Kinematics and dynamics of multibody system : a systematic approach to systems with arbitrary connections

KINEMATICS AND DYNAMICS OF MULTIBODY SYSTEMS

A SYSTEMATIC APPROACH TO SYSTEMS WITH ARBITRARY CONNECTIONS

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CONTENTS

ABSTRACT

The object of this study is to develop tools for the analysis of the kinematic and dynamic behaviour of multibody systems with arbitrary connections. The behaviour of such systems is described by sets of nonlinear algebraic and/or differential equations. Tools are available for the construction as well as for the solution of these equations. A severe limitation of the existing tools is that only simple connections are allowed. In this study a theory is described for systematically setting up the equations for multibody systems with arbitrary connections.

The first chapter is meant as an introduction to multibody theory in general. Chapter 2, on the kinematics and dynamics of a rigid body, is also intended as an introduction of the notation in the subsequent chapters. Chapter 3 brings in several important concepts concerning elements of connections, while the concept of the tree structure of bodies and hinges is discussed in chapter 4. This tree structure is used to describe the topology of a multibody system.

The tree-structure concept allows us to set up the relevant equations systematically. In chapter 5 the constraint equations describe the kinematics, and in chapter 6 the equations of motion are derived. These two chapters and chapter 7, which describes the assembly of simple connections to form complex ones, are the central chapters in this study.

The theory is used in chapter 8 to formulate the kinematic and dynamic simulation problem for multibody systems. In the next chapter an example of a multibody system is studied. This system, a fuel injection pump, contains three nonstandard connections, namely an elastohydrodynamic traction, a hydrodynamic bearing and a cam.

The last chapter contains the conclusions and a discussion on possible further research. In the appendix the mathematical notation used in the present study is discussed.

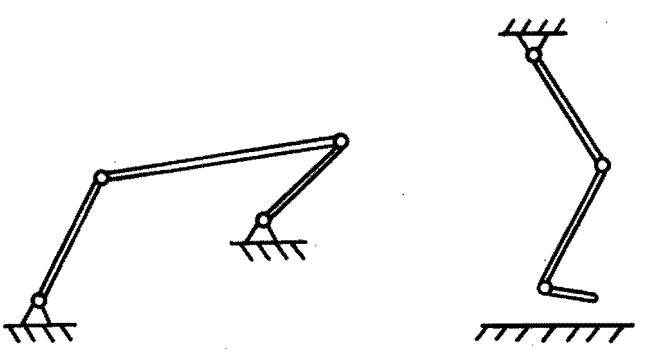
CHAPTER 1
INTRODUCTION

1.1 Scope of the study
1.2 Literature survey
1.3 Themes dealt with

This chapter is an introduction to the multibody theory as presented in the following chapters. It starts with a discussion on the scope of the present study. A literature survey is discussed next. This survey has been included for the benefit of the reader. Finally the main items of this study are mentioned as introduction to the following chapters.

1.1 Scope of the study

Multibody systems are considered as interconnected systems of rigid bodies. In the present instance a theory is presented for the analysis of multibody systems having arbitrary connections.

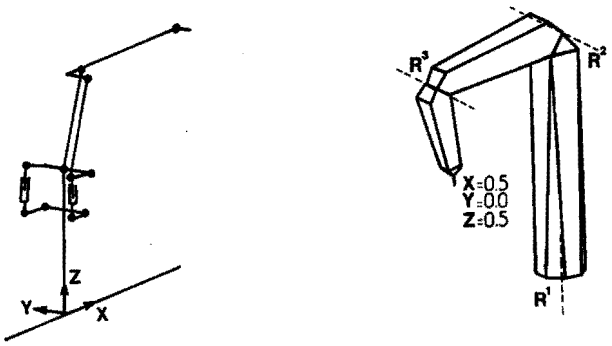


4-bar mechanism swing phase of leg

figure 1.1 Two simple multibody systems

This theory allows us to simulate the kinematic and dynamic behaviour of multibody systems like the simple planar 4-bar mechanism (see

figure 1.1) as well as complex, three-dimensional machines like industrial robots (see figure 1.2). In particular, the theory is developed for the analysis of multibody systems which have complicated connections. Such connections are to be found, for example in cam-mechanisms and in the human musculo-skeletal system.



ASEA robot 3R robot

figure 1.2 Two complicated multibody systems

The kinematic and dynamic behaviour of multibody systems can be described by means of mathematical equations. Multibody theories are therefore defined as theories or methods for the construction of these algebraic or differential equations. With the algebraic equations we analyse kinematic behaviour, with second order differential equations dynamic behaviour is analyzed. For very simple systems it is sometimes possible to solve the equations analytically. This is extremely difficult in the case of systems with several bodies. The equations become too complex and because they are highly nonlinear they have to be solved numerically.

Fischer [1906], for instance formulated the equations for the dynamic behaviour of a three body system representing a human limb. However, this was not very useful since he could not solve his equations at that time. At present powerful computers and well developed software solve these equations automatically. Working out the required equations for multibody systems by hand is a very difficult and error-prone job. When numerical methods for the solution of these equations

became available, research was initiated to enabel this to be done by computer. The results of this research were so-called multibody programs which automatically set up and solve the equations. Clearly this is a valuable development since it significantly simplifies the analysis of multibody systems.

The first multibody programs were written for systems with a fixed number of bodies. Only the lengths, body masses, stiffness of springs and similar parameters could be changed. Besides their value as a basis for further developments, these programs can be used for the design of a particular system. The second generation of programs was more useful since they allowed the behaviour of systems with an arbitrary number of bodies to be simulated in 2-dimensional (planar) and 3-dimensional spaces. In most programs prismatic, pin and/or ball-and-socket joints, as well as linear springs and dampers can be used to model the system.

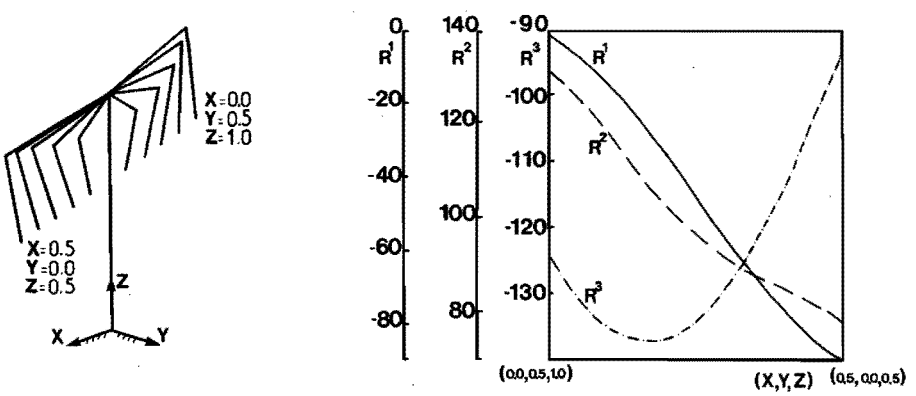


figure 1.3 Results for 3R robot of figure 1.2b

Commercially available programs can be used for the analysis of mechanisms in particular. These programs (IMP, ADAMS, DRAM, etc.) allow the use of several technically important connections, have extended graphic facilities and use improved numerical solvers [SDRC-IMP 1979] (the results in figure 1.3 are obtained with IMP from SDRC Ohio).

Limitations are encountered when using the presently available multibody programs for biomechanical research. The most serious limitation

is the small set of simple connections that can be used. It is of course possible to model the knee-joint as a pin-joint, but more realistic models of the knee are the 4-bar mechanism model or the 3-dimensional knee model of Wismans [et al. 1980]. In the last-named model the knee is represented as a 3-dimensional cam-mechanism whose surfaces are described by sets of polynomials. Another example of the limitation of presently available programs is the modelling of ligaments and muscles by connections having a straight line geometry and a linear constitutive behaviour. More realistic models should have an arbitrary geometry and a description of the behaviour by more appropriate constitutive and eventually state equations.

During the last decade, multibody programs were only used in biomechanical research for the simulation of human/vehicle interaction in injury prevention research. In sport biomechanics, but particularly in gait analyses the equations were still set up by hand. When we realize how complex the human musculo-skeletal system is, one may well question whether this approach results in useful and realistic models [Hatze 1980].

A remarkable example of an over-simplification is the modelling of the human leg during the swing phase of a stride. It is logical to model the leg as a double pendulum, but it is unrealistic to assume that only slight rotational changes take place. If we nevertheless assume that only slight changes take place, we can linearize the equations and determine eigenvalues and eigenvectors [Maillardet 1977]. But with changes of 50 degrees [Murray et al. 1964] the sine and cosine terms may not be linearized.

The purpose of the present study is to develop a tool for the analysis of the kinematic and dynamic behaviour of the musculo-skeletal system. In contrast to the femur project [Huiskes 1979] and the knee project [Wismans et al. 1980, Hamer 1982] it was decided not to start modelling another item of the musculo-skeletal system in more detail, but to develop a more adequate multibody theory for modelling the musculo-skeletal system or parts of it. When this study was started [Sol 1980] not much was known about realistic models for parts of the

musculo-skeletal system. It was therefore decided to develop a multibody theory for arbitrary connections.

1.2 Literature survey

Kinematics
Dynamics
Recent developments
Applications

The survey begins with a short discussion of the literature which deals with purely kinematic aspects. Then, based on different formalisms used to set up the equations for dynamic behaviour, several important multibody theories are mentioned. Some attention is also given to references on recent developments. Finally, application-orientated references are discussed. In particular, references in the fields of robotics and biomechanics are discussed. This survey makes no claim to completeness, its purpose is only to supply some background information on multibody theories in general. (See also figure 1.4)

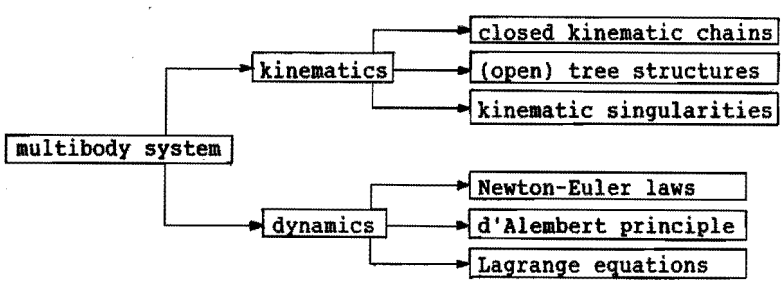


figure 1.4 Scheme with main items of multibody theories

Kinematics

Two approaches are important in describing the kinematic behaviour of multibody systems. The first is the closed-kinematic-chain approach. Closed kinematic chains are well known in the theory of mechanisms.

These chains can be modelled with the aid of loop-equations [Suh and Radcliffe 1978, Paul B 1979, Angeles 1982]. In American literature the Denavit-Hartenberg notation with the 4x4 transformation matrix is often used [Uicker, Denavit and Hartenberg 1964, Paul R 1981]. In the context of multibody theories Sheth's dissertation [Sheth 1972] gives a comprehensive treatment on the way this 4x4 notation is implemented in the IMP program.

The second approach to the description of the kinematic behaviour of multibody systems is that of the tree structure. It has been used to analyse spacecraft [Wittenburg 1977], industrial robots [Hollerbach 1980, Vukobratovic and Potkonjak 1982] and the human musculo-skeletal system [Hatze 1977]. These systems have a tree structure, while mechanisms generally have a closed chain. The theory which we develop in this study is based on the tree-structure approach. Nevertheless this theory is not restricted to systems with a tree structure. An extension to include closed kinematic chains is described too.

A very important, but often neglected problem is the occurence of kinematic singularities. What kind of checks are possible for detecting such singularities and how can the inherent problems be solved? Only Sheth [1972], in the context of closed kinematic chains, gives an exhaustive discussion on this subject. He also put forward a strategy to solve the inherent problems. For tree structures Whitney [1969, 1972] stated the problem in a completely different context and suggested some solutions. Based on the ideas of Sheth, a strategy for the detection and solution of kinematic singularities for multibody systems having an arbitrary topology is developed in the present study.

Dynamics

The literature on dynamic aspects is divided in three parts, that is the Newton-Euler laws, the virtual work principle of d'Alembert and the equations of Lagrange. Before discussing this literature some review articles will be mentioned.

There is an interesting article by Paul B [1975] on the use of the Newton-Euler laws and the Lagrange equations. He also dealt with numerical aspects as well as methods for the calculation of reaction forces. In his book 'Kinematics and dynamics of planar machinery' [1979] a large part is devoted to the description of the simulation of the kinematic and dynamic behaviour of multibody systems. Another survey can be found in the dissertation of Renaud [1975]. In this work all methods known at that time are discussed.

There are situations, to be discussed later on, where multibody programs with a minimum number of numerical operations are of prime importance. Hollerbach [1980] reviewed several multibody theories with regard to the number of required additions and multiplications. We also mention the survey by Kaufman [1978] on commercially available multibody programs for mechanisms and machine design and that by King and Chou [1976] on multibody programs for injury prevention research.

The Newton-Euler laws

The Newton-Euler laws are a combination of the second law of Newton (sum of forces equals change of momentum) and Euler's law for the change of angular momentum. Both laws lead for an n-body system to a set of 6n second-order differential equations describing the dynamic behaviour of the system.

The first publications on the computerized handling of the equations describing the dynamic behaviour of multibody systems are based on the Newton-Euler laws. These publications originate from spacecraft research [Fletcher et al. 1963, Hooker and Margulies 1965]. Particular progress was made by Roberson and Wittenburg [1966] for the description of the topology of systems with an arbitrary number of rigid bodies. In 1970 Wittenburg published results obtained with a program based on this approach.

Andrews and Kesavan [1975] also developed a Newton-Euler method for the analysis of multibody systems in a 3-dimensional space. Their program, called VECNET, is based on the combination of a formalism with vectors with ideas from network theories. Other programs based

on the Newton-Euler laws were developed during that time too. Here we mention MEDUSA [Dix and Lehman 1972] and the work of Gupta [1974] and Stepanenko and Vukobratovic [1976].

Some new publications have recently appeared in the field of robotics. In their paper, Luh, Walker and Paul R [1980a] develop a remarkably fast program based on the Newton-Euler laws. Hollerbach [1980] and Lee [1982] describe the same approach. The comparison by Luh et al. with regard to the computation time required by different programs is misleading: comparing a generally applicable program, based on the Lagrange equations and written in Fortran, with an optimized assembly program, based on the Newton-Euler laws and special written for a particular system, results in some exaggerated differences in computation time. The comparison by Hollerbach is more sensible.

The virtual work principle of d'Alembert

The principle of d'Alembert used in this study is based on the principle of virtual work. We will therefore call it the virtual work principle of d'Alembert [Renaud 1975]. Some authors, including Paul B [1979, p568] call this method the Lagrange-d'Alembert principle as Lagrange was the first to combine d'Alembert's inertial loads with Bernoulli's principle of virtual work [Rosenberg 1977, p125]. In this principle, generalized coordinates play a central role. The position and orientation of all bodies are described as function of such coordinates. As a result, a set of nq differential equations is found where nq, the number of generalized coordinates or Lagrange coordinates (see section 5.1), satisfies $0 < nq \leq 6n$.

There are several references in which the principle of d'Alembert is used differently. On the basis of relations between generalized coordinates and variables used to describe the position and orientation of all bodies, the Newton-Euler equations can be transformed into a smaller set of equations [Kane 1961, Hooker 1970, Langrana and Bartel 1975, Huston and Passerello 1979]. This approach finally results in exactly the same equations as the method mentioned earlier.

One of the first multibody programs based on the virtual work principle of d'Alembert was DYMAC, written by Paul and Krajcinovic [1970]. These authors only considered planar motions while large parts of the required equations had to be set up by hand. Mention should also be made to Williams and Seireg's work [1979] in which a generally applicable method is described. Lilov and Wittenburg [1977] also developed a general method. For a system with an arbitrary topology of bodies and connections they presented a theory especially suited for implementation in a multibody program. It is this theory and the improvements described in Wittenburg's book 'Dynamics of systems of rigid bodies' [1977] that we will use as a basis for our theory.

The Lagrange equations

To set up the equations of motion, the Lagrange method does not use the principle of virtual work but the Lagrange equations. These equations can be derived with the aid of the kinetic and potential (conservative) energy formulated as a function of some generalized coordinates (for example see Goldstein [1980, p20]. In the context of multibody systems, Bråt [1973] describes and illustrates this formalism for a simple example.

The Lagrange equations have been widely used in multibody theories. The first application of the Lagrange method in a multibody program was made by Wittenburg [1968, extracted from his disseration]. Probably because this work is written in German, hardly any references are ever made to it. A more cited work is that of Uicker [1967, 1969]. In 1972 he and Sheth developed the IMP program.

During the same time another well-known program was developed by Chance and Smith [Chance and Bayazitoglu 1971, Smith 1973]. Their program was first called DAMN, later DRAM. The program ADAMS, developed by Orlandea [Orlandea et al. 1977], makes extensive use of Lagrange multipliers, sparse matrix techniques and a special solver for stiff differential equations. IMP, DRAM and ADAMS are commercially available [Kaufman 1978]. They can be used for example to calculate the loads on wheel suspensions, while critical parts can be further analysed with finite element techniques.

It must be said, in fact, that the virtual work principle of d'Alembert and the Lagrange equations both result in exactly the same differential equations. It is probably just a matter of taste whether the Lagrange equations or the virtual work principle of d'Alembert is used. For example, for programs based on the Lagrange method the addition of Coulomb friction and intermittent motion have been described in literature [Threlfall 1978, Wehage and Haug 1982b]. To include these features in programs based on the virtual work principle of d'Alembert no significant different problems will be involved (for example see Wittenburg [1977 ch 6] on impact problems). The Lagrange equations have also been used to develop multibody programs for special systems. There are several examples in robotics and biomechanics especially.

Recent developments

Some new developments in multibody theories and programs must be mentioned. In this subsection we will first discuss some software-orientated developments and then discuss a number of theoretical developments.

Sometimes the differential equations describing the dynamic behaviour of multibody systems result in a problem with stiff differential equations. In these equations both very high, as well as very low eigenfrequencies occur. Such equations can only be solved with special implicit solvers [Gear 1971]. Orlandea et al. [1977] give much attention to this problem, while Cipra and Uicker [1981] discuss it too. Although most older multibody programs use the fourth order Runge-Kutta solver for the differential equations, recent articles [Hatze and Venter 1981, Allen 1981, Wehage and Haug 1982a] mention the use of the DE/STEP solver of Shampine and Gordon [1975]. This solver can be classified as a linear multistep solver with a variable order and variable step length. Such solvers are specially suited for use in problems in which the evaluation of the differential equations requires much computational effort.

The symbolic manipulation programs are another software development [Levinson 1977, Schiehlen and Kreuzer 1977]. These programs set up

the equations automatically in the form of analytical relations. Their drawback is probably the specialist knowledge required in their use. The recent hardware and software improvements for computer graphics are another noteworthy development. It should be realised that it is useless to analyse the behaviour of 3-dimensional multibody systems without adequate graphical facilities. Developments in this field seem very promising [Orlandea and Berenyi 1981].

The theoretical developments of importance for the analysis of multibody systems can be subdivided into two categories. The first category is the improvement of the present second-generation multibody programs for simulation studies. The second category concerns the development of a new generation of multibody programs for optimization studies. Improvements in the simulation programs are the addition of Coulomb friction, intermittent motion and impacts, nonrigid bodies and arbitrary connections. The first two features have already been mentioned. Nonrigid bodies are analysed by superposition of small deformations on the motion of rigid bodies [v.d. Werff 1977]. This approach is important for the analysis of spacecraft [Roberson 1972, Boland et al. 1977] and high-speed mechanisms and machines [Imam and Sandor 1973]. Improvements in multibody programs for the performance of very fast calculations have already been mentioned in the subsection on the Newton-Euler laws.

After the development of computer programs that automatically set up and solve the equations describing the behaviour of multibody systems, we may expect programs for automatically optimization of that behaviour. Two kinds of optimization can be considered, namely optimization of kinematic behaviour and optimization of dynamic behaviour.

Much work has already been done in the field of mechanism synthesis to optimize the kinematic behaviour of multibody systems [Freudenstein 1959, Kaufman 1973, Root and Ragsdell 1976]. It is characteristic for many developments in this field that the equations are still set up by hand [Suh and Radcliffe 1978, Haug and Arora 1979, Angeles 1982]. At present only Sohoni and Haug [1982a,b] and Langrana and Lee [1980] describe methods which are suitable for use as a basis for

multibody programs with optimization facilities. This last work uses a gradient solver which is more reliable and faster than the penalty solver used by Suh and Radcliffe. Based on this study the use of the more reliable and faster converging augmented Lagrange solver has been proposed [Sol et al. 1983].

If the aim is to optimize dynamic behaviour, two different kinds of problems are encountered. Examples of the simpler kind of problem are: optimum balancing of machines [Berkhof 1973, Sohoni and Haug 1982b], the (minimum) weight optimization [Thornton et al. 1979, Imam and Sandor 1973] and the design-sensitivity studies [Haug et al. 1981, Haug and Ehle 1982]. The second kind of problem is that of optimal control. In this case the purpose is to determine the optimal input or control variables as well as the optimal trajectories of the kinematic and force variables. Examples of performance criteria to be minimized are minimum time, minimum energy consumption etc. Optimal control problems result in nonlinear boundary-value problems which are very difficult to solve [Bryson and Ho 1975, Sage and White 1977]. In the next subsection on robotics and biomechanics some references will be made on optimal control.

Applications

An interesting application of multibody theories is robotics. Robots perform large movements in 3-dimensional space. Hence, the equations describing their kinematic and dynamic behaviour are highly nonlinear and coupled [Duffy 1980]. For example, if a position servo controls the rotation of a certain joint, the rotations of other joints can be influenced too. To solve this problem control engineers make use of multibody theories [Whitney 1972, Renaud 1975, Vukobratovic 1975 and Paul R 1981].

Control devices in industrial robots sample data at frequencies between 10-100 Hz. Based on the measured and the prescribed motion, the control device should be able to calculate and adjust a control signal within tenths of a second. Only recently, special multibody programs with which the required calculations can be performed in real-time have been developed [Luh et al. 1980b, Hollerbach 1980]. Until

that time it was necessary to calculate all necessary data in advance and to feed this data into the local memory of the control device [Albus 1975, Raibert and Horn 1978, Popov et al. 1981]. Another approach is to neglect several terms that are difficult to evaluate. However, for high speed motion, these terms cease to be negligible.

In this context it should be mentioned that instead of using the exact equations describing a multibody system, approximated or simplified equations can also be used. By means of an adaptive control these approximated equations should be updated each time [Liègeois 1977, Dubowsky and DesForges 1979, Hewit and Burdess 1981]. But if adaptive control is used, one should verify the stability. Multibody programs (with the exact equations) can be used for off-line simulation of the stability of such control devices.

Another case where off-line use of multibody programs is encountered is the elaborating of optimal control strategies. As we have said earlier, this problem results in a nonlinear two-point boundary-value problem which is difficult to solve. Kahn and Roth [1971] constructed the equations for a three-body system by hand and described a method to solve the minimum-time problem. This approach has also been dealt with by Vukobratovic and his co-workers [Cvetkovic and Vukobratovic 1981, Vukobratovic and Kircanski 1982, Vukobratovic and Stokic 1982 p69-95].

Biomechanical research is now using multibody programs more and more as a tool. For the "mathematical" simulation experiments in injury prevention research especially, much use is made of multibody programs because, compared with dummy experiments, parameters can be changed much more easily [Roberts and Thompson 1974, King and Chou 1976, Bacchetti and Maltha 1978, Reber and Goldsmith 1979, Schmid 1979, Huston and Kamman 1981]. For gait analysis, too, multibody programs are finding more and more application [Aleshinsky and Zatsiorsky 1978, Winter 1979 and Ramey and Yang 1981]. Most of these programs are used in simulation studies. As we will discuss below the application of multibody programs in biomechanical research, on the other hand, requires these programs to have optimization facilities incorporated.

An important biomechanical question is the magnitude of muscle and joint forces. To find an answer to this question several research-workers developed multibody programs in which the muscles are modelled as straight line connections with an unknown tensile force. The equations were mostly set up by hand, and to simplify this process, only static situations were considered [Paul J 1967, Barbenel 1972, Seireg and Arvikar 1975, Crowninshield 1978]. The number of unknown tensile forces and reaction loads in the joints exceeds the number of equations. As a result, there is an infinitely large number of possible solutions.

Several hypotheses have been formulated to approximate the real solution. With the aid of linear programming techniques one solution can been selected as the optimal solution as regards the hypothesis. According to the above-mentioned publications several hypotheses, such as minimum total tensile force, minimum average muscle tension, minimum total energy, etc., could be verified indirectly by EMG measurements [Hatze 1980]. Since it is not possible to measure the muscle force in the human body directly, the value of these verifications is doubtful, and more and more critism has been expressed in literature [Yeo 1976, Hardt 1978, Hatze 1980].

Similar to the work of Chow and Jacobson [1971] and Ghosh and Boykin [1976] some research workers [Hatze 1977 1981b, Hubbard 1981] started to use muscle-behaviour models in which (measurable) signals are included for motor-unit stimulation. Such models can be inserted into multibody systems of the musculo-skeletal system, resulting in realistic models in which dynamic aspects are also included. The number of unknown variables again exceeds the number of equations. But this time the stimulation signals are the unknown variables and not the forces. If the positions and velocities of the attachment points as well as the stimulation signals are known, it is possible to calculate the state of the muscles as well as the muscle forces. Based on minimum time [Hatze 1976], maximum jump height [Hubbard 1981] or maximum jump distance [Hatze 1981a] it is possible to find the optimal trajectories for the unknown stimulation (input) signals as well as the optimal initial conditions. Since stimulation signals are easier

to measure than muscle forces, it is possible to verify this approach.

1.3 Themes dealt with

We develop a multibody theory based on the work of Wittenburg [1977] which allows us to model arbitrary connections. An important feature is the assembly of arbitrary connections out of simpler, standard and/or user-defined elements. Since multibody theories have to be implemented in a computer program, much attention is given to the automatic detection and solution of problems caused by singularities. Furthermore, methods for and consequences of prescribing several kinematic variables are considered. Software questions as to the kinds of data and algorithm structure are not discussed. Only solvers for some crucial numerical aspects will be treated.

First we shall discuss three basic themes: the kinematics and dynamics of a rigid body (ch 2), the elements of connections (ch 3), and the topology (ch 4). Then the three main themes follow: the kinematics of a multibody system (ch 5), the dynamics of a multibody system (ch 6), and the arbitrary connections (ch 7). Finally we will concentrate in chapters 8 and 9 on a number of applications, namely: the simulation of the behaviour of a multibody system in general and the simulation of a fuel injection pump as an example of a multibody system.

Throughout this study a coordinate-free vector/tensor notation will be used. In appendix A a comprehensive presentation is given with regard to the notation. Those unacquainted with this notation are advised to read appendix A before proceeding to the next chapters. Readers who are not specialists in the field of multibody theories should be warned that the study is theoretical and its discussion here rather formal.

CHAPTER 2
KINEMATICS AND DYNAMICS OF A RIGID BODY

This chapter deals with the properties of one rigid body. After a definition of a rigid body, formulas for the kinematic and dynamic behaviour of a rigid body are presented. These formulas are used in the following chapters to develop the equations for the kinematic and dynamic behaviour for a multibody system. Another purpose of this chapter is to illustrate the abstract notation used in this study.

2.1 Definition

A body B^i, having the property that the distance between each set of two points remains constant, is called a rigid body. Rigid bodies cannot deform, e.g. cannot absorb deformation energy.

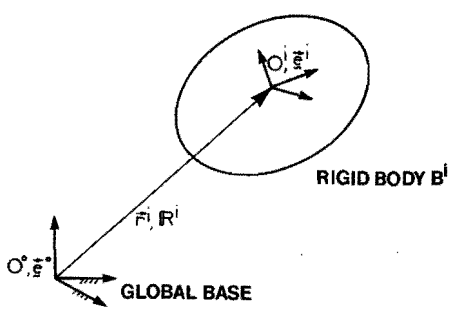


figure 2.1 A rigid body

Rigidly fixed to B^i we attach a vector base $\underset{\sim}{\vec{e}}^i$ with origin O^i, the local body-fixed base. The position and orientation of B^i in the Euclidian space S^3 is determined by the position of O^i with respect to

0^0 and the orientation of $\underset{\sim}{\vec{e}}^i$ with respect to $\underset{\sim}{\vec{e}}^0$ (see figure 2.1). The base $\underset{\sim}{\vec{e}}^0$ is the global or inertial base.

The Euclidian space S^3 is a vector space in which distances and angles are defined. To describe the position and orientation of B^i in S^3 we introduce attitude coordinates. Since the definition of these coordinates is complicated, we shall deal with this subject later on in this chapter. Since only one body is considered in this chapter, the superscript i will be dropped.

2.2 Kinematics

Orientation and derivatives
Position and derivatives
Formulas for an arbitrary point
Attitude coordinates

The discussion on the kinematics of B is divided into four subsections. First we describe the orientation of $\underset{\sim}{\vec{e}}$ with respect to $\underset{\sim}{\vec{e}}^0$, and then the position of 0 with respect to 0^0. Formulas for the position, velocity, etc. of an arbitrary point N on B are derived in the third subsection. Finally a definition of attitude coordinates is given.

Orientation and derivatives

The orientation of $\underset{\sim}{\vec{e}}$ with respect to $\underset{\sim}{\vec{e}}^0$ is described by an orthonormal, rotation tensor $\mathbb{R}$, defined by

$$\underset{\sim}{\vec{e}}^T = \mathbb{R} \cdot (\underset{\sim}{\vec{e}}^0)^T \qquad (2.2.1)$$

where

$$\mathbb{R} \cdot \mathbb{R}^T = \mathbb{I} \quad \text{and} \quad \det(\mathbb{R}) = +1$$

If B is free to move in S^3, we can write $\mathbb{R}$ as function of n variables $\varphi_1, \ldots, \varphi_n$ which have to fulfil n - 3 conditions while $n \geq 3$. For example, if we use all components of the matrix representation of $\mathbb{R}$ in $\underset{\sim}{\vec{e}}^0$,

the variables φ_i (i=1..9) have to satisfy n - 3 = 6 conditions. These conditions follow from the fact that $\mathbb{R} = \mathbb{R}(\underset{\sim}{\varphi})$ is orthonormal, so that

$$\mathbb{R}(\underset{\sim}{\varphi})\bullet\mathbb{R}(\underset{\sim}{\varphi})^T = \mathbb{I} \tag{2.2.2}$$

where $\underset{\sim}{\varphi}$ is a matrix with components $\varphi_1,\ldots,\varphi_n$. At the end of this subsection an example is given of a choice with n = 3. Although not strictly necessary, we assume in the rest of this study that $\mathbb{R}$ is expressed as a function of three variables φ_1, φ_2 and φ_3.

Differentiation of (2.2.1) with respect to time yields

$$\dot{\underset{\sim}{\vec{e}}}^T = \dot{\mathbb{R}}\bullet(\underset{\sim}{\vec{e}}^o)^T = \dot{\mathbb{R}}\bullet\mathbb{R}^T\bullet\underset{\sim}{\vec{e}}^T \tag{2.2.3}$$

Since $\mathbb{R}$ is orthonormal for each time t, after differentiation we find that

$$\dot{\mathbb{R}}\bullet\mathbb{R}^T + \mathbb{R}\bullet\dot{\mathbb{R}}^T = \mathbb{0} \tag{2.2.4}$$

from which it is easily seen that $\dot{\mathbb{R}}\bullet\mathbb{R}^T$ is a skew-symmetric tensor. For each skew-symmetric tensor $\mathbb{B} = -\mathbb{B}^T$ there exists an unambiguous vector $\vec{\omega}$ so that

$$\mathbb{B}\bullet\vec{u} = \vec{\omega}*\vec{u}\ , \qquad \forall\ \vec{u} \in S^3 \tag{2.2.5}$$

According to Chadwick [1976, p29], we will call $\vec{\omega}$ the axial vector of $\mathbb{B}$. Instead of (2.2.3) we write

$$\dot{\underset{\sim}{\vec{e}}}^T = \dot{\mathbb{R}}\bullet\mathbb{R}^T\bullet\underset{\sim}{\vec{e}}^T = \vec{\omega}*\underset{\sim}{\vec{e}}^T \tag{2.2.6}$$

where $\vec{\omega}$, the axial vector of $\dot{\mathbb{R}}\bullet\mathbb{R}^T$, is also called the angular velocity vector.

Since $\mathbb{R} = \mathbb{R}(\underset{\sim}{\varphi})$ and $\underset{\sim}{\varphi} = \underset{\sim}{\varphi}(t)$ we can express $\vec{\omega}$ as a function of $\dot{\underset{\sim}{\varphi}}$. With the definition of the axial vectors $\vec{w}^i_{\varphi}$ (i= 1..3) by

$$\frac{\partial R}{\partial \varphi_i} \cdot R^T \cdot \vec{u} = \vec{w}^i_\varphi * \vec{u}, \qquad \forall\, \vec{u} \in S^3 \tag{2.2.7}$$

it is seen that

$$\dot{R} \cdot R^T \cdot \vec{e}^T = \left(\sum_{i=1}^{3} \dot{\varphi}_i \vec{w}^i_\varphi \right) * \vec{e}^T \tag{2.2.8}$$

Comparison of (2.2.8) and (2.2.6) results in

$$\vec{\omega} = \vec{W}^T_{\varphi} \dot{\underset{\sim}{\varphi}} \tag{2.2.9}$$

where $\vec{W}_{\varphi}$ is a column with components $\vec{w}^1_\varphi$, $\vec{w}^2_\varphi$ and $\vec{w}^3_\varphi$. This column is a function of $\underset{\sim}{\varphi}$ but not of $\dot{\underset{\sim}{\varphi}}$.

From (2.2.9) follows that the angular acceleration vector is given by

$$\dot{\vec{\omega}} = \vec{W}^T_{\varphi} \ddot{\underset{\sim}{\varphi}} + \dot{\vec{W}}^T_{\varphi} \dot{\underset{\sim}{\varphi}} \tag{2.2.10}$$

Since $\vec{W}_{\varphi}$ depends only on φ, it can be shown that

$$\dot{\vec{W}}_{\varphi} = \vec{\underline{W}}_{\varphi} \dot{\underset{\sim}{\varphi}} \tag{2.2.11}$$

where the components of the square 3x3 matrix $\vec{\underline{W}}_{\varphi}(\underset{\sim}{\varphi})$ are given by

$$\left(\vec{\underline{W}}_{\varphi}\right)_{ij} = \frac{\partial \vec{w}^j_\varphi}{\partial \varphi_i} \tag{2.2.12}$$

The matrix $\vec{\underline{W}}_{\varphi}$ is not symmetric, although we can write

$$\vec{\underline{W}}_{\varphi} = \vec{\underline{W}}^T_{\varphi} - \vec{W}_{\varphi} * \vec{W}^T_{\varphi} \tag{2.2.13}$$

Finally, from (2.2.11) it follows that the angular acceleration vector becomes

$$\dot{\vec{\omega}} = \vec{W}^T_{\varphi} \ddot{\underset{\sim}{\varphi}} + \dot{\underset{\sim}{\varphi}}^T \vec{\underline{W}}^T_{\varphi} \dot{\underset{\sim}{\varphi}} \tag{2.2.14}$$

In the further discussions, variations $\delta\vec{e}_s$ of $\vec{e}_s$ caused by variations $\delta\varphi$ of φ play an important role. With

$$\delta\vec{e}_s^T = \delta R \cdot R^T \cdot \vec{e}_s^T = \sum_{i=1}^{3} \delta\varphi_i \left(\frac{\partial R}{\partial \varphi_i} \cdot R^T \cdot \vec{e}_s^T\right) \tag{2.2.15}$$

it follows that the variation $\delta\vec{e}_s$ caused by a variation $\delta\varphi$ of φ is given by

$$\delta\vec{e}_s^T = \delta\vec{\pi} * \vec{e}_s^T \tag{2.2.16}$$

In this formula $\delta\vec{\pi}$ is the angular variation vector. From (2.2.7) and (2.2.15) it is seen that

$$\delta\vec{\pi} = \vec{W}_{s\varphi}^T \delta\varphi \tag{2.2.17}$$

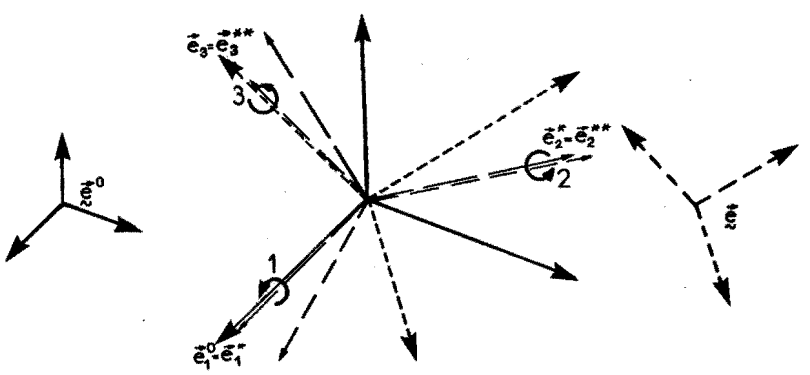


figure 2.2 The Bryant or Cardan angles

To illustrate the previously developed formulas, the Bryant or Cardan angles [Wittenburg 1977, p21-23] will be discussed in more detail. These angles form a sequence of three rotations in order to transform $\vec{e}_s^o$ into $\vec{e}_s$ (see figure 2.2). First we rotate $\vec{e}_s^o$ by an angle of φ_1 around $\vec{e}_1^o$. The result is named $\vec{e}_s^*$. Then we rotate $\vec{e}_s^*$ by an angle of φ_2 around $\vec{e}_2^*$ and name the result $\vec{e}_s^{**}$. Finally we rotate $\vec{e}_s^{**}$ by an angle φ_3 around $\vec{e}_3^{**}$ to obtain the desired $\vec{e}_s$. Since n = 3 there are

no constraints. The matrix representation of $\mathbf{R}$ in $\underset{\sim}{\vec{e}}^{O}$ as function of $\underset{\sim}{\varphi}$ is given by

$$ {}^{O}\underline{R} = \underset{\sim}{\vec{e}}^{O} \bullet \mathbf{R} \bullet (\underset{\sim}{\vec{e}}^{O})^{T} \qquad (2.2.18) $$

$$ = \begin{bmatrix} c_2c_3 & c_1s_3 + s_1s_2c_3 & s_1s_3 - c_1s_2c_3 \\ -c_2s_3 & c_1c_3 - s_1s_2s_3 & s_1c_3 + c_1s_2s_3 \\ s_2 & -s_1c_2 & c_1c_2 \end{bmatrix} $$

where c_i and s_i (i=1..3) represent $\cos(\varphi_i)$ and $\sin(\varphi_i)$. For $\underset{\sim\varphi}{\vec{w}}$ and $\underset{\sim\varphi}{\vec{w}}$ we can write:

$$ {}^{o}\underset{\sim\varphi}{w}^{1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad {}^{o}\underset{\sim\varphi}{w}^{2} = \begin{bmatrix} 0 \\ c_1 \\ s_1 \end{bmatrix}, \quad {}^{o}\underset{\sim\varphi}{w}^{3} = \begin{bmatrix} s_2 \\ -s_1c_2 \\ c_1c_2 \end{bmatrix} \qquad (2.2.19) $$

$$ {}^{o}\underset{\sim\varphi}{w}^{21} = \begin{bmatrix} 0 \\ -s_1 \\ c_1 \end{bmatrix}, \quad {}^{o}\underset{\sim\varphi}{w}^{31} = \begin{bmatrix} 0 \\ -c_1c_2 \\ -s_1c_2 \end{bmatrix}, \quad {}^{o}\underset{\sim\varphi}{w}^{32} = \begin{bmatrix} c_2 \\ s_1s_2 \\ -c_1s_2 \end{bmatrix} $$

while the other components of $\underset{\sim\varphi}{\vec{w}}$ are equal to $\vec{o}$.

Position and derivatives

The position of origin O of vector base $\underset{\sim}{\vec{e}}$ of a rigid body B is determined by the position vector $\vec{r}$ from O^O to O. If B can move freely in S^3, we can write $\vec{r}$ as a function of 3 independent variables u_1, u_2 and u_3, so that

$$ \vec{r} = \vec{r}(\underset{\sim}{u}) \ , \quad \underset{\sim}{u}^{T} = [u_1, u_2, u_3] \qquad (2.2.20) $$

These variables, for example can be the Cartesian coordinates of O in the global base $\underset{\sim}{\vec{e}}^{O}$, the spherical coordinates of O, etc.

The velocity vector of O is given by

$$ \dot{\vec{r}} = \underset{\sim u}{\vec{v}}^{T} \bullet \dot{\underset{\sim}{u}} \qquad (2.2.21) $$

the column $\underset{\sim}{\vec{v}}_u$ following from

$$\underset{\sim}{\vec{v}}_u^T = [\vec{v}_u^1, \vec{v}_u^2, \vec{v}_u^3], \quad \vec{v}_u^i = \frac{\partial \vec{r}}{\partial u_i} \qquad (i = 1..3) \qquad (2.2.22)$$

Sometimes this velocity vector is called the linear velocity in order to distinguish it from the angular velocity. Throughout this study the names velocity and angular velocity are used.

For the acceleration vector we find

$$\ddot{\vec{r}} = \underset{\sim}{\vec{v}}_u^T \underset{\sim}{\ddot{u}} + \underset{\sim}{\dot{u}}^T \underline{\vec{V}}_u \underset{\sim}{\dot{u}} \qquad (2.2.23)$$

with a square, symmetric matrix $\underline{\vec{V}}_u$ whose components are defined by

$$(\underline{\vec{V}}_u)_{ij} = \frac{\partial \vec{v}_u^j}{\partial u_i} = \frac{\partial}{\partial u_i} \frac{\partial \vec{r}}{\partial u_j} \qquad (i,j = 1...3) \qquad (2.2.24)$$

Note that both $\underset{\sim}{\vec{v}}_u$ as $\underline{\vec{V}}_u$ depend on $\underset{\sim}{u}$ but not on $\underset{\sim}{\dot{u}}$.

Finally, the variation $\delta\vec{r}$ of $\vec{r}$ caused by a variation $\delta\underset{\sim}{u}$ of $\underset{\sim}{u}$ is found to be

$$\delta\vec{r} = \underset{\sim}{\vec{v}}_u^T \delta\underset{\sim}{u} \qquad (2.2.25)$$

Formulas for an arbitrary point

The position vector $\vec{r}_n$ of an arbitrary point N in the body B is determined by the position vector $\vec{r}$ of the origin O of B and a vector $\vec{b}$ from O to N (see figure 2.3). Since B is a rigid body, the matrix representation $\underset{\sim}{b}$ of $\vec{b}$ in $\underset{\sim}{\vec{e}}$ will be constant. In other words

$$\underset{\sim}{b} = \underset{\sim}{\vec{e}} \bullet \vec{b} = \text{constant} \qquad (2.2.26)$$

Vectors with this property are called body-fixed vectors. To relate the orientation of a vector base $\underset{\sim}{\vec{e}}_n$ at N with the local body-fixed vector base $\underset{\sim}{\vec{e}}$ at O, we also introduce a body-fixed rotation tensor B

$$\underset{\sim}{\vec{e}}_n^T = B \bullet (\underset{\sim}{\vec{e}})^T \tag{2.2.27}$$

Since $\underset{\sim}{\dot{b}} = \underset{\sim}{0}$, it follows that $\dot{\vec{b}} = \vec{\omega} * \vec{b}$. Using this result and the relation for the position vector $\vec{r}_n$ from O^o to N, i.e.

$$\vec{r}_n = \vec{r} + \vec{b} \tag{2.2.28}$$

we find that the velocity vector $\dot{\vec{r}}_n$, the acceleration vector $\ddot{\vec{r}}_n$ and the variation vector $\delta\vec{r}_n$ are given by:

$$\begin{aligned} \dot{\vec{r}}_n &= \dot{\vec{r}} + \vec{\omega} * \vec{b} \\ \ddot{\vec{r}}_n &= \ddot{\vec{r}} + \dot{\vec{\omega}} * \vec{b} + \vec{\omega} * (\vec{\omega} * \vec{b}) \\ \delta\vec{r}_n &= \delta\vec{r} + \delta\vec{w} * \vec{b} \end{aligned} \tag{2.2.29}$$

while the orientation of the vector base $\underset{\sim}{\vec{e}}_n$ is given by

$$\underset{\sim}{\vec{e}}_n = B \bullet R \bullet \underset{\sim}{\vec{e}}^o \tag{2.2.30}$$

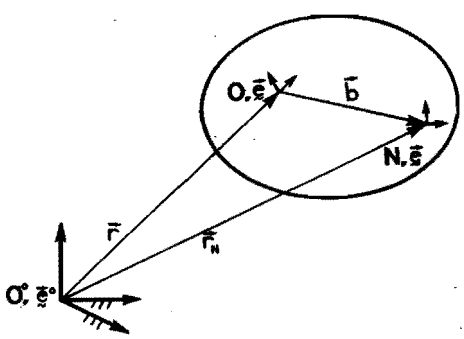


figure 2.3 A rigid body with an arbitrary point N

Attitude coordinates

To describe the position and orientation of B we sometimes prefer to use scalar variables instead of the position vector $\vec{r}$ and rotation

tensor $\mathbb{R}$. The matrix representations ${}^{O}\underset{\sim}{r}$ and ${}^{O}\underset{\sim}{R}$ of $\vec{r}$ and $\mathbb{R}$ in $\vec{\underset{\sim}{e}}^{O}$ contain in total 3 + 9 scalar quantities. These quantities can be stored in a column $\underset{\sim}{z}$ and are called attitude coordinates.

