Lectures on probability theory, dynamical systems and fractal geometry

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Lectures on Probability Theory, Dynamical Systems and Fractal Geometry

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LECTURES ON PROBABILITY THEORY, DYNAMICAL SYSTEMS, AND FRACTAL GEOMETRY

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INTRODUCTION

With the development of modern high-performance computers the 'chaos theory' in the form of fractal geometry has become a popular field of research. Simple iterative algorithms, i.e. simple repetetive step-by-step calculations, permit the composition of complicated shapes, which sometimes even appear to possess artistic value. One feature of the design underlying all these methods is self-similarity: manifestations, which when enlarged are visible to the naked eye, are repeated again and again in miniature and, given a corresponding enlargement, each step, no matter how small, will be similar to the next object as a whole. This explains how simple equations or geometrical forms can lead to complex and multilayered structures.

In recent years, the conceptual approach inherent in this theory has been extended to every realm of knowledge. Chaos theory has been applied in the attempt to explain complicated processes which cannot always be predicted, or at least not with precision. The best-known of these are the so-called Lorenz equations in physics, which describe the vertical flow of a gas. In medicine, scientists have conducted experiments aimed at modelling the generation of creativity in the brain through chaos. Electrical discharges or the creation of polymer compounds display clearly fractal structures. A further example is the modulation of population dynamics in biology, where the size of the population is subject to irregular fluctuations. The list is endless. This whole field of research has developed from problems met with when solving equations using the so-called Newton method. When employing this algorithm, initial numbers cannot just be selected at random if the aim is to achieve a useful solution to an equation. Experiments are now in train to use this system for solving equations with multiple variables. Julia sets in higher dimensions were investigated using geometric, analytic and probabilistic methods. These are structures which occur under simplest dynamics of four (and higher)-dimensional space, and which comprise just these points upon which the greatest chaos reigns.

For analytic endomorphisms of the Riemann sphere S^2 it is well known that the Julia sets of mappings of the form

$$f_c : z \mapsto z^2 + c$$

with $|c|$ small are Jordan curves (see Beardon, 1.6, 9.9 and Brolin, Theorem 8.1) and show similar dynamical behaviour as $\sigma_1 : z \mapsto z^2$. It is easy to see that this holds if $|c| \leq 1/4 - \varepsilon$ for some $\varepsilon \geq 0$. This is one example how - by means of geometric function theory - dynamical properties enforce certain geometric structures. The main point of our discussion here is concerned with a similar question in nature. Dynamics and probability created powerful methods to investigate the long time behaviour of stationary sequences. In case the time series have a geometric interpretation, these probabilistic and dynamical results sometimes force geometric constraints. For example, Makarov's result on the Hausdorff dimension of the harmonic measure on boundaries of Jordan domains uses the law of iterated logarithm for certain random processes. The dynamic analogon of this - using stationary processes - is also known.

Recall that an analytic endomorphism (or a rational function) $T : S^2 \to S^2$ on the Riemann sphere S^2 has always a nonempty, fully invariant Julia set, defined as the set of non-normal points for the family of functions $T^n : S^2 \to S^2$ $(n \geq 1)$.

The following theorem is a consequence of many people's efforts to develop the theory of rational functions, in particular of polynomials in $\mathbb{C}$:

Theorem:
For a polynomial map $P : \mathbb{C} \to \mathbb{C}$ of degree at least 2, the Julia set $J(P)$ equals each of the following sets:

(1) $\{z \in \mathbb{C} : \{P^k : k \geq 0\}$ is not normal at $z\}$.
(2) The boundary $\partial(K(P))$ where $K(P) = \{z \in \mathbb{C} : \sup_k |P^k(z)| < \infty\}$.
(3) The closure $\mathcal{R}$ of the set of repelling periodic points.
(4) The limit of the pull backs by P^{-k} of the boundary $\partial(\{z \in \mathbb{C} : |z| \leq r\})$ for sufficiently large $r > 0$.
(5) The support $\mathrm{supp}(\mu_P)$ of the measure of maximal entropy for P.

Before continuing, let us discuss the meaning of the 5 statements in the theorem briefly. Clearly, (1) is the description arising from geometric function theory and allows to introduce methods from complex function theory to study $J(P)$. In particular, bounded distortion properties play a fundamental role here. (2) tells us that the complement of $J(P)$ splits into connected components and hence the analysis on $J(P)$ can be studied using the theory of holomorphic functions on domains, in particular using harmonic analysis (Green's function). Certainly, this complements the description in (1). However, as is known, this boundary is equal to the Shilov boundary $\partial_{SH}(K(P))$, which is defined to be the minimal set Q with the property that holomorphic functions in some neighbourhood of $K(P)$ attain their maximum over $K(P)$ in Q. Certainly, $\partial_{SH}(K(P))$ is contained in $\partial K(P)$. Hence it is possible to introduce some abstract boundary theory to study the Julia set. (3) is a dynamic description. It tells that the dynamics is essentially determined by some hyperbolic behaviour. The tracing property by repelling period points, and the value of the topological entropy are an almost immediate consequence of it. (4) tells us that we may use the well known boundary theory for large centered balls and get the Julia sets as pull backs. Finally, we obtain the existence of a probability measure maximizing entropy. Intuitively it means that in addition to (3) we obtain a probabilistic structure which contains the maximum of randomness. Since the map P is not a homeomorphism, there is also a natural filtration given by the pull backs of the Borel σ-algebra. It is clear that this filtration can be used to introduce geometrically relevant martingale- and mixing structures.

What is described below is largely motivated by this example of rational functions. Indeed, the probabilistic results formulated below for various general types of dynamics find applications within this class.

In order to extend our motivating examples we discuss briefly methods to obtain a similar result for certain polynomials $f : \mathbb{C}^n \to \mathbb{C}^n$. We are interested in 'large' classes of polynomial maps satisfying Heinemann's program: The Julia set $J(P)$ can be characterized in the following ways:

(1) $J(P) := \{z \in \mathbb{C}^n : \{P^k : k \geq 0\}$ is not weakly normal at $z\}$.
(2) $J(P)$ is the Shilov boundary $\partial_{SH}(K(P))$ where $K(P) = \{z \in \mathbb{C}^n : \sup_k \|P^k(z)\| < \infty\}$.
(3) $J(P)$ is the closure $\mathcal{R}$ of the set of repelling periodic points.
(4) $J(P)$ is the limit of the pull backs by P^{-k} of the Shilov boundary $\partial_{SH}(\{z \in \mathbb{C}^n : \|z\| \leq r\})$ for sufficiently large $r > 0$.
(5) $J(P)$ is the support $\text{supp}(\mu_P)$ of the measure of maximal entropy for P.

Note that our program is using the term of weak normality instead of normality. This notion is as follows: A family of functions $\{f_k\}$ is called *weakly normal* in a point $z \in U$ if there are

- an open neighbourhood V of z;
- a family C_x of at least one-dimensional (complex) analytic sets indexed by the points $x \in V$,

such that

- each x lies in the corresponding analytic set C_x;
- for each $x \in V$ the family $\{f_k\}$ restricted to $C_x \cap V$ is normal (including convergence to infinity).

