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by

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Abstract

For a moving boundary problem modelling the motion of a semipermeable membrane by osmotic pressure and surface tension we prove the existence and uniqueness of classical solutions on small time intervals. Moreover, we construct solutions existing on arbitrary long time intervals, provided the initial geometry is close to an equilibrium. In both cases, our method relies on maximal regularity results for parabolic systems with inhomogeneous boundary data.

Keywords: moving boundary problem, maximal continuous regularity

MSC classification: 35R37, 35K55

1 Introduction

In this paper we consider the one phase version of a moving boundary problem modelling osmosis: A semipermeable membrane $\Gamma(t)$ moves freely in an incompressible fluid at rest. The membrane encloses a region $\Omega(t) \subset \mathbb{R}^N$, where a certain amount of a solute is dissolved. Its concentration at position $x \in \Omega(t)$ and at time t is denoted by $v = v(t, x)$. The evolution of the solute is given by linear diffusion:

$$\partial_t v - \mu \Delta v = 0 \qquad \text{in } \Omega(t),$$

where $\mu > 0$ is the constant diffusivity of the solvent.
The membrane is permeable to the solvent but impermeable to the solute. This fact combined with the local conservation of solute leads to the following condition on the free boundary $\Gamma(t)$:

$$v V_n + \mu \partial_n v = 0 \qquad \text{on } \Gamma(t),$$

where V_n is the normal velocity of the family $\{\Gamma(t)\}$, positive where $\Omega(t)$ grows, and $\partial_n v$ denotes the derivative in the direction of the outer unit normal field of $\Gamma(t)$. In particular, this condition ensures that the total amount of solute inside $\Omega(t)$ must be a conserved quantity.
The motion of the membrane is essentially governed by osmotic pressure and surface tension (cf [Pi2008]):

$$V_n = P(\psi H + \chi v) \qquad \text{on } \Gamma(t).$$

The positive constants P, ψ, χ are related to the permeability of the membrane, the (constant) surface tension coefficient and the osmotic pressure. By H we denote the $(N-1)$-fold mean curvature of $\Gamma(t)$, taken to be negative where $\Omega(t)$ is convex. More details about this model and its application in experimental cell biology can be found in [Pi2008], [Ve1992], [Ve2000], [Zaal08].
By nondimensionalizing the problem, all the constants but one appearing in the model can be normalized to the value one. We keep the same notation for the dimensionless variables. The remaining dimensionless parameter is denoted by κ. It can be interpreted as the ratio of the typical time scales of diffusion of solute and relaxation of the membrane without solute. Summarizing, we arrive at the following set of equations:

$$\left.\begin{array}{rcll} \partial_t v - \Delta v &=& 0 & \text{in } \Omega(t), \\ v V_n + \partial_n v &=& 0 & \text{on } \Gamma(t), \\ V_n &=& \kappa H + v & \text{on } \Gamma(t), \end{array}\right\} \tag{1.1}$$

or, equivalently,

$$\left.\begin{array}{rcll} \partial_t v - \Delta v &=& 0 & \text{in } \Omega(t), \\ \partial_n v + \kappa H v + v^2 &=& 0 & \text{on } \Gamma(t), \\ V_n &=& \kappa H + v & \text{on } \Gamma(t). \end{array}\right\} \tag{1.2}$$

These systems are complemented by appropriate initial conditions $v(0) = v_0$, $\Gamma(0) = \Gamma_0$.
To our knowledge, rigorous analysis of osmosis problems of the described type has been performed mainly in one space dimension situations ([MR1995]). In [Zaal08], problem (1.1) is considered in a radially symmetric setting.
In general, it has been a successful strategy in the context of free boundary problems to apply a suitable coordinate transformation in order to obtain local well posedness or stability results. In fact, by means of this transformation many moving boundary problems can be reformulated as a parabolic evolution equation over a fixed pair of Banach spaces. This equation can then be treated by abstract functional analytic methods. However, this is not true in the case of the osmosis model, because the boundary condition for the unknown function v cannot be 'hidden' in a *fixed* domain of definition for the transformed differential operators. This additional difficulty is also encountered, for example, in the case of the full Stefan problem with Gibbs-Thomson correction and kinetic undercooling which has been treated in [Kn2007]. In that work, the coordinate transformations lead to a single evolution equation for a function describing the moving boundary. This equation contains Volterra mappings, which are nonlocal in time, and it is solved with the help of the theory of maximal Hölder regularity.
In this paper we use an approach different from [Kn2007]. We consider the transformed system as an abstract operator equation which can be treated in a framework that resembles a maximal continuous regularity setting in a parabolic context. We obtain sharp regularity results this way. Nevertheless, the techniques of deriving estimates are influenced by those which are developed in [Kn2007].
In the second part of this paper we make a first approach to a stability result for the osmosis problem: We construct solutions near equilibria existing on arbitrary long time intervals and taking values in a prescribed arbitrarily small neighborhood of a given equilibrium. A main ingredient of the proof is the maximal regularity result for systems given in Theorem 2.1 in [DPZ08]. It enables us to identify the solution as the limit of a fixed point iteration with the help of the contraction mapping principle. A similar argument has already been used in [EPS03], where a Stefan problem with Gibbs-Thomson correction is considered.
More precisely, our main results are given in the following theorems. For the definition of the spaces used here we refer to Sections 2 and 4.

Theorem 1.1 *Let $0 < \alpha < \beta < 1$, and let $v_0 \in h^{2,\alpha}(\bar{\Omega}_0)$, $\Gamma_0 = \partial\Omega_0$ of class $h^{4,\beta}$ satisfy*

$$\partial_n v_0 + \kappa H(\Gamma_0) v_0 + v_0^2 = 0 \qquad \text{on } \Gamma_0.$$

Then there exists a positive time T^* and a unique classical solution of (1.1) on

$[0,T^*]$.

Observe that a pair $(\bar\Gamma,\bar v)$ is an equilibrium solution of (1.1) if and only if it is given by $(\bar\Gamma,\bar v)=(\partial\mathbb{B}(x,R),(N-1)\kappa/R)$ for some $x\in\mathbb{R}^N$ and $R>0$. Let $B:=\mathbb{B}(x,R)$, $S:=\partial B$, $\bar u:=(N-1)\kappa/R$. For suitable $\sigma\in C^1(S)$ let θ_σ denote the Hanzawa diffeomorphism and $S_\sigma:=\theta_\sigma[S]$, see Section 2, Section 4.

Theorem 1.2 *Let $p>N+2$, $T>0$ be given, and let $(v_0,\Gamma_0)=(\theta_*^{\rho_0}u_0,S_{\rho_0})$ for some $(u_0,\rho_0)\in Y:=W_p^{2-2/p}(B)\times W_p^{3-3/p}(S)$. If $\varepsilon>0$ is small enough, then there is a $\delta(\varepsilon)>0$ with the following property: If the conditions*

- $(u_0,\rho_0)\in\mathbb{B}_Y((\bar u,0),\delta)$;
- $\partial_n v_0+H(S_{\rho_0})+v_0^2=0$ *on* S_{ρ_0}

are satisfied, then (1.1) admits a unique strong solution $(v,\Gamma)=(\theta_*^\rho u,S_\rho)$ on $[0,T]$. Moreover,

$$\|(u-\bar u,\rho)\|_{\mathbb{Y}}<\varepsilon,\qquad \mathbb{Y}:=W_p^{2,1}(B_T)\times W_p^{3-1/p,(3-1/p)/2}(S_T).$$

This paper is organized as follows: In Section 2 we shall define the notion of a classical solution of (1.1) and prepare some preliminary material. In Section 3 the transformed version of the full problem is derived, and a proof of Theorem 1.1 is given. Strong solutions of (1.1) near equilibria are considered in Section 4, which contains a proof of Theorem 1.2. Finally, Section 5 contains the most technical part of the proof of Theorem 1.1.

2 The abstract setting

Throughout the article we keep the numbers $N\in\mathbb{N}$, $N\ge 2$, $0<\beta<\alpha<1$ fixed. From now on we shall focus on the equivalent model (1.2). We assume that

- $\Omega_0\subset\mathbb{R}^N$ is a domain and $\Gamma_0:=\partial\Omega_0$ is a closed compact hypersurface of regularity class $h^{4,\beta}$;
- $v_0\in h^{2,\alpha}(\overline{\Omega}_0)$.

