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Abstract

We discuss a pseudo-parabolic equation modelling two-phase flow in porous media, which includes a dynamic capillary pressure term. We extend results obtained previously for linear higher order terms and investigate the existence of travelling wave solutions in the non-linear and degenerate case. These cases may lead to non-smooth travelling waves, as well as to a discontinuous capillary pressure.

Keywords: Dynamic capillary pressure, non-linear, degenerate, pseudo-parabolic equations, sharp travelling waves.

1 Introduction

In this paper we discuss non-classical solutions of the Buckley-Leverett (BL) equation, from the perspective of a regularization derived from two-phase flow through porous media. The corresponding fundamental equation is:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \varepsilon \frac{\partial}{\partial x}\left(H(u) \frac{\partial p_c}{\partial x} \right). \tag{1.1}$$

Equation (1.1) is written in dimensionless form, and arises in two-phase flow in porous media as a model of oil recovery by water-drive in a one-dimensional horizontal flow. Here u stands for water saturation, and is expected to take values in [0,1].

Equation (1.1) results by coupling the mass conservation equations and Darcy laws for the water and the oil phases. Furthermore, p_c stands for the capillary pressure expressing the difference of phase pressures:

$$p_c = p_o - p_w, \tag{1.2}$$

where p_w and p_o are the water pressure and the oil pressure respectively. Typically, p_c is determined experimentally as a function of u. However, in this paper we consider the dynamic extension suggested by Hassanizadeh and Gray in [14], where the capillary pressure has a relaxation term:

$$p_c = p_c^{static} + p_c^{dynamic} = p_c^{static} + \varepsilon\tau L(u) u_t, \tag{1.3}$$

where p_c^{static} and $p_c^{dynamic}$ are the static and dynamic capillary pressures.

Furthermore, f and H are the water fractional flow function and the capillary induced diffusion function, which are given by

$$f(u) = \frac{\lambda_w(u)}{\lambda_w(u) + M\lambda_o(u)}, \qquad \text{and} \qquad H(u) = \frac{\lambda_w(u)\lambda_o(u)}{\lambda_w(u) + M\lambda_o(u)}, \tag{1.4}$$

where M is the water/oil viscosity ratio, while λ_w and λ_o are the normalized relative permeabilities. Commonly accepted in the engineering literature is the Brooks-Corey model:

$$\lambda_w(u) = u^{p+1}, \qquad \text{and} \qquad \lambda_o(u) = (1-u)^{q+1}, \qquad p > 0,\ q > 0, \tag{1.5}$$

leading to a function H that vanishes at $u = 0$ and $u = 1$. This degeneracy is an important feature of the model considered here. In this paper, we restrict to the specific functions in (1.5) to avoid non-essential technical difficulties. For simplicity we further take $p_c^{static} = u$ and $L(u) = 1$. Thus (1.1) becomes

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \varepsilon \frac{\partial}{\partial x}\left\{ H(u)(\frac{\partial u}{\partial x} + \varepsilon\tau \frac{\partial^2 u}{\partial x \partial t}) \right\}, \tag{1.6}$$

where τ is a positive constant, while the functions H and f are given by

$$H(u) = \frac{u^{p+1}(1-u)^{q+1}}{u^{p+1} + M(1-u)^{q+1}}, \qquad \text{and} \qquad f(u) = \frac{u^{p+1}}{u^{p+1} + M(1-u)^{q+1}}. \tag{1.7}$$

Their graphs are shown in Figure 1. Notice that

$$H(u) > 0, \quad \text{and} \quad f(u) > 0, \qquad \text{for} \quad 0 < u < 1.$$

Remark 1.1 *The definitions in (1.5) and (1.7) make sense only in the physically relevant regimes for* $0 \le u \le 1$. *For mathematical completeness we extend* λ_w *and* λ_o *by continuity with constant values* 0 *or* 1 *whenever* u *is outside* $[0,1]$. *The functions* f *and* H *are extended accordingly.*

For $\varepsilon = 0$, (1.1) becomes the non-viscous BL equation

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \tag{1.8}$$

a hyperbolic conservation law that can be seen as the limit ($\varepsilon \to 0$) of a family of extended equations of the form

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \mathcal{A}_\varepsilon(u), \quad \varepsilon > 0. \tag{1.9}$$

In the above, $\mathcal{A}_\varepsilon(u)$ is a regularization term involving higher order derivatives. Classical entropy solutions to the BL equation are constructed as limits of travelling wave (TW) solutions to (1.9), with $\mathcal{A}_\varepsilon(u)$ defined as

$$\mathcal{A}_\varepsilon(u) = \varepsilon \frac{\partial^2 u}{\partial x^2}.$$

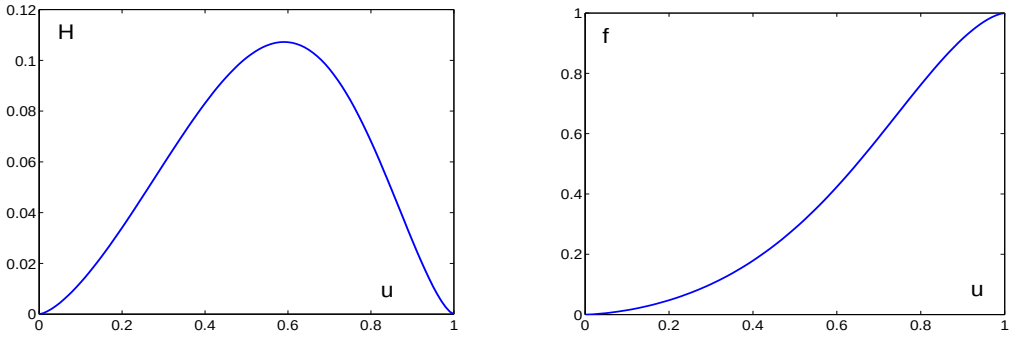

Figure 1: The functions H (left) and f (right) for $p = 0.5$, $q = 0.5$ and $M = 2.5$

A non-classical regularization is given in (1.6), which is motivated by dynamic capillarity effects, as mentioned in (1.3). For the case $H = 1$, we have the following linear pseudo-parabolic regularization of the BL equation

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2} + \varepsilon^2 \tau \frac{\partial^3 u}{\partial x^2 \partial t},$$

for which the existence of TW solution has been studied in [10]. In the limit $\varepsilon \to 0$, these TW solutions become shocks, which are weak solutions to the non-viscous BL equation. These shocks violate the Oleinik entropy conditions, and therefore are called non-classical. TW solutions to (1.9) and the relation with non-standard shock solutions to (1.8) are analyzed in [20], see also [2], [3], [15], [17]. The regularization there involves higher order spatial derivatives, but no mixed terms. Furthermore, non-local regularization operators and their effect on shocks solutions to hyperbolic conservation laws are studied in [19], [31]. The TW approach for degenerate pseudo-parabolic problems modelling one phase flow in porous media is considered in [4], [5], [12], [26]. In a similar context, a fourth order regularization is studied in [8] .

The present paper deals with (1.6), which is a non-linear and degenerate regularization of (1.8). In the spirit of [10], we seek TW solutions connecting a left state u_ℓ to a right state u_r. Here we only consider the case $u_l > u_r$, but in the degenerate context. As will be seen below, the degeneracy can lead to the so-called "sharp TW solutions", which are non-smooth, see [1], [21], [22], [32]. The case $u_l < u_r$ in the degenerate context is left for a future investigation. For linear regularization, this has been thoroughly analyzed in [10].

Qualitative properties of solutions to pseudo-parabolic problems are discussed in [18], where the small and waiting-time behavior of solutions to a Darcy type model involving a dynamic pressure saturation is analyzed. Short time existence is obtained in [11], whereas global existence results can be found in [24] and [25]. In particular, the model analyzed in [24] is very close to the present one. Besides, there it is shown that the solution is bounded essentially by the degeneracy values. A non-linear model involving memory terms is investigated in [30]. The present paper is also motivated by the experimental results in [9]. We further refer to [13] for a review of experimental work on dynamic effects in the pressure-saturation relationship, and to [23] for a dimensional analysis of such models.

This paper is organized in the following way: In Section 2 we investigate the non-linear, non-degenerate case where $u_r > 0$ and $u_\ell < 1$. Then the results are similar to the ones for a linear regularization (see [10]). In particular, a monotone and continuous dependence of τ on u_ℓ is shown. Section 3 includes degenerate cases, but is devoted to smooth TW, defined in the classical sense. The focus lies on the case $u_r = 0$. Depending on the parameters, two situations are encountered. These are described in terms of two constants $\alpha \in (u_r, 1)$ and $\beta \in (\alpha, 1]$ that will be defined below. In the first situation, $\beta < 1$ and for any $u_\ell \in (\alpha, \beta)$, there exists a $\tau > 0$ s.t. TW solutions connecting u_l to u_r are possible. Whenever $\beta = 1$, smooth TW solutions are only possible if $\tau < \tau^*$. In the limit case $\tau \nearrow \tau^*$, the corresponding u_l approaches 1. Section 4 continues the investigations in Section 3 by considering the case $\tau > \tau^*$, when smooth TW solutions are not possible. Then we consider the notion of TW in a larger sense, allowing for discontinuities in the derivatives and connecting $u_\ell = 1$ to $u_r > 0$. In the spirit of [1], [21], [22], [32], such solutions are called "sharp front solutions". These fronts are encountered due to the degeneracy at $u = 1$ and appear at the transition from regions where $u = 1$ to values of u below 1. At the end of Section 4 we also consider two degenerate points. Specifically, we take $u_\ell = 1$ and $u_r = 0$. Then we give a selection criterion leading to sharp TW that are continuously differentiable whenever $u < 1$. In particular, the transition $u > 0$ to $u = 0$ is smooth and encountered at some finite coordinate. Finally, Section 5 presents some numerical examples to illustrate the theoretical results.

