Large-time asymptotics of Stokes flow for perturbed balls with injection and suction

Large-time asymptotics of Stokes flow for perturbed balls with injection and suction

E. Vondenhoff

Department of Mathematics and Computer Science
Technische Universiteit Eindhoven
P.O. Box 513, 5600 MB Eindhoven, The Netherlands
E-mail: e.vondenhoff@tue.nl

Abstract

We discuss large-time behaviour of Stokes flow with surface tension and with injection or suction in one point. We consider domains in $\mathbb{R}^2$ or $\mathbb{R}^3$ that are initially small perturbations of balls. After a suitable time-dependent rescaling, a ball becomes a stationary solution. To prove stability of this solution, we derive a nonlinear non-local evolution equation describing the motion of perturbed domains. From spectral properties of the linearisation, we find global existence in time and decay properties for the injection case. For the suction case, we find that an arbitrarily large portion of liquid smaller than the entire domain can be removed if the initial domain is close enough to a ball.

AMS subject classifications: 35R35, 35K55, 76D07

Key words: moving boundary problem, non-local parabolic equation, vector spherical harmonics

1 Introduction

In the problem of Stokes flow with surface tension and with injection or suction in one point one seeks a family of domains $t \to \Omega(t)$ in $\mathbb{R}^N$ and two functions $p(\cdot, t) : \Omega(t) \to \mathbb{R}$ and $v(\cdot, t) : \Omega(t) \to \mathbb{R}^N$ that satisfy the following system of PDEs:

$$-\Delta v + \nabla p = 0 \qquad \text{in } \Omega(t), \tag{1.1}$$

$$\operatorname{div} v = \mu\delta \qquad \text{in } \Omega(t), \tag{1.2}$$

$$\left(\nabla v + \nabla v^T - pI\right) n = \gamma\kappa n \quad \text{on } \Gamma(t) := \partial\Omega(t). \tag{1.3}$$

The family $t \mapsto \Omega(t)$ models a liquid that moves under influence of injection or suction and surface tension. The functions v and p denote dimensionless velocity and pressure, respectively, μ is the injection speed ($\mu > 0$) or suction speed ($\mu < 0$), γ is a positive constant called surface tension coefficient, κ is the mean curvature (taken negative for convex domains), n is the outer normal on the boundary, I the identity matrix and δ denotes the delta distribution. The evolution of the boundary $t \mapsto \Gamma(t)$ is specified by the requirement that its normal velocity v_n satisfies

$$v_n = v \cdot n. \tag{1.4}$$

The velocity component in the fixed time problem (1.1)-(1.3) is determined only up to rigid body motions. The problem becomes uniquely solvable if we add two extra conditions, namely

$$\int_{\Omega(t)} v \; dx = 0, \tag{1.5}$$

which implies that the geometric centre of $\Omega(t)$ is constant in time (see Lemma 4.3), and

$$\int_{\Omega(t)} \operatorname{rot} v \; dx = 0. \tag{1.6}$$

Here, the operator rot in N dimensions should be interpreted in the following way. Let ω be any bijection between $\{(i,j) \in \mathbb{N}^2 : 1 \leq i < j \leq N\}$ and $\left\{1, 2, \ldots, \binom{N}{2}\right\}$. We define

$$\operatorname{rot} u = \sum_{1 \leq i < j \leq N} \left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j} \right) e_{\omega(i,j)},$$

where e_k is the k-th unit vector in $\mathbb{R}^{\binom{N}{2}}$.

The model can be derived from the Navier-Stokes equations if one assumes a fluid with low Reynolds number. A closely related model is used to study the growth of certain tumours, for which the tissue can be modeled as a fluid (see [3], [4] and [5]). The process of viscous sintering in glass technology is modeled by Stokes flow as well (see [10]). More industrial applications are given in [14].

Short-time existence of solutions for the problem without injection or suction is proved in [8]. In the same work, global existence results have been found for the case that the initial domain is close to a ball. The global existence results however, do not straightforwardly generalize to our situation with a source or sink since there is no stationary solution. Joint spatial and temporal analyticity of the moving boundary for the problem without injection or suction has been proved in [2].

For the problem with injection or suction, short-time existence results and smoothness of the boundary have been proved in [12]. Exact solutions for the suction problem are found in [1] from complex variable theory. In this paper we prove global existence results assuming initial domains that are perturbations of a ball. We use methods that are also used in [15] for Hele-Shaw flow.

We start by identifying the trivial solution, where $\Omega(0) = \mathbb{B}^N := \{x \in \mathbb{R}^N : |x| < 1\}$. For this special case, the functions v and p will be denoted by v_0 and p_0. It is easy to check from the divergence theorem that the volume $\mathfrak{V}(t)$ of the evolving domain $\Omega(t)$ that satisfies (1.1)-(1.6) changes linearly in time with rate μ:

$$\frac{d\mathfrak{V}}{dt} = \int_{\Gamma(t)} v \cdot n \; d\sigma = \int_{\Omega(t)} \operatorname{div} v \; dx = \mu.$$

Because of radial symmetry, the evolving domain will be a ball with radius $\alpha(t)$, given by

$$\alpha(t) := \sqrt[N]{\frac{\mu N t}{\sigma_N} + 1}.$$

Here σ_N is the surface area of the unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$. Note that the suction problem only makes sense on the interval $(0, T)$, where $T := -\frac{\sigma_N}{\mu N}$. It is easy to see that

$$v_0 = \frac{\mu}{\sigma_N |x|^N} x.$$

The mean curvature of $\mathbb{S}^{N-1}$ is $1 - N$. Therefore the mean curvature of $\Gamma(t)$ is equal to $\frac{1-N}{\alpha(t)}$ and we have

$$p_0 = \mu\delta + \gamma \frac{N-1}{\alpha(t)} - 2\mu \frac{N-1}{\sigma_N \alpha(t)^N}.$$

To investigate the stability of the trivial solution $\Omega(t) = \alpha(t)\mathbb{B}^N$, we rescale a moving domain $\Omega(t)$ that solves (1.1)-(1.6) by $\alpha(t)^{-1}$ such that the domain $\mathbb{B}^N$ becomes a stationary solution. We consider small star-shaped perturbations of this stationary solution. These perturbations will be described by a function $r(\cdot, t) : \mathbb{S}^{N-1} \to \mathbb{R}$.

In Section 2 we derive a nonlinear non-local evolution equation for r and linearise it around $r = 0$.

In Section 3, we determine the spectrum of the linearisation. We write it in terms of the Dirichlet-to-Neumann operator for the Laplacian on $\mathbb{B}^N$ for the cases $N = 2, 3$. This is done by solving a boundary value problem on $\mathbb{B}^N$ in terms of (scalar) spherical harmonics and vector spherical harmonics (see [9], [7] and [6]). For $N \geq 4$ calculating the spectrum becomes more complicated and the problem is less interesting for applications. Therefore we restrict ourselves to the cases $N = 2, 3$.

In Section 4 we derive global existence in time of solutions r for the case of injection. We also show that the corresponding moving domain converges to a ball as time goes to infinity. This is done by finding energy estimates in Sobolev spaces. A generalised chain rule for differential operators on functions on $\mathbb{S}^{N-1}$ is used to close a regularity gap.

In Section 5 we consider the case of suction. Because the eigenvalues of the linearisation go to infinity as t tends to T, we cannot derive global existence results. However, we show that an arbitrarily large portion of liquid smaller than the entire domain can be removed if the initial domain is close enough to a ball.

Comparison with Hele-Shaw flow

We end this introduction by comparing our problem to the corresponding problem for the related Hele-Shaw flow, where (1.1) is replaced by Darcy's law $v = -\nabla p$ and (1.3) is replaced by $p = -\gamma\kappa$ (see [15] and [16]). Again we have an elliptic system for each time and the evolution of a moving boundary that follows from the kinetic boundary condition $v_n = v \cdot n$. For both problems, the domain is rescaled such that a ball becomes stationary and existence results and decay properties are obtained from linearisation.

The fixed time problem for Hele-Shaw flow can be reduced to one scalar equation and a boundary condition for pressure only. The system (1.1)-(1.3) however cannot be decoupled. As a result, the evolution equation (2.20) for the motion of $\Omega(t)$ is more complicated than the equation for Hele-Shaw flow.

In both problems the linearisation is related to a solution operator for a boundary value problem on $\mathbb{B}^N$ (see (3.1)-(3.5)) and the (scalar) spherical harmonics (see Section 3) are eigenfunctions. This is not surprising since both evolution operators are equivariant with respect to rotations and therefore the eigenspaces have a corresponding invariance property. In contrast to Hele-Shaw flow, in order to solve the coupled Stokes system (3.1)-(3.5) we need to introduce vector-valued spherical harmonics as well.

Only for $N = 3$, the evolution problem for Hele-Shaw flow can be regarded as autonomous. Global existence results can be derived from the principle of linearised stability (see [16]). For Stokes flow, this only works for the uninteresting case $N = 1$. For other dimensions, we use the more complicated method of finding energy estimates. The existence results for $N = 2, 3$ (see Theorems 4.4 and 5.1) turn out to be similar to those for Hele-Shaw flow with $N \geq 4$.

The evolution operator in (2.20) is of first order whereas the operator for Hele-Shaw flow is of order three. Therefore one can apply a first order chain rule of differentiation (4.1) to obtain useful energy estimates in the existence proofs. For Hele-Shaw flow with $N \neq 3$ it is necessary to work with a second order chain rule.

2 An evolution equation for the motion of the domain

Let $\mathfrak{S}_k^N$ be the space of spherical harmonics of degree k on $\mathbb{S}^{N-1}$. Choose an $\mathbb{L}_2(\mathbb{S}^{N-1})$ orthonormal basis of $\mathfrak{S}_k^N$:

$$\left\{s_{k,1}, s_{k,2}, \ldots, s_{k,\nu(N,k)}\right\},$$

where $\nu(N,k) \in \mathbb{N}_0$. It is well-known (see e.g. [11] Lemma 2) that the spherical harmonics

$$\bigcup_{k=0}^{\infty} \left\{s_{k,1}, s_{k,2}, \ldots, s_{k,\nu(N,k)}\right\},$$

form an orthonormal basis for $\mathbb{L}_2(\mathbb{S}^{N-1})$. Let $(\cdot,\cdot)_0$ be the usual inner product on $\mathbb{L}_2(\mathbb{S}^{N-1})$. For each $r \in \mathbb{L}_2(\mathbb{S}^{N-1})$ define $r_{k,j}$ by

$$r_{k,j} := (r, s_{k,j})_0.$$

Equip for all $s > 0$, the Sobolev space $\mathbb{H}^s(\mathbb{S}^{N-1})$ with the inner product

$$(r,\tilde{r})_s = \sum_{k,j} (k^2+1)^s r_{k,j}\tilde{r}_{k,j}.$$

In this paper we use the Sobolev imbedding theorem: If $k \in \mathbb{N}_0$, $\alpha \in [0,1)$ and $s > \frac{N-1}{2} + k + \alpha$, then

$$\mathbb{H}^s(\mathbb{S}^{N-1}) \hookrightarrow \mathcal{C}^{k,\alpha}(\mathbb{S}^{N-1})$$

and

$$\mathbb{H}^{s+\frac{1}{2}}(\mathbb{B}^N) \hookrightarrow \mathcal{C}^{k,\alpha}(\overline{\mathbb{B}^N}).$$

We will also use the fact that for $s > \frac{N-1}{2}$, $\mathbb{H}^{s+\frac{1}{2}}(\mathbb{B}^N)$ and $\mathbb{H}^s(\mathbb{S}^{N-1})$ are Banach algebras.

Any continuous function $f : \mathbb{S}^{N-1} \to (-1, \infty)$ describes a domain Ω_f in the following way:

$$\Omega_f := \left\{ x \in \mathbb{R}^N \setminus \{0\} : |x| < 1 + f\left(\frac{x}{|x|}\right) \right\} \cup \{0\}.$$

We also introduce $\Gamma_f := \partial\Omega_f$. For a domain $\Omega(t)$ moving according to (1.1)-(1.6) we introduce a continuous function $R(\cdot, t) : \mathbb{S}^{N-1} \to (-1, \infty)$ satisfying $\Omega(t) = \Omega_{R(\cdot,t)}$. Here we need to restrict ourselves to star-shaped domains. Besides $R(\cdot, t)$ we introduce $r(\cdot, t)$ such that

$$r(t) = \frac{1 + R(t)}{\alpha(t)} - 1, \tag{2.1}$$

which is equivalent to

$$\Omega_{r(\cdot,t)} = \alpha(t)^{-1}\Omega_{R(\cdot,t)}.$$

Very often we will omit the argument t in $r(t)$. We assume that $r(t)$ is a small element of $\mathbb{H}^s(\mathbb{S}^{N-1})$ for each t with

$$s > \frac{N+5}{2}. \tag{2.2}$$

Introduce the function $\Phi : \mathbb{R}^N \to \mathbb{R}$ by

$$\Phi(x) := \begin{cases} -\dfrac{1}{2\pi} \ln |x| & N = 2, \\ \dfrac{1}{(N-2)\sigma_N |x|^{N-2}} - \dfrac{1}{(N-2)\sigma_N} & N \geq 3. \end{cases} \tag{2.3}$$

This function satisfies $\Delta\Phi = -\delta$ and it vanishes on $\mathbb{S}^{N-1}$. Define the functions V and P by

$$V := v + \mu\nabla\Phi = v - v_0 \tag{2.4}$$

and

$$P := p - \mu\delta.$$

If v and p satisfy (1.1)-(1.2) then we have

$$-\Delta V + \nabla P = 0 \qquad \text{on } \Omega_{\mathrm{R}}, \tag{2.5}$$

$$\operatorname{div} V = 0 \qquad \text{on } \Omega_{\mathrm{R}}. \tag{2.6}$$

The boundary condition (1.3) becomes

$$(\nabla V + \nabla V^T - PI)n = \gamma\kappa n + 2\mu H n \qquad \text{on } \Gamma_{\mathrm{R}}, \tag{2.7}$$

where $H : \mathbb{R}^N \to \mathbb{R}^{N \times N}$ is the Hessian of Φ given by

$$H(x) = \frac{1}{\sigma_N |x|^N} \left(-I + \frac{N}{|x|^2} x \otimes x \right),$$

where $x \otimes x$ denotes the matrix with coefficients $x_i x_j$. The extra conditions (1.5) and (1.6) translate to

$$\int_{\Omega_R} V \, dx = \int_{\Omega_R} \mu\nabla\Phi \, dx, \quad \int_{\Omega_R} \operatorname{rot} V \, dx = 0. \tag{2.8}$$

Define

- $(V_{1,f}, P_{1,f})^T : \Omega_f \to \mathbb{R}^N \times \mathbb{R}$ as the solution to (2.5) and (2.6) on the domain Ω_f with boundary condition
$$(\nabla V_{1,f} + \nabla V_{1,f}^T - P_{1,f} I)n = \kappa n \qquad \text{on } \Gamma_f$$
and extra conditions
$$\int_{\Omega_f} V_{1,f}\ dx = 0, \quad \int_{\Omega_f} \operatorname{rot} V_{1,f}\ dx = 0,$$
- $(V_{2,f}, P_{2,f})^T : \Omega_f \to \mathbb{R}^N \times \mathbb{R}$ as the solution to (2.5) and (2.6) on the domain Ω_f with boundary condition
$$(\nabla V_{2,f} + \nabla V_{2,f}^T - P_{2,f} I)n = 2Hn \qquad \text{on } \Gamma_f$$
and extra conditions
$$\int_{\Omega_f} V_{2,f}\ dx = \int_{\Omega_f} \nabla\Phi\ dx, \quad \int_{\Omega_f} \operatorname{rot} V_{2,f}\ dx = 0.$$

It is known (see e.g. [12] Chapter 3), that the solutions $(V_{1,f}, P_{1,f})^T$ and $(V_{2,f}, P_{2,f})^T$ exist and are unique for appropriate domains Ω_f. The solution $(V, P)^T$ to (2.5)-(2.8) can be written as $\gamma(V_{1,R}, P_{1,R})^T + \mu(V_{2,R}, P_{2,R})^T$.