As mentioned before, the nine quantities of $\underset{\sim}{\varphi}$ have to satisfy six orthonormality conditions. Instead of using nine quantities we can, for example also use the three Bryant angles which do not have to satisfy any condition. Although several choices are possible, we select Euler or Bryant angles since this results in as few conditions as possible and a matrix $\underset{\sim}{z}$ with six components, so that

$$\underset{\sim}{z}^{T} = [\ \underset{\sim}{u}^{T},\ \underset{\sim}{\varphi}^{T}] \tag{2.2.31}$$

2.3 Dynamics

Mass, inertia and momentum
Loads, forces and moments
The equations of motion

The discussion on the dynamics of a rigid body is divided into three subsections. First we discus the mass, the inertia tensor and the momentum and angular momentum vectors. Then follows a description of the load, force and moment vectors on a body and finally the equations of motion, based on the Newton-Euler laws, are given.

Mass, inertia and momentum

The total mass m of B is given by

$$m = \int_{V} \varrho dV \tag{2.3.1}$$

where ϱ and V are resp. the mass density and the volume of B. The vector from O^{O} to M, the centre of mass of B, is denoted by $\vec{r}_{m}$. With respect to O the position of M is determined by a vector $\vec{b}_{m}$, defined by

$$\vec{b}_m = \frac{1}{m} \int_V \rho \vec{b} dV \tag{2.3.2}$$

where $\vec{b}$ is the vector from O to an arbitrary point N of B and ρ the mass density in that point.

The inertia tensor J_O of B with respect to O is given by [Wittenburg 1977, p34]

$$J_O = \int_V \rho[(\vec{b}.\vec{b})I - \vec{b}\vec{b}]dV \tag{2.3.3}$$

This tensor is symmetric and positive-definite if $\vec{r}$ is not equal to zero everywhere in B.

The momentum $\vec{I}$ and the angular momentum $\vec{L}_O$ with respect to O are defined by:

$$\begin{aligned} \vec{I} &= \int_V \rho(\dot{\vec{r}} + \vec{\omega}*\vec{b})dV = m(\dot{\vec{r}} + \vec{\omega}*\vec{b}_m) \\ \vec{L}_O &= \int_V \vec{b}*\rho(\dot{\vec{r}} + \vec{\omega}*\vec{b})dV = m(\vec{b}_m*\dot{\vec{r}}) + J_O \cdot \vec{\omega} \end{aligned} \tag{2.3.4}$$

Differentiation with respect to time of these relations yields:

$$\begin{aligned} \dot{\vec{I}} &= m[\ddot{\vec{r}} + \dot{\vec{\omega}}*\vec{b}_m + \vec{\omega}*(\vec{\omega}*\vec{b}_m)] \\ \dot{\vec{L}}_O &= m(\vec{b}_m*\ddot{\vec{r}}) + m(\vec{\omega}*\vec{b}_m)*\vec{r} + \vec{\omega}*(J_O \cdot \vec{\omega}) + J_O \cdot \dot{\vec{\omega}} \end{aligned} \tag{2.3.5}$$

where the last two terms were obtained using

$$\begin{aligned} \frac{d}{dt}(J_O \cdot \vec{\omega}) &= \dot{J}_O \cdot \vec{\omega} + J_O \cdot \dot{\vec{\omega}} = \frac{d}{dt}(\vec{e}^T \underset{\sim}{J}_O \vec{e}) \cdot \vec{\omega} + J_O \cdot \dot{\vec{\omega}} \\ &= [(\vec{\omega}*\vec{e}^T)\underset{\sim}{J}_O\vec{e} - \vec{e}^T\underset{\sim}{J}_O(\vec{e}*\vec{\omega})] \cdot \vec{\omega} + J_O \cdot \dot{\vec{\omega}} \\ &= \vec{\omega}*(J_O \cdot \vec{\omega}) + J_O \cdot \dot{\vec{\omega}} \end{aligned} \tag{2.3.6}$$

Loads, forces and moments

We will divide the loads on B into internal and external loads. Internal loads are loads on B caused by the connections with other bodies of the multibody system. External loads are loads on B caused by the surroundings of the multibody system.

External loads

We assume that nf (nf > 0) external forces $\vec{F}^1_{ex},..,\vec{F}^{nf}_{ex}$ as well as nm (nm > 0) external moments $\vec{M}^1_{ex},..,\vec{M}^{nm}_{ex}$ are exerted on B. Furthermore, we consider the situation in which the attachment point of $\vec{F}^i_{ex}$ (i=1..nf) is always the same point of B. In other words, the vector $\vec{b}^i_{ex}$ from O to the point of attachment of $\vec{F}^i_{ex}$ is a body-fixed vector. In addition to these forces and moments, surface loads $\vec{p}$ and volume loads $\vec{q}$ are also possible. An example of such a volume load is the gravity load.

For the total external force $\vec{F}_{ex}$ and total external moment $\vec{M}_{ex,o}$ on B with respect to O we find:

$$\vec{F}_{ex} = \sum_{i=1}^{nf} \vec{F}^i_{ex} + \int_A \vec{p}\, dA + \int_V \vec{q}\, dV \qquad (2.3.7)$$

$$\vec{M}_{ex,o} = \sum_{j=1}^{nm} \vec{M}^j_{ex} + \sum_{i=1}^{nf} \vec{b}^i_{ex} * \vec{F}^i_{ex} + \int_A \vec{b} * \vec{p}\, dA + \int_V \vec{b} * \vec{q}\, dV$$

where A and V are the surface area and the volume of B, respectively.

The virtual work ΔW_{ex} of the external loads for a variation $\delta\vec{r}$ of the position of O and a variation $\delta\vec{\pi}$ of the orientation of $\vec{e}_{\underset{\sim}{}}$ is given by

$$\Delta W_{ex} = \delta\vec{r} \bullet \vec{F}_{ex} + \delta\vec{\pi} \bullet \vec{M}_{ex,o} \qquad (2.3.8)$$

With the aid of (2.2.7, 22 & 31) we can also write

$$\Delta W_{ex} = \delta z_{\underset{\sim}{}}^T \begin{bmatrix} \vec{v}_u \bullet \vec{F}_{ex} \\ \vec{w}_\varphi \bullet \vec{M}_{ex,o} \end{bmatrix} \qquad (2.3.9)$$

Instead of δW_{ex} we have deliberately written ΔW_{ex} because the notation δW_{ex} suggests that there is a function W_{ex} of $\underset{\sim}{u}$ and $\underset{\sim}{\varphi}$ with the property that the virtual work of the external force is obtained by variation of $\underset{\sim}{u}$ and $\underset{\sim}{\varphi}$. This is the case only for conservative loads, while in our system the loads may be nonconservative too.

Internal loads

The internal loads on B are forces and moments arising out of the connections of B with other bodies of the system. These forces and moments will be discussed in the following chapters. Here we only mention that the resultant internal force and resultant internal moment on B with respect to O will be denoted by $\vec{F}_{in}$ and $\vec{M}_{in,o}$. For the virtual work ΔW_{in} of the internal loads for variations $\delta\vec{r}$ and $d\vec{\pi}$ we find

$$\Delta W_{in} = \delta\vec{r}\bullet\vec{F}_{in} + \delta\vec{\pi}\bullet\vec{M}_{in,o} \tag{2.3.10}$$

and, like (2.3.9) we can rewrite this result in the form

$$\Delta W_{in} = \delta\underset{\sim}{z}^T \begin{bmatrix} \vec{v}_{\sim u}\bullet\vec{F}_{in} \\ \vec{w}_{\sim\varphi}\bullet\vec{M}_{in,o} \end{bmatrix} \tag{2.3.11}$$

The equations of motion

The equations of motion are those equations which relate kinematic variables of a body to the resulting loads on that body. As stated in chapter 1 we can use the Newton-Euler laws, the virtual work principle of d'Alembert or the Lagrange equations. Here we will illustrate the use of the Newton-Euler laws. The second law of Newton gives a relation between the resultant force on a body and the time derivative of the momentum of that body. Euler's law gives a relation between the resultant moment on B with respect to M and the time derivative of the angular momentum with respect to M, so that:

$$\vec{F}_{ex} + \vec{F}_{in} = \dot{\vec{I}}, \qquad \vec{M}_{ex,m} + \vec{M}_{in,m} = \dot{\vec{L}}_m \tag{2.3.12}$$

The subscript m indicates that the corresponding quantity is referred to M. Between $\vec{M}_{ex,m}$, $\vec{M}_{in,m}$ and $\vec{L}_m$ and the quantities $\vec{M}_{ex,o}$, $\vec{M}_{in,o}$ and $\vec{L}_o$ considered earlier with respect to O, the following relations obtain:

$$\vec{M}_{ex,m} = \vec{M}_{ex,o} - \vec{b}_m * \vec{F}_{ex}, \quad \vec{M}_{in,m} = \vec{M}_{in,o} - \vec{b}_m * \vec{F}_{in} \qquad (2.3.13)$$
$$\vec{L}_m = \vec{L}_o - m\vec{b}_m * \dot{\vec{r}}_m$$

After some manipulation of these relations we can rewrite Euler's law as

$$\vec{M}_{ex,o} + \vec{M}_{in,o} = \dot{\vec{L}}_o + m\,\dot{\vec{r}}_m * (\vec{\omega} * \vec{b}_m) \qquad (2.3.14)$$

Summary

In this chapter we considered several aspects of a rigid body. In the section on kinematics attention was given to the representation of position, orientation, velocities, etc. We also introduced the attitude coordinates and discussed how these coordinates are related to the position, orientation, velocities, etc. In the section on dynamics we introduced the notions mass, inertia, momentum and external and internal loads. Internal loads are internal with regard to the complete multibody system, while external loads were defined as loads on the bodies exerted from the surroundings of the multibody system. These notions are important in order to be able to set up the equations of motion. In chapters five and six we will discuss the kinematics and dynamics of systems of rigid bodies. In those chapters many aspects are considered which were introduced for one rigid body in the present chapter.

CHAPTER 3
ELEMENTS OF CONNECTIONS

A multibody system consists of several rigid bodies and connections between them. These connections are studied in more detail in this chapter. In particular we discuss elements of connections. In chapter 7 a method will be developed for the description of connections as assemblies of elements.

3.1 Introduction

A connection is a (material) part between two or more bodies of a system. It constitutes a relationship between kinematic variables of these bodies only, or between kinematic variables, force variables and eventually some other known external input variables. We restrict ourselves to massless connections, in other words, connections that make no contribution to the total kinetic energy of the system. It is also assumed that the mechanical behaviour of a connection can be described by kinematic and/or force variables in a finite number of points of the connection.

The concept of element (of connection) is introduced to describe the behaviour mathematically. An element includes all the arrangements as to the number of connection points, vector bases at these points, kinematic and force variables and eventually some other known input variables as well as an (explicitly given) relation between these variables. This relation will be called the constitutive equation of the element.

When studying an isolated element the connection points of the element are called endpoints (see figure 3.1).

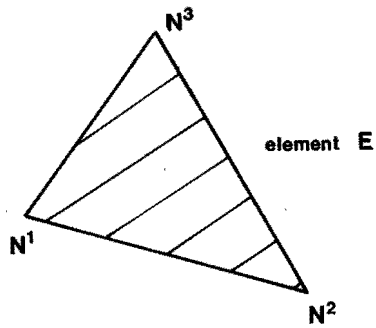


figure 3.1 Element with three endpoints N^1, N^2 and N^3.

It is assumed that the endpoints are rigidly attached to the surroundings of an element. Rigidly attached means that the kinematic variables of the points are coupled, while no work may be added or dissipated. The next section deals with some general aspects of elements. In the subsequent sections kinematic, energetic and active elements are discussed.

3.2 General aspects of elements

The kinematic variables
The force variables
The constitutive equation

Let E be an element with ne (ne $\geq$ 2) endpoints which are uniquely numbered from 1 to ne. The endpoint with number i (i=1..ne) is indicated as N^i. In the following three subsections the kinematic variables, the forces variables and the constitutive equations of E will be discussed in general.

The kinematic variables

The position vector of N^i with respect to the fixed origin O^o is called $\vec{r}^i$. In N^i an orthonormal, right-handed local base $\underset{\sim}{\vec{e}}^i$ is defined.

Its orientation with respect to the fixed global base $\underset{\sim}{\vec{e}}^{0}$ is determined by the rotation tensor $\mathbb{R}^{i}$, so that

$$(\underset{\sim}{\vec{e}}^{i})^{T} = \mathbb{R}^{i} \bullet (\underset{\sim}{\vec{e}}^{0})^{T}, \qquad i = 1..ne \tag{3.2.1}$$

It is often advantageous not to work with the absolute position vector $\vec{r}^{i}$ and rotation tensor $\mathbb{R}^{i}$, but with relative, element-bounded variables. We will therefore introduce at E a reference point N and a reference base $\underset{\sim}{\vec{e}}$. The position and orientation of N and $\underset{\sim}{\vec{e}}$ with respect to 0^{0} are described by a position vector $\vec{r}$ and a rotation tensor $\mathbb{R}$ in which

$$\underset{\sim}{\vec{e}}^{T} = \mathbb{R} \bullet (\underset{\sim}{\vec{e}}^{0})^{T} \tag{3.2.2}$$

For endpoint N^{i} (see figure 3.2) we can write

$$\vec{r}^{i} = \vec{r} + \vec{c}^{i}, \qquad \mathbb{R}^{i} = \mathbb{C}^{i} \bullet \mathbb{R}, \quad i = 1..ne \tag{3.2.3}$$

where $\vec{c}^{i}$ is the relative position or connection vector of N^{i} (i.e. the vector from N to N^{i}) and $\mathbb{C}^{i}$ is the relative rotation or connection tensor (i.e. the rotation tensor of $\underset{\sim}{\vec{e}}^{i}$ with respect to $\underset{\sim}{\vec{e}}$).

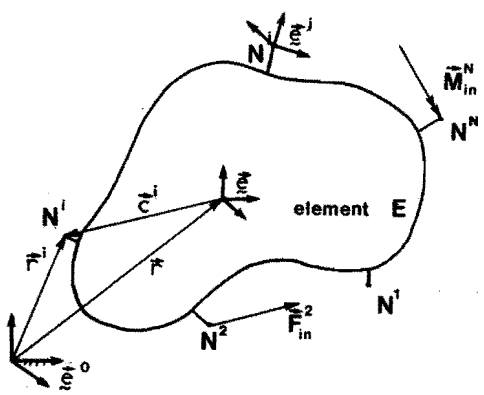


figure 3.2 Variables of an element

Unlike rigid bodies, an element can deform. As a result the matrix representation $\underset{\sim}{c}^{i}$ and $\underset{\sim}{\mathbb{C}}^{i}$ of $\vec{c}^{i}$ and $\mathbb{C}^{i}$ in $\underset{\sim}{\vec{e}}$ are not constant. From

$$(\underset{\sim}{\vec{e}}^i)^T = \underset{\sim}{C}^i \bullet \underset{\sim}{\vec{e}} = \underset{\sim}{\vec{e}}^T \underset{\sim}{C}^i \quad \text{and} \quad \underset{\sim}{\vec{e}}^T = R \bullet (\underset{\sim}{\vec{e}}^O)^T \tag{3.2.4}$$

it follows that the time derivative of $\underset{\sim}{\vec{e}}^i$ is given by

$$(\dot{\underset{\sim}{\vec{e}}}^i)^T = (\dot{R} \bullet R^T) \bullet (\underset{\sim}{\vec{e}}^i)^T + \underset{\sim}{\vec{e}}^T \underset{\sim}{\Omega}^i \tag{3.2.5}$$

The first term of the right hand side can be rewritten because $\dot{R} \bullet R^T$ is skew-symmetric. The corresponding axial vector, the angular velocity vector $\vec{\omega}$ of the element, satisfies

$$\dot{R} \bullet R^T \bullet \vec{u} = \vec{\omega} * \vec{u}, \qquad \forall\ \vec{u} \in S^3 \tag{3.2.6}$$

For the second term on the right hand side of (3.2.5) it is noted that $\underset{\sim}{C}^i (\underset{\sim}{C}^i)^T = \underset{\sim}{I}$ for all t. Hence, $\underset{\sim}{\vec{e}}^T \underset{\sim}{\Omega}^i (\underset{\sim}{C}^i)^T \underset{\sim}{\vec{e}}$ is skew-symmetric. The corresponding axial vector is the relative angular velocity vector $\vec{\Omega}^i$ of $\underset{\sim}{\vec{e}}^i$ with respect to $\underset{\sim}{\vec{e}}$, that is

$$\underset{\sim}{\vec{e}}^T \underset{\sim}{\Omega}^i (\underset{\sim}{C}^i)^T \underset{\sim}{\vec{e}} \bullet \vec{u} = \vec{\Omega}^i * \vec{u}, \quad \forall\ \vec{u} \in S^3 \tag{3.2.7}$$

Using (3.2.6) and (3.2.7), we finally obtain

$$(\dot{\underset{\sim}{\vec{e}}}^i)^T = (\vec{\omega} + \vec{\Omega}^i) * (\underset{\sim}{\vec{e}}^i)^T \tag{3.2.8}$$

If this relation is differentiated with respect to time we find the following expression

$$(\ddot{\underset{\sim}{\vec{e}}}^i)^T = (\dot{\vec{\omega}} + \dot{\vec{\Omega}}^i) * (\underset{\sim}{\vec{e}}^i)^T + (\vec{\omega} + \vec{\Omega}^i) * [(\vec{\omega} + \vec{\Omega}^i) * (\underset{\sim}{\vec{e}}^i)^T] \tag{3.2.9}$$

The term $\dot{\vec{\Omega}}^i$ needs some further investigation. From $\vec{\Omega}^i = \underset{\sim}{\vec{e}}^T \underset{\sim}{\Omega}^i$ it is seen that

$$\dot{\vec{\Omega}}^i = \vec{\omega} * \vec{\Omega}^i + \vec{\alpha}^i \tag{3.2.10}$$

where $\vec{\alpha}^i = \underset{\sim}{\vec{e}}^T \dot{\underset{\sim}{\Omega}}^i$ is the relative angular acceleration vector of $\underset{\sim}{\vec{e}}^i$ with respect to $\underset{\sim}{\vec{e}}$. Substituting this result in (3.2.9) after some manipulations yields

$$(\underset{\sim}{\ddot{e}}^{i})^{T} = (\dot{\vec{\omega}} + \dot{\vec{\alpha}}^{i})*(\underset{\sim}{e}^{i})^{T} + \vec{\omega}*[\vec{\omega}*(\underset{\sim}{e}^{i})^{T}] + \\ + (2\vec{\omega} + \vec{\Omega}^{i})*[\vec{\Omega}^{i}*(\underset{\sim}{e}^{i})^{T}] \qquad (3.2.11)$$

The absolute velocity vector $\dot{\vec{r}}^{i}$ of N^{i} follows from (3.2.3), hence

$$\dot{\vec{r}}^{i} = \dot{\vec{r}} + \dot{\vec{c}}^{i} = \dot{\vec{r}} + \underset{\sim}{\dot{e}}^{T}\vec{c}^{i} + \underset{\sim}{e}^{T}\dot{\vec{c}}^{i} \qquad (3.2.12)$$

and with $\underset{\sim}{\dot{e}}^{T} = \vec{\omega}*\underset{\sim}{e}^{T}$ we can write

$$\dot{\vec{r}}^{i} = \dot{\vec{r}} + \vec{\omega}*\vec{c}^{i} + \vec{v}^{i} \qquad (3.2.13)$$

where $\vec{v}^{i} = \underset{\sim}{e}^{T}\dot{\vec{c}}^{i}$ is the relative velocity vector of N^{i} with respect to N. The acceleration of N^{i} is obtained by differentiating (3.2.13):

$$\ddot{\vec{r}}^{i} = \ddot{\vec{r}} + \dot{\vec{\omega}}*\vec{c}^{i} + \vec{\omega}*(\vec{\omega}*\vec{c}^{i} + \vec{v}^{i}) + \dot{\vec{v}}^{i} \qquad (3.2.14)$$

and using $\dot{\vec{v}}^{i} = \vec{\omega}*\vec{v}^{i} + \underset{\sim}{e}^{T}\ddot{\vec{c}}^{i}$ it follows that

$$\ddot{\vec{r}}^{i} = \ddot{\vec{r}} + \dot{\vec{\omega}}*\vec{c}^{i} + \vec{\omega}*(\vec{\omega}*\vec{c}^{i} + 2\vec{v}^{i}) + \vec{a}^{i} \qquad (3.2.15)$$

where $\vec{a}^{i} = \underset{\sim}{e}^{T}\ddot{\vec{c}}^{i}$ is the relative acceleration vector of N^{i} with respect to N.

The variation of $\vec{r}^{i}$ and $\underset{\sim}{e}^{i}$ caused by variations of $\vec{r}$, $\underset{\sim}{e}$, $\underset{\sim}{c}^{i}$ and $\underline{c}^{i}$ can be determined in the same way as the time derivatives of $\vec{r}^{i}$ and $\underset{\sim}{e}^{i}$. With the angular variation vector $\delta\vec{\pi}$ of element E, defined by

$$\delta R \cdot R^{T}\cdot\vec{u} = \delta\vec{\pi}*\vec{u} \qquad \forall\ \vec{u} \in S^{3}, \qquad (3.2.16)$$

and the variation of the relative angular vector $\delta\vec{\Psi}^{i} = \underset{\sim}{e}^{T}\delta\underset{\sim}{\varrho}^{i}$ of $\underset{\sim}{e}^{i}$ with respect to $\underset{\sim}{e}$, defined by

$$\underset{\sim}{e}^{T}\delta\underline{c}^{i}(\underline{c}^{i})^{T}\underset{\sim}{e}\cdot\vec{u} = \delta\vec{\Psi}^{i}*\vec{u} \qquad \forall\ \vec{u} \in S^{3} \qquad (3.2.17)$$

it follows immediately that

$$\delta\underset{\sim}{\vec{e}}^{T} = \delta\vec{\pi}*\underset{\sim}{\vec{e}}^{T}, \qquad (\delta\underset{\sim}{\vec{e}}^{i})^{T} = (\delta\vec{\pi} + \delta\vec{\Pi}^{i})*(\underset{\sim}{\vec{e}}^{i})^{T} \tag{3.2.18}$$

Let $\delta\vec{r}$ be the variation of the position vector $\vec{r}$ of the reference point N. Then $\delta\vec{r}^{i}$ is seen to be equal to

$$\delta\vec{r}^{i} = \delta\vec{r} + \delta\vec{\pi}*\vec{r}^{i} + \delta\vec{v}^{i} \tag{3.2.19}$$

where $\delta\vec{v}^{i} = \underset{\sim}{\vec{e}}^{T}\delta\underset{\sim}{c}^{i}$ is the variation of the relative position vector.

The force variables

We assume that the interaction between the element and its surroundings takes place only at the endpoints. No external loads exert on the element elsewhere. The loads on endpoint N^{i} (i=1..ne), owing to the surroundings of the element, consist of the force vector $\vec{F}^{i}_{in}$ and the moment vector $\vec{M}^{i}_{in}$. As the element is assumed to be massless, $\vec{F}^{i}_{in}$ and $\vec{M}^{i}_{in}$ (i=1..ne) have to satisfy the equilibrium equations

$$\sum_{i=1}^{ne} \vec{F}^{i}_{in} = \vec{o} \quad \text{and} \quad \sum_{i=1}^{ne} (\vec{M}^{i}_{in} + \vec{c}^{i}*\vec{F}^{i}_{in}) = \vec{o} \tag{3.2.20}$$

If the position vector $\vec{r}^{i}$ of N^{i} and the orientation of the local base $\underset{\sim}{\vec{e}}^{i}$ in N^{i} are subjected to variations, $\vec{F}^{i}_{in}$ and $\vec{M}^{i}_{in}$ perform the virtual work ΔW, then

$$\Delta W = \sum_{i=1}^{ne} [\vec{F}^{i}_{in}\bullet\delta\vec{r}^{i} + \vec{M}^{i}_{in}\bullet(\delta\vec{\pi} + \delta\vec{\Pi}^{i})] \tag{3.2.21}$$

and, using (3.2.19) and (3.2.20), it is seen that

$$\Delta W = \sum_{i=1}^{ne} (\vec{F}^{i}_{in}\bullet\delta\vec{v}^{i} + \vec{M}^{i}_{in}\bullet\delta\vec{\Pi}^{i}) \tag{3.2.22}$$

In this relation the variation $\delta\vec{r}$ of the position vector of the reference point and the angular variation vector $\delta\vec{\pi}$ of the reference base do not occur any longer.

The constitutive equation

The behaviour of an element is mathematically characterized by the constitutive equation. We assume that this equation constitutes a relationship between the kinematic and force variables at the end-points, the history of these variables and a set of external input variables which are prescribed as a function of time.

It is assumed that the constitutive equation is invariant for rotation and translation of the element as a rigid body. Such translations and rotations can be described with the translation of the reference point N and the rotation of the reference base $\vec{e}_{\underset{\sim}{}}$, that is with the position vector $\vec{r}$ and the rotation tensor $\mathbb{R}$. This assumption implies that $\vec{r}$ and $\mathbb{R}$ play no role in the constitutive equation and also that the constitutive equation is invariant for the choice of the reference base. Therefore it is possible to formulate the constitutive equation in terms of the matrix representation of $\vec{c}^{\,i}$ and $\mathbb{C}^i$ as

$$\underset{\sim}{f}(\underset{\sim}{F}^i_{in}(\tau),\underset{\sim}{M}^i_{in}(\tau),\underset{\sim}{c}^i(\tau),\underline{\underset{\sim}{C}}^i(\tau),\underset{\sim}{i}(\tau),t|\ i=1..ne;\tau\varepsilon(-\infty,t]) = \underset{\sim}{o} \qquad (3.2.23)$$

where $\tau\varepsilon(-\infty,t]$ represents the history and the column $\underset{\sim}{i}$ contains the external input variables prescribed as a function of time.

A constitutive equation that contains only kinematic variables is called a kinematic constraint, hence

$$\underset{\sim}{f}(\underset{\sim}{c}^i(\tau),\underline{\underset{\sim}{C}}^i(\tau),t|\ i=1..ne;\tau\varepsilon(-\infty,t]) = \underset{\sim}{o} \qquad (3.2.24)$$

Elements with this constitutive equation are called kinematic elements. They are considered in more detail in the next section.

If the constitutive equation (3.2.23) contains both kinematic and force variables, it is called an energetic relation, thus

$$\underset{\sim}{f}(\underset{\sim}{F}^i_{in}(\tau),\underset{\sim}{M}^i_{in}(\tau),\underset{\sim}{c}^i(\tau),\underline{\underset{\sim}{C}}^i(\tau),t|\ i=1..ne;\tau\varepsilon(-\infty,t]) = \underset{\sim}{o} \qquad (3.2.25)$$

Elements with this constitutive equation are called energetic elements. We assume that, for an element of this type, the force variables at time t can be determined if the history of these force variables as well as of all other variables of (3.2.25) is known.

The last type of elements we will consider are the active elements. In the constitutive equation of these elements external input variables $\underset{\sim}{i}$ also appear. We restrict ourselves to active elements whose behaviour is described only by the current values of the variables. If the history is important it is assumed that, by introducing a finite number of state variables stored in a column $\underset{\sim}{x}$, it is still possible to describe the behaviour of an active element by current values alone. To determine the state variables $\underset{\sim}{x}$ at time t we have to solve a state equation of the kind

$$\dot{\underset{\sim}{x}}(\tau) = \underset{\sim}{s}(\underset{\sim}{x}(\tau),\underset{\sim}{i}(\tau),\tau) \qquad \underset{\sim}{x}_o = \underset{\sim}{x}(t_o) \qquad (3.2.26)$$

where t_o is a point of time between $-\infty$ and t at which a value for the state variables is known.

3.3 Kinematic elements

Constraint elements
Holonomic and nonholonomic constraint elements
Hinge elements
How to describe a (new) kinematic element
Examples

A kinematic connection between bodies is a connection which restricts the relative motions of these bodies. In the previous section it was stated that in the constitutive equation of a kinematic element E only the time and kinematic variables occur, that is

$$\underset{\sim}{f}(\underset{\sim}{c}^i(\tau),\underline{c}^i(\tau),t|\ i=1..ne,\ \tau\epsilon(-\infty,t]) = \underset{\sim}{o} \qquad (3.3.1)$$

The number of components of the column $\underset{\sim}{f}$, i.e. the number of equations in (3.3.1), will be denoted by np. We assume that the constitu-

tive equations, in the case of (3.3.1) also called the kinematic constraint equations, are independent. In that case the number np is equal to or lower than 6(ne - 1) where ne is the number of endpoints of E. If in the constraint equation the time t is explicit the element is called rheonomic, otherwise it is scleronomic.

Although it is possible to consider kinematic elements having three or more endpoints, most kinematic elements can be described as having two. We shall therefore restrict ourselves to kinematic elements with two endpoints N^1 and N^2 only. Furthermore, we choose the reference point N in endpoint N^1 and let the reference base coincide with the vector base at this endpoint. In that case only $\underset{\sim}{c}^2$ and $\underline{C}^2$ appear in (3.3.1) since $\underset{\sim}{c}^1 = \underset{\sim}{o}$ and $\underline{C}^1 = \underline{I}$. In the rest of this section we write $\underset{\sim}{c}$ and $\underline{C}$ instead of $\underset{\sim}{c}^2$ and $\underline{C}^2$.

According to (3.3.1) the complete history of the kinematic variables may be important but in practice this is not the case. Without any essential restriction it may be assumed that the constitutive behaviour of a kinematic element depends only on the current values of the kinematic variables and their first partial derivatives [Rosenberg, 1977, p43]. Instead of (3.3.1) we can write

$$\underset{\sim}{f}(\underset{\sim}{c},\underline{C},\dot{\underset{\sim}{c}},\dot{\underline{C}},t) = \underset{\sim}{o} \qquad (3.3.2)$$

and on the basis of the relations (3.2.13) and (3.2.7) also

$$\underset{\sim}{f}(\underset{\sim}{c},\underline{C},\underset{\sim}{v},\underline{\Omega},t) = \underset{\sim}{o} \qquad (3.3.3)$$

In the last-named formulas the dependence of the kinematic variables on time is not mentioned explicitly.

In the next two subsections we will describe two different types of kinematic elements: constraint elements and hinge elements.

Constraint elements

According to Rosenberg the relation (3.3.3) is still too general. He states that practically all relevant kinematic constraint equations

are linear in the velocity variables and can be written as Pfaff constraints [Rosenberg 1977, p38], so that

$$\underset{\sim}{P}_v \underset{\sim}{v} + \underset{\sim}{P}_Q \underset{\sim}{Q} + \underset{\sim}{P}_0 = \underset{\sim}{0} \tag{3.3.4}$$

where $\underset{\sim}{P}_v$ and $\underset{\sim}{P}_Q$ are matrices of order npx3 and $\underset{\sim}{P}_0$ is a column with np components. These matrices do not depend on $\underset{\sim}{v}$ and/or $\underset{\sim}{Q}$ but in general are functions of t, $\underset{\sim}{c}$ and $\underline{C}$.

In section 2.2 the position vector of an arbitrary point of a rigid body was expressed as function of three scalar variables, stored in column $\underset{\sim}{u}$. Here $\underset{\sim}{c}$ can be expressed as a function of a similarly defined column $\underset{\sim}{u}$, i.e. $\underset{\sim}{c} = \underset{\sim}{c}(\underset{\sim}{u})$. For $\underset{\sim}{v} = \dot{\underset{\sim}{c}}$ it then immediately follows that

$$\underset{\sim}{v} = \underline{\Psi}\dot{\underset{\sim}{u}}, \qquad \underline{\Psi} = \underline{\Psi}(\underset{\sim}{u}) \tag{3.3.5}$$

where $\underline{\Psi}$ is a square matrix of order 3x3 whose components depend only on $\underset{\sim}{u}$.

Similarly $\underline{C}$ can be expressed as a function of the three components of a column $\underset{\sim}{\varphi}$, so that $\underline{C} = \underline{C}(\underset{\sim}{\varphi})$. For $\underset{\sim}{Q}$ expressed as function of $\underset{\sim}{\varphi}$ and $\dot{\underset{\sim}{\varphi}}$ we find

$$\underset{\sim}{Q} = \underline{\Phi}\dot{\underset{\sim}{\varphi}}, \qquad \underline{\Phi} = \underline{\Phi}(\underset{\sim}{\varphi}) \tag{3.3.6}$$

with a square matrix $\underline{\Phi}$ of order 3x3, depending on $\underset{\sim}{\varphi}$ only.

To rewrite (3.3.4) more compact we introduce coordinates $\underset{\sim}{y}$, so that

$$\underset{\sim}{y}^T = [\underset{\sim}{u}^T, \underset{\sim}{\varphi}^T] \tag{3.3.7}$$

where it is assumed that $\underset{\sim}{y}$ describes the relative position and orientation, the connection vector $\vec{c}$ and tensor $\mathbb{C}$, of endpoint N^2 with respect to $N = N^1$. Note that these coordinates are similar to the attitude coordinates $\underset{\sim}{z}$ which describe the absolute position and orientation (see section 2.2). With this column $\underset{\sim}{y}$ the Pfaff constraints (3.3.4) are written as

$$\underset{\sim}{P}_y\dot{\underset{\sim}{y}} + \underset{\sim}{p}_{oy} = \underset{\sim}{o}, \qquad \underset{\sim}{P}_y = [\underset{\sim}{P}_v\underset{\sim}{V}, \underset{\sim}{P}_\Omega\underset{\sim}{\Phi}], \quad \underset{\sim}{p}_{oy} = \underset{\sim}{p}_o \tag{3.3.8}$$

where $\underset{\sim}{p}_{oy} = \underset{\sim}{p}_{oy}(\underset{\sim}{y},t)$ is a column with np components and $\underset{\sim}{P}_y = \underset{\sim}{P}_y(\underset{\sim}{y},t)$ is a matrix of order npx6. We will often refer to this equation as the Pfaff equation of a constraint element. The assumption that the constraint equations are independent implies that the rank of $\underset{\sim}{P}_y$, denoted as $r(\underset{\sim}{P}_y)$, must be equal to np. Note that, since $\underset{\sim}{P}_y$ is a function of $\underset{\sim}{y}$ and t, this rank can decrease if $\underset{\sim}{y}$ and/or t change. In chapters 7 and 8 this phenomemon will be given more attention.

Besides the Pfaff equation, its derivative with respect to time is required too. By differentiating (3.3.8) it is found that

$$\underset{\sim}{P}_y\ddot{\underset{\sim}{y}} + \underset{\sim}{p}_{ooy} = \underset{\sim}{o} \tag{3.3.9}$$

Here $\underset{\sim}{p}_{ooy}$ is a column with np components, given by

$$(\underset{\sim}{p}_{ooy})_i = \sum_{j=1}^{6}\sum_{k=1}^{6}\frac{\partial(\underset{\sim}{P}_y)_{ij}}{\partial y_k}\dot{y}_k\dot{y}_j + \sum_{j=1}^{6}\left[\frac{\partial(\underset{\sim}{P}_y)_{ij}}{\partial t} + \frac{\partial(\underset{\sim}{p}_{oy})_i}{\partial y_j}\right]\dot{y}_j + \frac{\partial(\underset{\sim}{p}_{oy})_i}{\partial t} \tag{3.3.10}$$

This column depends in general on $\underset{\sim}{y}$ and t and is a quadratic function of $\dot{\underset{\sim}{y}}$.

The variations $\delta\underset{\sim}{c}$ and $\delta\underset{\sim}{\Omega}$ caused by variations of $\underset{\sim}{u}$ and $\underset{\sim}{\varphi}$ are given by

$$\delta\underset{\sim}{c} = \underset{\sim}{V}\delta\underset{\sim}{u}, \qquad \delta\underset{\sim}{\Omega} = \underset{\sim}{\Phi}\delta\underset{\sim}{\varphi} \tag{3.3.11}$$

The variation $\delta\underset{\sim}{y}$ defined by

$$\delta\underset{\sim}{y}^T = [\delta\underset{\sim}{u}^T, \delta\underset{\sim}{\varphi}^T] \tag{3.3.12}$$

is said to be kinematically admissible if the condition

$$\underset{\sim}{P}_y\delta\underset{\sim}{y} = \underset{\sim}{o} \tag{3.3.13}$$

is satisfied.

In the constitutive equation of a kinematic element no force variables appear by definition. Nevertheless, at the endpoints the surroundings of the element will exert forces on the element. The virtual work done by these forces, due to variations $\delta\underset{\sim}{c}$ and $\delta\underset{\sim}{\Omega}$, is given by

$$\Delta W = \underset{\sim}{F}_{in}^T \delta\underset{\sim}{c} + \underset{\sim}{M}_{in}^T \delta\underset{\sim}{\Omega} \tag{3.3.14}$$

where $\underset{\sim}{F}_{in}$ and $\underset{\sim}{M}_{in}$ are the matrix representation in the reference base $\underset{\sim}{\vec{e}}$ of the force and moment vector on the endpoint N^2. Since the element is massless the corresponding force and moment vector on the element at $N = N^1$ can be determined easily.

Based on (3.3.11) we may rewrite (3.3.14) as

$$\Delta W = \delta\underset{\sim}{u}^T \underline{\Psi}^T \underset{\sim}{F}_{in} + \delta\underset{\sim}{\varphi}^T \underline{\Phi}^T \underset{\sim}{M}_{in} \tag{3.3.15}$$

or by using the more compact notation with $\underset{\sim}{y}$ to give

$$\Delta W = \delta\underset{\sim}{y}^T \underset{\sim}{\Lambda}, \qquad \underset{\sim}{\Lambda}^T = [\underset{\sim}{F}_{in}^T \underline{\Psi},\ \underset{\sim}{M}_{in}^T \underline{\Phi}] \tag{3.3.16}$$

where $\underset{\sim}{\Lambda}$ is a column with 6 components.

According to the fundamental principle of Lagrange mechanics ΔW is equal to zero for all kinematically admissible variations $\delta\underset{\sim}{y}$. Because $\underline{p}_y$ is a matrix of order np×6 with rank $r(\underline{p}_y)$ equal to np, it follows from (3.3.12, 13 & 16) that $\underset{\sim}{\Lambda}$ has to be a linear combination of the rows of $\underline{p}_y$ so that [Rosenberg 1977, p131]

$$\underset{\sim}{\Lambda} = \underline{p}_y^T \underset{\sim}{\lambda} \tag{3.3.17}$$

The components of the column $\underset{\sim}{\lambda}$ are a priori unknown, but they can be interpreted as Lagrange multipliers of the kinematic constraints.

Holonomic and nonholonomic constraint elements

The Pfaff constraints (3.3.4) are called holonomic if they can be integrated, i.e. if they satisfy the Frobenius conditions [Rosenberg

1977, p47]. Instead of (3.3.3) the constraint equations for holonomic constraints are written as

$$\underset{\sim}{f}(\underset{\sim}{c},\underset{\sim}{C},t) = \underset{\sim}{o} \tag{3.3.18}$$

or, using the coordinates $\underset{\sim}{y}$, as

$$\underset{\sim}{f}(\underset{\sim}{y},t) = \underset{\sim}{o} \tag{3.3.19}$$

Every kinematic constraint not of the form (3.3.18/19), or not reducible to this form, is called nonholonomic. A well known example of a nonholonomic Pfaff constraint is given by

$$\dot{y}_1 \sin(y_3) - \dot{y}_2 \cos(y_3) = 0 \tag{3.3.20}$$

where y_1 and y_2 represent the position of a skate and y_3 the direction as sketched in figure 3.3. In this case $\underset{\sim}{p} = [\tan(y_3), -1]$ while all components of the other Pfaff matrices are equal to zero. Other examples of nonholonomic constraints are inequality constraints of the form $\underset{\sim}{f}(\underset{\sim}{y},t) \geq \underset{\sim}{o}$.

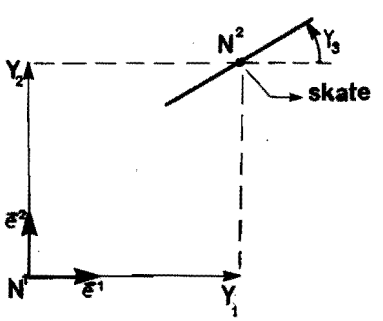


figure 3.3 The skate

Hinge elements

We consider independent holonomic constraint equations and assume that it is possible to write $\underset{\sim}{y}$ as a function of nq independent generalized coordinates $\underset{\sim}{q}$:

$$\underset{\sim}{q}^T = [q_1 \; .. \; q_{nq}], \qquad nq = 6 - np, \tag{3.3.21}$$

so that for all t and all $\underset{\sim}{q}$ the constraint equations are satisfied. Mathematically

$$\underset{\sim}{y} = \underset{\sim}{y}(\underset{\sim}{q},t), \qquad \underset{\sim}{f}(\underset{\sim}{y}(\underset{\sim}{q},t),t) = \underset{\sim}{0} \tag{3.3.22}$$

for all $\underset{\sim}{q}$ and all t.

Instead of characterizing a holonomic kinematic element by the constraint equation (3.3.22), it is also possible to characterize it by the functions

$$\underset{\sim}{c} = \underset{\sim}{c}(\underset{\sim}{q},t), \qquad \underline{C} = \underline{C}(\underset{\sim}{q},t) \tag{3.3.23}$$

Holonomic kinematic elements for which these functions instead of the constraint equations are supplied will be called hinge elements.

To describe a hinge element we have to supply $\underset{\sim}{c}$ and $\underline{C}$ as a function of $\underset{\sim}{q}$ and t, as well as the derivatives $\underset{\sim}{\Omega}$, $\underset{\sim}{a}$, $\underset{\sim}{v}$ and $\underset{\sim}{a}$ as functions of $\underset{\sim}{q}$, $\dot{\underset{\sim}{q}}$, $\ddot{\underset{\sim}{q}}$ and t.

From the definition (3.2.7) of the relative angular velocity $\underset{\sim}{\Omega}$ it follows that

$$\underset{\sim}{\Omega}^* \underset{\sim}{u} = \dot{\underline{C}}\underline{C}^T \underset{\sim}{u} \tag{3.3.24}$$

for all columns $\underset{\sim}{u}$ with three components. Because of (3.3.23) the matrix on the right hand side is equal to

$$\dot{\underline{C}}\underline{C}^T = \sum_{j=1}^{nq} \left[\frac{\partial \underline{C}}{\partial q_j}\underline{C}^T \dot{q}_j\right] + \frac{\partial \underline{C}}{\partial t}\underline{C}^T \tag{3.3.25}$$

where each of the terms on the right hand side is a skew-symmetric matrix because $\underline{C}\underline{C}^T = \underline{I}$ for all $\underset{\sim}{q}$ and t. Hence, there are columns $\underset{\sim}{w}_j$ (j=1..nq) and $\underset{\sim}{w}_0$ of such a nature that

$$\frac{\partial \underline{C}}{\partial q_j}\underline{C}^T\underline{u} = \underline{w}_j * \underline{u}, \qquad \frac{\partial \underline{C}}{\partial t}\underline{C}^T\underline{u} = \underline{w}_o * \underline{u} \tag{3.3.26}$$

for all $\underline{u}$. With this result it is seen that $\underline{\Omega}$ is given by

$$\underline{\Omega} = \sum_{j=1}^{nq} (\underline{w}_j \dot{q}_j) + \underline{w}_o \tag{3.3.27}$$

or, in a compact notation with the 3×nq matrix $\underline{w}^T = [\underline{w}_1 .. \underline{w}_{nq}]$, by

$$\underline{\Omega} = \underline{w}^T \dot{\underline{q}} + \underline{w}_o \tag{3.3.28}$$

Both $\underline{w}$ and $\underline{w}_o$ depend in general on $\underline{q}$ and t, but not on $\dot{\underline{q}}$.

Differentiating (3.3.28) results in a relation for $\underline{\alpha} = \dot{\underline{\Omega}}$, so that

$$\underline{\alpha} = \underline{w}^T \ddot{\underline{q}} + \underline{w}_{oo} \tag{3.3.29}$$

The column $\underline{w}_{oo}$ with three components $(\underline{w}_{oo})_i$ is defined by

$$(\underline{w}_{oo})_i = \sum_{j=1}^{nq}\sum_{k=1}^{nq} \frac{\partial w_{ji}}{\partial q_k}\dot{q}_k\dot{q}_j + \sum_{j=1}^{nq}\left[\frac{\partial w_{ji}}{\partial t} + \frac{\partial (\underline{w}_o)_i}{\partial q_j}\right]\dot{q}_j + \frac{\partial (\underline{w}_o)_i}{\partial t} \tag{3.3.30}$$

Besides $\underline{\Omega}$ and $\underline{\alpha}$ we also require relations for the relative velocity $\underline{v}$ and the relative acceleration $\underline{a}$ as a function of $\underline{q}$, $\dot{\underline{q}}$ and t. For $\underline{v}$ we find

$$\underline{v} = \dot{\underline{c}} = \underline{v}^T \dot{\underline{q}} + \underline{v}_o, \qquad \underline{v}_o = \frac{\partial \underline{c}}{\partial t} \tag{3.3.31}$$

where $\underline{v}$ is a matrix of order 3×nq, defined by

$$\underline{v}^T = [\underline{v}_1 .. \underline{v}_{nq}], \quad \underline{v}_j = \frac{\partial \underline{c}}{\partial q_j} \quad j = 1..nq \tag{3.3.32}$$

For $\underline{a}$ we get

$$\underline{a} = \underline{v}^T \ddot{\underline{q}} + \underline{v}_{oo} \tag{3.3.33}$$

with a column $\underset{\sim}{v}_{oo}$ with three components defined by

$$(\underset{\sim}{v}_{oo})_i = \sum_{j=1}^{nq}\sum_{k=1}^{nq} \frac{\partial^2 c_i}{\partial q_k \partial q_j}\dot{q}_k\dot{q}_j + \sum_{j=1}^{nq} 2\frac{\partial^2 c_i}{\partial q_j \partial t}\dot{q}_j + \frac{\partial^2 c_i}{\partial t \partial t} \qquad (3.3.34)$$

Note that both $\underset{\sim}{w}_{oo}$ and $\underset{\sim}{v}_{oo}$ are quadratic functions of $\dot{\underset{\sim}{q}}$.