Here, $h^{m,\gamma}$ denotes the *little Hölder space*, i.e. the closure of (sufficiently) smooth functions in the usual Hölder space $C^{m,\gamma}$, where m is a non-negative natural

number and $0 < \gamma < 1$.
We are going to transform system (1.2) into a set of equations given over a fixed and smooth reference domain. The unknown family of surfaces $\{\Gamma(t)\}$ will be described by a signed distance function with respect to that surface. In order to do these transformations, we need some preparation:
Given any surface Σ of class C^1, let $T_{[\Sigma]}$ be a *tubular neighborhood* of Σ, i.e. the diffeomorphic image of the map

$$X_{[\Sigma]} : \Sigma \times (-\delta, \delta) \to \mathbb{R}^N, \qquad (x, a) \mapsto x + a \cdot \nu_{[\Sigma]}(x),$$

where $\nu_{[\Sigma]}(x)$ is the outer unit normal vector at $x \in \Sigma$ and $\delta > 0$ is sufficiently small. It is convenient to decompose the inverse of $X_{[\Sigma]}$ into $X_{[\Sigma]}^{-1} = (P_{[\Sigma]}, \Lambda_{[\Sigma]})$, where $P_{[\Sigma]}$ is the metric projection of a point x onto Σ and $\Lambda_{[\Sigma]}$ is the signed distance function with respect to Σ. Let

$$\mathrm{Ad}_{\Sigma, T_{[\Sigma]}} := \{\sigma \in C^1(\Sigma);\ \|\sigma\|_{C(\Sigma)} < \delta/5\}.$$

It is well-known that, given $\sigma \in \mathrm{Ad}_{\Sigma, T_{[\Sigma]}}$, the map $\theta_\sigma(x) := x + \sigma(x) \cdot \nu_{[\Sigma]}(x)$ is a diffeomorphism mapping Σ onto $\Sigma_\sigma := \theta_\sigma[\Sigma]$.
Due to Theorem 4.2 in [BEL2011] we can fix a triple (Ω, S, ρ_0) in the following way:

- $\Omega \subset \Omega_0$ is a domain and $\Gamma := \partial\Omega$ is a closed compact real analytic hypersurface;
- S is a tubular neighborhood of Γ, $\Gamma_0 \subset S$;
- $\rho_0 \in h^{4,\beta}(\Gamma) \cap \mathrm{Ad}_{\Gamma,S}$ and the mapping $\theta_{\rho_0} : \Gamma \to \Gamma_0$ is a $h^{4,\beta}$ - diffeomorphism. In particular, $\Gamma_0 = \Gamma_{\rho_0}$.

From now on let (Ω, Γ, ρ_0) *be chosen as described above and let* $Ad := Ad_{\Gamma,S}$.

Suppose that $\sigma \in \mathrm{Ad} \cap h^{m,\gamma}(\Gamma)$ for some $(m, \gamma) \in \mathbb{N} \times (0, 1)$. It is not difficult to see that then $\theta_\sigma \in h^{m,\gamma}(\Gamma, \mathbb{R}^N)$ and $\theta_\sigma^{-1} \in h^{m,\gamma}(\Gamma_\sigma, \mathbb{R}^N)$. Moreover, given $\sigma \in \mathrm{Ad} \cap h^{m,\gamma}(\Gamma)$, the mapping θ_σ extends to a diffeomorphism

$$\theta_\sigma \in \mathrm{Diff}^{m,\gamma}(\mathbb{R}^N, \mathbb{R}^N), \qquad \theta_\sigma|_\Omega \in \mathrm{Diff}^{m,\gamma}(\Omega, \Omega_\sigma),$$

such that we have $\partial\Omega_\sigma = \Gamma_\sigma$, cf. [Es2004], Section 2. Note that for $\sigma \in \mathrm{Ad}$ the surface Γ_σ is the zero level set of the function φ_σ defined by

$$\varphi_\sigma(x) = \Lambda_{[\Gamma]}(x) - \sigma(P_{[\Gamma]}(x)),$$

$x \in S$, i.e. $\Gamma_\sigma = \varphi_\sigma^{-1}[\{0\}]$. For later use we set

$$L_\sigma(x) := |\nabla\varphi_\sigma|(\theta_\sigma(x)).$$

It can be shown that $L_\sigma > 0$ on Γ for all $\sigma \in \mathrm{Ad}$. Finally, if $\rho : [0,T] \to \mathrm{Ad}$ is time dependent, we use the notation

$$\Omega_{\rho,T} := \bigcup_{t\in(0,T)} \{t\} \times \Omega_{\rho(t)} \subset \mathbb{R}^{N+1}.$$

We are now ready to introduce the notion of a classical solution of (1.1):

Definition 2.1 *Let* $v_0 \in h^{2,\alpha}(\overline{\Omega}_0)$ *satisfy*

$$\partial_n v_0 + \kappa H(\Gamma_0)v_0 + v_0^2 = 0 \qquad \text{on } \Gamma_0$$

and let $\mathcal{O} := h^{4,\beta}(\Gamma) \cap Ad$ *inherit the topology of* $h^{4,\beta}(\Gamma)$*. A pair* $(v(t), \Gamma(t))$ *is said to be a classical solution of (1.1) on* $[0,T]$*, if there exists a function* $\rho \in C([0,T],\mathcal{O}) \cap C^1([0,T], h^{2,\beta}(\Gamma))$ *such that*

i) $\Gamma(t) = \Gamma_{\rho(t)}$, $t \in [0,T]$*;*

ii) $v(\cdot) \circ \theta_{\rho(\cdot)} \in C([0,T], h^{2,\alpha}(\overline{\Omega})) \cap C^1([0,T], h^\alpha(\overline{\Omega}))$*;*

iii) $(v(t), \Gamma(t))$ *satisfy the equations of (1.1) pointwise on* $[0,T]$.

Note that ii) in particular implies that

- $v \in C^{1,2}(\Omega_{\rho,T}, \mathbb{R}) \cap C(\overline{\Omega}_{\rho,T}, \mathbb{R})$;
- $v(t) \in h^{2,\alpha}(\overline{\Omega}_{\rho(t)})$ for $t \in [0,T]$.

3 The equations on a fixed domain - Quasilinear structure

Given $\sigma \in \mathrm{Ad}$, let $\theta_\sigma^*, \theta_*^\sigma$ denote the pull-back and push-forward operators induced by θ_σ, i.e. $\theta_\sigma^* f = f \circ \theta_\sigma$, $\theta_*^\sigma g = g \circ \theta_\sigma^{-1}$. If suitable functions b, ρ are time

dependent, i.e. $b = b(t,x)$, $\rho = \rho(t,x)$, we define $[\theta_\rho^* b](t,x) := [\theta_{\rho(t)}^* b(t,\cdot)](x)$, analogue for θ_*^ρ.
Using this notation, for suitable ρ we can introduce the transformed operators

$$\begin{array}{rcl} \mathcal{A}(\rho)u & := & \theta_\rho^*(\Delta(\theta_*^\rho u)) \\ \mathcal{B}(\rho)u & := & \theta_\rho^*(\nabla(\theta_*^\rho u) \cdot \nabla\varphi_\rho/|\nabla\varphi_\rho|) \\ H(\rho) & := & \theta_\rho^* H_{[\Gamma_\rho]}. \end{array}$$

We mention that $\mathcal{A}(\rho(t))$ is just the Laplace-Beltrami operator with respect to the Riemannian metric induced by $\theta_{\rho(t)}$. System (1.2) transforms into

$$\left.\begin{array}{rcll} \partial_t u - \mathcal{A}(\rho)u & = & R(\rho,u) & \text{in } \Omega, \\ \mathcal{B}(\rho)u & = & -\kappa u H(\rho) - u^2 & \text{on } \Gamma, \\ \partial_t\rho - \kappa L_\rho P(\rho)\rho & = & \kappa L_\rho Q(\rho) + L_\rho u & \text{on } \Gamma, \\ u(0) & = & u_0, & \\ \rho(0) & = & \rho_0, & \end{array}\right\} \tag{3.1}$$

where $u_0 := \theta_{\rho_0}^* v_0$. Here we used the splitting

$$H(\rho) = P(\rho)\rho + Q(\rho)$$

as introduced in [EsSi97]. The term R arises from the transformation of the time derivative v_t and is determined by

$$R(w,\sigma)(y) = r_0(\kappa L_\sigma[H(\sigma)+w], B_\mu(\sigma)w)(y), \qquad y \in \Omega,$$

where $w \in C^1(\bar{D})$, $\sigma \in \mathrm{Ad}$ and

$$r_0(h,k)(y) := \begin{cases} \chi(\Lambda(y)) \cdot h(P_{[\Gamma]}(y)) \cdot k(y), & \text{if } \; y \in \Omega \cap S \\ 0, & \text{if } \; y \in \Omega \setminus (\Omega \cap S), \end{cases} \tag{3.2}$$

$$B_\mu(\sigma)v(y) = \theta_\sigma^* \nabla(\theta_*^\sigma v)(y) \cdot (\mu_{[\Gamma]} \circ P_{[\Gamma]})(y), \qquad y \in S$$

(χ being a suitable cut-off function and $\mu_{[\Gamma]}$ being the exterior unit normal field of Γ, cf. [Kn2007], [Es2004]). The derivation of R is a straightforward calculation, cf. again [Kn2007], [Es2004].
Suppose that (u,ρ) is a solution of (3.1) in a sense to be made precise. We want $(\theta_*^\rho u, \Gamma_\rho)$ to be a classical solution of (1.2). For this we shall consider the following function spaces: if $T > 0$ is given and $J_T := [0,T]$, let

$$
\begin{array}{rcl}
E_0 & := & h^{\alpha}(\overline{\Omega}) \times h^{2+\beta}(\Gamma), \\
E_1 & := & h^{2+\alpha}(\overline{\Omega}) \times h^{4+\beta}(\Gamma), \\
\mathbb{E}_0(J_T) & := & BUC(J_T, E_0), \\
\mathbb{E}_1(J_T) & := & BUC^1(J_T, E_0) \cap BUC(J_T, E_1), \\
\mathbb{F}(J_T) & := & BUC(J_T, h^{1+\alpha}(\Gamma)) \cap h^{(1+\alpha)/2}(J_T, C(\Gamma)).
\end{array}
$$

In order to economize notation we drop the T - dependence, i.e. write $\mathbb{E}_1$ instead of $\mathbb{E}_1(J_T)$ etc. In the following, the interpolation embeddings

$$
\mathbb{E}_1 \hookrightarrow h^{\theta_1}(J_T, h^{l_1+\sigma_1}(\bar{\Omega})) \times h^{\theta_2}(J_T, h^{l_2+\sigma_2}(\Gamma)); \tag{3.3}
$$

$$
P_1\mathbb{E}_1 \hookrightarrow h^{(1+\alpha)/2}(J_T, C^1(\bar{\Omega})) \tag{3.4}
$$

(P_j, $j \in \{1,2\}$ denoting the projection) will be used repeatedly. Here,

$$
2\theta_1 + l_1 + \sigma_1 = 2 + \alpha, \quad 2\theta_2 + l_2 + \sigma_2 = 4 + \beta, \quad l_i \in \mathbb{N}, \sigma_i, \theta_i \in (0,1).
$$

Using standard interpolation results for the space variable, it is a basic computation to see that the estimates corresponding to the embeddings (3.3) and (3.4) do not depend on the length of the interval J_T.