Lemma 2.1. *If R and r are related via* (2.1), then
$$V_{1,r}(x) = V_{1,R}(\alpha(t)x), \tag{2.9}$$
$$P_{1,r}(x) = \alpha(t)P_{1,R}(\alpha(t)x), \tag{2.10}$$
$$V_{2,r}(x) = \alpha(t)^{N-1} V_{2,R}(\alpha(t)x), \tag{2.11}$$
$$P_{2,r}(x) = \alpha(t)^N P_{2,R}(\alpha(t)x). \tag{2.12}$$

Proof. Let $\hat{V}_{1,r}(x)$, $\hat{P}_{1,r}(x)$, $\hat{V}_{2,r}(x)$ and $\hat{P}_{2,r}(x)$ be the right-hand side of (2.9)-(2.12). We must prove that $\hat{V}_{1,r} = V_{1,r}$, $\hat{V}_{2,r} = V_{2,r}$, $\hat{P}_{1,r} = P_{1,r}$ and $\hat{P}_{2,r} = P_{2,r}$. Suppressing the time argument in $\alpha(t)$, we have for $x \in \Omega_r$
$$-\Delta \hat{V}_{1,r}(x) + \nabla \hat{P}_{1,r}(x) = \alpha^2 \left(-\Delta V_{1,R}(\alpha x) + \nabla P_{1,R}(\alpha x)\right) = 0.$$
For $\hat{V}_{2,r}$ and $\hat{P}_{2,r}$ this can be done in a similar way. Let $x \in \Gamma_r$, such that $\alpha x \in \Gamma_R$. Let $\kappa_r : \Gamma_r \to \mathbb{R}$ and $\kappa_R : \Gamma_R \to \mathbb{R}$ be the mean curvature of these boundaries. We have
$$\begin{aligned}\big(\nabla \hat{V}_{1,r}(x) + \nabla \hat{V}_{1,r}^T(x) - \hat{P}_{1,r}(x)\big)n &= \alpha\big(\nabla V_{1,R}(\alpha x) + \nabla V_{1,R}^T(\alpha x) - P_{1,R}(\alpha x)\big)n \\ &= \alpha\gamma\kappa_R(\alpha x)n = \gamma\kappa_r(x)n.\end{aligned}$$
For the boundary condition for $\hat{V}_{2,r}$ and $\hat{P}_{2,r}$ this can be done in a similar way, using the fact that $H(x) = \alpha^N H(\alpha x)$. From scaling properties of $\nabla\Phi$ we get
$$\begin{aligned}\int_{\Omega_r} \hat{V}_{2,r}(x)\ dx &= \int_{\Omega_r} \alpha^{N-1} V_{2,R}(\alpha x)\ dx = \int_{\Omega_R} \alpha^{-1} V_{2,R}(x)\ dx \\ &= \int_{\Omega_R} \alpha^{-1}\nabla\Phi(x)\ dx = \int_{\Omega_r} \alpha^{N-1}\nabla\Phi(\alpha x)\ dx = \int_{\Omega_r} \nabla\Phi(x)\ dx.\end{aligned}$$

Verifying the other conditions is straightforward. $\square$

Introduce

- $\tilde{z}(r,\cdot):\mathbb{S}^{N-1}\to\Gamma_r$ by
$$\tilde{z}(r,\xi)=(1+r(\xi))\,\xi,$$
- $n(r,\cdot)$ by the function that maps an element $\xi\in\mathbb{S}^{N-1}$ to the exterior unit normal vector on Γ_r at the point $\tilde{z}(r,\xi)$,
- $\kappa(r,\cdot)$ by the function that maps an element $\xi\in\mathbb{S}^{N-1}$ to the mean curvature of Γ_r at $\tilde{z}(r,\xi)$.

We will often use the notations $\tilde{z}(r)$, $n(r)$ and $\kappa(r)$ instead of $\tilde{z}(r,\cdot)$, $n(r,\cdot)$ and $\kappa(r,\cdot)$. On a neighbourhood $\mathcal{U}$ of zero in $\mathbb{H}^s(\mathbb{S}^{N-1})$ for $s>\frac{N+4}{2}$, the mappings $n:\mathcal{U}\to\left(\mathbb{H}^{s-1}(\mathbb{S}^{N-1})\right)^N$ and $\kappa:\mathcal{U}\to\mathbb{H}^{s-2}(\mathbb{S}^{N-1})$ are analytic (see [12] Chapter 3 Lemma 16). From [12] Chapter 3 we have the following evolution equation for R:
$$\frac{\partial R}{\partial t}(\xi)=\frac{v(\tilde{z}(R,\xi))\cdot n(R,\xi)}{n(R,\xi)\cdot\xi}.$$
Combining this and (2.4) we get
$$\frac{\partial R}{\partial t}=\gamma\frac{(V_{1,R}\circ\tilde{z}(R))\cdot n(R)}{n(R)\cdot\mathrm{id}}+\mu\left(\frac{(V_{2,R}\circ\tilde{z}(R))\cdot n(R)}{n(R)\cdot\mathrm{id}}+\frac{1}{\sigma_N(1+R)^{N-1}}\right).$$
Because
$$\frac{\partial r}{\partial t}=\frac{1}{\alpha}\frac{\partial R}{\partial t}-\frac{\alpha'}{\alpha}(1+r)=\frac{1}{\alpha}\frac{\partial R}{\partial t}-\mu\frac{1+r}{\sigma_N\alpha^N},$$
we get from Lemma 2.1 and the fact that $n(R,x)=n(r,x)$
$$\frac{\partial r}{\partial t}=\frac{\gamma}{\alpha(t)}\frac{(V_{1,r}\circ\tilde{z}(r))\cdot n(r)}{n(r)\cdot\mathrm{id}}+\frac{\mu}{\alpha(t)^N}\left(\frac{(V_{2,r}\circ\tilde{z}(r))\cdot n(r)}{n(r)\cdot\mathrm{id}}+\frac{1}{\sigma_N(1+r)^{N-1}}-\frac{1+r}{\sigma_N}\right).\tag{2.13}$$

We see two terms on the right-hand side of this evolution equation. One term describes the effect of surface tension. Here time dependence occurs as a multiplication by $\alpha(t)^{-1}$. In the other term, describing the effect of injection/suction, time dependence occurs as a multiplication by $\alpha(t)^{-N}$. Only for $N=1$ the two effects scale in the same way. For Hele-Shaw flow (see [16]) this is the case for $N=3$. From the structure of the evolution equation we expect that the results that we get for Stokes flow in dimension two or higher are similar to those for Hele-Shaw flow in dimension four or higher (see [15]).

Now we transform our moving boundary problem to the fixed reference domain $\mathbb{B}^N$ and write the right-hand side of the evolution equation (2.13) as an operator on a function space on $\mathbb{S}^{N-1}$. By [13] Theorem 6.108 and interpolation, there exists an extension operator $E\in\mathcal{L}(\mathbb{H}^s(\mathbb{S}^{N-1}),\mathbb{H}^{s+\frac{1}{2}}(\mathbb{B}^N))$, such that for all $r\in\mathbb{H}^s(\mathbb{S}^{N-1})$
$$Er|_{\mathbb{S}^{N-1}}=r.\tag{2.14}$$
Define $z:\mathbb{H}^s(\mathbb{S}^{N-1})\to\left(\mathbb{H}^{s+\frac{1}{2}}(\mathbb{B}^N)\right)^N$ by
$$z(r,x)=(1+(Er)(x))\,x,$$
identifying $z(r,\cdot)$ and $z(r)$.

Lemma 2.2. *Let $s > \frac{N+5}{2}$. There exists a $\delta > 0$ such that if $\|r\|_s < \delta$, then $z(r) : \overline{\mathbb{B}^N} \to \overline{\Omega}_r$ is bijective and $z(r)^{-1} \in \left(\mathcal{C}^2(\overline{\Omega_r})\right)^N$.*

Proof. From the Sobolev imbedding theorem we get $\mathbb{H}^s(\mathbb{S}^{N-1}) \hookrightarrow \mathcal{C}^3(\mathbb{S}^{N-1})$. The bijectivity follows from [16] Lemma 2.1. Using [16] Lemma 2.2, we can prove the other statement as well. ☐

Introduce the bilinear mapping $\star : \mathbb{R}^{\binom{N}{2}} \times \mathbb{R}^N \to \mathbb{R}^N$ in the following way:

$$u \star v = \sum_{i=1}^{N} \left(\sum_{j=1}^{i-1} u_{\omega(j,i)} v_j - \sum_{j=i+1}^{N} u_{\omega(i,j)} v_j \right) e_i.$$

Here ω is the bijection that we introduced to define the operator rot in (1.6).

On a neighbourhood $\mathcal{U}$ of zero in $\mathbb{H}^s(\mathbb{S}^{N-1})$ with $s > \frac{N+5}{2}$ we define the following mappings:

- $\mathcal{A} : \mathcal{U} \to \mathcal{L}\left(\left(\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)\right)^N, \left(\mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N)\right)^N\right)$, $\mathcal{Q} : \mathcal{U} \to \mathcal{L}\left(\mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^N), \left(\mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N)\right)^N\right)$, $\mathcal{Q}^+ : \mathcal{U} \to \mathcal{L}\left(\left(\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)\right)^N, \left(\mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^N)\right)^{N\times N}\right)$, $b : \mathcal{U} \to \mathcal{L}\left(\left(\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)\right)^N, \mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^N)\right)$ and $\mathcal{R} : \mathcal{U} \to \mathcal{L}\left(\left(\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)\right)^N, \left(\mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^N)\right)^{\binom{N}{2}}\right)$ by

$$\mathcal{A}(r)u := \left(\Delta\left(u \circ z(r)^{-1}\right)\right) \circ z(r) = \sum_{i,k,l} j^{i,l}(r) \frac{\partial}{\partial x_i}\left(j^{k,l}(r)\frac{\partial u}{\partial x_k}\right),$$

$$\mathcal{Q}(r)u := \left(\nabla\left(u \circ z(r)^{-1}\right)\right) \circ z(r) = \sum_{i,k} j^{k,i}(r) \frac{\partial u}{\partial x_k} e_i,$$

$$\mathcal{Q}^+(r)u := \left(\nabla\left(u \circ z(r)^{-1}\right)\right) \circ z(r) = \sum_{i,k,l} j^{k,i}(r) \frac{\partial u_l}{\partial x_k} e_l \otimes e_i,$$

$$b(r)u := \left(\operatorname{div}\left(u \circ z(r)^{-1}\right)\right) \circ z(r) = \sum_{i,k} j^{k,i}(r) \frac{\partial u_i}{\partial x_k},$$

$$\mathcal{R}(r)u := \left(\operatorname{rot}\left(u \circ z(r)^{-1}\right)\right) \circ z(r) = \sum_{1\le i<k\le N} \sum_{l} \left(j^{l,i}(r)\frac{\partial u_k}{\partial x_l} - j^{l,k}(r)\frac{\partial u_i}{\partial x_l}\right) e_{\omega(i,k)},$$

  where $j^{k,i}(r)$ are the coefficients of the inverse of the matrix

$$\mathcal{J}(r) = \frac{\partial z(r)}{\partial x} \in \left(\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)\right)^{N\times N}.$$

  The elements $j^{k,i}(r)$ are in $\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)$ for $\|r\|_s$ small because of continuity of inversion near the identity, which is equal to $\mathcal{J}(0)$, in the Banach algebra $\left(\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)\right)^{N\times N}$. Note that we need assumption (2.2) here to define $\mathcal{A}$.

- $\mathcal{S} : \mathcal{U} \to \mathcal{L}(\mathcal{X}_s, \mathcal{Y}_s)$, where

$$\mathcal{X}_s := \left(\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)\right)^N \times \mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^N) \times \mathbb{R}^N \times \mathbb{R}^{\binom{N}{2}},$$

$$\mathcal{Y}_s := \left(\mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N)\right)^N \times \mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^N) \times \left(\mathbb{H}^{s-2}(\mathbb{S}^{N-1})\right)^N \times \mathbb{R}^N \times \mathbb{R}^{\binom{N}{2}},$$

by

$$\mathcal{S}(r)(\tilde{v}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2) = \begin{pmatrix} -\mathcal{A}(r)\tilde{v} + \mathcal{Q}(r)\tilde{p} + \tilde{\eta}_1 \\ b(r)\tilde{v} \\ \mathrm{Tr}(\mathcal{Q}^+(r)\tilde{v} + \mathcal{Q}^+(r)\tilde{v}^T - \tilde{p}I)n(r) + \tilde{\eta}_2 \star n(r) \\ \int_{\mathbb{B}^N} \tilde{v} \det \mathcal{J}(r) dx \\ \int_{\mathbb{B}^N} (\mathcal{R}(r)\tilde{v}) \det \mathcal{J}(r) dx \end{pmatrix}. \tag{2.15}$$

- $h : \mathcal{U} \to \left(\mathbb{H}^s(\mathbb{S}^{N-1})\right)^{N\times N}$ by

$$h(r, \xi) = H(\tilde{z}(r, \xi)) = \frac{1}{\sigma_N(1 + r(\xi))^N} \left(-I + N\xi \otimes \xi\right). \tag{2.16}$$

We identify $h(r, \cdot)$ and $h(r)$.

- $m : \mathcal{U} \to \mathbb{R}^N$ by

$$m(r) = \int_{\Omega_r} \nabla\Phi \; dx = -\frac{1}{\sigma_N} \int_{\mathbb{S}^{N-1}} r(x)x \; d\sigma.$$

If we combine Lemma 2.2 and [12] Chapter 3 Lemma 11, then we see that for small $r \in \mathcal{U}$, the operator $\mathcal{S}(r)$ is bijective. In the definition of $\mathcal{X}_s$ and $\mathcal{Y}_s$, we use the vectors $\tilde{\eta}_1$ and $\tilde{\eta}_2$ because the equation $\mathcal{S}(r)f = g$ does not have a solution $f \in \mathcal{X}_s$ of the type $(\tilde{v}, \tilde{p}, 0, 0)$ for all $g \in \mathcal{Y}_s$ and the range of the mapping $(\tilde{v}, \tilde{p}) \mapsto \mathcal{S}(r)(\tilde{v}, \tilde{p}, 0, 0)$ depends on r. It is also known ([12] Chapter 3 Lemma 17) that $\mathcal{S}$ is analytic near zero.