Finally, the variations $\delta\underset{\sim}{c}$ and $\delta\underset{\sim}{Q}$ caused by infinitesimal, but arbitrary variations $\delta\underset{\sim}{q}$ are given by

$$\delta\underset{\sim}{c} = \underline{v}^T\delta\underset{\sim}{q}, \qquad \delta\underset{\sim}{Q} = \underline{w}^T\delta\underset{\sim}{q} \qquad (3.3.35)$$

where $\underline{v}$ and $\underline{w}$ have been defined in (3.3.32) and (3.3.28)

We have to check whether the generalized coordinates are independent and whether all generalized coordinates are necessary. Before investigating this question we ought to realize that, according to (3.3.35) $\delta\underset{\sim}{c}$ and $\delta\underset{\sim}{Q}$ should be equal to $\underset{\sim}{o}$ if $\delta\underset{\sim}{q} = \underset{\sim}{o}$. We say that the generalized coordinates are necessary and independent if the opposite holds too. In other words, if for all $\underset{\sim}{q}$ and t $\delta\underset{\sim}{c} = \underset{\sim}{o}$ and $\delta\underset{\sim}{Q} = \underset{\sim}{o}$, it follows that $\delta\underset{\sim}{q} = \underset{\sim}{o}$. This statement is equivalent to the statement

$$\underline{y}^T\delta\underset{\sim}{q} = \underset{\sim}{o} \quad \langle == \rangle \quad \delta\underset{\sim}{q} = \underset{\sim}{o} \qquad (3.3.36)$$

with $\underline{y} = [\underline{v}, \underline{w}]$. This implies that the matrix $\underline{y}$ has maximum rank equal to nq. Because $\underline{y}$ is a function of $\underset{\sim}{q}$ and t, its rank can change if $\underset{\sim}{q}$ and t change. This problem is discussed in more detail in chapter 7.

How to describe a (new) kinematic element?

To describe a kinematic element we must first choose the two endpoints with their bases and select one of the endpoints as the reference point. In the case of a constraint element, the Pfaff matrices $\underline{p}_y$, $\underset{\sim}{p}_{oy}$, $\underset{\sim}{p}_{ooy}$ and in that of a holonomic constraint also the equation $\underset{\sim}{f}$ should be specified as a function of $\underset{\sim}{y}$, $\dot{\underset{\sim}{y}}$ and t. In the case of a hinge element we must choose generalized coordinates and specify the

matrix representation with respect to the reference base of $\underset{\sim}{c}$ and $\underset{\sim}{C}$ as well as the columns $\underset{\sim}{w}_{o}$, $\underset{\sim}{w}_{oo}$, $\underset{\sim}{v}_{o}$ and $\underset{\sim}{v}_{oo}$ and the matrices $\underset{\sim}{w}$ and $\underset{\sim}{v}$ as a function of $\underset{\sim}{q}$, $\dot{\underset{\sim}{q}}$ and t. The matrices $\underset{\sim}{w}_{o},..,\underset{\sim}{v}$ are called the partial derivatives of a hinge element.

Examples

To illustrate the preceding theory two examples of kinematic elements are discussed, namely a rigid, massless bar and a ball-and-socket joint. These elements will be described as a constraint element and as a hinge element respectively. The rigid, massless bar from figure 3.4 can be modelled as a holonomic constraint element. Since this element is scleronomic the time t and the partial derivatives with respect to t do not occur. With $\underset{\sim}{y}^{T} = [\underset{\sim}{u}^{T}, \underset{\sim}{\varphi}^{T}]$, where $\underset{\sim}{u}$ contains the relative Cartesian coordinates of N^2 with respect to N and where $\underset{\sim}{\varphi}$ contains the three Euler or Bryant angles for the rotation from $\underset{\sim}{\vec{e}}$ to $\underset{\sim}{\vec{e}}^{2}$, we find

$$\underset{\sim}{f}(\underset{\sim}{y},t) = \underset{\sim}{y} - [1,0,0,0,0,0]^{T} = \underset{\sim}{o}, \qquad \underset{y}{p} = \underline{I} \qquad (3.3.37)$$

where $\underline{I}$ has an order of 6x6 and the components of all other Pfaff matrices are equal to zero. Note that np = 6 and that $\underset{y}{p}$ has a full rank.

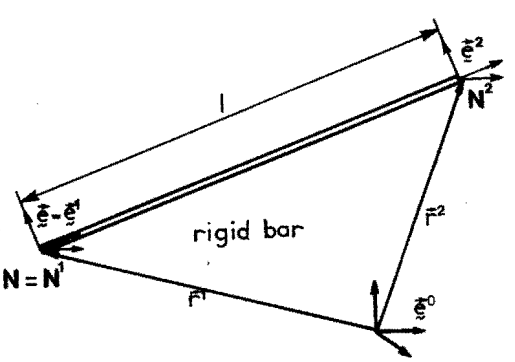


figure 3.4 A rigid, massless bar

If the bar is modelled as a hinge element the number of generalized coordinates becomes nq = 6 - np = 0 and (3.3.23) reduces to

$$\underset{\sim}{c}^T = [1,\ 0,\ 0], \qquad \underset{\sim}{C} = \underset{\sim}{I} \tag{3.3.38}$$

The components of all partial derivatives of this hinge element are all equal to zero.

The ball-and-socket joint, see figure 3.5, is also a scleronomic, holonomic kinematic element. The endpoints and the reference point are placed in the rotation centre of the joint, in other words their position coincides. Modelled as a constraint element the holonomic constraint equation becomes

$$\underset{\sim}{f}(\underset{\sim}{y},t) = \underset{\sim}{u} = \underset{\sim}{o} \tag{3.3.39}$$

where $\underset{\sim}{u} = \underset{\sim}{c}$ is the column with the relative Cartesian coordinates of the position of N^2 with respect to N. For the Pfaff matrix we find

$$\underset{\sim}{P}_y = [\ \underset{\sim}{I}\ \ \underset{\sim}{0}\ ] \tag{3.3.40}$$

while the components of the columns $\underset{\sim}{p}_{oy}$ and $\underset{\sim}{p}_{ooy}$ are zero. Note that this constraint element is scleronomic and that $\underset{\sim}{P}_y$ has a full rank $r(\underset{\sim}{P}_y) = np = 3$.

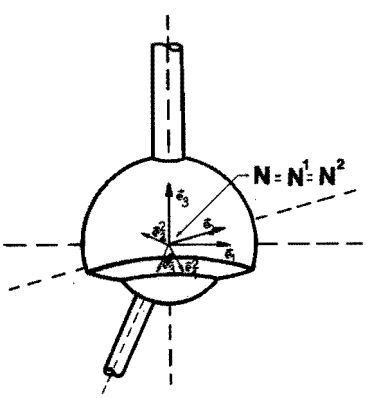


figure 3.5 A ball-and-socket joint

The choice of the endpoints implies that, if the ball-and-socket joint is modelled as a hinge element, $\underset{\sim}{c} = \underset{\sim}{o}$ and all components of $\underset{\sim}{v}$, $\underset{\sim o}{v}$ and $\underset{\sim oo}{v}$ are zero. If Bryant angles are chosen as generalized coordinates, we find for $\underline{C}$ that

$$\underline{C} = \begin{bmatrix} c_2c_3 & c_1s_3 + s_1s_2c_3 & s_1s_3 - c_1s_2c_3 \\ -c_2s_3 & c_1c_3 - s_1s_2s_3 & s_1c_3 + c_1s_2s_3 \\ s_2 & -s_1c_2 & c_1c_2 \end{bmatrix} \qquad (3.3.41)$$

where $c_i = \cos(q_i)$, $s_i = \sin(q_i)$. The corresponding partial derivatives are found to be

$$\underline{w}^T = [\underset{\sim 1}{w}, \underset{\sim 2}{w}, \underset{\sim 3}{w}]$$

with:

$$\underset{\sim 1}{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \underset{\sim 2}{w} = \begin{bmatrix} 0 \\ c_1 \\ s_1 \end{bmatrix}, \quad \underset{\sim 3}{w} = \begin{bmatrix} s_2 \\ -s_1c_2 \\ c_1c_2 \end{bmatrix} \qquad (3.3.42)$$

while the column $\underset{\sim o}{w}$ contains zero components and $\underset{\sim oo}{w}$ is equal to

$$\underset{\sim oo}{w} = \begin{bmatrix} c_2\dot{q}_2 \\ -s_1\dot{q}_1\dot{q}_2 - c_1c_2\dot{q}_1\dot{q}_3 + s_1s_2\dot{q}_2\dot{q}_3 \\ c_1\dot{q}_1\dot{q}_2 - s_1c_2\dot{q}_1\dot{q}_3 - c_1s_2\dot{q}_2\dot{q}_3 \end{bmatrix} \qquad (3.3.43)$$

3.4 Energetic elements

The constitutive equation
Endpoint variables and relevant variables
Examples

The behaviour of kinematic elements is only completely described by relations between kinematic variables. These elements do not perform any virtual work when the kinematic variables undergo a kinematically admissible variation. Therefore force variables play a role of minor importance. On the other hand, energetic elements can perform virtual

work if the kinematic variables are varied. As a result force variables play an important role.

We will not formulate the constitutive equation of an energetic element in terms of all kinematic and force variables of the endpoints but in terms of a smaller set of relevant variables. The relevant kinematic variables will be stored in a column $\underset{\sim}{\varepsilon}$ of order nvx1 with $0 \leq nv \leq 6xne$ where ne is the number of endpoints. The components of $\underset{\sim}{\varepsilon}$ must be independent and necessary. Generally they are interpreted as strains or displacements.

In connection with the relevant kinematic variables $\underset{\sim}{\varepsilon}$, relevant force variables $\underset{\sim}{F}$ can be defined so that the virtual work ΔW caused by a variation $\delta\underset{\sim}{\varepsilon}$ of $\underset{\sim}{\varepsilon}$ is given by

$$\Delta W = \underset{\sim}{F}^T \delta\underset{\sim}{\varepsilon} \qquad (3.4.1)$$

Note that column $\underset{\sim}{F}$ has as many components as $\underset{\sim}{\varepsilon}$. The interpretation of $\underset{\sim}{F}$ follows from (3.4.1) and the choice of $\underset{\sim}{\varepsilon}$. For example, if component ε_i of $\underset{\sim}{\varepsilon}$ represents the elongation of a spring, the corresponding component F_i of $\underset{\sim}{F}$ is the tensile force in that spring.

In the following subsections we first discuss the constitutive equation of an energetic element as a function of appropriately chosen relevant variables. Then the relationship between the relevant variables and the variables of the endpoints is dicussed.

The constitutive equation

The constitutive equation of an energetic element E with ne endpoints has already been mentioned earlier in section 3.2. We assume that this equation can be rewritten with $\underset{\sim}{\varepsilon}$ and $\underset{\sim}{F}$ as

$$\underset{\sim}{f} = \underset{\sim}{f}(\underset{\sim}{F}, \underset{\sim}{\varepsilon}, \tau \mid \tau \in (-\infty, t]) = \underset{\sim}{0} \qquad (3.4.2)$$

and that, if the trajectories of $\underset{\sim}{\varepsilon}(\tau)$ for $-\infty < \tau \leq t$ and of $\underset{\sim}{F}(\tau)$ for

$-\infty<\tau<t$ are known, $\underset{\sim}{F}(t)$ can be calculated from (3.4.2). In other words we consider energetic elements where the force variables depend not only on current values of $\underset{\sim}{\varepsilon}$ but on the history of $\underset{\sim}{\varepsilon}$ and $\underset{\sim}{F}$ as well.

An example of a constitutive equation of this kind is given by the integral equation

$$\underset{\sim}{F}(t) = \int_{\tau=-\infty}^{t} \underline{G}(t-\tau) \frac{\partial \underline{E}}{\partial \underset{\sim}{\varepsilon}} \frac{\partial \underset{\sim}{\varepsilon}}{\partial \tau} \, d\tau \qquad (3.4.3)$$

where $\underline{G}$ is a matrix with relaxation functions and $\underline{E}(\underset{\sim}{\varepsilon}(t))$ represents the elastic response. This constitutive equation has been introduced by Fung [1972] and is often used to describe the behaviour of biological tissue. For a special group of linear visco-elastic elements this integral equation may be replaced by a differential equation [for example see Findley et al. 1976] to give

$$\dot{\underset{\sim}{F}}(t) = \underline{K}_{f} \underset{\sim}{F}(t) + \underline{K}_{\varepsilon} \underset{\sim}{\varepsilon} + \underline{K}_{\dot{\varepsilon}} \dot{\underset{\sim}{\varepsilon}} \qquad (3.4.4)$$

Constitutive equations where only current values of the relevant variables are found, take the form

$$\underset{\sim}{f} = \underset{\sim}{f}(\underset{\sim}{F}, \underset{\sim}{\varepsilon}, t) = \underset{\sim}{0}, \qquad \underset{\sim}{f} = \underset{\sim}{f}(\underset{\sim}{F}, \dot{\underset{\sim}{\varepsilon}}, t) = \underset{\sim}{0} \qquad (3.4.5)$$

and a special case is given by the explicit equations

$$\underset{\sim}{F} = \underset{\sim}{F}(\underset{\sim}{\varepsilon}, t), \qquad \underset{\sim}{F} = \underset{\sim}{F}(\dot{\underset{\sim}{\varepsilon}}, t) \qquad (3.4.6)$$

Elements with such constitutive equations are called elastic elements or viscous elements. If their constitutive equations are linear we can write

$$\underset{\sim}{F} = \underline{K}(t) \underset{\sim}{\varepsilon}, \qquad \underset{\sim}{F} = \underline{B}(t) \dot{\underset{\sim}{\varepsilon}} \qquad (3.4.7)$$

where $\underline{K}$ and $\underline{B}$ are called the stiffness and damping matrices respectively.

It may be clear that equations of the type (3.4.2) are more difficult to evaluate than those of type (3.4.5). For example, to represent the

integral of (3.4.3) numerically, a summation over the time interval $(-\infty,t]$ has to be used, the relevant variables having to be stored for each point of time. Implicit equations are generally more difficult to solve than explicit ones as they require an iterative solver.

Endpoint variables and relevant variables

To describe the constitutive equation we used relevant variables in stead of the variables of the endpoints. We still have to relate these variables to each other. For this purpose a relationship between the relevant kinematic variables $\underset{\sim}{\varepsilon}$ and the kinematic variables $\underset{\sim}{c}^i$, $\underline{c}^i$ (i=1..ne) of the endpoints has to be given. We restrict ourselves to relations like

$$\underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}(\underset{\sim}{c}^i,\underline{c}^i \mid i=1..ne) \tag{3.4.8}$$

If this equation is differentiated, a relation for $\dot{\underset{\sim}{\varepsilon}}$ is found, so that

$$\dot{\underset{\sim}{\varepsilon}} = \sum_{i=1}^{ne} (\underline{G}_v^i \underset{\sim}{v}^i + \underline{G}_\Omega^i \underset{\sim}{\Omega}^i) \tag{3.4.9}$$

while variations $\delta\underset{\sim}{\varepsilon}$ of $\underset{\sim}{\varepsilon}$ are given by

$$\delta\underset{\sim}{\varepsilon} = \sum_{i=1}^{ne} (\underline{G}_v^i \delta\underset{\sim}{c}^i + \underline{G}_\Omega^i \delta\underset{\sim}{\Omega}^i) \tag{3.4.10}$$

To check whether the components of $\underset{\sim}{\varepsilon}$ are independent we follow the same strategy as in section 3.3 between (3.3.4) and (3.3.8).

The virtual work done by the forces $\underset{\sim}{F}^i_{in}$ and moments $\underset{\sim}{M}^i_{in}$ (i=1..ne) in the endpoints is given by

$$\Delta W = \sum_{i=1}^{ne} [(\underset{\sim}{F}^i_{in})^T \delta\underset{\sim}{c}^i + (\underset{\sim}{M}^i_{in})^T \delta\underset{\sim}{\Omega}^i] \tag{3.4.11}$$

Since no other loads are exerted on the element and because the element is massless, this virtual work should be equal to the virtual

work arising out of variations of $\underset{\sim}{\varepsilon}$. From (3.4.1) and (3.4.10) it follows that

$$\underset{\sim}{F}^i_{in} = (\underset{-V}{G}^i)^T \underset{\sim}{F}, \qquad \underset{\sim}{M}^i_{in} = (\underset{-Q}{G}^i)^T \underset{\sim}{F} \tag{3.4.12}$$

If the reference point N coincides with endpoint N^i, the corresponding $\underset{-V}{G}^i$ and $\underset{-Q}{G}^i$ are not defined. In that case it is not possible to obtain $\underset{\sim}{F}^i_{in}$ and $\underset{\sim}{M}^i_{in}$ from (4.3.12). However, as the element is massless these forces and moments can be obtained from the equilibrium equations (3.2.20).

Examples

The first example of an energetic element is a homogeneous bar element with two endpoints.

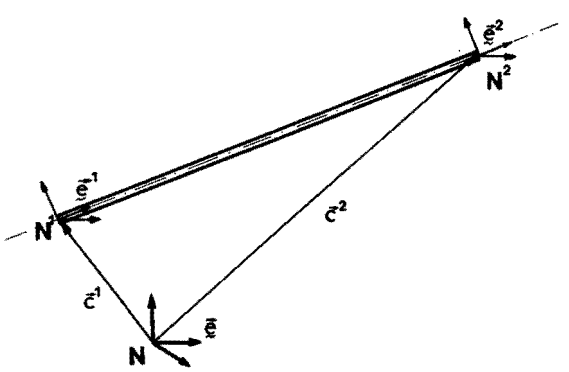


figure 3.6 The bar element

The element and the vector bases at the endpoints are shown in figure 3.6. During deformation the axis of the bar, defined as the line between N^1 and N^2, remains straight. Furthermore, there is no torsion and planes perpendicular to the axis of the bar remain perpendicular. The only relevant kinematic variable necessary for describing the mechanical behaviour of the bar is its elongation ε, this being the difference between the present length l(t) and the unloaded length l_o, so that

$$\underset{\sim}{\varepsilon} = [\varepsilon], \qquad \varepsilon = l(t) - l_o, \qquad l(t) = \| \underset{\sim}{c}^2 - \underset{\sim}{c}^1 \| \tag{3.4.13}$$

For the relationship between $\dot{\underset{\sim}{\varepsilon}}$ and $\dot{\underset{\sim}{c}}^2$, $\dot{\underset{\sim}{c}}^1$ we find:

$$\dot{\varepsilon} = -\underset{\sim}{n}^T \dot{\underset{\sim}{c}}^1 + \underset{\sim}{n}^T \dot{\underset{\sim}{c}}^2 \quad , \qquad \underset{\sim}{n} = \frac{\underset{\sim}{c}^2 - \underset{\sim}{c}^1}{l} \tag{3.4.14}$$

Therefore the matrices $\underline{G}_v^i$ (i=1,2) are row matrices, given by

$$\underline{G}_v^1 = -\underset{\sim}{n}^T, \qquad \underline{G}_v^2 = \underset{\sim}{n}^T \tag{3.4.15}$$

while $\underline{G}_Q^i$ (i=1,2) are zero row matrices. The relevant force variable is the tensile force F along the axis of the bar. If F is known, the internal forces vectors are given by

$$\underset{\sim}{F}_{in}^1 = -F\underset{\sim}{n}, \qquad \underset{\sim}{F}_{in}^2 = F\underset{\sim}{n} \tag{3.4.16}$$

while the moments $\underset{\sim}{M}_{in}^i$ (i=1,2) are zero. In the case of linear elastic behaviour the constutitive equation becomes

$$F = k\varepsilon \tag{3.4.17}$$

where k is the stiffness of the bar.

Assume that $N^1 = N$. In this case $\underset{\sim}{c}^1 = \underset{\sim}{o}$, $\dot{\underset{\sim}{c}}^1 = \underset{\sim}{o}$ and the components of $\underline{G}_v^1$ and $\underline{G}_Q^1$ are not defined. Knowing that the sum of all forces on the element should be equal to zero, it is found that $\underset{\sim}{F}_{in}^1 = -\underset{\sim}{F}_{in}^2$ and $\underset{\sim}{M}_{in}^1 = \underset{\sim}{M}_{in}^2 = \underset{\sim}{o}$.

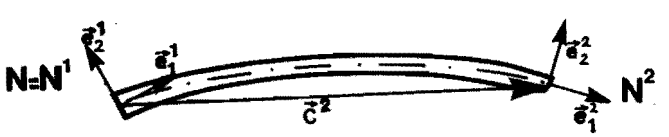


figure 3.7 The beam element

The second example concerns an initially straight beam element with two endpoints. Planes perpendicular to the beam axis remain perpendicular and torsion is left out of consideration. For simplicity it is assumed that the deformations remain small and that the beam axis deforms in one plane only. We will choose the reference base so that $\vec{e}_3$ is normal to this plane of deformation. The vector bases at the endpoints are chosen with $\vec{e}_3^{\,1}$ and $\vec{e}_3^{\,2}$ in the same direction as $\vec{e}_3$, while $\vec{e}_1^{\,1}$ and $\vec{e}_1^{\,2}$ are, at each moment, the tangents to the beam axis at the endpoints (see figure 3.7).

For the relevant kinematic variables we take the elongation ε_1 in the $\vec{e}_1$ direction, the displacement ε_2 of N^2 in the $\vec{e}_2$ direction and the rotation ε_3 of $\vec{\underset{\sim}{e}}^{\,2}$ around $\vec{e}_3$ (see figure 3.8).

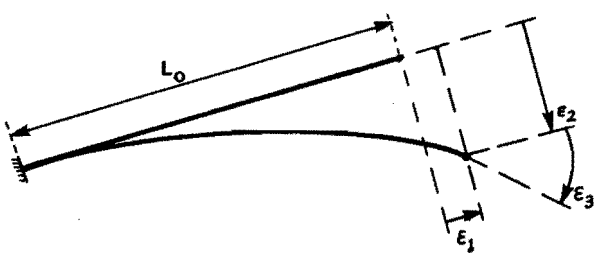


figure 3.8 The relevant kinematic variables of a beam element

The corresponding relevant force variables are the tensile force F_1 in the same direction as ε_1, the shear force F_2 in the same direction as ε_2 and the bending moment F_3 at endpoint N^2 in the same direction as ε_3. In the case of linear elastic behaviour the constitutive equations between the relevant kinematic and force variables become the elementary beam formulas.

If $N = N^1$ the relation between the relevant variables $\underset{\sim}{\varepsilon}$ and the variables $\underset{\sim}{c}^2$ and $\underset{\sim}{C}^2$ becomes

$$\varepsilon_1 = c_1^2 - l_o, \quad \varepsilon_2 = c_2^2, \quad \varepsilon_3 = \arcsin(C_{12}^2), \tag{3.4.18}$$

where c_1^2, c_2^2 and C_{12}^2 are components of the matrix representation of $\vec{c}^{\,2}$ and $\mathbb{C}^2$ with respect to the reference base. The matrices $\underset{\sim}{G}_v^1$ and $\underset{\sim}{G}_Q^1$ do not exist in this case since $N = N^1$ while $\underset{\sim}{G}_v^2$ and $\underset{\sim}{G}_Q^2$ are given by

$$\underline{G}_V^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \underline{G}_Q^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.4.19}$$

When $\underset{\sim}{F}^T = [F_1, F_2, F_3]$ is known we can easily determine $\underset{\sim}{F}^2_{in}$ and $\underset{\sim}{M}^2_{in}$ by using $\underline{G}_V^2$ and $\underline{G}_Q^2$ in (3.4.12). As already mentioned $\underset{\sim}{F}^1_{in}$ and $\underset{\sim}{M}^1_{in}$ are obtained by using the known $\underset{\sim}{F}^2_{in}$ and $\underset{\sim}{M}^2_{in}$ and the equilibrium equations.

The final example of an energetic element is an element with 3 endpoints. It is assumed that only small deformations due to plane strain/stress occur. The reference base is chosen so that $\vec{e}_3$ is normal to the plane spanned by the three endpoints and that $\vec{e}_1$ lies on the line from $N^1 = N$ to N^2. The relevant kinematic variables are shown in figure 3.9.

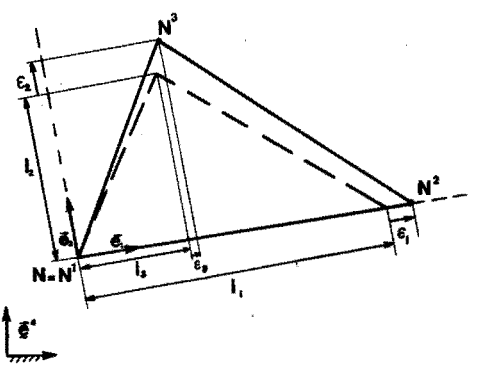


figure 3.9 Element with 3 endpoints

The relation $\underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}(\underline{c}^i, \underline{C}^i | i=1..3)$ in this case yields

$$\varepsilon_1 = c_1^2 - l_1, \qquad \varepsilon_2 = c_2^3 - l_2, \qquad \varepsilon_3 = c_1^3 - l_3 \tag{3.4.20}$$

while $\underline{G}_Q^i = \underline{o}$ for i=1..3 and $\underline{G}_V^i$ are defined by

$$\underline{G}_V^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \underline{G}_V^3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{3.4.21}$$

Note that $\underline{G}_v^1$ is not defined because $N^1 = N$. In the case of linearly elastic behaviour the constitutive equation becomes

$$\underset{\sim}{f} = \underline{K}\underset{\sim}{\varepsilon} \tag{3.4.22}$$

where $\underline{K}$ is called the stiffness matrix.

In the literature on finite element methods one can find such stiffness matrices for this 3-point element as well as many other elements [Zienkiewicz 1977].

3.5 Active elements

The constitutive equation
Examples
Concluding remarks

We have up to now considered kinematic and energetic elements where only kinematic and force variables played a role. In active elements other variables also occur. These will be called external input variables since active elements generally supply energy from external sources to a multibody system. Active elements are very similar to energetic elements and therefore only the new aspects will be discussed in this section.

The constitutive equation

Again we introduce relevant kinematic variables $\underset{\sim}{\varepsilon}$ and their corresponding forces variables $\underset{\sim}{F}$. In the case of active elements we also introduce external input variables, known as function of time, and stored in column $\underset{\sim}{i}(t)$. Examples of these variables are prescribed pressure differences for hydraulic actuators, voltage differences for electric motors, set-point values for position servos and recruitment and firing rate of the motor units in muscles.

For the constitutive equation of an active element we can write

$$\underset{\sim}{f}(\underset{\sim}{F}(\tau),\underset{\sim}{\varepsilon}(\tau),\underset{\sim}{i}(\tau)|\ \tau\in(-\infty,t]) = \underset{\sim}{0} \qquad (3.5.1)$$

If $\underset{\sim}{i} = \underset{\sim}{i}(\tau)$ and $\underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}(\tau)$ are known for $\tau \leq t$ and $\underset{\sim}{F} = \underset{\sim}{F}(\tau)$ is known for $\tau < t$, it should be possible to calculate the current value $\underset{\sim}{F}(t)$. In practice this constitutive equation is difficult to apply. Therefore we restrict ourselves to active elements where the influence of the history on the current value of $\underset{\sim}{F}$ is a function of the current value of the relevant kinematic variables, their first derivatives to time and a finite number of state variables. With the state variables stored in a column $\underset{\sim}{x}$, the constitutive equation becomes

$$\underset{\sim}{f}(\underset{\sim}{F}(t),\underset{\sim}{\varepsilon}(t),\dot{\underset{\sim}{\varepsilon}}(t),\underset{\sim}{x}(t),\underset{\sim}{i}(t),t) = \underset{\sim}{0} \qquad (3.5.2)$$

To obtain the value of the state variable at time t, a state equation must be solved. We restrict ourselves to state equations in the form of a differential equation with initial conditions. Mathematically

$$\dot{\underset{\sim}{x}}(t) = \underset{\sim}{s}(\underset{\sim}{x}(t),\underset{\sim}{\varepsilon}(t),\dot{\underset{\sim}{\varepsilon}}(t),\underset{\sim}{i}(t),t), \qquad \underset{\sim}{x}_0 = \underset{\sim}{x}(t_0) \qquad (3.5.3)$$

where t_0 is a point of time and where a value for the state variables is known. If the element is linear, the constitutive equation and the state equations become

$$\underset{\sim}{F} = \underline{C}\underset{\sim}{x} + \underline{D}\begin{bmatrix}\underset{\sim}{i}\\ \underset{\sim}{\varepsilon}\\ \dot{\underset{\sim}{\varepsilon}}\end{bmatrix}, \qquad \dot{\underset{\sim}{x}} = \underline{A}\underset{\sim}{x} + \underline{B}\begin{bmatrix}\underset{\sim}{i}\\ \underset{\sim}{\varepsilon}\\ \dot{\underset{\sim}{\varepsilon}}\end{bmatrix}, \quad \underset{\sim}{x}(t_0) = \underset{\sim}{x}_0 \qquad (3.5.4)$$

where $\underline{A}$, $\underline{B}$, $\underline{C}$ and $\underline{D}$ are called the system, the input, the output and the driving matrix respectively.

Sometimes only current values of the variables determine the behaviour of an active element. For the constitutive equations of such elements we write

$$\underset{\sim}{f}(\underset{\sim}{F}(t),\underset{\sim}{\varepsilon}(t),\dot{\underset{\sim}{\varepsilon}}(t),\underset{\sim}{i}(t),t) = \underset{\sim}{0} \qquad (3.5.5)$$

Note that no state variables are necessary to describe the behaviour of these elements. To distinguish these elements from the active elements where state variables are used, we speak of active elements without or with memory.

Examples

The first example of an active element is a hydraulic actuator. Being an active element without memory we consider the prescribed pressure as the external input variable i(t). The reference and endpoints are chosen as shown in figure 3.10.

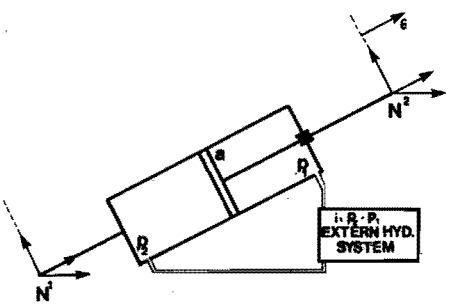


figure 3.10 An actuator as an active element

As relevant kinematic variable ε we choose the displacement of N^2 along the axis of the element. In this case $\underline{G}_V = [1,0,0]$ and $\underline{G}_Q = \underline{0}$. The corresponding relevant force variable is the axial force on the actuator and is given by

$$F(t) = ai(t) \qquad (3.5.6)$$

where a is the effective pressure area.

The second example we consider is an electric motor with a prescribed voltage V(t). This is an element with memory since the produced torque depends on the history of the voltage. The endpoints are shown in figure 3.11 with $N = N^1$. We consider the rotation of the base at N^2 with respect to the reference base as a relevant kinematic variable, while the corresponding relevant force variable is the torque about the rotation axis. In this case $\underline{G}_V = \underline{0}$ and $\underline{G}_Q = [1,0,0]$. We assume that the produced torque depends linearly on the electric current

through the motor. With the parameter c for the 'e.m.f.' constant of the motor this results in the constitutive equation

$$F(t) = cx(t) \tag{3.5.7}$$

where the state variable x represents the electric current.

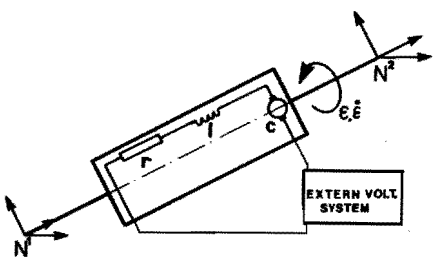


figure 3.11 An electric motor

For the state equation of a simple motor we use

$$\dot{x}(t) = Ax(t) + \underset{\sim}{B}[i, \varepsilon, \dot{\varepsilon}]^T, \qquad x_o = x(t_o) \tag{3.5.8}$$

where $A = -r/l$ and $\underset{\sim}{B} = [1, 0, c]/l$ with r and l representing the resistance and inductance of the electronic circuit.

For a position servo whose voltage V(t) depends on the difference between a set-point value $i(t) = i_o$ and the actual rotation of the shaft $\varepsilon(t)$ some changes should be made (see figure 3.12).

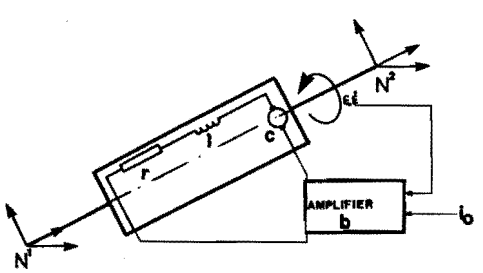


figure 3.12 The position servo

With b the amplification factor of the difference or error signal the state equation can be written, with A = -r/l and $\underset{\sim}{B}$ = [b,-b,c]/l, as

$$\dot{x} = Ax + \underset{\sim}{B}[i,\varepsilon,\dot{\varepsilon}]^T, \qquad x(t_o) = x_o \tag{3.5.9}$$

Concluding remarks

Active elements transport energy from an external source to a multibody system. All active elements described so far are elements similar to energetic elements. Active elements similar to kinematic elements have not been discussed, although it is possible to supply energy from an external source by prescribing the motion of a body. In this case we do not introduce a new active element where only kinematic variables occur, but we shall use a constraint element as discussed in 3.3.

In realistic systems more complex elements will be encountered, such as the electro-hydraulic actuator as described by Vukobratovic and Potkonjak [1982, p196]. Another example is the model of the human muscle of Hatze [1981] while containing two external input variables and five state variables. Each muscle or part of a muscle, represented by such an active element, involves 5 complicated nonlinear state equations with roughly 30 parameters.

Muscles are also a good example of connections which have more than two attachment points to their surroundings. There are several muscles in the human body that split into two or more parts, while there are also muscles which span two joints. These muscles can only be modelled by assemblies of elements. Assembling a connection out of several elements will be discussed in chapter 7. For the time being it is assumed that a connection consists of one element only.

CHAPTER 4
TOPOLOGY

A multibody system comprises a number of bodies and connections. In the previous chapters bodies and connections were discussed in detail. In this chapter consideration is given to a method for describing the topology of the bodies and connections in an arbitrary multibody system. In particular, the construction of a tree structure of bodies and hinges will be discussed. This tree structure plays an important role in chapter 5 where the kinematics of a multibody system are described.

4.1 Introduction

A graph is an abstract representation of a discrete system where certain parts of the system are represented by vertices while others are represented by edges between the vertices. The purpose of a graph is to determine a topology, an order, in the system without referring to physical properties of the parts of that system. In our multibody systems we will construct a graph in which the bodies are represented by vertices and the connections by edges.

The set of vertices will be noted as $\{V^0..V^{nv}\}$ and the set of edges as $\{A^1..A^{na}\}$. The edges can be considered as relationships between several vertices. In a formal notation:

$$A^k = A^k(V^i,..,V^j), \quad k = 1..na \tag{4.1.1}$$

This type of relations will be used in chapter 7 to describe the topology of energetic and active connections in a multibody system as

well as the topology of the elements in an connection. In the remaining part of this chapter energetic and active connections are not taken into account. This implies that we represent the kinematic connections of a system by edges in the graph of that system while energetic and active connections are not represented in that graph at all.

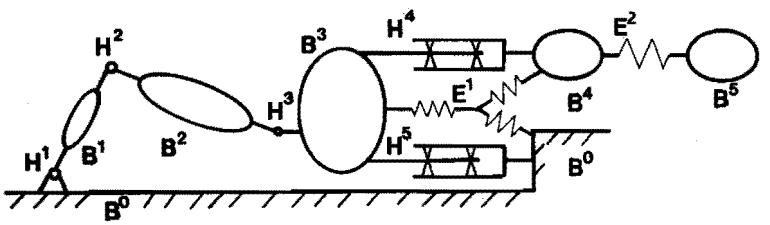


figure 4.1 An example of a multibody system

For example, the springs E^1 and E^2 of the multibody system of figure 4.1 do not occur in the graph (figure 4.2) of this system. The ball-and-socket joint H^1, the two pin-joints H^2 and H^3, as well as the two prismatic joints H^4 and H^5 are represented in this graph by the edges $A^1,..,A^5$ respectively.

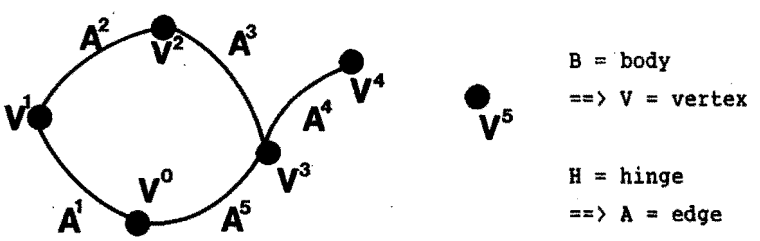


figure 4.2 The graph of the system of figure 4.1

4.2 The tree structure

To describe the kinematics of an arbitrary multibody system it is useful to introduce the notion of a graph with a tree structure. How-

ever, before a definition of a tree structure is given, it is necessary first to introduce the notions arc, branch, chord, path, degree of a vertex and isolated and closed subgraphs.

A kinematic connection is attached to the bodies at two attachment points only. As a result, each edge in the graph of a system of bodies and kinematic connections can be represented by a line between two vertices. More formally, edge A^k constitutes a relationship between two different vertices V^i and V^j, hence

$$A^k = A^k(V^i,V^j) \qquad (4.2.1)$$

It is possible to assign a direction to such an edge. For the edge in (4.2.1) the positive direction is defined from V^i towards V^j. If V^i and V^j in (4.2.1) are interchanged, the positive direction is from V^j towards V^i. Directed edges will be called arcs. A graph which consists of vertices and arcs can be represented by a figure in which each vertex is denoted by a dot and each arc by a line between two dots with an arrow to indicate the chosen direction.

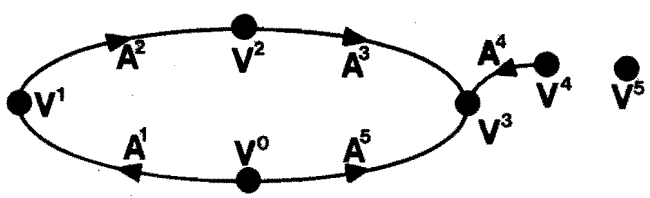


figure 4.3 A graph of vertices (●) and arcs (➔)

It is advantageous to distinguish between arcs which represent hinges and arcs which represent kinematic constraints. An arc which represents a hinge will be called a branch and will be drawn in the graph as a solid line. An arc which represents a kinematic constraint is called a chord and is drawn in the graph as a dotted line. For the system of figure 4.1 all arcs are branches (see figure 4.3).

The path from V^i to V^j is defined as the ordered set of branches (A^k, .., A^l) so that it is possible to start in V^i and end in V^j while

each branch of this set is passed only once and no jumps are made in the figure. In figure 4.3 both (A^1, A^2) and (A^5, A^3) are paths from V^o to V^2. The degree of a vertex is the number of branches starting or ending in that vertex. In figure 4.2 degree(V^5)=0, degree(V^4)=1 and degree(V^o)=2. The vertex with number o, V^o, will be called the reference vertex.

If there is no path between the reference vertex and a subset of other vertices, the vertices and the branches between them form an isolated subgraph. If there are two or more paths from a vertex to another vertex, we speak of a closed subgraph. In figure 4.1, V^5 is an isolated subgraph, while the vertices and arcs V^o, A^1, V^1, A^2, V^2, A^3, V^3, A^5 form a closed subgraph. A graph has a tree structure if there is an unambiguous path between the reference vertex and each of the other vertices. If degree(V^o) = nt with nt > 1 we will speak of a tree structure with nt trees.

The graph of an arbitrary multibody system will generally not have a tree structure. In that case we have to define a modified graph with a tree structure for this system. To obtain such a graph we first introduce a branch between one vertex of each isolated subgraph and one vertex of the rest of the graph and, second, open each closed subgraph by replacing one of the branches of that subgraph by a chord. For the system this means that, for each additional branch between an isolated subgraph and the rest of the graph, a hinge with six generalized coordinates is introduced, while one of the hinges in a closed kinematic chain have to be replaced by a kinematic constraint. In general it will be advisable to replace the hinge with the highest number (nq) of generalized coordinates as the hinge will be replaced by a kinematic constraint with np = 6 - nq constraint equations.

Finally the number of branches, nb, in the modified graph with tree structure is equal to the total number of vertices nv (excluding V^o). Note that the number of arcs in the original graph, na, is in general not equal to the sum of all branches and chords, nb + nc. This is due to the fact that additional branches must be introduced between isolated subgraphs and the rest of the graph.

As an example we define a tree structure for the multibody system in figure 4.1. The graph in figure 4.3 of this system contains one closed and one isolated subgraph. To connect the isolated subgraph we introduce a branch between V^5 and V^0. This branch is the representation of a hinge with six generalized coordinates between body B^5 and the ground.

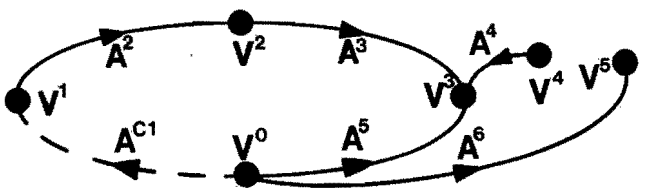


figure 4.4 A tree structure of figure 4.3

The closed subgraph represents the set of three bodies, a ball-and-socket joint (A^1), two pin-joints (A^2 and A^3) and a prismatic joint (A^5). To obtain a graph with tree structure we replace branch A^1 by a chord A^{C1}. This means that this ball-and-socket joint is modelled as a kinematic constraint with three constraint equations.

In the final graph with tree structure, all branches (solid lines) represent hinges and all chords (dotted lines) represent kinematic constraints. One of the consequences is that, if the generalized coordinates of all hinges are known, the kinematic quantities of the multibody system can be determined.

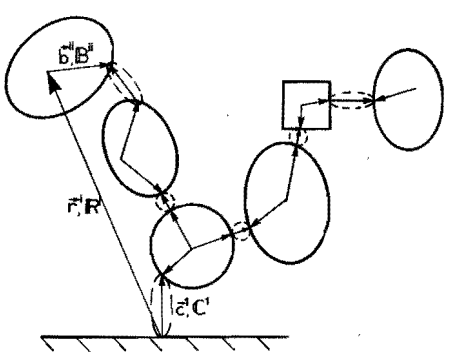


figure 4.5 A system of bodies and hinges in a tree structure

For example, knowing the generalized coordinates, it is possible to calculate for each hinge H^k the matrix representation of its connection vector $\vec{c}^k$ and tensor $\mathbb{C}^k$ (k = 1..nh). Together with the known, constant matrix representations of the body-fixed vectors and tensors ($\vec{b}^{ik}$, $\mathbb{B}^{ik}$) of each body B^i (i = 0..nb), all position vectors $\vec{r}^i$ and rotation tensors $\mathbb{R}^i$ can be calculated (see figure 4.5).

To store the topological data of a tree structure in graph matrices (see next section), we will always number (or renumber) the vertices and branches of a tree structure and direct the branches as follows:

- for V^i on the path from V^0 to V^j ($V^j \neq V^0$)
  the number i must be lower then j (i < j)
- A^k is the last branch on the path from V^0 to V^k
  and is directed towards V^k

If the graph contains nt trees then the vertices of the first tree are numbered from 1 to nv1, the vertices of the second tree from nv1 + 1 to nv1 + nv2, etc. Here nvi is the number of vertices in i^{th} tree (i=1..nt). The nc chords will be numbered from 1 to nc. A tree structure that is numbered according to these rules will be called a regular-numbered tree structure. From now on we assume that graphs with tree structures are always regular numbered. In figure 4.6 a regular numbering for the graph of figure 4.4 is shown.

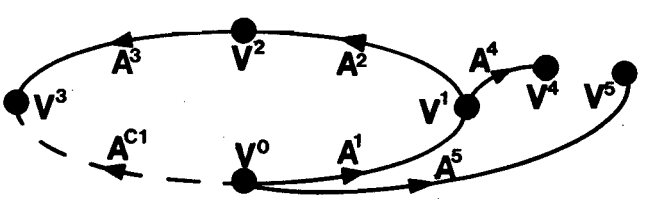


figure 4.6 A regular numbered tree structure

4.3 The graph matrices

Graph matrices are introduced in this section for graphs of vertices and arcs and in particular for graphs with tree structures. The purpose of these matrices is to store the topological data systematically.