We define the sets

$$
\widetilde{\mathrm{Ad}} = \{(\nu,\psi) \in E_1 \,|\, \nu \in \mathrm{Ad}\}, \quad \widehat{\mathrm{Ad}} = \{w \in \mathbb{E}_1 \,|\, w(t) \in \widetilde{\mathrm{Ad}},\, t \in [0,T]\}
$$

which are open subsets of E_1 and $\mathbb{E}_1$, respectively. Our goal is to write system (3.1) as a single operator equation. For this we define the mappings

$$
\begin{array}{rcl}
\mathbb{A}(w)(t) & = & \begin{pmatrix} \mathcal{A}(\rho(t)) & 0 \\ 0 & \kappa L_\rho(t) P(\rho(t)) \end{pmatrix}, \\
\tilde{\mathbb{B}}(\nu,\psi)(\zeta,\chi) & = & \mathcal{B}(\psi)\zeta, \\
\big(\mathbb{B}(w)(v,\sigma)\big)(t) & = & \tilde{\mathbb{B}}(w(t))(v(t),\sigma(t)), \\
\mathbb{L}(w) & = & (\partial_t - \mathbb{A}(w), \mathbb{B}(w), \gamma_t),
\end{array}
$$

where $w = (u,\rho) \in \mathbb{E}_1$, $(v,\sigma) \in \mathbb{E}_1$, $(\nu,\psi), (\zeta,\chi) \in E_1$ and $\gamma_t \in \mathcal{L}(\mathbb{E}_1, E_1)$ denotes the time trace map $w \mapsto w(0)$. It is not difficult to see that

$$
\begin{aligned}
\mathbb{A} &\in C^\infty\big(\widehat{\mathrm{Ad}}, \mathcal{L}(\mathbb{E}_1, \mathbb{E}_0)\big),\\
\tilde{\mathbb{B}} &\in C^\infty\big(\widetilde{\mathrm{Ad}}, \mathcal{L}(E_1, h^{1+\alpha}(\Gamma))\big),\\
\mathbb{B} &\in C^\infty\big(\widehat{\mathrm{Ad}}, \mathcal{L}(\mathbb{E}_1, \mathbb{F})\big),\\
\mathbb{L} &\in C^\infty\big(\widehat{\mathrm{Ad}}, \mathcal{L}(\mathbb{E}_1, \mathbb{E}_0 \times \mathbb{F} \times E_1)\big),
\end{aligned}
$$

cf. [Kn2007], [Es2000], [EsSi97] for example, where the mappings $\mathcal{A}$, $\mathcal{B}$, P, Q and L are studied in detail. Observe that the mapping properties for $\mathbb{B}$ follow from (3.4) and the fact that additionally

$$
\mathcal{B} \in C^\infty\big(\widetilde{\mathrm{Ad}}, \mathcal{L}(C^1(\bar{\Omega}), C(\Gamma))\big).
$$

Let $w_0 = (u_0, \rho_0)$. For given, fixed M (to be determined later) we define the closed set

$$
\mathcal{C} = \mathcal{C}(M,T) := \{w \in \mathbb{E}_1 \,|, w(0) = w_0, \|w\|_{\mathbb{E}_1} \leq M\}.
$$

Furthermore, we introduce the subspace

$$
\mathbb{Z} = \{(f,g,h) \in \mathbb{E}_0 \times \mathbb{F} \times E_1 \,|\, \gamma_t g = \tilde{\mathbb{B}}(w_0)h\}.
$$

Lemma 3.1 *There is a $T^* = T^*(w_0, M)$ such that if $T \in (0, T^*]$ then $\mathcal{C} \subset \widehat{Ad}$. In this case $\mathbb{L}(\mathcal{C}) \subset \mathcal{L}(\mathbb{E}_1, \mathbb{Z})$.*

Proof: Let $w = (u, \rho) \in \mathcal{C}$. As $w_0 \in \widetilde{\mathrm{Ad}}$ and

$$
\|w(t) - w_0\|_{E_0} = \|w(t) - w(0)\|_{E_0} \leq MT, \quad t \in [0,T]
$$

we get $\|\rho(t) - \rho_0\|_{h^{2+\alpha}(\Gamma)} \leq MT$ and therefore $\rho(t) \in \mathrm{Ad}$ for T small as Ad is open in $h^{2+\alpha}(\Gamma)$. The second statement is a consequence of $\gamma_t \circ \mathbb{B}_0 = \tilde{\mathbb{B}}(w_0) \circ \gamma_t$. ■

Assume $T \in (0, T^*]$ and define $\hat{w}_0 \in \mathbb{E}_1$ to be the constant function on J_T with value w_0.

Lemma 3.2 *(Maximal regularity for frozen coefficients)*

We have

$$
\mathbb{L}_0 = \mathbb{L}(\hat{w}_0) \in \mathcal{L}_{is}(\mathbb{E}_1, \mathbb{Z}).
$$

Proof: This is a consequence of Theorem 1.4 in [Lunardi89] and standard results concerning the uniformly elliptic operator $L_{\rho_0}P(\rho_0)$ on the closed compact manifold Γ. ∎

Our problem can now be reformulated as

$$\mathbb{L}(w)w = F(w) := (\mathcal{R}(w), \mathcal{G}(w), w_0), \quad w \in \mathcal{C}, \tag{3.5}$$

where

$$\begin{aligned} \mathcal{R}(w)(t) &= \begin{pmatrix} R(w(t)) \\ L_{\rho(t)}(\kappa Q(\rho(t)) + u(t)) \end{pmatrix}, \\ \mathcal{G}(w)(t) &= -\kappa u(t)H(\rho(t)) - u(t)^2, \end{aligned}$$

$w = (u, \rho)$. It is not hard to check that

$$F \in C^\infty(\widehat{\mathrm{Ad}}, \mathbb{E}_0 \times \mathbb{F} \times E_1).$$

An obviously necessary solvability condition is the compatibility demand $F(\mathcal{C}) \subset \mathbb{Z}$, i.e.

$$\tilde{\mathbb{B}}(w_0)w_0 = \mathcal{G}(w_0). \tag{3.6}$$

This will be assumed from now on. The following lemma will be proved in the appendix.

Lemma 3.3 *(Quasilinear character)*

Let $\varepsilon > 0$ be given. There is a $T^* = T^*(\varepsilon, M, w_0)$ such that if $T \in (0, T^*]$, $w_1, w_2 \in \mathcal{C}$, then

$$\begin{aligned} \|\mathbb{L}(w_1) - \mathbb{L}(w_2)\|_{\mathcal{L}(\mathbb{E}_1, \mathbb{Z})} &\leq \varepsilon \|w_1 - w_2\|_{\mathbb{E}_1}; \qquad (3.7) \\ \|F(w_1) - F(w_2)\|_{\mathbb{Z}} &\leq \varepsilon \|w_1 - w_2\|_{\mathbb{E}_1}. \qquad (3.8) \end{aligned}$$

Lemma 3.4 *(Maximal regularity for variable coefficients)*

There is a $T^* = T^*(M, w_0)$ and a $C = C(w_0)$ such that if $T \in (0, T^*]$ then $\mathbb{L}(\mathcal{C}) \subset \mathcal{L}_{is}(\mathbb{E}_1, \mathbb{Z})$ and

$$\|\mathbb{L}(w)^{-1}\|_{\mathcal{L}(Z, \mathbb{E}_1)} \leq C, \qquad w \in \mathcal{C}.$$

Proof: This is an immediate consequence of Lemma 3.2, (3.7), and standard perturbation results for linear isomorphisms. Note, in particular, that the bound C depends essentially only on $\|\mathbb{L}(w_0)^{-1}\|_{\mathcal{L}(Z, \mathbb{E}_1)}$ and can therefore be chosen independently of M. ∎