It is clear that this definition selects a set of maximal randomness. It is known from Heinemann's work that torus like maps of the form

$$P(x,y) = (x^2 + k(y), y^2 + l(x)) \qquad (x,y) \in \mathbb{C}^2$$

satisfy this characterization as long as the norms of the polynomials $k(y)$ and $l(x)$ are small enough in some neighbourhood of 0. Also certain polynomial skew products of the form

$$P(x,y) = (p(x), q_x(y))$$

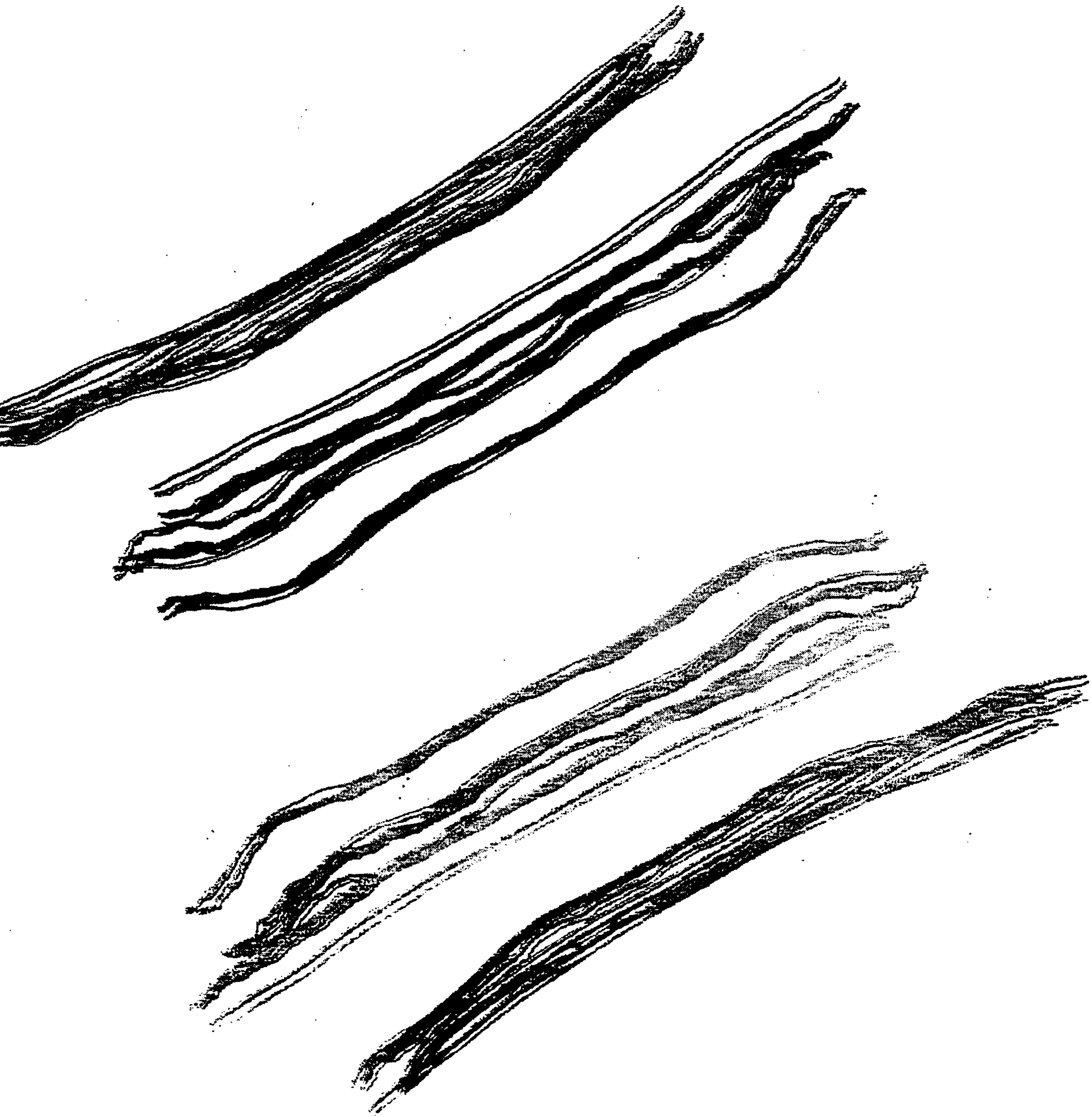
fall into this category. These are subclasses of polynomial maps $\mathbb{C} \mapsto \mathbb{C}$ satisfying the regularity condition

$$\begin{gathered}\exists R > 0, s \in \mathbb{N}, t \in \mathbb{Q}, k_1, k_2 \in \mathbb{R} \text{ such that} \\ k_1\|z\|^t \leq \|P(z)\| \leq k_2\|z\|^s \quad \forall \|z\| > R.\end{gathered}$$

We do not know whether these new classes of maps give rise to some new results concerning their probabilistic structures. Certainly, the thermodynamic formalism as part of the dynamic behavior is different, at least for the skew products. Thus we can expect some more refined probabilistic theorems respecting canonical filtrations given by the system.

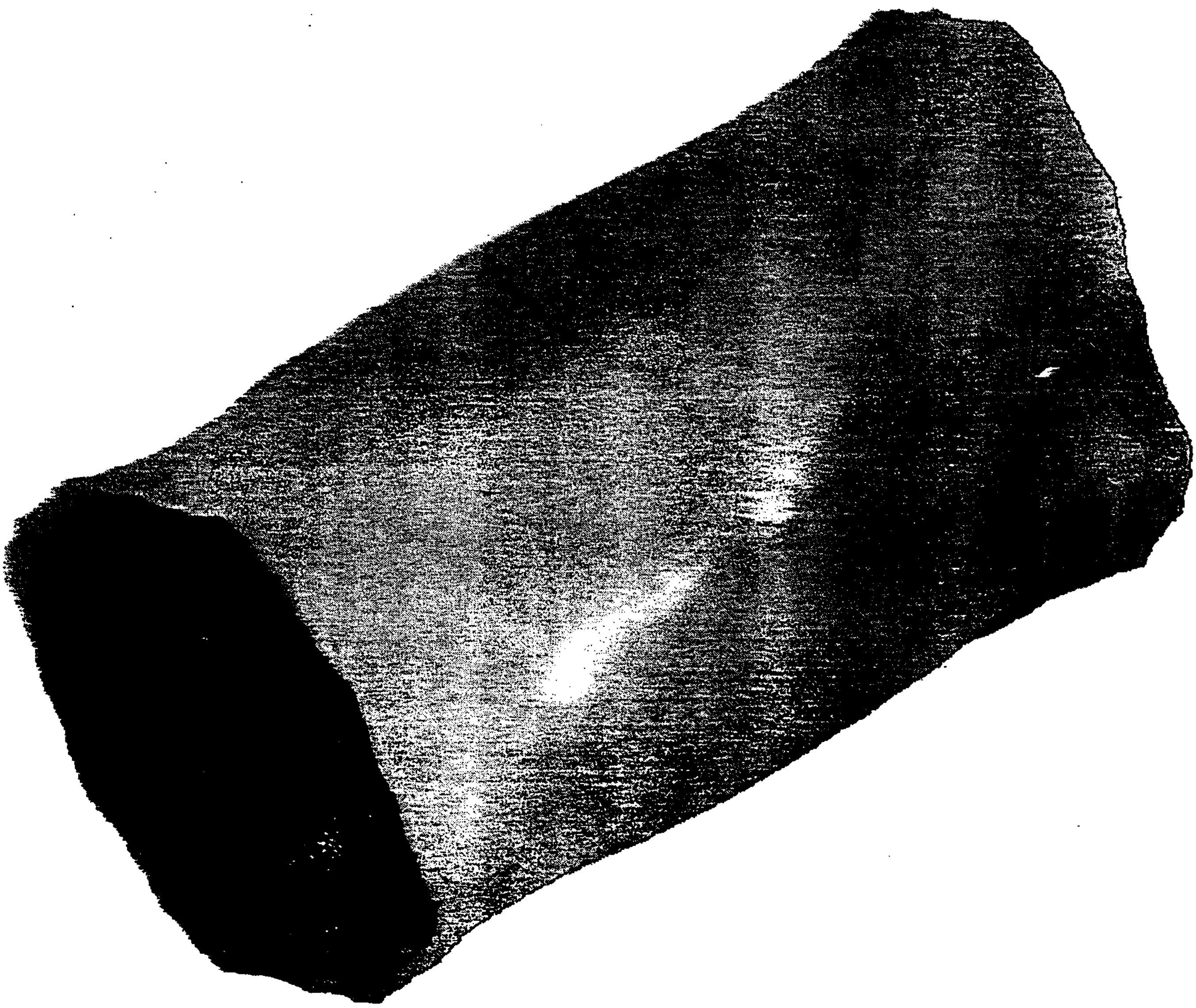
Julia sets for two-dimensional polynomials:

Spaghetti-type skew product ©*S.-M. Heinemann*



Julia sets for two-dimensional polynomials:

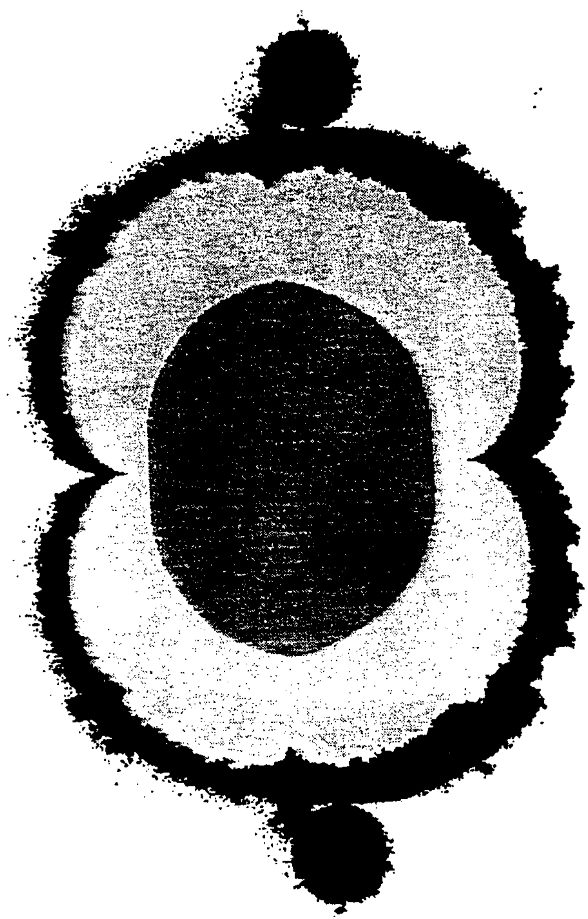
Canneloni-type skew product ©*S.-M. Heinemann*



Julia sets for two-dimensional polynomials:

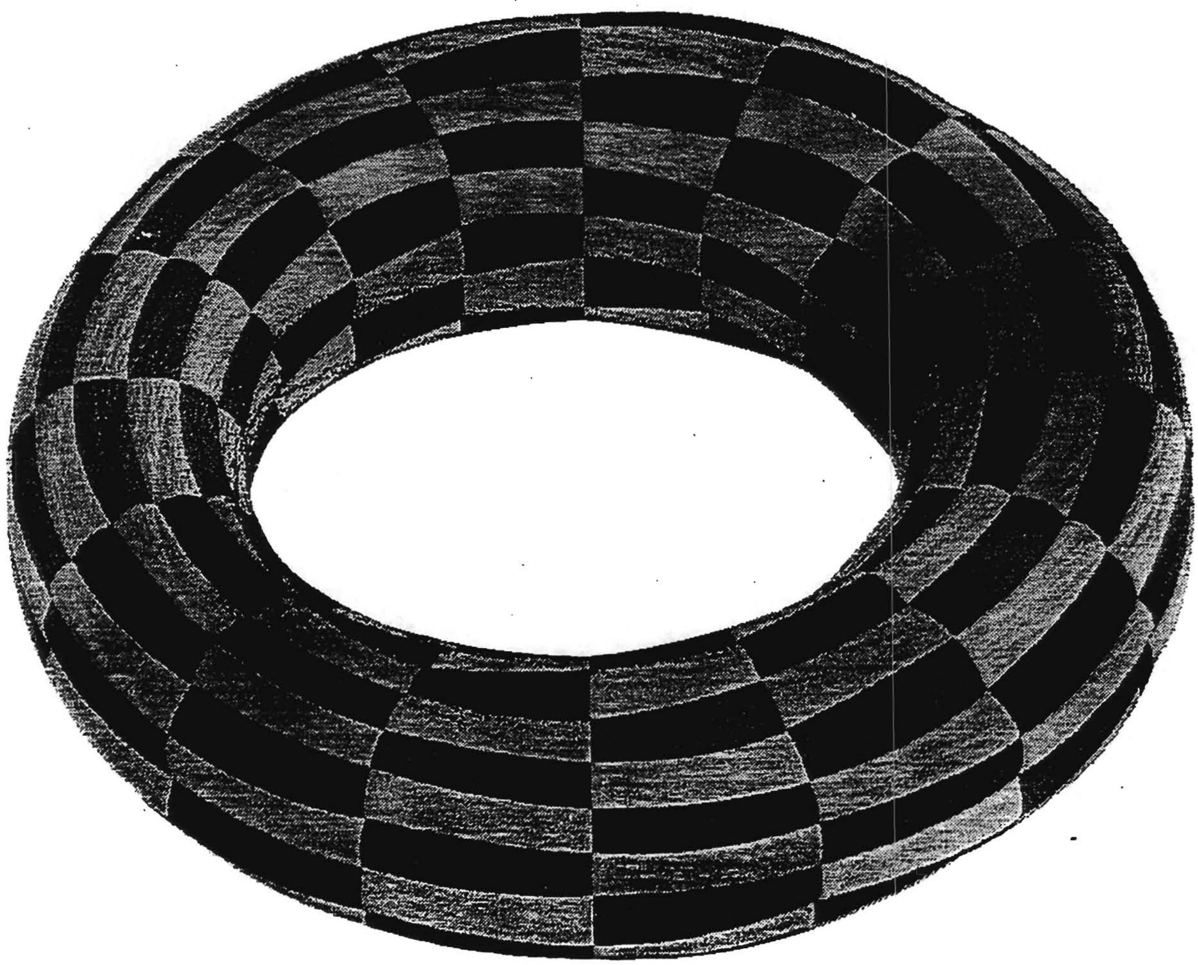
Mandelbrot set of the family

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 + \lambda/10 + \lambda^2 \\ y^2 + F \end{pmatrix}$$



Julia sets for two-dimensional polynomials:

Regular hyperbolic skew product and its Markov partition ©*S.-M. Heinemann*



BASIC DEFINITIONS AND NOTATIONS

$(\Omega, \mathcal{A}, \mu)$ σ-finite measure space $\quad T : \Omega \to \Omega$ measurable

T *invertible on* $B \in \mathcal{A} \Leftrightarrow$ T is 1-1 on B & $TB \in \mathcal{A}$ & $T^{-1} : TB \to B$ is measurable

T *nonsingular on* $B \in \mathcal{A}$ if $\forall C \in TB \cap \mathcal{A}$

$$\mu((T^{-1}C) \cap B) = 0 \Leftrightarrow \mu(C) = 0$$

T *measure preserving* if $\mu(T^{-1}C) = \mu(C)$ $\forall C \in \mathcal{A}$

Lemma: Let T be measure preserving and $T^{-1}\{x\}$ countable $\forall x \in \Omega$. Then there exists a countable partition α of Ω such that $T : a \to Ta$ is nonsingular and invertible on a, $\forall a \in \alpha$.

Let T be nonsingular. T is called *ergodic* if $A \subset T^{-1}A$ a.s. implies that $\mu(A) = 0$ or $\mu(A^c) = 0$.

Let T be locally invertible. A probability measure μ is called *f-conformal* if

$$\frac{d\mu \circ T}{d\mu}|_A = f$$

on each measurable set A on which T is invertible.

MEAN ERGODIC THEOREM (v. Neumann)

Let $U : H \to H$ be a linear contraction on the Hilbertspace H and let $f \in H$,

$$S_n f = \sum_{i=0}^{n-1} U^i f, \text{ and } A_n f = \frac{1}{n} S_n f.$$

Then $A_n f$ converges in norm to the orthogonal projection of f onto the subspace of U-invariant vectors.