The first graph matrices to be introduced are the location matrices $\underline{L}$ and $\underline{L}^c$. These matrices are defined for a graph with vertices, branches and chords such as:

$$\begin{aligned} L_{ik} &= +1 \quad \text{if branch } A^k \text{ ends in } V^i \\ &= -1 \quad \text{if branch } A^k \text{ starts at } V^i \\ &= \ 0 \quad \text{if branch } A^k \text{ does not start or end in } V^i \end{aligned}$$

or (4.3.1)

$$\begin{aligned} L^c_{ik} &= +1 \quad \text{if chord } A^k \text{ ends in } V^i \\ &= -1 \quad \text{if chord } A^k \text{ starts at } V^i \\ &= \ 0 \quad \text{if chord } A^k \text{ does not start or end in } V^i \end{aligned}$$

In these location matrices we distinguish four submatrices $\underset{\sim o}{S}$, $\underline{S}$, $\underset{\sim o}{S}^c$ and $\underline{S}^c$:

$$\underline{L} = \begin{bmatrix} \underset{\sim o}{S} \\ \hline \underline{S} \end{bmatrix} \begin{matrix} \\ \hline \\ \end{matrix} \begin{matrix} 1 \\ \hline nv \end{matrix}, \qquad \underline{L}^c = \begin{bmatrix} \underset{\sim o}{S}^c \\ \hline \underline{S}^c \end{bmatrix} \begin{matrix} 1 \\ \hline nv \end{matrix} \tag{4.3.2}$$

$$\begin{matrix} | \ nb \ | \end{matrix} \qquad \begin{matrix} | \ nc \ | \end{matrix}$$

Note that $\underline{S}$ is square since nv = nb. If degree(V^o) = nt > 1, then $\underset{\sim o}{S}$ and $\underline{S}$ are subdivided into rows $\underset{\sim o}{S}^i$ and matrices $\underline{S}^i$ with i = 1..nt, given by

$$\underset{\sim o}{S} = [\underset{\sim o}{S}^1 .. \ \underset{\sim o}{S}^{nt}], \qquad \underline{S} = \begin{bmatrix} \underline{S}^1 & & \underline{0} \\ & \ddots & \\ \underline{0} & & \underline{S}^{nt} \end{bmatrix} \tag{4.3.3}$$

The location matrices of the graph with two trees from figure 4.6 become

$$\underline{L} = \left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & -1 \\ \hline +1 & -1 & 0 & -1 & 0 \\ 0 & +1 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 \\ \hline 0 & 0 & 0 & 0 & +1 \end{array}\right] \begin{array}{c} V^0, \\ V^1 \\ V^2 \\ V^3 \\ V^4 \\ V^5 \end{array} \qquad \underline{L}^c = \left[\begin{array}{c} -1 \\ \hline +1 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right] \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array} \qquad (4.3.4)$$

$$\begin{array}{cccccc} A^1 & A^2 & A^3 & A^4 & A^5 & \end{array} \qquad\qquad A^1$$

Instead of storing all data in the sparse matrices $\underset{\sim}{S}_{vo}$ and $\underline{S}$ we sometimes use a location column $\underset{\sim}{l}$. This column is defined as

$$l_j = i \quad \text{if} \quad L_{ij} = -1 \qquad (i = 0..nv,\ j = 1..nv) \qquad (4.3.5)$$

so that the j^{th} component of $\underset{\sim}{l}$ is equal to the number of the last but one vertex on the path from V^0 to V^j. In the case of our example (see figure 4.6) the values of the components of this column become 0, 1, 2, 1, 0 respectively.

It is also possible to store the topological data of a tree in a tree matrix $\underline{T}$, defined as:

$$T_{ki} = \begin{cases} +1 & \text{if } A^k \text{ is a branch on the path from } V^0 \text{ to } V^i \text{ and is directed towards } V^i \\ -1 & \text{if } A^k \text{ is a branch on the path from } V^0 \text{ to } V^i \text{ and is directed back to } V^0 \\ 0 & \text{if } A^k \text{ is not a branch on the path from } V^0 \text{ to } V^i \end{cases} \qquad (4.3.6)$$

For the two tree matrices $\underline{T}^1$ and $\underline{T}^2$ of the trees in the graph of figure 4.6 we find

$$\begin{array}{cccc} V^1 & V^2 & V^3 & V^4 \end{array} \qquad\qquad V^5$$

$$\underline{T}^1 = \left[\begin{array}{cccc} +1 & +1 & +1 & +1 \\ 0 & +1 & +1 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{array}\right] \begin{array}{c} A^1, \\ A^2 \\ A^3 \\ A^4 \end{array} \qquad \underline{T}^2 = \left[\begin{array}{c} +1 \end{array}\right] A^5 \qquad (4.3.7)$$

The matrices $\underset{\sim}{S}^i_o$, $\underline{S}^i$ and $\underline{T}^i$ of a regular numbered tree structure have a regular structure. The $\underset{\sim}{S}^i_o$ matrices, except for the first component, contain zero components only. The $\underline{S}^i$ and $\underline{T}^i$ matrices have a main diagonal with components +1 and lower triangular matrices with zero components only. The upper triangular matrices only contain components equal to -1, 0 and +1, 0. These properties are lost if the numbering is not regular.

An important relationship exists between the submatrices $\underset{\sim}{S}^i_o$ and $\underline{S}^i$ of the location matrix and the tree matrix $\underline{T}^i$ (see Wittenburg [1977, p88]):

$$\underset{\sim}{S}^i_o\underline{T}^i = -\underset{\sim}{1}^T_{nvi}, \qquad \underline{S}^i\underline{T}^i = \underline{T}^i\underline{S}^i = \underline{I}_{nvi} \qquad i = 1..nt \qquad (4.3.8)$$

where $-\underset{\sim}{1}_n$ is a column with n components equal to -1 and $\underline{I}_n$ is the identity matrix of order nxn.

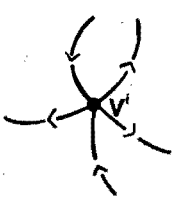
To illustrate the importance of the location matrix $\underline{L}$ we consider two columns $\underset{\sim}{y}$ and $\underset{\sim}{x}$ where component y_i of $\underset{\sim}{y}$ is a variable defined at vertex V^i and component x_k of $\underset{\sim}{x}$ is a corresponding variable of branch A^k. The relation

$$\underset{\sim}{y} = \underline{L}\underset{\sim}{x} \qquad (4.3.9)$$

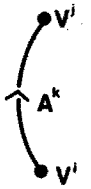
then expresses that y_i is the sum over k = 1..nb of the variables x_k of those branches that end at V^i minus the variables x_k of those branches that start from V^i (see figure 4.7a). For example if $\underset{\sim}{x}$ contains internal forces in the attachment points of the connections then $\underset{\sim}{y}$ is the column with the resulting internal forces exerted on the bodies. The opposite relation

$$\underset{\sim}{x} = \underline{L}^T\underset{\sim}{y} \qquad (4.3.10)$$

expresses that the variable x_k, defined for branch A^k, is determined by the variables y_i and y_j defined in the vertices V^i and V^j of A^k. An example of x_k is the connection vector $\vec{c}_k$ of two points while the



a



b

figure 4.7 Vertex with arcs Arc between two vertices

corresponding variables y_i, y_j represent the absolute position of these points (see figure 4.7b).

Summary

Graphs are used to represent the topology of multibody systems as well as assemblies of elements. In this chapter attention is given only to graphs of systems of bodies and kinematic connections. For such graphs it is possible to define a tree structure in which branches represent hinges and chords represent kinematic constraints.

CHAPTER 5
KINEMATICS OF A MULTIBODY SYSTEM

This chapter deals with the formulas describing the kinematic behaviour of multibody systems. In order to study this behaviour in a system with an arbitrary topology we first define a tree structure of bodies and hinges and then add the kinematic constraints (sections 5.2 & 5.3). In the first section Lagrange coordinates will be introduced as coordinates to describe the kinematic behaviour of a multibody system. Section 5.4 deals with prescribed Lagrange and/or attitude coordinates.

Up to now we have not discussed the assembly of elements into one complex connection. It is therefore assumed that each connection contains one element only. Since this chapter deals with kinematic behaviour, energetic and active connections are not considered. These connections only become relevant when we study the dynamic behaviour of a multibody system.

5.1 The Lagrange coordinates

Consider a multibody system of nb + 1 bodies in which body B^0 is the reference body with the global vector base $\underset{\sim}{\vec{e}}^0$. This system of bodies, hinges and kinematic constraints may have a graph with an arbitrary topology. Nevertheless a tree structure of vertices (bodies) and branches (hinges) can be defined in the graph. As discussed in the previous chapter, first of all the kinematic constraints are represented in the graph by chords. In the graph so obtained there may be closed and isolated subgraphs. In the case of an isolated part an additional hinge with six generalized coordinates is introduced between

a body of the isolated part and the rest of the system. In the case of a closed kinematic chain of bodies and hinges the chain is opened by removing one hinge. Instead of that hinge a kinematic constraint is introduced.

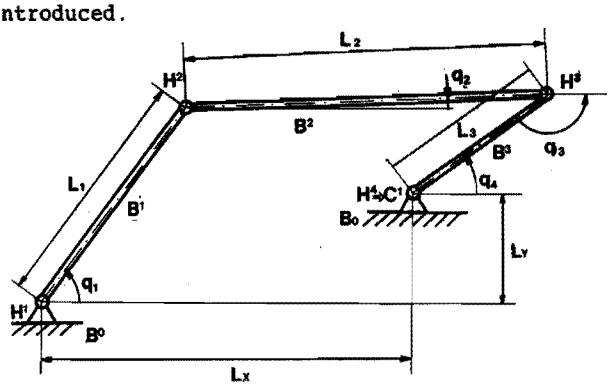


figure 5.1 A four-bar mechanism

To illustrate this process a four-bar mechanism with four pin-joints (hinges) is considered (see figure 5.1). There is a closed kinematic chain in this system. The chain is opened at hinge H^4 and the pin-joint replaced by a kinematic constraint C^1. The graph of the four-bar mechanism then has a tree structure with three bodies, three pin-joints and one kinematic constraint (see figure 5.2).

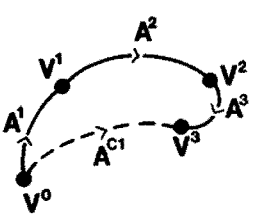


figure 5.2 Graph with tree structure of figure 5.1

Each of the pin-joints H^i (i = 1..3) is described as a function of one generalized coordinate q_i, while the kinematic constraint is represented by the two following constraint equations:

$$f_1: l_1\sin(q_1) + l_2\sin(q_2) + l_3\sin(q_3) - l_x = 0 \qquad (5.1.1)$$
$$f_2: l_1\cos(q_1) + l_2\cos(q_2) + l_3\cos(q_3) - l_y = 0$$

To describe the kinematics of this system we can use the generalized coordinates q_1, q_2 and q_3, which have to satisfy the two constraint equations.

Usually the kinematics of a multibody system with a tree structure are described by means of the generalized coordinates of the hinges in the tree structure. We use these coordinates as the primary unknowns. Position vectors, rotation tensors, attitude coordinates etc. can be determined as a function of these coordinates. On the other hand we could use the attitude coordinates $\underset{\sim}{z}_i$ (i = 1..nb) of all bodies B^i as primary unknowns.

One important reason for choosing the generalized coordinates of the hinges in the tree structure as primary unknowns is because the use of attitude coordinates generally results in a larger number of unknowns and a larger set of constraint equations. If we define a column $\underset{\sim}{q}$ with the generalized coordinates of all hinges in the tree structure we obtain

$$\underset{\sim}{q}^T = [(\underset{\sim}{q}^1)^T \; .. \; (\underset{\sim}{q}^{nh})^T], \qquad nq = \sum_{k=1}^{nh} nq^k \tag{5.1.2}$$

and for a column with all attitude coordinates we have

$$\underset{\sim}{z}^T = [(\underset{\sim}{z}^1)^T \; .. \; (\underset{\sim}{z}^{nb})^T], \tag{5.1.3}$$

then the number nq of the components of $\underset{\sim}{q}$ is generally much smaller than the number nz = 6nb of components of $\underset{\sim}{z}$.

As the coordinates $\underset{\sim}{q}$ have to satisfy constraint equations arising out of kinematic constraints (the chords in the tree), they are in general dependent. In accordance with the literature we want to reserve the name generalized coordinates for sets of coordinates which are independent. Therefore the nq coordinates of all hinges in the tree structure will be called Lagrange coordinates (see e.g. Paul [1979, p267]). In the next section kinematic formulas will be constructed as a function of the Lagrange coordinates and time.

5.2 The kinematic formulas for a tree structure

Orientation and derivatives
Position and derivatives
Attitude coordinates and derivatives
Combining the formulas for several trees
Example

The position vectors and the rotation tensors of all bodies, their first and second order derivatives and their variation will be expressed as functions of $\underset{\sim}{q}$, $\dot{\underset{\sim}{q}}$, $\ddot{\underset{\sim}{q}}$, $\delta\underset{\sim}{q}$ and t. First of all the formulas for one tree are constructed. Then we briefly discuss the modifications of the formulas for a structure with several trees. Last of all an example is given.

Orientation and derivatives

The rotation tensor $\mathbb{R}^j$ of body B^j is determined by using the recursive relation (see figure 5.3):

$$\begin{aligned} &\text{for } j = 0 && \mathbb{R}^j = \mathbb{I} \\ &\text{for } j = 1..nb && \mathbb{R}^j = (\mathbb{B}^{jk})^T \bullet \mathbb{C}^k \bullet \mathbb{B}^{ik} \bullet \mathbb{R}^i \end{aligned} \qquad (5.2.1)$$

where i and k are the numbers of the last body and hinge on the path from the ground B^o to B^j. The numbers i and k depend on j. Since the numbering of the bodies and hinges is regular, k = j while i is obtained from the location column $\underset{\sim}{l}$ as $i = l_j$.

As can be seen from figure 5.3, the relationship between the relative angular velocity of the two connected bodies B^i and B^j can be expressed as

$$\vec{\Omega}^k = \vec{\omega}^j - \vec{\omega}^i = S_{jk}\vec{\omega}^j - -S_{ik}\vec{\omega}^i \qquad (5.2.2)$$

where S_{ik} and S_{jk} are components of the submatrices $\underset{\sim}{S}_o$ and $\underset{\sim}{S}$, introduced in the previous chapter.

This relationship holds true for all bodies B^j, $j > 1$, in the tree structure. We can therefore write (Wittenburg [1977, ch5])

$$\vec{\underset{\sim}{Q}} = \underline{S}^T \vec{\underset{\sim}{\omega}} \tag{5.2.3}$$

Here the angular velocities $\vec{\omega}^j$ of all bodies B^j, j=1..nb, are stored in a column $\vec{\underset{\sim}{\omega}}$, given by $\vec{\underset{\sim}{\omega}}^T = [\vec{\omega}^1 .. \vec{\omega}^{nb}]$, while the relative angular velocities $\vec{\Omega}^k$ of all hinges H^k, k=1..nh, are stored in a column $\vec{\underset{\sim}{Q}}$, given by $\vec{\underset{\sim}{Q}}^T = [\vec{\Omega}^1 .. \vec{\Omega}^{nh}]$.

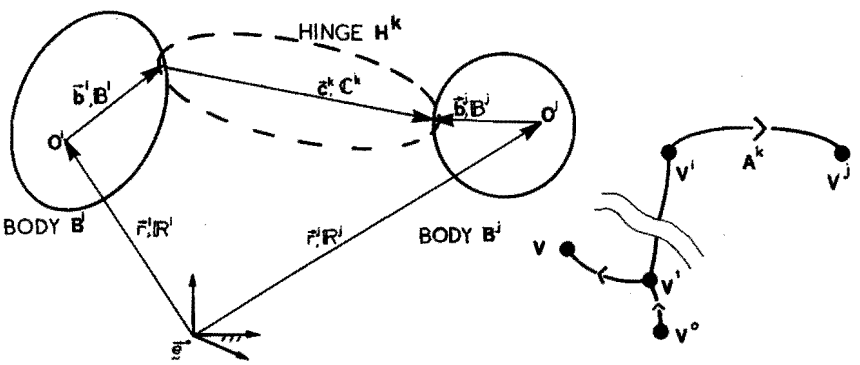


figure 5.3 Two bodies connected by a hinge

In chapter 3 we derived relation (3.3.28) for $\vec{\Omega}^k$, k=1..nk,

$$\vec{\Omega}^k = (\vec{\underset{\sim}{w}}^k)^T \vec{\underset{\sim}{q}}^k + \vec{w}^k_o, \qquad (\vec{\underset{\sim}{w}}^k)^T = [\vec{w}^k_1 .. \vec{w}^k_{nq}] \tag{5.2.4}$$

If we introduce a diagonal matrix $\vec{\underline{W}}$ and a column $\vec{\underset{\sim}{W}}_o$, defined by

$$\vec{\underline{W}} = \begin{bmatrix} \vec{\underset{\sim}{w}}^1 & & \vec{0} \\ & \ddots & \\ \vec{0} & & \vec{\underset{\sim}{w}}^{nh} \end{bmatrix}, \qquad \vec{\underset{\sim}{W}}_o = \begin{bmatrix} \vec{w}^1_o \\ \vdots \\ \vec{w}^{nh}_o \end{bmatrix} \tag{5.2.5}$$

then we can combine it with relation (5.2.4) for k=1..nh in one matrix equation for $\vec{\underset{\sim}{Q}}$ into

$$\vec{\underset{\sim}{Q}} = \vec{\underline{W}}^T \vec{\underset{\sim}{q}} + \vec{\underset{\sim}{W}}_o \tag{5.2.6}$$

If (5.2.3) and (5.2.6) are combined and the relation $\underline{S}\underline{T} = \underline{T}\underline{S} = \underline{I}$ is used, the final relation for the angular velocities of the bodies becomes

$$\underset{\sim}{\vec{\omega}} = (\underline{\vec{W}}\underline{T})^T \dot{\underset{\sim}{q}} + \underline{T}^T \underset{\sim}{\vec{W}}_0 \qquad (5.2.7)$$

The angular accelerations of the bodies can be determined by differentiation of (5.2.3), yielding

$$\dot{\vec{\omega}}^j - \dot{\vec{\omega}}^i = \dot{\vec{\Omega}}^k = \vec{\omega}^i * \vec{\Omega}^k + \vec{\alpha}^k \qquad (5.2.8)$$

where $\vec{\alpha}^k$ is the relative angular acceleration in hinge H^k for which, we found in (3.3.29)

$$\vec{\alpha}^k = (\underset{\sim}{\vec{w}}^k)^T \ddot{\underset{\sim}{q}}^k + \vec{w}^k_{00} \qquad (5.2.9)$$

If a column $\underset{\sim}{\vec{W}}_{00}$ is defined by

$$\underset{\sim}{\vec{W}}^T_{00} = [\vec{W}^1_{00} .. \vec{W}^{nh}_{00}], \qquad \vec{W}^k_{00} = \vec{w}^k_{00} + \vec{\omega}^i * \vec{\Omega}^k \qquad (5.2.10)$$

where i is the value of the k^{th} component of $\underset{\sim}{l}$, we can, for the angular accelerations write

$$\dot{\underset{\sim}{\vec{\omega}}} = (\underline{\vec{W}}\underline{T})^T \ddot{\underset{\sim}{q}} + \underline{T}^T \underset{\sim}{\vec{W}}_{00} \qquad (5.2.11)$$

where $\underline{\vec{W}}$ is a function of $\underset{\sim}{q}$ and t, while $\underset{\sim}{\vec{W}}_{00}$ is a function of $\underset{\sim}{q}$, $\dot{\underset{\sim}{q}}$, t.

The variation of the orientation of the body-fixed bases, thanks to variations of the Lagrange coordinates $\underset{\sim}{q}$, can be determined in a manner similar to $\underset{\sim}{\vec{\omega}}$. This yields

$$\delta \underset{\sim}{\vec{\pi}} = (\underline{\vec{W}}\underline{T})^T \delta \underset{\sim}{q}, \qquad \delta \underset{\sim}{\vec{\pi}}^T = [\delta \vec{\pi}^1 .. \delta \vec{\pi}^{nb}] \qquad (5.2.12)$$

Position and derivatives

The position of body B^j with respect to the global base is determined by using the recursive relation:

$$\begin{aligned} &\text{for } j = 0 && \vec{r}^{\,j} = \vec{o} \\ &\text{for } j = 1..nb && \vec{r}^{\,j} = \vec{r}^{\,i} + \vec{b}^{\,ik} + \vec{c}^{\,k} - \vec{b}^{\,jk} \end{aligned} \qquad (5.2.13)$$

where i and k, with $i = \underset{\sim}{l}(j)$ and $k = j$, are the numbers of the last body and hinge on the path from the ground B^0 to B^j (see figure 5.3).

The body-fixed vectors $\vec{b}^{\,ik}$ and $\vec{b}^{\,jk}$ as well as the connection vector $\vec{c}^{\,k}$ will be stored in matrices with the same structure as the sub-matrices $\underset{\sim}{S}_o$ and $\underset{\sim}{S}$ of the location matrix $\underset{\sim}{L}$. These matrices, denoted by $\vec{\underset{\sim}{B}}_o$, $\vec{\underset{\sim}{B}}$ and $\vec{\underset{\sim}{C}}_o$, $\vec{\underset{\sim}{C}}$, contain the body-fixed vectors and connection vectors and are given by:

$$\begin{aligned} \vec{B}_{ik} &= \vec{b}^{\,ik}, & \vec{C}_{ik} &= \vec{o} && \text{if } L_{ik} = 1 \\ \vec{B}_{ik} &= -\vec{b}^{\,ik}, & " &= -\vec{c}^{\,k} && \text{if } " = -1 \\ \vec{B}_{ik} &= \vec{o}, & " &= \vec{o} && \text{if } " = 0 \end{aligned} \qquad (5.2.14)$$

where i = 0..nb and k = 1..nh (nb=nh). Using these matrices we can rewrite (5.2.13) as

$$S_{jk}\vec{r}^{\,j} - -S_{ik}\vec{r}^{\,i} = -\vec{B}_{ik} - \vec{C}_{ik} - \vec{B}_{jk}, \qquad k=1..nh \qquad (5.2.15)$$

and, in a more compact matrix notation, also as

$$\underset{\sim}{S}^T\vec{\underset{\sim}{r}} = -(\vec{\underset{\sim}{B}}_o + \vec{\underset{\sim}{C}}_o)^T - (\vec{\underset{\sim}{B}} + \vec{\underset{\sim}{C}})^T 1_{nb} \qquad (5.2.16)$$

Here the position vectors $\vec{r}^{\,j}$ of all bodies excluding the ground are stored in the column $\vec{\underset{\sim}{r}}^T = [\vec{r}^{\,1}..\vec{r}^{\,nb}]$. Multiplication of (5.2.16) by $\underset{\sim}{T}$ and substitution of $\underset{\sim}{S}\underset{\sim}{T} = \underset{\sim}{I}$ finally yields

$$\vec{\underset{\sim}{r}} = -((\vec{\underset{\sim}{B}}_o + \vec{\underset{\sim}{C}}_o)\underset{\sim}{T})^T - ((\vec{\underset{\sim}{B}} + \vec{\underset{\sim}{C}})\underset{\sim}{T})^T 1_{nb} \qquad (5.2.17)$$

If we differentiate (5.2.13) we find

$$\dot{\vec{r}}^{j} - \dot{\vec{r}}^{i} = \vec{\omega}^{i} * \vec{b}^{ik} + \vec{\omega}^{i} * \vec{c}^{k} + \vec{v}^{k} - \vec{\omega}^{j} * \vec{b}^{jk} \qquad (5.2.18)$$

In chapter 3 we derived relation (3.3.31) for the relative velocity vector $\vec{v}^{k}$ of hinge H^{k} (k = 1..nh), so that

$$\vec{v}^{k} = (\underset{\sim}{\vec{v}}^{k})^{T} \dot{\underset{\sim}{q}}^{k} + \vec{v}_{o}^{k}, \qquad (\underset{\sim}{\vec{v}}^{k})^{T} = [\vec{v}_{1}^{k} .. \vec{v}_{nq}^{k}] \qquad (5.2.19)$$

Let $\underline{\vec{V}}$ and $\underset{\sim}{\vec{V}}_{o}$ be a diagonal matrix and a column, defined by:

$$\underline{\vec{V}} = \begin{bmatrix} \underset{\sim}{\vec{v}}^{1} & & \underset{\sim}{\vec{0}} \\ & \ddots & \\ \underset{\sim}{\vec{0}} & & \underset{\sim}{\vec{v}}^{nh} \end{bmatrix}, \qquad \underset{\sim}{\vec{V}}_{o} = \begin{bmatrix} \vec{v}_{o}^{1} \\ \vdots \\ \vec{v}_{o}^{nh} \end{bmatrix} \qquad (5.2.20)$$

then instead of (5.2.19) we can write

$$\underset{\sim}{\vec{v}} = \underline{\vec{V}}^{T} \dot{\underset{\sim}{q}} + \underset{\sim}{\vec{V}}_{o}, \qquad \underset{\sim}{\vec{v}}^{T} = [\vec{v}^{1} .. \vec{v}^{nb}] \qquad (5.2.21)$$

Using the matrices $\underline{\vec{B}}$ and $\underline{\vec{C}}$ and the relation $\underline{S}\underline{T} = \underline{I}$ it is easily shown that the velocities of the bodies are given by

$$\dot{\underset{\sim}{\vec{r}}} = ((\underline{\vec{B}} + \underline{\vec{C}})\underline{T})^{T} * \underset{\sim}{\vec{\omega}} + (\underline{\vec{V}}\underline{T})^{T} \dot{\underset{\sim}{q}} + \underline{T}^{T} \underset{\sim}{\vec{V}}_{o} \qquad (5.2.22)$$

If (5.2.7) is substituted for $\underset{\sim}{\vec{\omega}}$ we finally arrive at

$$\dot{\underset{\sim}{\vec{r}}} = (\underline{\vec{U}}\underline{T})^{T} \dot{\underset{\sim}{q}} + \underline{T}^{T} \underset{\sim}{\vec{U}}_{o} \qquad (5.2.23)$$

with

$$\underline{\vec{U}} = \underline{\vec{V}} - \underline{\vec{W}}\underline{T} * (\underline{\vec{B}} + \underline{\vec{C}}), \qquad \underset{\sim}{\vec{U}}_{o} = \underset{\sim}{\vec{V}}_{o} + (\underline{\vec{B}} + \underline{\vec{C}})^{T} * \underline{T}^{T} \underline{\vec{W}}_{o}$$

Although the formulas become very abstract, it is possible to give a global interpretation of $\underline{\vec{U}}$ and $\underset{\sim}{\vec{U}}_{o}$. This matrix and column determine the absolute velocity as the sum of the relative velocities $\underline{\vec{V}}$ of the hinges and the (out-)product of angular velocities $\underline{\vec{W}}$ of the bodies and their absolute positions $\underline{T}(\underline{\vec{B}}+\underline{\vec{C}})$.

The acceleration of the bodies will be determined in a way similar to the velocities. If we differentiate (5.2.18) we again find

$$\vec{r}^{j} - \vec{r}^{i} = \dot{\vec{\omega}}^{i} * \vec{b}^{ik} + \vec{\omega}^{i} * (\vec{\omega}^{i} * \vec{b}^{ik}) - \dot{\vec{\omega}}^{j} * \vec{b}^{jk} - \vec{\omega}^{j} * (\vec{\omega}^{j} * \vec{b}^{jk}) + \dot{\vec{\omega}}^{i} * \vec{c}^{k} + \vec{\omega}^{i} * (\vec{\omega}^{i} * \vec{c}^{k} + 2\vec{v}^{k}) + \vec{a}^{k} \qquad (5.2.24)$$

where $\vec{a}^{k}$ is the relative acceleration of hinge H^{k}, k=1..nh. We have found in (3.3.33) that

$$\vec{a}^{k} = (\underset{\sim}{\vec{v}}^{k})^{T} \ddot{\underset{\sim}{q}} + \vec{v}^{k}_{oo}, \qquad k = 1..nh \qquad (5.2.25)$$

Now $\underset{\sim}{\vec{v}}^{T}_{oo} = [\vec{v}^{1}_{oo} .. \vec{v}^{nh}_{oo}]$ is defined with components

$$\vec{v}^{k}_{oo} = \vec{\omega}^{i} * (\vec{\omega}^{i} * \vec{b}^{ik}) - \vec{\omega}^{j} * (\vec{\omega}^{j} * \vec{b}^{jk}) + \vec{\omega}^{i} * (\vec{\omega}^{i} * \vec{c}^{k} + 2\vec{v}^{k}) + \vec{v}^{k}_{oo} \qquad (5.2.26)$$

With this definition the relation for the accelerations is given in matrix notation as

$$\ddot{\underset{\sim}{\vec{r}}} = ((\underline{\vec{B}} + \underline{\vec{C}})\underline{T})^{T} * \dot{\underset{\sim}{\vec{\omega}}} + (\underline{\vec{T}}\underline{T})^{T} \ddot{\underset{\sim}{q}} + \underline{T}^{T} \underset{\sim}{\vec{V}}_{oo} \qquad (5.2.27)$$

After substitution of (5.2.11) for $\dot{\underset{\sim}{\vec{\omega}}}$ we can, for the accelerations finally write

$$\ddot{\underset{\sim}{\vec{r}}} = (\underline{\vec{U}}\underline{T})^{T} \ddot{\underset{\sim}{q}} + \underline{T}^{T} \underset{\sim}{\vec{U}}_{oo} \qquad (5.2.28)$$

where $\underline{\vec{U}}$ is defined in (5.2.23) and $\underset{\sim}{\vec{U}}_{oo}$ by:

$$\underset{\sim}{\vec{U}}_{oo} = \underset{\sim}{\vec{V}}_{oo} + (\underline{\vec{B}} + \underline{\vec{C}})^{T} * \underline{T}^{T} \underset{\sim}{\vec{W}}_{oo}$$

The variation of the position vectors of the bodies, by variation of the Lagrange coordinates, can be derived in a way similar to the velocities (5.2.23), yielding

$$\delta\underset{\sim}{\vec{r}} = (\underline{\vec{U}}\underline{T})^{T} \delta\underset{\sim}{q}, \qquad \delta\underset{\sim}{\vec{r}}^{T} = [\delta\vec{r}^{1} .. \delta\vec{r}^{nb}] \qquad (5.2.29)$$

Attitude coordinates and derivatives

As soon as $\underline{R}$ and $\underset{\sim}{\vec{r}}$ are obtained as functions of the Lagrange coordinates $\underset{\sim}{q}$ and time, we can determine the attitude coordinates $\underset{\sim}{z}(\underset{\sim}{q},t)$.

For the first and second order derivatives with respect to time and the variation of the attitude coordinates we find:

$$\dot{\underset{\sim}{z}} = \underline{Z}^T \dot{\underset{\sim}{q}} + \underset{\sim}{Z}_o, \qquad \ddot{\underset{\sim}{z}} = \underline{Z}^T \ddot{\underset{\sim}{q}} + \underset{\sim}{Z}_{oo}, \qquad \delta\underset{\sim}{z} = \underline{Z}^T \delta\underset{\sim}{q} \tag{5.2.30}$$

where the matrix $\underline{Z}$ and the columns $\underset{\sim}{Z}_o$ and $\underset{\sim}{Z}_{oo}$ are combinations of $\vec{\underline{U}}$, $\vec{\underline{W}}$, etc. For example $\underline{Z}$ is defined by

$$[\, z_{i(l+1)} \cdots z_{i(l+6)}] = [\, (\vec{\underset{\sim}{e}}^k \bullet (\vec{\underline{U}}\underline{T})_{ik})^T,\ (\vec{\underset{\sim}{e}}^k \bullet (\vec{\underline{W}}\underline{T})_{ik})^T] \tag{5.2.31}$$

for i = 1..nq, k = 1..nb and l = 6(k - 1).

If we study a tree structure of bodies and hinges and disregard eventual kinematic constraints, we assume that the Lagrange coordinates are independent, i.e. that $\underline{Z}$ has a full rank. Then there is no $\delta\underset{\sim}{q} \neq \underset{\sim}{o}$ for which $\delta\underset{\sim}{z} = \underset{\sim}{o}$, i.e. $\delta\vec{\underset{\sim}{r}} = \vec{\underset{\sim}{o}}$ and $\delta\vec{\underset{\sim}{\pi}} = \vec{\underset{\sim}{o}}$. Without proof we state that $\underline{Z}$ will have a deficient rank if the Lagrange coordinates $\underset{\sim}{q}$ are dependent. Therefore we have to make sure that the Lagrange coordinates, i.e. the generalized coordinates of the hinges, never become dependent (sections 3.3 and 7.2).

Combining the formulas for several trees

The preceding kinematic analysis for one tree is applicable to every tree in a structure with nt (nt>1) trees. The matrices $\underset{\sim}{S}^i_o$, $\underline{S}^i$ and $\underline{T}^i$ can be used for each tree T^i (i=1..nt). To analyse the kinematics of the structure as a whole these graph matrices can be combined as in (4.3.3), while the Lagrange and attitude coordinates of the nt trees can be combined as:

$$\underset{\sim}{q}^T = [(\underset{\sim}{q}^1)^T .. (\underset{\sim}{q}^{nt})^T], \qquad \underset{\sim}{z}^T = [(\underset{\sim}{z}^1)^T .. (\underset{\sim}{z}^{nt})^T] \tag{5.2.32}$$

From now on it is assumed that all columns of the different trees are combined as in (5.2.32). For all matrices it is assumed that the corresponding matrices are placed on the diagonal of the combined matrix. For example:

$$\underset{\sim}{\vec{r}} = \begin{bmatrix} \underset{\sim}{\vec{r}}^{1} \\ \cdot \\ \underset{\sim}{\vec{r}}^{nt} \end{bmatrix}, \qquad \underline{\vec{U}} = \begin{bmatrix} \underline{\vec{U}}^{1} & & \underline{\vec{o}} \\ & \cdot & \\ \underline{\vec{o}} & & \underline{\vec{U}}^{nt} \end{bmatrix} \qquad (5.2.33)$$

With these modifications we have completed the description of the kinematics of a multibody system having a tree structure.

Example

Consider the multibody system used in chapter 4 (figure 4.1). The numbering of the bodies and hinges is regular as shown in figure 5.4.

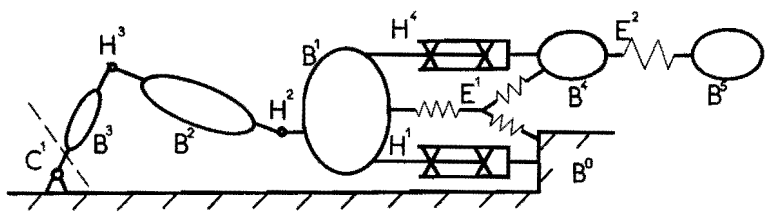


figure 5.4 A multibody system

The $\underset{\sim}{S}_{o}$, $\underset{\sim}{S}$ and $\underline{T}$ matrices are given in chapter 4. The matrices with the body-fixed vectors and connection vectors become:

$$\underline{\vec{B}} = \begin{bmatrix} \vec{b}^{11} & -\vec{b}^{12} & \vec{o} & -\vec{b}^{14} & \vec{o} \\ & \vec{b}^{22} & -\vec{b}^{23} & \vec{o} & \vec{o} \\ & & \vec{b}^{33} & \vec{o} & \vec{o} \\ & & & \vec{b}^{44} & \vec{o} \\ & & & & \vec{b}^{55} \end{bmatrix}, \quad \underline{\vec{C}} = \begin{bmatrix} \vec{o} & -\vec{c}^{2} & \vec{o} & -\vec{c}^{4} & \vec{o} \\ & \vec{o} & -\vec{c}^{3} & \vec{o} & \vec{o} \\ & & \vec{o} & \vec{o} & \vec{o} \\ & & & \vec{o} & \vec{o} \\ & & & & \vec{o} \end{bmatrix}$$

$$\underset{\sim}{\vec{B}}_{o} = [\, -\vec{b}^{o1} \quad \vec{o} \quad \vec{o} \quad \vec{o} \quad -\vec{b}^{o5} \,], \quad \underset{\sim}{\vec{C}}_{o} = [\, -\vec{c}^{1} \quad \vec{o} \quad \vec{o} \quad \vec{o} \quad -\vec{c}^{5} \,]$$

The hinges H^1 and H^4 are prismatic joints while H^2 and H^3 are pin-joints. In that case $\vec{\Omega}^1 = \vec{\omega}^1 = \vec{o}$, $\vec{\Omega}^4 = \vec{o}$ and $\vec{c}^2 = \vec{c}^3 = \vec{o}$, while $\underline{\vec{W}}$ and $\underline{\vec{V}}$ can be written as

$$\underset{\sim}{\vec{W}} = \begin{bmatrix} \vec{0} & \vec{0} & \vec{0} & \vec{0} & \vec{0} \\ & \vec{w}^2 & \vec{0} & \vec{0} & \vec{0} \\ & & \vec{w}^3 & \vec{0} & \vec{0} \\ & & & \vec{0} & \vec{0} \\ & & & & \vec{w}^5 \end{bmatrix}, \qquad \underset{\sim}{\vec{V}} = \begin{bmatrix} \vec{v}^1 & \vec{0} & \vec{0} & \vec{0} & \vec{0} \\ & \vec{0} & \vec{0} & \vec{0} & \vec{0} \\ & & \vec{0} & \vec{0} & \vec{0} \\ & & & \vec{v}^4 & \vec{0} \\ & & & & \vec{v}^5 \end{bmatrix}$$

Since the hinges are all scleronomic, the columns $\underset{\sim}{\vec{W}}_{o}$ and $\underset{\sim}{\vec{V}}_{o}$ only contain components equal to zero.

The motion of B^2 will be described as an example. For the orientation of $\underset{\sim}{\vec{e}}^2$ with respect to $\underset{\sim}{\vec{e}}^o$ we find

$$R^2 = (B^{22})^T \cdot C^2 \cdot B^{12} \cdot R^1, \qquad R^1 = (B^{11})^T \cdot C^1, \qquad C^1 = I$$

For the angular velocity and angular acceleration we find from (5.2.7) and (5.2.11) that

$$\vec{\omega}^2 = \vec{w}^2 \dot{q}^2, \qquad \dot{\vec{\omega}}^2 = \vec{w}^2 \ddot{q}^2$$

as $\vec{\omega}^1 * \vec{\Omega}^2 = \vec{o} * \vec{\Omega}^2 = \vec{o}$ and $\vec{w}^1_{oo} = \vec{w}^2_{oo} = \vec{o}$. The position vector $\vec{r}^2$ of body B^2 becomes

$$\vec{r}^2 = \vec{b}^{o1} + \vec{c}^1 - \vec{b}^{11} + \vec{b}^{12} - \vec{b}^{22}$$

while the velocity and the acceleration are given by:

$$\dot{\vec{r}}^2 = \vec{U}_{21} \dot{q}^1 + \vec{U}_{22} \dot{q}^2, \qquad \ddot{\vec{r}}^2 = \vec{U}_{21} \ddot{q}^1 + \vec{U}_{22} \ddot{q}^2$$

with

$$\vec{U}_{21} = \vec{v}^1, \qquad \vec{U}_{22} = -\vec{b}^{22} * \vec{w}^2$$

$\vec{V}^1_{oo}$ and $\vec{V}^2_{oo}$ do not occur in the relation for the acceleration of B^2. This is due to the fact that the hinges H^1 and H^2 are scleronomic, the derivatives of $\vec{v}^1$ and $\vec{v}^2$ with respect to q are zero and because $\vec{\omega}^1 = \vec{o}$.

5.3 The kinematic constraints

Holonomic constraints
Relationship between coordinates $\underset{\sim}{y}$ and $\underset{\sim}{q}$
Implicit and explicit constraints
The final constraint equations

The kinematic behaviour of a multibody system is not completely determined by the tree structure of bodies and hinges. In chapter 4 it was shown that, if no energetic and active connections are considered, the graph of a multibody system will, in general, contain branches as well as chords. These chords represent the kinematic constraints which also determine the kinematic behaviour. In this section the constraints will be studied in detail. In particular we will combine all constraint equations in one matrix equation for the Lagrange coordinates $\underset{\sim}{q}$.

Holonomic constraints

The chords in the graph of a multibody system are used to represent Pfaff constraints as well as hinges replaced by a holonomic constraint in order to open closed kinematic chains. Pfaff constraints are generally described by their Pfaff equations (3.3.8). Some Pfaff constraints can be integrated, and can be described by a holonomic constraint equation (3.3.19). The kinematic constraints arising from the replaced hinges always result in holonomic constraint equations. For the moment we assume that all nc chords represent kinematic constraints described by holonomic constraint equations. In the last subsection we mention the modifications required to include the non-holonomic Pfaff constraints.

To combine all constraint equations into one matrix equation we introduce a column $\underset{\sim}{y}$ with the columns $\underset{\sim}{y}^k$, k=1..nc, as components, so that

$$(\underset{\sim}{y})^T = [(\underset{\sim}{y}^1)^T \; .. \; (\underset{\sim}{y}^{nc})^T] \qquad (5.3.1)$$

Note that this $\underset{\sim}{y}$ differs from $\underset{\sim}{z}$, but that both columns contain the same kind of coordinates. Since each component $\underset{\sim}{y}^k$ has six components, the order of $\underset{\sim}{y}$ is 6nc×1. Using this column, the constraint equation $\underset{\sim}{f}^k(\underset{\sim}{y}^k,t) = \underset{\sim}{o}$ for the holonomic constraints C^k, k=1..nc, can be written in one equation as

$$\underset{\sim}{f}_y(\underset{\sim}{y},t) = \underset{\sim}{o} \qquad (5.3.2)$$

where $\underset{\sim}{f}_y$ is a column with $\underset{\sim}{f}^k$, k=1..nc, as components

$$\underset{\sim}{f}_y^T = [(\underset{\sim}{f}^1)^T \; .. \; (\underset{\sim}{f}^{nc})^T]$$

If we differentiate (5.3.2) with respect to time, the obtained set of equations becomes a set of Pfaff equations, so that

$$\underline{P}_y(\underset{\sim}{y},t)\dot{\underset{\sim}{y}} + \underset{\sim}{P}_{oy}(\underset{\sim}{y},t) = \underset{\sim}{o} \qquad (5.3.3)$$

where $\underline{P}_y$ and $\underset{\sim}{P}_{oy}$ with components $\underline{p}_y^k$ and $\underset{\sim}{p}_{oy}^k$ (3.3.8), k=1..nc, are defined by

$$\underline{P}_y = \begin{bmatrix} \underline{p}_y^1 \\ \vdots \\ \underline{p}_y^{nc} \end{bmatrix}, \qquad \underset{\sim}{P}_{oy} = \begin{bmatrix} \underset{\sim}{p}_{oy}^1 \\ \vdots \\ \underset{\sim}{p}_{oy}^{nc} \end{bmatrix}$$

Let np be the total number of Pfaff equations, then the order of $\underline{P}_y$ and $\underset{\sim}{P}_{oy}$ becomes np×6nc resp. np×1. For $\ddot{\underset{\sim}{y}}$ and $\delta\underset{\sim}{y}$ similar equations hold, hence

$$\underline{P}_y(\underset{\sim}{y},t)\ddot{\underset{\sim}{y}} + \underset{\sim}{P}_{ooy}(\underset{\sim}{y},\dot{\underset{\sim}{y}},t) = o, \qquad \underline{P}_y(\underset{\sim}{y},t)\delta\underset{\sim}{y} = o \qquad (5.3.4)$$

where $\underset{\sim}{P}_{ooy}$ is defined in a way similar to $\underset{\sim}{P}_{oy}$.

As mentioned in section 5.1, we use the Lagrange coordinates $\underset{\sim}{q}$ as the primary unknowns. This means that the equations (5.3.2-4) have to be rewritten as a function of $\underset{\sim}{q}$, $\dot{\underset{\sim}{q}}$, $\ddot{\underset{\sim}{q}}$, $\delta\underset{\sim}{q}$ and t. The relationship between the coordinates $\underset{\sim}{y}$ and the Lagrange coordinates $\underset{\sim}{q}$ and their derivatives as well, will be discussed in the next subsection, while

the transformation of the constraints for $\underset{\sim}{y}$ to constraints for $\underset{\sim}{q}$ is the main subject of the subsection after.