In view of Lemma 3.4 we can rewrite (3.5) as a fixed point equation

$$w = \Phi(w) := \mathbb{L}(w)^{-1}F(w), \qquad w \in \mathcal{C}. \tag{3.9}$$

Lemma 3.5 *(Contraction)*

For given $M > 0$, there is a $T^* = T^*(M, w_0)$ such that for all $T \in (0, T^*]$ we have

$$\|\Phi(w_1) - \Phi(w_2)\|_{\mathbb{E}_1} \leq \frac{1}{2}\|w_1 - w_2\|_{\mathbb{E}_1}. \tag{3.10}$$

Proof: Choose T^* small enough to be in the situation of Lemma 3.4, and by (3.8), to have

$$\|F(w)\|_{\mathbb{Z}} \leq C(w_0), \qquad w \in \mathcal{C}.$$

According to Lemma 3.3, for any $\varepsilon > 0$ we have, for $w_1, w_2 \in \mathcal{C}$ and T sufficiently small,

$$\begin{aligned}
& \|\Phi(w_1) - \Phi(w_2)\|_{\mathbb{E}_1} \\
\leq\ & \|\mathbb{L}(w_2)^{-1}\|_{\mathcal{L}(\mathbb{Z},\mathbb{E}_1)}\|\mathbb{L}(w_1) - \mathbb{L}(w_2)\|_{\mathcal{L}(\mathbb{E}_1,\mathbb{Z})}\|\mathbb{L}(w_1)^{-1}\|_{\mathcal{L}(\mathbb{Z},\mathbb{E}_1)}\|F(w_1)\| \\
& +\|\mathbb{L}(w_2)^{-1}\|_{\mathcal{L}(\mathbb{Z},\mathbb{E}_1)}\|F(w_1) - F(w_2)\|_{\mathbb{Z}} \\
\leq\ & C(w_0)\varepsilon\|w_1 - w_2\|_{\mathbb{E}_1}.
\end{aligned}$$

This implies the assertion if we choose $\varepsilon < 1/2C(w_0)$. ■

Lemma 3.6 *(Mapping into $\mathcal{C}$)*

There are constants $M = M(w_0)$, $T^* = T^*(w_0)$ such that for any $t \in (0, T^*]$ $\Phi(\mathcal{C}) \subset \mathcal{C}$ and (3.10) is satisfied.

Proof: Let $M \geq 2\|\mathcal{L}(\hat{w}_0)^{-1}F(\hat{w}_0)\|_{\mathbb{E}_1} + \|\hat{w}_0\|_{\mathbb{E}_1}$. Then, for small T (depending only on w_0) and any $w \in \mathcal{C}$

$$\|\phi(w)\|_{\mathbb{E}_1} \leq \|\phi(\hat{w}_0)\|_{\mathbb{E}_1} + \frac{1}{2}(\|w\|_{\mathbb{E}_1} + \|\hat{w}_0\|_{\mathbb{E}_1}) \leq M.$$

■

By Banach's fixed point theorem we get from this

Theorem 3.7 *(Short-time wellposedness)*

Let $w_0 = (u_0, \rho_0) \in h^{2+\alpha}(\bar{\Omega}) \times (h^{4+\beta}(\Gamma) \cap Ad)$ satisfy the compatibility condition

$$\mathcal{B}(\rho_0)u_0 = -\kappa u_0 H(\rho_0) - u_0^2.$$

Then there are constants $M, T^* > 0$ such that (3.5), or, equivalently, (3.1) has precisely one solution in $\mathcal{C}$ for any $T \in (0, T^*]$.

From this one deduces the statement of Theorem 1.1 easily.

4 Long time existence near equilibria

In this section we shall construct solutions close to equilibria living on arbitrary long time intervals. Remember

A pair $(\bar{\Gamma}, \bar{v})$ is an equilibrium solution of (1.1) if and only if it is given by $(\bar{\Gamma}, \bar{v}) = (\partial\mathbb{B}(x, R), (N-1)\kappa/R)$,

where $x \in \mathbb{R}^N$ and $R > 0$, i.e. each equilibrium is given by a sphere and a certain constant concentration. Without loss of generality we restrict ourselves to the treatment of the case $x = 0$, $R = 1$. Near this equilibrium we can simplify our abstract setting by choosing the unit sphere as reference domain:
Let $\Omega := \mathbb{B}(0,1) \subset \mathbb{R}^N$, $S := \partial\Omega$. Fix $\rho \in C^2(S, (-1/4, \infty)$, $\chi \in C^\infty([0,1])$ such that $0 \leq \chi' \leq 4$, $\chi \equiv 0$ on $[0, 1/3]$, $\chi \equiv 1$ on $[2/3, 1]$. For $x \in \mathbb{R}^N \setminus \{0\}$ we will write $r = |x|$ and $\omega = \omega(x) = x/r$. Defining

$$\Omega_\rho := \{x \in \mathbb{R}^n \setminus \{0\} \,|\, r < \rho(\omega)\} \cup \{0\},$$

the Hanzawa diffeomorphism $\theta_\rho \in \mathrm{Diff}^2(\bar{\Omega}, \bar{\Omega}_\rho)$ is given by the formula

$$\theta_\rho(x) := (1 + \rho(\omega)\chi(r))x.$$

Since $\bar{v} = (N-1)\kappa \neq 0$, near $\bar{v}$ the original problem (1.1) is equivalent to

$$\left.\begin{array}{rcll} \partial_t v - \Delta v &=& 0 & \text{in } \Omega(t), \\ \partial_n v + \kappa v H + u^2 &=& 0 & \text{on } \Gamma(t), \\ V_n &=& v^{-1}\partial_n v & \text{on } \Gamma(t). \end{array}\right\} \tag{4.1}$$

In this simplified situation the problem transform to

$$\left.\begin{array}{rcll} \partial_t u - \mathcal{A}(\rho)u - \mathcal{R}(u,\rho) &=& 0 & \text{in } \Omega, \\ \partial_t \rho + u^{-1} L_\rho \mathcal{B}(\rho) u &=& 0 & \text{on } S, \\ \mathcal{B}(\rho)u + \alpha u H(\rho) + u^2 &=& 0 & \text{on } S, \\ u(0) &=& u_0, & \\ \rho(0) &=& \rho_0, & \end{array}\right\} \tag{4.2}$$

where

$$
\begin{aligned}
\mathcal{A}(\rho) &= \theta_\rho^* \Delta \theta_*^\rho, \\
\mathcal{B}(\rho)u &= \theta_\rho^*\big(n_\rho \cdot (\nabla \theta_*^\rho u)|_{S_\rho}\big), \\
L_\rho &= (\theta_\rho^* n_\rho \cdot n_0)^{-1}, \\
\mathcal{R}(u,\rho) &= -\chi(r)\, u^{-1} L_\rho \mathcal{B}(\rho) u\, (g \cdot \theta_\rho^* \nabla \theta_*^\rho u), \qquad g(y) := y, \\
H(\rho) &= \theta_\rho^* (\Delta_{S_\rho} x_i \cdot n_\rho^i), \\
x_i(\theta_\rho \nu) &= (1+\rho(\nu))\nu_i, \quad \nu \in S.
\end{aligned}
$$

Here, n_ρ denotes the exterior unit normal on $S_\rho := \partial\Omega_\rho$, and Δ_{S_ρ} is the Laplace-Beltrami operator on S_ρ. In the equation defining $H(\rho)$, summation over $i = 1 \ldots n$ has to be performed. Here and in the sequel, trace operators from Ω to S will be suppressed in the notation if no confusion is likely.
As already mentioned in the introduction, we want to apply Theorem 2.1 in [DPZ08] to system (4.2). For this we must design a framework in the scale of Besov spaces.:

We fix $p \in (n+2, \infty)$, $T > 0$ and let $J := [0,T]$, $Q_T := \Omega \times J$, $\Sigma_T := S \times J$. For $s > 0$, $M \in \{\Omega, S, J\}$, and X a Banach space we define

$$
W_p^s(M,X) = \begin{cases} H_p^s(M,X) & s \in \mathbb{N}, \\ B_{pp}^s(M,X) & s \notin \mathbb{N}, \end{cases} \qquad W_p^s(M) := W_p^s(M,\mathbb{R}),
$$

and for $M \in \{\Omega, S\}$ we set $M_T := M \times J$ and

$$
W_p^{s,s/2}(M_T) := L^p(J, W_p^s(M)) \cap W_p^{s/2}(J, L^p(M)).
$$

These spaces are given their usual norms. Recall the standard interpolation result

$$
W_p^{s,s/2}(M_T) \hookrightarrow W_p^\tau(J, W_p^\sigma(M)) \tag{4.3}
$$

whenever $2\tau + \sigma \le s$.