In the sequel we use the notation $\Pi_i f$ for the i-th component of any f. On a suitable neighbourhood $\mathcal{U}$ of zero in $\mathbb{H}^s(\mathbb{S}^{N-1})$ we define
$\mathcal{E} : \mathcal{U} \to \mathcal{L}\left(\left(\mathbb{H}^{s-2}(\mathbb{S}^{N-1})\right)^N \times \mathbb{R}^N, \mathbb{H}^{s-1}(\mathbb{S}^{N-1})\right)$ by

$$\left(\mathcal{E}(r)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\right) := \frac{\left(\mathrm{Tr}\Pi_1 \mathcal{S}(r)^{-1}\left(0, 0, \psi_1, \psi_2, 0\right)^T\right) \cdot n(r)}{n(r) \cdot \mathrm{id}}.$$

The evolution equation (2.13) can be written in the following way:

$$\frac{\partial r}{\partial t} = \frac{\gamma}{\alpha(t)}\mathcal{F}_1(r) + \frac{\mu}{\alpha(t)^N}\mathcal{F}_2(r), \tag{2.17}$$

where $\mathcal{F}_1 : \mathcal{U} \to \mathbb{H}^{s-1}(\mathbb{S}^{N-1})$ and $\mathcal{F}_2 : \mathcal{U} \to \mathbb{H}^{s-1}(\mathbb{S}^{N-1})$ are given by

$$\mathcal{F}_1(r) = \mathcal{E}(r)\begin{pmatrix} \kappa(r)n(r) \\ 0 \end{pmatrix}$$

and

$$\mathcal{F}_2(r) = \mathcal{E}(r)\begin{pmatrix} 2h(r)n(r) \\ m(r) \end{pmatrix} + \frac{1}{\sigma_N(1+r)^{N-1}} - \frac{1+r}{\sigma_N}.$$

Lemma 2.3. *If $\psi_1 = \kappa(r)n(r)$ or $\psi_1 = 2h(r)n(r)$ and ψ_2 is any element of $\mathbb{R}^N$, then*

$$\tilde{\eta}_1 := \Pi_3\mathcal{S}(r)^{-1}\left(0,0,\psi_1,\psi_2,0\right)^T = 0, \quad \tilde{\eta}_2 := \Pi_4\mathcal{S}(r)^{-1}\left(0,0,\psi_1,\psi_2,0\right)^T = 0.$$

Proof. Let κ_r and n_r be the mean curvature and the normal on Γ_r. From the variational formulation of (1.1)-(1.3), (1.5) and (1.6) (see [12] (3.24)) we have for all velocity fields w corresponding to rigid body motions in $\mathbb{R}^N$

$$\int_{\Omega_r} \tilde{\eta}_1 \cdot w + \tilde{\eta}_2 \cdot \operatorname{rot} w \; dx = \int_{\Gamma_r} (\psi_1 \circ \tilde{z}(r)^{-1}) \cdot w \; d\sigma.$$

Therefore, to prove this lemma it is sufficient to show that for all rigid body motions w we have

$$\int_{\Gamma_r} \kappa_r n_r \cdot w \; d\sigma = \int_{\Gamma_r} H n_r \cdot w \; d\sigma = 0.$$

Let Δ_r be the Laplace-Beltrami operator on Γ_r and let ∇_r be defined by

$$\nabla_r f = \nabla f - (\nabla f \cdot n_r)n_r,$$

for any differentiable $f : \Omega_r \to \mathbb{R}$. From the formula $\kappa_r n_r = \Delta_r \mathrm{id}$ and Green's formula for closed surfaces we derive

$$\begin{aligned}
\int_{\Gamma_r} \kappa_r n_r \cdot w \; d\sigma &= \int_{\Gamma_r} \Delta_r \mathrm{id} \cdot w \; d\sigma = -\int_{\Gamma_r} \sum_i \nabla_r x_i \cdot \nabla_r w_i \; d\sigma = -\int_{\Gamma_r} \sum_i \nabla x_i \cdot \nabla_r w_i \; d\sigma \\
&= -\int_{\Gamma_r} \Big(\operatorname{div} w - \sum_{i,j} \frac{\partial w_i}{\partial x_j} (n_r \cdot e_i)(n_r \cdot e_j) \Big) \; d\sigma = 0.
\end{aligned}$$

In the last step we used anti-symmetry of ∇w and $\operatorname{div} w = 0$. Because H is symmetric we get

$$\begin{aligned}
\int_{\Gamma_r} H n_r \cdot w \; d\sigma &= \int_{\Gamma_r} H w \cdot n_r \; d\sigma \\
&= \int_{\Omega_r} \operatorname{div}(Hw) \; dx = \int_{\Omega_r} (\Delta \nabla \Phi) \cdot w + \operatorname{tr}(H \nabla w) \; dx \\
&= \int_{\Omega_r} -\nabla \delta \cdot w + \operatorname{tr}(H \nabla w) \; dx = 0.
\end{aligned}$$

In the last step we used $\operatorname{div} w = 0$ and the fact that the trace of the product of a symmetric matrix and an anti-symmetric matrix is always zero. $\square$

We introduce a new time variable $\tau = \tau(t)$ such that $\tau(0) = 0$ and

$$\frac{d\tau}{dt} = \frac{1}{\alpha(t)}. \tag{2.18}$$

From this we get for $N \geq 2$

$$\tau(t) = \frac{\sigma_N}{\mu(N-1)} \left(\left(\frac{\mu N t}{\sigma_N} + 1 \right)^{1-\frac{1}{N}} - 1 \right). \tag{2.19}$$

For the original time variable t, the injection problems are defined on an infinite time interval and the suction problems on a finite interval. For the new time variable τ this is still the case, because

$$\lim_{t\to\infty} \tau(t) = \infty, \qquad \text{for } \mu > 0,$$

$$\lim_{t\to T} \tau(t) = \frac{\sigma_N}{|\mu|(N-1)}, \qquad \text{for } \mu < 0.$$

Regarding r as a function of τ we get

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r,\tau) := \gamma \mathcal{F}_1(r) + \mu \alpha(\tau)^{1-N} \mathcal{F}_2(r). \tag{2.20}$$

For convenience we write here and in the sequel $\alpha(\tau)$ instead of $\alpha(t(\tau))$.

Lemma 2.4. *Suppose that $s > \frac{N+5}{2}$. The mappings $\mathcal{F}_1$ and $\mathcal{F}_2$ are both analytic from a neighbourhood $\mathcal{U}$ of zero in $\mathbb{H}^s(\mathbb{S}^{N-1})$ to $\mathbb{H}^{s-1}(\mathbb{S}^{N-1})$.*

Proof. In [12] Chapter 3 Lemma 19, a proof is given for $\mathcal{F}_1$. Analyticity of $\mathcal{F}_2$ can be proven is a similar way. The proof is based on analyticity of $\mathcal{S}$, bijectivity of $\mathcal{S}(0) : \mathcal{X}_s \to \mathcal{Y}_s$ and the Implicit Function theorem. $\square$

Now we determine the linearisation of the operators $\mathcal{F}_1$ and $\mathcal{F}_2$ around $r = 0$.

Lemma 2.5. *We have*

$$\mathcal{F}_1'(0)[r] = \mathcal{E}(0) \begin{pmatrix} \kappa'(0)[r]n \\ 0 \end{pmatrix},$$

$$\mathcal{F}_2'(0)[r] = \mathcal{E}(0) \begin{pmatrix} \frac{2N(1-N)}{\sigma_N} rn + \frac{2N}{\sigma_N} \nabla_0 r \\ m(r) \end{pmatrix} - \frac{N}{\sigma_N} r,$$

with

$$\nabla_0 r := \nabla \tilde{r} - (\nabla \tilde{r} \cdot n(0)) n(0),$$

where $\tilde{r}$ is any smooth extension of r near the unit sphere.

Proof. First we show that

$$n'(0)[r] = -\nabla_0 r.$$

Fréchet differentiation of the expression $n(r) \cdot n(r) = 1$ around $r = 0$ leads to

$$n'(0)[r] \cdot n(0) = 0. \tag{2.21}$$

Let $\Xi = \Xi(\omega)$ be a parametrisation of a part of $\mathbb{S}^{N-1}$. Note that for $i = 1, 2, \ldots, N-1$ we have

$$0 = n(r) \cdot \frac{\partial \tilde{z}(r)}{\partial \omega_i} = n(r) \cdot \left((1+r) \frac{\partial \mathrm{id}}{\partial \omega_i} + \frac{\partial r}{\partial \omega_i} \mathrm{id} \right).$$

Taking Fréchet derivatives with respect to r around $r=0$ leads to

$$\begin{aligned}0 &= n'(0)[r]\cdot\frac{\partial \mathrm{id}}{\partial\omega_i} + n(0)\cdot\left(r\frac{\partial \mathrm{id}}{\partial\omega_i} + \frac{\partial r}{\partial\omega_i}\mathrm{id}\right)\\ &= n'(0)[r]\cdot\frac{\partial \mathrm{id}}{\partial\omega_i} + \frac{\partial r}{\partial\omega_i}.\end{aligned}$$

Here we used the fact that $\frac{\partial \mathrm{id}}{\partial\omega_i}$ is orthogonal to $n(0)=\mathrm{id}$. The vector fields $\frac{\partial \mathrm{id}}{\partial\omega_i}$, $i=1,2,\dots N-1$, can be chosen pointwise orthogonal and therefore

$$n'(0)[r] = \sum_{i=1}^{N-1}\left(n'(0)[r]\cdot\frac{\partial \mathrm{id}}{\partial\omega_i}\right)\left|\frac{\partial \mathrm{id}}{\partial\omega_i}\right|^{-2}\frac{\partial \mathrm{id}}{\partial\omega_i} = -\sum_{i=1}^{N-1}\frac{\partial r}{\partial\omega_i}\left|\frac{\partial \mathrm{id}}{\partial\omega_i}\right|^{-2}\frac{\partial \mathrm{id}}{\partial\omega_i} = -\nabla_0 r.$$

This follows from (2.21) and the fact that $\frac{\partial \mathrm{id}}{\partial\omega_i}\perp n(0)$.

To shorten notation we introduce

$$\mathcal{G}_1(r)\psi_1 := \mathcal{S}(r)^{-1}\left(0,0,\psi_1,0,0\right)^T, \quad \mathcal{G}_2(r)\psi_2 := \mathcal{S}(r)^{-1}\left(0,0,0,\psi_2,0\right)^T.$$

Define $v_1 : \mathcal{U}\to\left(\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)\right)^N$ and $p_1 : \mathcal{U}\to\mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^N)$ by

$$\left(v_1(r),p_1(r),0,0\right)^T := \mathcal{G}_1(r)\left(\kappa(r)n(r)\right)$$

Note that $\mathcal{F}_1(r) = \frac{v_1(r)\cdot n(r)}{n(r)\cdot\mathrm{id}}$. It is easy to check that $v_1(0)\equiv 0$ and $p_1(0)\equiv -\kappa(0)$ and therefore

$$\begin{aligned}v_1'(0)[r] &= \Pi_1\mathcal{G}_1'(0)[r]\big(\kappa(0)n(0)\big) + \Pi_1\mathcal{G}_1(0)\big(\kappa'(0)[r]n(0)+\kappa(0)n'(0)[r]\big)\\ &= -\Pi_1\mathcal{S}(0)^{-1}\mathcal{S}'(0)[r]\mathcal{G}_1(0)\big(\kappa(0)n(0)\big) + \Pi_1\mathcal{G}_1(0)\big(\kappa'(0)[r]n(0)+\kappa(0)n'(0)[r]\big)\\ &= -\Pi_1\mathcal{S}(0)^{-1}\mathcal{S}'(0)[r](0,-\kappa(0),0,0)^T + \Pi_1\mathcal{G}_1(0)\big(\kappa'(0)[r]n(0)+\kappa(0)n'(0)[r]\big)\\ &= -\Pi_1\mathcal{G}_1(0)\big(\kappa(0)n'(0)[r]\big) + \Pi_1\mathcal{G}_1(0)\big(\kappa'(0)[r]n(0)+\kappa(0)n'(0)[r]\big)\\ &= \Pi_1\mathcal{G}_1(0)\big(\kappa'(0)[r]n(0)\big).\end{aligned}$$

Because $v_1(0)=0$ and $n(0)=\mathrm{id}$ we get

$$\mathcal{F}_1'(0)[r] = \mathrm{Tr}v_1'(0)[r]\cdot n(0)$$

and the result follows.

Now we calculate $\mathcal{F}_2'(0)[r]$ in a similar way. Define

$$\left(v_2(r),p_2(r),0,0\right)^T := \mathcal{G}_1(r)\big(2h(r)n(r)\big) + \mathcal{G}_2(r)m(r).$$

From a simple calculation we obtain $v_2(0)\equiv 0$ and $p_2(0)\equiv 2\frac{1-N}{\sigma_N}$. Because m is linear we have $m(0) = 0$ and $m'(0)[r] = m(r)$. Therefore

$$\begin{aligned}
v_2'(0)[r] &= -\Pi_1 \mathcal{S}(0)^{-1}\mathcal{S}'(0)[r]\mathcal{G}_1(0)\big(2h(0)n(0)\big) \\
&\quad +\Pi_1\mathcal{G}_1(0)\big(2h'(0)[r]n(0) + 2h(0)n'(0)[r]\big) + \Pi_1\mathcal{G}_2(0)m(r) \\
&= -\Pi_1 \mathcal{S}(0)^{-1}\mathcal{S}'(0)[r](0, 2\frac{1-N}{\sigma_N}, 0, 0)^T \\
&\quad +\Pi_1\mathcal{G}_1(0)\left(\frac{2N(1-N)}{\sigma_N} rn(0) - \frac{2}{\sigma_N} n'(0)[r]\right) + \Pi_1\mathcal{G}_2(0)m(r) \\
&= -\Pi_1\mathcal{G}_1(0)\left(2\frac{N-1}{\sigma_N} n'(0)[r]\right) \\
&\quad +\Pi_1\mathcal{G}_1(0)\left(\frac{2N(1-N)}{\sigma_N} rn(0) - \frac{2}{\sigma_N} n'(0)[r]\right) + \Pi_1\mathcal{G}_2(0)m(r) \\
&= \Pi_1\mathcal{G}_1(0)\left(\frac{2N(1-N)}{\sigma_N} rn(0) - \frac{2N}{\sigma_N} n'(0)[r]\right) + \Pi_1\mathcal{G}_2(0)m(r).
\end{aligned}$$

Here we also used (2.21). The lemma follows from this and the fact that $\frac{1}{\sigma_N(1+r)^{N-1}} - \frac{1+r}{\sigma_N} = -\frac{N}{\sigma_N} r + \mathcal{O}(r^2)$ around $r = 0$. $\square$

Lemma 2.6. *We have*

$$\mathcal{E}(0)\begin{pmatrix} 0 \\ m(r) \end{pmatrix} = -\frac{1}{\sigma_N}\sum_{j=1}^{N}(r, s_{1,j})s_{1,j}.$$

Proof. Let $\tilde{v}$ and $\tilde{p}$ be defined by

$$(\tilde{v}, \tilde{p}, 0, 0)^T := \mathcal{G}_2(0)m(r).$$

It is easy to check that $\tilde{p}$ is zero and $\tilde{v}$ is constant. Therefore $\tilde{v}\cdot n$ can be written as a linear combination of spherical harmonics of degree one $(s_{1,j})_{j=1}^N$. If we choose $s_{1,j} = \sqrt{\frac{N}{\sigma_N}} x_j$, then we get

$$\begin{aligned}
\int_{\mathbb{S}^{N-1}} (\tilde{v}\cdot n)s_{1,j}\, d\sigma &= \sqrt{\frac{N}{\sigma_N}}\int_{\mathbb{B}^N} \operatorname{div}(x_j\tilde{v})\, dx = \sqrt{\frac{N}{\sigma_N}}\int_{\mathbb{B}^N} \tilde{v}_j\, dx \\
&= \sqrt{\frac{N}{\sigma_N}} m(r)_j = -\frac{1}{\sigma_N} r_{1,j}.
\end{aligned}$$

This proves the lemma. $\square$

3 Explicit solution for the linearised problem

In this section we describe the linearisations of $\mathcal{F}_1$ and $\mathcal{F}_2$ around $r = 0$ that we found in Lemma 2.5 in terms of the Dirichlet-to-Neumann operator $\mathcal{N}$ for the Laplacian on

$\mathbb{B}^N$. The spectrum and the eigenfunctions of $\mathcal{F}_1'(0)$ and $\mathcal{F}_2'(0)$ are easily derived from the spectral properties of $\mathcal{N}$. We restrict ourselves to the cases $N = 2$ and $N = 3$.