A.S. ERGODIC THEOREM (Birkhoff)

Let $T : \Omega \to \Omega$ be an endomorphism of the probability space $(\Omega, \mathcal{A}, \mu)$ and J be the σ-algebra of T–invariant sets. Let $Uf = f \circ T$. Then $A_n f$ converges a.s. to $E(f|J)$ for every $f \in L_1(\mu)$.

SUBADDITIVE ERGODIC THEOREM (Kingman)

Let $T : \Omega \to \Omega$ be an endomorphism of the probability space $(\Omega, \mathcal{A}, \mu)$ and $F = \{F_{ik} : i, k \in \mathbb{Z}, 0 \in i < k\}$ be a subadditive process. If

$$\gamma(F) = \inf_n \frac{1}{n} \int F_{0n} d\mu > -\infty$$

then

$$\frac{1}{n} F_{0n}$$

converges a.s. to a function f satisfying

$$\int f d\mu = \gamma(F).$$

MULTIPLICATIVE ERGODIC THEOREM (Oseledets)

Let $T : \Omega \to \Omega$ be an endomorphism of the probability space $(\Omega, \mathcal{A}, \mu)$ and $A : \Omega \to L(\mathbb{R}^d, \mathbb{R}^d) = M(d, \mathbb{R})$ measurable so that $\log^+ ||A|| \in L_1(\mu)$. Then the following holds for almost all $\omega \in \Omega$:
The limit

$$\lim_{n\to\infty} [(A(T^{n-1}(\omega))(A(T^{n-1}(\omega))...A(\omega))]^{\frac{1}{n}} = \Delta(\omega)$$

exists.
$\exists -\infty \leq \ \lambda_1(\omega) < .. < \lambda_{r(\omega)}(\omega)$
$\exists$ subspaces $E_{r(\omega)}(\omega), ..., \ E_1(\omega) \subset \mathbb{R}^d$
with the following properties:

[i.] $e^{\lambda_j(\omega)}$ $(j = 1, .., r(\omega))$ are the different eigenvalues of $\Delta(\omega)$.
[ii.] $\mathbb{R}^d = E_1(\omega) + ... + E_{r(\omega)}(\omega)$
[iii.] $E_j(\omega)$ is the eigenspace belonging to $e^{\lambda_j(\omega)}$.
[iv.] $\lim_{n\to\infty} \frac{1}{n} \log ||A(T^{n-1}(\omega))..A(\omega)x|| = \lambda_j(\omega)$ $\forall x \in U_j \backslash U_{j-1}$ and $U_e = \langle E_1(\omega), ..., E_e(\omega)\rangle$

$\omega \to \dim E_j(\omega), \omega \to r(\omega), \omega \to \lambda_j(\omega)$ are T-invariant
If T is ergodic, $\det A(w) = 1$ and $\overline{\lim}\frac{1}{n} \inf \log ||A(T^{n-1}(\omega))..A(\omega)||d\mu > 0 \Rightarrow \lambda_1 \leq 0 \& \lambda_{r(\omega)} \geq 0$

RATIO ERGODIC THEOREM (Chacon Ornstein)

Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space and

$$U : L_1(\mu) \to L_1(\mu), U^*1 = 1$$

a positive contraction:
Then , for $f, g \in L_1^+(\mu)$,

$$\frac{S_n f}{S_n g}$$

converges a.s. to a finite limit on the set $\{x : \sum_{k \geq 0} U^k gx > 0\}$.

UNIFORM ERGODIC THEOREM

[I.] (Yosida-Kakutani)

Let $U : E \to E$ be a power bounded and quasicompact operator. Then there are $\lambda_i \in \mathbb{R}$, $|\lambda_i| = 1$, $P_i : E \to E$ such that

[1.] $U^n = \sum_{i=1}^k \lambda_i^n P_i + S^n \quad \forall n$
[2.] $P_i^2 = P_i \; P_i P_j = 0 \; \forall i \neq j \; P_i S = S P_i = 0 \; \forall i$
[3.] $\dim P_i(E) < \infty$
[4.] $\exists \; M > 0 \; \exists \; 0 < q < 1$ such that $||S^n|| \leq M q^n$.

[II.] (Jonescu-Tulcea and Marinescu)

Let $B \subset E$ be Banach spaces with norms $\| \;\|$ and $| \;|$ respectively. Let $U : B \to B$ a continuous, linear operator.
Assume that

[1.] $x_n \in B$, $\|x_n\| \leq K$, $|x_n - x| \to 0 \Rightarrow x \in B, \|x\| \leq K$
[2.] $\sup_n |U^n| < \infty$
[3.] $\forall x \in B : \|Ux\| \leq r\|x\| + R|x|$ for some $0 < r < 1$ and $R > 0$.
[4.] $A \subset B$ is $\| \;\|$-bounded $\Rightarrow UA$ is relatively compact in $| \cdot |$

Then U is $\| \;\|$ power bounded and quasicompact.

[3.] is called the *ITM inequality.*

FURTHER ERGODIC THEOREMS

(Hopf, Dunford-Schwarz)
$U : L_1(\mu) \to L_1(\mu), \mu(\Omega) < \infty$, $||U||_{L_1}, ||U||_{L_\infty} \le 1 \Rightarrow$ $A_n f \to$ a.e for every $f \in L_1(\mu)$

(Riesz, Eberlein, Yosida, Kakutani, Lorch...)
$U : E \to E$ power bounded, E reflexive $\Rightarrow$

$$A_n x \to y(x),\ Uy(x) = y(x)\ \forall x \in E$$

(Furstenberg multiple recurrence theorem)
$T : \Omega \to \Omega$ weakly mixing, $\mu(\Omega) < \infty \Rightarrow$ $\forall A_1, ...A_k \in \mathcal{A}$

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M+1}^{N} |\mu(A_1 \cap T^{-n} A_2 \cap ... \cap T^{-n(k-1)} A_k) - \prod_{j=1}^{k} \mu(A_k)| = 0$$

(Second order ergodic theorem) (Aaronson, Denker, Fisher)
T pointwise dual ergodic, $\mu(\Omega) = \infty$, $a(n)$ return sequence, $a(n) = n^\alpha L(n)$, $\alpha > 0$, L slowly varying. Then

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{na(n)} S_n f = \int f d\mu \quad a.s.$$

From J. Aaronson's book:
The story is told about a disappointed angler who caught a large dolphin which escaped. The angler comforted himself with the thought that history repeats itself and hence at some time in the future, he would catch the same dolphin again. The dolphin had the same impression and lived in fear.

but after pondering the matter, realized that if history repeats itself, he would escape again.

RECURRENCE

$T : \Omega \to \Omega$ nonsingular. $W \in \mathcal{A}$ is called *wandering* if $\{T^{-n}W : n = 0, 1, 2...\}$ is pairwise disjoint.

Theorem: (Halmos)
Let T be nonsingular, $A \in \mathcal{A}, \mu(A) > 0$. Then $\mu(A \cap W) = 0 \ \forall$ wandering W implies that

$$\sum_{n=1}^{\infty} 1_B \circ T^n = \infty \qquad \text{a.s. on } B \ \forall B \in A \cap \mathcal{A}, \mu(B) > 0.$$

The *dissipative part* of Ω (relative to the transformation T): is the measurable union of the dissipative sets of Ω, denoted by $D(T)$.

The *conservative part* is defined to be $C(T) = \Omega \backslash D(T)$.

T is called *conservative* if $C(T) = \Omega \mod \mu$.

T is called *dissipative* if T is not conservative.

T is called *completely dissipative* if $\Omega = D(T) \mod \mu$.