Let

$$
\begin{aligned}
\mathbb{E}_1 &:= W_p^{2,1}(Q_T) \times W_p^{3-\frac{1}{p}, \frac{3}{2}-\frac{1}{2p}}(\Sigma_T), \\
\mathbb{E}_0 &:= L^p(Q_T) \times \Big(W_p^{1-\frac{1}{p}, \frac{1}{2}-\frac{1}{2p}}(\Sigma_T)\Big)^2.
\end{aligned}
$$

Observe that for the time trace operator

$$
\gamma_0 = \big[(u,\rho) \mapsto (u(0), \rho(0))\big]
$$

we have

$$\gamma_0 \in \mathcal{L}\big(\mathbb{E}_1, E_1\big), \qquad E_1 := W_p^{2-2/p}(\Omega) \times W_p^{3-3/p}(S).$$

Thus, the following definition is meaningful:

Definition 4.1 *Let $(u_0, \rho_0) \in E_1$. If $(u, \rho) \in \mathbb{E}_1$ is a solution of (4.2), then $(v, \Gamma) := (\theta_*^{\rho_0} u, S_\rho)$ is said to be a strong solution of (1.1).*

Let $m := N - 1$. Recall that $(\bar{u}, 0)$ with $\bar{u} \equiv \kappa m$ is an equilibrium solution to (4.2). We will show that near this point, the operator F given by (cf. (4.2))

$$F(u, \rho) = \begin{pmatrix} \partial_t u - \mathcal{A}(\rho)u - \mathcal{R}(u, \rho) \\ \partial_t \rho + u^{-1} L_\rho \mathcal{B}(\rho) u \\ \mathcal{B}(\rho)u + \kappa u H(\rho) + u^2 \end{pmatrix} \tag{4.4}$$

is well-defined and smooth with respect to the spaces just defined.

Lemma 4.2 *(Local Analyticity)*
There is an $\mathbb{E}_1$-neighborhood $\mathcal{V}$ of $(\bar{u}, 0)$ such that we have

$$F \in C^\omega(\mathcal{V}, \mathbb{E}_0)$$

for F defined by (4.4).

Proof: One has to check the mapping properties separately for all terms contained in F. To economize our notation we will occasionally write $\mathbb{E}_j^{(i)}$, $\mathcal{V}^{(i)}$ for the i-th component of $\mathbb{E}_j$ and $\mathcal{V}$.

When X, Y, and Z are spaces of functions on the same domain of definition for which a pointwise multiplication is defined we will write

$$X \cdot Y \hookrightarrow Z$$

for the fact that the map $(f, g) \mapsto fg$ is a continuous bilinear map from $X \times Y$ to Z. In particular, X will be called a multiplication algebra if $X \cdot X \hookrightarrow X$.

By the definition of the spaces we have

$$\begin{aligned} u \mapsto \partial_t u \quad &\in \quad \mathcal{L}\big(\mathbb{E}_1^{(1)}, \mathbb{E}_0^{(1)}\big), \\ \rho \mapsto \partial_t \rho \quad &\in \quad \mathcal{L}\big(\mathbb{E}_1^{(2)}, W_p^{\frac{1}{2} - \frac{1}{2p}}(J, L^p(S))\big), \end{aligned}$$

and, using (4.3) with $\tau = 1$, $\sigma = 1 - 1/p$,

$$\rho \mapsto \partial_t \rho \in \mathcal{L}\big(\mathbb{E}_1^{(2)}, L^p(J, W_p^{1-\frac{1}{p}}(S))\big).$$

The term $A(\rho)u$ can be written as a sum of terms

$$c_\beta\big(\nabla(\rho(\omega)\chi(r)), \nabla^2(\rho(\omega)\chi(r))\big)D^\beta u, \quad 1 \le |\beta| \le 2,$$

where the coefficient functions c_β are analytic functions on a neighborhood of $(0,0)$ in $\mathbb{R}^n \times \mathbb{R}^{(n^2)}$. Due to $p > n+2$ it is possible to choose $\tau > 1/p$, $\sigma > 2 + (n-1)/p$ such that $2\tau + \sigma \le 3 - 1/p$ and therefore by (4.3) and Sobolev embedding theorems

$$\mathbb{E}_1^{(2)} \hookrightarrow W_p^\tau(J, W_p^\sigma(S)) \hookrightarrow BUC(J, BUC^2(S)).$$

Consequently,

$$\rho \mapsto \big(\nabla(\rho(\omega)\chi(r)), \nabla^2(\rho(\omega)\chi(r))\big) \in \mathcal{L}\big(\mathbb{E}_1^{(2)}, BUC(Q_T)^{n+n^2}\big).$$

Together with the facts that $BUC(Q_T)$ is a multiplication algebra and

$$BUC(Q_T) \cdot L^p(Q_T) \hookrightarrow L^p(Q_T)$$

this implies

$$(u, \rho) \mapsto \mathcal{A}(\rho)u \in C^\omega\big(\mathcal{V}, \mathbb{E}_0^{(1)}\big)$$

for $\mathcal{V}$ sufficiently small. Similarly one shows

$$\mathcal{R} \in C^\omega\big(\mathcal{V}, \mathbb{E}_0^{(1)}\big).$$

The boundary term $u^{-1}L_\rho\mathcal{B}(\rho)u$ is a sum of terms

$$(u|_S)^{-1}c_i(\cdot, \rho, \nabla_S\rho)(D_i u)|_S, \qquad i = 1, \dots n$$

where the c_i are analytic functions on the bundle

$$\bigcup_{\omega' \in S} \big(\{\omega'\} \times (-1/4, \infty) \times T_{\omega'}S\big)$$

where $T_{\omega'}S$ is the tangent space in ω' at S. Note first that

$$u \mapsto (D_i u)|_S \in \mathcal{L}\big(\mathbb{E}_1^{(1)}, L^p(J, W_p^{1-\frac{1}{p}}(S))\big)$$

and that $W_p^{1-\frac{1}{p}}(S)$ is a multiplication algebra. By interpolation and embedding, we have

$$\mathbb{E}_1^{(2)} \hookrightarrow BUC(J, W_p^{2-\frac{1}{p}}(S))$$

and therefore (with TS denoting the tangent bundle over S and slightly abused notation)

$$\rho \mapsto (\rho, \nabla_S\rho) \in \mathcal{L}\big(BUC(J, W_p^{1-\frac{1}{p}}(S) \times W_p^{1-\frac{1}{p}}(S, TS))\big).$$

Together with the fact that $BUC(J, W_p^{1-\frac{1}{p}}(S))$ is a multiplication algebra this implies

$$\rho \mapsto c_i(\cdot, \rho, \nabla_S \rho) \in C^\omega\big(\mathcal{V}^{(2)}, BUC(J, W_p^{1-\frac{1}{p}}(S))\big).$$

Finally,

$$u \mapsto u|_S \in \mathcal{L}\big(\mathbb{E}_1^{(1)}, W_p^{2-\frac{1}{p}, 1-\frac{1}{2p}}(\Sigma_T)\big)$$

and $W_p^{2-\frac{1}{p}, 1-\frac{1}{2p}}(\Sigma_T) \hookrightarrow BUC(J, W_p^{1-\frac{1}{p}}(S))$. In the multiplication algebra $BUC(J, W_p^{1-\frac{1}{p}}(S))$, $\bar{u}$ is invertible, and therefore the inversion $v \mapsto v^{-1}$ is an analytic mapping near $\bar{u}$ in this space. Together with the multiplication property

$$BUC(J, W_p^{1-\frac{1}{p}}(S)) \cdot L^p(J, W_p^{1-\frac{1}{p}}(S)) \hookrightarrow L^p(J, W_p^{1-\frac{1}{p}}(S))$$

these facts imply

$$(u, \rho) \mapsto u^{-1} L_\rho \mathcal{B}(\rho) u \in C^\omega\big(\mathcal{V}, L^p(J, W_p^{1-\frac{1}{p}}(S))\big). \tag{4.5}$$

On the other hand, we also have

$$\begin{aligned} u \mapsto (D_i u)|_S \quad &\in \quad \mathcal{L}\big(\mathbb{E}_1^{(1)}, W_p^{\frac{1}{2}-\frac{1}{2p}}(J, L^p(S))\big), \\ \mathbb{E}_1^{(2)} \quad &\hookrightarrow \quad W_p^{\frac{1}{2}-\frac{1}{2p}}(J, BUC^1(S)), \\ u \mapsto u|_S \quad &\in \quad \mathcal{L}\big(\mathbb{E}_1^{(1)}, W_p^{\frac{1}{2}-\frac{1}{2p}}(J, BUC(S))\big). \end{aligned}$$

It follows from the results given in [Am1991], Sect. 4, that $W_p^{\frac{1}{2}-\frac{1}{2p}}(J, BUC(S))$ is a multiplication algebra and

$$W_p^{\frac{1}{2}-\frac{1}{2p}}(J, BUC(S)) \cdot W_p^{\frac{1}{2}-\frac{1}{2p}}(J, L^p(S)) \hookrightarrow W_p^{\frac{1}{2}-\frac{1}{2p}}(J, L^p(S)).$$

Using these results, we find in analogy to the proof of (4.5)

$$(u, \rho) \mapsto u^{-1} L_\rho \mathcal{B}(\rho) u \in C^\omega\big(\mathcal{V}, W_p^{\frac{1}{2}-\frac{1}{2p}}(J, L^p(S))\big)$$

and therefore together with (4.5)

$$(u, \rho) \mapsto u^{-1} L_\rho \mathcal{B}(\rho) u \in C^\omega\big(\mathcal{V}, \mathbb{E}_0^{(2)}\big).$$