We investigate for which $f : \mathbb{S}^{N-1} \to \mathbb{R}^N$, the system

$$-\Delta v + \nabla p = 0, \quad \text{on } \mathbb{B}^{\mathrm{N}} \tag{3.1}$$

$$\operatorname{div} v = 0, \quad \text{on } \mathbb{B}^{\mathrm{N}} \tag{3.2}$$

$$(\nabla v + \nabla v^T - pI)n = f, \quad \text{on } \mathbb{S}^{\mathrm{N}-1} \tag{3.3}$$

$$\int_{\mathbb{B}^N} v \ dx = 0, \tag{3.4}$$

$$\int_{\mathbb{B}^N} \operatorname{rot} v \ dx = 0, \tag{3.5}$$

has a solution (v, p). This solution will always be unique. For suitable f we solve the system.

3.1 The two dimensional boundary value problem

For the two-dimensional problem we introduce polar coordinates ρ and θ and unit vectors e_ρ and e_θ. Define for all $k \in \mathbb{Z}$ the functions $s_k : \mathbb{S}^1 \to \mathbb{C}$ by

$$s_k := \frac{1}{\sqrt{2\pi}} e^{ik\theta}.$$

Complexifying the spaces $\mathfrak{S}_k^2$ in Section 2, one can identify these functions with the spherical harmonics $s_{k,j}$ in the following way:

$$s_{k,1} := s_k, \quad s_{k,2} := s_{-k},$$

for $k > 0$ and $s_{0,1} = s_0$. We have $\nu(2,k) = \dim \mathfrak{S}_k^2 = 2$, for $k \neq 0$ and $\nu(2,0) = \dim \mathfrak{S}_0^2 = 1$. We write

$$f = f^\rho e_\rho + f^\theta e_\theta, \quad f^\rho(\theta) = \sum_{k=-\infty}^{\infty} f_k^\rho s_k(\theta), \quad f^\theta(\theta) = \sum_{k=-\infty}^{\infty} f_k^\theta s_k(\theta),$$

$$v = v^\rho e_\rho + v^\theta e_\theta, \quad v^\rho(\rho,\theta) = \sum_{k=-\infty}^{\infty} v_k^\rho(\rho) s_k(\theta), \quad v^\theta(\rho,\theta) = \sum_{k=-\infty}^{\infty} v_k^\theta(\rho) s_k(\theta).$$

for $f^\rho, f^\theta : \mathbb{S}^1 \to \mathbb{R}$, $v^\rho, v^\theta : \mathbb{B}^2 \to \mathbb{R}$, $f_k^\rho, f_k^\theta \in \mathbb{C}$ and $v_k^\rho, v_k^\theta : [0,1] \to \mathbb{C}$. Because p is harmonic we have

$$p = \sum_{k=-\infty}^{\infty} p_k \rho^{|k|} s_k(\theta), \tag{3.6}$$

for certain $p_k \in \mathbb{C}$.

Lemma 3.1. *For $N = 2$, the system (3.1)-(3.5) is solvable if and only if $f_0^\theta = 0$, $f_1^\rho = i f_1^\theta$ and $f_{-1}^\rho = -i f_{-1}^\theta$. For the components of the normal velocity on $\mathbb{S}^1$ we have for $k \notin \{-1, 0, 1\}$*

$$v_k^\rho(1) = \frac{|k|}{2(k^2-1)} f_k^\rho - \frac{i \operatorname{sgn} k}{2(k^2-1)} f_k^\theta. \tag{3.7}$$

For $k \in \{-1, 0, 1\}$ we have $v_k^\rho = 0$.

Proof. Parallel to [7], we write (3.1) in polar coordinates and get

$$v_k^{\rho\prime\prime} + \frac{1}{\rho} v_k^{\rho\prime} - \frac{k^2+1}{\rho^2} v_k^{\rho} - \frac{2ik}{\rho^2} v_k^{\theta} = |k| p_k \rho^{|k|-1},$$

$$v_k^{\theta\prime\prime} + \frac{1}{\rho} v_k^{\theta\prime} - \frac{k^2+1}{\rho^2} v_k^{\theta} + \frac{2ik}{\rho^2} v_k^{\rho} = ik p_k \rho^{|k|-1}.$$

Any solution for v_k^{ρ} and v_k^{θ} in terms of p_k can be written as

$$v_k^{\rho} = \frac{1}{2} p_k \rho^{|k|+1} + V_k^{\rho},$$

$$v_k^{\theta} = V_k^{\theta},$$

where V_k^{ρ} and V_k^{θ} satisfy

$$V_k^{\rho\prime\prime} + \frac{1}{\rho} V_k^{\rho\prime} - \frac{k^2+1}{\rho^2} V_k^{\rho} - \frac{2ik}{\rho^2} V_k^{\theta} = 0,$$

$$V_k^{\theta\prime\prime} + \frac{1}{\rho} V_k^{\theta\prime} - \frac{k^2+1}{\rho^2} V_k^{\theta} + \frac{2ik}{\rho^2} V_k^{\rho} = 0.$$

For $k \neq 0$, the general regular solution to these equations is given by

$$V_k^{\rho} = A_k \rho^{|k|+1} + B_k \rho^{|k|-1},$$

$$V_k^{\theta} = i \operatorname{sgn} k \big(- A_k \rho^{|k|+1} + B_k \rho^{|k|-1} \big),$$

for some constants A_k and B_k. The result is

$$v_k^{\rho} = \frac{1}{2} p_k \rho^{|k|+1} + A_k \rho^{|k|+1} + B_k \rho^{|k|-1}, \tag{3.8}$$

$$v_k^{\theta} = i \operatorname{sgn} k \big(- A_k \rho^{|k|+1} + B_k \rho^{|k|-1} \big). \tag{3.9}$$

For $k = 0$ we get

$$V_0^{\rho} = A_0 \rho, \quad V_0^{\theta} = B_0 \rho$$

and

$$v_0^{\rho} = \frac{1}{2} p_0 \rho + A_0 \rho, \tag{3.10}$$

$$v_0^{\theta} = B_0 \rho. \tag{3.11}$$

For each $k \in \mathbb{Z}$ we have to determine three constants: p_k, A_k and B_k. These follow from the boundary conditions (3.3), the incompressibility condition (3.2) and extra conditions (3.4) and (3.5). In polar coordinates conditions (3.3) and (3.2) are given by

$$2 \frac{\partial v^{\rho}}{\partial \rho} - p = f^{\rho}, \tag{3.12}$$

$$\frac{\partial v^{\theta}}{\partial \rho} + \frac{\partial v^{\rho}}{\partial \theta} - v^{\theta} = f^{\theta} \tag{3.13}$$

and

$$\frac{\partial v^\rho}{\partial \rho} + \frac{1}{\rho} v^\rho + \frac{1}{\rho} \frac{\partial v^\theta}{\partial \theta} = 0. \tag{3.14}$$

We distinguish between three cases: $k = 0$, $k = \pm 1$ and $k \notin \{-1, 0, 1\}$.

1. For $k = 0$, (3.6), (3.10), (3.11) and (3.12)-(3.14) give the underdetermined system

$$\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} p_0 \\ A_0 \\ B_0 \end{pmatrix} = \begin{pmatrix} f_0^\rho \\ f_0^\theta \\ 0 \end{pmatrix}.$$

From this system, B_0 cannot be determined. However, condition (3.5) implies

$$\int_{\mathbb{S}^1} v^\theta \, d\sigma = \pm \int_{\mathbb{B}^2} \operatorname{rot} v \, dx = 0.$$

From (3.11) we get $B_0 = 0$. We conclude that

$$p_0 = -f_0^\rho, \quad A_0 = \frac{1}{2} f_0^\rho, \quad B_0 = 0.$$

Combining this and (3.10) we get $v_0^\rho = 0$. There is also a condition on f, namely

$$f_0^\theta = 0. \tag{3.15}$$

2. For $k = \pm 1$, we derive from (3.6), (3.8), (3.9) and (3.12)-(3.14)

$$\begin{pmatrix} 1 & 4 & 0 \\ \pm 1 & 0 & 0 \\ 3 & 8 & 0 \end{pmatrix} \begin{pmatrix} p_{\pm 1} \\ A_{\pm 1} \\ B_{\pm 1} \end{pmatrix} = \begin{pmatrix} f_{\pm 1}^\rho \\ -2i f_{\pm 1}^\theta \\ 0 \end{pmatrix}. \tag{3.16}$$

The first and second equation in the system (3.16) give

$$p_{\pm 1} = \mp 2i f_{\pm 1}^\theta, \quad A_{\pm 1} = \frac{1}{4} f_{\pm 1}^\rho \pm \frac{1}{2} i f_{\pm 1}^\theta.$$

We cannot determine $B_{\pm 1}$ from (3.16). However, (3.2) and (3.4) imply

$$\begin{aligned} \int_{\mathbb{S}^1} x_i v \cdot n \, d\sigma &= \int_{\mathbb{B}^2} \operatorname{div}(x_i v) \, dx = \int_{\mathbb{B}^2} x_i \operatorname{div} v \, dx + \int_{\mathbb{B}^2} \nabla x_i \cdot v \, dx \\ &= \int_{\mathbb{B}^2} v_i \, dx = 0. \end{aligned} \tag{3.17}$$

This implies

$$v_{\pm 1}^\rho = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{S}^1} (x_1 \pm i x_2) v \cdot n \, d\sigma = 0. \tag{3.18}$$

Combining this and (3.8) gives us

$$B_{\pm 1} = -\frac{1}{4} f_{\pm 1}^\rho \pm \frac{1}{2} i f_{\pm 1}^\theta.$$

From the third equation in (3.16) we derive

$$f_{\pm 1}^\rho = \pm i f_{\pm 1}^\theta. \tag{3.19}$$

This gives two more conditions on f.

3. For $k \notin \{-1,0,1\}$ we get from (3.6), (3.8), (3.9) and (3.12)-(3.14) the following system of equations:

$$\begin{pmatrix} |k| & 2(|k|+1) & 2(|k|-1) \\ k & 0 & 4(k-\operatorname{sgn} k) \\ |k|+2 & 4(|k|+1) & 0 \end{pmatrix} \begin{pmatrix} p_k \\ A_k \\ B_k \end{pmatrix} = \begin{pmatrix} f_k^\rho \\ -2if_k^\theta \\ 0 \end{pmatrix}. \tag{3.20}$$

The matrix on the left-hand side is invertible for $k \notin \{-1,0,1\}$ and the solution to (3.20) is given by

$$\begin{aligned} p_k &= -f_k^\rho - i\operatorname{sgn} k f_k^\theta, \\ A_k &= \frac{(|k|+2)(f_k^\rho + i\operatorname{sgn} k f_k^\theta)}{4(|k|+1)}, \\ B_k &= \frac{k f_k^\rho + i(|k|-2)f_k^\theta}{4(k-\operatorname{sgn} k)}. \end{aligned}$$

We are interested in the normal component of the velocity v^ρ on $\mathbb{S}^1$. For $k \notin \{-1,0,1\}$ we get from (3.8)

$$v_k^\rho(1) = \frac{1}{2}p_k + A_k + B_k$$

and (3.7) follows.

□

We introduce the Dirichlet-to-Neumann operator $\mathcal{N} : \mathbb{H}^\sigma(\mathbb{S}^{N-1}) \to \mathbb{H}^{\sigma-1}(\mathbb{S}^{N-1})$, $\sigma > 1$, by the operator that maps a function r to $\mathcal{N}r := \frac{\partial u}{\partial n}$, where u satisfies

$$\begin{aligned} \Delta u &= 0 \quad \text{on } \mathbb{B}^{\mathrm{N}}, \\ u &= r \quad \text{on } \mathbb{S}^{\mathrm{N}-1}. \end{aligned}$$

This is a first order pseudodifferential operator on $\mathbb{S}^{N-1}$ with spectrum $\mathbb{N}_0$. It is well known that for $N = 2$ the functions s_k, for $k \in \mathbb{Z}$, form a complete orthonormal set of eigenfunctions in $\mathbb{L}_2(\mathbb{S}^1)$ for $\mathcal{N}$, with

$$\mathcal{N}s_k = |k|s_k. \tag{3.21}$$

Now we write for $N = 2$ the linearisations of $\mathcal{F}_1$ and $\mathcal{F}_2$ around zero that we found in Lemma 2.5 in terms of $\mathcal{N}$.

- Consider (3.1)-(3.5) with $f = \kappa'(0)[r]n$. From [16] we have

$$\kappa'(0)[r] = -\mathcal{N}^2 r + r,$$

which implies

$$\begin{aligned} f_k^\rho &= (-k^2+1)r_k, \\ f_k^\theta &= 0, \end{aligned}$$

with $r_k = (r, s_k)_0$. Note that (3.15) and (3.19) are satisfied in this case. For $k \notin \{-1, 0, 1\}$ we get from (3.7)

$$v_k^\rho(1) = -\frac{1}{2}|k|r_k$$

and we have $v_0^\rho(1) = v_{\pm 1}^\rho(1) = 0$. Lemma 2.5 and (3.21) imply

$$\mathcal{F}_1'(0)[r] = -\frac{1}{2}\mathcal{N}\mathcal{P}_1 r, \tag{3.22}$$

where $\mathcal{P}_1 : \mathbb{L}_2(\mathbb{S}^1) \to \mathbb{L}_2(\mathbb{S}^1)$ is the orthogonal projection along $\langle s_{-1}, s_0, s_1\rangle = \mathfrak{S}_0^2 \oplus \mathfrak{S}_1^2$.