$\Omega = D(T) \cup C(T)$ is called the *Hopf-decomposition.*

Poincaré Recurrence Theorem:
T conservative, nonsingular. Let (Z, d) separable, metric, $f : \Omega \to Z$ measurable. Then

$$\liminf_{n \to \infty} d(f(x), f(T^n x)) = 0 \text{ a.s. on } \Omega.$$

In particular: $f = 1_A \Rightarrow T^n x \in A$ ∞-often.

Characterisation of conservativity.
Let T measure preserving, μ σ-finite.

(1) $f \in L_1^+(\mu) \Rightarrow \{\sum_{n\geq 0} fT^n = \infty\} \subset \mathcal{C}(T)$.

(2) $f \in L_1^+(\mu), f > 0 \Rightarrow \{\sum_{n\geq 0} fT^n = \infty\} = \mathcal{C}(T)$ a.s.

(3) $\mu(\Omega) < \infty$ implies that $\mathcal{C}(T) = \Omega$.

(4) $\mu(A) < \infty, \Omega = \bigcup_{n=0}^{\infty} T^{-n}A \Rightarrow$ T conservative. (Maharams recurrence theorem)

(5) T invertible, ergodic, μ nonatomic $\Rightarrow$ $\mathcal{C}(T) = \Omega$

(6) T conservative, ergodic $\Leftrightarrow$ $\sum_{n=1}^{\infty} 1_A \circ T^n = \infty$ a.s. $\forall A \in \mathcal{A}^+$.

(7) T_1 measure preserving, $\mu_1(\Omega_1) < \infty, T_2$ conservative $\Rightarrow$ $T_1 \times T_2$ conservative.
T_1, T_2 conservative ergodic $\Rightarrow$ $T_1 \times T_2$ conservative or completely dissipative.

INDUCED TRANSFORMATION

T conservative, nonsingular, $\mu(A) > 0$.

$$\varphi_A(x) = \min\{n \geq 1 : T^n x \in A\}$$

is called the *return time* to A (and is a stopping time).

$T_A : A \to A, T_A(x) = T^{\varphi_A(x)}(x)$ is called the *induced transformation.*

$$\mu_A(B) = \frac{\mu(A \cap B)}{\mu(A)}$$

is called the *induced measure* on A.

Properties:

(1) $\mu_A \circ T_A^{-1} \ll \mu_A$.

(2) $T_A^k(x) = T^{\varphi_k(x)}(x)$ where $\varphi_k(x) = \sum_{j=0}^{k-1} \varphi_A \circ T_A^j$.

(3) T_A is conservative and nonsingular.

(4) T ergodic $\Rightarrow$ T_A ergodic. T_A ergodic and $\bigcup_{n\geq 0} T^{-n}A = \Omega \Rightarrow$ T ergodic.

(5) μ is T-invariant $\Rightarrow$ μ_A is T_A-invariant.

(6) (Kac formula) $\int_A \varphi_A d\mu = \mu(\Omega)$ if T ergodic, measure preserving and $0 < \mu(A) < \infty$.

(7) Let T conservative, nonsingular, and $q \ll \mu_A$ be T_A-invariant. Then $m(B) = \int_A \sum_{k=0}^{\varphi_A - 1} 1_B \cdot T^k dq$ is T-invariant.

THE DUAL OPERATOR

$T : \Omega \to \Omega$ nonsingular. $Uf = f \circ T$ is an isometry $U : L_\infty(\mu) \to L_\infty(\mu)$
The 'dual operator'

$$V = \widehat{T} : L_1(\mu) \to L_1(\mu)$$

defined by

$$\int_\Omega \widehat{T} f \cdot g d\mu = \int_\Omega f \cdot g \circ T d\mu \qquad (f \in L_1(\mu),\ g \in L_\infty(\mu))$$

is called *Frobenius-Perron operator.*

Properties:
If T is invertible then $\widehat{T}f = \frac{d\mu \circ T^{-1}}{d\mu} \cdot f \circ T^{-1}$.

If $f \in L_1(\mu), f > 0$ then $\mathcal{C}(T) = \{\sum_{n=1}^\infty \widehat{T}^n f = \infty\}$.

If T is conservative ergodic, then $\sum_{n=1}^\infty \widehat{T}^n f = \infty$ a.s. $\forall f \in L_1^+(\mu),\ \int f d\mu > 0$

If T exact (i.e. $A \in \bigcap_{n=1}^\infty T^{-n}\mathcal{A} \Rightarrow \mu(A)\mu(A^c) = 0$), then $||\widehat{T}^n f||_1 \to 0$ $\forall f \in L_1(\mu),\ \int f d\mu = 0$.

Aaronson's Theorem: Let T be conservative ergodic, $a(n) \uparrow \infty$, $\frac{a(n)}{n} \downarrow 0$. Then
If there is $A \in \mathcal{A}, 0 < \mu(A) < \infty$ mit $\int_A a(\varphi_A) d\mu < \infty$ then $\frac{1}{a(n)} S_n(f) \to \infty$ $\forall f \in L_1(\mu)$.

In the other case, $\liminf \frac{S_n(f)}{a(n)} = 0$ $\forall f \in L_1(\mu)$.

Let T be conservative, ergodic, measure preserving.

T is called *rational ergodic* if there exists $A \in \mathcal{A},\ 0 < \mu(A) < \infty$ such that

$$(*)\ \sup_n \frac{\int_A (S_n 1_A)^2 d\mu}{(\int_A S_n 1_A d\mu)^2} < \infty$$

Theorem: Let T be rational ergodic and let A satisfy $(*)$. Then there exists a sequence $a_n \uparrow \infty$ such that:

$$\forall B, C \in A \cap \mathcal{A} \Rightarrow \lim_{n\to\infty} \frac{1}{a_n} \sum_{k=0}^{n-1} \mu(B \cap T^{-k}C) = \mu(B)\mu(C)$$

The sequence $a_n = a_n(T)$, $(n \geq 1)$, with the property in the preceding theorem is called the *return sequence (of T)*. It is uniquely determined up to a proportionality factor. $A(T) = \{(a'_n)_n : \lim \frac{a'_n}{a_n(T)} \in \mathbb{R}\}$ is called the *asymptotic type.*

T is called *pointwise dual ergodic* if there exists a sequence $(a_n)_{n\geq 1}$, such that

$$\frac{1}{a_n} \sum_{k=0}^{n-1} \widehat{T}^k f \to \int f d\mu \text{ a.s. } \forall f \in L_1(\mu)$$

A set $A \in \mathcal{A}$ is called *Darling-Kac set* (DK set) if
- $0 < \mu(A) < \infty$
- $\exists (a_n)_{n\geq 1} : \frac{1}{a_n} \sum_{k=0}^{n-1} \widehat{T}^k 1_A \to \mu(A)$ uniformly on A.

Theorem: $\exists A$ DK set for $T \Rightarrow T$ is pointwise dual ergodic $\Rightarrow T$ is rational ergodic. The corresponding sequences a_n agree (asymptotically up to a factor).

SHIFTSPACES AND MARKOV FIBRED SYSTEMS

Let $A (= \{1, 2, 3, ...\})$ be a finite or countable alphabet, $\Sigma = A^{\mathbb{N}}$ (or $A^{\mathbb{Z}}$) is called a *shiftspace.*

$$S : \Sigma \to \Sigma$$

$$S((x_n)_{n\in I}) = (x_{n+1})_{n\in I} \quad I = \mathbb{N} \text{ or } \mathbb{Z}$$

is called the *shift.*

A closed, S-invariant $\Omega \subset \Sigma$ is called a *subshift.* If A is finite, $M = (m_{ij})_{i,j \in A}$ a $0-1$–matrix, then the subshift

$$\Omega = \Omega_M = \{(x_n)_{n \in I} \in \Sigma : m_{x_n x_{n+1}} = 1 \; \forall n \in I\}$$

is called a *subshift of finite type* or a *topological Markov chain.*

Let A be finite, M a $0-1$–matrix and μ an invariant probability measure. μ is called a *Gibbs measure* if there exist a measurable function $f : \Omega_M \to \mathbb{R}$, constants $C > 0$ and $P \in \mathbb{R}$, such that

$$C^{-1} \leq \frac{\mu([a_0, ..., a_{n-1}])}{\exp[-nP + S_n f(x)]} \leq C$$

for all $x \in [a_0, ...a_{n-1}], n \in \mathbb{N}, a_j \in A \; j = 0, .., n-1.$

T is called a *Markov-map* if there exists a generating partition α such that

$$Ta \in \sigma(\alpha) \; \forall a \in \alpha.$$

and a *Bernoulli map* if

$$Ta = \Omega \; \forall a \in \alpha.$$

α as above is a *Markov partition* if it is finite.