The remaining nonlinear terms can be treated in the same fashion, using the fact that H is a quasilinear second-order differential operator, i.e. in local coordinates it can be written as a sum of terms of the form

$$d_{\mu\nu}(\rho, \nabla_s(\rho)) \partial_{\mu\nu} \rho, \quad d(\rho, \nabla_s(\rho)),$$

where d and $d_{\mu\nu}$ are analytic. ■

To determine the linearization of F at the equilibrium, we note

$$\begin{aligned} \mathcal{A}(0) &= \Delta, & \mathcal{A}'(0)[\sigma]\bar{u} &= 0, \\ \mathcal{R}(\bar{u},0) &= 0, & \partial_1\mathcal{R}(\bar{u},0)[w] &= 0, \\ & & \partial_2\mathcal{R}(\bar{u},0)[\sigma] &= 0, \\ \mathcal{B}(0) &= \partial_n, & \mathcal{B}'(0)[\sigma]\bar{u} &= 0, \\ H(0) &= -m, & H'(0)[\sigma] &= \Delta_S\sigma + m\sigma, \end{aligned}$$

which can be checked by straightforward calculation, see e.g. [EsSi97a] for the curvature term. This yields

$$F'(\bar{u},0)[w,\sigma] = \begin{pmatrix} \partial_t w - \Delta w \\ \partial_t\sigma + (\kappa m)^{-1}\partial_n w \\ \mathcal{B}_1 w + \mathcal{C}_1\sigma \end{pmatrix},$$

where

$$\mathcal{B}_1 w = \partial_n w + \kappa m w, \qquad \mathcal{C}_1\sigma = \kappa^2 m\Delta_S\sigma + \kappa^2 m^2\sigma. \tag{4.6}$$

Remember that for the time trace operator

$$\gamma_0 = \big[(u,\rho) \mapsto (u(0),\rho(0))\big]$$

we have

$$\gamma_0 \in \mathcal{L}\big(\mathbb{E}_1, E_1\big), \qquad E_1 := W_p^{2-2/p}(\Omega) \times W_p^{3-3/p}(S).$$

Define the space

$$\mathbb{Z} := \{((f,g_0,g_1),(z,\zeta)) \in \mathbb{E}_0 \times E_1 \,|\, \mathcal{B}_1 z + \mathcal{C}_1\zeta = g_1(0)\}$$

with its natural norm, and the operator $L \in \mathcal{L}(\mathbb{E}_1,\mathbb{Z})$ by

$$L = (F'(\bar{u},0),\gamma_0).$$

Lemma 4.3 *(Maximal regularity)*

We have

$$L \in \mathcal{L}_{is}(\mathbb{E}_1,\mathbb{Z}).$$

Proof: It is sufficient to show that the problem

$$L(u,\rho) = ((f,g_0,g_1),(z,\zeta)) \tag{4.7}$$

has precisely one solution $(u,\rho) \in \mathbb{E}_1$ for any

$$((f,g_0,g_1),(z,\zeta)) \in \mathbb{E}_0 \times E_1$$

if and only if the compatibility condition $\mathcal{B}_1 z + \mathcal{C}_1 \zeta = g_1(0)$ is satisfied. This is assured by Theorem 2.1 in [DPZ08] once we check that (4.7) satisfies all assumptions of that theorem.

Indeed, setting (in [DPZ08]) $m = 1$, $E = F = \mathbb{C}$, $\mathcal{A} = -\Delta$, $\mathcal{B}_0 = (\alpha m)^{-1}\partial_n$, $\mathcal{C}_0 = 0$, and $\mathcal{B}_1$, $\mathcal{C}_1$ as in (4.6) brings (4.7) in the framework of [DPZ08] with

$$Y_0 = Y_1 = W_p^{1-\frac{1}{p},\frac{1}{2}-\frac{1}{2p}}(\Sigma_T), \quad Z_u \times Z_\rho = \mathbb{E}_1, \quad \pi Z_u \times \pi Z_\rho = E_1.$$

(More precisely, the problem belongs to Case 1 there. Observe, in particular, that the compatibility condition involving $\mathcal{B}_0 z$ and g_0 is automatically satisfied.) Conditions (E)–(SC) are obviously satisfied. It remains to verify the Lopatinskii-Shapiro condition (LS). For this purpose, one has to consider solutions $v : [0,\infty) \longrightarrow \mathbb{C}$, $\sigma \in \mathbb{C}$ to the initial value problem

$$\left.\begin{array}{rcll} (\lambda + |\xi'|^2 - \partial_y^2)v(y) & = & 0 & (y > 0), \\ \lambda\sigma - (\kappa m)^{-1}\partial_y v(0) & = & 0, & \\ -\kappa^2 m|\xi'|^2\sigma - \partial_y v(0) & = & 0, & \end{array}\right\} \tag{4.8}$$

depending on the parameters $\xi' \in \mathbb{R}^m$, $\lambda \in \bar{\mathbb{C}}_+$, $|\xi'| + |\lambda| \neq 0$. Only solutions decaying for large $|\xi'|$ are admissible, hence

$$v(y) = v(0)e^{-\omega y}, \qquad \omega = \sqrt{\lambda + |\xi'|^2}$$

(using the principal value of the square root), and thus

$$\begin{pmatrix} \lambda & (\kappa m)^{-1}\omega \\ -\kappa^2 m|\xi'|^2 & \omega \end{pmatrix} \begin{pmatrix} \sigma \\ v(0) \end{pmatrix} = 0.$$

As we have

$$\begin{vmatrix} \lambda & (\kappa m)^{-1}\omega \\ -\kappa^2 m|\xi'|^2 & \omega \end{vmatrix} = \omega(\lambda + \kappa|\xi'|^2) \neq 0$$

this shows that (4.8) has only the trivial solution, and (LS) is satisfied. ∎

Remark: For $\kappa = 1$ the statement of Lemma 4.3 (up to lower order terms) is just Example 3.5 in [DPZ08].

We are ready now to prove the main result on existence of solutions to (4.2) near equilibria.

Theorem 4.4 *Let $p > n+2$, $T > 0$ be given. There is a $\varepsilon_0 = \varepsilon_0(p, T, \kappa) > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ there is a $\delta = \delta(\varepsilon, p, T, \kappa) > 0$ such that for all $(u_0, \rho_0) \in E_1$ satisfying*

$$\|(u_0 - \bar{u}, \rho_0)\|_{E_1} < \delta \tag{4.9}$$

and the compatibility condition

$$\mathcal{B}(\rho_0)u_0 + \kappa u_0 H(\rho) + u_0^2 = 0 \tag{4.10}$$

the problem (4.2) *has precisely one solution* (u^*, ρ^*) *that satisfies*

$$\|(u^* - \bar{u}, \rho^*)\|_{\mathbb{E}_1} \leq \varepsilon.$$

Proof: Define $K \in C^\omega(\mathcal{V}, \mathbb{E}_0)$ by

$$K(u, \rho) = F(u, \rho) - F'(\bar{u}, 0)(u - \bar{u}, \rho).$$

Then $K(\bar{u}, 0) = 0$, $K'(\bar{u}, 0) = 0$.

For $\eta > 0$ let

$$B_\eta := \{(u, \rho) \in \mathbb{E}_1 \mid \|(u - \bar{u}, \rho)\|_{E_1} \leq \eta\}$$

and choose $\varepsilon_0 > 0$ small enough to ensure $B_\varepsilon \subset \mathcal{V}$ and

$$\|K'(u, \rho)\|_{\mathcal{L}(\mathbb{E}_1, \mathbb{E}_0)} \leq 1/\big(2\|L^{-1}\|_{\mathcal{L}(\mathbb{Z}, \mathbb{E}_1)}\big), \quad (u, \rho) \in B_\varepsilon \tag{4.11}$$

for $\varepsilon \in (0, \varepsilon_0]$.

Let $\mathcal{E} \in \mathcal{L}(E_1, \mathbb{E}_1)$ denote a fixed right inverse of the trace operator γ_0. (Such an operator can be constructed along the lines described in [DPZ08], Sect. 4.1.) Fix $\varepsilon \in (0, \varepsilon_0]$ and let

$$\delta = \varepsilon / \max\{4\|L^{-1}\|_{\mathcal{L}(\mathbb{Z}, \mathbb{E}_1)}, \|\mathcal{E}\|_{\mathcal{L}(E_1, \mathbb{E}_1)}\}.$$

Pick $(u_0, \rho_0) \in E_1$ such that (4.9) and (4.10) are satisfied. Define the closed convex set

$$M_\varepsilon = \{(u, \rho) \in B_\varepsilon \mid (u, \rho)(0) = (u_0, \rho_0)\}$$

Due to our choice of δ we have $\mathcal{E}(u_0, \rho_0) \in M_\varepsilon$ so that this set is nonempty.