- Consider (3.1)-(3.5) with $f = \frac{2N(1-N)}{\sigma_N} rn + \frac{2N}{\sigma_N}\nabla_0 r = -\frac{2}{\pi}rn + \frac{2}{\pi}\nabla_0 r$. Because $\nabla_0 s_k = \frac{\partial s_k}{\partial\theta}e_\theta = iks_k e_\theta$ we get

  $$f_k^\rho = -\frac{2}{\pi}r_k$$

  and

  $$f_k^\theta = \frac{2ik}{\pi}r_k.$$

  We see that (3.15) and (3.19) are satisfied. From (3.7) we get for all $k \in \mathbb{Z}$

  $$v_k^\rho(1) = 0.$$

  Lemmas 2.5 and 2.6 give us

  $$\mathcal{F}_2'(0)[r] = -\frac{1}{\pi}r - \frac{1}{2\pi}(r_1 s_1 + r_{-1}s_{-1}). \tag{3.23}$$

3.2 The three dimensional boundary value problem

For the three dimensional problem we introduce the spherical harmonics $Y_{km} : \mathbb{S}^2 \to \mathbb{C}$ for each $k \in \mathbb{N}_0$ and $m \in \{-k, -k+1, \dots, 0, \dots, k-1, k\}$ by means of spherical coordinates in the following way:

$$Y_{km} = (-1)^m \sqrt{\frac{(2k+1)(k-m)!}{4\pi(k+m)!}}\, P_k^m(\cos\theta)e^{im\phi},$$

where θ is the polar coordinate, ϕ the azimuthal coordinate and P_k^m the Legendre polynomial given by

$$P_k^m(x) = \frac{\sqrt{(1-x^2)^m}}{2^k k!}\frac{d^{k+m}}{dx^{k+m}}\left[(x^2-1)^k\right].$$

Complexifying the spaces $\mathfrak{S}_k^3$ in Section 2, one can identify Y_{km} with the spherical harmonics $s_{k,j}$ with $j = m + k + 1$. We also introduce the vector spherical harmonics $\vec{V}_{km}$,

$\vec{X}_{km}, \vec{W}_{km} : \mathbb{S}^2 \to \mathbb{C}^3$ conform [6] and [9] in the following way:

$$\begin{aligned}\vec{V}_{km} :&= -\sqrt{\frac{k+1}{2k+1}}Y_{km}e_\rho + \frac{1}{\sqrt{(k+1)(2k+1)}}\frac{\partial Y_{km}}{\partial\theta}e_\theta \\ &\quad + \frac{im}{\sqrt{(k+1)(2k+1)}\sin\theta}Y_{km}e_\phi, \\ \vec{X}_{km} :&= -\frac{m}{\sqrt{k(k+1)}\sin\theta}Y_{km}e_\theta - \frac{i}{\sqrt{k(k+1)}}\frac{\partial Y_{km}}{\partial\theta}e_\phi, \\ \vec{W}_{km} :&= \sqrt{\frac{k}{2k+1}}Y_{km}e_\rho + \frac{1}{\sqrt{k(2k+1)}}\frac{\partial Y_{km}}{\partial\theta}e_\theta + \frac{im}{\sqrt{k(2k+1)}\sin\theta}Y_{km}e_\phi,\end{aligned}$$

for $k \in \mathbb{N}_0$ and $m \in \{-k, -k+1, \dots, 0, \dots, k-1, k\}$. The functions Y_{km} form a complete orthonormal set in $\mathbb{L}_2(\mathbb{S}^2)$ and $\vec{V}_{km}$, $\vec{X}_{km}$, $\vec{W}_{km}$, excluding $\vec{X}_{00} \equiv \vec{W}_{00} \equiv 0$, form a complete orthonormal set in $\left(\mathbb{L}_2(\mathbb{S}^2)\right)^3$. The functions $\vec{W}_{1,-1}$, $\vec{W}_{1,0}$ and $\vec{W}_{1,1}$ are three independent constant vector fields. Therefore, if we take $\mathbb{L}_2(\mathbb{S}^2)$-inner products of the constant vector fields $e_1 = (1,0,0)^T$, $e_2 = (0,1,0)^T$ and $e_3 = (0,0,1)^T$ with other vector spherical harmonics we get

$$\int_{\mathbb{S}^2} \vec{W}_{km}\, d\sigma = 0, \quad k \in \mathbb{N}_0 \setminus \{1\}, \quad \int_{\mathbb{S}^2} \vec{W}_{1m}\, d\sigma \neq 0 \tag{3.24}$$

$$\int_{\mathbb{S}^2} \vec{V}_{km}\, d\sigma = \int_{\mathbb{S}^2} \vec{X}_{km}\, d\sigma = 0, \quad k \in \mathbb{N}_0. \tag{3.25}$$

In this paper we use the following identities:

$$Y_{km}e_\rho = -\sqrt{\frac{k+1}{2k+1}}\vec{V}_{km} + \sqrt{\frac{k}{2k+1}}\vec{W}_{km}, \tag{3.26}$$

$$\nabla_0 Y_{km} = k\sqrt{\frac{k+1}{2k+1}}\vec{V}_{km} + (k+1)\sqrt{\frac{k}{2k+1}}\vec{W}_{km}, \tag{3.27}$$

$$\mathrm{rot}(g(\rho)V_{km}(\theta,\phi)) = i\sqrt{\frac{k}{2k+1}}\left(\frac{dg}{d\rho} + \frac{k+2}{\rho}g\right)\vec{X}_{km}, \tag{3.28}$$

$$\mathrm{rot}(g(\rho)X_{km}(\theta,\phi)) = i\sqrt{\frac{k}{2k+1}}\left(\frac{dg}{d\rho} - \frac{k}{\rho}g\right)\vec{V}_{km} + i\sqrt{\frac{k+1}{2k+1}}\left(\frac{dg}{d\rho} + \frac{k+1}{\rho}g\right)\vec{W}_{km}, \tag{3.29}$$

$$\mathrm{rot}(g(\rho)W_{km}(\theta,\phi)) = i\sqrt{\frac{k+1}{2k+1}}\left(\frac{dg}{d\rho} - \frac{k-1}{\rho}g\right)\vec{X}_{km}, \tag{3.30}$$

for any g depending only on ρ (see [6] or [9]). Introduce functions $\alpha_{km}, \beta_{km}, \gamma_{km} : [0,1] \to \mathbb{C}$ such that

$$v(\rho,\theta,\phi) = \sum_{k,m} \alpha_{km}(\rho)\vec{V}_{km}(\theta,\phi) + \beta_{km}(\rho)\vec{X}_{km}(\theta,\phi) + \gamma_{km}(\rho)\vec{W}_{km}(\theta,\phi) \tag{3.31}$$

and introduce $f_{km}^V, f_{km}^X, f_{km}^W \in \mathbb{C}$ such that

$$f(\theta,\phi) = \sum_{k,m} f_{km}^V \vec{V}_{km}(\theta,\phi) + f_{km}^X \vec{X}_{km}(\theta,\phi) + f_{km}^W \vec{W}_{km}(\theta,\phi).$$

Here and in the sequel summations are over all $k \in \mathbb{N}_0$ and $m \in \{-k, -k+1, \dots, 0, \dots, k-1, k\}$, excluding terms containing $\vec{X}_{00}$ and $\vec{W}_{00}$. Because p is harmonic, there exist $p_{km} \in \mathbb{C}$ such that

$$p(\rho,\theta,\phi) = \sum_{k,m} p_{km} \rho^k Y_{km}(\theta,\phi). \tag{3.32}$$

Lemma 3.2. *For $N = 3$, the system (3.1)-(3.5) is solvable if and only if $f_{1m}^X = f_{1m}^W = 0$ for $m \in \{-1, 0, 1\}$. Furthermore*

$$v \cdot n = \sum_{k \neq 1, m} \left[-\frac{k}{2k^2+4k+3} \sqrt{\frac{k+1}{2k+1}} f_{km}^V + \frac{1}{2(k-1)} \sqrt{\frac{k}{2k+1}} f_{km}^W \right] Y_{km}. \tag{3.33}$$

Proof. Combining (3.26), (3.27), (3.31) and (3.32) (see also [6] equations (2.16) and (3.5)) we get

$$\nabla p = \sum_{k,m} p_{km} \rho^{k-1} \sqrt{k(2k+1)} \vec{W}_{km}$$

and

$$\Delta v = \sum_{k,m} (\Lambda_{k+1} \alpha_{km}) \vec{V}_{km} + (\Lambda_k \beta_{km}) \vec{X}_{km} + (\Lambda_{k-1} \gamma_{km}) \vec{W}_{km},$$

where

$$\Lambda_k : \psi \mapsto \frac{\partial^2 \psi}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial \psi}{\partial \rho} - \frac{k(k+1)}{\rho^2} \psi.$$

From (3.1) we get

$$\begin{aligned} \Lambda_{k+1} \alpha_{km} &= 0, \\ \Lambda_k \beta_{km} &= 0, \\ \Lambda_{k-1} \gamma_{km} &= p_{km} \rho^{k-1} \sqrt{k(2k+1)}. \end{aligned}$$

The general regular solutions to these equations are given by

$$\begin{aligned} \alpha_{km}(\rho) &= A_{km} \rho^{k+1}, \\ \beta_{km}(\rho) &= B_{km} \rho^k, \\ \gamma_{km}(\rho) &= C_{km} \rho^{k-1} + \frac{1}{2} \sqrt{\frac{k}{2k+1}} p_{km} \rho^{k+1}. \end{aligned}$$

For each pair (k, m) we have to determine four constants: p_{km}, A_{km}, B_{km} and C_{km}. As in the two dimensional case, these constants follow from the boundary conditions (3.3), the incompressibility condition (3.2) and extra conditions (3.4) and (3.5). In [6] equations (4.3)-(4.6), conditions (3.3) and (3.2) are written in terms of α_{km}, β_{km} and

γ_{km}. If we substitute the expressions above, then we get for $k \in \mathbb{N}_0$ and $m \in \{-k, -k+1, \ldots, 0, \ldots, k-1, k\}$

$$\begin{pmatrix} \left(\frac{k+1}{2k+1}\right)^{\frac{3}{2}} & \frac{2k^2+3k+2}{2k+1} & 0 & 0 \\ 0 & 0 & k-1 & 0 \\ \sqrt{\frac{k}{2k+1}}\frac{2k^2-1}{2k+1} & -\frac{\sqrt{k}\sqrt{k+1}(2k+3)}{2k+1} & 0 & 2(k-1) \\ \frac{k}{2k+1} & -\sqrt{\frac{k+1}{2k+1}}(2k+3) & 0 & 0 \end{pmatrix} \begin{pmatrix} p_{km} \\ A_{km} \\ B_{km} \\ C_{km} \end{pmatrix} = \begin{pmatrix} f^V_{km} \\ f^X_{km} \\ f^W_{km} \\ 0 \end{pmatrix}. \tag{3.34}$$

Only for $k = 1$ the matrix on the left-hand side is not invertible. In this case we get

$$\begin{pmatrix} (\frac{2}{3})^{\frac{3}{2}} & \frac{7}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{3\sqrt{3}} & -\frac{5}{3}\sqrt{2} & 0 & 0 \\ \frac{1}{3} & -5\sqrt{\frac{2}{3}} & 0 & 0 \end{pmatrix} \begin{pmatrix} p_{1m} \\ A_{1m} \\ B_{1m} \\ C_{1m} \end{pmatrix} = \begin{pmatrix} f^V_{1m} \\ f^X_{1m} \\ f^W_{1m} \\ 0 \end{pmatrix}. \tag{3.35}$$

The rank of the matrix on the left-hand side of (3.35) is two. Therefore we have six restrictions on f and six degrees of freedom. From the last three equations in the system (3.35), we obtain the following conditions:

$$f^X_{1m} = f^W_{1m} = 0, \tag{3.36}$$

for $m = -1, 0, 1$. From the first and the fourth equation in (3.35) we derive

$$A_{1m} = \frac{1}{9} f^V_{1m} \tag{3.37}$$

and

$$p_{1m} = \frac{5}{3}\sqrt{\frac{2}{3}} f^V_{1m}. \tag{3.38}$$

The six constants B_{1m} and C_{1m}, for $m = -1, 0, 1$, cannot be determined from (3.35). In order to calculate these constants we consider (3.4) and (3.5). Writing condition (3.4) in spherical coordinates and substituting (3.31) we get

$$\begin{aligned} 0 &= \int_{\mathbb{B}^3} v \; dx \\ &= \sum_{k,m} \int_{\mathbb{S}^2} \int_0^1 \rho^2 \alpha_{km}(\rho) \vec{V}_{km} + \rho^2 \beta_{km}(\rho) \vec{X}_{km} + \rho^2 \gamma_{km}(\rho) \vec{W}_{km} \; d\rho d\sigma \\ &= \sum_{k,m} \int_{\mathbb{S}^2} \int_0^1 A_{km} \rho^{k+3} \vec{V}_{km} + B_{km} \rho^{k+2} \vec{X}_{km} \\ &\quad + \left(C_{km} \rho^{k+1} + \frac{1}{2}\sqrt{\frac{k}{2k+1}} p_{km} \rho^{k+3} \right) \vec{W}_{km} \; d\rho d\sigma \\ &= \sum_{k,m} \int_{\mathbb{S}^2} \frac{A_{km}}{k+4} \vec{V}_{km} + \frac{B_{km}}{k+3} \vec{X}_{km} + \left(\frac{C_{km}}{k+2} + \frac{1}{2}\sqrt{\frac{k}{2k+1}} \frac{p_{km}}{k+4} \right) \vec{W}_{km} \; d\sigma. \end{aligned}$$

From (3.24) and (3.25) we get

$$\frac{1}{3}C_{1m} + \frac{1}{10\sqrt{3}}p_{1m} = 0,$$

for $m = -1, 0, 1$. Combining this and (3.38) we get

$$C_{1m} = -\frac{\sqrt{2}}{6} f^V_{1m}, \tag{3.39}$$

for $m = -1, 0, 1$.