2 Travelling waves: non-linear, non-degenerate case

Entropy shock solutions to BL equation are based on TW solutions to (1.8) and their limits as $\varepsilon \to 0$. A TW solution has the form

$$u(x,t) = u(\eta), \qquad \text{where} \qquad \eta = \frac{x - st}{\varepsilon}. \tag{2.1}$$

Notice that the TW solution is still denoted by u to avoid unnecessary notations. Applying this into (1.6) we obtain

$$-su' + (f(u))' = \{H(u)(u' - \tau s u'')\}'. \tag{2.2}$$

With given $0 \le u_r < u_\ell \le 1$, these waves are connecting the left state u_ℓ to the right state u_r,

$$u(-\infty) = u_\ell, \qquad \text{and} \qquad u(+\infty) = u_r. \tag{2.3}$$

Remark 2.1 *Values outside* $[0,1]$ *are physically unrealistic. From the mathematical point of view, if one of the states is outside* $[0,1]$*, the problem degenerates and the solution remains constant, so no connection with different states is possible. This is why we only consider the case* $0 \le u \le 1$*.*

Integrating (2.2) over $\mathbb{R}$ and assuming $u'(\eta) \to 0$ as $\eta \to \pm\infty$ yields

$$-s(u_r - u_\ell) + \{f(u_r) - f(u_\ell)\} = 0.$$

This leads to the Rankine-Hugoniot (RH) condition:

$$s = s(u_r, u_\ell) = \frac{f(u_r) - f(u_\ell)}{u_r - u_\ell}. \tag{2.4}$$

Furthermore, integrating (2.2) over $(\eta, +\infty)$ and using the condition at $\eta = +\infty$, we obtain

$$s(u - u_r) - \{f(u) - f(u_r)\} = -H(u)(u' - \tau s u''), \tag{2.5}$$

or

$$s\tau u'' - u' - g(u; u_r, u_\ell) = 0, \tag{2.6}$$

where

$$g(u; u_r, u_\ell) = \frac{1}{H(u)} \left\{ s(u_r, u_\ell)(u - u_r) - [f(u) - f(u_r)] \right\}. \tag{2.7}$$

With s defined in (2.4), we seek a solution u of the problem

$$(\mathrm{TW}_1) \begin{cases} s\tau u'' - u' - g(u; u_r, u_\ell) = 0, & \eta \in \mathbb{R}, \\ u(-\infty) = u_\ell, \qquad u(+\infty) = u_r. \end{cases} \tag{2.8}$$

With

$$\xi = -\frac{1}{\sqrt{s\tau}}\,\eta, \qquad u(\eta) = u(\xi) \quad \text{and} \quad c = \frac{1}{\sqrt{s\tau}}, \tag{2.9}$$

Problem (TW_1) becomes:

$$(\mathrm{TW}_2) \begin{cases} u'' + cu' - g(u; u_r, u_\ell) = 0, & \xi \in \mathbb{R}, \\ u(-\infty) = u_r, \qquad u(+\infty) = u_\ell. \end{cases} \tag{2.10}$$

We have the following:

Lemma 2.1 *A necessary condition for the existence of a travelling wave solution of Problem $(TW_{1,2})$ is*

$$\int_{u_r}^{u_\ell} g(u; u_r, u_\ell)\, du > 0. \tag{2.11}$$

Proof Multiplying the differential equation in (2.10) by u' and integrating over $\mathbb{R}$ yields

$$c \int_{\mathbb{R}} (u')^2 d\xi - \int_{u_r}^{u_\ell} g(u; u_r, u_\ell)\, du = 0.$$

Since $c = 1/\sqrt{s\tau} > 0$, the assertion follows. □

In the standard case, when $\tau = 0$, an elementary analysis shows that Problem (TW_1) has a solution if and only if f and two states u_ℓ and u_r satisfy (i) the RH condition (2.4), and (ii) the Oleinik entropy condition:

$$\text{(E)} \quad \frac{f(u_\ell) - f(u)}{u_\ell - u} \geq \frac{f(u_\ell) - f(u_r)}{u_\ell - u_r} \quad \text{for} \quad u \quad \text{between} \quad u_\ell \quad \text{and} \quad u_r.$$

So it is natural to define $\alpha(u_r) > u_r$ as the unique value of u such that

$$f'(\alpha) = \frac{f(\alpha) - f(u_r)}{\alpha - u_r}. \tag{2.12}$$

To investigate the non-standard case $\tau > 0$, we start by introducing

$$\beta(u_r) = \sup \left\{ \alpha < u < 1 : \int_{u_r}^{u} g(t; u_r, u)\, dt > 0 \right\}. \tag{2.13}$$

Remark 2.2 *Recalling Lemma 2.1, solutions are possible only for left states u_ℓ for which*

$$\int_{u_r}^{u_\ell} g(u; u_r, u_\ell)\, du > 0.$$

These left states are bounded from above by $\beta(u_r)$.

In view of (1.7), the regularization terms are vanishing whenever $u = 0$ or $u = 1$, where the equation becomes degenerate. We start by discussing the non-degenerate case. Specifically, in the remaining of this section we consider:

$$0 < u_r < u_\ell < 1.$$

For $u_r > 0$, $\alpha < u_\ell < \beta$ and $s = s(u_r, u_\ell)$, the function $g(u; u_r, u_\ell)$ (as shown in Figure 2) has three positive zeros:

$$u = u_r, \qquad u = u_m \quad \text{and} \qquad u = u_\ell,$$

where

$$g'(u_r; u_r, u_\ell) > 0, \qquad g'(u_m; u_r, u_\ell) < 0, \qquad \text{and} \quad g'(u_\ell; u_r, u_\ell) > 0.$$

Here primes denote differentiation with respect to u. We have the following:

Lemma 2.2 *Let $\lambda_w(u)$ and $\lambda_o(u)$ be given in (1.5) and $u_r > 0$. If $q \geq 1$, then $\beta < 1$.*

Proof Suppose that $\beta = 1$, then

$$\int_{u_r}^{1} g(t; u_r, 1)\, dt \geq 0. \tag{2.14}$$

However, as $t \to 1$, g behaves asymptotically as $-(1-t)^{-q}$. Therefore we can find two constants $C_1 > C_2 > 0$ and a $t_0 < 1$ such that

$$-C_1(1-t)^{-q} \leq g(t; u_r, 1) \leq -C_2(1-t)^{-q} \qquad \text{for all} \quad t \in [t_0, 1),$$

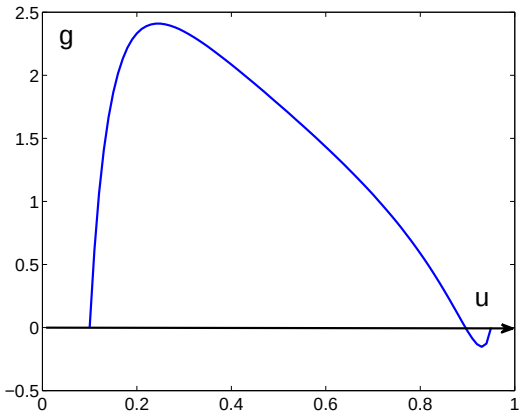

Figure 2: The function g for $p = q = 0.5$, $M = 2.5$, $s = 1.125$, $u_r = 0.1$ and $u_\ell = 0.95$

Since $q \geq 1$,

$$\int_{t_0}^{1} -(1-t)^{-q} dt = -\infty.$$

Here and below the equalities should be understood in the sense that

$$\lim_{\mu \to 0} \int_{t_0}^{1-\mu} -(1-t)^{-q} = -\infty.$$

Therefore

$$\int_{u_r}^{1} g(t; u_r, 1) dt = \int_{u_r}^{t_0} g(t; u_r, 1) dt + \int_{t_0}^{1} g(t; u_r, 1) dt = C - \infty = -\infty,$$

contradicting (2.14). This means that $\beta < 1$. □

Remark 2.3 *The same result holds in a degenerate context. Specifically, if $u_r = 0$ and $q \geq 1$, then $\beta < 1$. This means that the left state u_ℓ can not be 1, so degeneracy may only occur at $u = 0$.*

Whenever $\beta < 1$, the following existence result can be obtained. Its proof goes along the lines of Lemma 4.1 in [10]. We omit the details here.

Theorem 2.1 *Given $u_r > 0$, $p, q > 0$ and with α, β introduced in (2.12) and (2.13), for every $u_\ell \in (\alpha, \beta)$ there exists a unique value of $\tau > 0$ such that Problem (TW_1) admits a solution. This solution is unique, decreasing and travels with speed $s(u_\ell, u_r)$ given in (2.4).*

Remark 2.4 *For fixed $u_r > 0$, $p, q > 0$, given $u_\ell \in (\alpha, \beta)$, Theorem 2.1 provides a unique $\tau = \tau(u_l)$. In this way, we define the function*

$$\tau : (\alpha, \beta) \to \mathbb{R}^+.$$

Lemma 2.3 *The function τ is continuous and strictly increasing on (α, β).*

Proof We follow the proof of Lemma 4.2 in [10] and show first the monotonicity of τ. Taking two left states $\alpha < u_{\ell,1} < u_{\ell,2} < \beta$, define

$$s_i = \frac{f(u_{\ell,i}) - f(u_r)}{u_{\ell,i} - u_r}, \qquad \text{and} \qquad g_i(u) = \frac{1}{H(u)}\{s_i(u - u_r) - [f(u) - f(u_r)]\}, \quad i = 1, 2.$$

Observe that

$$\frac{d}{du}\left(\frac{f(u) - f(u_r)}{u - u_r}\right) < 0 \qquad \text{for} \qquad \alpha < u < \beta,$$

therefore

$$s_1 > s_2 \qquad \text{and} \qquad g_1(u) > g_2(u) \qquad \text{for all} \quad u \in (u_r, u_{\ell,1}).$$

Rewriting Problem (TW_2) as the first order system

$$\begin{cases} u' = w, \\ w' = -c_i w + g_i(u), \end{cases} \tag{2.15}$$

we denote by Γ_i $(i = 1, 2)$ the orbits emerging from the saddle $(u_r, 0)$. They do so under an angle θ_i given by

$$\theta_i = \frac{1}{2}\left(\sqrt{c_i^2 + 4g_i'(u_r)} - c_i\right).$$

Plainly,

$$g_i'(u_r) = \frac{1}{H(u_r)}\{s_i - f'(u_r)\}.$$

Define the function

$$\theta(c, s) \stackrel{\text{def}}{=} \frac{1}{2}\left(\sqrt{c^2 + 4g'(u_r)} - c\right).$$

Then

$$\frac{\partial \theta}{\partial c} < 0, \qquad \text{and} \qquad \frac{\partial \theta}{\partial s} > 0.$$

and we conclude that $c_1 > c_2$ as in [10]. This gives

$$\tau_2 s_2 > \tau_1 s_1, \quad \text{therefore} \quad \tau_2 > \frac{s_1}{s_2}\tau_1 > \tau_1,$$

as asserted. For the continuity of τ, one can follow the ideas in Lemma 4.3 in [10]. □

3 Smooth travelling waves for degenerate case

In this section, we seek standard, smooth TW solutions for the degenerate case $u_r = 0$, where H vanishes, whereas g becomes unbounded. We start with a non-existence result.

Theorem 3.1 *If $p \geq 1$ and $u_r = 0$, then no TW solutions are possible.*

Proof As in Lemma 2.2, if $p \geq 1$ we have

$$\int_0^{u_\ell} g(t;0,u_\ell)dt = -\infty, \quad \text{for} \quad u_\ell > 0.$$

According to Lemma 2.1, no TW solutions exist. □

Remark 3.1 *In a similar fashion, no TW solutions exist for $u_\ell = 1$ whenever $q \geq 1$.*

In view of the above, no TW solutions exist for $p \geq 1$ and $u_r = 0$, or $q \geq 1$ and $u_\ell = 1$. Therefore in this section we restrict to the cases $0 < p < 1$ and $0 < q < 1$.