A nonsingular (T, μ) (μ a probability) is called a *Markov fibred system,* if $T|_a$ is nonsingular and invertible and T is a Markov map.

A measure preserving system (T, μ) (μ a probability) is called *Gibbs-Markov,* if it is Markov and

- $\exists M > 0 \; |\frac{v_a'(x)}{v_a'(y)} - 1| \leq Md(x, y) \; \forall n \geq 1 \; \forall a \in (\alpha)_0^{n-1} \; \forall x, y \in T^n a$

Here $v_a : T^n a \to a$ denotes the local inverse and

$$v_a' = \frac{d\mu \circ v_a}{d\mu}$$

μ has *bounded metric distortion,* if $\exists M > 0$

$$\frac{\frac{d\mu o T^n}{d\mu}(x)}{\frac{d\mu o T^n}{d\mu}(y)} \leq M \quad \forall n \geq 1 \; \forall a \in (\alpha)_0^{n-1} \; \forall x, y \in a$$

FROBENIUS–PERRON THEORY

$T : \Omega \to \Omega$ countable to one. ($\exists \alpha : T : a \to Ta$ invertible $\forall a \in \alpha$.)

$Pf(x) = \sum_{Ty=x} f(y)e^{\varphi(y)}$ is the *Frobenius-Perron-Operator*, where $\varphi : \Omega \to \mathbb{R}(\mathbb{C})$ is measurable.

P acts on measures by P^*m defined by

$$\int f dP^*m = \int Pf dm$$

Properties:

$m = P^*m \Leftrightarrow m$ is $e^{-\varphi}$ conformal.

Equivalent to $e^{-\varphi}$ conformality:
$\int gdm = \int g \circ Te^{-\varphi}dm \; \forall g : TA \to \mathbb{R}$
$\int hdm = \int h \circ T_*^{-1}e^{\varphi \circ T_*^{-1}} \; \forall h : A \to \mathbb{R}, T_*^{-1} : TA \to A$
$\int fdm = \int Pfdm$
$\frac{d\mu \circ T}{d\mu} = e^{-\varphi}$

$[g \cdot Pf](x) = P(f \cdot g \circ T)(x)$ i.e. P is "dual" to $Uf = f \circ T$ on L_∞.

Let $\mathcal{B}_n = T^{-n}\mathcal{A}$, $L_2(\mathcal{B}_n) = \{f \in L_2 : f$ is $\mathcal{B}_n$-measurable $\}$. Then $P1 = 1 \Rightarrow U^nP^n$ is the orthogonal projection onto $L_2(\mathcal{B}_n)$.

$P1 = 1 \Leftrightarrow m \circ T^{-1} = m$
$E(f|\mathcal{B}_n) = U^nP^nf = [P^nf] \circ T^n$
$h \in L_2(\mathcal{A}) \ominus L_2(\mathcal{B}_1) \Rightarrow h \circ T^k$ is a reversed martingale difference sequence.

EXISTENCE OF MARKOV PARTITIONS

Ω compact, metric, with metric d, T continuous.
T is called *expansive* if there exists $\delta > 0$ such that

$$x, y \in \Omega, d(T^nx, T^ny) < \delta \; \forall n \geq 0 \Rightarrow x = y$$

T is called *expanding* if there exist $\epsilon > 0$, $\Lambda > 1$ and $n \in \mathbb{N}$ such that

$$d(T^nx, T^ny) \geq \Lambda d(x, y) \; \forall x, y \in \Omega \text{ with } d(x, y) < \epsilon.$$

T is called *R-expanding* if T is expanding and open.

Theorem: If T is expansive, then there exists a metric ρ on Ω, such that T is ρ-expanding.
Theorem: If T is expansive and open, then there exists a finite Markov partition. The sequence metric of the associated topological Markov chain is equivalent to d only if T is R-expanding.

Let Ω be compact metric with metric d, $T : \Omega \to \Omega$ expanding and open and $f : \Omega \to \mathbb{R}$ Hölder continuous with exponent α. Then

$$C_f := \sup_{\{x,y\in\Omega,x\neq y\}} \frac{(f(x)-f(y))}{d(x,y)^\alpha} < \infty$$

$\mathcal{H}^\alpha = \{f : C_f < \infty\}$ is a Banach space with norm $||f||_\alpha = C_f + ||f||_\infty$.
Theorem: Let $\varphi \in \mathcal{H}^\alpha$ and $Pf(x) = \sum_{Ty=x} f(y)e^{\varphi(y)}$. Then
(a) $P : C(\Omega) \to C(\Omega)$ $P : \mathcal{H}^s \to \mathcal{H}^s$ $\forall s \leq \alpha$
(b) bounded sets in $\mathcal{H}^s$ are relatively compact in $C(\Omega)$.
(c) P is power bounded on $C(\Omega)$ and $\mathcal{H}^s$ $(s \leq \alpha)$
(d) $\exists \rho < 1 \exists C \exists n$ such that

$$||P^n f||_\alpha \leq \rho||f||_\alpha + C||f||_\infty$$

(e) $P = \sum P_i + Q, ||Q||_\alpha < 1, P_iP_j = 0, P_iQ = QP_i = 0, P_i : \mathcal{H}^\alpha \to E_i$ projection, E_i finite dimensional.

Theorem: Let $\phi \in \mathcal{H}^\alpha$. Then there exists a Gibbs measure μ with respect to ϕ.

FROBENIUS–PERRON THEORY FOR MFS

Let $(\Omega, \mathcal{A}, T, \mu)$ be a MFS with partition $\alpha = \{a_s : s \in A\}$. Define the partition β by $\sigma(T\alpha) = \sigma(\beta)$.

$$P^n f = \sum_{b\in\beta_b} \sum_{a\in(\alpha)_0^{n-1}, T^n a\supset b} v_a' f \circ v_a$$

$$Lip_{q,\gamma} \subset L_q(\mu)$$

is defined by $f \in Lip_{q,\gamma} \Leftrightarrow ||f||_{Lip_{q,\gamma}} = ||f||_q + D_\gamma f < \infty$ where

$$D_\gamma f = \sup_{a\in\gamma} C_{f|_a}$$

Theorem: Let T be mixing,
$\inf_{a\in\alpha} \mu(Ta) > 0$
$\exists M > 0 : |\frac{v_a'(x)}{v_a'(y)} - 1| \leq Md(x,y) \quad \forall n \geq 0 \forall a \in (\alpha)_0^{n-1} \forall x, y \in T^n a$
Then
$P : Lip_{1,\beta} \to L = Lip_{\infty,\beta}$

$||P^n f||_L = O(r^n D_\beta f + ||f||_1)$ (where $d(x,y) = \sum r^n 1_{x \neq y}$)
$||P^n f||_{L_1} = ||f||_{L_1}$
P is an ITM operator and $P = P_0 + Q$
Let $\psi : \Omega \to \mathbb{R}^d$, $P_t = P(e^{i<t,\psi>})$. Then

$$\int P_t^n 1 d\mu = \int e^{i<t,S_n\psi>} d\mu$$

(charactaristic function operator)
$t \to P_t$ is continuous in norm in Hom(L, L).