Now we consider condition (3.5). From (3.28) and (3.30) we see that $\mathrm{rot}(\alpha_{km}(\rho)\vec{V}_{km})$ and $\mathrm{rot}(\gamma_{km}(\rho)\vec{W}_{km})$ have no $\vec{W}_{km}$-components. Therefore, from (3.25) we see that integrals of $\mathrm{rot}(\alpha_{km}(\rho)\vec{V}_{km})$ and $\mathrm{rot}(\gamma_{km}(\rho)\vec{W}_{km})$ over $\mathbb{S}^2$ vanish. Using (3.29) as well we get

$$\begin{aligned}
0 &= \int_{\mathbb{B}^3} \mathrm{rot}\, v \; dx \\
&= \sum_{k,m} \int_{\mathbb{S}^2} \int_0^1 \rho^2 \,\mathrm{rot}(\alpha_{km}(\rho)\vec{V}_{km}) + \rho^2 \,\mathrm{rot}(\beta_{km}(\rho)\vec{X}_{km}) + \rho^2 \,\mathrm{rot}(\gamma_{km}(\rho)\vec{W}_{km}) \; d\rho d\sigma \\
&= \sum_{k,m} \int_{\mathbb{S}^2} \int_0^1 \rho^2 \,\mathrm{rot}(\beta_{km}(\rho)\vec{X}_{km}) \; d\rho d\sigma \\
&= \sum_{k,m} \int_{\mathbb{S}^2} \int_0^1 \rho^2 i \sqrt{\frac{k}{2k+1}} \left(\frac{\partial \beta_{km}}{\partial \rho} - \frac{k}{\rho}\beta_{km} \right) \vec{V}_{km} \\
&\quad + \rho^2 i \sqrt{\frac{k+1}{2k+1}} \left(\frac{\partial \beta_{km}}{\partial \rho} + \frac{k+1}{\rho}\beta_{km} \right) \vec{W}_{km} \; d\rho d\sigma \\
&= \sum_{k,m} \int_{\mathbb{S}^2} \int_0^1 \rho^2 i \sqrt{\frac{k+1}{2k+1}} \left(\frac{\partial \beta_{km}}{\partial \rho} + \frac{k+1}{\rho}\beta_{km} \right) \vec{W}_{km} \; d\rho d\sigma.
\end{aligned}$$

Since in the last expression, only terms for $k = 1$ are unequal to zero (see (3.24)) we get

$$B_{1m} = 0. \tag{3.40}$$

For $k \neq 1$ the solution to (3.34) is given by

$$\begin{aligned}
p_{km} &= \frac{(2k+3)\sqrt{(2k+1)(k+1)}}{2k^2+4k+3} f^V_{km}, \\
A_{km} &= \frac{k}{2k^2+4k+3} f^V_{km}, \\
B_{km} &= \frac{1}{k-1} f^X_{km}, \\
C_{km} &= \frac{1}{2(k-1)} \left[-\frac{\sqrt{k}\sqrt{k+1}(2k+3)(k-1)}{2k^2+4k+3} f^V_{km} + f^W_{km} \right].
\end{aligned}$$

For the normal component of v on the unit sphere we get

$$
\begin{aligned}
v\cdot n &= \sum_{k,m} \alpha_{km}(1)\vec{V}_{km}\cdot e_\rho + \beta_{km}(1)\vec{X}_{km}\cdot e_\rho + \gamma_{km}(1)\vec{W}_{km}\cdot e_\rho \\
&= \sum_{k,m}\left[-\alpha_{km}(1)\sqrt{\frac{k+1}{2k+1}} + \gamma_{km}(1)\sqrt{\frac{k}{2k+1}}\right] Y_{km} \\
&= \sum_{k,m}\left[-\sqrt{\frac{k+1}{2k+1}}A_{km} + \sqrt{\frac{k}{2k+1}}C_{km} + \frac{1}{2}\frac{k}{2k+1}p_{km}\right] Y_{km} \\
&= \sum_{k\neq 1,m}\left[-\frac{k\sqrt{\frac{k+1}{2k+1}}}{2k^2+4k+3}f^V_{km} + \frac{1}{2(k-1)}\sqrt{\frac{k}{2k+1}}f^W_{km}\right] Y_{km}.
\end{aligned}
$$

In the last step we omitted the terms for $k=1$. This is possible because from (3.37), (3.38) and (3.39) it follows that

$$
-\sqrt{\frac{2}{3}}A_{1m} + \sqrt{\frac{1}{3}}C_{1m} + \frac{1}{6}p_{1m} = 0,
$$

for $m=-1,0,1$. Note that the fact that the terms for $k=1$ in (3.33) vanish also follows if we repeat calculations (3.17) and (3.18) for the three dimensional case. □

Consider the Dirichlet-to-Neumann operator $\mathcal{N}$ that we introduced earlier. It is known that the spherical harmonics Y_{km} form a complete orthonormal set of eigenfunctions of $\mathcal{N}$ in $\mathbb{L}_2(\mathbb{S}^2)$, with

$$
\mathcal{N}Y_{km} = kY_{km}. \tag{3.41}
$$

Now we write $\mathcal{F}_1'(0)$ and $\mathcal{F}_2'(0)$ in terms of the Dirichlet-to-Neumann operator for the case $N=3$. As for the two dimensional case, we do this by considering two special cases for f in the system (3.1)-(3.5).

- Let us consider the special case

  $$
  f = \kappa'(0)[r]n,
  $$

  where $n=e_\rho$. Write

  $$
  r = \sum_{k,m} r_{km}Y_{km},
  $$

  with $r_{km}=(r,Y_{km})_0$. Since for $N=3$ we have from [16]

  $$
  \kappa'(0)[r]n = \left(-\mathcal{N}^2 r - \mathcal{N}r + 2r\right)n = \sum_{k,m}(-k^2-k+2)r_{km}Y_{km}n,
  $$

  we get from (3.26)

  $$
  \kappa'(0)[r]n = \sum_{k,m}(-k^2-k+2)\left(-\sqrt{\frac{k+1}{2k+1}}\vec{V}_{km} + \sqrt{\frac{k}{2k+1}}\vec{W}_{km}\right) r_{km}.
  $$

We have

$$f^V_{km} = -\sqrt{\frac{k+1}{2k+1}}(-k^2-k+2)r_{km},$$
$$f^X_{km} = 0,$$
$$f^W_{km} = \sqrt{\frac{k}{2k+1}}(-k^2-k+2)r_{km}.$$

Note that (3.36) is satisfied. From (3.33) we get

$$v\cdot n = \sum_{k\neq 1,m} -\frac{k(k+2)(k+\frac{1}{2})}{2k^2+4k+3} r_{km}Y_{km}.$$

From Lemma 2.5 and (3.41) we get

$$\mathcal{F}_1'(0)[r] = -\mathcal{N}\left(\mathcal{N}+2\mathcal{I}\right)\left(\mathcal{N}+\frac{1}{2}\mathcal{I}\right)\left(2\mathcal{N}^2+4\mathcal{N}+3\mathcal{I}\right)^{-1}\mathcal{P}_1 r, \tag{3.42}$$

where $\mathcal{P}_1 : \mathbb{L}_2(\mathbb{S}^2) \to \mathbb{L}_2(\mathbb{S}^2)$ is the orthogonal projection along $\mathfrak{S}^3_0 \oplus \mathfrak{S}^3_1 = \langle Y_{0,0}, Y_{1,-1}, Y_{1,0}, Y_{1,1}\rangle$.

- Now we consider the case

$$f = \frac{2N(1-N)}{\sigma_N} rn + \frac{2N}{\sigma_N}\nabla_0 r = -\frac{3}{\pi} rn + \frac{3}{2\pi}\nabla_0 r.$$

From formulas (3.26) and (3.27) we get

$$\begin{aligned}
f &= \sum_{k,m} -\frac{3}{\pi} r_{km}\left[-\sqrt{\frac{k+1}{2k+1}}\vec{V}_{km} + \sqrt{\frac{k}{2k+1}}\vec{W}_{km}\right] \\
&\quad + \frac{3}{2\pi} r_{km}\left[k\sqrt{\frac{k+1}{2k+1}}\vec{V}_{km} + (k+1)\sqrt{\frac{k}{2k+1}}\vec{W}_{km}\right] \\
&= \sum_{k,m} \frac{3}{2\pi}(k+2)\sqrt{\frac{k+1}{2k+1}} r_{km}\vec{V}_{km} + \frac{3}{2\pi}(k-1)\sqrt{\frac{k}{2k+1}} r_{km}\vec{W}_{km}.
\end{aligned}$$

In this case we have

$$f^V_{km} = \frac{3}{2\pi}(k+2)\sqrt{\frac{k+1}{2k+1}} r_{km},$$
$$f^X_{km} = 0,$$
$$f^W_{km} = \frac{3}{2\pi}(k-1)\sqrt{\frac{k}{2k+1}} r_{km}.$$

From (3.33) we get

$$v\cdot n = \sum_{k\neq 1,m} -\frac{3}{4\pi}\frac{k}{2k^2+4k+3} r_{km}Y_{km}.$$

Lemmas 2.5 and 2.6 imply

$$\mathcal{F}_2'(0)[r] = -\frac{3}{4\pi}\mathcal{N}\left(2\mathcal{N}^2+4\mathcal{N}+3\mathcal{I}\right)^{-1}\mathcal{P}_1 r - \frac{3}{4\pi}r - \frac{1}{4\pi}\left(r_{1,-1}Y_{1,-1}+r_{1,0}Y_{1,0}+r_{1,1}Y_{1,1}\right). \tag{3.43}$$

From the linearisation (3.22) and (3.23) we see that for $N=2$ we have

$$(s_k, \mathcal{F}_j'(0)[s_k])_0 = -p_j(|k|) \tag{3.44}$$

for $j=1,2$, $k\in\mathbb{Z}$ and

$$p_1(k) = \begin{cases} \dfrac{k}{2} & k\neq 1 \\ 0 & k=1 \end{cases}$$

$$p_2(k) = \begin{cases} \dfrac{1}{\pi} & k\neq 1 \\ \dfrac{3}{2\pi} & k=1. \end{cases}$$

For $N=3$, we see from (3.42) and (3.43) that

$$(Y_{km}, \mathcal{F}_j'(0)[Y_{km}])_0 = -p_j(k), \tag{3.45}$$

for $j=1,2$, $k\in\mathbb{Z}$, $m\in\{-k,-k+1,\dots,0,\dots,k-1,k\}$ and

$$p_1(k) = \begin{cases} \dfrac{k(k+2)(k+\frac{1}{2})}{2k^2+4k+3} & k\neq 1 \\ 0 & k=1 \end{cases}$$

$$p_2(k) = \begin{cases} \dfrac{3}{4\pi}\dfrac{k}{2k^2+4k+3}+\dfrac{3}{4\pi} & k\neq 1 \\ \dfrac{1}{\pi} & k=1. \end{cases}$$

Lemma 3.3. *Let $N=2$ or $N=3$. There exists a $C_1>0$ such that for all $r\in\mathbb{H}^s(\mathbb{S}^{N-1})$*

$$(\tilde{r}, \mathcal{F}_1'(0)[\tilde{r}])_{s-1} \le -C_1\|\tilde{r}\|^2_{s-\frac{1}{2}}$$

and

$$(\tilde{r}, \mathcal{F}_2'(0)[\tilde{r}])_{s-1} \le 0,$$

where $\tilde{r}$ is the $\mathbb{L}_2(\mathbb{S}^{N-1})$-projection of r along $\mathfrak{S}_0^N\oplus\mathfrak{S}_1^N$.

Proof. Define

$$C_1 := \inf_{k\in\mathbb{N}\setminus\{1\}}\frac{\gamma p_1(k)}{(k^2+1)^{\frac{1}{2}}}.$$

All values for $p_1(k)$ with $k\in\{2,3,4,\dots\}$ are positive and $\lim_{k\to\infty}\frac{p_1(k)}{(k^2+1)^{\frac{1}{2}}}$ is positive. Therefore $C_1>0$. The lemma follows from this and the inequality $p_2(k)\ge 0$. □

4 Global Existence results for the injection problems via energy estimates

In this section we find a global existence result and decay properties for solutions to (2.20) with $\mu > 0$. From the energy estimates for the linearisation of the evolution operator given in Lemma 3.3 and smoothness properties we get an energy estimate for the evolution operator itself. Combining this with a local existence result from [12] we get global existence. To obtain the estimates, we need a chain rule for differential operators on $\mathbb{S}^{N-1}$ defined by

$$D_{\omega(l,m)} := x_l \frac{\partial}{\partial x_m} - x_m \frac{\partial}{\partial x_l}, \quad 1 \le l < m \le N.$$

Here ω is the bijection that we introduced to define the operator rot in (1.6). These differential operators generate one-parameter groups:

$$e^{hD_i} f = f \circ g_h,$$

where $g_h : \mathbb{S}^{N-1} \to \mathbb{S}^{N-1}$ are rotations.

Lemma 4.1. *Let $k = 1, 2$. For $r \in \mathbb{H}^{s+1}(\mathbb{S}^{N-1})$ with $\|r\|_s$ small and $s > \frac{N+5}{2}$ we have the generalised chain rule of differentiation:*

$$D_i \mathcal{F}_k(r) = \mathcal{F}_k'(r)[D_i r], \quad 1 \le i \le \tbinom{N}{2}. \tag{4.1}$$

Proof. Because $\mathcal{F}_k(r)$ commutes with g_h we get

$$\begin{aligned} D_i \mathcal{F}_k(r) &= \lim_{h\to 0} \frac{1}{h}(e^{hD_i} - \mathcal{I})\mathcal{F}_k(r) = \lim_{h\to 0} \frac{1}{h}\left(\mathcal{F}_k(r) \circ g_h - \mathcal{F}_k(r)\right) \\ &= \lim_{h\to 0} \frac{1}{h}\left(\mathcal{F}_k(r \circ g_h) - \mathcal{F}_k(r)\right) = \lim_{h\to 0} \frac{1}{h}\mathcal{F}_k'(r)[r \circ g_h - r] \\ &= \mathcal{F}_k'(r)\left[\lim_{h\to 0} \frac{r \circ g_h - r}{h}\right] = \mathcal{F}_k'(r)[D_i r], \end{aligned}$$

where $\mathcal{I}$ is the identity. □

Lemma 4.2. *If $r \in \mathfrak{S}_k^N$ then $D_i r \in \mathfrak{S}_k^N$.*

Proof. This follows from the fact that the spaces $\mathfrak{S}_k^N \subseteq \mathcal{C}^\infty(\mathbb{S}^{N-1})$ are invariant under rotations. □

Let for $\sigma > 1$ the norm $\|\cdot\|_{\sigma-1,1}$ on $\mathbb{H}^\sigma(\mathbb{S}^{N-1})$ be induced by the inner product

$$(r, \tilde{r})_{\sigma-1,1} := (r, \tilde{r})_{\sigma-1} + \sum_i (D_i r, D_i \tilde{r})_{\sigma-1}.$$

This norm is equivalent to the norm $\|\cdot\|_\sigma$ that we introduced earlier (see [8] Section 4).

Define

$$\mathfrak{M}_1^N := \left\{ r \in \mathcal{C}^0(\mathbb{S}^{N-1}) : \int_{\Omega_r} dx = \frac{\sigma_N}{N}, \int_{\Omega_r} x \, dx = 0 \right\}.$$

Note that $\mathfrak{M}_1^N$ is the set of continuous functions on $\mathbb{S}^{N-1}$, for which the corresponding domains Ω_r have the same volume as the unit ball and a geometric centre that coincides with the origin.