Since $u_l > u_r$ is assumed, we seek monotone decreasing TW solutions of Problem (TW_1). If such solutions exist, a bijection $\eta \to u$, where $\eta \in \mathbb{R}$ and $u \in (u_r, u_\ell)$, can be defined. Therefore the functions $\eta : (u_r, u_\ell) \to \mathbb{R}$, as well as $w : (u_r, u_\ell) \longrightarrow \mathbb{R}$, $w(u) = -u'(\eta(u))$ can be defined. Further, since u is decreasing, we have $w > 0$ on (u_r, u_ℓ). Nevertheless, $w(u_r)$ and $w(u_\ell)$ still have to fixed. To do so we recall that the waves sought in this section are smooth, monotone, and approaching u_ℓ and u_r asymptotically. Therefore we have $\lim\limits_{\eta \to \pm\infty} u'(\eta) = 0$ yielding $w(u_r) = w(u_l) = 0$. In terms of w, Problem (TW_1) introduced in (2.8) becomes

$$\begin{cases} \tau s w w' + w = g(u), & \text{for } u \in (u_r, u_\ell), \\ w(u_r) = 0, w(u_\ell) = 0. \end{cases} \tag{3.1}$$

Notice that this first order problem has two boundary conditions, which will be fixed by the parameter τ. When seeking monotone TW solutions to Problem (TW_1) we in fact seek a pair $(w, \tau) \in C^1(u_r, u_\ell) \times (0, \infty)$ for which (3.1) holds. Once w is known, u can be obtained by integration in η:

$$\eta(u) = \int_u^{\frac{u_\ell + u_r}{2}} \frac{dz}{w(z)}, \tag{3.2}$$

defining a TW satisfying $u(0) = \frac{u_\ell + u_r}{2}$. This choice of $u(0)$ is a possible normalization of the TW, any value in (u_r, u_ℓ) is possible.

3.1 Existence results

Theorem 3.2 *Let $0 < p < 1$, $u_r = 0$ and $\alpha < u_\ell < \beta$. Then there exists a unique τ for which Problem (TW_1) admits a solution.*

Proof . The proof is divided into two steps:
Step 1: Existence. We prove that there exists a unique pair (τ, w) such that (3.1) is satisfied. By (3.2), this also provides a solution to Problem (TW_1).

First observe that, since $u_r = 0$ and $u_\ell < 1$, we have $g(u) \to +\infty$ as $u \to 0$ and $g(u_m) = g(u_\ell) = 0$ for some $u_m \in (0, u_\ell)$. Clearly, for any $\tilde{w} > 0$, there exists a unique $\tilde{u} \in (0, u_m)$ such that $g(\tilde{u}) = \tilde{w}$. For proving the theorem, we consider two cases $u > \tilde{u}$, and $u < \tilde{u}$.

Consider the initial value problem

$$\begin{cases} \tau s w w' + w = g(u), & \text{for } u > \tilde{u}, \\ w(\tilde{u}) = \tilde{w}. \end{cases} \tag{3.3}$$

Notice that (3.3) is defined only for non-negative values of w. Furthermore, we are interested in values of w for $u \leq u_\ell$, and in particular in $w(u_\ell)$.

We start by proving that for any $\tilde{w}$, there exists a $\tau = \tau(\tilde{u})$ such that the solution of (3.3) also satisfies $w(u_\ell) = 0$. To this end, define $\nu(\tau) = \sup\{u \leq u_\ell \mid w(u) > 0\}$ and notice that $\tau \to \infty$ implies $w' = 0$ uniformly on $(\tilde{u}, \nu)$. Therefore if τ is large enough we have $w(u_\ell) > 0$ and $\nu(\tau) = u_\ell$. On the other hand, $\tau \to 0$ gives $w \to g$, thus $\nu(\tau) < u_\ell$ for τ small enough. Due to continuous dependence on the data, there exists a $\tau = \tau(\tilde{u})$ such that $w(u_\ell) = 0$.

Further, taking $C = \int_{u_m}^{u_\ell} g(u)du$, from (3.3) we have

$$C < \int_{\tilde{u}}^{u_\ell} g(u)du = -\frac{s\tau(\tilde{u})}{2}\tilde{w}^2 + \int_{\tilde{u}}^{u_\ell} w(u)du < -\frac{s\tau(\tilde{u})}{2}\tilde{w}^2 + (u_\ell - \tilde{u})\tilde{w},$$

since w is decreasing if $u > \tilde{u}$. As $\tilde{u} \to 0$ implies $\tilde{w} \to +\infty$, we also conclude that $\tau(\tilde{u}) \to 0$.

Next, from (3.3) we obtain $w' = \frac{g(u)-w}{s\tau(\tilde{u})w}$ and therefore

$$w'' = \frac{g'w - gw'}{s\tau(\tilde{u})w^2}.$$

Clearly, g is negative and has a minimum on (u_m, u_ℓ). Let u_e be the minimum point attained. On (u_m, u_e) we have $g' < 0$, $w > 0$ and $w' < 0$, giving $w'' < 0$. At $u = u_m$, by (3.3) we have $w'(u_m) = -\frac{1}{s\tau(\tilde{u})}$. Since $w'' \leq 0$ on (u_m, u_e), we have $0 < w(u_e) < w(u_m) - \frac{1}{s\tau(\tilde{u})}(u_e - u_m)$ giving

$$w(u_m) > \frac{1}{s\tau(\tilde{u})}(u_e - u_m). \tag{3.4}$$

As seen before, $\tau(\tilde{u}) \to 0$ as $\tilde{u} \to 0$ showing that $w(u_m) \to +\infty$ in this case.

Once these are known, we seek for $\tilde{u} \in (0, u_m)$ s.t. w solving (3.3) for $\tau = \tau(\tilde{u})$ obtained above and for $u < \tilde{u}$ also satisfies $w(0) = 0$. We do this by employing similar ideas, namely by showing that $w > 0$ everywhere on $(0, \tilde{u})$ for $\tilde{u}$ small enough, and that $w(u) = 0$ for some $u > 0$ for larger values of $\tilde{u}$.

Consider (3.3) for $\tau = \tau(\tilde{u})$, but with $u < \tilde{u}$ and define $\mu(\tilde{u}) = \inf\{0 < u \leq \tilde{u} \mid w(u) > 0\}$. Below we prove that $\mu(\tilde{u}) = 0$ and $w(0) = 0$ for some $\tilde{u} \in (0, u_m)$, yielding a $\tau = \tau(\tilde{u})$ and a w solving (3.1). To this end we integrate (3.3) from $\mu(\tilde{u})$ to u_ℓ and obtain

$$\frac{\tau(\tilde{u})s}{2}(w^2(u_\ell) - w^2(\mu(\tilde{u}))) + \int_{\mu(\tilde{u})}^{u_\ell} w(u)du = \int_{\mu(\tilde{u})}^{u_\ell} g(u)du.$$

If $\mu(\tilde{u}) > 0$, then $w(\mu(\tilde{u})) = 0$, leading to

$$\int_{\mu(\tilde{u})}^{u_\ell} g(u)du = \int_{\mu(\tilde{u})}^{u_\ell} w(u)du > \int_{\tilde{u}}^{u_m} w(u)du > (u_m - \tilde{u})w(u_m).$$

As $\tilde{u} \to 0$, $w(u_m) \to +\infty$ and the term on the right side in the above becomes unbounded. However, $\int_{\mu(\tilde{u})}^{u_\ell} g(u)du$ is bounded, showing that $\mu(\tilde{u}) = 0$ whenever $\tilde{u}$ is small enough.

It remains to rule out the possibility of having $\mu(\tilde{u}) = 0$ and $w(0) > 0$ for all $\tilde{u} \in (0, u_m)$. Then a $\tilde{u}$ exists s.t. $\mu(\tilde{u}) = 0$ and $w(0) = 0$. To prove this assertion above, we assume that for all pairs $(\tilde{u}, \tau(\tilde{u}))$, the corresponding w satisfies $w(0) > 0$. Integrating (3.3) from 0 to u_ℓ gives

$$-\frac{1}{2}s\tau(\tilde{u})w^2(0) + \int_0^{u_\ell} w(u)du = \int_0^{u_\ell} g(u)du =: B > 0.$$

Notice that $w \le \tilde{w}$ for all $u \in (0, u_\ell)$, therefore

$$0 < B \le -\frac{1}{2}s\tau(\tilde{u})w^2(0) + u_\ell\tilde{w} < -\frac{1}{2}s\tau(\tilde{u})w^2(0) + \tilde{w}.$$

Choosing $\tilde{u}$ s.t. $\tilde{w} < B$ we have $0 < B - \tilde{w} < -\frac{1}{2}s\tau(\tilde{u})w^2(0) < 0$, which is a contradiction.

Step 2: Uniqueness. We assume the existence of (τ_1, w_1) and (τ_2, w_2) satisfying (3.1), where $\tau_1 > \tau_2 > 0$. Integrating (3.1) over $(0, u)$ yields

$$\frac{s\tau_1}{2}w_1(u)^2 + \int_0^u w_1(t)dt = \int_0^u g(t)dt = \frac{s\tau_2}{2}w_2(u)^2 + \int_0^u w_2(t)dt. \tag{3.5}$$

Since $\tau_1 > \tau_2$, this gives $w_1 < w_2$ for u small enough (see the left picture of Figure 3). If w_1 and w_2 do not intersect inside $(0, u_\ell)$, then $w_1 > w_2$ everywhere. Taking $u = u_\ell$ in (3.5) gives

$$\int_0^{u_\ell} w_1(u)du = \int_0^{u_\ell} g(u)du = \int_0^{u_\ell} w_2(u)du$$

which contradicts the ordering in the $w's$.

Thus w_1 and w_2 must have at least one intersection point inside $(0, u_\ell)$, where w_1 and w_2 are both positive. No intersection can occur at points where w_1 or w_2 is increasing. To see this we let $u_0 \in (0, u_\ell)$ be the first intersection point, so $w_1(u_0-) < w_2(u_0-)$ implying that $w_1'(u_0) \ge w_2'(u_0)$. However, since

$$w_1'(u_0) = \frac{g(u_0) - w_1(u_0)}{s\tau_1 w_1(u_0)}, \quad \text{and} \quad w_2'(u_0) = \frac{g(u_0) - w_2(u_0)}{s\tau_2 w_2(u_0)},$$

if $w_1(u_0) < g(u_0)$, then

$$w_2'(u_0) = \frac{g(u_0) - w_2(u_0)}{s\tau_2 w_2(u_0)} = \frac{g(u_0) - w_1(u_0)}{s\tau_2 w_1(u_0)} > \frac{g(u_0) - w_1(u_0)}{s\tau_1 w_1(u_0)} = w_1'(u_0).$$

which contradicts the above. Next, if $w_1(u_0) = g(u_0)$, then $w_1'(u_0) = w_2'(u_0) = 0$ and there exists $\delta > 0$ small enough such that

$$w_1'(u_0 - \delta) > w_2'(u_0 - \delta) > 0, \quad \text{and} \quad w_2(u_0 - \delta) > w_1(u_0 - \delta) > 0,$$

Hence

$$(w_1' - w_2')|_{u=u_0-\delta} = \left[\frac{1}{s\tau_1}\left(\frac{g}{w_1} - 1\right) - \frac{1}{s\tau_2}\left(\frac{g}{w_2} - 1\right)\right]\Big|_{u=u_0-\delta} > 0,$$

therefore

$$\frac{\tau_2}{\tau_1} > \frac{\frac{g}{w_2} - 1}{\frac{g}{w_1} - 1}\Big|_{u=u_0-\delta}.$$

As $\delta \to 0$, l'Hôpital's rule gives

$$\lim_{\delta\to 0} \frac{\frac{g}{w_2} - 1}{\frac{g}{w_1} - 1}\Big|_{u=u_0-\delta} = \lim_{\delta\to 0} \frac{\frac{g'w_2 - gw_2'}{w_2^2}}{\frac{g'w_1 - gw_1'}{w_1^2}}\Big|_{u=u_0-\delta} = 1,$$

thus

$$\frac{\tau_2}{\tau_1} \geq 1,$$

contradicting with $\tau_1 > \tau_2$. Thus $w_1(u_0) > g(u_0)$, implying $w_1'(u_0) < 0$ and $w_2'(u_0) < 0$.