MIXING CONDITIONS

Let $(\Omega, \mathcal{A}, \mu)$ be a probability space $\mathcal{B}_t \subset \mathcal{A}$, $\mathcal{B}_n^m = \sigma(\mathcal{B}_t : n \leq t < m)$ $(0 \leq n < m \leq \infty)$.

$$\sup_{A \in \mathcal{B}_0^k, B \in \mathcal{B}_{k+n}^\infty, k \geq 0} \left| \frac{\mu(A \cap B) - \mu(A)\mu(B)}{\mu(A)^r \mu(B)^s} \right| =: \alpha_{rs}(n)$$

$\mathcal{B}_t$ (or μ with respect to $\mathcal{B}_t$ or a process) $(t \geq 0)$ is called
α-mixing or strongly mixing if $\alpha_{rs}(n) \to 0$ as $n \to \infty$ for some $r + s < 1$.
ρ-mixing if $\alpha_{r1-r}(n) \to 0$ as $n \to \infty$.
φ-mixing if $\alpha_{10}(n) \to 0$ as $n \to \infty$.
φ^*-mixing if $\alpha_{01}(n) \to 0$ as $n \to \infty$.
ψ-mixing if $\alpha_{11}(n) \to 0$ as $n \to \infty$.

Remark:
ψ-mixing $\Rightarrow \varphi - (\varphi^*-)$mixing $\Rightarrow \rho$-mixing (absolutely regular) $\Rightarrow \alpha$-mixing.
$\mathcal{B}_t$ is called *absolutely regular* if

$$E \sup_{k \geq 0, B \in \mathcal{B}_{k+n}^\infty} |\mu(B|\mathcal{B}_0^k) - \mu(B)| \to 0.$$

Let α be countable measurable partition of Ω and $\mathcal{F} = \sigma(\alpha)$. α is called *continued fraction mixing* (c.f.m.) if there exists $n_0 \in \mathbb{N}$ a sequence $\epsilon_n \downarrow 0$, such that $\forall A \in \mathcal{F}_0^{n-1} \forall B \in \mathcal{A}$
- $\mu(A \cap T^{-k-n}B) \leq (1 + \epsilon_n)\mu(A)\mu(B)$ $\forall n \geq 1$
- $\mathcal{F}_n^m$ is ψ-mixing with $\psi(n) = \epsilon_n$ for $n \geq n_0$.

Theorem: Let $0 < \mu(A) < \infty$, $\alpha \subset A \cap \mathcal{A}$ a generating partition for T_A and φ_A α-measurable.
If α is c.f.m. for $T_A : A \to A$, then A is a DK-set for T.

PROBABILITY FOR (CLASSICAL) MARKOV PARTITIONS

Theorem: Let $T : \Omega \to \Omega$ be exact (i.e. the Frobenius-Perron-operator satisfies $P^n g \to$ const.) Let α be a Markov partition and μ be a Gibbs measure. Then $\mathcal{B}_n = \sigma(T^{-n}\alpha)$ is ψ-mixing with

$$\psi(n) = \alpha_{11}(n) \leq M\rho^n \text{ (for some } 0 < \rho < 1, M > 0).$$

Theorem: (Central limit theorem) Let (Ω, B, T, μ) have a Markov partition α so that μ is Gibbs. Then for every $f \in L_2(\mu)$, satisfying $\sum_{n=1}^{\infty} ||P^n f||_2 < \infty$ one has

$$\sigma^2 = \int f^2 d\mu + 2\sum_{n=1}^{\infty} \int f \cdot f \circ T^n d\mu < \infty$$

and if $\sigma^2 > 0$

$$\mu(\{\omega \in \Omega : \frac{1}{\sqrt{n}\sigma} \sum_{j=0}^{\infty} f(T^j\omega) \leq x\}) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du$$

In particular, every centered Hölder continuous function satisfies the CLT.

Theorem: (Invariance principle) Let $f \in \mathcal{H}^s$ (Hölder with exponent s) and $\int f d\mu = 0$. If $\sigma^2 > 0$, then there exists w.l.o.g. a standard-Brownian motion $B_t (t \geq 0)$ such that

$$\sum_{n=0}^{n-1} f \circ T^k - \sigma B_n \ll n^{1/2-\gamma} \quad \text{a.s.}$$

for some $\gamma > 0$.

Corollary: Upper and lower-class results.

Let (Ω_M, T, μ) be a subshift of finite type with Gibbs measure μ and Hölder continuous potential ϕ. Let

$$\Omega = \{(\omega, s) : 0 \leq s \leq l(\omega)\}$$

where $l : \Omega_M \to \mathbb{R}_+$ is Hölder continuous. The flow (T_t) on Ω ($T_t(\omega, s) = (\omega, t + s)$ and identification by T) serves as a model for C^2-Anosov flows and geodesic flows on compact manifolds of negative curvature.

Theorem: Let $f : \Omega \to \mathbb{R}$ be measurable, centered and have finite $2 + \delta$ moment where $\delta > 0$. Assume that

$$||f - E(f|\sigma(\{a \times \mathbb{R} : a \in (\alpha)_{-n+1}^{n-1}\})|| = O(n^{-2-7/\delta}).$$

If $\sigma^2 > 0$ then there exists a Brownian motion B_t on Ω_M such that

$$\sup_{0 \leq s \leq l(\omega)} \int_0^u f(T_t(\omega, s))dt - B_u(\omega) = O(u^{1/2-\lambda}) \quad \text{a.s.}$$

for some $\lambda > 0$.

Theorem: Let T be the suspension flow over the natural extension of the continued fraction map. Let f be non-lattice and Hölder continuous so that T and f are flow-independent. Then there exists a function $H : \mathbb{R} \to [0, \infty]$, such that H is real analytic, surjective and strictly convex on $I_F = \{H < \infty\}$. Moreover, for any compact non-empty set $K \subset \mathbb{R}$ and $a \in I_F$

$$m(x : \int_0^u f(T_t(x)dt - ua \in K) \sim C(a)(\int_K e^{-H'(a)t}dt)\sqrt{\frac{H''(a)}{2\pi}}\frac{e^{-uH(a)}}{\sqrt{u}}.$$

Flow independence means: Let $G(y) = t_0 + t_1 f(y)$, and $G^t = \int_0^t G(T_\tau)d\tau$. If the flow $S_t^G : S^1 \times \Omega \to S^1 \times \Omega$,

$$S_t^G(z, y) = (z \exp[2\pi i G^t(y)], T_t(y))$$

is not topologically ergodic then $t_0 = t_1 = 0$.

Corollaries: Large deviation, local limit theorem and central limit theorem.

For $\phi \in \mathcal{H}^s$ its free energy function

$$c(t) = \lim_{n\to\infty} \frac{1}{n} \log \int \exp(tS_n\phi)dm \qquad (t \in \mathbb{R})$$

is well defined for Gibbs measures m with potential f.

The *pressure of* f is

$$P(T, f) = \sup\{h_m(T) + \int f dm : \ m \circ T^{-1} = m, \ m(\Omega) = 1\},$$

where $h_m(T)$ denotes the entropy of m.

Proposition:

$$c(t) = P(T, f + t\phi) - P(T, f) \qquad (t \in \mathbb{R}).$$

$$c'(t) = \int \phi \, dm_t \qquad (t \in \mathbb{R}),$$

where m_t denotes the Gibbs measure with potential $f + t\phi$, and

$$c''(t) = \lim_{n\to\infty} \frac{1}{n} \int (S_n(\phi - m_t(\phi)))^2 dm_t = \lim_{n\to\infty} c_n''(t).$$

Moreover, $c''(t) = 0$ if and only if ϕ is cohomologuous to a constant (everywhere).

Let T denote a hyperbolic or parabolic rational function, and let m denote the Gibbs measure for the Hölder continuous potential f. In case of a parabolic T assume in addition that $P(T,f) > \sup_{z\in\mathbb{C}} f(z)$. Denote the point mass in $z \in \mathbb{C}$ by δ_z and the space of probability measures on $J(T)$ by $\mathcal{M}(J(T))$ and its subspace of T-invariant measures by $\mathcal{M}_T(J(T))$.

