theta) - f(\xi(t), \eta(\lambda^*)) + \mathbf{f}(\xi, \eta(\lambda^*)) - \mathbf{f}(\xi, \eta(\lambda(\sigma(t)))).$$

The function $\mathbf{f}(\cdot)$ is locally bounded and $\eta(\cdot)$ is Lipschitz, and then

$$\|\mathbf{f}(\xi, \theta) - \mathbf{f}(\xi, \eta(\lambda^*))\| \leq D_f \|\epsilon_\theta\| + \Delta_f = \Delta_\theta + \Delta_f,$$

where Δ_θ is arbitrarily small. Hence

$$(104) \quad \begin{aligned} \|\mathbf{f}(\xi, \eta(\lambda^*)) - \mathbf{f}(\xi, \eta(\lambda(\sigma(t))))\| &\leq D_f \|\eta(\lambda^*) - \eta(\lambda(\sigma(t)))\| + \Delta_f + \Delta_\theta \\ &\leq D_f \cdot D_\eta \|\lambda^* - \lambda(\sigma(t))\| + \Delta_f + \Delta_\theta. \end{aligned}$$

Noticing that $\lambda^* = \lambda(\tau^*, \lambda_0) = \lambda(\sigma(\tau^*), \lambda_0)$ and taking into account the Poisson stability of (51), we can always choose $\lambda^*(\sigma^*, \lambda_0)$ such that $\sigma^* > \sigma(t_0) = \tau_0$ for any $\tau_0 \in \mathbb{R}_+$. Hence, according to (104) the following estimate holds:

$$(105) \quad \begin{aligned} \|\mathbf{f}(\xi, \eta(\lambda^*)) - \mathbf{f}(\xi, \eta(\lambda(\sigma(t))))\| &\leq D_f \cdot D_\eta \left\| \int_{\sigma(t)}^{\sigma^*} S(\lambda(\sigma(\tau))) d\tau \right\| + \Delta_f + \Delta_\theta \\ &\leq D_f \cdot D_\eta \cdot \max_{\lambda \in \Omega_\lambda} \|S(\lambda)\| |\sigma^* - \sigma(t)| = \mathcal{D} \cdot |\sigma^* - \sigma(t)| + \Delta_f + \Delta_\theta, \\ \mathcal{D} &= D_f \cdot D_\eta \cdot \max_{\lambda \in \Omega_\lambda} \|S(\lambda)\|. \end{aligned}$$

Denoting $\mathbf{u}(t) = \mathbf{f}(\xi(t), \theta) - \mathbf{f}(\xi(t), \hat{\theta}) + \varepsilon(t)$ we can now conclude that

$$(106) \quad \begin{aligned} \|\mathbf{u}(t)\| &\leq \Delta_\epsilon + \Delta_f + \|\mathbf{f}(\xi(t), \theta) - \mathbf{f}(\xi(t), \eta(\lambda^*))\| + \mathcal{D} \cdot |\sigma^* - \sigma(t)| \\ &\leq \Delta_\epsilon + 2\Delta_f + \Delta_\theta + D_f \|\theta - \eta(\lambda^*)\| + \mathcal{D} \cdot |\sigma^* - \sigma(t)|. \end{aligned}$$

Notice that due to the denseness of $\lambda(t, \lambda_0)$ in Ω_λ it is always possible to choose λ^* such that

$$D_f \|\theta - \eta(\lambda^*)\| = D_f \|\eta(\lambda_\theta) - \eta(\lambda^*)\| \leq D_f D_\eta \|\lambda_\theta - \eta(\lambda^*)\| \leq \Delta_\lambda.$$

Hence, according to (106), we have

$$\|\mathbf{u}(t)\|_{\infty,[t_0,t]} \leq 2\Delta_f + \Delta_\varepsilon + \delta + \mathcal{D} \cdot \|\sigma^* - \sigma(t)\|_{\infty,[t_0,t]},$$

where the term $\delta > \Delta_\theta + \Delta_\lambda$ can be made arbitrarily small.