Relationship between coordinates $\underset{\sim}{y}$ and $\underset{\sim}{q}$

We consider a connection C^k as shown in figure 5.5, in which C^k is a kinematic constraint and not a hinge. In a regular numbered tree, C^k is

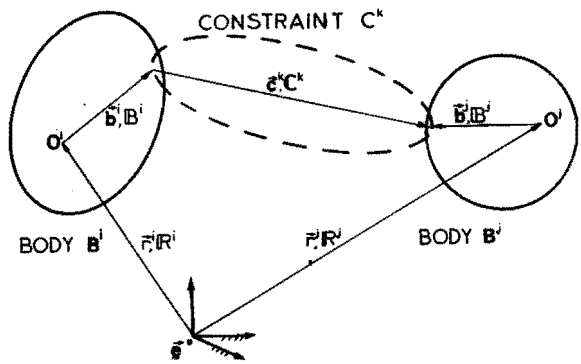


figure 5.5 A kinematic constraint

represented by a chord with a number $1 \leq k \leq nc$. If the chord C^k is directed from body B^i to B^j, the vector $\vec{c}^k$ and rotation tensor $\mathbb{C}^k$ are given by:

$$\vec{c}^k = \vec{r}^j + \vec{b}^{jk} - \vec{b}^{ik} - \vec{r}^i \qquad (5.3.5)$$

$$\mathbb{C}^k = \mathbb{B}^{jk} \cdot \mathbb{R}^j \cdot (\mathbb{R}^i)^T \cdot (\mathbb{B}^{ik})^T$$

All terms on the right-hand side of these relations are functions of $\underset{\sim}{q}$ and t. This implies that $\vec{c}^k$ and $\mathbb{C}^k$, and thus the column $\underset{\sim}{y}^k$, are functions of $\underset{\sim}{q}$ and t. By using (5.3.5) we can easily determine $\underset{\sim}{y}$ as a function of $\underset{\sim}{q}$ and t, hence

$$\underset{\sim}{y} = \underset{\sim}{y}(\underset{\sim}{q},t) \qquad (5.3.6)$$

To express $\dot{\underset{\sim}{y}}$ as a function of $\underset{\sim}{q}$, $\dot{\underset{\sim}{q}}$ and t we differentiate (5.3.6). This yields

$$\dot{\underset{\sim}{y}} = \underset{\sim}{Y}(\underset{\sim}{q},t)^T\dot{\underset{\sim}{q}} + \underset{\sim o}{Y}(\underset{\sim}{q},t) \qquad (5.3.7)$$

In order to construct $\underset{\sim}{Y}$ and $\underset{\sim o}{Y}$ the relation (5.3.5) will be differentiated with respect to time and rewritten in matrix notation. After differentiation of (5.3.5), we find

$$\vec{v}^k = \dot{\vec{r}}^j + \vec{\omega}^j * \vec{b}^{jk} - \dot{\vec{r}}^i - \vec{\omega}^i * \vec{b}^{ik} - \vec{\omega}^i * \vec{c}^k \qquad (5.3.8)$$

$$\vec{\Omega}^k = \vec{\omega}^j - \vec{\omega}^i$$

The body-fixed vectors $\vec{b}^{ik}$ and $\vec{b}^{jk}$ and the connection vectors $\vec{c}^k$ are stored in the matrices $\underset{\sim o}{\vec{B}}^C$, $\underset{\sim}{\vec{B}}^C$, $\underset{\sim o}{\vec{C}}^C$ and $\underset{\sim}{\vec{C}}^C$. These matrices are defined in a way similar to those defined in (5.2.14) and have the same structure as $\underset{\sim o}{S}^C$ and $\underset{\sim}{S}^C$. For (5.3.8), expressed in matrix notation, it follows that

$$\underset{\sim}{\vec{v}}^C = -(\underset{\sim}{\vec{B}}^C + \underset{\sim}{\vec{C}}^C)^T * \underset{\sim}{\vec{\omega}} + (\underset{\sim}{S}^C)^T \dot{\underset{\sim}{\vec{r}}}, \qquad \underset{\sim}{\vec{\Omega}}^C = (\underset{\sim}{S}^C)^T \underset{\sim}{\vec{\omega}} \qquad (5.3.9)$$

where $\underset{\sim}{\vec{v}}^C$ and $\underset{\sim}{\vec{\Omega}}^C$ are defined by:

$$(\underset{\sim}{\vec{v}}^C)^T = [\ \vec{v}^1 .. \vec{v}^{nc}], \qquad (\underset{\sim}{\vec{\Omega}}^C)^T = [\ \vec{\Omega}^1 .. \vec{\Omega}^{nc}]$$

After substitution of the formulas for $\underset{\sim}{\vec{\omega}}$ and $\dot{\underset{\sim}{\vec{r}}}$ (5.2.7/23) we find:

$$\underset{\sim}{\vec{v}}^C = (\underset{\sim}{\vec{U}}^C)^T\dot{\underset{\sim}{q}} + (\underset{\sim}{T}\underset{\sim}{S}^C)^T\underset{\sim o}{\vec{U}} - (\underset{\sim}{\vec{B}}^C + \underset{\sim}{\vec{C}}^C)^T * \underset{\sim}{T}^T\underset{\sim o}{\vec{W}}$$

$$\underset{\sim}{\vec{\Omega}}^C = (\underset{\sim}{\vec{W}}\underset{\sim}{T}\underset{\sim}{S}^C)^T\dot{\underset{\sim}{q}} + (\underset{\sim}{T}\underset{\sim}{S}^C)^T\underset{\sim o}{\vec{W}} \qquad (5.3.10)$$

with

$$\underset{\sim}{\vec{U}}^C = \underset{\sim}{\vec{U}}\underset{\sim}{T}\underset{\sim}{S}^C + \underset{\sim}{\vec{W}}\underset{\sim}{T} * (\underset{\sim}{\vec{B}}^C + \underset{\sim}{\vec{C}}^C)$$

If the coordinates $\dot{y}^k$ for all constraints are related to $\vec{\Omega}^k$ and $\vec{v}^k$ in accordance with

$$\dot{\underset{\sim}{y}}^k = \begin{bmatrix} \underset{\sim}{\vec{e}}^k \bullet \vec{v}^k \\ \underset{\sim}{\vec{e}}^k \bullet \vec{\Omega}^k \end{bmatrix}, \qquad \underset{\sim}{\vec{e}}^k = \mathbb{B}^{ik} \bullet \mathbb{R}^i \bullet \underset{\sim}{\vec{e}}^o \qquad (5.3.11)$$

then $\underset{\sim}{Y}$ and $\underset{\sim}{Y}_o$ become

$$[\ Y_{i(l+1)} \cdots Y_{i(l+6)}] = [\ (\vec{\underset{\sim}{e}}^k \bullet (\vec{\underset{\sim}{U}}^C)_{ik})^T,\ (\vec{\underset{\sim}{e}}^k \bullet (\vec{\underset{\sim}{W}}\underline{\underline{TS}}^C)_{ik})^T]$$

$$\begin{bmatrix} (\underset{\sim}{Y}_o)_{l+1} \\ \vdots \\ (\underset{\sim}{Y}_o)_{l+6} \end{bmatrix} = \begin{bmatrix} \vec{\underset{\sim}{e}}^k \bullet ((\underline{\underline{TS}}^C)^T \vec{\underset{\sim}{U}}_o - (\vec{\underline{B}}^C + \vec{\underline{C}}^C)^T * \underline{T}^T \vec{\underset{\sim}{W}}_o)_k \\ \vec{\underset{\sim}{e}}^k \bullet ((\underline{\underline{TS}}^C)^T \vec{\underset{\sim}{W}}_o)_k \end{bmatrix} \qquad (5.3.12)$$

for i = 1..nq, k = 1..nc and l = 6(k - 1).

Besides the relation for $\dot{\underset{\sim}{y}}$ we also need the one for $\ddot{\underset{\sim}{y}}$. To obtain this relation we diffentiate (5.3.7) again, yielding

$$\ddot{\underset{\sim}{y}} = \underline{Y}^T \ddot{\underset{\sim}{q}} + \underset{\sim}{Y}_{oo} \qquad (5.3.13)$$

To determine $\underset{\sim}{Y}_{oo}$ we must differentiate (5.3.10) to time. As a result, in matrix notation we find:

$$\vec{\underset{\sim}{a}}^C = (\vec{\underline{U}}^C)^T \ddot{\underset{\sim}{q}} + \vec{\underset{\sim}{U}}^C_{oo} + (\underline{\underline{TS}}^C)^T \vec{\underset{\sim}{U}}_{oo} - (\vec{\underline{B}}^C + \vec{\underline{C}}^C)^T * \underline{T}^T \vec{\underset{\sim}{W}}_{oo}$$

$$\dot{\vec{\underset{\sim}{Q}}}^C = (\vec{\underline{\underline{WTS}}}^C)^T \ddot{\underset{\sim}{q}} + (\underline{\underline{TS}}^C)^T \vec{\underset{\sim}{W}}_{oo} \qquad (5.3.14)$$

where $\vec{\underset{\sim}{a}}^C$ is defined in a way similar to $\vec{\underset{\sim}{v}}^C$ and the component k (k = 1..nc) of the column $\vec{\underset{\sim}{U}}^C_{oo}$ is given by

$$(\vec{\underset{\sim}{U}}^C_{oo})_k = \vec{\omega}^j * (\vec{\omega}^j * \vec{b}^{jk}) - \vec{\omega}^i * (\vec{\omega}^i * (\vec{c}^k + \vec{b}^{ik}) + 2\vec{v}^k)$$

From this result we can easily obtain the column $\underset{\sim}{Y}_{oo}$, yielding

$$\begin{bmatrix} (\underset{\sim}{Y}_{oo})_{l+1} \\ \vdots \\ (\underset{\sim}{Y}_{oo})_{l+6} \end{bmatrix} = \begin{bmatrix} \vec{\underset{\sim}{e}}^k \bullet (\vec{\underset{\sim}{U}}^C_{oo} + (\underline{\underline{TS}}^C)^T \vec{\underset{\sim}{U}}_{oo} - (\vec{\underline{B}}^C + \vec{\underline{C}}^C)^T * \underline{T}^T \vec{\underset{\sim}{W}}_{oo})_k \\ \vec{\underset{\sim}{e}}^k \bullet ((\underline{\underline{TS}}^C)^T \vec{\underset{\sim}{W}}_{oo})_k \end{bmatrix} \qquad (5.3.15)$$

where i = 1..nq, k = 1..nc and l = 6(k - 1).

Finally we mention the relationship between the variations of $\underset{\sim}{y}$ and these of $\underset{\sim}{q}$. With $\underline{Y}$ as in (5.3.12) we find

$$\delta \underset{\sim}{y} = \underline{Y}^T \delta \underset{\sim}{q} \qquad (5.3.16)$$

Implicit and explicit constraints

Constraints which are specified as functions of the Lagrange coordinates are called explicit constraints (see e.g. Paul [1979, p263]). If constraints are specified as functions of other coordinates we speak of implicit constraints. Explicit constraints have the form:

$$\underset{\sim}{f}(\underset{\sim}{q},t) = \underset{\sim}{o}, \qquad \underline{P}_h(\underset{\sim}{q},t)\dot{\underset{\sim}{q}} + \underset{\sim}{P}_{oh}(\underset{\sim}{q},t) = \underset{\sim}{o} \qquad (5.3.17)$$

while its variation and its derivative with respect to time become

$$\underline{P}_h(\underset{\sim}{q},t)\delta \underset{\sim}{q} = \underset{\sim}{o}, \qquad \underline{P}_h(\underset{\sim}{q},t)\ddot{\underset{\sim}{q}} + \underset{\sim}{P}_{ooh}(\underset{\sim}{q},\dot{\underset{\sim}{q}},t) = \underset{\sim}{o}$$

The constraints for $\underset{\sim}{y}$, as given in (5.3.2/3), are implicit constraints. The names explicit and implicit constraints indicate that constraint equations for $\underset{\sim}{q}$ can be used directly while constraints for $\underset{\sim}{y}$ have to be transformed before they can be used.

In the previous subsection the following transformation formulas were derived:

$$\dot{\underset{\sim}{y}} = \underline{Y}^T\dot{\underset{\sim}{q}} + \underset{\sim}{Y}_o, \qquad \ddot{\underset{\sim}{y}} = \underline{Y}^T\ddot{\underset{\sim}{q}} + \underset{\sim}{Y}_o, \qquad \delta \underset{\sim}{y} = \underline{Y}^T\delta \underset{\sim}{q} \qquad (5.3.18)$$

Substitution of these formulas in the implicit constraints (5.3.2/3) yields explicit constraints of the form (5.3.17) in which:

$$\underline{P}_h = \underline{P}_y\underline{Y}^T, \quad \underset{\sim}{P}_{oh} = \underset{\sim}{P}_{oy} + \underline{P}_h\underset{\sim}{Y}_o, \quad \underset{\sim}{P}_{ooh} = \underset{\sim}{P}_{ooy} + \underline{P}_y\underset{\sim}{Y}_{oo} \qquad (5.3.19)$$

The terms on the right-hand side can be calculated when $\underset{\sim}{q}$, $\dot{\underset{\sim}{q}}$ and t are known.

The final constraint equations

In the previous subsections we have studied kinematic constraints described exclusively by holonomic equations. In general the kinematic

constraints can be divided into constraints with holonomic constraint equations and constraints with nonholonomic constraint equations. Let nhc be the number of kinematic constraints described by holonomic constraint equations. The number of kinematic constraints described by nonholonomic constraint equations is then equal to nnc = nc-nhc. We assume that the constraints described by holonomic constraint equations are numbered from 1 to nhc inclusive, while the other constraints are numbered from nhc + 1 to nc inclusive.

The holonomic constraints result in a set of constraint equations for $\underset{\sim}{y}$ or $\underset{\sim}{q}$ of the form (5.3.2/17). Nonholonomic constraints do not result in restrictions for $\underset{\sim}{y}$ or $\underset{\sim}{q}$. As discussed in section 3.3, these constraints result in a set of Pfaff equations for $\dot{\underset{\sim}{y}}$ or $\dot{\underset{\sim}{q}}$. After transformation of all implicit Pfaff equations into explicit Pfaff equations we can write for the set of equations for $\dot{\underset{\sim}{q}}$, obtained from these nonholonomic constraints

$$\underline{\underline{P}}_{n}\dot{\underset{\sim}{q}} + \underline{\underline{P}}_{on} = \underset{\sim}{o} \tag{5.3.20}$$

while the time derivative $\ddot{\underset{\sim}{q}}$ and the variation $\delta\underset{\sim}{q}$ have to satisfy:

$$\underline{\underline{P}}_{n}\ddot{\underset{\sim}{q}} + \underline{\underline{P}}_{oon} = \underset{\sim}{o}, \qquad \underline{\underline{P}}_{n}\delta\underset{\sim}{q} = \underset{\sim}{o}$$

Finally, all constraint equations can be combined. Let the nhc holonomic constraints result in a set of nhp explicit constraint equations for $\underset{\sim}{q}$. For these equations we write

$$\underset{\sim}{f}(\underset{\sim}{q},t) = \underset{\sim}{o} \tag{5.3.21}$$

Let the nnc nonholonomic constraints result in a set of nnp explicit Pfaff equations for $\dot{\underset{\sim}{q}}$. Differentiation with respect to time of the holonomic constraint equations (5.3.21) results in a set of nhp Pfaff equations for $\dot{\underset{\sim}{q}}$ as well. Combining these equations yields the final set of np = nhp + nnp Pfaff equations for $\dot{\underset{\sim}{q}}$ as well as a set for $\ddot{\underset{\sim}{q}}$ and $\delta\underset{\sim}{q}$ so that

$$\underline{\underline{P}}\dot{\underset{\sim}{q}} + \underline{\underline{P}}_{o} = \underset{\sim}{o}, \qquad \underline{\underline{P}}\ddot{\underset{\sim}{q}} + \underline{\underline{P}}_{oo} = \underset{\sim}{o}, \qquad \underline{\underline{P}}\delta\underset{\sim}{q} = \underset{\sim}{o} \tag{5.3.22}$$

in which $\underline{P}$, $\underset{\sim}{P}_{o}$ and $\underset{\sim}{P}_{oo}$ are defined as:

$$\underline{P} = \begin{bmatrix} \underline{P}_h \\ \underline{P}_n \end{bmatrix}, \qquad \underset{\sim}{P}_{o} = \begin{bmatrix} \underset{\sim}{P}_{oh} \\ \underset{\sim}{P}_{on} \end{bmatrix}, \qquad \underset{\sim}{P}_{oo} = \begin{bmatrix} \underset{\sim}{P}_{ooh} \\ \underset{\sim}{P}_{oon} \end{bmatrix} \tag{5.3.23}$$

The matrix $\underline{P}$ is of the order npxnq and must have full rank np with $np \leq nq$. If $\underline{P}$ does not have full rank one or more constraints are dependent (see (3.3.8)). Even if the Pfaff equations of all constraints are themselves independent, the combination of all Pfaff equations in one equation can still result in a dependent set of constraint equations. Since $\underline{P}$ is a function of $\underset{\sim}{q}$ and t, the rank of $\underline{P}$ may change whenever $\underset{\sim}{q}$ and t change. Therefore we must check the rank of $\underline{P}$ for each value of $\underset{\sim}{q}$ and t. In chapter 8 attention will be given to this problem.

5.4 Prescribed Lagrange and/or attitude coordinates

Sometimes several Lagrange and/or attitude coordinates are prescribed as a function of time. The nq Lagrange coordinates will be divided into ns prescribed and nf = nq - ns free coordinates. If needed, we assume that the first and second derivatives to time of the prescribed coordinates are given too. The values for the remaining nf Lagrange coordinates are a priori unknown. Since our formulas were developed with the Lagrange coordinates as primary unknowns, prescribing Lagrange coordinates raise no problems. They are easily substituted in the formulas.

Prescribing attitude coordinates is less trivial. In order to prescribe attitude coordinates we use kinematic constraints. For the reference point of these constraints the origin of the global base ($O^0 = N = N^1$) is used, while the other attachment points (N^2) will be connected to the origin of the vector base of the bodies for which the attitude coordinates are prescribed. In that case the constraint equations become

$$\underset{\sim}{f}(\underset{\sim}{y},t) = \underset{\sim}{y} - \underset{\sim}{z}_s(t) = \underset{\sim}{0} \tag{5.4.1}$$

in which $\underset{\sim}{z}_s$ represents the subset of prescribed attitude coordinates. These constraint equations can be treated as holonomic constraint equations in exactly the same way as discussed in the preceding section.

If we prescribed ns Lagrange coordinates, the column $\underset{\sim}{q}$ and the Pfaff matrix $\underline{P}$ will be rearranged as follows:

$$\underset{\sim}{q} = \begin{bmatrix} \underset{\sim}{q}_s \\ \underset{\sim}{q}_f \end{bmatrix} \qquad \underline{P} = \begin{bmatrix} \underline{P}_{hs} & \underline{P}_{hf} \\ \underline{P}_{ns} & \underline{P}_{nf} \end{bmatrix} \qquad (5.4.2)$$

Summary

In this chapter we derived the formulas describing the kinematic behaviour of a multibody system with bodies, hinges and kinematic constraints. To set up these formulas a tree structure of bodies and hinges was defined. The unknown quantities in these formulas are the Lagrange coordinates. These coordinates were defined in the first section as the generalized coordinates of the hinges in the tree structure.

CHAPTER 6
DYNAMICS OF A MULTIBODY SYSTEM

In dealing with the dynamic behaviour of multibody systems we first discuss three methods of obtaining the equations of motion. These equations are set up in section 6.2 as a function of the Lagrange coordinates for a system with a tree structure and without kinematic constraints. The equations for a system with kinematic constraints, obtained by using Lagrange multipliers, are given in section 6.3. In the same section we discuss the modifications for those cases in which some of the Lagrange and/or attitude coordinates are prescribed.

6.1 Methods for deriving the equations of motion

In chapter 2 we formulated the equations of motion for one rigid body by using the Newton-Euler laws. In addition to these laws, the virtual work principle of d'Alembert and the Lagrange equations can also be used to obtain the equations of motion (see literature review in section 1.2). Although there are other methods like the principle of Hamilton [Rosenberg 1977, p169] or the Appel equations [Vukobratovic and Potkonjak 1982, p128], we will only discuss the three methods mentioned above. For the sake of simplicity we consider a multibody system with a tree structure and without kinematic constraints.

The Newton-Euler laws

Using the Newton-Euler laws, each body is isolated from the rest of the system by means of a free-body diagram (see figure 6.1). The interaction of a body with the rest of the system and its surroundings is taken into account by means of the internal loads at the attachment points and the external loads on the body.

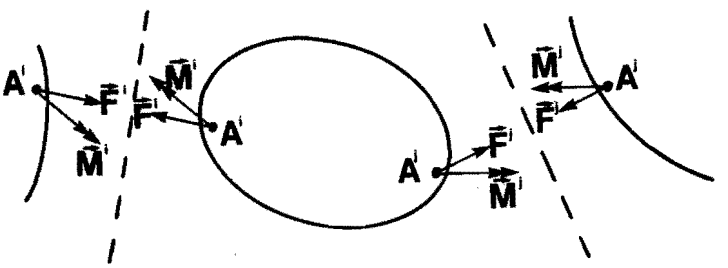


figure 6.1 A free-body diagram

For each body the Newton-Euler laws yield two vector equations. The first equation states that, for each time t, the resulting force on a body is equal to the time derivative of the momentum of that body. The resulting force on a body comprises external forces as well as internal forces (see section 2.3). The second vector equation states that the resulting moment on a body with respect to its centre of mass equals the time derivative of the angular momentum around the centre of mass. For a multibody system with nb bodies, we can write in two matrix equations

$$\vec{\underset{\sim}{F}}_{ex} + \vec{\underset{\sim}{F}}_{in} = \dot{\vec{\underset{\sim}{I}}} \quad \text{and} \quad \vec{\underset{\sim}{M}}_{ex,m} + \vec{\underset{\sim}{M}}_{in,m} = \dot{\vec{\underset{\sim}{L}}}_{m} \qquad (6.1.1)$$

where $\vec{\underset{\sim}{F}}_{ex}$, $\vec{\underset{\sim}{F}}_{in}$, etc. are columns with the resulting external forces, internal forces, etc. on the bodies as components, hence

$$\vec{\underset{\sim}{F}}^{T}_{ex} = [\vec{\underset{\sim}{F}}^{1}_{ex} \; .. \; \vec{\underset{\sim}{F}}^{nb}_{ex}], \qquad \vec{\underset{\sim}{F}}^{T}_{in} = [\vec{\underset{\sim}{F}}^{1}_{in} \; .. \; \vec{\underset{\sim}{F}}^{nb}_{in}]$$

These matrix equations can be transformed into a set of scalar equations by using the matrix representation of all vectors in a vector base. These scalar equations form a set of 6nb coupled, nonlinear second order differential equations. The unknowns in these equations are the attitude coordinates, as well as the unknown components of the internal force and moment vectors of the kinematic connections.

The virtual work principle of d'Alembert

For a multibody system the principle of d'Alembert states that

$$\delta\vec{r}_{\underset{\sim}{m}}^{T}\bullet(\vec{F}_{\underset{\sim}{ex}}+\vec{F}_{\underset{\sim}{in}}-\dot{\vec{I}}_{\underset{\sim}{}})+\delta\vec{\pi}_{\underset{\sim}{}}^{T}\bullet(\vec{M}_{\underset{\sim}{m,ex}}+\vec{M}_{\underset{\sim}{m,in}}-\dot{\vec{L}}_{\underset{\sim}{m}})=o \qquad (6.1.2)$$

for all kinematically admissible $\delta\vec{r}_{\underset{\sim}{m}}$ and $\delta\vec{\pi}_{\underset{\sim}{}}$. As was shown in the preceding chapter, kinematically admissible variations $\delta\vec{r}_{\underset{\sim}{m}}$ and $\delta\vec{\pi}_{\underset{\sim}{}}$ can be written as a linear combination of variations of the nq Lagrange coordinates of the tree structure.

Using this principle, we obtain a set of nq scalar equations of motion for a multibody system with a tree structure. Internal loads due to the hinges in the tree structure do not contribute to the virtual work for any kinematically admissible $\delta\vec{r}_{\underset{\sim}{m}}$ and $\delta\vec{\pi}_{\underset{\sim}{}}$ [Rosenberg 1977, p122].

The Lagrange equations

The third of methods for obtaining the equations of motion is based on the Lagrange equations. With this method the kinetic and potential energy of the bodies must be expressed as a function of the Lagrange coordinates $\underset{\sim}{q}$,

$$E_{kin}=E_{kin}(\underset{\sim}{q},\dot{\underset{\sim}{q}},t), \qquad E_{pot}=E_{pot}(\underset{\sim}{q},t), \qquad (6.1.3)$$

and the virtual work ΔW caused by the external loads as well as the internal loads arising out of the energetic and active connections must be expressed as a function of $\delta\underset{\sim}{q}$, so that

$$\Delta W = (\underset{\sim}{Q}_{ex}^T + \underset{\sim}{Q}_{in}^T)\delta\underset{\sim}{q} \qquad (6.1.4)$$

where the columns $\underset{\sim}{Q}_{ex}$ and $\underset{\sim}{Q}_{in}$ have nq components. The Lagrange equations then yield

$$\frac{d}{dt}\left(\frac{\partial E_{kin}}{\partial \dot{q}_i}\right) - \frac{\partial E_{kin}}{\partial q_i} + \frac{\partial E_{pot}}{\partial q_i} = Q_i, \quad i = 1..nq \qquad (6.1.5)$$

The functions $E_{kin} = E_{kin}(\underset{\sim}{q},\dot{\underset{\sim}{q}},t)$ and $E_{pot} = E_{pot}(\underset{\sim}{q},t)$ are difficult to set up by using the formulas given in section 5.2. According to (6.1.5) these functions must be differentiated with respect to $\underset{\sim}{q}$, $\dot{\underset{\sim}{q}}$ and t in order to obtain the final set of nq equations of motion. The resulting set of equations of motion is exactly the same as the set obtained by using the virtual work principle of d'Alembert.

Comparing the three methods

Comparison of the method based on the Newton-Euler laws with the two others shows that the most important difference is the final set of equations of motion. The set obtained by using the Newton-Euler laws contains 6nb nonlinear second-order differential equations. The set obtained by using the virtual work principle of d'Alembert or the Lagrange equations contains only nq nonlinear differential equations. As the number nq is in general much smaller than 6nb, we prefer to use the virtual work principle of d'Alembert or the Lagrange equations. There is hardly any difference between using the virtual work principle of d'Alembert and the Lagrange equations, but as we prefer to work with vector variables, we will use the virtual work principle of d'Alembert.

The drawback in not using the Newton-Euler laws is that the internal loads caused by the kinematic connections are not considered. This is not felt to be serious. When the Lagrange coordinates and their derivatives are calculated the unknown internal loads are easily determined afterwards (see section 8.5).

6.2 The virtual work principle of d'Alembert

Kinematically admissible variations
The inertia loads
The external loads
The internal loads
The equations of motion for a tree structure

The virtual work principle of d'Alembert was discussed in the previous section in general terms. The method will now be discussed in detail. First, kinematically admissible variations are considered. Then we deal with the contributions of the inertia, external, and internal loads. Finally, we set up the equations of motion as a function of the Lagrange coordinates and time.

Let us assume for the moment that there are no kinematic constraints and that the multibody system has a tree structure of bodies and hinges with nq Lagrange coordinates. In this section no prescibed coordinates are considered. The only prescribed variables are the external forces and the external input variables of active connections.

Kinematically admissible variations

The virtual work principle of d'Alembert as described in (6.1.2) contained the condition "for all kinematical admissible variations $\delta\vec{r}_{\nu m}$ and $\delta\vec{\pi}_{\nu}$". For simplicity we assume that the centre of mass of each body coincides with the origin of the local body-fixed vector base of that body or that it has been translated such that $\vec{r}^{\,i} = \vec{r}^{\,i}_{m}$ for all $i \in (1..nb)$. In that case the variation in position and orientation of all bodies as a function of the variation of the Lagrange coordinates are given by

$$\delta\vec{r}_{\nu m} = \delta\vec{r}_{\nu} = (\vec{\underline{U}}\underline{T})^{T}\delta q, \qquad \delta\vec{\pi}_{\nu} = (\vec{\underline{W}}\underline{T})^{T}\delta q \tag{6.2.1}$$

Substituting this result in (6.1.2) for the virtual work principle of d'Alembert we obtain

$$\delta q^T[(\vec{\underline{\underline{U}}}^T)\bullet(\vec{\underline{F}}_{ex} + \vec{\underline{F}}_{in} - \dot{\vec{\underline{l}}}) + (\vec{\underline{\underline{W}}}^T)\bullet(\vec{\underline{M}}_{ex} + \vec{\underline{M}}_{in} - \dot{\vec{\underline{L}}})] = 0$$

for all kinematically admissible δq (6.2.2)

Here the subscript m in the symbols for the moment vector and the angular momemtum vectors is dropped.

For multibody systems with a tree structure and without kinematic constraints the Lagrange coordinates q are independent as long as the matrix $\underline{Z}$ (see section 5.2) has a full rank. This is assumed to be the case. Then the term in (6.2.2) between the square brackets must be equal to o. As a result we obtain the set of nq equations of motion

$$(\vec{\underline{\underline{U}}}^T)\bullet(\vec{\underline{F}}_{ex} + \vec{\underline{F}}_{in} - \dot{\vec{\underline{l}}}) + (\vec{\underline{\underline{W}}}^T)\bullet(\vec{\underline{M}}_{ex} + \vec{\underline{M}}_{in} - \dot{\vec{\underline{L}}}) = o \qquad (6.2.3)$$

It has been stated that the equations of motions are a set of second-order differential equations. The second-order derivatives will appear after the inertia loads, $-\dot{\vec{l}}$ and $-\dot{\vec{L}}$, are expressed as a function of the Lagrange coordinates. These inertia loads as well as the external and internal loads are discussed in detail in the remaining part of this section .

The inertia loads

The momentum $\vec{l}$ and angular momentum $\vec{L}$ for a rigid body were introduced in (2.3.4). For all bodies of a multibody system these vectors are stored in two columns $\vec{l}$ and $\vec{L}$ with nb components. For the multibody system we can thus write:

$$\vec{\underline{l}} = \underline{m}\dot{\vec{\underline{r}}}, \qquad \vec{\underline{L}} = \underline{\underline{J}}\bullet\vec{\omega} \qquad (6.2.4)$$

where $\underline{m}$ and $\underline{\underline{J}}$ are diagonal matrices with the masses and the inertia tensors of the bodies respectively as diagonal components. In (6.2.4) it is assumed that the centre of mass and the origin of the local body-fixed reference base coincides for each of the bodies in the system.

To obtain the inertia loads we must differentiate (6.2.4) with respect to time. This yields

$$\dot{\vec{\underset{\sim}{i}}} = \underline{m}\ddot{\vec{\underset{\sim}{r}}}, \qquad \dot{\vec{\underset{\sim}{L}}} = \vec{\underset{\sim}{H}} + \underline{J}\bullet\dot{\vec{\underset{\sim}{\omega}}} \tag{6.2.5}$$

where $\vec{\underset{\sim}{H}}$ is a column with nb components, defined by

$$\vec{H}_i = \vec{\omega}^i * (\underline{J}^i \bullet \vec{\omega}^i), \qquad i = 1..nb \tag{6.2.6}$$

To express the inertia loads as an explicit function of the second-order derivatives of the Lagrange coordinates we substitute the kinematic formulas as derived in section 5.2. This results in

$$\dot{\vec{\underset{\sim}{i}}} = \underline{m}[(\vec{\underline{U}}\vec{\underline{T}})^T\ddot{\underset{\sim}{q}} + \underline{T}^T\vec{\underset{\sim}{U}}_{OO}], \quad \dot{\vec{\underset{\sim}{L}}} = \vec{\underset{\sim}{H}} + \underline{J}\bullet[(\vec{\underline{W}}\vec{\underline{T}})^T\ddot{\underset{\sim}{q}} + \underline{T}^T\vec{\underset{\sim}{W}}_{OO}] \tag{6.2.7}$$

The external loads

In section 2.3 we discussed extensively how the resulting external load on a body is to be determined. Columns with the resulting external forces and moments exerted on the bodies were defined in relation (6.1.1). In general, not all external loads on a multibody system are external loads on the bodies. Some external loads may be exerted on internal points of energetic and active connections (see chapter 7).

External loads may be expressed as a function of the Lagrange coordinates and time or as a function of the attitude coordinates and time. However, these loads are no functions of second or higher order derivatives. Since the attitude coordinates are a function of the Lagrange coordinates we can write

$$\vec{\underset{\sim}{F}}_{ex} = \vec{\underset{\sim}{F}}_{ex}(\underset{\sim}{q},\dot{\underset{\sim}{q}},t), \qquad \vec{\underset{\sim}{M}}_{ex} = \vec{\underset{\sim}{M}}_{ex}(\underset{\sim}{q},\dot{\underset{\sim}{q}},t) \tag{6.2.8}$$

The gravity load is an illustrative example of an external load expressed as a function of the attitude coordinates. If the gravity field is directed along the $-\vec{e}_3^{\,o}$ base vector, then the i^{th} component of $\vec{\underset{\sim}{F}}_{ex}$ contains at least the contribution

$$\vec{F}^{i}_{ex} = .. + -gm^{i}\vec{e}^{i\to o}_{3} + .. \qquad (6.2.9)$$

where m^i is the mass of body B^i.

The internal loads

Internal loads caused by kinematic connections do not have to be considered if the virtual work principle of d'Alembert is used. These loads do not contribute to the virtual work. Internal loads caused by energetic connections were discussed in section 3.4 for the special case of elements and will be discussed in a more general context in section 7.3. Active connections without memory are treated in exactly the same way as energetic connections, the only difference being the introduction of the external input variables $\underset{\sim}{i}(t)$ in the constitutive equation. Since $\underset{\sim}{i}$ is assumed to be known as a function of time no problems are encountered. In the case of active connections with memory we must consider additional state variables and state equations. In the next chapter more attention will be given to such connections. Since we are interested here in the equations of motion themselves, we will assume for the moment that only active connections without memory are used.

As stated earlier in chapter 3, the constitutive behaviour of an element never depends on second or higher order derivatives of the kinematic variables. Hence the resulting internal loads due to the energetic and active connections are in general a function of $\underset{\sim}{q}$, $\dot{\underset{\sim}{q}}$, $\underset{\sim}{i}$ and time:

$$\vec{\underset{\sim}{F}}_{in} = \vec{\underset{\sim}{F}}_{in}(\underset{\sim}{q},\dot{\underset{\sim}{q}},\underset{\sim}{i},t), \qquad \vec{\underset{\sim}{M}}_{in} = \vec{\underset{\sim}{M}}_{in}(\underset{\sim}{q},\dot{\underset{\sim}{q}},\underset{\sim}{i},t) \qquad (6.2.10)$$

The equations of motion for a tree structure

To obtain the final equations of motion the results of the last three subsections are substituted in the set of equations (6.2.3). This yields

$$\underline{A}(\underset{\sim}{q},t)\ddot{\underset{\sim}{q}} = \underset{\sim}{Q}(\underset{\sim}{q},\dot{\underset{\sim}{q}},\underset{\sim}{i},t) \qquad (6.2.11)$$

where the generalized mass matrix $\underline{A}$ and the generalized load column $\underset{\sim}{Q}$ follow from:

$$\underline{A} = (\vec{\underline{U}}\underline{T})\bullet\underline{m}(\vec{\underline{U}}\underline{T})^T + (\vec{\underline{W}}\underline{T})\bullet\underline{J}\bullet(\vec{\underline{W}}\underline{T})^T$$

$$\underset{\sim}{Q} = \underset{\sim}{Q}_{ex} + \underset{\sim}{Q}_{in} + \underset{\sim}{Q}_{oo}$$

while $\underset{\sim}{Q}_{ex}$, $\underset{\sim}{Q}_{in}$ and $\underset{\sim}{Q}_{oo}$ are defined by: (6.2.12)

$$\underset{\sim}{Q}_{ex} = (\vec{\underline{U}}\underline{T})\bullet\vec{\underset{\sim}{F}}_{ex} + (\vec{\underline{W}}\underline{T})\bullet\vec{\underset{\sim}{M}}_{ex}$$

$$\underset{\sim}{Q}_{in} = (\vec{\underline{U}}\underline{T})\bullet\vec{\underset{\sim}{F}}_{in} + (\vec{\underline{W}}\underline{T})\bullet\vec{\underset{\sim}{M}}_{in}$$

$$\underset{\sim}{Q}_{oo} = -(\vec{\underline{U}}\underline{T})\bullet\underline{m}\underline{T}^T\vec{\underset{\sim}{U}}_{oo} - (\vec{\underline{W}}\underline{T})\bullet(\vec{\underset{\sim}{H}} + \underline{J}\bullet\underline{T}^T\vec{\underset{\sim}{W}}_{oo})$$

For a further evaluation of $\underline{A}$ we consider the kinematic energy E_{kin} of the multibody system. This energy is given by

$$E_{kin} = \frac{1}{2}(\dot{\vec{\underset{\sim}{r}}}^T\bullet\underline{m}\dot{\vec{\underset{\sim}{r}}} + \vec{\underset{\sim}{\omega}}^T\bullet\underline{J}\bullet\vec{\underset{\sim}{\omega}}) \qquad (6.2.13)$$

As stated in chapter 2, the components $\dot{\vec{r}}^j$ and $\vec{\omega}^j$ (j = 1..nb) of $\dot{\vec{\underset{\sim}{r}}}$ and $\vec{\underset{\sim}{\omega}}$ can be written as linear combinations of the time derivatives of the attitude coordinates of B^j. Storing these coordinates in the column $\dot{\underset{\sim}{z}}$, we can write

$$E_{kin} = \frac{1}{2}\dot{\underset{\sim}{z}}^T\underline{J}\dot{\underset{\sim}{z}} \qquad (6.2.14)$$

where $\underline{J}$ is the mass matrix with respect to the attitude coordinates $\underset{\sim}{z}$. Using (5.2.30) for the relationship between $\dot{\underset{\sim}{z}}$ and $\dot{\underset{\sim}{q}}$ yields

$$E_{kin} = \frac{1}{2}\dot{\underset{\sim}{q}}^T\underline{Z}\underline{J}\underline{Z}^T\dot{\underset{\sim}{q}} + \dot{\underset{\sim}{q}}^T\underline{Z}\underline{J}\underline{z}_{\sim o} + \frac{1}{2}\underset{\sim o}{z}^T\underline{J}\underline{z}_{\sim o} \qquad (6.2.15)$$

and it can be shown that

$$\underline{A} = \underline{Z}\underline{J}\underline{Z}^T \qquad (6.2.16)$$

In chapter 5 it was stated that the transformation matrix $\underline{Z}$ of order nqx(6nb) has a full rank: $r(\underline{Z}) = nq < 6nb$. Hence, the matrix $\underline{A}$ will be positive definite if and only if $\underline{J}$ is a positive definite matrix of order (6nb)x(6nb), that is if $m^j \neq o$ and J^j is positive definite for all bodies B^j (j = 1..nb). To ensure that the matrix $\underline{J}$ is positive definite we will only use bodies with a given mass and inertia.

6.3 The final equations of motion

Kinematic constraints
Prescribed Lagrange and/or attitude coordinates

The equations of motion as derived in the previous section hold for a tree structure of bodies and hinges and without kinematic constraints and no prescribed coordinates. In order to obtain the equations of motions for a multibody system with an arbitrary topology we must modify the equations obtained so far. Some modifications are also necessary to allow for prescribed Lagrange and/or attitude coordinates.

Kinematic constraints

The Lagrange coordinates of a multibody system are no longer indepedent if the system contains kinematic constraints. This means that kinematic admissible variations of these coordinates have to satisfy the Pfaff equation

$$\underline{P}\delta\underline{q} = \underline{o} \qquad (6.3.1)$$

To construct the equations of motion in this case we use Lagrange multipliers. The Lagrange multipliers rule (see e.g. [Rosenberg 1977, p132]) states that the equation (6.2.2) in which the variations $\delta\underline{q}$ must satisfy (6.3.1) may be replaced by the requirement that

$$\delta\underline{q}^T[(\vec{\underline{U}}\underline{T})\bullet(\vec{\underline{F}}_{ex} + \vec{\underline{F}}_{in} - \dot{\vec{\underline{I}}}) + (\vec{\underline{W}}\underline{T})\bullet(\vec{\underline{M}}_{ex} + \vec{\underline{M}}_{in} - \dot{\vec{\underline{L}}}) - \underline{P}^T\underline{\lambda}] = 0 \qquad (6.3.2)$$

holds for every $\delta\underset{\sim}{q}$. Here the column $\underset{\sim}{\lambda}$ contains np Lagrange multipliers. This equation is equivalent to

$$(\vec{\underline{U}}\underline{T})\bullet(\vec{\underset{\sim}{F}}_{ex} + \vec{\underset{\sim}{F}}_{in} - \dot{\vec{\underset{\sim}{i}}}) + (\vec{\underline{W}}\underline{T})\bullet(\vec{\underset{\sim}{M}}_{ex} + \vec{\underset{\sim}{M}}_{in} - \dot{\vec{\underset{\sim}{L}}}) - \underline{P}^T\underset{\sim}{\lambda} = \underset{\sim}{0} \tag{6.3.3}$$

Similarly to section 6.2 we can also write

$$\underline{A}(\underset{\sim}{q},t)\ddot{\underset{\sim}{q}} = \underset{\sim}{Q}(\underset{\sim}{q},\dot{\underset{\sim}{q}},\underset{\sim}{i},t) + \underline{P}^T(\underset{\sim}{q},t)\underset{\sim}{\lambda} \tag{6.3.4}$$

This set of nq equations contains nq unknown components of $\ddot{\underset{\sim}{q}}$ and np unknown components of $\underset{\sim}{\lambda}$. To determine these unknowns we must add the set of nq constraint equations to (6.3.4). Since $\ddot{\underset{\sim}{q}}$ is unknown we will not use (6.3.1) itself, but its derivative with respect to time

$$\underline{P}(\underset{\sim}{q},t)\ddot{\underset{\sim}{q}} + \underset{\sim}{P}_{oo}(\underset{\sim}{q},\dot{\underset{\sim}{q}},t) = \underset{\sim}{0} \tag{6.3.5}$$

If $\underline{A}$ is positive definite, it is possible to solve (6.3.4) for $\ddot{\underset{\sim}{q}}$ and substitute this result in (6.3.5). After rearrangment of the terms we obtain the equations of motion for a multibody system with an arbitrary topology (and no prescribed Lagrange coordinates) as

$$\underset{\sim}{\lambda} = -(\underline{P}\underline{A}^{-1}\underline{P}^T)^{-1}[\underline{P}\underline{A}^{-1}\underset{\sim}{Q} + \underset{\sim}{P}_{oo}] \tag{6.3.6}$$

$$\ddot{\underset{\sim}{q}} = \underline{A}^{-1}(\underline{P}^T\underset{\sim}{\lambda} + \underset{\sim}{Q})$$

In this set of equations all terms on the right-hand side are functions of $\underset{\sim}{q}$, $\dot{\underset{\sim}{q}}$, $\underset{\sim}{i}$ and t. The matrix product $\underline{P}\underline{A}^{-1}\underline{P}^T$ is regular if $\underline{P}$ has full rank. In chapter 8 the rank of $\underline{P}$ is studied in more detail.

Prescribed Lagrange and/or attitude coordinates

The equations of motion can be used to calculate the trajectories of the Lagrange coordinates, i.e. the motions of the bodies. These trajectories are functions of external loads and external input variables of active connections. Since prescribed attitude coordinates $\underset{\sim}{z}_s(t)$ are modelled as (implicit) kinematic constraints (see section 5.4), the obtained trajectories may be a function of some prescribed

attitude coordinates too. Prescribing attitude coordinates results in a larger set of constraint equations as well as a higher number of unknown Lagrange multipliers. Prescribed Lagrange coordinates $\underset{\sim}{q}_s(t)$ can be brought into play by introducing additional (explicit) kinematic constraints. This results in an even larger set of constraint equations and higher number of unknown multipliers. This can be avoided if we use another method to deal with prescribed Lagrange coordinates.

For those situations in which some Lagrange coordinates are prescribed as a function of time we will modify the equations of motion. First of all the Lagrange coordinates are stored in $\underset{\sim}{q}$ as described in section 5.4. Next we permutate $\underline{A}$ and $\underset{\sim}{Q}$ in a similar way in order to obtain:

$$\underline{A} = \begin{bmatrix} \underline{A}_{ss} & \underline{A}_{sf} \\ \underline{A}_{fs} & \underline{A}_{ff} \end{bmatrix}, \qquad \underset{\sim}{Q} = \begin{bmatrix} \underset{\sim}{Q}_s \\ \underset{\sim}{Q}_f \end{bmatrix} \tag{6.3.7}$$

Using these submatrices we can rewrite the equations of motion in the same manner as we did to obtain (6.3.6). After some manipulations we find

$$\begin{aligned} \underset{\sim}{\lambda} &= -(\underline{P}_f \underline{A}_{ff}^{-1} \underline{P}_f^T)^{-1} [\underline{P}_f \underline{A}_{ff}^{-1} (\underset{\sim}{Q}_f - \underline{A}_{fs} \ddot{\underset{\sim}{q}}_s(t)) + \underline{P}_s \ddot{\underset{\sim}{q}}_s(t) + \underline{P}_{oo}] \\ \ddot{\underset{\sim}{q}}_f &= \underline{A}_{ff}^{-1} [\underline{P}_f^T \underset{\sim}{\lambda} + \underset{\sim}{Q}_f - \underline{A}_{fs} \ddot{\underset{\sim}{q}}_s(t)] \\ \underset{\sim}{Q}_s &= \underline{A}_{sf} \ddot{\underset{\sim}{q}}_f - \underline{P}_s^T \underset{\sim}{\lambda} + \underline{A}_{ss} \ddot{\underset{\sim}{q}}_s(t) \end{aligned} \tag{6.3.8}$$

We require the submatrix $\underline{A}_{ff}$ to have full rank. Then $\underline{A}_{ff}$ is positive definite while the same holds for the product $\underline{P}_f \underline{A}_{ff}^{-1} \underline{P}_f^T$ if $\underline{P}_f$ has full rank. The components of $\underset{\sim}{Q}_s$ represent the required load for prescribing the corresponding Lagrange coordinates.

Summary

In this chapter we used the virtual work principle of d'Alembert to set up the equations of motion for a multibody system with an arbitrary topology. External loads as well as (Lagrange and/or attitude) coordinates may be prescribed. External input variables of active connections may be prescribed too, although active connections with memory have not been considered yet. These connections will be included in chapter 8.