Lemma 4.3. *If* $\Omega_{R(t)}$ *satisfies* (1.1)-(1.6)*, then*

$$M(t) = \int_{\Omega_{R(t)}} x\ dx$$

is constant in t*.*

Proof. Let M_i be the ith component of M. Combining the divergence theorem, (1.2) and (1.5) we get

$$\dot{M}_i(t) = \int_{\Gamma_{R(t)}} (v \cdot n_R) x_i\ d\sigma = \int_{\Omega_{R(t)}} v_i\ dx + \int_{\Omega_{R(t)}} x_i \operatorname{div} v\ dx = 0,$$

where n_R is the normal on $\Gamma_{R(t)}$. $\square$

From this lemma we see that if r is a solution to (2.20) with $r_0 \in \mathfrak{M}_1^N$, then $r(t) \in \mathfrak{M}_1^N$ for all t. Introduce the Hilbert spaces $\mathbb{H}_1^\sigma(\mathbb{S}^{N-1})$ by

$$\mathbb{H}_1^\sigma(\mathbb{S}^{N-1}) = \{r \in \mathbb{H}^\sigma(\mathbb{S}^{N-1}) : (r,s)_0 = 0, \quad \forall s \in \mathfrak{S}_0^N \oplus \mathfrak{S}_1^N\}.$$

Define on a suitable neighbourhood $\mathcal{U}$ of zero in $\mathbb{H}^s(\mathbb{S}^{N-1})$ the operator $f_1 : \mathcal{U} \to \mathbb{R} \times \mathbb{R}^N$ by

$$f_1(r) = \left(\int_{\Omega_r} dx - \frac{\sigma_N}{N}, \int_{\Omega_r} x\ dx \right)^T .$$

Let $\mathcal{P}_1 : \mathbb{H}^s(\mathbb{S}^{N-1}) \to \mathbb{H}_1^s(\mathbb{S}^{N-1})$ be the orthogonal projection onto $\mathbb{H}_1^s(\mathbb{S}^{N-1})$ with respect to the $\mathbb{L}_2(\mathbb{S}^{N-1})$-inner product and let $\phi : \mathcal{U} \to \mathbb{R} \times \mathbb{R}^N \times \mathbb{H}_1^s(\mathbb{S}^{N-1})$ be defined by

$$\phi(r) = (f_1(r), \mathcal{P}_1 r)^T .$$

The vector $f_1'(0)[r]$ consists of inner products of r with spherical harmonics of order zero and one. Therefore, by the Implicit function Theorem, ϕ is an analytic bijection between a neighbourhood of zero in $\mathbb{H}^s(\mathbb{S}^{N-1})$ and a neighbourhood of zero in $\mathbb{R} \times \mathbb{R}^N \times \mathbb{H}_1^s(\mathbb{S}^{N-1})$. This result can be obtained in the same way as in [16] for different function spaces. On a suitable neighbourhood $\mathcal{U}$ of zero in $\mathbb{H}_1^s(\mathbb{S}^{N-1})$ we define $\psi : \mathcal{U} \to \mathfrak{M}_1^N$ by

$$\psi(r) = \phi^{-1}(0, r).$$

Using Lemma 4.3 it is easy to calculate that for a solution r to (2.20) we have

$$f_1(r(\tau)) = \left(\frac{V_0}{\alpha(\tau)^N}, \frac{1}{\alpha(\tau)^{N+1}} m_0 \right)^T =: (V_\tau, m_\tau)^T ,$$

where

$$(V_0, m_0)^T := f_1(r(0)).$$

For notational convenience we introduce $q_\tau := (V_\tau, m_\tau)^T$.

Theorem 4.4. *Let $N = 2$ or $N = 3$ and $\mu > 0$. Suppose that $s > \frac{N+6}{2}$. There exist a $\delta > 0$ and an $M > 0$ such that if $r_0 \in \mathbb{H}^s(\mathbb{S}^{N-1})$ with $\|r_0\|_s < \delta$ then the problem*

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r, \tau), \quad r(0) = r_0, \tag{4.2}$$

has a solution $r \in \mathcal{C}([0,\infty), \mathbb{H}^s(\mathbb{S}^{N-1})) \cap \mathcal{C}^1([0,\infty), \mathbb{H}^{s-1}(\mathbb{S}^{N-1}))$. Furthermore, $r \in \mathcal{C}^\infty(\mathbb{S}^{N-1} \times (0,\infty))$. If we regard r as a function of the original time-variable t, then

$$\|r(t)\|_s \le \frac{M}{\frac{\mu N t}{\sigma_N} + 1} \|r_0\|_s. \tag{4.3}$$

Proof. We follow the lines of the proof of a similar theorem for the Hele-Shaw flow in dimensions higher or equal to four, see [15]. Let $\lambda_0 \in (0, \frac{C_1}{2})$ and $\varepsilon := \frac{C_1}{2} - \lambda_0$, with C_1 as defined in Lemma 3.3.

1. If r satisfies (2.20), then $\tilde{r} := \mathcal{P}_1 r$ satisfies

$$\frac{\partial \tilde{r}}{\partial \tau} = \mathcal{P}_1 \mathcal{F}\left(\phi^{-1}\left(q_\tau, \tilde{r}\right), \tau\right). \tag{4.4}$$

First we prove solvability of this equation. Below, we will find an estimate of the type

$$\begin{aligned} &\gamma\left(\tilde{r}, \mathcal{P}_1 \mathcal{F}_1\left(\phi^{-1}\left(q_\tau, \tilde{r}\right)\right)\right)_{s-1,1} + \mu\alpha(\tau)^{1-N}\left(\tilde{r}, \mathcal{P}_1 \mathcal{F}_2\left(\phi^{-1}\left(q_\tau, \tilde{r}\right)\right)\right)_{s-1,1} \\ &\le -\lambda_0 \|\tilde{r}\|^2_{s-1,1} + C\frac{|q_0|^2}{\alpha(\tau)^{2N}}, \end{aligned}$$

for some $C > 0$, assuming that $|q_0|$ is small, $\tilde{r} \in \mathbb{H}^{s+1}(\mathbb{S}^{N-1})$ and $\|\tilde{r}\|_s < \delta'$, with δ' small enough. From this estimate we derive existence of solutions to (4.4) and find an estimate for the decay of $\tilde{r}$ as a function of time. The symbol C is used for a constant that may vary throughout the proof.

2. By Lemma 3.3 we have

$$\gamma(\tilde{r}, \mathcal{F}_1'(0)[\tilde{r}])_{s-1} + \mu\alpha(\tau)^{1-N}(\tilde{r}, \mathcal{F}_2'(0)[\tilde{r}])_{s-1} \le -C_1 \|\tilde{r}\|^2_{s-\frac{1}{2}}. \tag{4.5}$$

3. Because of Lipschitz continuity of $\mathcal{P}_1 \circ \mathcal{F}_k \circ \phi^{-1}$, for $k = 1, 2$, and because $\psi = \phi^{-1}(0, \cdot)$ we have

$$\|\mathcal{P}_1 \mathcal{F}_k\left(\phi^{-1}\left(q_\tau, \tilde{r}\right)\right) - \mathcal{P}_1 \mathcal{F}_k(\psi(\tilde{r}))\|_{s-1} \le C|q_\tau|. \tag{4.6}$$

From a straightforward calculation we see that $\psi'(0)$ is the identity on $\mathbb{H}_1^{s-\frac{1}{2}}(\mathbb{S}^{N-1})$ and $\psi(0) = 0$. Therefore the restriction of $\mathcal{F}_k'(0)$ to $\mathbb{H}_1^{s-\frac{1}{2}}(\mathbb{S}^{N-1})$ is the Fréchet derivative at zero of the mapping $\mathcal{P}_1 \circ \mathcal{F}_k \circ \psi$ that is analytic near zero in $\mathbb{H}_1^{s-\frac{1}{2}}(\mathbb{S}^{N-1})$. This gives us

$$\|\mathcal{P}_1 \mathcal{F}_k(\psi(\tilde{r})) - \mathcal{F}_k'(0)[\tilde{r}]\|_{s-\frac{3}{2}} \le C\|\tilde{r}\|^2_{s-\frac{1}{2}}. \tag{4.7}$$

Note that here the demand $s > \frac{N+6}{2}$ is necessary for analyticity (see Lemma 2.4). Combining (4.6) and (4.7) we get

$$\begin{aligned} &\gamma\left\{\left(\tilde{r}, \mathcal{P}_1\mathcal{F}_1\left(\phi^{-1}\left(q_\tau, \tilde{r}\right)\right)\right)_{s-1} - (\tilde{r}, \mathcal{F}_1'(0)[\tilde{r}])_{s-1}\right\} \\ &+\mu\alpha(\tau)^{1-N}\left\{\left(\tilde{r}, \mathcal{P}_1\mathcal{F}_2\left(\phi^{-1}\left(q_\tau, \tilde{r}\right)\right)\right)_{s-1} - (\tilde{r}, \mathcal{F}_2'(0)[\tilde{r}])_{s-1}\right\} \\ \le\; &C\left(|q_\tau|\|\tilde{r}\|_{s-1} + \|\tilde{r}\|^3_{s-\frac{1}{2}}\right). \end{aligned} \tag{4.8}$$

Here we used $\alpha(\tau)^{1-N} \le 1$.

4. From the chain rule (4.1) we get

$$\left(\tilde{r}, \mathcal{P}_1\mathcal{F}\left(\phi^{-1}\left(q_\tau, \tilde{r}\right), \tau\right)\right)_{s-1,1} = \gamma(F_1 + G_1) + \mu\alpha(\tau)^{1-N}(F_2 + G_2), \tag{4.9}$$

where for $k = 1, 2$

$$\begin{aligned} F_k &:= \left(\tilde{r}, \mathcal{P}_1\mathcal{F}_k\left(\phi^{-1}\left(q_\tau, \tilde{r}\right)\right)\right)_{s-1}, \\ G_k &:= \sum_i (D_i\tilde{r}, \mathcal{P}_1\mathcal{F}_k'\left(\phi^{-1}\left(q_\tau, \tilde{r}\right)\right)[D_i\phi^{-1}(q_\tau, \tilde{r})])_{s-1}. \end{aligned}$$

We will estimate the terms F_k and G_k, for $k = 1, 2$, separately.

5. From (4.5) and (4.8), we get

$$\begin{aligned} \gamma F_1 + \mu\alpha(\tau)^{1-N}F_2 &\le -C_1\|\tilde{r}\|^2_{s-\frac{1}{2}} + C\left(|q_\tau|\|\tilde{r}\|_{s-1} + \|\tilde{r}\|^3_{s-\frac{1}{2}}\right) \\ &\le (-C_1 + C\delta')\|\tilde{r}\|^2_{s-\frac{1}{2}} + C|q_\tau|\|\tilde{r}\|_{s-1}. \end{aligned} \tag{4.10}$$

6. Now we find an estimate for G_1. We have

$$G_1 = \sum_i (D_i\tilde{r}, I_i + J_i)_{s-1}, \tag{4.11}$$

where

$$\begin{aligned} I_i &:= \mathcal{P}_1\mathcal{F}_1'\left(\phi^{-1}\left(q_\tau, \tilde{r}\right)\right)[D_i\phi^{-1}(q_\tau, \tilde{r})] - \mathcal{P}_1\mathcal{F}_1'\left(\phi^{-1}\left(0, \tilde{r}\right)\right)[D_i\phi^{-1}(0, \tilde{r})] \\ J_i &:= \mathcal{P}_1\mathcal{F}_1'\left(\psi(\tilde{r})\right)[\psi'(\tilde{r})[D_i\tilde{r}]]. \end{aligned}$$

Here we applied the chain rule on $\psi = \phi^{-1}(0, \cdot)$. This is possible because ψ commutes with rotations. For detail we refer to [15]. Because $(q_\tau, \tilde{r}) \mapsto \mathcal{P}_1\mathcal{F}_1'(\phi^{-1}(q_\tau, \tilde{r}))[D_i\phi^{-1}(q_\tau, \tilde{r})]$ is Lipschitz continuous on a neighbourhood of zero in $\mathbb{R} \times \mathbb{R}^N \times \mathbb{H}^s_1(\mathbb{S}^{N-1})$ we have

$$\|I_i\|_{s-1} \le C|q_\tau|.$$

Therefore we get from the Cauchy-Schwarz inequality

$$(D_i\tilde{r}, I_i)_{s-1} \le C|q_\tau| \|D_i\tilde{r}\|_{s-1}. \tag{4.12}$$

Because of analyticity of $\mathcal{P}_1 \circ \mathcal{F}_1 \circ \psi$ on a neighbourhood of zero in $\mathbb{H}_1^{s-\frac{1}{2}}(\mathbb{S}^{N-1})$ and the fact that $\psi'(0)$ is the identity we have

$$\begin{aligned}(D_i\tilde{r}, J_i)_{s-1} &\le (D_i\tilde{r}, \mathcal{F}_1'(0)[D_i\tilde{r}])_{s-1} + C\|\tilde{r}\|_{s-\frac{1}{2}}\|D_i\tilde{r}\|^2_{s-\frac{1}{2}} \\ &\le -C_1\|D_i\tilde{r}\|^2_{s-\frac{1}{2}} + C\|\tilde{r}\|_{s-\frac{1}{2}}\|D_i\tilde{r}\|^2_{s-\frac{1}{2}} \\ &\le (-C_1 + C\delta')\|D_i\tilde{r}\|^2_{s-\frac{1}{2}}. \end{aligned} \tag{4.13}$$

Combining (4.12) and (4.13) we get from (4.11)

$$G_1 \le \sum_i (-C_1 + C\delta')\|D_i\tilde{r}\|^2_{s-\frac{1}{2}} + C|q_\tau|\|D_i\tilde{r}\|_{s-1}.$$

We can estimate G_2 in the same way, replacing C_1 by zero (see Lemma 3.3). Because $\alpha(\tau)^{1-N} \le 1$ we get

$$\gamma G_1 + \mu\alpha(\tau)^{1-N} G_2 \le \sum_i (-C_1 + C\delta')\|D_i\tilde{r}\|^2_{s-\frac{1}{2}} + C|q_\tau|\|D_i\tilde{r}\|_{s-1}. \tag{4.14}$$

7. Adding (4.10) and (4.14), we get from (4.9) and Cauchy's inequality

$$\begin{aligned}\left(\tilde{r}, \mathcal{P}_1\mathcal{F}\left(\phi^{-1}\left(q_\tau, \tilde{r}\right), \tau\right)\right)_{s-1,1} &\le (-C_1 + C\delta')\|\tilde{r}\|^2_{s-\frac{1}{2},1} + C|q_\tau|\|\tilde{r}\|_{s-1,1} \\ &\le (-C_1 + C\delta')\|\tilde{r}\|^2_{s-\frac{1}{2},1} + \frac{C_1}{2}\|\tilde{r}\|^2_{s-1,1} + C|q_\tau|^2 \\ &\le -\frac{C_1}{2}\|\tilde{r}\|^2_{s-\frac{1}{2},1} + C\left(\delta'\|\tilde{r}\|^2_{s-\frac{1}{2},1} + |q_\tau|^2\right).\end{aligned}$$

Choose $\delta' < \frac{\varepsilon}{C}$. Then

$$\begin{aligned}\left(\tilde{r}, \mathcal{P}_1\mathcal{F}\left(\phi^{-1}\left(q_\tau, \tilde{r}\right), \tau\right)\right)_{s-1,1} &\le -\lambda_0\|\tilde{r}\|^2_{s-1,1} + C|q_\tau|^2 \\ &\le -\lambda_0\|\tilde{r}\|^2_{s-1,1} + C\frac{|q_0|^2}{\alpha(\tau)^{2N}}.\end{aligned} \tag{4.15}$$

8. Define $\tilde{r}_0 := \mathcal{P}_1 r_0$. From the local existence result in time from [12] Chapter 6 Proposition 9 and 10 we get, diminishing δ' if necessary, an $S > 0$ such that if $|q_0|$ is small and $\|\tilde{r}_0\|_{s-1,1} < \delta'$ then there exists a solution $\tilde{r} \in \mathcal{C}([0,S], \mathbb{H}^s(\mathbb{S}^{N-1})) \cap \mathcal{C}^1([0,S], \mathbb{H}^{s-1}(\mathbb{S}^{N-1}))$ to (4.4) with $\tilde{r}(0) = \tilde{r}_0$. We also have $\tilde{r} \in \mathcal{C}^1((0,S], \mathcal{C}^\infty(\mathbb{S}^{N-1}))$ and therefore $\tilde{r} \in \mathcal{C}^\infty(\mathbb{S}^{N-1} \times (0,S])$. Furthermore, from (4.15) we see that for small $\vartheta \in (0,S)$ and $\tau \in (\vartheta, S]$ we have $\|\tilde{r}(\tau)\|^2_{s-1,1} \le y(\tau)$ where $y : [\vartheta, \infty) \to \mathbb{R}$ satisfies

$$\frac{dy}{d\tau} = -2\lambda_0 y + C\frac{|q_0|^2}{\alpha(\tau)^{2N}},$$

with $y(\vartheta) = \|\tilde{r}(\vartheta)\|_{s-1,1}^2$, for some $C > 0$. For the solution y to this ODE we derive from the variation of constants formula that

$$y(\tau) \le e^{-2\lambda_0(\tau-\vartheta)} y(\vartheta) + C \frac{|q_0|^2}{\alpha(\tau)^{2N}}.$$

For detail, see the proof of Theorem 3.6 in [15]. We get

$$\|\tilde{r}(\tau)\|_{s-1,1} \le C e^{-\lambda_0(\tau-\vartheta)} \|\tilde{r}(\vartheta)\|_{s-1,1} + \frac{C}{\alpha(\tau)^N} |q_0|, \tag{4.16}$$

where C can be chosen independently of ϑ. Letting ϑ go to zero we get for $\tau \in [0, S]$

$$\|\tilde{r}(\tau)\|_{s-1,1} \le C e^{-\lambda_0 \tau} \|\tilde{r}_0\|_{s-1,1} + \frac{C}{\alpha(\tau)^N} |q_0|. \tag{4.17}$$

This implies that $\|\tilde{r}(\tau)\|_{s-1,1} \le C(\|\tilde{r}_0\|_{s-1,1} + |q_0|)$ for $\tau \in [0, S]$. One can show by induction over $k \in \mathbb{N}$ that if $\|\tilde{r}_0\|_{s-1,1} < \frac{1}{2} C^{-1} \delta'$ and $|q_0| < \frac{1}{2} C^{-1} \delta'$ then a solution $\tilde{r}$ to (4.4) on $[0, kS]$ exists with $\tilde{r}(0) = \tilde{r}_0$. This solution satisfies (4.17) on $[0, kS]$ and it has the desired smoothness properties.