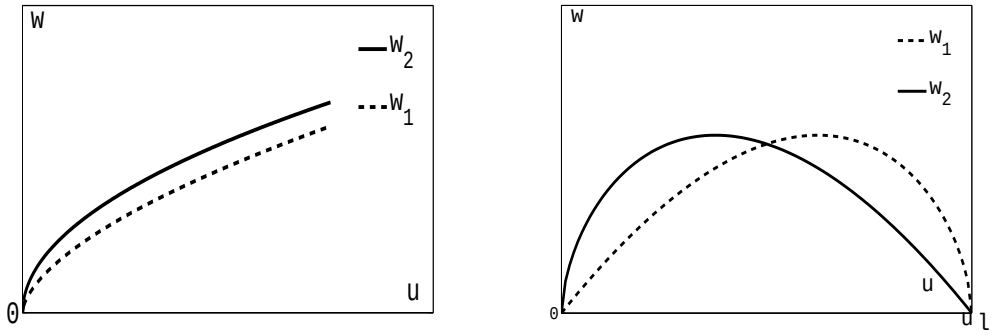

Figure 3: Sketched solutions of (3.1) for $\tau_1 > \tau_2$: behavior close to $u = 0$ (left), and global behavior assuming a unique intersection point inside $(0, u_l)$

We assume now there exist at least two intersection points inside $(0, u_\ell)$, and let u_0 and u_1 be the first two of them. We have $0 > w_1'(u_0) \geq w_2'(u_0)$ and $w_1'(u_1) \leq w_2'(u_1) < 0$. However

$$w_1'(u_1) = \frac{g(u_1) - w_1(u_1)}{s\tau_1 w_1(u_1)} = \frac{g(u_1) - w_2(u_1)}{s\tau_1 w_2(u_1)} > \frac{g(u_1) - w_2(u_1)}{s\tau_2 w_2(u_1)} = w_2'(u_1),$$

contradicting the above.

It only remains to rule out the possibility of having exactly one intersection point $u_0 \in (0, u_\ell)$ (see the right picture of Figure 3). Then $w_1 > w_2$ for $u \in (u_0, u_\ell)$. Since $w_1(u_\ell) = w_2(u_\ell)$ there exists u_2 close to u_ℓ such that $w_1'(u_2) < w_2'(u_2)$. However, since $\tau_1 > \tau_2$, $w_1(u_2) > w_2(u_2)$ and $g(u_2) < 0$,

$$w_1'(u_2) = \frac{g(u_2)}{s\tau_1 w_1(u_2)} - \frac{1}{s\tau_1} > \frac{g(u_2)}{s\tau_2 w_2(u_2)} - \frac{1}{s\tau_2} = w_2'(u_2),$$

which is a contradiction and shows the uniqueness. □

Remark 3.2 *The dependence of τ on u_ℓ is monotone. Specifically, with $u_r = 0$ and given two pairs $(\tau_i, u_{\ell,i})$, $i = 1, 2$, satisfying $0 < \tau_1 < \tau_2$, then $u_{\ell,1} < u_{\ell,2}$. To see this we first notice that the functions g appearing on the right of (3.1) are ordered: $g_1(u) < g_2(u)$ for $u > 0$. Then by Theorem 3.2, the solutions w_i of (3.1) are uniquely defined, and that $u_{\ell,1} \neq u_{\ell,2}$. Clearly, $w_1 > w_2$ close to the origin. Assuming $u_{\ell,1} > u_{\ell,2}$, then $w_1 > w_2$ close to $u_{\ell,2}$ as well. The arguments in the second step of the proof above rule out the possibility of having two intersection points of w_1 and w_2 at strictly positive u. This implies that $w_1 > w_2$ everywhere. Integrating (3.1) on $(0, u_{ell,i})$ gives*

$$\int_0^{u_{\ell,i}} w_i(u)du = \int_0^{u_{\ell,i}} g_i(u)du.$$

This contradicts the ordering in w_i and g_i, namely $w_1 > w_2$ and $g_1 < g_2$ on $(0, u_{\ell,2})$.

Remark 3.3 *The proof of Theorem 3.2 also provides bounds for τ. First notice that the maximum of w satisfies*

$$\tilde{w} = \max_{u\in[0,u_\ell]} w(u) > \frac{1}{u_\ell}\int_0^{u_\ell} g(u)du =: A.$$

This immediately gives an upper bound for τ:

$$\tau < \frac{2}{s\tilde{w}^2}\int_0^{\tilde{u}} g(u)du < \frac{2}{s\ A^2}\int_0^{u_m} g(u)du.$$

To obtain a lower bound we start with

$$\int_0^{u_\ell} g(u)du = \int_0^{u_\ell} w(u)du > \int_{\tilde{u}}^{u_m} w(u)du > (u_m - \tilde{u})w(u_m).$$

Taking $u_3 \in (0, u_m)$ s.t. $g(u_3) = A$ we notice that $A < \tilde{w} = g(\tilde{u})$, so $\tilde{u} < u_3$. From the above we get $\int_0^{u_\ell} g(u)du > (u_m - u_3)w(u_m)$. This, together with (3.4) gives

$$\tau > \frac{u_e - u_m}{sw(u_m)} > \frac{(u_e - u_m)(u_m - u_3)}{s\int_0^{u_\ell} g(u)du}.$$

For the non-degenerate case, we can proceed in the same manner to get similar bounds.

The proof of Theorem 3.2 can be extended without major differences to the case $u_\ell = 1$, giving the following result:

Theorem 3.3 *Let $u_r \geq 0$, $u_\ell = 1$ and assume $\int_{u_r}^1 g(t; u_r, 1)\, dt > 0$ (thus $\beta = 1$). Then there exists a unique pair (τ, w) solving (3.1).*

3.2 Non-existence results

We focus on $u_r = 0$, the case $u_r > 0$ being similar. We carry out the phase plane analysis of (2.10) by taking $\varphi = u, \psi = \varphi'$:

$$(\mathrm{P_c}) \begin{cases} \varphi' = \psi, \\ \psi' = -c\psi + g(\varphi; u_r, u_\ell). \end{cases} \tag{3.6}$$

Theorem 3.4 *Suppose $\beta = 1$, and assume $\int_0^1 g(t;0,1)\,dt > 0$. For $\tau > 0$ large enough, there exists no heteroclinic orbit connecting $(0,0)$ and $(1,0)$.*

Proof . We give the proof for the generic case mentioned in (1.5), $\lambda_w(u) = u^{p+1}$ and $\lambda_o(u) = (1-u)^{q+1}$. In the general situation one can proceed as in Lemma 2.2.

Suppose that exists a heteroclinic orbit (φ, ψ) connecting $(0,0)$ to $(1,0)$, then $u_\ell = 1$ and $s(1) = 1$. Hence $\tau = 1/(sc^2) = 1/c^2$. Assume $c = 0$. Then we have

$$\varphi'' = g(\varphi;0,1), \qquad \text{on} \qquad \xi \in \mathbb{R}. \tag{3.7}$$

Note that

$$g(t;0,1) = \frac{1}{t^p(1-t)^q}\{-t^p + M(1-t)^q\}.$$

Because of the singularity at $t = 0, 1$ and the continuity of φ, we conclude that for some $a \in \mathbb{R}$ and $b > a$,

$$\varphi(\xi) = 0, \quad \text{for} \quad \xi \le a, \qquad \text{and} \qquad \varphi(\xi) = 1, \quad \text{for} \quad \xi \ge b,$$

and

$$\varphi(\xi) \in (0,1), \quad \text{for} \quad a < \xi < b.$$

In addition, since φ' is continuous, we conclude

$$\varphi'(a) = 0, \qquad \text{and} \qquad \varphi'(b) = 0. \tag{3.8}$$

Without loss of generality we take $a = 0$.

We multiply (3.7) by $2\varphi'$ and integrate over $(0,\xi)$, where $\xi \in (0,b)$. Using the conditions at $\xi = 0$ we obtain

$$(\varphi')^2(\xi) = 2\int_0^\xi g(\varphi;0,1)\varphi'(t)\,dt = 2\int_0^{\varphi(\xi)} g(t;0,1)\,dt.$$

As $\xi \to b$, we have $\varphi(\xi) \to 1$ yielding

$$(\varphi')^2(b) = 2\int_0^1 g(t;0,1)\,dt > 0,$$

contradicting (3.8). Therefore, for $c = 0$, the orbit emerging from $(0,0)$ will intersect the half line $\{(\varphi,\psi) : \varphi = 1, \psi > 0\}$ for some positive value of ψ. By continuous dependence on the data, the same holds whenever c is sufficiently small. Since $\tau = 1/c^2$, we conclude that there exists no heteroclinic orbit if τ is large enough. □

Remark 3.4 *A similar result can be obtained for $u_r > 0$. Specifically, if $\int_{u_r}^1 g(u;u_r,1) > 0$ there exists no heteroclinic orbit connecting $(u_r,0)$ and $(1,0)$ whenever $\tau > 0$ is large enough for $u_r > 0$. This allows defining the following*

$$\tau^* = \{\tau > 0 : \textit{there exists a homoclinic orbit of } (P_c) \textit{ connecting} \quad (u_r,0) \quad \textit{and} \quad (1,0)\}.$$

In view of the above, one has $\tau^* < \infty$. Notice that the maximal value τ^* depends on u_r, so we denote it by $\tau^*(u_r)$.

4 Non-smooth travelling waves

The results proven yet are obtained for smooth C^2 travelling wave solutions to (1.6). Following Theorem 3.3 (see also Remarks 3.4), whenever $\beta = 1$ such solutions are only possible for finite values of τ. Specifically, given a $u_r \geq 0$ and a $\tau < \tau^*(u_r)$, a unique $u_\ell < 1$ exists allowing for smooth travelling waves connecting the two states u_ℓ and u_r. The limit situation is for $\tau = \tau^*(u_r)$, when $u_\ell = 1$. The following question appears naturally: what does it happen whenever $\tau > \tau^*(u_r)$?

Before answering this question we notice that given a right state u_r, Theorem 2.1 shows the monotone dependence of τ on the the related left state u_ℓ. Therefore the case $\tau > \tau^*(u_r)$ can only lead to travelling waves with $u_\ell \geq 1$. However, the case $u_\ell > 1$ is ruled out by the degeneracy $H(1) = 0$, therefore one has to find waves connecting $u_\ell = 1$ to u_r.

The existence of smooth and monotone TW to (2.2) is obtained by involving the function $w(u) = -u'(\eta(u))$. Given two states $u_\ell = 1$ and u_r, Theorem 3.3 provides a unique pair $(w, \tau) \in C^1(u_r, u_\ell) \times (0, \infty)$ for which (3.1) is valid. For the construction of u, we refer to (3.2).