CHAPTER 7
ARBITRARY CONNECTIONS

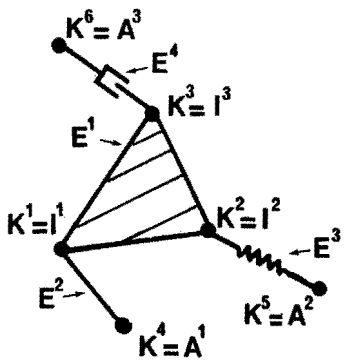
The connections in the previous chapters contained one element only. In this chapter we discuss an element-assembly process for modelling connections with an arbitrary topology and geometry.

7.1 General aspects

Until now all endpoints of elements were rigidly attached to points of the surrounding bodies. In a connection C^k, consisting of nk elements $E^1 .. E^{nk}$, one or more endpoints of the elements can be rigidly attached to endpoints of other elements. A nodal point of C^k is a point where endpoints of two or more elements are rigidly attached to each other. Furthermore, each free endpoint, that is each endpoint of an element not attached to another endpoint, is a nodal point of the connections. External points of C^k are those nodal points of C^k that are rigidly attached to the surrounding bodies. These points are denoted by $A^1 .. A^{na}$ where na is the number of external points of C^k. All other nodal points of C^k are internal points. They are denoted by $I^1 .. I^{ni}$, ni being the number of internal points. The nodal points are so numbered that nodal point K^i corresponds to internal point I^i if $i \leq ni$, and to external point A^j if $j = i - ni > 0$. In figure 7.1 a connection is shown with four elements (nk = 4), three internal points (ni = 3) and three external points (na = 3).

The topology of the elements in a connection is described by means of location matrices. As will be seen in the next section we can use the location matrix as defined in chapter 4 for kinematic connections. These connections consist of kinematic elements with two endpoints

per element. For connections consisting of energetic and active elements with an arbitrary number of endpoints per element, the topological data is stored in a location matrix $\underline{L}$.



$$\underline{L} = \left[\begin{array}{cccc|ccc} I^1 & I^2 & I^3 & A^1 & A^2 & A^3 & \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 0 & 0 & E^1 \\ 0 & 0 & 1 & 0 & 0 & 0 & \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & E^2 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & E^3 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & E^4 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & \end{array}\right]$$

figure 7.1 Connection of four (energetic) elements

Let ne^k be the number of endpoints of element E^k, k = 1..nk and let ne be the total number of endpoints in the connection, that is

$$ne = \sum_{k=1}^{nk} ne^k \qquad (7.1.1)$$

Then $\underline{L}$ is a matrix with ne rows (one for each endpoint in the connection) and ni + na columns (one for each of the internal and external points in the connection). Row 1 to row ne^1 refer to the endpoints of E^1, row ne^1 + 1 to row ne^1 + ne^2 refer to the endpoints of E^2, etc. Each row of $\underline{L}$ contains one component equal to 1, all other components being equal to zero. If endpoint N^i of element E^k is attached to nodal point K^l, then the corresponding row $j = ne^1 + .. + ne^{k-1} + i$ of $\underline{L}$ has a component equal to one in column l, that is $L_{jl} = 1$ and $L_{js} = 0$ for $s \neq l$ and s = 1..ni+na, see figure 7.1.

This location matrix can be partitioned in a matrix $\underline{L}_{in}$ with the first ni columns of $\underline{L}$ and a matrix $\underline{L}_{ex}$ with the remaining columns

$$\underline{L} = [\, \underline{L}_{in} \mid \underline{L}_{ex} \,] \qquad (7.1.2)$$

Each column of $\underline{\underline{L}}_{in}$ refers to an internal point and each column of $\underline{\underline{L}}_{ex}$ refers to an external point.

Let $\vec{c}^{\,i}_{in}$ and $\vec{c}^{\,j}_{ex}$ be the relative position vectors of internal point I^i (i = 1..ni) and external point A^j (j = 1..na) with respect to a reference base of the connection. These vectors are stored in columns $\underset{\sim}{\vec{c}}_{in}$ and $\underset{\sim}{\vec{c}}_{ex}$, defined by:

$$\underset{\sim}{\vec{c}}_{in} = \begin{bmatrix} \vec{c}^{\,1}_{in} \\ \vdots \\ \vec{c}^{\,ni}_{in} \end{bmatrix}, \qquad \underset{\sim}{\vec{c}}_{ex} = \begin{bmatrix} \vec{c}^{\,1}_{ex} \\ \vdots \\ \vec{c}^{\,na}_{ex} \end{bmatrix} \tag{7.1.3}$$

The relative position vector of endpoint N^i of element E^k with respect to the reference base is denoted by $\vec{u}^{\,j}$ where $j = ne^1 + .. + ne^{k-1} + i$ ($i = 1..ne^k$, $k = 1..nk$). If the column with the position vectors of all endpoints of the elements in the connection is denoted by $\underset{\sim}{\vec{u}}$, i.e.

$$\underset{\sim}{\vec{u}}^T = [\, \vec{u}^{\,1}, \, .., \, \vec{u}^{\,ne} \,] \tag{7.1.4}$$

then the relationship between $\underset{\sim}{\vec{u}}$ and $\underset{\sim}{\vec{c}}_{in}$ and $\underset{\sim}{\vec{c}}_{ex}$ is given by

$$\underset{\sim}{\vec{u}} = \underline{\underline{L}}_{ex}\underset{\sim}{\vec{c}}_{ex} + \underline{\underline{L}}_{in}\underset{\sim}{\vec{c}}_{in} \tag{7.1.5}$$

When $\underset{\sim}{\vec{c}}_{ex}$ and $\underset{\sim}{\vec{c}}_{in}$ are known, the relative position vector of each endpoint of each element follows from (7.1.5). Knowing these vectors, we can determine the relative position or connection vectors for element E^k (k = 1..nk) with respect to the reference base of that element.

A local base is defined at each nodal point of a connection. Let $\mathbb{C}^i_{in}$ and $\mathbb{C}^j_{ex}$ be the rotation tensors of the local base at internal point I^i (i = 1..ni) and at external point A^j (j = 1..na) with respect to the reference base of the connection. As mentioned earlier in chapter 3, a local base is defined at each endpoint of each element as well. Let endpoint N^i of element E^k be rigidly attached to nodal point K^l. The (constant) rotation tensor of the local base at N^i to the base at K^l must be specified. Using the location matrix $\underline{\underline{L}}$ it is possible to

determine the orientation of the local base at all endpoints of the elements in the connection if $\mathbb{C}_{in}^{i}$ and $\mathbb{C}_{ex}^{j}$ are known for all internal and external points. Hence the connection tensors of each element E^{k} (k = 1..nk) can be determined.

The constitutive equation of a connection is the relationship between kinematic variables at the external points (for kinematic connections) or between kinematic and force variables at these points (for energetic and active connections). This constitutive equation is not known a priori, but has to be derived from the constitutive equations of the elements in the connection. The elimination process of the variables at the internal points will be discussed in the next sections for kinematic, energetic and active connections.

7.2 Kinematic connections

A kinematic connection may consist of hinge elements as well as constraint elements. These elements, considered in chapter 3, each have two endpoints. We restrict ourselves to kinematic connections with two external points and any number of internal points.

For the kinematic connection a tree structure is defined in which each nodal point is represented by a vertex and each hinge element by an arc. This tree structure is similar to that of a multibody system (see chapter 4) with the difference that each vertex now represents a nodal point instead of a rigid body. The reference vertex represents one of the two external points, the other point being represented by a vertex of degree one. In general, not all hinge elements can be represented by an arc in the tree structure. As in chapter 4, the remaining hinge elements are considered as constraint elements and are represented by chords in the graph.

The formulas derived in chapter 5 for the kinematic behaviour of a multibody system can also be used, with some minor modifications, for kinematic connections. Since nodal points are infinitely small, all body-fixed vectors are zero vectors and all components of the matri-

ces $\underset{\sim}{\vec{B}}_O$, $\underline{\vec{B}}$, $\underset{\sim}{\vec{B}}^C_O$ and $\underline{\vec{B}}^C$ are equal to $\vec{o}$. Similar to the column $\underset{\sim}{q}$, introduced in chapter 5, the generalized coordinates of all hinge elements in the tree structure are stored in a column $\underset{\sim}{\gamma}$ with n_γ components. The constraint elements, including those due to the hinge elements modelled as chords, result in constraint equations for the components of $\underset{\sim}{\gamma}$. In the case of implicit constraint equations, that are constraint equations for the relative attitude coordinates of the nodal points, the equations are transformed into explicit constraint equations by means of the techniques described in section 5.3. After this tranformation the holonomic constraint elements result in a set of nhc explicit constraint equations of the same type as (5.3.17), i.e.

$$\underset{\sim}{f}(\underset{\sim}{\gamma},t) = \underset{\sim}{o}, \tag{7.2.1}$$

while the nonholonomic constraint elements result in a set of nnc constraint equations of the same type as (5.3.20), that is

$$\underset{\sim}{B}_{n\gamma}\dot{\underset{\sim}{\gamma}} + \underset{\sim}{B}_{on\gamma} = \underset{\sim}{o}. \tag{7.2.2}$$

For further use (7.2.1) is differentiated with respect to time. This results in a set of equations, similar to (7.2.2), given by

$$\underset{\sim}{B}_{h\gamma}\dot{\underset{\sim}{\gamma}} + \underset{\sim}{B}_{oh\gamma} = \underset{\sim}{o} \tag{7.2.3}$$

To determine the constitutive equation of a kinematic connection, we are interested in the relationship between the kinematic variables of the two external points. Noting that one of these points is chosen as the reference point, it is sufficient to derive from (7.2.1) and (7.2.2) the constraint equations for the relative attitude coordinates of the second external point with respect to the reference point. These coordinates are stored in a column $\underset{\sim}{y}$ with six components. By means of the methods from section 5.2 it is possible to determine $\underset{\sim}{y}$ as a function of t and of the generalized coordinates $\underset{\sim}{\gamma}$ of the hinge elements in the tree structure of the connection, yielding

$$\underset{\sim}{y} = \underset{\sim}{y}(\underset{\sim}{\gamma},t) \tag{7.2.4}$$

while differentiation with respect to time yields

$$\dot{\underset{\sim}{y}} = \underline{Y}_{\gamma}(\underset{\sim}{\gamma},t)\dot{\underset{\sim}{\gamma}} + \underset{\sim}{Y}_{o\gamma}(\underset{\sim}{\gamma},t) \qquad (7.2.5)$$

where $\underset{\sim}{\gamma}$ has to satisfy the holonomic constraint equations (7.2.1). These matrix equations are similar to equations (5.3.6 & 7) for a multibody system. The matrix $\underline{Y}_{\gamma}$ of order 6x$n\gamma$ and the column $\underset{\sim}{Y}_{o\gamma}$ with six components can be determined with the techniques from section 5.3

We assume that the holonomic constraint equations are independent, that the rank of the matrix $\underline{P}_{h\gamma}$ of order nhcx$n\gamma$ is equal to nhc. If this is not the case, nhc - rank($\underline{P}_{h\gamma}$) equations are redundant and may be neglected, while the set of equations (7.2.1) may be reduced to a smaller set of independent equations. Using the (reduced) set it is possible to determine nhc (= rank($\underline{P}_{h\gamma}$)) components of $\underset{\sim}{\gamma}$ as a function of the $n\gamma$ - nhc other components of $\underset{\sim}{\gamma}$. Defining a column $\underset{\sim}{\varrho}$ with these $n\gamma$ - nhc = $n\varrho$ components, we can write

$$\underset{\sim}{\gamma} = \underset{\sim}{\gamma}(\underset{\sim}{\varrho},t) \qquad (7.2.6)$$

where the holonomic constraint equations $\underset{\sim}{f}(\underset{\sim}{\gamma},t) = \underset{\sim}{f}(\underset{\sim}{\gamma}(\underset{\sim}{\varrho},t),t) = \underset{\sim}{o}$ are satisfied for all $\underset{\sim}{\varrho}$ and t. With this result and using (7.2.4) and (7.2.5) we can determine the relationship between $\underset{\sim}{y}$, $\dot{\underset{\sim}{y}}$ and $\underset{\sim}{\varrho}$, $\dot{\underset{\sim}{\varrho}}$, t. This yields

$$\underset{\sim}{y} = \underset{\sim}{y}(\underset{\sim}{\gamma},t) = \underset{\sim}{y}(\underset{\sim}{\gamma}(\underset{\sim}{\varrho},t),t) \qquad (7.2.7)$$

$$\dot{\underset{\sim}{y}} = \underline{Y}_{\varrho}\dot{\underset{\sim}{\varrho}} + \underset{\sim}{Y}_{o\varrho} \qquad (7.2.8)$$

where the matrix $\underline{Y}_{\varrho}$ of order 6x$n\varrho$ and the column $\underset{\sim}{Y}_{o\varrho}$ with components are in general functions of $\underset{\sim}{\varrho}$ and t.

The rank of $\underline{Y}_{\varrho}$ is at most equal to the lowest of the numbers 6 and $n\varrho$. If rank($\underline{Y}_{\varrho}$) = 6 the holonomic constraint equations (7.2.1) do not imply any holonomic constraint for the attitude coordinates $\underset{\sim}{y}$. If rank($\underline{Y}_{\varrho}$) < 6 we have 6 - rank($\underline{Y}_{\varrho}$) holonomic constraint equations for the coordinates $\underset{\sim}{y}$. In what follows we restrict ourselves to this case. If rank($\underline{Y}_{\varrho}$) < 6 and $n\varrho$ > rank($\underline{Y}_{\varrho}$), we are confronted with the situation that (currently) $\underset{\sim}{y}$ is not a function of $n\varrho$ - rank($\underline{Y}_{\varrho}$) components of $\underset{\sim}{\varrho}$. This implies that these components of $\underset{\sim}{\varrho}$ are (currently)

redundant. If we are interested in the value of $\underline{\gamma}$ at a certain time, for example the initial time, these redundant components may be fixed at any value. If we are interested in the trajectories, the redundant components will be determined by integration of $\dot{\underline{q}}$.

A trivial example of a redundant set of coordinates $\underline{q}$ occurs in a kinematic connection with more then six generalized coordinates ($n\gamma > 6$) and no constraint elements ($n\varphi = n\gamma$). Another example is given in figure 7.2. The connection in this figure consists of a ball-and-socket joint (E^1), a rigid bar element (E^2) and a second ball-and-socket joint (E^3). In this case all hinge elements are represented by branches in the tree structure of the connection and there are no chords. Therefore $\underline{\gamma} = \underline{q}$ and $n\gamma = n\varphi = 6$. However, the rank of the matrix $\underline{Y}_{\varphi}$ is equal to 5, since the attitude coordinates $\underline{\gamma}$ of external point A^2 with respect to the reference point A^1 are independent of the rotation along the longitudional axis of hinge element E^2. The column of $\underline{Y}_{\varphi}$ corresponding to this component of $\underline{q}$ is a zero column.

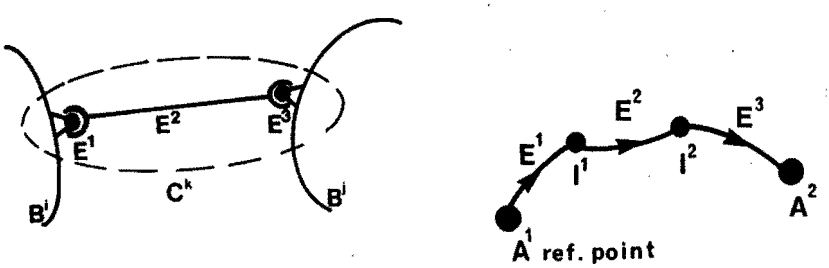


figure 7.2 An assembly of hinge elements

We have until now studied the holonomic constraints for the relative attitude coordinates $\underline{\gamma}$. To obtain the constraint equations for $\dot{\underline{\gamma}}$ we must also consider the nonholonomic constraint equations (7.2.2). From (7.2.6) we obtain a relationship between $\dot{\underline{\gamma}}$ and $\dot{\underline{q}}$ and with this result the nonholonomic equations (7.2.2) can be written as

$$\underline{P}_{n\varphi}\dot{\underline{q}} + \underline{P}_{on\varphi} = \underline{0} \qquad (7.2.9)$$

Assuming that $\text{rank}(\underline{P}_{n\varphi}) = nnc$ it is possible to determine nnc components of $\dot{\underline{q}}$ as functions of the remaining $n\varphi - nnc$ components of $\dot{\underline{q}}$. If $\underline{P}_{n\varphi}$ has a full rank, this expression yields

$$\dot{\underset{\sim}{q}}_1 = -\underset{\sim}{p}_{n\varrho 1}^{-1}(\underset{\sim}{p}_{n\varrho 2}\dot{\underset{\sim}{q}}_2 - \underset{\sim}{p}_{on\varrho}) \tag{7.2.10}$$

where $\dot{\underset{\sim}{q}}_1$ and $\dot{\underset{\sim}{q}}_2$ are columns of order nnc×1 and (nϱ - nnc)×1 respectively, while $\underset{\sim}{p}_{n\varrho 1}$ and $\underset{\sim}{p}_{n\varrho 2}$ are the partitioned submatrices of $\underset{\sim}{p}_{n\varrho}$ such that $\underset{\sim}{p}_{n\varrho 1}$ is a regular matrix of order nnc×nnc. Partitioning of $\underset{\sim}{Y}_\varrho$ similarly as in $\underset{\sim}{p}_{n\varrho}$ and substitution of $\dot{\underset{\sim}{q}}_1$ results in the following expression for $\dot{\underset{\sim}{y}}$,

$$\dot{\underset{\sim}{y}} = (\underset{\sim}{Y}_{\varrho 2} - \underset{\sim}{Y}_{\varrho 1}\underset{\sim}{p}_{n\varrho 1}^{-1}\underset{\sim}{p}_{n\varrho 2})\dot{\underset{\sim}{q}}_2 + (\underset{\sim}{Y}_{o\varrho} - \underset{\sim}{Y}_{\varrho 1}\underset{\sim}{p}_{n\varrho 1}^{-1}\underset{\sim}{p}_{on\varrho}) \tag{7.2.11}$$

Introducing an orthogonal matrix $\underset{\sim}{M} = [\,\underset{\sim}{M}_1,\ \underset{\sim}{M}_2]$ the matrix in front of $\dot{\underset{\sim}{q}}$ can be decomposed and be written as

$$\underset{\sim}{Y}_{\varrho 2} - \underset{\sim}{Y}_{\varrho 1}\underset{\sim}{p}_{n\varrho 1}^{-1}\underset{\sim}{p}_{n\varrho 2} = [\,\underset{\sim}{M}_1\ \ \underset{\sim}{M}_2]\begin{bmatrix}\underset{\sim}{N}\\ \underset{\sim}{0}\end{bmatrix}, \qquad \underset{\sim}{N} = \begin{bmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{bmatrix} \tag{7.2.12}$$

where $\underset{\sim}{N}$ is an upper triangular matrix of order (nϱ - nnc)*(nϱ - nnc) while $\underset{\sim}{M}_1$ and $\underset{\sim}{M}_2$ are matrices of order 6×(nϱ - nnc) and 6×(6 - (nϱ - nnc)), assuming that $0 \leq n\gamma - nnc \leq 6$. If (7.2.11) is premultiplied by $\underset{\sim}{M}^T$, we obtain the two equations

$$\underset{\sim}{M}_1^T\dot{\underset{\sim}{y}} = \underset{\sim}{N}\dot{\underset{\sim}{q}}_2 + \underset{\sim}{M}_1^T(\underset{\sim}{Y}_{o\varrho} - \underset{\sim}{Y}_{n\varrho 1}\underset{\sim}{p}_{n\varrho 1}^{-1}\underset{\sim}{p}_{on\varrho}) \tag{7.2.13}$$

and

$$\underset{\sim}{M}_2^T\dot{\underset{\sim}{y}} + \underset{\sim}{M}_2^T(\underset{\sim}{Y}_{\varrho 1}\underset{\sim}{p}_{n\varrho 1}^{-1}\underset{\sim}{p}_{on\varrho} - \underset{\sim}{Y}_{o\varrho}) = \underset{\sim}{0} \tag{7.2.14}$$

Equation (7.2.14) has the form of the desired nonholonomic constraint equation for the coordinates $\dot{\underset{\sim}{y}}$. By means of (7.2.13) we can determine $\dot{\underset{\sim}{q}}_2$ and, by means of (7.2.10) $\dot{\underset{\sim}{q}}_1$ as well.

7.3 Energetic connections

In this section on energetic connections we consider assemblies of energetic elements. Combinations of energetic and kinematic elements in one connection are not considered. The energetic connections may have an arbitrary topology and geometry as long as there are at least two external points. As stated in section 3.4, the forces and moments

exerted at the endpoints of an energetic element can be determined at time t if the current value and the history of the relevant kinematic variables and the history of the relevant force variables are known. For a connection in a multibody system the current values of the kinematic variables of the external points follow from the attitude coordinates of the bodies in the system. However, the current values of the kinematic variables of the internal points are, a priori, unknown. Since the resulting force and moment vector on each internal nodal point in the connection must be equal to $\vec{o}$, it is possible to determine these unknowns.

The determination of the unknown kinematic variables of the internal points is a well known subject in the 'finite element method' [e.g. Zienkiewicz 1977]. Some aspects of this method will be described roughly. To simplify the discussion we assume that the constitutive equation for all elements in the connection depends on the current position and force variables only, i.e. $\underset{\sim}{f} = \underset{\sim}{f}(\underset{\sim}{F},\underset{\sim}{\varepsilon},t)$ and $\underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}(\vec{\underset{\sim}{u}})$. The complications caused by constitutive equations which depend on velocity variables and/or history are not discussed here.

The known kinematic variables of the external points are the connection vectors $\vec{c}^{1}_{ex}$ .. $\vec{c}^{na}_{ex}$ and the connection tensors $\mathbb{C}^{1}_{ex}$ .. $\mathbb{C}^{na}_{ex}$. From equation (7.1.5) we can obtain $\vec{\underset{\sim}{u}}$ and determine the matrix representation of the connection vectors of all endpoints of each element with respect to the reference base of the element. From these matrix representations the relevant kinematic variables of the elements, i.e. $\underset{\sim}{\varepsilon}$, are obtained. Substitution of these variables in the constitutive equation of the energetic element results in the forces and moments at the endpoints of the elements. From these the resulting force and moment vectors $\vec{F}^{i}$ and $\vec{M}^{i}$ at each nodal point K^{i} (i = 1..nk) can be determined.

If the kinematic variables of the internal points are estimates, these forces and moments are estimates, too. However, since the elements are massless, the resulting force and moment vector on each internal point of the connection must be equal to $\vec{o}$. As long as this is not the case, we must improve the estimated values for the kinematic

variables of the internal points. In general this requires an iterative process. Only for connections with elements having a linear constitutive equation and small deformations, the solution can be determined in one step.

External loads may be applied at the internal points of a connection but not on the elements. In that case the resulting forces $\vec{F}^{i}$ and moment $\vec{M}^{i}$ on internal nodal point I^{i} (and arising from the forces and moments at the endpoints of the element that are rigidly attached to that nodal point) are not equal to zero but to the external force and the external moment vectors at that nodal point I^{i} (i = 1..ni). The process for the determination of the kinematic variables of the internal points as indicated above is only slightly modified by these external loads.

7.4 Active connections

Connections with active elements are treated similarly to energetic connections. It is permitted to combine active and energetic elements in one active connection. For example, a muscle may be modelled as an active connection with an (active) contraction element and one or more (passive) elastic and/or viscous elements. For active connections the same problems, concerning the determination of the a priori unknown values of the kinematic variables of the internal points, arise as for energetic connections.

In the case of active connections with memory, we must solve both the constitutive and the state equations:

$$\dot{\underset{\sim}{x}} = \underset{\sim}{s}(\underset{\sim}{x}(t),\underset{\sim}{\varepsilon}(t),\dot{\underset{\sim}{\varepsilon}}(t),\underset{\sim}{i}(t),t) \qquad \underset{\sim}{x}_{o} = \underset{\sim}{x}(t_{o}) \qquad (7.4.1)$$

$$\underset{\sim}{f}(\underset{\sim}{F}(t),\underset{\sim}{\varepsilon}(t),\dot{\underset{\sim}{\varepsilon}}(t),\underset{\sim}{x}(t),\underset{\sim}{i}(t),t) = \underset{\sim}{o}$$

The state equations must be solved simultaneously with the equations of motion. As a result the state equations must be combined with the equations of motion as described in chapter six.

Summary

Several aspects of the process of assembling a complex connection of simple elements are discussed. In each connection it is assumed that only elements of the same type appear, except for active connections which may contain energetic elements. Restrictions with regard to topology are only made for kinematic connections.

Based on this approach we can describe simple elements and assemble them in systematically into complex connections. However, to achieve an efficient numerical implementation in a general-purpose computer program many aspects, not discussed here, require further investigation.

CHAPTER 8
SIMULATING THE BEHAVIOUR OF MULTIBODY SYSTEMS

In this chapter we describe some aspects of the simulation of the behaviour of multibody systems. The first section contains a summary of those subjects from the previous chapters that will be used in this one. The determination of the degrees of freedom of a multibody system and the simulation of the kinematic behaviour are discussed in the second and third sections. In the fourth and fifth sections the simulation of the dynamic behaviour and calculation of the a priori unknown forces are studied.

8.1 Introduction

The theory presented in the previous chapter can be used for simulating the behaviour of systems with nb (nb>1) bodies and an arbitrary number of kinematic, energetic and active connections. The first step in the simulation of the behaviour of such a system involves the definition of a tree structure of bodies and hinges and results in the introduction of a column $\underset{\sim}{q}$ of nq Lagrange coordinates. These coordinates determine the position and orientation of each body in the system uniquely, that is the column $\underset{\sim}{y}$ of the attitude coordinates of the bodies is known as a function of $\underset{\sim}{q}$. In this step of the simulation process only holonomic kinematic connections are taken into account.

As discussed in section 5.3 the holonomic constraints and eventually prescribed attitude coordinates (implicit constraint equations) yield a set of nhp equations for the Lagrange coordinates and are given by

$$\underset{\sim}{f}(\underset{\sim}{q},t) = \underset{\sim}{0} \tag{8.1.1}$$

Differentiation with respect to time results in a set of equations for the time derivative $\dot{\underset{\sim}{q}}$

$$\underline{P}_h\dot{\underset{\sim}{q}} + \underset{\sim}{P}_{oh} = \underset{\sim}{0} \tag{8.1.2}$$

where $\underline{P}_h$ is a matrix of order nhpxnq while $\underset{\sim}{P}_{oh}$ is a column with nhp components. In general the components of both $\underline{P}_h$ and $\underset{\sim}{P}_{oh}$ depend on $\underset{\sim}{q}$ and t.

Besides the holonomic constraints, nonholonomic constraints can also occur. These are represented by a set of nnp = np - nhp equations in $\dot{\underset{\sim}{q}}$ of the same type as (8.1.2) and are given (see section 5.3) by

$$\underline{P}_n\dot{\underset{\sim}{q}} + \underset{\sim}{P}_{on} = \underset{\sim}{0} \tag{8.1.3}$$

Combining equations (8.1.2.) and (8.1.3) in one matrix equation yields

$$\underline{P}\dot{\underset{\sim}{q}} + \underset{\sim}{P}_o = \underset{\sim}{0} \tag{8.1.4}$$

where the matrix $\underline{P}$ of order npxnq and the column $\underset{\sim}{P}_o$ with np components are given by:

$$\underline{P} = \begin{bmatrix} \underline{P}_h \\ \underline{P}_n \end{bmatrix}, \qquad \underset{\sim}{P}_o = \begin{bmatrix} \underset{\sim}{P}_{oh} \\ \underset{\sim}{P}_{on} \end{bmatrix} \tag{8.1.5}$$

Since (8.1.4) must hold for every time t, we can derive from (8.1.4) an equation for the second derivatives of the Lagrange coordinates

$$\underline{P}\ddot{\underset{\sim}{q}} + \underset{\sim}{P}_{oo} = \underset{\sim}{0} \tag{8.1.6}$$

In this equation the components of $\underline{P}$ depend on t and $\underset{\sim}{q}$ only, while the components of $\underset{\sim}{P}_{oo}$ are a function of t, $\underset{\sim}{q}$ and $\dot{\underset{\sim}{q}}$:

$$\underline{P} = \underline{P}(\underset{\sim}{q},t), \qquad \underset{\sim}{P}_{oo} = \underset{\sim}{P}_{oo}(\underset{\sim}{q},\dot{\underset{\sim}{q}},t) \tag{8.1.7}$$

The equations of motion for a multibody system have been derived in chapter 6. The final set is of the form

$$\underline{\underline{A}}\ddot{\underline{q}} = \underline{Q} + \underline{\underline{P}}^T\underline{\lambda} \tag{8.1.8}$$

The matrix $\underline{\underline{A}}$ of order nq×nq, the generalized mass matrix associated with the Lagrange coordinates $\underline{q}$, depends on t and $\underline{q}$. The column $\underline{Q}$ contains the contributions of the external loads, the loads on the bodies exerted by the energetic and active connections as well as a rest term arising out of the inertia loads. The components of $\underline{Q}$ can be a function of t, $\underline{q}$ and $\dot{\underline{q}}$, but not of $\ddot{\underline{q}}$. If active connections occur in the multibody system the generalized loads will also depend on the state variables $\underline{x}$ of these connections. In turn, these variables are determined by a set of state equations

$$\dot{\underline{x}} = \underline{s}(\underline{x}, \underline{\varepsilon}, \dot{\underline{\varepsilon}}, \underline{i}, t) \tag{8.1.9}$$

where $\underline{i}$ is a set of known external input variables and $\underline{\varepsilon}$ contains the relevant kinematic variables of the active connections. Finally, the np components of the column $\underline{\lambda}$ are Lagrange multipliers which take into account the holonomic and nonholonomic kinematic constraints (8.1.1) and (8.1.3).

In many problems ns (ns>1) Lagrange coordinates are prescribed. The prescribed coordinates are stored in a column $\underline{q}_s$ with ns components, the remaining coordinates being stored in a column $\underline{q}_f$ with nf = nq - ns components. It is always possible to renumber the Lagrange coordinates so that $\underline{q}$ is partitioned in $\underline{q}_s$ and $\underline{q}_f$

$$\underline{q} = \begin{bmatrix} \underline{q}_s \\ \underline{q}_f \end{bmatrix} \tag{8.1.10}$$

Furthermore the holonomic constraints (8.1.1) can be rewritten to yield an equation of the form

$$\underline{f}(\underline{q}_s, \underline{q}_f, t) = \underline{o} \tag{8.1.11}$$

while the Pfaff equations (8.1.4) and (8.1.5) can be permutated to give

$$\begin{bmatrix} \underline{P}_{hs} & \underline{P}_{hf} \\ \underline{P}_{ns} & \underline{P}_{nf} \end{bmatrix} \begin{bmatrix} \dot{\underline{q}}_s \\ \dot{\underline{q}}_f \end{bmatrix} + \begin{bmatrix} \underline{P}_{oh} \\ \underline{P}_{on} \end{bmatrix} = \underline{0} \qquad (8.1.12)$$

For the analysis of the dynamic behaviour in particular, it turns out to be advantageous to write the last set of equations in the form

$$\underline{P}_s \dot{\underline{q}}_s + \underline{P}_f \dot{\underline{q}}_f + \underline{P}_o = \underline{0} \qquad (8.1.13)$$

where the matrices $\underline{P}_s$ of order np×ns and $\underline{P}_f$ of order np×nf are defined by:

$$\underline{P}_s = \begin{bmatrix} \underline{P}_{hs} \\ \underline{P}_{ns} \end{bmatrix}, \qquad \underline{P}_f = \begin{bmatrix} \underline{P}_{hf} \\ \underline{P}_{nf} \end{bmatrix} \qquad (8.1.14)$$

Using these matrices, the equation (8.1.6) for the second derivatives $\ddot{\underline{q}}$ can be transformed into

$$\underline{P}_s \ddot{\underline{q}}_s + \underline{P}_f \ddot{\underline{q}}_f + \underline{P}_{oo} = \underline{0} \qquad (8.1.15)$$

Finally, the equations of motion can be rewritten as

$$\begin{bmatrix} \underline{A}_{ss} & \underline{A}_{sf} \\ \underline{A}_{sf}^T & \underline{A}_{ff} \end{bmatrix} \begin{bmatrix} \ddot{\underline{q}}_s \\ \ddot{\underline{q}}_f \end{bmatrix} = \begin{bmatrix} \underline{Q}_s \\ \underline{Q}_f \end{bmatrix} + \begin{bmatrix} \underline{P}_s^T \\ \underline{P}_f^T \end{bmatrix} \underline{\lambda} \qquad (8.1.16)$$

where the columns $\underline{Q}_s$ and $\underline{Q}_f$ represent the generalized loads associated with the prescribed Lagrange coordinates $\underline{q}_s$ and the a priori unknown Lagrange coordinates $\underline{q}_f$, respectively.

In the following sections we discuss four problems that are linked up with the simulation of the behaviour of multibody systems. The first problem is the determination of the number of degrees (nd) of freedom of a multibody system. This number is important when simulating the kinematic behaviour of a multibody system. To simulate this behaviour

we must prescribe exactly nd Lagrange coordinates. As will be discussed, the number nd is not a constant as suggested by the often used Grübler rule.

The kinematic simulation problem is the second problem to be discussed. Its solution enables the positions, orientations, velocities, etc. of all bodies to be obtained as function of the prescribed Lagrange coordinates. The third problem is the simulation of the dynamic behaviour to obtain the trajectories of the Lagrange coordinates and attitude coordinates. These trajectories are functions of the prescribed external loads, the prescribed coordinates and, in the case of active connections, also of the prescribed external input variables. The fourth problem to be discussed deals with the calculation of the internal forces arising out of the kinematic connections.

The formulation of each problem results in a set of equations. Some (numerical) methods for solving these equations are mentioned. Sometimes they fail and attention is given to the background of such failures as well as methods to correct or avoid this.

8.2 Degrees of freedom of a multibody system

An important characteristic of the (kinematic) behaviour of a multibody system is its number of degrees of freedom. An easy way to calculate this number would appear to be given by the Grübler rule. However, there are several examples where the Grübler rule goes wrong (see Sheth [1972, p166-178] and Paul [1979, p276-287]).

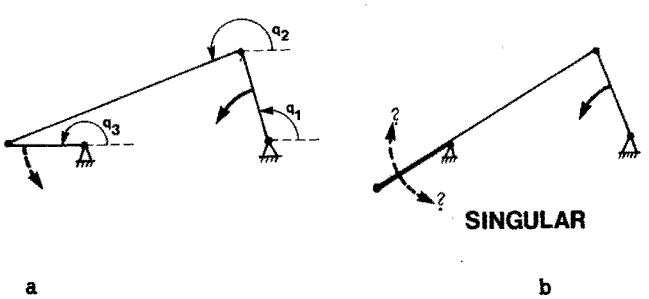


figure 8.1 Four-bar mechanism in singular attitude

Several adjustments to the Grübler rule have been suggested, but even with these adjustments the Grübler and similar rules are not applicable if the number of degrees of freedom (temporarily) changes. This happens, for example in the system given in figure 8.1b. Normally the four-bar mechanism in this figure has one degree of freedom (figure 8.1a), but in the attitude shown in figure 8.1b there are two degrees of freedom. Here the crank can continue or reverse its motion.

According to the Grübler rule the number of degrees of freedom (nd) is equal to the total number of Lagrange coordinates (nq) minus the number of holonomic constraint equations (nhp). The reason why the Grübler rule sometimes results in wrong answers is that it is not the number of holonomic constraint equations that should be used, but the rank of the Jacobian of these equations, the rank of $\underline{P}_h$. In agreement with Sheth [1972, p178, 232] the number of degrees of freedom nd is defined as

$$nd = nq - \text{rank}(\underline{P}_h) \qquad (8.2.1)$$

Since $\underline{P}_h = \underline{P}_h(\underline{q},t)$ is a function of $\underline{q}$ and t, the rank of $\underline{P}_h$ (and hence nd) can change in time. While $\underline{P}_h$ is a matrix of order nhp×nq and nhp≤nq, the rank of $\underline{P}_h$ is equal to or smaller than nhp. If $\text{rank}(\underline{P}_h) < nhp$, the constraints are dependent and a kinematic singularity occurs. The Grübler rule gives the correct answer for nd if and only if $\text{rank}(\underline{P}_h) = nhp$.

In a general purpose program the determination of the rank of a matrix requires the use of a numerical method. The singular value decomposition described by Golub and Reinsch [1971] is an adequate method and probably the most reliable one. Another is the Gauss elimination process with complete pivoting as used in the IMP program (Sheth [1972, p226]). In numerical subroutine libraries these methods are generally available (e.g. the MFGR routine in the SSP library of IBM [1970, p127]).

Note that the number of degrees of freedom is determined by the holonomic constraints only. Rosenberg [1977, p37] associates the number of degrees of freedom with the dimension of the space of accessible

attitudes (configurations). The dimension of this space is determined by holonomic constraints only and is not affected by the nonholonomic constraints. Nonholonomic constraints result in a space of admissible variations with a lower dimension than the space of accessible attitudes. In the second part of the next section, where we study the velocities and accelerations, this aspect becomes important.

8.3 The kinematic simulation problem

The positions and orientations (KSP/0)
The velocities and accelerations (KSP/1, KSP/2)
The calculation of trajectories

The kinematic simulation problem (abbreviated: KSP) leads to a set of nonlinear algebraic equations which, in general, can only be solved numerically. Usually the KSP involves the determination of the attitude coordinates, the velocities and the accelerations of the bodies in the system. This process will be split into three parts, denoted by KSP/0, KSP/1 and KSP/2 respectively. Each will be discussed in the following subsections. In the last subsection we will deal with the calculation of trajectories.

The positions and orientations (KSP/0)

In this subsection we discuss the case in which the aim is to determine the Lagrange coordinates $\underset{\sim}{q}$ at a given time t (KSP/0). Hence we are interested in the derivatives $\dot{\underset{\sim}{q}}$ and $\ddot{\underset{\sim}{q}}$. The relevant equation for the KSP/0 is given by (8.1.1), i.e. by $\underset{\sim}{f}(\underset{\sim}{q},t) = \underset{\sim}{o}$. This is a set of nhp nonlinear algebraic equations for the nq (> nhp) Lagrange coordinates. For the moment we assume that these equations are independent, that the rank of the Jacobian matrix $\underline{P}_h$ is equal to nhp. From (8.2.1) we know that the number of degrees of freedom of the system at the given time t is given by nd = nq - nhp = nq - rank($\underline{P}_h$).

An unique solution of $\underset{\sim}{f}(\underset{\sim}{q},t) = \underset{\sim}{o}$ is possible only if exactly nd components of $\underset{\sim}{q}$ are prescribed at time t. From a physical point of view this is trivial. To define the state of a system having nd degrees of

freedom it is necessary to prescribe exactly nd of these degrees of freedom. Let the prescribed Lagrange coordinates be the components of a column q_s. The remaining coordinates, the components of the column q_f, must be calculated from (8.1.1) or from the equivalent set (8.1.11)

$$\underset{\sim}{f}(q_s, q_f, t) = \underset{\sim}{0} \tag{8.3.1}$$

We assume that there is at least one solution for q_f. There may be more solutions, but we will not go into this any further.

To solve (8.3.1) for q_f an iterative solver of the Newton type can be used. Let q_f^i be an estimate for the solution q_f and let $\Delta^i q_f$ be the error in this estimate, i.e. $q_f^i = q_f - \Delta^i q_f$. Substitution in (8.3.1) yields

$$\underset{\sim}{f}(q_s, q_f^i + \Delta^i q_f, t) = \underset{\sim}{0} \tag{8.3.2}$$

From this set of equations an estimate for $\Delta^i q_f$ is calculated, so that a new estimate for q_f follows from

$$q_f^{i+1} = q_f^i + \Delta^i q_f \tag{8.3.3}$$

In the Newton-Raphson method the estimate for $\Delta^i q_f$ is determined by linearization of (8.3.2), resulting in

$$\underline{P}_{hf}^i \Delta^i q_f = -\underset{\sim}{f}^i \tag{8.3.4}$$

where $\underline{P}_{hf}^i = \underline{P}_{hf}(q_s, q_f^i, t)$ and $\underset{\sim}{f}^i = \underset{\sim}{f}(q_s, q_f^i, t)$, while the matrix $\underline{P}_{hf}$ has been introduced earlier in (8.1.12). If $\underline{P}_{hf}^i$ is regular, it can be proved that the Newton-Raphson method converges quadratic in the neighbourhood of a solution. In literature several modifications of this method are given (see e.g. Stoer and Bulirsch [1980, p257] or Gill et al. [1981, p305]).

To start the iterative solver an initial estimate q_f^o is required. Usually the solution for q_f must be calculated at a number of discrete times t_o, t_1, t_2, .. . Based on the assumption that q_f is a smooth

function of time we use the obtained solution $\underset{\sim}{q}_f(t_j)$ as the initial estimate for time $t_{j+1} = t_j + \Delta t$. The time step Δt may not be chosen too large, otherwise situations as shown in figure 8.2 can occur.

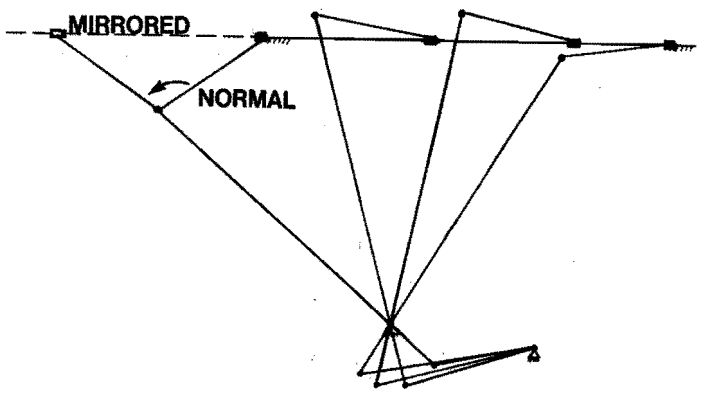


figure 8.2 Normal and mirrored attitude

We have to supply an initial estimate for $\underset{\sim}{q}_f$ only at time $t = t_o$. If this estimate is far away from the desired solution the solver can converge to another solution (figure 8.2) or diverge. In both cases a better estimate must be supplied. However, there is another way to determine an initial estimate for $\underset{\sim}{q}_f$ at time $t = t_o$. Assume that at time t_o the attitude coordinates of all bodies in the system are known approximately. Using least squares techniques and the kinematic relations between attitude coordinates and Lagrange coordinates, given in chapter 5, a very accurate estimate for all Lagrange coordinates $\underset{\sim}{q}$, and hence for $\underset{\sim}{q}_f$, can be calculated. This subject, to be classified as an optimization problem, will not be discussed any further.

The Newton-Raphson method fails if the matrix $\underline{P}^i_{hf}$ is singular. This matrix is part of the matrix $\underline{P}_h$ of order nhp×nq (see 8.1.11). Since we assumed that the holonomic constraint equations are independent, rank($\underline{P}_h$) = nhp and there is at least one regular submatrix $\underline{P}_{hf}$ of order nhp×nhp. For a given matrix $\underline{P}_h$, the submatrix $\underline{P}_{hf}$ is completely determined by the choice of the nd prescribed Lagrange coordinates.

Each choice that results in a regular matrix $\underline{P}_{hf}$ is called a suitable choice. In practice it will in general be no problem to find a suitable choice.

An important assumption in the preceding part of this section is that $\underline{P}_h$ is a matrix with full rank, that all holonomic constraints in $\underset{\sim}{f}(\underset{\sim}{q},t) = \underset{\sim}{0}$ are independent. Then rank($\underline{P}_h$) = nhp and nd = nq - nhp. The number of degrees of freedom and therefore also the number of Lagrange coordinates that must be prescribed is then known a priori. Usually nd is constant, but in the case of a kinematic singularity the rank of $\underline{P}_h$ and thus nd changes.

As an example we again consider the four-bar mechanism in figure 8.1. This mechanism has two holonomic constraint equations (nhp = 2) and three Lagrange coordinates (nq = 3). If $q_1 - q_2 \neq n\pi$ (integer n) the constraint equations are independent and the number of degrees of freedom is equal to one (nd = 1). Using the rotation of the rocker q_3 as the prescribed Lagrange coordinate it is possible to simulate the kinematic behaviour of the system. However, if $q_1 - q_2 = n\pi$ the constraint equations become dependent and the rank of $\underline{P}_h$ reduces to one. As a result, nd becomes equal to two and we have to specify two Lagrange coordinates instead of one. From a physical point of view this is trivial. In the situation shown in figure 8.1b we have to specify the direction in which the crank will rotate if q_3 is changed.

In general, a numerical procedure is used to determine the rank of $\underline{P}_h$. If rank($\underline{P}_h$) < nhp, that is if a kinematic singularity occurs, these procedures can detect the constraint equations which are dependent and the Lagrange coordinates which should be prescribed extra. This results in an extended column $\underset{\sim}{q}_s$ with nq - rank($\underline{P}_h$) = nd instead of nq - nhp components, as well as a reduced column $\underset{\sim}{q}_f$ with rank($\underline{P}_h$) components. Deleting the dependent constraint equations and using these columns, the earlier sketched procedure for the determination of $\underset{\sim}{q}_f$ is applicable again. In the rest of this section no further attention is paid to kinematic singularities and it is assumed that each dependent set of constraint equations is modified to an independent set.