9. If we construct a solution r to the original problem via

$$r(\tau) := \phi^{-1}(q_\tau, \tilde{r}(\tau)),$$

then we find, regarding r and α as functions of the original time variable t, that

$$\|r(t)\|_s \le C(\|\tilde{r}(t)\|_s + |q_t|) \le \frac{C}{\alpha(t)^N} \|r_0\|_s.$$

Here, we used the fact that there exists a $C > 0$ such that $e^{-\lambda_0 \tau} \le \frac{C}{\alpha(\tau)^N}$ for all $\tau \ge 0$.

□

Note that from (4.17) we see that if we start with a domain Ω_{r_0} for which the zeroth and first Richardson moments vanish, i.e. $q_0 = (0,0)^T$, then convergence will be faster.

In contrast to the problem of Hele-Shaw flow with injection (see [16]), we cannot treat the case of zero surface tension for Stokes flow by the methods of the proof of Theorem 4.4. The order of $\mathcal{F}_2'(0)$ is lower than the order of $\mathcal{F}_2$. Therefore, energy estimates of the linearisation, $(r, \mathcal{F}_2'(0)[r])_s \le -C\|r\|_s^2$, for some $C > 0$, can no longer control energy estimates for the nonlinear part, $(r, \mathcal{F}_2(r) - \mathcal{F}_2'(0)[r])_s \le \epsilon \|r\|_{s+\frac{1}{2}}^2$, for some $\epsilon > 0$.

5 Almost global existence results for the suction problems

In this section we use energy estimates to get an existence result for the suction problems in 2D and 3D. Starting close enough to the unit ball, an arbitrarily large portion of liquid can be removed.

Theorem 5.1. *Let $N = 2$ or $N = 3$, $\mu < 0$, $T_+ \in [0, \frac{\sigma_N}{|\mu|(N-1)})$ and $s > \frac{N+6}{2}$. There exists a $\delta > 0$ such that if $\|r_0\|_s < \delta$, then there exists a solution $r \in \mathcal{C}([0, T_+), \mathbb{H}^s(\mathbb{S}^{N-1})) \cap \mathcal{C}^1([0, T_+), \mathbb{H}^{s-1}(\mathbb{S}^{N-1}))$ to*

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r, \tau), \quad r(0) = r_0. \tag{5.1}$$

Furthermore $r \in \mathcal{C}^\infty(\mathbb{S}^{N-1} \times (0, T_+))$.

Proof. We assume that $r \in \mathbb{H}^{s+1}(\mathbb{S}^{N-1})$ with $\|r\|_s < \delta'$ for δ' small enough.

1. If $\mu < 0$, then $\alpha(\tau)^{1-N}$ goes to infinity as τ approaches $\frac{\sigma_N}{|\mu|(N-1)}$. Nevertheless, on the time interval $[0, T_+]$, $\alpha(\tau)^{1-N} < A$ for some $A > 0$. Choose $K \in \mathbb{N}$ such that for $k \geq K$
$$-\gamma p_1(k) + |\mu| A p_2(k) < 0.$$
Define $C_2 > 0$ by
$$C_2 := \inf_{k \geq K} \frac{\gamma p_1(k) - |\mu| A p_2(k)}{(k^2+1)^{\frac{1}{2}}}.$$
The positivity of C_2 follows from the fact that $\frac{\gamma p_1(k) - |\mu| A p_2(k)}{(k^2+1)^{\frac{1}{2}}}$ converges to $\frac{\gamma}{2}$ as k tends to infinity.

2. Let $\mathcal{P}_K : \mathbb{L}_2(\mathbb{S}^{N-1}) \to \mathbb{L}_2(\mathbb{S}^{N-1})$ be the orthogonal projection with respect to the $\mathbb{L}_2(\mathbb{S}^{N-1})$-inner product along $\bigoplus_{k=0}^K \mathfrak{S}_k^N$ and define $r_{k,j} := (r, s_{k,j})_0$. We get
$$\begin{aligned}
&\gamma(r, \mathcal{F}_1'(0)[r])_{s-1} + \mu\alpha(\tau)^{1-N}(r, \mathcal{F}_2'(0)[r])_{s-1} \\
&= \sum_{k<K} (k^2+1)^{s-1+\frac{1}{2}} \frac{-\gamma p_1(k) + |\mu|\alpha(\tau)^{1-N} p_2(k)}{(k^2+1)^{\frac{1}{2}}} r_{k,j}^2 \\
&\quad + \sum_{k \geq K} (k^2+1)^{s-1+\frac{1}{2}} \frac{-\gamma p_1(k) + |\mu|\alpha(\tau)^{1-N} p_2(k)}{(k^2+1)^{\frac{1}{2}}} r_{k,j}^2 \\
&\leq C\|r\|_0^2 - C_2\|\mathcal{P}_K r\|_{s-\frac{1}{2}}^2 \\
&= C\|r\|_0^2 + C_2\|(\mathcal{I} - \mathcal{P}_K) r\|_{s-\frac{1}{2}}^2 - C_2\|r\|_{s-\frac{1}{2}}^2 \\
&\leq C\|r\|_0^2 - C_2\|r\|_{s-\frac{1}{2}}^2.
\end{aligned} \tag{5.2}$$
Here we used boundedness of $\mathcal{I} - \mathcal{P}_K : \mathbb{H}^{s-\frac{1}{2}}(\mathbb{S}^{N-1}) \to \mathbb{L}_2(\mathbb{S}^{N-1})$.

3. By analyticity of $\mathcal{F}_1$ and $\mathcal{F}_2$ and boundedness of $\alpha(\tau)^{1-N}$ on $[0, T_+]$ we have
$$\gamma(r, \mathcal{F}_1(r) - \mathcal{F}_1'(0)[r])_{s-1} + \mu\alpha(\tau)^{1-N}(r, \mathcal{F}_2(r) - \mathcal{F}_2'(0)[r])_{s-1} \leq C\|r\|_{s-\frac{1}{2}}^3. \tag{5.3}$$

4. By the chain rule (4.1) we get

$$\begin{aligned}&\gamma(r,\mathcal{F}_1(r))_{s-1,1}+\mu\alpha(\tau)^{1-N}(r,\mathcal{F}_2(r))_{s-1,1}\\ =&\,\gamma(r,\mathcal{F}_1(r))_{s-1}+\mu\alpha(\tau)^{1-N}(r,\mathcal{F}_2(r))_{s-1}\\ &+\gamma\sum_i(D_ir,\mathcal{F}_1'(r)[D_ir])_{s-1}+\mu\alpha(\tau)^{1-N}\sum_i(D_ir,\mathcal{F}_2'(r)[D_ir])_{s-1}\end{aligned}$$

As in the previous proof, we distinguish between two parts, that will be estimated separately.

5. For the first part we have by (5.2) and (5.3)

$$\gamma(r,\mathcal{F}_1(r))_{s-1}+\mu\alpha(\tau)^{1-N}(r,\mathcal{F}_2(r))_{s-1}\leq(C\delta'-C_2)\|r\|^2_{s-\frac{1}{2}}+C\|r\|_0^2.$$

6. For the other part we have

$$\begin{aligned}&\gamma(D_ir,\mathcal{F}_1'(r)[D_ir])_{s-1}+\mu\alpha(\tau)^{1-N}(D_ir,\mathcal{F}_2'(r)[D_ir])_{s-1}\\ =&\,\gamma(D_ir,\mathcal{F}_1'(0)[D_ir])_{s-1}+\mu\alpha(\tau)^{1-N}(D_ir,\mathcal{F}_2'(0)[D_ir])_{s-1}\\ &+\gamma(D_ir,\left\{\mathcal{F}_1'(r)-\mathcal{F}_1'(0)\right\}[D_ir])_{s-1}\\ &+\mu\alpha(\tau)^{1-N}(D_ir,\left\{\mathcal{F}_2'(r)-\mathcal{F}_2'(0)\right\}[D_ir])_{s-1}\\ \leq&\,(C\delta'-C_2)\|D_ir\|^2_{s-\frac{1}{2}}+C\|D_ir\|_0^2.\end{aligned}$$

7. Combining these two results and taking $\delta'<\frac{C_2}{C}$ we get

$$\begin{aligned}\gamma(r,\mathcal{F}_1(r))_{s-1,1}+\mu\alpha(\tau)^{1-N}(r,\mathcal{F}_2(r))_{s-1,1}&\leq(C\delta'-C_2)\|r\|^2_{s-\frac{1}{2},1}+C\|r\|^2_{0,1}\\ &\leq C\|r\|^2_{0,1}\leq C\|r\|^2_{s-1,1}.\end{aligned}\tag{5.4}$$

From [12] Chapter 6 Proposition 9 we get, diminishing δ' if necessary, an $S>0$ such that if $\|r_0\|_{s-1,1}<\delta'$, then there exists a solution $r\in\mathcal{C}([0,S],\mathbb{H}_1^s(\mathbb{S}^{N-1}))\cap\mathcal{C}^1([0,S],\mathbb{H}_1^{s-1}(\mathbb{S}^{N-1}))$ to (5.1), with $r(0)=r_0$. We also have $r\in\mathcal{C}^\infty(\mathbb{S}^{N-1}\times(0,S])$. Using the same methods as in the previous proof we can show that from (5.4) it follows that $\|r(\tau)\|_{s-1,1}\leq e^{C\tau}\|r_0\|_{s-1,1}$. Choose $\delta<\delta'e^{-CT_+}$ and suppose that $\|r_0\|_s<\delta$. By induction over $k\in\mathbb{N}$, one can show existence of a solution r with $r(0)=r_0$ on the interval $[0,kS]\cap[0,T_+]$, like in step 9 of the proof of Theorem 3.5 in [15].

□

Funding

This research is sponsored by NWO (Nederlandse Organisatie voor Wetenschappelijk Onderzoek) grant 613.000.433.

Acknowledgements

Special thanks to Georg Prokert, Jan Draisma and Remco Duits for their input in this work and many fruitful discussions.

References

[1] Cummings, L. J., and Howison, S. D. Two-dimensional Stokes flow with suction and small surface tension. *European J. Appl. Math. 10*, 6 (1999), 681–705. Hele-Shaw flows and related problems (Oxford, 1998).

[2] Escher, J., and Prokert, G. Analyticity of solutions to nonlinear parabolic equations on manifolds and an application to Stokes flow. *J. Math. Fluid Mech. 8*, 1 (2006), 1–35.

[3] Friedman, A. A free boundary problem for a coupled system of elliptic, hyperbolic, and Stokes equations modeling tumor growth. *Interfaces Free Bound. 8*, 2 (2006), 247–261.

[4] Friedman, A., and Hu, B. Bifurcation for a free boundary problem modeling tumor growth by Stokes equation. *SIAM J. Math. Anal. 39*, 1 (2007), 174–194.

[5] Friedman, A., and Hu, B. Bifurcation from stability to instability for a free boundary problem modeling tumor growth by Stokes equation. *J. Math. Anal. Appl. 327*, 1 (2007), 643–664.

[6] Friedman, A., and Reitich, F. Quasi-static motion of a capillary drop. II. The three-dimensional case. *J. Differential Equations 186*, 2 (2002), 509–557.

[7] Friedman, A., and Reitich, F. Quasistatic motion of a capillary drop. I. The two-dimensional case. *J. Differential Equations 178*, 1 (2002), 212–263.

[8] Günther, M., and Prokert, G. Existence results for the quasistationary motion of a free capillary liquid drop. *Z. Anal. Anwendungen 16*, 2 (1997), 311–348.

[9] Hill, E. L. The theory of vector spherical harmonics. *Amer. J. Phys. 22* (1954), 211–214.

[10] Kuiken, H. Viscous sintering: the surface-tension-driven flow of a liquid form under the influence of curvature gradients at its surface. *J. Fluid Mech. 214* (1990), 503–515.

[11] Müller, C. *Spherical harmonics*, vol. 17 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, (1966).

[12] Prokert, G. *Parabolic evolution equations for quasistationary free boundary problems in capillary fluid mechanics*. Technische Universiteit Eindhoven, Eindhoven, 1997. Dissertation, Technische Universiteit Eindhoven, Eindhoven, 1997.

[13] Renardy, M., and Rogers, R. C. *An introduction to partial differential equations*, vol. 13 of *Texts in Applied Mathematics*. Springer-Verlag, New York, (1993).

[14] Tanveer, S., and Vasconcelos, G. L. Time-evolving bubbles in two-dimensional Stokes flow. *J. Fluid Mech. 301* (1995), 325–344.

[15] Vondenhoff, E. *Large-time behaviour of Hele-Shaw flow with injection or suction for perturbations of balls in* $\mathbb{R}^N$ *(submitted), electronic version:* `ftp://ftp.win.tue.nl/pub/rana/rana08-04.pdf`.

[16] Vondenhoff, E. *Long-time asymptotics of Hele-Shaw flow for perturbed balls with injection and suction (submitted), electronic version:* `ftp://ftp.win.tue.nl/pub/rana/rana06-19.pdf`, `ftp://ftp.win.tue.nl/pub/rana/rana06-42.pdf`.