Note that (4.10) implies

$$(-K(u, \rho), (u_0 - \bar{u}, \rho_0)) \in \mathbb{Z}, \qquad (u, \rho) \in M_\varepsilon.$$

In view of this we define the operator $\Phi : M_\varepsilon \longrightarrow \mathbb{E}_1$ by

$$\Phi(u, \rho) = (\bar{u}, 0) + L^{-1}(-K(u, \rho), (u_0 - \bar{u}, \rho_0)).$$

Observe further that $(u, \rho) \in M_\varepsilon$ is a solution to (4.2) if and only if it is a fixed point of Φ. We establish the existence and uniqueness of such a fixed point by the Banach Contraction Principle. For $(u, \rho) \in M_\varepsilon$ we have

$$\|\Phi(u, \rho) - (\bar{u}, 0)\|_{\mathbb{E}_1} \leq \|L^{-1}\|_{\mathcal{L}(\mathbb{Z}, \mathbb{E}_1)}\big(\|K(u, \rho)\|_{\mathbb{E}_0} + \|(u_0 - \bar{u}, \rho_0)\|_{E_1}\big) \leq \varepsilon$$

due to (4.9) and (4.11), hence $\Phi[M_\varepsilon] \subset M_\varepsilon$. Furthermore, Φ is a contraction because, again by (4.11),

$$\begin{aligned} & \|\Phi(u_1,\rho_1) - \Phi(u_2,\rho_2)\|_{\mathbb{E}_1} \\ \leq\ & \|L^{-1}\|_{\mathcal{L}(\mathbb{Z},\mathbb{E}_1)} \|(K(u_2,\rho_2) - K(u_1,\rho_1),(0,0))\|_{\mathbb{E}_0\times E_1} \\ \leq\ & \tfrac{1}{2}\|(u_1,\rho_1)-(u_2,\rho_2)\|_{\mathbb{E}_1} \end{aligned}$$

for $(u_1,\rho_1),(u_2,\rho_2) \in M_\varepsilon$. This completes the proof. ∎

Remark: The arguments in the proof are completely parallel to those in the proof of [EPS03], Theorem 7.5.

Appendix: Proof of Lemma 3.3

Assume $T \leq 1$ without loss of generality. Let $w_i = (u_i,\rho_i)$, $i = 1,2$. To show (3.7), for $z = (v,\sigma) \in \mathbb{E}_1$ we have to show

$$\|(\mathbb{A}(w_1) - \mathbb{A}(w_2))z\|_{\mathbb{E}_0} \leq \varepsilon \|w_1 - w_2\|_{\mathbb{E}_1}\|z\|_{\mathbb{E}_1}, \tag{.12}$$

$$\|(\mathbb{B}(w_1) - \mathbb{B}(w_2))z\|_{\mathbb{F}} \leq \varepsilon \|w_1 - w_2\|_{\mathbb{E}_1}\|z\|_{\mathbb{E}_1}. \tag{.13}$$

Note first that there is a $h^{1+\alpha}$-neighborhood $\mathcal{U}$ of u_0, a $h^{3+\alpha}$-neighborhood $\mathcal{V} \subset \mathrm{Ad}$ of ρ_0, and a constant C such that for $\nu_1,\nu_2,\nu \in \mathcal{U}$, $\psi,\psi_1,\psi_2 \in \mathcal{V}$ and $j \in \{0,1\}$

$$\|\mathcal{A}(\psi_1) - \mathcal{A}(\psi_2)\|_{\mathcal{L}(h^{2+\alpha}(\bar\Omega),h^{\alpha}(\bar\Omega))} \leq C\|\psi_1-\psi_2\|_{h^{3+\beta}(\Gamma)}, \tag{.14}$$

$$\|L_{\psi_1}P(\psi_1) - L_{\psi_2}P(\psi_2)\|_{\mathcal{L}(h^{4+\beta}(\Gamma),h^{2+\beta}(\Gamma))} \leq C\|\psi_1-\psi_2\|_{h^{3+\beta}(\Gamma)}, \tag{.15}$$

$$\|\mathcal{B}(\psi_1) - \mathcal{B}(\psi_2)\|_{\mathcal{L}(h^{2+\alpha}(\bar\Omega),h^{1+\alpha}(\Gamma))} \leq C\|\psi_1-\psi_2\|_{h^{3+\beta}(\Gamma)}, \tag{.16}$$

$$\|\mathcal{B}(\psi_1) - \mathcal{B}(\psi_2)\|_{\mathcal{L}(C^{1}(\bar\Omega),C(\Gamma))} \leq C\|\psi_1-\psi_2\|_{h^{2+\beta}(\Gamma)}, \tag{.17}$$

$$\|\partial\mathcal{B}(\psi_1) - \partial\mathcal{B}(\psi_2)\|_{\mathcal{L}(h^{2+\beta}(\Gamma),\mathcal{L}(C^{1}(\bar\Omega),C(\Gamma)))} \leq C\|\psi_1-\psi_2\|_{h^{2+\beta}(\Gamma)}, \tag{.18}$$

$$\|\partial\mathcal{B}(\psi)\|_{\mathcal{L}(h^{2+\beta}(\Gamma),\mathcal{L}(C^{1}(\bar\Omega),C(\Gamma)))} \leq C, \tag{.19}$$

$$\|R(\nu_1,\psi_1) - R(\nu_2,\psi_2)\|_{h^{\alpha}(\bar\Omega)} \leq C\Big(\|\nu_1-\nu_2\|_{h^{1+\alpha}(\bar\Omega)} + \|\psi_1-\psi_2\|_{h^{2+\alpha}(\Gamma)}\Big), \tag{.20}$$

$$\|L_{\psi_1}Q(\psi_1) - L_{\psi_2}Q(\psi_2)\|_{h^{2+\beta}(\Gamma)} \leq C\|\psi_1-\psi_2\|_{h^{3+\beta}(\Gamma)}, \tag{.21}$$

$$\|H(\psi_1) - H(\psi_2)\|_{h^{j+\alpha}(\Gamma)} \leq C\|\psi_1-\psi_2\|_{h^{j+2+\alpha}(\Gamma)}, \tag{.22}$$

$$\|H(\psi_1) - H(\psi_2)\|_{C(\Gamma)} \leq C\|\psi_1-\psi_2\|_{C^{2}(\Gamma)}, \tag{.23}$$

$$\|\partial H(\psi_1) - \partial H(\psi_2)\|_{\mathcal{L}(C^{2}(\Gamma),C(\Gamma))} \leq C\|\psi_1-\psi_2\|_{C^{2}(\Gamma)}, \tag{.24}$$

$$\|\partial H(\psi)\|_{\mathcal{L}(C^{2}(\Gamma),C(\Gamma))} \leq C. \tag{.25}$$

Using (3.3) we find that for $w_i \in \mathcal{C}$, $t \in J$,

$$\begin{aligned} \|u_i(t) - u(0)\|_{h^{1+\alpha}(\bar\Omega)} = \|u_i(t) - u_0\|_{h^{1+\alpha}(\bar\Omega)} &\leq CT^{1/2}M, \\ \|\rho_i(t) - \rho(0)\|_{h^{3+\alpha}(\Gamma)} = \|\rho_i(t) - \rho_0\|_{h^{3+\alpha}(\Gamma)} &\leq CT^{(1+\beta-\alpha)/2}M, \end{aligned}$$

and therefore for T sufficiently small $(u_i(t), \rho_i(t) \in \mathcal{U} \times \mathcal{V}$.

Using (.14), (3.3), and the fact that $\rho_1(0) = \rho_2(0) = \rho_0$ we can estimate now for T small

$$\begin{aligned} & \big\|\big(\mathcal{A}(\rho_1) - \mathcal{A}(\rho_2)\big)v\big\|_{C(J,h^\alpha(\bar\Omega))} \\ = \; & \sup_{t\in J} \big\|\big(\mathcal{A}(\rho_1(t)) - \mathcal{A}(\rho_2(t))\big)v(t)\big\|_{h^\alpha(\bar\Omega)} \\ \leq \; & \sup_{t\in J} \|\mathcal{A}(\rho_1(t)) - \mathcal{A}(\rho_2(t))\|_{\mathcal{L}(h^{2+\alpha}(\bar\Omega),h^\alpha(\bar\Omega))} \|v(t)\|_{h^{2+\alpha}(\bar\Omega)} \\ \leq \; & C \sup_{t\in J} \|\rho_1(t) - \rho_2(t)\|_{h^{3+\beta}(\bar\Omega)} \|z\|_{\mathbb{E}_1} \\ \leq \; & C \sup_{t\in J} \|(\rho_1 - \rho_2)(t) - (\rho_1 - \rho_2)(0)\|_{h^{3+\beta}(\bar\Omega)} \|z\|_{\mathbb{E}_1} \\ \leq \; & C\|\rho_1 - \rho_2\|_{h^{1/2}(J,h^{3+\beta}(\bar\Omega))} T^{1/2} \|z\|_{\mathbb{E}_1} \\ \leq \; & CT^{1/2}\|w_1 - w_2\|_{\mathbb{E}_1}\|z\|_{\mathbb{E}_1}. \end{aligned}$$

Similarly, using (.15),

$$\big\|\big(L_{\rho_1}P(\rho_1) - L_{\rho_2}P(\rho_1)\big)\sigma\big\|_{C(J,h^{2+\beta}(\bar\Omega))} \leq CT^{1/2}\|w_1 - w_2\|_{\mathbb{E}_1}\|z\|_{\mathbb{E}_1}.$$

These two estimates imply (.12) for T small.