The velocities and accelerations (KSP/1, KSP/2)

In the preceding subsection on the determination of the a priori unknown Lagrange coordinates, only the holonomic constraint equations (8.1.1) were relevant. If we are also interested in time derivatives of $\underset{\sim}{q}$ at that time, the time derivatives of the holonomic constraint equations (8.1.2) and the nonholonomic constraint equations (8.1.3) must be considered, too. These equations are combined in one matrix equation in (8.1.12). $\underline{P}_{hf}$ is a square, regular matrix and therefore it is possible to express $\dot{\underset{\sim}{q}}_f$ in $\dot{\underset{\sim}{q}}_s$ and the column $\underset{\sim}{P}_{oh}$ as

$$\dot{\underset{\sim}{q}}_f = -\underline{P}_{hf}^{-1}\underline{P}_{hs}\dot{\underset{\sim}{q}}_s - \underline{P}_{hf}^{-1}\underset{\sim}{P}_{oh} \tag{8.3.5}$$

With this result we can eliminate $\dot{\underset{\sim}{q}}_f$ in the lower part of (8.1.12), yielding an equation for $\dot{\underset{\sim}{q}}_s$

$$(\underline{P}_{ns} - \underline{P}_{nf}\underline{P}_{hf}^{-1}\underline{P}_{hs})\dot{\underset{\sim}{q}}_s = \underline{P}_{nf}\underline{P}_{hf}^{-1}\underset{\sim}{P}_{oh} - \underset{\sim}{P}_{on} \tag{8.3.6}$$

where the term between brackets represents a matrix of order nnp×nd. Apart from special cases, this matrix has full rank nnp. This means that it is possible to express nnp components of $\dot{\underset{\sim}{q}}_s$ in the remaining (nd - nnp) components. Hence, if we want to determine all the time derivatives of the Lagrange coordinates of a system at a time t, exactly nd - nnp suitable chosen components of $\dot{\underset{\sim}{q}}_s$ must be specified.

The number of time derivatives to be specified is equal to the number of Lagrange coordinates (nq) minus the number of independent holonomic and nonholonomic constraint equations (nhp + nnp). For the KSP/0 nd Lagrange coordinates have to be prescribed, while for the KSP/1 nd - nnp time derivatives have to be prescribed. Rosenberg associates the number nd with the dimension of the space of accessible attitudes and the smaller number nd - nnp with that of the space of admissible variations.

Determination of the unknown second time derivatives of the Lagrange coordinates $\ddot{\underset{\sim}{q}}$ is similar to the determination of the unknown components of $\dot{\underset{\sim}{q}}$. Therefore this process, abbreviated as the KSP/2, will not be discussed any further.

The calculation of trajectories

The KSP/0 and KSP/1, /2, as discussed so far, deal with the simulation of the kinematic behaviour at a certain time t. At that time nd Lagrange coordinates q_{js} and nd - nnd coordinates $\dot{q}_{js}$, $\ddot{q}_{js}$ must be prescribed. If we are interested in the trajectories of the Lagrange coordinates, as well as the motions of the bodies, we must solve the KSP at several points of time in a time interval $[t_o, t_f]$ from the origin time t_o to the final time t_f. We have already mentioned these points of time in the discussion of the estimates for the Newton solver.

If the system contains only holonomic constraints, nnp = 0, then we assume that the nd components of $q_{js}(t)$ and their derivatives are given. If it contains nnp nonholonomic constraints, nd-nnp components of $\dot{q}_{js}(t)$ and $\ddot{q}_{js}(t)$ may be prescribed. In that case we prescribe only nd - nnp (suitably chosen) components of q_{js}, $\dot{q}_{js}$ and $\ddot{q}_{js}$. The remaining components of $\dot{q}_{js}$ and $\ddot{q}_{js}$ are obtained by solving (8.3.6), the remaining components of q_{js} being obtained by integration of $\dot{q}_{js}$. Initial values for the remaining components of q_{js} must be specified only at time t_o.

8.4 The dynamic simulation problem

Formulation of the DSP
Solution of the DSP

The third problem to be discussed is the simulation of the dynamic behaviour of multibody systems (abbreviated: DSP). This problem leads to a set of nonlinear differential equations which, again, must be solved numerically. In this section we start with the formulation of the DSP, while the second subsection discusses the numerical solution of the DSP.

Formulation of the DSP

The DSP is described by the constraint equations (8.1.11, 12 and 15) and the equations of motion (8.1.16). If the system contains active connections too, the state equations (8.1.9) must also be taken into account. The unknown quantities in these equations are the second derivatives $\ddot{\underset{\sim}{q}}_f$, the generalized loads $\underset{\sim}{Q}_s$, the Lagrange multipliers $\underset{\sim}{\lambda}$ and the state variables $\underset{\sim}{x}$ for the active connections. As shown in chapter 6 it is possible to derive explicit relations for these unknowns:

$$\underset{\sim}{\lambda} = -(\underset{=}{P}_f\underset{=}{A}_{ff}^{-1}\underset{=}{P}_f^T)^{-1}[\underset{=}{P}_f\underset{=}{A}_{ff}^{-1}(\underset{\sim}{Q}_f - \underset{=}{A}_{sf}^T\ddot{\underset{\sim}{q}}_s) + \underset{=}{P}_s\ddot{\underset{\sim}{q}}_s + \underset{\sim}{P}_{oo}] \qquad (8.4.1)$$

$$\dot{\underset{\sim}{x}} = \underset{\sim}{s}(\underset{\sim}{x}, \underset{\sim}{\varepsilon}, \dot{\underset{\sim}{\varepsilon}}, \underset{\sim}{i}, t) \qquad (8.4.2)$$

$$\ddot{\underset{\sim}{q}}_f = \underset{=}{A}_{ff}^{-1}(\underset{=}{P}_f^T\underset{\sim}{\lambda} + \underset{\sim}{Q}_f - \underset{=}{A}_{sf}^T\ddot{\underset{\sim}{q}}_s) \qquad (8.4.3)$$

$$\underset{\sim}{Q}_s = \underset{=}{A}_{sf}\ddot{\underset{\sim}{q}}_f - \underset{=}{P}_s^T\underset{\sim}{\lambda} + \underset{=}{A}_{ss}\ddot{\underset{\sim}{q}}_s \qquad (8.4.4)$$

Because of the differential equations for $\dot{\underset{\sim}{x}}$ and $\ddot{\underset{\sim}{q}}_f$ we must also specify initial conditions for $\underset{\sim}{x}$, $\underset{\sim}{q}_f$ and $\dot{\underset{\sim}{q}}_f$ at the starting time t_o.

A basic assumption in the derivation of these equations is that the matrices $\underset{=}{A}_{ff}$ and $\underset{=}{P}_f\underset{=}{A}_{ff}^{-1}\underset{=}{P}_f^T$ are regular. For the generalized mass matrix $\underset{=}{A}_{ff}$ this will always be true (see section 6.2). Then $\underset{=}{P}_f\underset{=}{A}_{ff}^{-1}\underset{=}{P}_f^T$ is regular if and only if the matix $\underset{=}{P}_f$ of order npxnq has rank np. This will be true if and only if the matrix $\underset{=}{P}$ has rank np, in other words if all holonomic and nonholonomic constraint equations are independent. In the case of kinematic singularities, the actions described in the previous section are applied. In the rest of the present section we assume that $\underset{=}{P}_f\underset{=}{A}_{ff}^{-1}\underset{=}{P}_f^T$ is regular.

Solution of the DSP

Assume that at a certain time t_i that the values of all prescribed variables, as well as the value of $\underset{\sim}{q}_f$, $\dot{\underset{\sim}{q}}_f$ and $\underset{\sim}{x}$ are known. With (8.4.1-4) the values at that time for $\underset{\sim}{\lambda}$, $\ddot{\underset{\sim}{q}}_f$, $\dot{\underset{\sim}{x}}$ and $\underset{\sim}{Q}_s$ are obtained

and, using an integration scheme a value for $\underset{\sim}{q}_f$, $\dot{\underset{\sim}{q}}_f$ and $\underset{\sim}{x}$ can be determined at the next time $t_i + \Delta t_i$. This process starts at the origin time t_o and ends when the final time t_f is reached.

To evaluate the equations for $\underset{\sim}{\lambda}$, $\ddot{\underset{\sim}{q}}_f$, $\dot{\underset{\sim}{x}}$ and $\underset{\sim}{Q}_s$, a large number of calculations must be performed. Each time we have to calculate all local-to-global rotation tensors, evaluate the positions, orientations and (angular) velocities of the bodies as well as the behaviour of the energetic and active connections, determine the components of all matrices, test for kinematic singularities and finally perform the matrix operations in the equations (8.4.1-4).

To reduce the computation effort, the time step $\Delta t = t_{i+1} - t_i$ is chosen as large as possible. However, the maximum length of the time step is restricted by an accuracy condition of the form

$$C_1 \, \Delta t^p < \delta \qquad (8.4.5)$$

and a stability condition of the form

$$L \, \Delta t < C_2 \qquad (8.4.6)$$

The constants C_1, C_2 and the exponent p depend on the integration scheme, whereas δ and L represent the required accuracy and the Lipschitz constant (Stoer and Bulirsch [1980, p406]). The last constant depends on the equations to be solved.

Given a certain integration scheme with known C_1, C_2 and p, as well as a certain required accuracy δ, no problems arise if the accuracy conditions restrict the maximum length of the time step. This happens, for example if C_2 is large and/or L is small. For small C_2 and/or large L the maximum length of Δt is restricted by the stability condition. In that case the length of the time step must be kept short in order to prevent the solution from becoming unstable (see figure 8.3). Differential equations with a high value of L are called stiff differential equations.

Until a few years ago, Runge-Kutta schemes were mainly used as integration schemes in multibody programs. These schemes are easy to implement and C_2 is reasonably large. Their drawback is that in general they require more evaluations of the differential equations than other schemes do, especially if high accuracy is demanded (Shampine et al. [1976]). At present linear multistep schemes with a variable-order/variable-steplength are becoming more and more popular. Compared to the single-step Runge-Kutta schemes, which are based on Taylor approximations, the linear multistep schemes are based on linear interpolation polynomials. Changing both the order and the steplength requires some overhead calculations to adapt the polynomials (Shampine and Gordon [1975]), but compared to the number of calculations required for one evaluation of our equations this overhead is negligible.

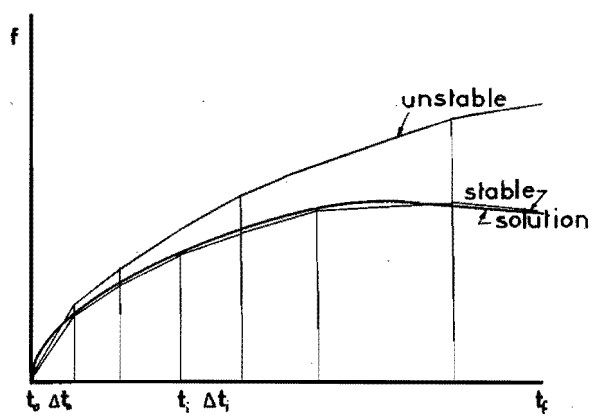


figure 8.3 An unstable solution

During the numerical integration of differential equations the stability of the solution must be verified. In general the solution may become unstable because of stiff differential equations and because of constraint equations which are only satisfied approximately. For the integration of stiff differential equations there are special implicit integration schemes which are unconditionally stable (Gear, 1971]). These schemes do not have to satisfy a stability condition of the form (8.4.6) and they allow the use of long time steps. The disadvantage of these schemes is that, at each time step, they require the solution of a set of nonlinear algebraic equations.

To avoid instability caused by violated constraints we mention the possibility of augmenting the constraint equations. As $\underset{\sim}{q}_f$ is an approximation, the actual kinematic constraint equations for $\underset{\sim}{q}_f$, $\dot{\underset{\sim}{q}}_f$ and $\ddot{\underset{\sim}{q}}_f$ may be violated. Therefore, instead of using the equation

$$\underset{\sim}{\underset{\sim}{P}}_f \ddot{\underset{\sim}{q}}_f + \underset{\sim}{\underset{\sim}{P}}_s \ddot{\underset{\sim}{q}}_s + \underset{\sim}{P}_{oo} = \underset{\sim}{0} \tag{8.4.7}$$

the stability of the solution can be improved by using the augmented equation defined by

$$\underset{\sim}{\underset{\sim}{P}}_f \ddot{\underset{\sim}{q}}_f + \underset{\sim}{P}_\beta = \underset{\sim}{0} \tag{8.4.8}$$

where $\underset{\sim}{P}_\beta$ is given by (Wittenburg [1977, p176]; Baumgarte [1972])

$$\underset{\sim}{P}_\beta = \underset{\sim}{\underset{\sim}{P}}_s \ddot{\underset{\sim}{q}}_s + \underset{\sim}{P}_{oo} + 2\beta(\underset{\sim}{\underset{\sim}{P}}_f \dot{\underset{\sim}{q}}_f + \underset{\sim}{\underset{\sim}{P}}_s \dot{\underset{\sim}{q}}_s + \underset{\sim}{P}_o) + \beta^2 \begin{bmatrix} \underset{\sim}{f} \\ \underset{\sim}{0} \end{bmatrix} \tag{8.4.9}$$

In this equation β must have a low, positive value so that the extra terms do not become dominant. If the constraint equations are satisfied the extra term is zero. If the constraints are violated at some later time step, the augmented equation results in damping the deviations, whereas the original constraint equations magnify these deviations linearly.

8.5 The unknown internal loads

The constraint loads
Loads caused by the hinges

The loads in a multibody system have been divided into inertia and external loads as well as internal loads caused by connections. The inertia and external loads and the generalized loads $\underset{\sim}{Q}_s$ are known after solution of the dynamic simulation problem . Also known are the internal loads caused by energetic and active connections, but the loads caused by the kinematic constraints and the hinges in the tree structure are still unknown. Sometimes we are interested in these

loads, for example in order to determine bearing forces. In this section a method for the calulation of these loads will be discussed. First attention is given to the unknown loads caused by the kinematic constraints. By using the properties of the tree structure and the Newton-Euler laws we finally determine the unknown loads caused by the hinges in the tree structure.

The constraint loads

The kinematic constraints are taken into account in the virtual work principle of d'Alembert by using Lagrange multipliers. As discussed in section 3.3 these multipliers are related to the constraint forces by

$$\underset{\sim}{\Lambda} = \underline{P}_y^T \underset{\sim}{\lambda} \tag{8.5.1}$$

where $\underset{\sim}{\Lambda}$ is defined as

$$\underset{\sim}{\Lambda}^T = [\ (\underset{\sim}{\Lambda}^1)^T\ ..\ (\underset{\sim}{\Lambda}^{nc})^T\ ], \qquad \underset{\sim}{\Lambda}^k = \begin{bmatrix} (\underline{\Psi}^k)^T \underset{\sim}{F}^k_{C,in} \\ (\underline{\Phi}^k)^T \underset{\sim}{M}^k_{C,in} \end{bmatrix} \tag{8.5.2}$$

Here $\underset{\sim}{F}^k_{C,in}$ and $\underset{\sim}{M}^k_{C,in}$ (k=1..nc) are the matrix representation of the force and moment vectors on the body attached to point N^2 of constraint C^k, while $\underline{\Psi}^k$ and $\underline{\Phi}^k$ define the relation between $\delta\underset{\sim}{y}^k$ and $\delta\underset{\sim}{c}^k$, $\delta\underset{\sim}{\Omega}^k$.

There are two problems to be mentioned. To determine the unknown loads caused by the kinematic constraints, all kinematic constraints must be specified as implicit constraints. If, however, one or more constraints are specified as explicit constraints, the corresponding rows in $\underline{P}_y$ are unknown and must be specified additionally. A second problem occurs after $\underset{\sim}{\Lambda}$ has been calculated. To determine the $\underset{\sim}{F}^k_{C,in}$ and $\underset{\sim}{M}^k_{C,in}$ the following two equations must be solved

$$(\underline{\Psi}^k)^T \underset{\sim}{F}^k_{C,in} = \begin{bmatrix} \Lambda^k_1 \\ \Lambda^k_2 \\ \Lambda^k_3 \end{bmatrix}, \qquad (\underline{\Phi}^k)^T \underset{\sim}{M}^k_{C,in} = \begin{bmatrix} \Lambda^k_4 \\ \Lambda^k_5 \\ \Lambda^k_6 \end{bmatrix} \tag{8.5.3}$$

Normally both $\underline{\psi}^k$ and $\underline{\phi}^k$ have an inverse, but in certain situations no inverse exists. Using Bryant angles, the inverse of $\underline{\phi}^k$ does not exist if $\varphi_2^k = \frac{\pi}{2} \pm n\pi$ (n = 0,1,..). For a comprehensive discussion of this problem see Wittenburg [1977, ch 2].

Loads caused by the hinges

We assume as known the attitude coordinates, the first and second derivatives of these coordinates, the inertia and external loads and the internal loads caused by all connections except the hinges in the tree structure.

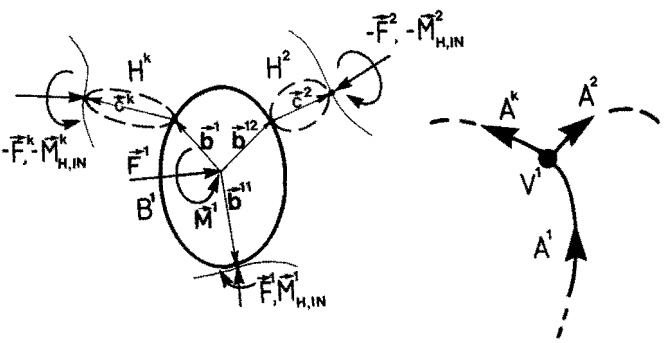


figure 8.4 The 'free-body' diagram for body B^1

Let $\vec{F}^i$ and $\vec{M}^i$ be the resulting known force and moment on body B^i (i=1..nb), defined by

$$\vec{F}^i = \vec{F}^i_{ex} + \vec{F}^i_{in} - \dot{\vec{I}}^i, \qquad \vec{M}^i = \vec{M}^i_{ex} + \vec{M}^i_{in} - \dot{\vec{L}}^i \qquad (8.5.4)$$

and let $\vec{F}^i_{h,in}$ and $\vec{M}^i_{h,in}$ be the unknown force and moment exerted by hinge H^i on body B^i in the regular numbered tree structure. To determine the unknown loads each body is isolated and all loads (known and unknown) on these bodies are introduced as shown in figure 8.4.The unknown loads then follow from the requirement that the Newton-Euler laws must hold for each body. This leads to a set of 2nh = 2nb vector equations for $\vec{F}^i_{h,in}$ and $\vec{M}^i_{h,in}$. In accordance with Wittenburg [1977, p92], these equations can be formulated in matrix notation as:

$$\vec{\underset{\sim}{F}} + \underline{\underline{S}}\vec{\underset{\sim}{F}}_{h,in} = \vec{\underset{\sim}{0}}, \qquad \vec{\underset{\sim}{M}} + (\vec{\underline{B}} + \vec{\underline{C}})^{*}\vec{\underset{\sim}{F}}_{h,in} + \underline{\underline{S}}\vec{\underset{\sim}{M}}_{h,in} = \vec{\underset{\sim}{0}} \qquad (8.5.5)$$

Using the relationship $\underline{\underline{T}}\underline{\underline{S}} = \underline{\underline{I}}$ this finally yields:

$$\vec{\underset{\sim}{F}}_{h,in} = \underline{\underline{T}}\vec{\underset{\sim}{F}}, \qquad \vec{\underset{\sim}{M}}_{h,in} = \underline{\underline{T}}(\vec{\underset{\sim}{M}} - (\vec{\underline{B}} + \vec{\underline{C}})^{*}\vec{\underset{\sim}{F}}_{h,in}) \qquad (8.5.6)$$

These are simple algebraic equations with known right-hand sides.

Summary

The simulation of the kinematic and dynamic behaviour of multibody systems was discussed in this chapter. In the first section the relevant formulas were repeated. Of all formulas derived in the previous chapters only the constraint equations and the equations of motion remain. Before the kinematic simulation problem was formulated, we first defined the number nd of degrees of freedom of a multibody system. For the KSP we have to prescribe nd Lagrange coordinates. Since nd is not constant, care should be taken when simulating the kinematic behaviour. The formulation of the dynamic simulation problem is straightforward. However, several problems can be encountered during the solution of the equations of motion. In this chapter we have given some attention to stiff differential equations and the stabilization of kinematic constraints. Finally the calculation of the unknown internal loads caused by the hinges and the kinematic constraints was studied.

Several numerical solvers and possible failures of these solvers were mentioned in this chapter. Some of these aspects will appear in the next chapter, but systematic experiments with these solvers have not been performed. All comments on the behaviour of these solvers are based on literature studies and some adhoc applications.

CHAPTER 9
THE THEORY APPLIED TO AN EXAMPLE

To demonstrate the theory an example of a multibody system will be studied in this chapter. The example is a fuel injection pump and was chosen as it consists of several nonstandard connections. The pump and, in particular the nonstandard connections, will be described in section 9.1. The simulation of its kinematic and dynamic behaviour is studied in sections 9.2 and 9.3, respectively.

9.1 The description of the fuel injection pump

Introduction

Fuel injection pumps are used to inject fuel at high pressure into the combustion chambers of diesel engines. The pump in figure 9.1, a P-pump of Bosch (Stuttgart), consists of a pumpcasing, six plungers and a driving shaft with six cams. Between each cam and plunger there is a roller with an additional floating ring between roller and plunger. The pumpcasing is partly filled with oil to create a hydrodynamic fluid film between roller, ring and plunger. In addition, the oil also brings about an elastohydrodynamic lubrication film between the cam and the roller.

In this chapter we will analyse the kinematics and dynamics of the pump in general and the lubrication characteristics and the Hertzian stresses between the cam and roller in particular. The main purpose

here is to illustrate the use of the theory of the preceding chapters in a systematic analysis of a multibody system with some nonstandard connections. From this point of view the numerical results finally obtained are of minor importance and are therefore not verified experimentally.

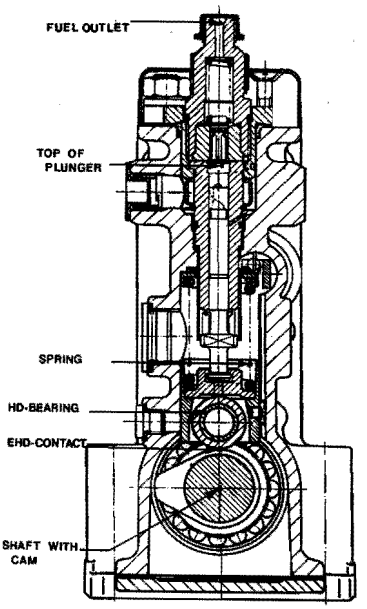


figure 9.1 The pump

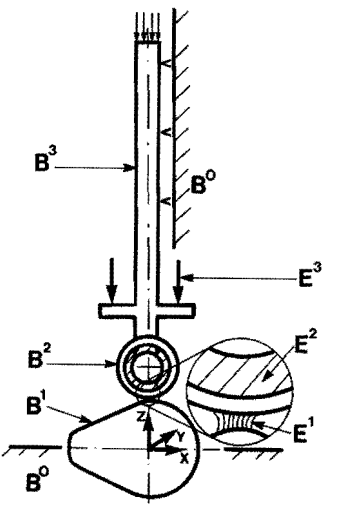


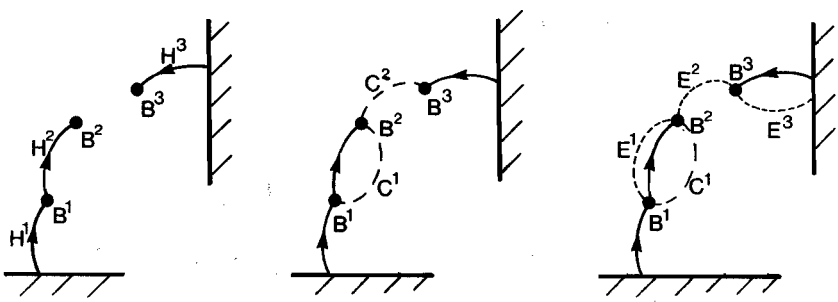
figure 9.2 The model

The multibody model of the pump

The model of the pump in figure 9.1 consists of a fixed world B^0, a driving shaft with a cam B^1, one roller B^2 and one plunger B^3. These bodies are all considered to be rigid. The floating ring and the inertia of the oil are left out of consideration. The model, together with the global base (xyz), is shown in figure 9.2. Only displacements in the xz-plane and rotations along the y-axis will be studied. Therefore three instead of six coordinates are necessary for the description of the (relative) attitudes of the bodies.

We distinguish three kinematic connections: a pin-joint between B^0 and B^1, a connection between B^1 and B^2 and a prismatic joint between

B^3 and B^0. Because of the complicated camprofile, we will model the connection between B^1 and B^2 by a kinematic constraint C^1, while the pin-joint and the prismatic joint are modelled as hinges. In the graph of this system these hinges are represented by the branches H^1 and H^3, while the constraint C^1 is represented by the chord C^1. As mentioned in chapter 4, only the branches (solid lines in figure 9.3a) are considered in the definition of a tree structure in this graph. Because B^2 is an isolated vertex, we introduce between B^1 and B^2 a hinge with three generalized coordinates. This yields the regularly numbered graph with a tree structure of bodies and hinges as shown in figure 9.3a. The first tree consists of B^0, the pin-joint H^1, the shaft B^1, an extra hinge H^2 and the roller B^2. The other tree contains B^0, the prismatic joint H^3 and the plunger B^3.



a) tree b) kinematic simulation c) dynamic simulation

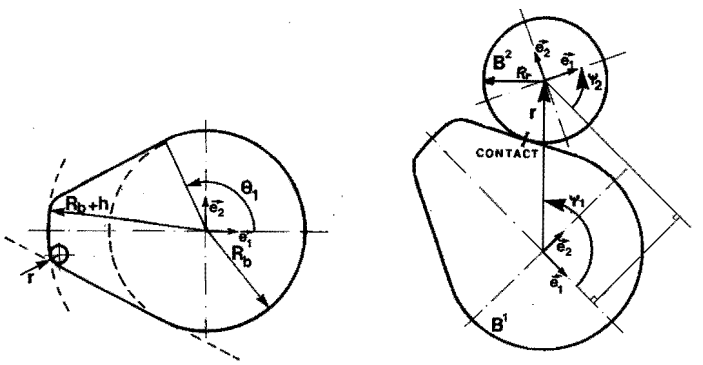
figure 9.3 Graph of the pump

A constraint C^2 is introduced between B^2 and B^3 for the simulation of the kinematic behaviour (figure 9.3b). This constraint must ensure that the roller and the pin at the bottom of the plunger stay in contact. For the simulation of the dynamic behaviour we distinguish three energetic connections: the elastohydrodynamic (EHD) fluid film E^1, the hydrodynamic (HD) fluid film E^2 and the spring E^3 between B^0 and B^3. The HD fluid film E^2 replaces constraint C^2 used in simulation of the kinematic behaviour (see figure 9.3c). Between B^3 and B^0 another energetic connection can be introduced to represent the viscous friction in the prismatic joint. Since the estimated magnitude of this friction force was very small (less then 1 Newton), it was decided to leave this connection out of consideration.

In the following subsections the nonstandard connections, that is the HD fluid film and the cam-roller connection will shortly be described. Only the endpoints and relevant variables of these connections are defined. A comprehensive discussion of the constitutive equations is given in an intern report (Sol [1983]).

The cam-roller connection

The relative motion between cam and roller is described by three generalized coordinates, these being the polar coordinates φ_1 and r of the centre of the roller with respect to the centre of the shaft and the counter-clockwise rotation φ_2 of the roller (see figure 9.4b). The camprofile consists of a basic circle, two tangent lines, a top circle and two small circle segments on the transition from the tangent lines to the top circle.



a) geometry b) kinematics

figure 9.4 The cam

For the simulation of the kinematic behaviour we assume that the cam and roller are in contact and that the roller rolls (without slip) on the cam. The contact condition results in a holonomic constraint of the form

$$f_1(\varphi_1) - r = 0 \qquad (9.1.1)$$

while the assumption of pure rolling is expressed mathematically by a nonholonomic constraint, of the form

$$P_{21}(\varphi_1)\dot{\varphi}_1 + P_{22}(\varphi_1)\dot{r} - R_r\dot{\varphi}_2 = 0 \qquad (9.1.2)$$

For $f_1 = f_1(\varphi_1)$, $P_{21}(\varphi_1)$ and $P_{22}(\varphi_1)$ different expressions apply to different parts of the camprofile. For example, the expressions for the basic circle are given by:

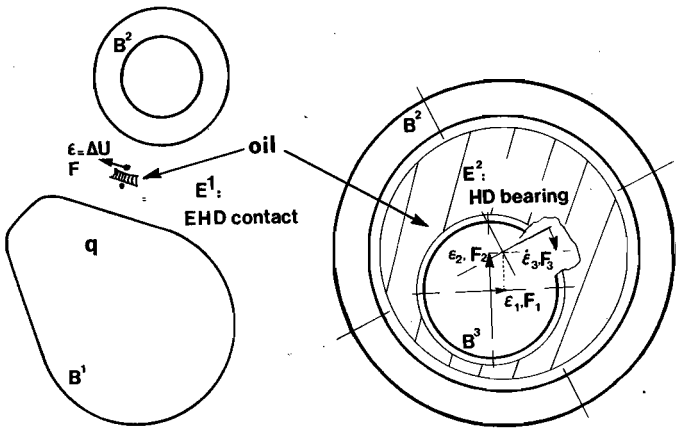
$$f_1 = R_b + R_r, \qquad P_{21} = R_b + R_r, \qquad P_{22} = 0.0 \qquad (9.1.3)$$

where R_b and R_r are the radius of the basic circle and the radius of the roller respectively. The more complicated functions for the tangent lines and the transition segments are not discussed here.

For the simulation of the dynamic behaviour we have to consider the EHD fluid film between cam and roller (Johnson, 1970). In this case there is no direct contact between cam and roller and some slip occurs. As a result we have to drop the nonholonomic constraint (9.1.2). The slip in the EHD fluid film gives rise to a traction force which causes the roller to rotate. This fluid film will be modelled by an energetic connection. The relevant kinematic variable $\dot{\varepsilon}$ of this energetic connection is the slip, while the traction force is the relevant force variable F (see figure 9.5a). The constitutive equation is a relationship of the kind

$$F = f_t\Lambda, \qquad f_t = f_t(\varepsilon, \dot{\varepsilon}, \Lambda) \qquad (9.1.4)$$

where f_t is the friction (also called traction) coefficient and Λ the (a priori) unknown normal load between cam and roller. This load can be determined after the equations of motion have been solved. In section 9.3 some attention will be given to the iterative solution to obtain the unknown load Λ. For the function f_t we use the expression derived by Houpert [1980]. This expression accounts for the nonlinear viscous effects produced in the EHD fluid film.



a) the EHD fluid film b) the HD fluid film

figure 9.5 Relevant variables

Although the EHD fluid film has a certain thickness, the constraint (9.1.1) is not modified. Compared to the displacements in this constraint, the thickness of the film (> 1 μm) is negligible. Finally it should be said that the normal load between cam and roller can become negative. In that case the roller loses contact and the constraint (9.1.1) is no longer active.

The hydrodynamic (HD) fluid film

The HD fluid film between the roller and the plunger is considered as an energetic connection too. Its relevant kinematic variables are the eccentricities in the x and z directions, ε_1 and ε_2, and the angular velocity of the roller, ε_3 (see figure 9.5b). As a result of the rotation of the roller and the eccentricity, the HD fluid film produces a bearing force, denoted by the relevant force variables F_1 and F_2, as well as a (low) viscous friction moment, denoted by F_3. For the constitutive equation of the HD fluid film a relation of the following kind applies

$$\underset{\sim}{F}^2 = \underset{\sim}{F}^2(\underset{\sim}{\varepsilon}^2, \dot{\underset{\sim}{\varepsilon}}^2), \qquad (\underset{\sim}{F}^2)^T = [\, F_1, F_2, F_3] \qquad (\underset{\sim}{\varepsilon}^2)^T = [\, \varepsilon_1, \varepsilon_2, \varepsilon_3] \qquad (9.1.5)$$

In the literature on HD bearings this type of equation is known as the impedance formula [Childs, Moes and van Leeuwen, 1977]. For the constitutive equations for F_1 and F_2 we use the impedance formula for w-bearings of finite length [Moes and Bosma, 1981]. For the determination of the friction moment F_3 a simplified formula is used as suggested by van Leeuwen [1983].

9.2 The simulation of the kinematic behaviour

To simulate the kinematic behaviour we consider the system without the energetic connections. As stated in the previous section, it is assumed that the roller rolls without slip on the cam and the roller stays in contact with the plunger. In this case we assume that the position of the centre of the roller coincides with the position of the pin at the bottom of the plunger. The system consists of three bodies, three hinges, two constraints and two trees.

Topology

The graph of the system is shown in figure 9.3b. From this figure we can determine the location (sub)matrices and the tree matrix. This yields:

$$\underset{\sim}{S}_o = [-1 \quad 0 \quad -1\ ], \quad \underset{\sim}{S}^c_o = [-1 \quad 0\ ], \quad \underline{l}^T = [\ 0 \quad 1 \quad 0\ ] \tag{9.2.1}$$

$$\underset{\sim}{S} = \begin{bmatrix} +1 & -1 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & +1 \end{bmatrix}, \quad \underset{\sim}{S}^c = \begin{bmatrix} +1 & 0 \\ 0 & -1 \\ 0 & +1 \end{bmatrix}, \quad \underline{T} = \begin{bmatrix} +1 & +1 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & +1 \end{bmatrix}$$

Selection of the Lagrange coordinates

Of the three hinges H^1 has one generalized coordinate (φ), H^2 has three (φ_1, r, φ_2) coordinates and H^3 has one (z). As Lagrange coordinates we therefore use

$$\underset{\sim}{q}^T = [q_1, q_2, q_3, q_4, q_5] = [\varphi, \varphi_1, r, \varphi_2, z] \tag{9.2.2}$$

Description of the (elements of) connections

The three hinges are characterized each by the matrix representation of the connection tensor and vector ($\mathbb{C}$, $\vec{c}$) and their derivatives ($\vec{w}$, .., $\vec{v}_{oo}$). For a pin-joint like H^1 and a prismatic joint like H^3 these quantities are given in section 3.3. For hinge H^2 with the Lagrange coordinates $q_2 = \varphi_1$, $q_3 = r$ and $q_4 = \varphi_2$ we obtain:

$$
{}^1\underline{C}^2 = \begin{bmatrix} c_4 & 0 & -s_4 \\ 0 & 1 & 0 \\ s_4 & 0 & c_4 \end{bmatrix}, \quad \vec{w}_2^2 = \vec{o}, \quad \vec{w}_3^2 = \vec{o}, \quad {}^1\underset{\sim}{w}_4^2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{w}_{oo}^2 = \vec{o},
$$

$$
{}^1\underset{\sim}{c}^2 = \begin{bmatrix} q_3 c_2 \\ 0 \\ q_3 s_2 \end{bmatrix}, \quad {}^1\underset{\sim}{v}_2^2 = \begin{bmatrix} -q_3 s_2 \\ 0 \\ q_3 c_2 \end{bmatrix}, \quad {}^1\underset{\sim}{v}_3^2 = \begin{bmatrix} c_2 \\ 0 \\ s_2 \end{bmatrix}, \quad \vec{v}_4^2 = \vec{o},
$$

$$
{}^1\underset{\sim}{v}_{oo}^2 = \begin{bmatrix} -c_2 q_3 \dot{q}_2 \dot{q}_2 - 2 s_2 \dot{q}_2 \dot{q}_3 \\ 0 \\ -s_2 q_3 \dot{q}_2 \dot{q}_2 + 2 c_2 \dot{q}_2 \dot{q}_3 \end{bmatrix} \qquad (9.2.3)
$$

where c_i and s_i stand for $\cos(q_i)$ and $\sin(q_i)$ respectively. The kinematic constraint C^1 has already been mentioned in section 9.1. As functions of the Lagrange coordinates the constraint equations are given by

$$
f_1(q_2) - q_3 = 0, \qquad P_{22}\dot{q}_2 + P_{23}\dot{q}_3 - R_r\dot{q}_4 = 0 \qquad (9.2.4)
$$

The constraint equations for the second constraint, introduced to replace the HD fluid film in the simulation of the kinematic behaviour, become

$$
(\vec{r}^3 - \vec{r}^2)\bullet\vec{e}_x^o = 0 \quad \text{and} \quad (\vec{r}^3 - \vec{r}^2)\bullet\vec{e}_z^o = 0 \qquad (9.2.5)
$$

where $\vec{r}^2$ is the position of the centre of the roller and $\vec{r}^3$ the position of the centre of the pin at the bottom of the plunger.

Kinematics of the tree structure

The next step is to set up the tree matrices $\vec{\underline{W}}\underline{T}$, $\underline{T}^T\vec{\underset{\sim}{W}}_o$, $\underline{T}^T\vec{\underset{\sim}{W}}_{oo}$, and $\vec{\underline{U}}\underline{T}$, $\underline{T}^T\vec{\underset{\sim}{U}}_o$, $\underline{T}^T\vec{\underset{\sim}{U}}_{oo}$ and substitute these matrices in the kinematic formulas

of section 5.2. The $\vec{\underline{UT}}$ matrix is given as an example, both in coordinate free form and in its matrix representation with respect to the global base, yielding

(9.2.6)

$$\vec{\underline{UT}} = \begin{bmatrix} \vec{o} & \vec{w}_1^1 * \vec{c}^2 & \vec{o} \\ \vec{o} & \vec{v}_2^2 & \vec{o} \\ \vec{o} & \vec{v}_3^2 & \vec{o} \\ \vec{o} & \vec{o} & \vec{o} \\ \vec{o} & \vec{o} & \vec{v}_5^3 \end{bmatrix}, \qquad {}^{o}\underline{UT} = \begin{bmatrix} 0\,0\,0 & -s_{12}q_3 & 0 & c_{12}q_3 & 0\,0\,0 \\ 0\,0\,0 & -s_{12}q_3 & 0 & c_{12}q_3 & 0\,0\,0 \\ 0\,0\,0 & c_{12} & 0 & s_{12} & 0\,0\,0 \\ 0\,0\,0 & 0 & 0 & 0 & 0\,0\,0 \\ 0\,0\,0 & 0 & 0 & 0 & 0\,0\,1 \end{bmatrix}$$

where c_{12}, s_{12} denote $\cos(q_1+q_2)$ and $\sin(q_1+q_2)$ respectively. After substitution of the tree matrices we obtain the columns with all the angular velocity vectors, the position vectors etc. For example, for $\vec{w}_{ss}$ and $\vec{r}_{ss}$ this yields

$$\vec{w}_{ss} = \begin{bmatrix} \vec{w}_1^1 \dot{q}_1 \\ \vec{w}_1^1 \dot{q}_1 + \vec{w}_4^2 \dot{q}_4 \\ \vec{o} \end{bmatrix}, \qquad \vec{r}_{ss} = \begin{bmatrix} \vec{o} \\ \vec{c}^2 \\ \vec{b}^{o3} + \vec{c}^3 - \vec{b}^{33} \end{bmatrix} \tag{9.2.7}$$

while their matrix representation with respect to the global base becomes:

$$ {}^{o}w_{ss}^1 = \begin{bmatrix} 0 \\ -\dot{q}_1 \\ 0 \end{bmatrix}, \quad {}^{o}w_{ss}^2 = \begin{bmatrix} 0 \\ -\dot{q}_1 - \dot{q}_4 \\ 0 \end{bmatrix}, \quad {}^{o}w_{ss}^3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{9.2.8}$$

$$ {}^{o}r_{ss}^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad {}^{o}r_{ss}^2 = \begin{bmatrix} c_{12}q_3 \\ 0 \\ s_{12}q_3 \end{bmatrix}, \quad {}^{o}r_{ss}^3 = \begin{bmatrix} 0 \\ 0 \\ b_o - b_3 + q_5 \end{bmatrix} $$

where b_o and b_3 are the z-coordinates of the matrix representations of $\vec{b}^{o3}$ and $\vec{b}^{33}$ in the global base.

Kinematic constraints

The implicit constraint equations can be rewritten as explicit equations. The constraint equations (9.2.4) are already expressed as

functions of the Lagrange coordinates and therefore only the constraint equations (9.2.5) have to be rewritten. For these constraint equations we obtain:

$$c_{12}q_3 = 0, \qquad s_{12}q_3 - b_o + b_3 - q_5 = 0 \tag{9.2.9}$$

The holonomic constraint equations can be written in matrix form. This yields

$$\underset{\sim}{f}(q,t) = \begin{bmatrix} R_1(q_2) - q_3 \\ c_{12}q_3 \\ s_{12}q_3 - b_o + b_3 - q_5 \end{bmatrix} = \underset{\sim}{o} \tag{9.2.10}$$

Differentiation with respect to time furnishes us with a relation of the form

$$\underline{P}_h\dot{\underset{\sim}{q}} + \underset{\sim}{P}_{oh} = \underset{\sim}{o}, \qquad \underline{P}_h\ddot{\underset{\sim}{q}} + \underset{\sim}{P}_{ooh} = \underset{\sim}{o} \tag{9.2.11}$$

where $\underset{\sim}{P}_{oh} = \underset{\sim}{o}$ and $\underline{P}_h$ and $\underset{\sim}{P}_{ooh}$ are given by

$$\underline{P}_h = \begin{bmatrix} 0 & P_{12} & -1 & 0 & 0 \\ UT_{14} & UT_{24} & UT_{34} & 0 & 0 \\ UT_{16} & UT_{26} & UT_{36} & 0 & -1 \end{bmatrix}, \qquad \underline{P}_{ooh} = \begin{bmatrix} P_{oo1} \\ TU_{oo4} \\ TU_{oo6} \end{bmatrix}$$

The terms P_{12} and P_{oo1} are derived form (9.1.1) and the terms UT_{14}, .., UT_{oo6} are components of $\underline{\vec{U}}\vec{T}$ and $\underline{T}^T\vec{\underset{\sim}{U}}_{oo}$. Furthermore, the nonholonomic constraint equations in (9.2.4) can be written in matrix form as

$$\underline{P}_n\dot{\underset{\sim}{q}} + \underset{\sim}{P}_{on} = \underset{\sim}{o}, \qquad \underline{P}_n\ddot{\underset{\sim}{q}} + \underset{\sim}{P}_{oon} = \underset{\sim}{o} \tag{9.2.12}$$

where $\underset{\sim}{P}_{on} = \underset{\sim}{o}$ and $\underline{P}_n$ and $\underset{\sim}{P}_{oon}$ are given by

$$\underline{P}_n = [\ 0 \quad P_{22} \quad P_{23} \quad -R_r \quad 0\ ], \qquad \underline{P}_{oon} = [\ P_{oo2}\ ]$$

The terms P_{22}, P_{23} and P_{oo2} derived from (9.1.2)

Formulation of the kinematic simulation problem

The simulation of the kinematic behaviour includes the determination of the trajectories of the positions, orientations, velocities and

accelerations of the bodies. The number of degrees of freedom, nd, is found as

$$nd = nq - \text{rank}(\underline{P}_h) = 5 - 3 = 2 \tag{9.2.13}$$

With q_1 representing the prescribed rotation of the driving shaft, we still must prescribe one of the other Lagrange coordinates. For this we select the fourth Lagrange coordinate, that is the rotation of the roller. As a result of this choice the columns $\underline{q}_s$ and $\underline{q}_f$ are defined by

$$\underline{q}_s^T = [\, q_1,\ q_4 \,], \qquad \underline{q}_f^T = [\, q_2,\ q_3,\ q_5 \,] \tag{9.2.14}$$

However, there is the nonholonomic constraint for pure rolling, resulting in a Pfaff matrix $\underline{P}_n$ of rank one. This implies that only nd - nnp = 1 component of $\dot{\underline{q}}_s$ and $\ddot{\underline{q}}_s$ may be prescribed. For this we choose $\dot{q}_1$ and $\ddot{q}_1$, while the derivatives $\dot{q}_4$ and $\ddot{q}_4$ follow from (8.3.6). Thus the value of q_4 at a certain time remains undetermined. Since we are interested in the trajectory of q_4, we will integrate $\dot{q}_4$ in order to obtain the trajectory of q_4. Only at the origin time t_o must a value for q_4 be specified.

For the determination of the trajectories of the Lagrange coordinates, also called the KSP/0, we must solve for a given q_1 the values for the coordinates of $\underline{q}_f$ from the constraint equations (9.2.10). Instead of solving these equations numerically, we can rewrite the constraint equations in this special case in an explicit form. Since $q_3 \neq 0$, the second constraint equation yields $\cos(q_1+q_2) = 0$. This implies that $q_2 = \frac{\pi}{2} + n\pi - q_1$ (integer n). Substitution of this result in the third constraint equation yields $q_3 - b_o + b_3 - q_5 = 0$. Instead of (9.2.10) we can therefore formulate the KSP/0 as

$$\begin{aligned} q_2 &= 90^o - q_1 \\ q_3 &= R_1(q_2) \\ q_5 &= q_3 - b_o + b_3 \end{aligned} \tag{9.2.15}$$

The coordinate q_4 does not appear in this set of equations. As stated above, the value for this Lagrange coordinate is determined by integration.