Therefore Assumption 3 implies that the following inequality holds:

$$(107) \quad \|\mathbf{x}(t)\|_{\mathcal{A}_{\Delta(M)}} \leq \beta(t - t_0) \|\mathbf{x}(t_0)\|_{\mathcal{A}_{\Delta(M)}} + c \cdot \mathcal{D} \cdot \|\sigma^* - \sigma(t)\|_{\infty,[t_0,t]}.$$

Let us now define $\sigma(t)$ as follows:

$$(108) \quad \sigma(t) = \int_{t_0}^t \gamma \|\psi(\mathbf{x}(\tau))\|_{\mathcal{A}_{\Delta(M)}} d\tau.$$

Moreover, let us introduce the following notation:

$$h(t) = \sigma^* - \sigma(t) = \sigma^* - \int_{t_0}^t \gamma \|\psi(\mathbf{x}(\tau))\|_{\mathcal{A}_{\Delta(M)}} d\tau;$$

then for all $t', t \geq t_0, t \geq t'$ we have that

$$h(t') - h(t) = \int_{t'}^t \gamma \|\psi(\mathbf{x}(\tau))\|_{\mathcal{A}_{\Delta(M)}} d\tau.$$

Taking into account (104), (105), equality

$$\frac{\partial \boldsymbol{\lambda}(\sigma(t), \boldsymbol{\lambda}_0)}{\partial t} = \frac{\partial \sigma(t)}{\partial t} S(\boldsymbol{\lambda}(\sigma(t), \boldsymbol{\lambda}_0)) = \gamma \|\psi(\mathbf{x}(\tau))\|_{\mathcal{A}_{\Delta(M)}} S(\boldsymbol{\lambda}(\sigma(t), \boldsymbol{\lambda}_0)),$$

and (107), and denoting $D_\lambda = c\mathcal{D}$, we can conclude that the following holds along the trajectories of (52):

$$(109) \quad \begin{aligned} \|\mathbf{x}(t)\|_{\mathcal{A}_{\Delta(M)}} &\leq \beta(t - t_0) \|\mathbf{x}(t_0)\|_{\mathcal{A}_{\Delta(M)}} + D_\lambda \|h(t)\|_{\infty,[t_0,t]}, \\ h(t_0) - h(t) &= \int_{t_0}^t \gamma \|\psi(\mathbf{x}(\tau))\|_{\mathcal{A}_{\Delta(M)}} d\tau. \end{aligned}$$

Hence, according to Corollary 8, the limit relation (54) holds for all $|h(t_0)|, \|\mathbf{x}(t_0)\|_{\mathcal{A}_{\Delta(M)}}$ which belong to the domain

$$\begin{aligned} \Omega_\gamma : \gamma &\leq \left(\beta_t^{-1} \left(\frac{d}{\kappa} \right) \right)^{-1} \frac{\kappa - 1}{\kappa} \\ &\times \frac{h(t_0)}{\beta_t(0) \|\mathbf{x}(t_0)\|_{\mathcal{A}_{\Delta+\delta}} + \beta_t(0) \cdot D_\lambda \cdot |h(t_0)| \left(1 + \frac{\kappa}{1-d} \right) + D_\lambda |h(t_0)|} \end{aligned}$$

for some $d < 1, \kappa > 1$. Notice, however, that $\|\mathbf{x}(t)\|_{\mathcal{A}_{\Delta+\delta}}$ is always bounded, as $\mathbf{f}(\cdot)$ is Lipschitz in θ and both $\boldsymbol{\theta}$ and $\hat{\boldsymbol{\theta}}$ are bounded ($\boldsymbol{\eta}(\cdot)$ is Lipschitz and $\boldsymbol{\lambda}(t, \boldsymbol{\lambda}_0)$ is bounded according to assumptions of the corollary). Moreover, due to the Poisson stability of (51) it is always possible to choose a point $\boldsymbol{\lambda}^*$ such that $h(t_0) = \sigma^*$ is arbitrarily large. Hence the choice of γ in (109), as (53) suffices to ensure that $h(t)$ is bounded. Moreover, it follows that $h(t)$ converges to a limit as $t \rightarrow \infty$. This implies

that $\gamma \int_{t_0}^t \|\mathbf{x}(\tau)\|_{\mathcal{A}_{\Delta(M)}}$ also converges as $t \rightarrow \infty$ and, consequently, $\boldsymbol{\lambda}(t, \boldsymbol{\lambda}_0)$ converges to some $\boldsymbol{\lambda}' \in \Omega_\lambda$. Hence the following holds:

$$\lim_{t \rightarrow \infty} \hat{\boldsymbol{\theta}}(t) = \boldsymbol{\theta}'$$

for some $\boldsymbol{\theta}' \in \Omega_\theta$. According to the corollary conditions, system (50) has steady-state characteristics with respect to $\hat{\boldsymbol{\theta}}$. Then, in the same way as in the proof of Lemma 6, we can show that (54) holds. \square

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