To show (.13) we estimate in an analogous way, using (.16),

$$\big\|\big(\mathcal{B}(\rho_1) - \mathcal{B}(\rho_2)\big)v\big\|_{C(J,h^{1+\alpha}(\Gamma))} \leq CT^{1/2}\|w_1 - w_2\|_{\mathbb{E}_1}\|z\|_{\mathbb{E}_1}. \tag{.26}$$

Finally,

$$\begin{aligned} & \big\|\big(\mathcal{B}(\rho_1) - \mathcal{B}(\rho_2)\big)v\big\|_{h^{(1+\alpha)/2}(J,C(\Gamma))} \\ = \; & \big\|\big(\mathcal{B}(\rho_1) - \mathcal{B}(\rho_2)\big)v\big\|_{C(J,C(\Gamma))} \\ & + \sup_{s,t\in J} |t-s|^{-(1+\alpha)/2} \\ & \qquad \big\|\big(\mathcal{B}(\rho_1(t)) - \mathcal{B}(\rho_2(t))\big)v(t) - \big(\mathcal{B}(\rho_1(s)) - \mathcal{B}(\rho_2(s))\big)v(s)\big\|_{C(\Gamma)}. \end{aligned}$$

The first term can be estimated by (.26). For the second term we use

$$\begin{aligned} & \big\|\big(\mathcal{B}(\rho_1(t)) - \mathcal{B}(\rho_2(t))\big)v(t) - \big(\mathcal{B}(\rho_1(s)) - \mathcal{B}(\rho_2(s))\big)v(s)\big\|_{C(\Gamma)} \\ \leq \; & \big\|\big(\mathcal{B}(\rho_1(t)) - \mathcal{B}(\rho_2(t))\big)\big(v(t) - v(s)\big)\big\|_{C(\Gamma)} \\ & + \big\|\big[\mathcal{B}(\rho_1(t)) - \mathcal{B}(\rho_2(t)) - \big(\mathcal{B}(\rho_1(s)) - \mathcal{B}(\rho_2(s))\big)\big]v(s)\big\|_{C(\Gamma)} \end{aligned}$$

and estimate the terms on the right separately.

For the first of these terms we use (.17) and (3.4) to get

$$\begin{aligned}
&\big\|\big(\mathcal{B}(\rho_1(t))-\mathcal{B}(\rho_2(t))\big)\big(v(t)-v(s)\big)\big\|_{C(\Gamma)}\\
\leq\;& \|\mathcal{B}(\rho_1(t))-\mathcal{B}(\rho_2(t))\|_{\mathcal{L}(C^1(\bar\Omega),C(\Gamma))}\|v(t)-v(s)\|_{C^1(\bar\Omega)}\\
\leq\;& C\|\rho_1(t)-\rho_2(t)\|_{h^{2+\beta}(\Gamma)}\|v\|_{C^{(1+\alpha)/2}(J,C^1(\bar\Omega)}|t-s|^{(1+\alpha)/2}\\
\leq\;& C\|(\rho_1-\rho_2)(t)-(\rho_1-\rho_2)(0)\|_{h^{2+\beta}(\Gamma)}\|z\|_{\mathbb{E}_1}|t-s|^{(1+\alpha)/2}\\
\leq\;& CT\|\rho_1-\rho_2\|_{C^1(J,h^{2+\beta}(\Gamma))}\|z\|_{\mathbb{E}_1}|t-s|^{(1+\alpha)/2}\\
\leq\;& CT\|w_1-w_2\|_{\mathbb{E}_1}\|z\|_{\mathbb{E}_1}|t-s|^{(1+\alpha)/2}.
\end{aligned}$$

Finally, for the second term we use (.18) and (.19) together with (3.4), to get

$$\begin{aligned}
&\big\|\big[\mathcal{B}(\rho_1(t))-\mathcal{B}(\rho_2(t))-\big(\mathcal{B}(\rho_1(s))-\mathcal{B}(\rho_2(s))\big)\big]v(s)\big\|_{C(\Gamma)}\\
\leq\;& \Big\|\int_s^t \partial_\xi\big[\mathcal{B}(\rho_1(\xi))-\mathcal{B}(\rho_2(\xi))\big]\,d\xi\Big\|_{\mathcal{L}(C^1(\bar\Omega),C(\Gamma))}\|v(s)\|_{C^1(\bar\Omega)}\\
\leq\;& \Big(\|\partial\mathcal{B}(\rho_1)-\partial\mathcal{B}(\rho_2)\|_{C(J,\mathcal{L}(h^{2+\beta}(\Gamma),\mathcal{L}(C^1(\bar\Omega),C(\Gamma)))}\|\rho_1'\|_{C(J,h^{2+\beta}(\Gamma))}\\
&+\|\partial\mathcal{B}(\rho_2)\|_{C(J,\mathcal{L}(h^{2+\beta}(\Gamma),\mathcal{L}(C^1(\bar\Omega),C(\Gamma)))}\|\rho_1'-\rho_2'\|_{C(J,h^{2+\beta}(\Gamma))}\Big)|t-s|\|z\|_{\mathbb{E}_1}\\
\leq\;& CT^{1-(1+\alpha)/2}\Big(\|\rho_1-\rho_2\|_{C(J,h^{2+\beta}(\Gamma))}+\|\rho_1-\rho_2\|_{C^1(J,h^{2+\beta}(\Gamma))}\Big)\cdot\\
&\cdot|t-s|^{(1+\alpha)/2}\|z\|_{\mathbb{E}_1}\\
\leq\;& CT^{1-(1+\alpha)/2}\|w_1-w_2\|_{\mathbb{E}_1}\|z\|_{\mathbb{E}_1}|t-s|^{(1+\alpha)/2}. \qquad (.27)
\end{aligned}$$

If T is chosen sufficiently small, these estimates imply (.13).

To show (3.8) we use (.20) and (3.3) and estimate for $t\in J$

$$\begin{aligned}
&\|R(u_1(t),\rho_1(t))-R(u_2(t),\rho_2(t))\|_{h^\alpha(\bar\Omega)}\\
\leq\;& C\big(\|u_1(t)-u_2(t)\|_{h^{1+\alpha}(\bar\Omega)}+\|\rho_1(t)-\rho_2(t)\|_{h^{2+\alpha}(\Gamma)}\big)\\
=\;& C\big(\|(u_1-u_2)(t)-(u_1-u_2)(0)\|_{h^{1+\alpha}(\bar\Omega)}\\
&\quad+\|(\rho_1-\rho_2)(t)-(\rho_1-\rho_2)(0)\|_{h^{2+\alpha}(\Gamma)}\big)\\
\leq\;& C\big(T^{1/2}\|u_1-u_2\|_{h^{1/2}(J,h^{1+\alpha}(\bar\Omega))}+T^{(1-\alpha+\beta)/2}\|\rho_1-\rho_2\|_{h^{(1-\alpha+\beta)/2}(J,h^{2+\alpha}(\Gamma))}\big)\\
\leq\;& CT^{(1-\alpha+\beta)/2}\|w_1-w_2\|_{\mathbb{E}_1}. \qquad (.28)
\end{aligned}$$

Similarly, using (.21), the Banach algebra property of little Hölder spaces, and

$$\|u_1(t)-u_2(t)\|_{h^{2+\beta}(\bar\Omega)}\leq CT^{\alpha-\beta}\|u_1-u_2\|_{C(J,h^{2+\alpha}(\bar\Omega))}$$

we get

$$\begin{aligned}
& \|L_{\rho_1(t)}(Q(\rho_1(t))+u_1(t)) - L_{\rho_2(t)}(Q(\rho_2(t))+u_2(t))\|_{h^{2+\beta(\Gamma)}} \\
\leq \;& C(T^{\alpha-\beta}+T^{1/2})\|w_1-w_2\|_{\mathbb{E}_1} \qquad (.29)
\end{aligned}$$

and using (.22)

$$\begin{aligned}
& \|(u_1(t)H(\rho_1(t))-u_1(t)^2)-(u_2(t)H(\rho_2(t))-u_2(t)^2)\|_{h^{1+\alpha}(\Gamma)} \\
\leq \;& CT^{(1-\alpha+\beta)/2}\|w_1-w_2\|_{\mathbb{E}_1}. \qquad (.30)
\end{aligned}$$

Finally, the estimate

$$\begin{aligned}
& \|(u_1H(\rho_1)-u_1^2)-(u_2H(\rho_2)-u_2^2)\|_{h^{(1+\alpha)/2}(J,C(\Gamma))} \\
\leq \;& CT^{1-(1+\alpha)/2}\|w_1-w_2\|_{\mathbb{E}_1} \qquad (.31)
\end{aligned}$$

can be shown by using

$$\begin{aligned}
& \|u_1(t)^2-u_2(t)^2-(u_1(s)^2-u_2(s)^2)\|_{C(\Gamma)} \\
= \;& 2\left\|\int_s^t (u_1u_1'-u_2u_2')(\xi)\,d\xi\right\|_{C(\Gamma)} \\
\leq \;& 2|t-s|\cdot \\
& \cdot\Big(\|u_1-u_2\|_{C(J,C(\bar\Omega))}\|u_1'\|_{C(J,C(\bar\Omega))}+\|u_2\|_{C(J,C(\bar\Omega))}\|u_1'-u_2'\|_{C(J,C(\bar\Omega))}\Big) \\
\leq \;& 2CT^{1-(1+\alpha)/2}|t-s|^{(1+\alpha)/2}\|w_1-w_2\|_{\mathbb{E}_1},
\end{aligned}$$

$s,t\in J$, as well as arguments similar to the ones used in (.27), based on (.23)–(.25). The estimate (3.8) follows now from (.28)–(.31) if T is chosen small enough. ∎

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