To formulate the KSP/1, /2, that is to determine the derivatives of the Lagrange coordinates $\underset{\sim}{q}_f$, we must set up the matrices $\underline{P}_{hf}$ and $\underline{P}_{hs}$ (see 8.3.5). In this case these matrices consist of the second, third and fifth column and the first and fourth column of the matrix $\underline{P}_h$. The submatrix $\underline{P}_{hf}$ can easily be inverted analytically. With the abbreviations $\underset{\sim}{P}_1$ and $\underset{\sim}{P}_2$ given by

$$\begin{aligned}(\underset{\sim}{P}_1)^T &= [0,\ UT_{14}\dot{q}_1,\ UT_{16}\dot{q}_1,\ 0] \\ (\underset{\sim}{P}_2)^T &= [P_{100},\ UT_{14}\ddot{q}_1 + TU_{004},\ UT_{16}\ddot{q}_1 + TU_{006},\ P_{002}]\end{aligned} \tag{9.2.16}$$

the KSP/1 and KSP/2, as formulated in section 8.3, become

$$\begin{aligned}\dot{q}_2 &= -UT_{14}\dot{q}_1/(UT_{24} + UT_{34}P_{12}) \\ \dot{q}_3 &= P_{12}\dot{q}_2 \\ \dot{q}_4 &= (P_{22}\dot{q}_2 + P_{23}\dot{q}_3)/R_r \\ \dot{q}_5 &= UT_{16}\dot{q}_1 + UT_{26}\dot{q}_2 + UT_{36}\dot{q}_3\end{aligned} \tag{9.2.17}$$

and

$$\begin{aligned}\ddot{q}_2 &= -(UT_{14}\ddot{q}_1 + TU_{004} + UT_{34}P_{001})/(UT_{24} + UT_{34}P_{12}) \\ \ddot{q}_3 &= P_{12}\ddot{q}_2 + P_{001} \\ \ddot{q}_4 &= (P_{22}\ddot{q}_2 + P_{23}\ddot{q}_3 + P_{002})/R_r \\ \ddot{q}_5 &= UT_{16}\ddot{q}_1 + TU_{006} + UT_{26}\ddot{q}_2 + UT_{36}\ddot{q}_3\end{aligned} \tag{9.2.18}$$

The equations for $\dot{q}_4$ and $\ddot{q}_4$ are obtained from the (linear) nonholomic constraint equation itself. Using (8.3.6) results in the same equation, but now formulated in $\dot{q}_1$ and $\ddot{q}_1$.

Solution of the kinematic simulation problem

The equations (9.2.13-16) have been programmed so that the solution for $\underset{\sim}{q}$, $\dot{\underset{\sim}{q}}$ and $\ddot{\underset{\sim}{q}}$ is obtained each time for a prescribed rotation of the driving shaft. The results of a simulation with $\dot{q}_1 = -104.7$ rad/sec (1000 rpm) is shown in figures 9.6 and 9.7.

In these figures the results for q_4, that is the rotation of the roller, and q_5, the displacement of the plunger, as well as their derivatives are shown for q_1, the angle of the cam, ranging from 100^o

(t = 0 msec, just before the beginning of one of the tangent lines) to 260° (t = 27 msec, just after returning to the basic circle).

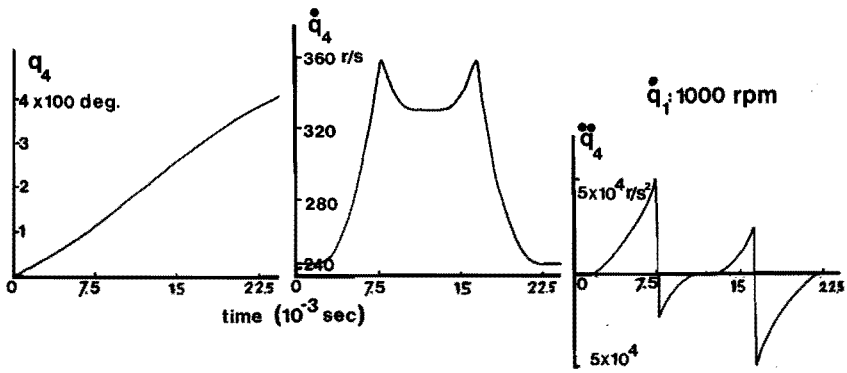


figure 9.6 Trajectories of q_4 and its derivatives

From figure 9.6 it follows that the angular velocity of the roll, $\dot{q}_4$, increases as soon as the plunger is lifted. From an angular velocity of 245 r/s on the basic circle to a value of 332 r/s at the top of the cam, the angular velocity reaches a maximum value of 360 r/s. This value is attained at the moment the camprofile changes from the tangent lines to the transition circle segments.

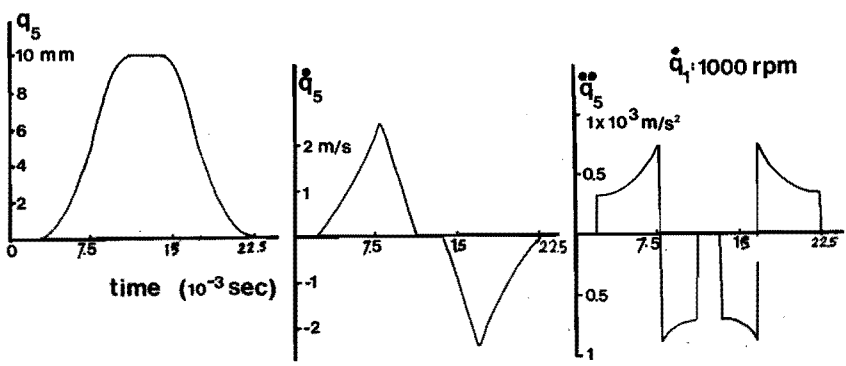


figure 9.7 Trajectories of q_5 and its derivatives

The trajectory for q_5 illustrates the camprofile. The changes from the basic circle to the tangent lines, to the transition circle segments and to the top circle are clearly illustrated by the trajectories of the derivatives of q_5. The deceleration of the plunger on the

transition circles is higher than the acceleration on the tangent lines. By chosing another camprofile these peaks may be reduced.

9.3 The simulation of the dynamic behaviour

The equations of motion
Two serious problems
The results

Simulation of the dynamic behaviour is of particular interest for the analysis of the lubrication processes and the Hertzian stresses in the contact between cam and roller. However, simulation of the dynamic behaviour is more difficult than of the kinematic behaviour. First of all we have to deal with the normal load between cam and roller unknown a priori, second, the displacements in the HD fluid film are extremely slight compared to the other displacements. The first problem requires an iterative solver which must be used each time the equations of motions are evaluated. The second problem restricts the length of the time step significantly and requires a special integration scheme suitable for stiff differential equations.

The equations of motion

Compared to the kinematic simulation problem we drop the constraint for pure rolling as well as constraint C^2 for the relative position of the centre of the roller and the pin of the plunger. These kinematic connections are replaced by the elastohydrodynamic (EHD) fluid film and the hydrodynamic (HD) fluid film. As mentioned in section 9.1, we denote these energetic connections by E^1 and E^2 respectively. Only the kinematic constraint for contact between cam and roller remains. This constraint results in the Pfaff matrices given below

$$\underline{P}_s = \underline{0} = [\ 0\ ], \qquad \underline{P}_f = [\ P_{12},\ -1,\ 0,\ 0\ ] \tag{9.3.1}$$

and implies the following partition of the Lagrange coordinates

$$\underline{q}_s^T = [\ q_1\ ], \qquad \underline{q}_f^T = [\ q_2,\ q_3,\ q_4,\ q_5\ ] \tag{9.3.2}$$

For the equations of motion we have to determine the generalized mass matrix $\underline{A}$ and the generalized loads Q_{oo}, Q_{ex} and Q_{in}. The mass and inertia tensors of the bodies are easy to determine. $\underline{A}$ and Q_{oo} are formulated with the known matrices of the tree structure. After partition of $\underline{A}$ and Q_{oo} we obtain:

(9.3.3)

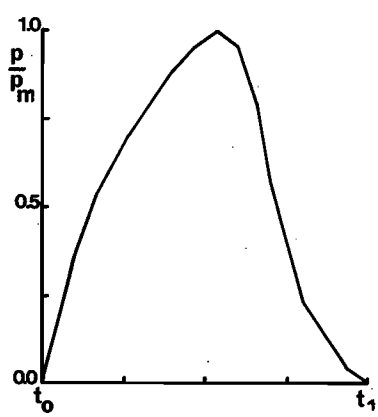
$$\underline{A}_{ss} = m_2(q_3)^2 + J_y^1 + J_y^2, \quad \underline{A}_{fs}^T = \underline{A}_{sf} = [\, m_2(q_3)^2,\ 0,\ J_y^2,\ 0\,]$$

$$\underline{A}_{ff} = \begin{bmatrix} m_2(q_3)^2 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & J_y^2 & 0 \\ 0 & 0 & 0 & m_3 \end{bmatrix}, \quad Q_{oo,f} = \begin{bmatrix} -2m_2 q_3 \dot{q}_3 (\dot{q}_1 + \dot{q}_2) \\ m_2 q_3 (\dot{q}_1 + \dot{q}_2)^2 \\ 0 \\ 0 \end{bmatrix}$$

The generalized external load Q_{ex} is completely determined by the pressure p on top of the plunger. From $\vec{F}_{ex}^T = [\, \vec{o},\ \vec{o},\ \vec{p}\,]$ and $\vec{M}_{ex} = \vec{o}$ it follows that

$$Q_{ex}^T = [0,\ 0,\ 0,\ 0, -A_p p] \tag{9.3.4}$$

where A_p is the area on top of the plunger. The experimentally determined pressure $p = p(q_1(t))$ is given as a function of the angle of the driving shaft and is therefore known as a function of time. A normed pressure curve is shown in figure 9.8



p_m = max. pressure

t_o = starting time
t_1 = final time

figure 9.8 Normed pressure curve

The actual curve starts (point t_o of normed curve) as soon as the angle q_1 of the driving shaft reaches a value of approximately -148^o. A few degrees later, somewhere between -156^o and -153^o, the actual curve ends (point t_1 of normed curve). The starting point, final point and the maximum are determined as functions of the angular velocity $\dot{q}_1$ (500 - 1500 rpm) and a control variable for the fuel inlet (i_e= 9 - 15 mm). The function p = p(t) is chosen so that the experimental data fits as accurately as possible.

From the relation $Q_{in} = (\vec{\underline{U}}\underline{T})\bullet\vec{F}_{in} + (\vec{\underline{W}}\underline{T})\bullet\vec{M}_{in}$ it follows that the subcolumn $Q_{in,f}$ depends on the internal load vectors $\vec{F}^2_{in}$, $\vec{M}^2_{in}$ and $\vec{F}^3_{in}$ only. These internal loads on the bodies arise out of the energetic connections E^1, E^2 and E^3. The positive directions of these loads are shown in figure 9.9. Note that the direction of $\vec{F}^1$ is determined by the normal vector of the cam profile at the contact point. The direction of this normal vector is a function of the cam angle q_2. On the basic and top circle segments the angles α and q_2 are equal, on the tangent lines and the transition circle segments these angles differ.

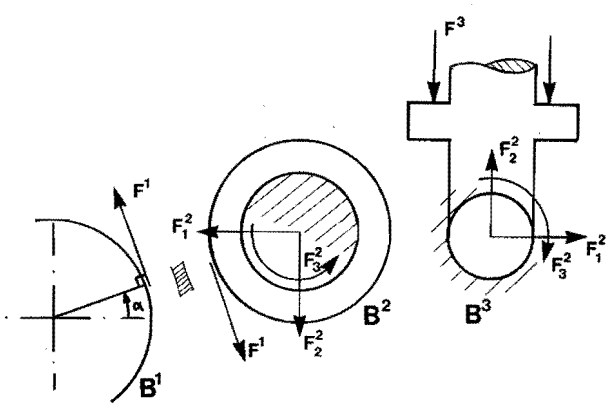


figure 9.9 The positive directions of the internal loads

From these arrangements, one can derive the generalized internal load $Q_{in,f}$ as

$$\underset{\sim}{Q}_{in,f} = \begin{bmatrix} q_3(-F^1 c_{\alpha 2} + F_1^2 s_{12} + F_2^2 c_{12}) \\ F^1 s_{\alpha 2} + F_1^2 c_{12} - F_2^2 s_{12} \\ R_2 F^1 + F_3^2 \\ F_2^2 - k_3(q_5 - l_3) \end{bmatrix} \qquad (9.3.5)$$

where $c_{\alpha 2} = \cos(\alpha - q_2)$ and $s_{\alpha 2} = \sin(\alpha - q_2)$.

For the numerical solution of the equations of motion, the constraint stabilization technique with the Baumgarte constant was used (see section 8.4). The corresponding Pfaff term yields

$$\underset{\sim}{P}_{\beta} = P_{100} + 2\beta\{P_{12}\dot{q}_2 - \dot{q}_3\} + \beta^2\{R_1(q_2) - q_3\} \qquad (9.3.6)$$

Based on these results, the final equations of motion become:

$$\lambda = \{m_2(q_3)^2(P_{12}\ddot{q}_1 - P_\beta) + (q_3)^2 Q_3 - P_{12}Q_2\}/\{(P_{12})^2 + (q_3)^2\}$$

$$\begin{aligned} \ddot{q}_2 &= -\ddot{q}_1 + (P_{12}\lambda + Q_2)/m_2(q_3)^2 \\ \ddot{q}_3 &= (Q_3 - \lambda)/m_2 \\ \ddot{q}_4 &= -\ddot{q}_1 + Q_4/J_y^2 \\ \ddot{q}_5 &= Q_5/m_3 \end{aligned} \qquad (9.3.7)$$

$$Q_1 = \{m_2(q_3)^2 + J_y^1 + J_y^2\}\ddot{q}_1 + m_2(q_3)^2\ddot{q}_2 + J_y^2\ddot{q}_4$$

The equations of motion as described above can be solved numerically. Before any results are shown, some problems must be mentioned.

Two serious problems

The value of the traction force F^1 depends on the traction coefficient f and the value of the normal load between cam and roller, indicated by Λ. The relationship is given by

$$F^1 = f(\Lambda)\Lambda \qquad (9.3.8)$$

The normal load Λ is an internal load arising out of the kinematic constraint. As discussed in section 8.5 the normal load Λ and the Lagrange multiplier λ are related to each other. In this case the normal load and the internal load caused by the constraint have opposite directions

$$\lambda = -\Lambda \tag{9.3.9}$$

The line of action of both loads intersects the contact point and the centre of the roller. From this property and the relationships

$$Q_2 = Q_2(F_1(\lambda)), \qquad Q_3 = Q_3(F_1(\lambda)) \tag{9.3.10}$$

the equation for the Lagrange multiplier λ becomes an implicit, nonlinear algebraic equation of the kind

$$\lambda + \{m_2(q_3)^2(P_\beta - P_{12}\ddot{q}_1) + P_{12}Q_2(\lambda) - (q_3)^2Q_3(\lambda)\}/\{(P_1)^2 + (q_3)^2\} = 0 \tag{9.3.11}$$

Since this equation must be solved each time the equations of motion are evaluated, it was decided to use a fast converging Newton solver. This solver requires the partial derivatives of Q_2 and Q_3 with respect to λ, given by:

$$\frac{\partial Q_2}{\partial \lambda} = -(\lambda + \lambda\frac{\partial f(\lambda)}{\partial \lambda})c_{\alpha 2}, \qquad \frac{\partial Q_3}{\partial \lambda} = -(\lambda + \lambda\frac{\partial f(\lambda)}{\partial \lambda})s_{\alpha 2} \tag{9.3.12}$$

The derivative of the friction coefficient $f(\lambda)$ with respect to λ was obtained by differentiation of the expression of Houpert.

Another problem is caused by the slight displacements in the HD bearing. During the solution process, when the plunger and roller are raised 10 mm, displacements in the HD fluid film are of order of 2 μm. Since slight displacements in this fluid film introduce large hydrodynamic forces, we are confronted with a set of stiff differential equations. To solve these equations a Gear solver has been used (subroutine D02EAF of the NAG library [NAG, 1981]). Again because of the great difference in displacements, all calculations have been done with double precision and finally with an accuracy of 10^{-7}.

The results

The results discussed in this subsection apply to a pump with a fuel inlet i_e = 15 mm and $\dot{q}_1$ = -104.7 rad/s (1000 rpm). It is assumed that the radial clearance in the HD fluid film has a value of 6 μm, which implies a radius/clearance ratio of 1000. For the oil parameters we have used data for a common engine oil. At a temperature of 90° C these parameters have the following values:

$$\alpha = 1.85\ 10^{-8}\ Pa^{-1},\quad \tau_o = 5.0\ 10^{6}\ Pa,\quad \eta_o = 1.7\ 10^{-2}\ Pa\ s$$

All values have been scaled to 10^{-3} metres, 10^{-2} sec and 1 kg to achieve that during the solution of the differential equations all variables attain values in the neigbourhood of 1. For the stabilization parameter β, occuring in the constraint equation, a value of .001 gave the best results.

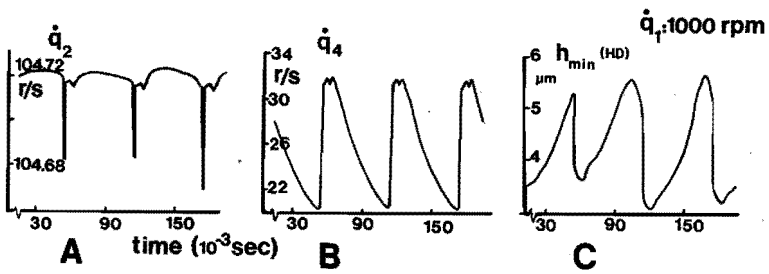


figure 9.10 The trajectories of $\dot{q}_2$, $\dot{q}_4$ and h_{min} at the HD bearing

In figure 9.10 $\dot{q}_2$, $\dot{q}_4$ and thickness h_{min} of the HD fluid film are shown for three revolutions of the cam. From figure 9.10a it can be seen that the initial values for $\dot{q}_2$ (and q_2) have to be very accurate (5 digits). The three spikes are caused by the applied fuel pressure on top of the plunger. When figure 9.10b for $\dot{q}_4$ is compared to figure 9.6, it can be concluded that the roller slips on the cam. At top of the cam this slip approaches a value of 33% of the average velocity of the EHD fluid film. From a cam angle of -145° to 0° the slip is in

the opposite direction in order to decelerate the rotation of the roller. The thickness in the HD fluid film is shown in figure 9.10c. The irregularity is caused by the transient response. It is of interest to note that the eccentricity ratio in this HD bearing is rather slight (10% .. 50%). This is an undesired situation since it results in less stability of the bearing. If the radius/clearance ratio is decreased from 1000 to 500 the eccentricity ratio is increased to 50% .. 75%.

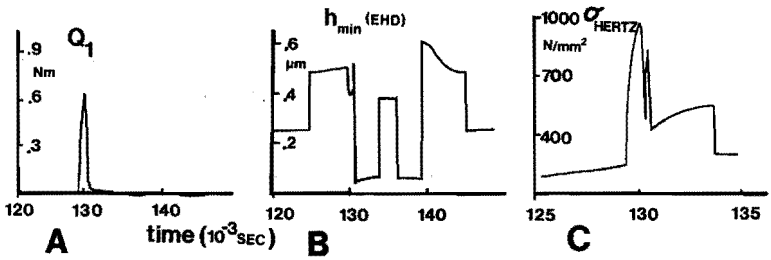


figure 9.11 The trajectories of Q_1, h_{min} at EHD and Hertzian stress

In figures 9.11a and 9.11b some results are shown for a cam angle between 100^o and 260^o degrees, while figure 9.10c gives the Hertzian stress for a cam angle between 100^o and 180^o. The load Q_1 (figure 9.11a) is the load required at the driving shaft in order to realise a constant angular rotation. As can be seen from figure 9.11b the EHD fluid film thickness has an average value of .26 µm, while its minimum value, obtained on the transition circle segments, is .06 µm. The latter value implies that the surfaces of the cam and roller should be very smooth, so as to avoid metalic contact. In figure 9.11c the Hertzian stress in the contact between the cam and roller is shown. The value of 975 N/mm^2 is a reasonable one for a line contact. The additional (second) spike is caused by the change in curvature at the intersection of the tangent line and the transition circle.

Summary

In this chapter the behaviour of a simple multibody system with some nonstandard connections is studied. The example, a fuel injection pump, is described in the first section. This section also describes the nonstandard connections, that are the cam, the EHD and the HD fluid film. The kinematic behaviour is studied in the second section. After deriving the constraint equations, they can be solved analytically. The construction of the equations of motion is as straightforward as that of the constraint equations. The solution, on the other hand has required more effort. Because of the difference between slight displacements in the HD fluid film and large displacements of the plunger, we had to cope with stiff differential equations, while the inclusion of friction introduced a nonlinear algebraic equation.

CHAPTER 10
SUMMARY AND CONCLUSIONS

In the simulation of kinematic and dynamic behaviour of multibody systems the study describes an approach. As stated in chapter 1 it is the purpose of the study to develop a multibody theory which allows us to model connections of arbitrary geometry and/or complex constitutive behaviour. After a literature survey we have concluded that no such theory is available, but that the work of Wittenburg provides a suitable approach to develop such a theory.

To develop a multibody theory one requires a suitable mathematical notation to deal with the complicated equations. A notation with matrices of vectors and tensors is therefore introduced in the appendix and its use is illustrated in chapter 2. In the subsequent chapters this notation has been found to be useful.

A systematic approach is required to set up the equations by means of a computer program. Two concepts are introduced to achieve this goal. The concept of elements of connections is introduced to handle complex connections. These elements are characterized by their endpoint variables and constitutive equations. It is possible to define elements, store them in a library of elements and model complex connections as a set of elements. This approach, which is based on the finite element technique, allows us to model geometrically complex connections, such as human ligaments and muscles.

The second concept is the description of the topology of a multibody system by means of a tree structure of bodies and hinges. In chapter 4 it is shown that it is allways possible to define such a tree structure in a multibody system. This tree structure is used in the following chapter to define the Lagrange coordinates, these being the generalized coordinates of the hinges of the tree. By using this tree structure we can formulate the equations describing the kinematics of multibody systems with open and closed kinematic chains as well as holonomic and nonholonomic constraints. It is shown that it is also

possible to automatically transform implicit constraint equations into explicit constraint equations. The equations of motion are derived by using the virtual work principle of d'Alembert. These equations apply to systems with open and closed kinematic chains, with prescribed external loads as well as prescribed kinematic variables and active connections, characterized by external input variables.

Before the equations for the kinematic and dynamic simulation problem are formulated, we give some attention to the number of degrees of freedom of a multibody system. In general this number is not constant. If it changes, special precautions should be taken to avoid problems with the numerical solution process. The kinematic simulation problem generally results in a set of implicit, nonlinear algebraic equations which can be solved with Newton-type solvers. To solve the dynamic simulation problem, the equations of motion and eventually the state equations of active elements must be integrated. This results in the trajectories of the Lagrange coordinates as functions of time. As soon as these trajectories are obtained, we can determine the unknown internal loads in the kinematic connections (bearing loads, etc.).

Finally, the theory is illustrated by the analysis of the behaviour of a fuel injection pump. The example shows that the theory allows us to derive the equations systematically. The constitutive equations of the connections in this example are derived independently of the rest of the system. In this case it is possible to manipulate the constraint equations into a set of explicit algebraic equations. The simulation of the dynamic behaviour requires the solution of a set of stiff differential equations. In this study the numerical results are not checked experimentally.

Subjects of further research

The theory as presented here is primarily meant as a set of specifications for a general-purpose program for the analysis of complicated multibody systems. A general-purpose program, in addition to the selection of appropriate numerical solvers, requires a well designed

(engineering) data base to store and retrieve the data and extensive graphic facilities. The development of a general-purpose program which allows the use of arbitrary connections was not possible in the time available for this study .

However, as long as no general-purpose program is available, the theory can serve as a basis for special-purpose programs or be used in symbolic manipulation programs. The results shown in chapter 9 were obtained with a special-purpose program. Such special-purpose programs can also be developed for the design of industrial robots or biomechanical studies of the musculo-skeletal system.

Besides the development of a general-purpose program, other topics of further research have to do with extensions to the presented theory. Examples of extensions are the introduction of flexible bodies, contact and impact problems and the construction of partial derivatives required for integration schemes for stiff differential equations. The optimization of the desired behaviour of multibody systems is of great importance, especially for the design of new systems like such as spacecrafts, robots, wheel suspensions etc. Multibody programs with optimization facilities are also required for more realistic studies of the musculo-skeletal system. At present optimization studies require many simulations in order to find an optimum in a heuristic way. It may be expected that a subsequent study will concentrate on optimization problems.

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APPENDIX
MATHEMATICAL NOTATION

The purpose of this appendix is to explain the notations, names and operations used in this study. The use of symbols and some typographical arrangements for these symbols are dealt with first. Next we discuss some operation rules for vectors and tensors, introduce matrices of scalars, vectors and tensors and present the rules for operations on such matrices. Finally we discuss the matrix representation of vectors and tensors with regard to a vector base.

A.1 The use of symbols

The symbol for a quantity is in general the first letter of the name of that quantity. A scalar quantity will be indicated by a simple character, a vector by a character with an arrow on top of it and a tensor by a shadowed capital. For example,

$$i,\ j;\quad \vec{v},\ \vec{F};\quad \mathbb{B},\ \mathbb{R} \tag{A.1}$$

A superscripted index on the right-hand side of a symbol will generally mean a number, for example body B^i. A subscripted index on the right is generally used to indicate a subset of a certain type. For example, the forces on a body exerted at the points O (origin of vector base) or M (centre of mass) are indicated as $\vec{F}_o$ or $\vec{F}_m$. This is not a strict arrangement since components of columns and matrices are also indicated by subscripted indices. If we are dealing with the matrix representation of a vector or tensor (see section A.4) a superscripted index on the left of the symbol is used. For example, the

matrix representation of a force $\vec{F}$ with regard to a vector base $\vec{e}^{\,o}$ is given by ${}^{o}\underset{\sim}{F}$.

A.2 Vectors and tensors

We use vectors as abstract quantities in the Euclidian space S^3. The notions of distance and angle are defined in S^3. Several operations are defined for vectors. Well known operation rules are the addition $\vec{v} + \vec{u}$, the scalar multiplication $\alpha\vec{v}$, the inproduct $\vec{v}\cdot\vec{u}$ and the vector product $\vec{v}*\vec{u}$. The tensor product $\vec{v}\vec{u}$ is defined as

$$(\vec{v}\vec{u})\cdot\vec{w} = \vec{v}(\vec{u}\cdot\vec{w}) \qquad \forall\ \vec{w} \in S^3 \tag{A.2}$$

and has, for example, the following properties:

$$\begin{aligned} \vec{v}(\alpha\vec{u}) &= \alpha(\vec{v}\vec{u}) && \forall\ \alpha \in R \\ \vec{v}(\vec{u} + \vec{w}) &= \vec{v}\vec{u} + \vec{v}\vec{w} && \forall\ \vec{v},\ \vec{u} \text{ and } \vec{w} \in S^3 \end{aligned} \tag{A.3}$$

A (second order) tensor $\mathbb{B}$ is a linear transformation that maps each vector $\vec{u}$ onto a vector $\vec{v}$. This mapping is called the inproduct of $\mathbb{B}$ and $\vec{u}$ and is written as

$$\vec{v} = \mathbb{B}\cdot\vec{u} \qquad \vec{v},\ \vec{u} \in S^3 \tag{A.4}$$

Some operation rules for tensors are:

$$\begin{aligned} \mathbb{B}\cdot(\vec{v} + \vec{u}) &= \mathbb{B}\cdot\vec{v} + \mathbb{B}\cdot\vec{u}, & \alpha\mathbb{B}\cdot\vec{v} &= \mathbb{B}\cdot(\alpha\vec{v}) \\ (\mathbb{B} + \mathbb{C})\cdot\vec{v} &= \mathbb{B}\cdot\vec{v} + \mathbb{C}\cdot\vec{v}, & \mathbb{B}\cdot(\mathbb{C}\cdot\vec{v}) &= (\mathbb{B}\cdot\mathbb{C})\cdot\vec{v} \end{aligned} \tag{A.5}$$

The identity tensor $\mathbb{I}$ and the zero tensor $\mathbb{O}$ are defined by:

$$\mathbb{I}\cdot\vec{v} = \vec{v}, \qquad \mathbb{O}\cdot\vec{v} = \vec{o} \qquad \forall\ \vec{v} \in S^3 \tag{A.6}$$

The transpose $\mathbb{B}^T$ of a tensor $\mathbb{B}$ is defined by the requirement that $\vec{v}\cdot(\mathbb{B}\cdot\vec{u}) = \vec{u}\cdot(\mathbb{B}^T\cdot\vec{v})$ holds for each $\vec{u}$ and $\vec{v}$. The tensor $\mathbb{B}$ is orthogonal if $\mathbb{B}\cdot\mathbb{B}^T = \mathbb{I}$. If $\mathbb{B} = \mathbb{B}^T$, resp. $\mathbb{B} = -\mathbb{B}^T$ we say that $\mathbb{B}$ is symmetric resp. skew symmetric. The inverse of a tensor is defined by $\mathbb{B}\cdot\mathbb{B}^{-1} = \mathbb{I}$, at least if $\mathbb{B}$ has an inverse at all.

A.3 Columns and matrices

Sets of scalar quantities, vectors and tensors can be stored in matrices. In order to improve the readability of formulas we use the notations $\underset{\sim}{A}$ and $\underline{A}$ to indicate a one-dimensional matrix or column with components A_i ($i \in 1..ni$) and a two-dimensional matrix with components A_{ij} ($i \in 1..ni$, $j \in 1..nj$):

$$\underset{\sim}{A} = \begin{bmatrix} A_1 \\ \vdots \\ A_{ni} \end{bmatrix}, \qquad \underline{A} = \begin{bmatrix} A_{11} & .. & A_{1nj} \\ \vdots & & \vdots \\ A_{ni1} & .. & A_{ninj} \end{bmatrix} \tag{A.7}$$

For columns and matrices with scalars, the operation rules are well known. For example, for the transposed $\underline{A}^T$ of a matrix $\underline{A}$ holds

$$(\underline{A}^T)_{ij} = A_{ji} \qquad i = 1..ni, \quad j = 1..nj \tag{A.8}$$

In addition to operation rules for scalar matrices we define operation rules for matrices with vectors. The product $\underline{\alpha}\vec{\underline{U}}$ of a scalar matrix $\underline{\alpha}$ of order ni×nk and a matrix $\vec{\underline{U}}$ of order nk×nj with vectors as components yields a matrix $\vec{\underline{V}}$ of order n×m with components $\vec{V}_{ij}$ in which

$$\vec{V}_{ij} = \sum_{k=1}^{nk} \alpha_{ik} \vec{U}_{kj} \qquad i = 1..ni, \; j = 1..nj \tag{A.9}$$

The inproduct $\vec{\underline{V}} \bullet \vec{\underline{U}}$ of two matrices $\vec{\underline{V}}$ (of order ni×nk) and $\vec{\underline{U}}$ (of order nk×nj) yields a scalar matrix $\underline{W}$ of order ni×nj in which

$$W_{ij} = \sum_{k=1}^{nk} \vec{V}_{ik} \bullet \vec{U}_{kj} \qquad i = 1..ni, \; j = 1..nj \tag{A.10}$$

The vector product $\vec{\underline{V}} * \vec{\underline{U}}$ of these matrices $\vec{\underline{V}}$ and $\vec{\underline{U}}$ results in a vectorial matrix $\vec{\underline{W}}$ with components $\vec{W}_{ij}$, defined by

$$\vec{W}_{ij} = \sum_{k=1}^{nk} \vec{V}_{ik} * \vec{U}_{kj} \qquad i = 1..ni, \; j = 1..nj \tag{A.11}$$

From this definition it is easily shown that

$$(\vec{\underline{V}}*\vec{\underline{U}})^T = -\vec{\underline{U}}^T*\vec{\underline{V}}^T \qquad (A.12)$$

Operation rules similar to (A.9) and (A.10) apply to the product of a tensorial matrix (i.e. a matrix with tensors as components) with a scalar matrix, with a vectorial matrix or with another tensorial matrix.

A.4 Matrix representation of vectors and tensors

Let $\vec{\underline{e}}$ be an orthonormal, right-handed vector base. Then:

$$\vec{\underline{e}}^T = [\ \vec{e}_1,\ \vec{e}_2,\ \vec{e}_3\ ], \qquad \vec{e}_3 = \vec{e}_1*\vec{e}_2 \qquad (A.13)$$

$$\vec{\underline{e}}^T\vec{\underline{e}} = \mathbb{I}, \qquad \vec{\underline{e}}\bullet\vec{\underline{e}}^T = \underline{I}_3$$

Vectors and tensors can be represented in such a vector base by scalar matrices. These scalar matrices are called the matrix representations in that vector base. Each vector $\vec{v} \in S^3$ can be written as

$$\vec{v} = \underline{v}^T\vec{\underline{e}}, \qquad \underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \qquad \vec{\underline{e}} = \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix} \qquad (A.14)$$

The column $\underline{v}$ is the matrix representation of $\vec{v}$ in $\vec{\underline{e}}$. Since the vector base $\vec{\underline{e}}$ is orthonormal, i.e. $\vec{\underline{e}}\bullet\vec{\underline{e}}^T = \underline{I}$, we can write

$$\underline{v} = \vec{\underline{e}}\bullet\vec{v} \qquad (A.15)$$

A similar definition applies to tensors. For the matrix representation $\underline{B}$ of a tensor $\mathbb{B}$ in an orthonormal base $\vec{\underline{e}}$ we write

$$\mathbb{B} = \vec{\underline{e}}^T\underline{B}\vec{\underline{e}} \qquad \underline{B} = \vec{\underline{e}}\bullet\mathbb{B}\bullet\vec{\underline{e}}^T \qquad (A.16)$$

Working with the matrix representation of vectors and tensors requires to indicate the base used. Since we work with vectors and tensors in this study and not with the matrix representations of such quanti-

ties, we do not have to specify which base is used. This is especially useful for multibody theories in which, in addition to the inertial or global base $\underset{\sim}{\vec{e}}^{O}$, several body-fixed bases occur. For examples of this coordinate-free method of notation see Wittenburg [1977, p12-14] or Veldpaus [1980].

LIST OF SYMBOLS

Not every symbol used in the text is included in this list. Only those symbols essential for the theory and useful in a multibody program are mentioned.

symbol	description	page
$\vec{a}$	relative acceleration vector	35
$\vec{b}$	body-fixed vector	23
$\vec{c}$	connection or relative position vector	33
$\underset{\sim}{\vec{e}}$	vector base	17, A.4
$\underset{\sim}{f}$	constitutive equation	37
$\underset{\sim}{i}$	external input variables of active element	37, 57
$\vec{i}$	momentum	26
$\underset{\sim}{l}$	location column	70
m	mass of a body	25, 100
na	number of edges (arcs)	63
na	number of external points	109
nb	number of branches	66
nb	number of bodies	68, 73
nc	number of kinematic constraints (chords)	85
ne	number of endpoints	32, 110
nf	number of forces on a body	27
nf	number of free Lagrange coordinates	92
nh	number of hinges	68
nhc	number of holonomic constraints	91
nhp	number of holonomic Pfaff equations	91
ni	number of external input variables	57
ni	number of internal points	109
nk	number of elements	109
nnc	number of nonholonomic constraints	91
nnp	number of nonholonomic Pfaff equations	91
nm	number of moments on a body	27
np	number of Pfaff equations	38, 86
nq	number of Lagrange coordinates	75

ns	number of prescribed Lagrange coordinates	92
nt	number of trees	66
nv	number of vertices	63
n_γ	number of generalized coord. in kin. conn.	113
n_ϱ	number of reduced gen. coord. in kin. conn.	114
$\vec{p}$	surface load on body	27
$\underline{p}_y$, $\underset{\sim}{p}_{oy}$, $\underset{\sim}{p}_{ooy}$	Pfaff matrices for implicit constraints, see (3.3.8-10)	41
$\underline{p}_\gamma$, $\underset{\sim}{p}_{o\gamma}$, $\underset{\sim}{p}_{oo\gamma}$	Pfaff matrices for constraint elements in a kinematic connection, see (7.2.2/3/9)	113, 114
$\underline{p}_\varrho$, $\underset{\sim}{p}_{o\varrho}$, $\underset{\sim}{p}_{oo\varrho}$		
$\vec{q}$	volume load at body	27
$\underset{\sim}{q}$	generalized coordinates of hinge or Lagrange coordinates of a multibody system	43, 75
$\vec{r}$	(absolute) position vector	22
$\underset{\sim}{s}$	state equation of active element	38
t	time	19
$\vec{u}$	relative positions of endpoints, (7.1.4)	111
u	scalar position variable (attitude coord.)	22
$\vec{v}$	relative velocity vector	35
$\vec{v}_u$	see (2.2.22)	23
$\underline{v}$, $\underset{\sim}{v}_o$, $\underset{\sim}{v}_{oo}$	partial derivatives of hinge element see (3.3.31-34)	45, 46
$\vec{v}^C$	see (5.3.9)	88
$\vec{w}_\varphi$	see (2.2.7)	20
$\underline{w}$, $\underset{\sim}{w}_o$, $\underset{\sim}{w}_{oo}$	partial derivatives of hinge element see (3.3.28-30)	45
$\underset{\sim}{x}$	state variables	38
$\underset{\sim}{y}$	relative attitude coordinates	40
$\underline{y}$	see (3.3.36)	46
$\underset{\sim}{z}$	(absolute) attitude coordinates	25
A	external point (attached to a body)	109
A	surface area of body	27
A^i	arc, edge, branch with number i	63
$\underline{\underline{A}}$	generalized mass matrix	103
B	rigid body	17
$\mathbb{B}$	body-fixed tensor	23
$\underline{\vec{B}}$, $\underset{\sim}{\vec{B}}_o$	tree matrices with body-fixed vectors	79

C	connection, kinematic constraints	109, 87
$\mathbb{C}$	connection or relative rotation tensor	33
$\underline{\vec{C}}$, $\underset{\sim}{\vec{C}}_O$	tree matrices with connection vectors	79
E	element of connection	32
$\vec{F}$	force vector	27
$\underset{\sim}{F}$	relevant force variables	50
$\underline{G}_v$, $\underline{G}_Q$	see (3.4.9)	52
H	hinge	64
$\underset{\sim}{\vec{H}}$	see (6.2.5-6)	101
I	internal point of a connection	109
$\mathbb{J}$	inertia tensor	26, 100
K	nodal point of a connection	109
L	Lipschitz constant, see (8.4.6)	134
$\vec{L}$	angular momentum of a rigid body	26
$\underline{L}$, $\underline{L}^C$, $\underline{L}_{ex}$, $\underline{L}_{in}$	location matrices	69, 110
M	centre of mass of rigid body	25
$\vec{M}$	moment vector	27
$\underline{\underline{M}}$	orthogonal matrix, see (7.2.12)	116
N, N^i	reference point, i^{th} endpoint	33, 32
$\underline{\underline{N}}$	see (7.2.12)	116
O	origin	23
$\underline{P}$, $\underset{\sim}{P}_O$, $\underset{\sim}{P}_{OO}$	Pfaff matrices for explicit constraints	91
$\underline{P}_y$, $\underset{\sim}{P}_{Oy}$, $\underset{\sim}{P}_{OOy}$	Pfaff matrices for implicit constraints	86
$\underset{\sim}{Q}$	generalized loads	103
R	rotation tensor	18
S^3	Euclidian space, 3-dimensional	17, A.2
$\underline{S}$, $\underset{\sim}{S}_O$, $\underline{S}^C$, $\underset{\sim}{S}^C_O$	submatrices of location matrices $\underline{L}$, $\underline{L}^C$	69
$\underline{\underline{T}}$	tree matrix with topology of tree struct.	70
$\underline{\vec{U}}$, $\underset{\sim}{\vec{U}}_O$, $\underset{\sim}{\vec{U}}_{OO}$	tree matrices with (absolute) velocities of bodies, see (5.2.23, 5.2.28)	80, 81
$\underset{\sim}{\vec{U}}^C$	see (5.3.10)	88
V	volume of rigid body	25
V^O, V^i	reference vertex, vertex with number i	66, 63
$\underline{\vec{V}}_u$	see (2.2.24)	23
$\underline{\vec{V}}$, $\underset{\sim}{\vec{V}}_O$, $\underset{\sim}{\vec{V}}_{OO}$	tree matrices with relative velocities of bodies, see (5.2.20, 5.2.26)	80, 81
ΔW	virtual work	27
$\underline{\vec{W}}_\varphi$	see (2.2.12)	20

$\vec{\underline{\tilde{W}}}$, $\vec{\underset{\sim}{\tilde{W}}}_O$, $\vec{\underset{\sim}{\tilde{W}}}_{OO}$	tree matrices with angular velocities of bodies, see (5.2.5, 5.2.10)	77, 78
$\underline{Y}$, $\underset{\sim}{Y}_O$, $\underset{\sim}{Y}_{OO}$	transformation matrices, see (5.3.7-15)	88, 89
$\underline{Y}_\gamma$, $\underline{Y}_\varrho$, $\underset{\sim}{Y}_{O\gamma}$, $\underset{\sim}{Y}_{O\varrho}$	transformation matrices, see (7.2.5 & 8)	114
$\underline{Z}$, $\underset{\sim}{Z}_O$, $\underset{\sim}{Z}_{OO}$	transformation matrices, see (5.2.30,31)	82
$\vec{\alpha}$	relative angular acceleration vector	34
β	Baumgarte stabilization constant	136
γ	generalized coordinates, see (7.2.1)	114
$\underset{\sim}{\varepsilon}$	relevant kinematic variables	50
$\underset{\sim}{\lambda}$	Lagrange multipliers	42
$\delta\vec{\pi}$	angular variation vector	21, 35
ϱ	mass density	25
$\underset{\sim}{\varrho}$	generalized coordinates, see (7.2.6)	114
τ	element of time interval $(\infty, t]$, history	37
$\delta\vec{v}$	variation of the relative position vector	36
φ	scalar orientation variables (attitude co.)	18
$\vec{\omega}$	angular velocity vector	19
$\underset{\sim}{\Lambda}$	constraint loads, see (8.5.2)	42, 137
$\delta\vec{\Pi}$	relative angular variation vector	35
$\underline{\Psi}$	see (3.3.5)	40
$\underline{\Phi}$	see (3.3.6)	40
$\vec{\Omega}$	relative angular velocity vector	34
$\underset{\sim}{\vec{Q}}^C$	see (5.3.9)	88

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STELLINGEN

behorende bij het proefschrift

KINEMATICS AND DYNAMICS OF MULTIBODY SYSTEMS

1. De coordinaat-vrije formulering tezamen met het opbergen van vectoren en tensoren in matrices bevordert de overzichtelijkheid bij de presentatie van theorieën voor de analyse van systemen met veel lichamen.

   - hoofdstuk 5 en 6 van dit proefschrift

2. Bij de beschrijving van de topologie van een "multibody" systeem is het begrip boomstructuur bruikbaar zowel voor systemen met een open als met een gesloten kinematische structuur.

   - hoofdstuk 5 van dit proefschrift

3. De definitie van het begrip aantal graden van vrijheid in termen van de dimensie van de ruimte van mogelijke standen van een "multibody" systeem is behept met dezelfde beperking als de veel oudere regel van Grübler omdat beide geen rekening houden met kinematische singulariteiten.

   - hoofdstuk 8 van dit proefschrift
   - Rosenberg, R.M.: Analytical Dynamics of Discrete Systems. Plenum Press, New York, 1977.

4. De in dit proefschrift toegepaste formulering voor de analyse van kinematische en dynamische "multibody" systemen is een goed uitgangspunt voor de optimalisering van dergelijke systemen.

5. De arm en de hand van de mens kunnen op een groot aantal manieren een lichaam van de ene stand naar een andere stand transporteren. Een realisering in robot systemen van soortgelijke mogelijkheden zal voordelen bieden bijv. bij het vermijden van obstakels en kinematische singulariteiten, bij optimalisering alsmede bij het uitvoeren van compenserende bewegingen in het geval van defecten.

6. Maillardet stelt dat het been van de mens tijdens de zwaaifase van het lopen voor de analyse van het mechanische gedrag kan worden geschematiseerd tot een dubbele slinger en concludeert vervolgens dat de bewegingsvorm van het been overeen komt met de tweede eigentrillingsvorm. Deze conclusie is een te vergaande simplificatie van het niet-lineaire dynamisch gedrag van het spier-skelet stelsel.

   Maillardet,F.J.: The swing phase of locomotion.
   Engineering in Medicine, ImechE, 6: 67-75 & 101-106, 1977.

7. Optimaliseren van het gedrag van een complex systeem zoals het spier-skelet-stelsel van de mens leidt tot een momenteel nauwelijks oplosbaar optimaliseringsprobleem.

8. Het gebruik van de 4x4 notatie van Denavit en Hartenberg voor de beschrijving van de positie en orientatie van een lichaam in een drie-dimensionale ruimte is omslachtig en onnodig.

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   - Hollerbach,J.M.: A recursive Lagrangian formulation of manipulator dynamics and a comparative study of dynamics formulation complexity.
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9. Vanaf het begin dient in de opleiding tot werktuigbouwkundig ingenieur veel intensiever dan tot nu toe aandacht besteed te worden aan het gebruik van (micro-)computers en van (real-time) software.

10. Een zeilwagen met een vleugel kan een hogere topsnelheid bereiken dan een normale zeilwagen doordat de drifthoek in het loopvlak van de banden geringer wordt ten gevolge van de kleinere zijwaarts gerichte aerodynamische kracht van de vleugel.

11. Geheel aangepast aan zijn gecomputeriseerde omgeving zal de species Homo Terminum-Videns zich in de evolutie kenmerken door vierkante groene ogen.

Geldrop, 1983-sept-22 Egbert Jan Sol