To deal with the case $\tau > \tau^*(u_r)$, we proceed in a similar manner and seek positive solution to

$$\begin{cases} \tau s w w' + w = g(u), \quad u \in (u_r, 1], \\ w(u_r) = 0. \end{cases} \tag{4.1}$$

Since $\tau > \tau^*(u_r)$, we give up the condition $w(u_\ell) = 0$. As follows from the uniqueness result in Theorem 3.3, a positive solution to (4.1) can never end up in 0 for some $u \in (u_r, 1)$, yielding $w(1) > 0$. This means that u has a strictly negative slope when approaching 1, so this value is attained for some finite $\eta = \eta_1$. Clearly, one can extend u by 1 at the left of η_1, by giving up the continuity of u'. In this way, we extend the concept of TW solutions to cover the case $\tau > \tau^*(u_r)$, where classical waves are ruled out.

In view of the above, any $\tau > \tau^*(u_r)$ cannot provide smooth travelling waves, and this requires interpreting this concept in a broader sense that allows for discontinuities in the derivatives. The generalized TW concept used here have much in common with the sharp waves in [32]. To define such travelling waves we rewrite (2.2) as a system involving three unknown quantities: the saturation u, the total flux F and the capillary pressure p. With s given in (2.4), a sharp TW solution to (1.6) is a triple (u, F, p) satisfying

$$\begin{cases} s u' = F', \\ H(u) p' = f(u) - F, \\ p = u - \tau s u', \end{cases} \tag{4.2}$$

almost everywhere in $\mathbb{R}$. This system is complemented by (2.3):

$$u(-\infty) = u_\ell, \qquad \text{and} \qquad u(+\infty) = u_r,$$

describing the behavior at infinity. Clearly, any smooth solution to (2.2) provides a smooth solution triple to (4.2) and reciprocally. The difference between the two cases appears

whenever a smooth TW solution to (2.2) fails to exist, which can happen in the degenerate case, i. e. if $u_\ell = 1$ or $u_r = 0$.

We start by investigating the case involving a single value of degeneracy $u_\ell = 1$, whereas $u_r > 0$. Then we consider the doubly degenerate case, $u_\ell = 1$ and $u_r = 0$. In this case, the TW solutions are constructed as the limit $\delta \searrow 0$ of the case $u_\ell = 1$ and $u_r = \delta$. Notice that only the physically relevant cases $u_\ell, u_r \in [0, 1]$ are considered. As shown in [24], weak solutions to (1.6) are essentially bounded by 0 and 1, the degeneracy values. This property is inherited by TW solutions.

4.1 The case $u_r > 0$ and $u_\ell = 1$

Here we take $u_r > 0$ and $\tau > \tau^*(u_r)$. As mentioned above, this yields $u_\ell = 1$. We seek for solutions to (4.2) where u and F are continuous:

Definition 4.1 *Let $\eta_1 \in \mathbb{R}$ be a fixed coordinate. A triple (u, F, p) is a sharp travelling wave solution to (2.8) if*

$$\begin{cases} su' = F', \\ H(u)p' = f(u) - F, \\ p = u - \tau s u', \end{cases} \tag{4.3}$$

for all $\eta > \eta_1$, whereas $u(\eta) = 1$, $F(\eta) = s$, and $p(\eta) = 1$ for all $\eta < \eta_1$.

Clearly, such solutions are determined up to a translation. One can normalize the TW by taking e. g. $\eta_1 = 0$, or by assuming that $u(0) = (u_r + 1)/2$.

Remark 4.1 *Whenever $u \in (0, 1)$ in some interval $N \subset \mathbb{R}$, one has $H(u) > 0$ there. Since u and F are continuous, by (4.3_2) we obtain a continuous p'. Then (4.3_3) yields $u \in C^2(N)$, implying the same regularity for F. In this way we conclude the smoothness of the solution triple for arguments η where u remains between 0 and 1. Moreover, if $H \in C^1$ - as in the case investigated in [10], where $H \equiv 1$ - the argument can be continued to provide higher regularity for p, u, and F.*

Therefore, if $u_r, u_\ell \in (0, 1)$, or if H is globally smooth, the existence of a solution triple (u, F, p) would also provide a classical TW solution to (2.8). However, classical TW solutions are ruled out in the present situation by the choice $\tau > \tau^*(u_r)$.

Remark 4.2 *In the above we have disregarded the case $\eta_1 = \pm\infty$. The case $\eta_1 = \infty$ implies $u \equiv 1$ and is trivially obtained for $u_\ell = u_r = 1$. If $\eta_1 = -\infty$, the solution triple is continuous on the entire $\mathbb{R}$. This means that $u \in (0, 1)$ everywhere, thus $H(u) > 0$. As above, this situation is equivalent to the case of a smooth TW and is only possible for $\tau \leq \tau^*(u_r)$.*

Remark 4.3 *The pressure component p in Definition 4.1 is discontinuous at η_1, where it only has left and right limits. To give a physical interpretation, we notice that the saturation u becomes 1 at η_1, meaning that only one phase (oil) is present. Since the capillary pressure is defined as the pressure difference between the two phases (oil and water), this pressure cannot be defined clearly in the absence of one phase. To extend p*

into regions where $u = 1$, one can follow (4.3$_3$) and end up with $p = u = 1$ if $\eta < \eta_1$. Moreover, this extension has no physical motivation, and does not take into account any dunamic effect encountered for $\eta > \eta_1$.

The argument in Remark 4.1 provides the smoothness of u, F and p for all $\eta > \eta_1$. The existence of a solution as introduced in Definition 4.1 is provided by (4.1). Specifically, with $w = -u'(\eta)$ solving (4.1), $u \in (u_r, 1)$ is defined by implicitly on $(\eta_1, +\infty)$ by:

$$\eta(u) = \eta_1 + \int_u^1 \frac{dz}{w(z)}. \tag{4.4}$$

Furthermore, we have $u(\eta) = 1$ for all $\eta \leq \eta_1$. The above can be summarized as follows:

Lemma 4.1 *Let $u_r > 0$ and $\tau > \tau^*(u_r)$, therefore $u_\ell = 1$. Then Problem (TW_1) admits a sharp TW solution in the sense of Definition 4.1.*

4.2 The case $u_r = 0$ and $u_\ell = 1$

In the previous section we have dealt with one degeneracy point, $u_\ell = 1$. Here we extend the results to the doubly degenerate case: $u_r = 0$ and $u_\ell = 1$. We focus on the case when smooth TW solutions are not possible, i. e. if $\beta = 1$ (see (2.13)) and $\tau > \tau^*(0)$. As explained in Remark 4.1, u' becomes discontinuous only at degeneracy points. Therefore constructing the TW solutions for $u_\ell = 1$ and $u_r > 0$ is based on investigating w solving (4.1), and starting at $w(u_r) = 0$. This implies the smoothness of u' whenever $u < 1$. If a second degeneracy point $u_r = 0$ is involved, there is no particular reason to start at $w(0) = 0$. In fact, as will be seen below, for each non-negative value of $w(0)$ a solution w of (4.1$_1$) can be obtained. This solution satisfies $w(1) > 0$. Based on (4.4), each of the $w's$ provide a non-smooth TW solution having possibly two discontinuities in u', whenever $u = 0$ or $u = 1$. At this point it is not clear how to select a relevant solution among all these waves. Below we give a selection criterium providing the TW having a smooth transition to $u_r = 0$. To this aim we start with the following ordering result.

Lemma 4.2 *Let $u_r = 0$, $\tau > \tau^*(0)$, while w and $\widetilde{w}$ solve (4.1) with initial data w_0 and $\widetilde{w}_0$. If $\widetilde{w}_0 > w_0 \geq 0$, then $\widetilde{w} > w$ for all $u \in [0, 1]$.*

Proof . Assume w and $\widetilde{w}$ intersect. Let $\overline{u}$ be the smallest intersection point. Since $\widetilde{w}_0 > w_0$, we know $\overline{u} > 0$. We distinguish two cases:

Case 1: $\overline{u} < 1$, from

$$w'(\overline{u}) = \frac{g(\overline{u}) - w(\overline{u})}{\overline{\tau} w(\overline{u})}, \quad \text{and} \quad \widetilde{w}'(\overline{u}) = \frac{g(\overline{u}) - \widetilde{w}(\overline{u})}{\overline{\tau}\widetilde{w}(\overline{u})},$$

we obtain

$$w'(\overline{u}) = \widetilde{w}'(\overline{u}).$$

(i) If $w'(\overline{u}) \geq 0$, then $w'(\overline{u} + \delta) < \widetilde{w}'(\overline{u} + \delta)$, $w(\overline{u} + \delta) < \widetilde{w}(\overline{u} + \delta)$ and $g(\overline{u} + \delta) > 0$ for $\delta > 0$ small enough. However,

$$w'(\overline{u} + \delta) = \frac{g(\overline{u} + \delta)}{\tau w(\overline{u} + \delta)} - \frac{1}{\tau} > \frac{g(\overline{u} + \delta)}{\tau \widetilde{w}(\overline{u} + \delta)} - \frac{1}{\tau} = \widetilde{w}'(\overline{u} + \delta),$$

which is a contradiction.
(ii) If $w'(\overline{u}) < 0$, then $w'(\overline{u}+\delta) < \widetilde{w}'(\overline{u}+\delta)$ and $w(\overline{u}+\delta) < \widetilde{w}(\overline{u}+\delta)$ for $\delta > 0$ small enough. If $g(\overline{u}) \geq 0$, then

$$w'(\overline{u}+\delta) = \frac{g(\overline{u}+\delta)}{\tau w(\overline{u}+\delta)} - \frac{1}{\tau} \geq \frac{g(\overline{u}+\delta)}{\tau \widetilde{w}(\overline{u}+\delta)} - \frac{1}{\tau} = \widetilde{w}'(\overline{u}+\delta),$$

which is a contradiction again. If $g(\overline{u}) < 0$, we know $w'(\overline{u}-\delta) > \widetilde{w}'(\overline{u}-\delta)$ and $w(\overline{u}-\delta) < \widetilde{w}(\overline{u}-\delta)$ for $\delta > 0$ small enough. However

$$w'(\overline{u}-\delta) = \frac{g(\overline{u}-\delta)}{\tau w(\overline{u}-\delta)} - \frac{1}{\tau} < \frac{g(\overline{u}-\delta)}{\tau \widetilde{w}(\overline{u}-\delta)} - \frac{1}{\tau} = \widetilde{w}'(\overline{u}-\delta),$$

contradicting the inequalities above.

Case 2: $\overline{u} = 1$, there exists $u_0 < 1$ close enough to 1 such that $w(u_0) < \widetilde{w}(u_0)$ and $w'(u_0) > \widetilde{w}'(u_0)$. Notice that $w'(u)$, $\widetilde{w}'(u)$ and $g(u)$ are negative when u is close enough to 1. But

$$w'(u_0) = \frac{g(u_0)}{\tau w(u_0)} - \frac{1}{\tau} < \frac{g(u_0)}{\tau \widetilde{w}(u_0)} - \frac{1}{\tau} = \widetilde{w}'(u_0),$$

meaning $\widetilde{w} > w$ for all $u \in [0,1]$. □

Since $\tau > \tau^*(0)$, for w solving (4.1) with $u_r = 0$ one has $w(1) > 0$. By Lemma 4.2, starting with $w(0) > 0$ still gives $w(1) > 0$. This property is determining the selection criterium for the TW solution connecting 1 to 0. This is based on regularization, a commonly used approach in dealing with degenerate problems.

One possible regularization is to perturb the data such that the solution stays away from the degeneracy values. For the analysis of the *porous medium equation*, this techinque has been applied in [27]; a numerical scheme based on this approach is investigated in [29]. The solutions obtained stay away from the degeneracy values and therefore have better regularity. As the regularization parameter approaches 0, the sequence of regularized solutions converges to the relevant solution in the degenerate case. With $\delta > 0$ being a small regularization parameter, one has the following possibilities:
a) $u_r = \delta$ and w solving (4.1), yielding $w(1) > 0$;
b) $u_r = 0$ and w solving (4.1) on $(0, 1-\delta)$, but choosing $w(0)$ s.t. $w(1-\delta) = 0$.

Then the regularized solutions will be smooth whenever $u < 1$, respectively $u > 0$. However, Lemma 4.2 rules out the second possibility. As follows from the ordering result proven there, any solution w satisfying $w(0) \geq 0$ cannot reach 0 within $(0,1]$, implying $w(1) > 0$. Therefore u still lacks smoothness when approaching 1. This is why we only consider the first possibility, $u_r = \delta, u_\ell = 1$, and investigate the limit $\delta \searrow 0$. We start with an elementary result:

Proposition 4.1 *Let $u_r = \delta > 0, u_\ell = 1$, and define*

$$s_\delta := \frac{1-f(\delta)}{1-\delta}, \quad g_\delta := g(u;\delta,1),$$

Then

$$s_\delta \searrow 1, \quad \text{and} \quad g_\delta \nearrow g = \frac{u-f(u)}{H(u)}, \quad \text{as} \quad \delta \to 0,$$

the convergence for g is pointwise on $(0,1)$.

Proposition 4.2 *Let $\tau > \tau^*(0)$, consider the following two initial value problems,*

$$\begin{cases} \tau w w' + w = g(u), & u \in (0,1], \\ w(0) = 0, \end{cases} \tag{4.5}$$

and

$$\begin{cases} \tau s_\delta w_\delta w_\delta' + w_\delta = g_\delta(u), & u > \delta, \\ w_\delta(\delta) = 0. \end{cases} \tag{4.6}$$

with $\tilde{u} \in (0,1)$ such that $w(\tilde{u}) = g(\tilde{u})$, we have:

$$w_\delta(u) < w(u) \quad \text{for all} \quad u \in (\delta, \tilde{u}). \tag{4.7}$$

Proof . Assume there exist u^* such that $w_\delta(u^*) = w(u^*)$, and u^* is the first one. Clearly, we have $w_\delta(u^*-) < w(u^*-)$, therefore we have $w_\delta'(u^*) \geq w'(u^*)$. From

$$\tau w(u^*)w'(u^*) + w(u^*) = g(u^*) > g_\delta(u^*) = \tau s_\delta w_\delta(u^*)w_\delta'(u^*) + w_\delta(u^*),$$

we obtain $w'(u^*) > s_\delta w_\delta'(u^*)$. However, since $u^* \in (\delta, \tilde{u})$, we have $w'(u^*) \geq 0$ and $w_\delta'(u^*) \geq w'(u^*) \geq 0$. As $s_\delta > 1$, this implies $w'(u^*) \leq s_\delta w_\delta'(u^*)$, contradicting the previous inequality. □

The function w in (4.5) is defined on $[0,1]$, whereas w_δ is only defined on $[\delta, c(\delta)]$ for some $c(\delta)$ defined by

$$c(\delta) = \sup\{\tilde{u} < u < 1 \mid w_\delta(u) > 0\}.$$

For practical reasons we extend w_δ by 0 on $[0,\delta]$, and on $[c(\delta),1]$ if $c(\delta) < 1$, and investigate its behavior as $\delta \searrow 0$. We do so by considering two intervals, $[0,\tilde{u}]$ and $[\tilde{u},1]$.

Proposition 4.3 *Let w and w_δ solve (4.5) and (4.6). Along any sequence $\delta \to 0$, w_δ converges pointwise to w on $[0,\tilde{u}]$.*

Proof . Integrating the equations in (4.5) and (4.6), we obtain

$$\frac{\tau}{2}w^2(u) + \int_0^u w(z)dz = \int_0^u g(z)dz,$$

and

$$\frac{s_\delta \tau}{2}w_\delta^2(u) + \int_\delta^u w_\delta(z)dz = \int_\delta^u g_\delta(z)dz.$$

By (4.7) we have $w_\delta(u) < w(u)$. From the above we obtain

$$\begin{aligned} 0 \quad \leq \quad & \frac{\tau}{2}(w^2(u) - w_\delta^2(u)) = \frac{\tau}{2}(s_\delta - 1)w_\delta^2(u) - \int_0^\delta w_\delta(z)dz \\ & + \int_0^\delta g(z)dz - \int_\delta^u (w(z) - w_\delta(z))dz + \int_\delta^u (g(z) - g_\delta(z))dz. \end{aligned}$$

This leads to

$$0 \le \frac{\tau}{2}(w^2(u) - w_\delta^2(u)) \le \frac{\tau}{2}(s_\delta - 1)w_\delta^2(u) + \int_0^\delta g(z)dz + \int_\delta^u (g(z) - g_\delta(z))dz,$$

Notice that the two integrals in the above are vanishing, since

$$\int_0^\delta g(z)dz = \int_0^\delta (Mz^{-p} - (1-z)^{-q})dz = \frac{M}{1-p}\delta^{1-p} + \frac{1}{1-p}((1-\delta)^{1-p} - 1) \longrightarrow 0,$$

$$\int_\delta^u (g(z) - g_\delta(z))dz \le C\delta^{1-p} \longrightarrow 0,$$

with a positive constant C. Furthermore, according to Proposition 4.1, $s_\delta \searrow 1$ giving $w_\delta^2(u) \nearrow w^2(u)$ as $\delta \to 0$. Since w and w_δ are non-negative, this gives the pointwise convergence of w_δ towards w on the compact interval $[0, \tilde{u}]$. □

Now we consider the interval $[\tilde{u}, 1]$, where the following holds

Proposition 4.4 *Along any sequence $\delta \to 0$, w_δ converges pointwise to w on $[\tilde{u}, 1]$.*

Proof . Let $\delta > 0$ and $u < c(\delta)$. Integrating (4.5) and (4.6) from $\tilde{u}$ to u, we have

$$\frac{\tau}{2}(w^2(u) - w(\tilde{u})^2) + \int_{\tilde{u}}^u w(z)dz = \int_{\tilde{u}}^u g(z)dz, \tag{4.8}$$

$$\frac{s_\delta\tau}{2}(w_\delta^2(u) - w_\delta(\tilde{u})^2) + \int_{\tilde{u}}^u w_\delta(z)dz = \int_{\tilde{u}}^u g_\delta(z)dz. \tag{4.9}$$

Subtracting (4.9) by (4.8), we have

$$\begin{aligned} &\tfrac{s_\delta\tau}{2}(w_\delta^2(u) - w^2(u)) \\ &= \textstyle\int_{\tilde{u}}^u (g_\delta(z) - g(z))dz - \int_{\tilde{u}}^u (w_\delta(z) - w(z))dz + \frac{\tau}{2}(s_\delta w_\delta(\tilde{u})^2 - w(\tilde{u})^2) \\ &\quad -\tfrac{\tau}{2}(s_\delta - 1)w^2(u) \ =: T_1 - T_2 + T_3 - T_4. \end{aligned} \tag{4.10}$$

By Proposition 4.1, T_1 vanishes as δ approaches 0. Furthermore, Proposition 4.3 gives the convergence of $w_\delta(\tilde{u})$ to $w(\tilde{u})$. Using Proposition 4.1 again, since w is bounded we obtain

$$T_4 = \frac{\tau}{2}w^2(u)(s_\delta - 1) \longrightarrow 0, \tag{4.11}$$

as well as

$$T_3 = \frac{\tau}{2}\big((s_\delta - 1)w_\delta(\tilde{u})^2 + (w_\delta(\tilde{u})^2 - w(\tilde{u})^2)\big) \longrightarrow 0. \tag{4.12}$$

Next, with $M := \max\limits_{z\in[\tilde{u},u]} |w_\delta(z) - w(z)|$ one has

$$|T_2| = \left|\int_{\tilde{u}}^u (w_\delta(z) - w(z))dz\right| \le (u - \tilde{u})M. \tag{4.13}$$

Further, since w is decreasing on $[\tilde{u}, 1]$ we have

$$\max_{z\in[\tilde{u},u]} |w_\delta^2(z) - w^2(z)| \geq w(1)M.$$

Since $s_\delta \geq 1$, by (4.10) – (4.13)

$$\frac{\tau}{2}w(1)M \leq \frac{s_\delta \tau}{2}(w_\delta^2(u) - w^2(u)) \leq |T_1| + (u-\tilde{u})M + |T_3| + |T_4|, \tag{4.14}$$

Taking $u = c(\delta)$ and with δ small enough, from (4.14) we get

$$c(\delta) > \tilde{u} + \frac{\tau}{4}w(1).$$

Further, (4.14) also gives

$$\left(\frac{\tau}{2}w(1) - (u-\tilde{u})\right) M \leq |T_1| + |T_3| + |T_4|, \tag{4.15}$$

whenever δ is small enough. As $\delta \to 0$, all limits on the right side in (4.15) go to 0, which gives $w_\delta(u) \to w(u)$ pointwisely for $u \in [\tilde{u}, \tilde{u} + \frac{\tau}{4}w(1)]$. Let $\Delta u = \frac{\tau}{4}w(1)$. If $\tilde{u} + \Delta u \geq 1$, then the conclusion is shown. Otherwise, if $\tilde{u} + \Delta u < 1$, notice that Δu does not dependent on δ, therefore we can continue the same procedure for $u \in [\tilde{u} + \Delta u, \tilde{u} + 2\Delta u]$ and further until reaching 1. □

Combining Proposition 4.3 and 4.4, we have the following theorem:

Theorem 4.1 *Let $\tau > \tau^*(0)$, for any $\delta > 0$, w_δ solves (4.1) with $u_r = \delta$. Along any sequence $\delta \searrow 0$, the sequence $\{w_\delta\}$ approaches w solving (4.5). In particular, the limit w satisfies $w(1) > 0$.*

Remark 4.4 *Theorem 4.1 provides a selection criterium for the TW solution to (1.6), in the doubly degenerate case. Specifically, each w_δ solving (4.1) provides a TW solution to (1.6) with $u_r = \delta$ and $u_\ell = 1$. Passing $\delta \searrow 0$, the limit w provides a TW solution to (1.6) connecting $u_l = 1$ to $u_r = 0$, as limit of regularized travelling waves.*

Recalling the connection between the *TW* solution u and the solution w of (4.1) and since $w(1) > 0$, u has a discontinuous derivative (a kink) at the transition $u = 1$ to $u < 1$. This point separates a fully saturated region, when only one phase is present, from a partially saturated one, when both phases are present. Since the capillary pressure is defined as $p = u + \tau u'$, it becomes discontinuous there as well. Specifically, in the fully saturated region where $u = 1$, one has $p = 1$, implying the same value for its limit from this side to the point where the partially saturated regime starts. At the same time, in the unsaturated region we have $p = u + \tau u'$ bounded below from 1, therefore its limit from this side stays below 1. However, this does not contradict the concept of the capillary pressure, defined as a difference between the pressures inside the two phases. Its limit from the partially saturated region is defined by continuity as the difference of the phase pressures. In the fully saturated region such a definition does not make sense since only one phase is present. In the latter case p does not include any dynamic effects, and can therefore not be seen as a continuous extension of the dynamic capillary pressure from the former case.

As seen above, u has a kink at the transition from $u = 1$ to $u < 1$. In what follows we study the transition to the other degenerate value, $u = 0$.

Theorem 4.2 *Let $u_r = 0$ and $u_\ell = 1$. If $\tau > \tau^*(0)$, then the travelling wave selected by Theorem 4.1 becomes 0 at a finite $\eta_0 \in \mathbb{R}$, and has a smooth derivative there.*

Proof . By Theorem 4.1, u is provided by $w = \lim_{\delta \searrow 0} w_\delta$. Integrating (4.6) from δ to $u > \delta$ gives

$$\frac{s_\delta \tau}{2} w_\delta(u)^2 + \int_\delta^u w_\delta(v)dv = \int_\delta^u g_\delta(v)dv \leq \int_\delta^u g(v)dv \leq \int_0^u g(v)dv.$$

Recalling the asymptotic behavior of g as $u \searrow 0$, we have

$$w_\delta(u)^2 \leq \frac{2}{s_\delta \tau} \int_0^u g(v)dv \leq \frac{2}{\tau} \int_0^u g(v)dv \leq \frac{2}{\tau} C_1 u^{1-p},$$

for some $C_1 > 0$ not dependent on δ. Therefore, with $C_2 = \sqrt{\frac{2C_1}{\tau}}$, we get $w_\delta(u) \leq C_2 u^{\frac{1-p}{2}}$, yielding

$$-u'(\eta) = w(u) \leq C_2 u^{\frac{1-p}{2}}.$$

Integrating from 0 to η and $u(0)$ gives

$$\frac{2}{p+1}(u(\eta)^{\frac{p+1}{2}} - u(0)^{\frac{p+1}{2}}) \geq -C_2 \eta,$$

yielding

$$u(\eta) \geq \{u(0)^{\frac{p+1}{2}} - \frac{(p+1)C_2}{2}\eta\}^{\frac{2}{1+p}}. \tag{4.16}$$

Notice that $u(0)$ can be taken arbitrarily small by choosing η_1 conveniently. To be more precise, since w is known, one can use it in (4.4) to define $\eta_1 = -\int_{u(0)}^1 \frac{dz}{w(z)}$ leading to $\eta(u(0)) = 0$.
Similarly, if $u_r = 0$,

$$\frac{\tau}{2} w(u)^2 + \int_0^u w(v)dv = \int_0^u g(v)dv,$$

With u small enough, we have $g(u) \geq C_3 u^{-p}$ for some $C_3 > 0$, giving

$$\frac{\tau}{2} w(u)^2 + \int_0^u w(v)dv \geq \frac{C_3}{1-p} u^{1-p}.$$

Since $w(u) \leq C_2 u^{\frac{1-p}{2}}$, we obtain

$$\frac{\tau}{2} w(u)^2 + \frac{2C_2}{3-p} u^{\frac{3-p}{2}} \geq \frac{C_3}{1-p} u^{1-p}.$$

For $u(0)$ small enough, one has

$$\frac{2C_2}{3-p} u^{\frac{3-p}{2}} < \frac{C_3}{2(1-p)} u^{1-p}, \quad \text{for all} \quad u \in (0, u(0)],$$

implying

$$\frac{\tau}{2} w(u)^2 \geq \frac{C_3}{2(1-p)} u^{1-p}, \quad \text{or} \quad w(u) \geq \sqrt{\frac{C_3}{\tau(1-p)}} u^{\frac{1-p}{2}}.$$

This gives

$$-u'(\eta) \geq C_4 u^{\frac{1-p}{2}}, \quad \text{where} \quad C_4 = \sqrt{\frac{C_3}{\tau(1-p)}}.$$

Similarly, we obtain

$$u(\eta) \leq \{u(0)^{\frac{p+1}{2}} - \frac{(p+1)C_4}{2}\eta\}^{\frac{2}{1+p}}. \tag{4.17}$$

By (4.16) and (4.17), we obtain that u is bounded from above and below by two curves behaving like $(A - D\eta)^{\frac{2}{1+p}}$, where $A = u(0)^{\frac{p+1}{2}}$ and $D = \frac{(p+1)C_i}{2}$ with $i = 2$ or 4. Moreover, as $u(0) \searrow 0$, C_2 and C_4 can be chosen arbitrarily close to each other, showing that u behaves asymptotically like $(A-D\eta)^{\frac{2}{1+p}}$ close to the coordinate η where it becomes 0. In particular, there exists an η_0 such that $u(\eta) > 0$ whenever $\eta < \eta_0$ and $u(\eta \geq \eta_0) \equiv 0$, as well as u' is continuous at η_0 and $u'(\eta_0) = 0$. □

5 Numerical results

In this section, we provide some numerical experiments. We solve the full problem (1.6), using a semi-implicit Euler finite volume scheme. This scheme is similar to the ones investigated in [4], [7], or [16]. There a particular attention is paid to heterogeneities and the conditions at the interface between two homogeneous sub-domains. We also mention [28] for a review of different numerical methods for pseudo-parabolic equations.

We consider the problem (1.6) in the domain $S = R \times R^+$:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \varepsilon \frac{\partial}{\partial x}\left\{H(u)(\frac{\partial u}{\partial x} + \varepsilon\tau \frac{\partial^2 u}{\partial x \partial t})\right\}, \tag{5.1}$$

with initial value

$$u(x,0) = (u_B - u_r) * \widetilde{H}(-x) + u_r, \tag{5.2}$$

where u_r is the right state, u_B is the inflow value and $\widetilde{H}(x)$ is a smooth monotone approximation of the Heaviside function H. By using $\widetilde{H}$ instead of H we avoid unnecessary technical difficulties due to discontinuities in the initial conditions. As shown in [6], if the initial data has jumps, these will persist for all $t > 0$, at the same location. This would require an adapted and more complicated numerical approach for ensuring the continuity in flux and pressure (see for example [4], Chapter 3, or [7]).

Remark 5.1 *We emphasize that u_B is an inflow value, which in general is not equal to the value associated to τ, $u_\ell = \bar{u}(\tau)$. This value will be an outcome of the calculations.*

Since the scaling

$$x \to \frac{x}{\varepsilon}, \quad t \to \frac{t}{\varepsilon}, \tag{5.3}$$

removes the parameter ε from (5.1), we fix $\varepsilon = 1$ here. In the absence of analytic solutions, for verifying the numerical solution we recall the transformation $w(u) = -u'(\eta(u))$, based

on which a relation between τ and an admissible left state u_ℓ can be established. As shown in Section 3, given $u_r \geq 0$, a value $\tau^* \in (0, \infty]$ exists such that to any $\tau < \tau^*$ a unique left state $u_\ell = u_\ell(\tau) \leq 1$ can be associated. This left state can be connected to u_r through a smooth TW solution to (5.1). Whenever $\tau^* < \infty$, if $\tau > \tau^*$ no smooth travelling waves are possible, but sharp ones connecting $u_\ell = 1$ to u_r, and having a kink at the point when u becomes less than 1. Figure 4 below presents the diagrams u_ℓ - τ for $p = q = 0.5$, $M = 2.5$, and for two values of u_r: $u_r = 0.1$ (non-degenerate), and $u_r = 0$ (degenerate). To obtain these diagrams we have solved (3.1) numerically with fixed u_r, but for several left states u_ℓ, providing different the pairs (w, τ) such that $w(u_\ell) = 0$. We start with $u_\ell = \alpha$, a minimal value of that corresponds to the point where the line through $(u_r, f(u_r))$ becomes tangent to the graph of the water fractional flow function f. In terms of hyperbolic conservation laws, the shock $\{\alpha, u_r\}$ is an admissible entropy solution to the non-viscous (BL) equation (obtained for $\varepsilon = 0$). We have $\alpha = 0.926$ if $u_r = 0.1$, respectively $\alpha \approx 0.936$ if $u_r = 0$. Starting with $u_\ell = \alpha$, for which a lower value $\tau = \tau_*$ is obtained, we increase u_ℓ by a small Δu_ℓ (in this case $5 * 10^{-4}$) and determine the corresponding τ value either until $u_\ell = 1$ (yielding a finite upper limit τ^* to τ), or up to a maximal value less than one, which is attained asymptotically as $\tau \nearrow \infty$. The pairs (u_ℓ, τ) obtained in this way are included in the diagram.

Both cases considered here give $\tau^* < \infty$: $\tau^* \approx 1.37$ for $u_r = 0.1$ and $\tau^* \approx 0.22$ for $u_r = 0$. For the lower limits we get $\tau_* \approx 0.067$, respectively $\tau_* \approx 0.054$. Below we will present numerical solutions to (5.1) for two values of τ: $\tau_1 = 0.1$, and $\tau_2 = 2$. For both right states u_r mentioned above they satisfy $\tau_* < \tau_1 < \tau^* < \tau_2$. As resulting from the diagrams, $\tau_1 = 0.1$ is associated to the left state $u_\ell = 0.9475$ if $u_r = 0.1$, respectively to $u_\ell = 0.977$ if $u_r = 0$.

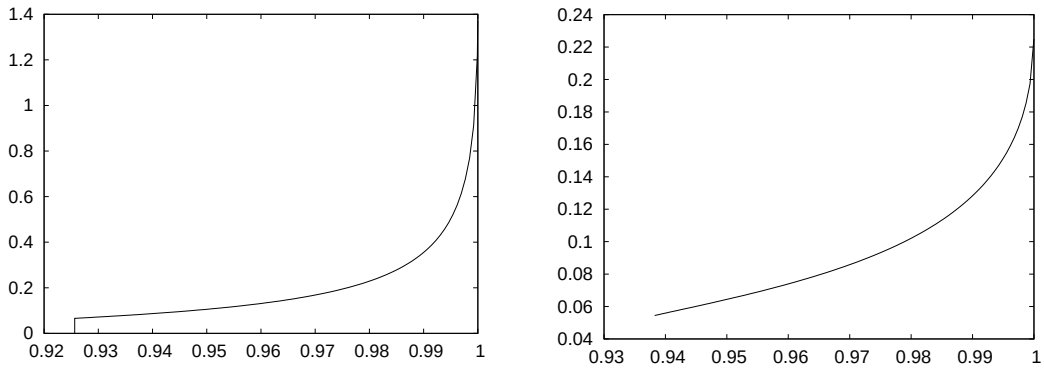

Figure 4: The diagrams u_ℓ - τ computed for $p = q = 0.5$, $M = 2.5$ and with $u_r = 0.1$ (left) respectively $u_r = 0$. Numerically we obtain $\tau^* \approx 1.37$, respectively $\tau^* \approx 0.22$.

To discretize (5.1) we take a fixed time step $\Delta t = t_{n+1} - t_n$ and apply a semi-implict first order method:

$$\frac{u^{n+1} - u^n}{\Delta t} + \frac{d}{dx} F^n(u) = 0.$$

Here $F^n(u)$ is the time discrete flux function at $t = t_n$,

$$F^n(u) := f^n(u) - H^n(u)\left(\partial_x u^{n+1} + \tau \frac{\partial_x u^{n+1} - \partial_x u^n}{\Delta t}\right).$$

Similarly, the functions f^n and H^n are time discrete variants of f and H. These are not defined explicitly since we are interested here only in their fully discrete counterparts. Notice that the scheme is explicit in the convective terms, and semi-implicit in the higher order ones.

For the space discretization, we use a finite volume scheme on a dual mesh. Taking a uniform grid with mesh size $\Delta x = x_n - x_{n-1}$ and defining $u_i = \frac{1}{\Delta x}\int_{i-1/2}^{i+1/2} u(x)dx$, the fully discretized equation becomes

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{F^n(u_i, u_{i+1}) - F^n(u_{i-1}, u_i)}{\Delta x} = 0. \tag{5.4}$$

Here the numerical flux $F^n(u_i, u_{i+1})$ is defined by

$$F^n(u_i, u_{i+1}) = f(u_i^n) - H_{i+1/2}^n \frac{u_{i+1}^{n+1} - u_i^{n+1}}{\Delta x} - \tau H_{i+1/2}^n \frac{u_{i+1}^{n+1} - u_i^{n+1} - u_{i+1}^n + u_i^n}{\Delta x \Delta t}.$$

For the coefficient $H_{i+1/2}^n$, we use the upwind value:

$$H_{i+1/2}^n = H(u_i^n).$$

This approach is important when doing calculations with degenerate outflow value, $u_r = 0$. The numerical diffusion added in this way has regularizing effects, leading to a numerical solution fulfilling the selection criterion in Remark 4.4.

In what follows we present the numerical solutions of (5.1) obtained on a spatial interval $(-1, 19)$ and at time $T = 5$. As mentioned above, we take $p = q = 0.5, M = 2.5$, and consider two right states, $u_r = 0.1$ and $u_r = 0$, as well as two values for τ: $\tau_1 = 0.1$ and $\tau_2 = 2$. The discretization parameters are $\Delta x = 5 * 10^{-4}$ and $\Delta t = 10^{-4}$, providing stable numerical results. On the endpoints of the interval we take value that are compatible with the ones appearing in (5.2): u_B at the inflow, and u_r at the outflow. All calculations are carried out for $u_B = 1$, which is not necessary equal to the value u_ℓ related to τ. Therefore the numerical solution of the degenerate pseudo-parabolic problem (5.1)–(5.2) does not necessary have a TW profile, but instead will feature a "plateau" region of constant value $\bar{u}$ corresponding to u_ℓ related to τ.

The solutions presented in Figure 5, computed for $\tau_1 = 0.1$ are clearly presenting this situation: they both decay from 1 to the plateau value $u = \bar{u} < 1$. This value is taken over an interval that is delimited on the right by a front going down from $\bar{u}$ to u_r. This front travels with a constant speed provided by the RH condition in (2.4), written for the states $\bar{u}$ and u_r. A similar situation is obtained in [10] for the non-degenerate case $H = 1$. As in that paper, we associate the plateau value $\bar{u}$ with the value $u_\ell = \bar{u}(\tau)$. The plateau value $\bar{u}$ exhibited by the numerical solution is $\bar{u} = 0.9467$ for $u_r = 0.1$, whereas $\bar{u} = 0.979$ for $u_r = 1$. This agrees well with the value $u_\ell = \bar{u}(\tau)$ predicted at $\tau = 0.1$ by the u_ℓ - τ diagrams discussed above. There we obtained $u_\ell = 0.9475$ if $u_r = 0.1$, respectively to $u_\ell = 0.977$ if $u_r = 0$.

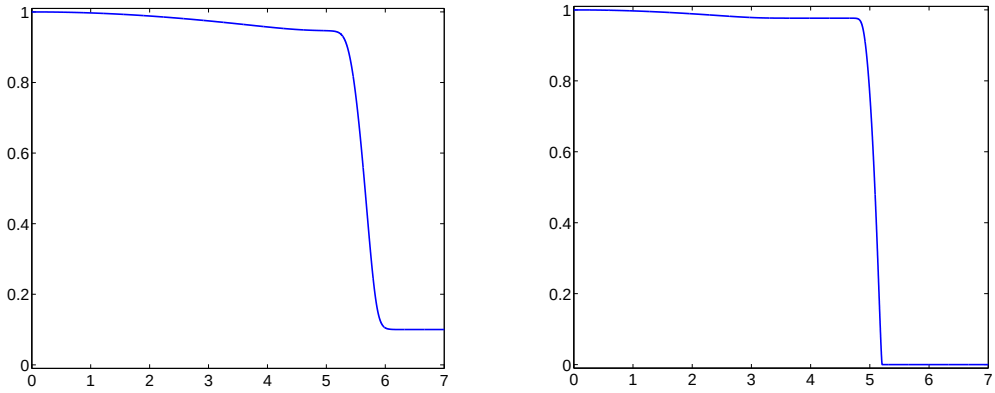

Figure 5: Left: graph of u for $u_r = 0.1$ and $\tau = 0.1 < \tau^*$, containing a plateau at $\bar{u} = 0.9467$. Right: graph of u for $u_r = 0$, and the plateau value $\bar{u} = 0.979$.

The next numerical results are obtained for $\tau_2 = 2$, exceeding τ^* up to which smooth travelling waves are possible. Therefore the u_ℓ - τ diagrams are not providing any information that can be used for testing the numerical solutions. However, as discussed in Section 4, waves connecting the left state $u_\ell = 1$ to u_r are still possible, but these have a discontinuous derivative (a kink) at the transition point from $u = 1$ to $u < 1$. Correspondingly, the transformed w solving (4.1) on $(u_r, 1]$ will remain strictly positive at $u = 1$. The value $w(1)$ gives the slope of u at the right of the kink. In this case we compare this (numerical) slope to $w(1) = -u'(\eta_1 + 0)$.

The left pictures in Figures 6 and 7 are presenting the numerical results for $u_r = 0.1$, respectively $u_r = 0$. The kinks encountered at the transition from $u = 1$ to $u < 1$ are estimated to -0.27, for $u_r = 0.1$, and to -1.27 for $u_r = 0$. For w we obtain $w(1) = 0.266$ in the first case, and $w(1) = 1.266$ in the second one. The two functions w are presented in the right pictures of Figures 6 and 7.

Finally, we recall that in the doubly degenerate case $u_\ell = 1$ and $u_r = 0$ the sharp waves are not unique. Theorem 4.1 provides a selection criterion. As follows from Theorem 3.7, this particular sharp wave is smooth everywhere away from the transition from $u = 1$ to $u < 1$. The smoothness includes the transition $u > 0$ to $u = 0$, which is achieved for a finite η. The same is featured by the numerical solution: Figure 8 presents two zoomed views of it. We clearly see a kink also in the left picture, whereas the transition to $u = 0$ is smooth, as displayed in the right picture.

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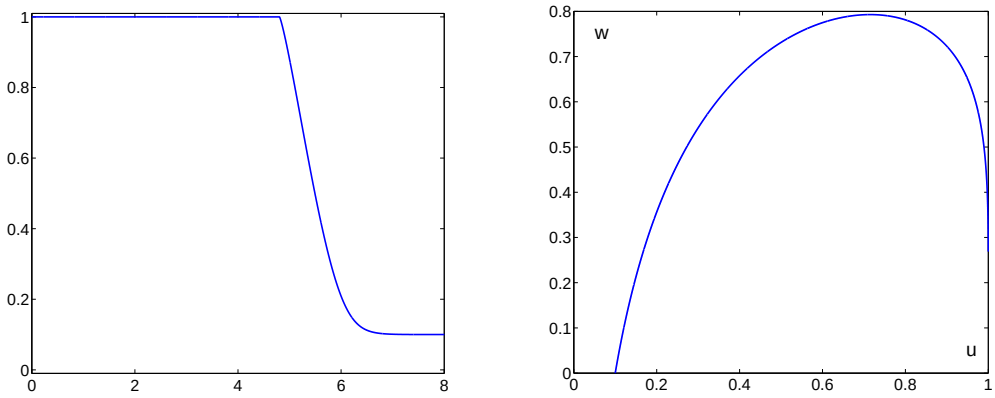

Figure 6: The graph of u for $u_r = 0.1$, and $\tau = 2 > \tau^*$, presenting a kink at the transition $u = 1$ to $u < 1$ (left); the slope at the right of the kink is $u' = -0.27$. The corresponding w (right), where $w(1) = 0.266$.

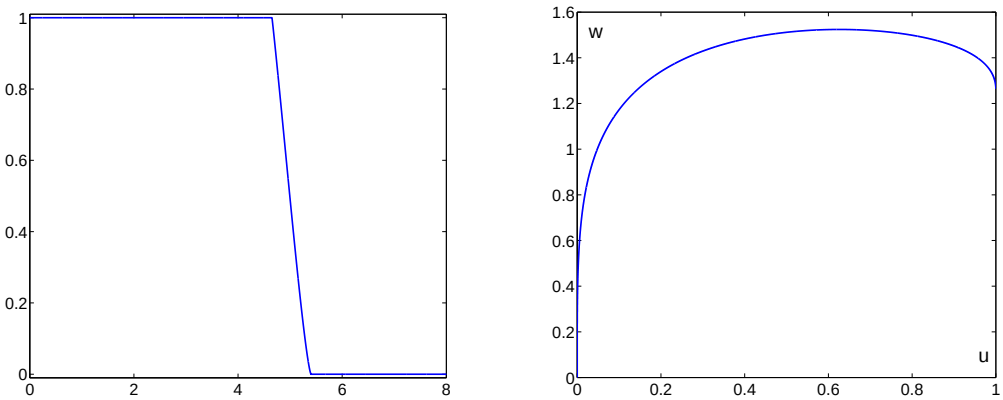

Figure 7: The graph of u for $u_r = 0$, and $\tau = 2 > \tau^*$, presenting a kink at the transition $u = 1$ to $u < 1$ (left); the slope at the right of the kink is $u' = -1.27$. The corresponding w (right) where $w(1) = 1.26$.

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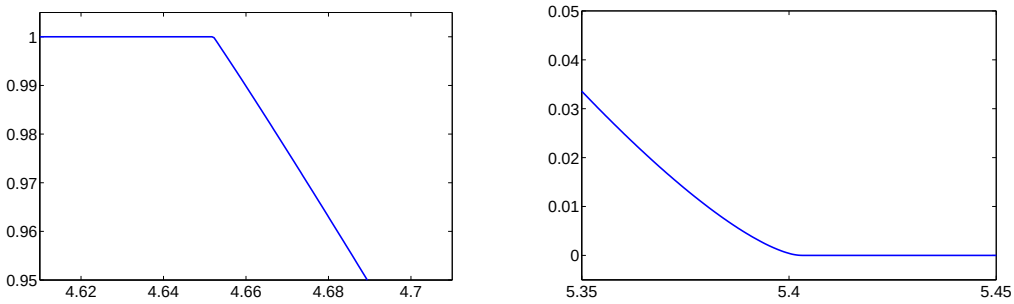

Figure 8: Zoomed view of u for $u_r = 0$, $\tau = 2 > \tau^*$: a kink appears at the transition to $u < 1$ (left), whereas the transition to $u = 0$ is smooth (right).

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