Theorem: $(J(T), \mathcal{A}, m, T)$ satisfies the large deviation principle at level 2 with rate function

$$I^{(2)}(\nu) = \begin{cases} P(T,f) - \nu(f) - h_\nu(T) & \text{if } \nu \in \mathcal{M}_T(J(T)) \\ \infty & \text{if } \nu \notin \mathcal{M}_T(J(T)), \end{cases}$$

that is: For any closed (compact) set $K \subset \mathcal{M}(J(T))$ and any open set $G \subset \mathcal{M}(J(T))$,

$$\limsup_{n\to\infty} \frac{1}{n} \log m\Big(\Big\{z \in J(T) : \ \frac{1}{n}\sum_{k=0}^{n-1} \delta_{T^k(z)} \in K\Big\}\Big) \le -\inf\{I^{(2)}(\nu) : \ \nu \in K\}$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \log m\Big(\Big\{z \in J(T) : \ \frac{1}{n}\sum_{k=0}^{n-1} \delta_{T^k(z)} \in G\Big\}\Big) \ge -\inf\{I^{(2)}(\nu) : \ \nu \in G\}.$$

Theorem: The free energy function

$$c(\phi) := \lim_{n\to\infty} \frac{1}{n} \log \int_{J(T)} \exp\Big[\sum_{k=0}^{n-1} \phi \circ T^k\Big]\, dm$$

exists for any continuous function f and equals

$$c(\phi) = P(T, f+\phi) - P(T,f).$$

c is continuous on $C(J(T))$. Moreover, if T is hyperbolic, c is Gateaux differentiable at each Hölder continuous function ϕ with derivative

$$\frac{d}{dt}\{P(T, f+t\phi) - P(T,f)\}_{|t=0} = m(\phi).$$

Theorem: Let T be hyperbolic and let m denote the measure of maximal entropy. Then there exists $\rho > 0$ such that for each $0 < \alpha < \rho$ there are constants d_* and d^* satisfying

$$\lim_{n\to\infty} \min_{0\le i\le n-d_* \log n} \frac{1}{d_* \log n} \sum_{i\le k\le i-1+d_* \log n} \log|T'(T^k(z))| = \chi_m - \alpha$$

$$\lim_{n\to\infty} \max_{0\le i\le n-d^* \log n} \frac{1}{d^* \log n} \sum_{i\le k\le i-1+d^* \log n} \log|T'(T^k(z))| = \chi_m + \alpha,$$

where $\chi_m = m(\log|T'|)$ denotes the Lyapunov exponent of T with respect to m.

PROBABILITY THEORY FOR MFS

Let $(\Omega, \mathcal{A}, T, \mu, \alpha)$ be a MFS with the Schweiger property with respect to

$$\mathcal{R}(C,T) = \{A \in (\alpha)_0^n \text{ has bounded metric distortion by } C\}$$

(i.e. generates $\mathcal{A}$ and subsets of $A \in \mathcal{R}(C,T)$ inherit the distortion property). Let $\mathcal{R}^*$ denote the partition generated by $\mathcal{R}(C,T)$, $N_C : \Omega \to \mathbb{N}$.

$$N_C(\omega) = \inf\{n \geq 1 : \omega \in a \in (\alpha)_0^{n-1} \cap \mathcal{R}(C,T)\}$$

$T^* = T^{N_C}$ is called the *jump transformation*

Assumptions:
α is aperiodic
T is parabolic (i.e. $N_C \cdot T = N_C - 1$ on $\{N_C \geq 2\}$, $|(\alpha)_0^1 \cap \{N_C = 2\}| < \infty$, $T(\{N_C = 1\} \backslash T\{N_C = 2\}) = \Omega$ and $T : \{N_C \geq 2\} \to T\{N_C \geq 2\}$ invertible)

Lemma: $\exists m \sim \mu, m \circ T^{-1} = m$.
$\exists q \sim \mu, q \cdot {T^*}^{-1} = q$. m is finite $\Leftrightarrow A = \int N_C dm < \infty$

For $f : \Omega \to \mathbb{R}$ define $f^* = f + f \circ T + .. + f \circ T^{N_C - 1} - N_C \int f dm$. Then $\int f^* dq = 0$.
Theorem: Let $(T^*, \mathcal{R}^*)$ be absolutely regular such that $\sum_{n=1}^{\infty} \beta(n)^{1/2+\delta} < \infty$ for some $\delta > 0$. Let $f^* \in L_\infty(q)$, $\xi_n = ||f^* - E_q(f^*|(\mathcal{R}^*)_0^n)||_{2+\eta} \leq Cn^{-2-\delta}$ for some $\eta > 0$. Then $c_f^2 = \int f^* dq + 2 \sum_{n=1}^{\infty} \int f^* \cdot f^* \circ T^n dq < \infty$
and if $c_f > 0$

$$m(\{\frac{\sqrt{A}}{\sqrt{n} c_f} \sum_{k=0}^{n} f \circ T^k - \int f dm \leq x\}) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du$$

Let $(\Omega, \mathcal{A}, T, \mu, \alpha)$ be a mixing Gibbs-Markov map.

$$\phi : \Omega \to \mathbb{R}, \quad D_\alpha \phi = \sup_{a \in \alpha} C_{\phi|a} < \infty$$

$$\mathcal{L}(\phi) \in DA(p) \quad \text{(domain of attraction to stable)}$$

i.e.

$$\mu(\phi > x) = (c_1 + o(1)) \frac{L(x)}{x^p} \quad \text{as } x \to \infty$$

$$\mu(\phi < -x) = (c_2 + o(1)) \frac{L(x)}{x^p} \quad \text{as } x \to \infty$$

Let P denote the Frobenius-Perron operator for μ and
$P_t f = P(f \cdot \exp(it\phi))$
$\lambda_t =$ the maximal eigenvalue of P_t, g_t the eigenvector

Theorem: Let $p < 2$. Then

$$Re \log \lambda_t = -c|t|^p L(\frac{1}{|t|})(1 + o(1))$$

$$Im \log \lambda_t = \begin{cases} t\gamma + c\beta|t|^p \mathrm{sgn}(t) \tan \frac{p\pi}{2} + O(|t|^p L(\frac{1}{|t|})) & \text{if } p \neq 1 \\ \gamma t + \frac{2\beta c}{\pi} CtL(\frac{1}{|t|}) + f(H_1(\frac{1}{|t|} - H_2(\frac{1}{|t|})) + O(|t|L(\frac{1}{|t|})) & \text{if } p = 1. \end{cases}$$

$H_j(\lambda) = \int_0^\lambda \frac{xc_j L(x)}{1+x^2} dx + 0(L(\lambda))$ $(j = 1, 2)$
$C = \int_0^\infty (\cos y - \frac{1}{1+y^2}) \frac{dy}{y}$
$\beta = \frac{c_1 - c_2}{c_1 + c_2}$

$$c = \begin{cases} (c_1 + c_2)\Gamma(1-p) \cos \frac{p\pi}{2} & \text{if } p \neq 1 \\ \frac{c_1 + c_2}{2} \pi & \text{if } p = 1. \end{cases}$$

$$\gamma = \begin{cases} 0 & \text{if } p < 1 \\ \int_{-\infty}^\infty (\frac{x}{1+x^2} + \mathrm{sgn}(x) \int_0^{|x|} \frac{2u^2}{(1+u^2)^2} du) d\mu(x) & \text{if } p = 1 \\ \int_{-\infty}^\infty x d\mu(x) & \text{if } p > 1. \end{cases}$$

Corollaries:

$$\frac{S_n\phi - A_n}{B_n} \to X_p$$

weakly, where X_p is p-stable. Here:
$nL(B_n) = B_n^p$,

$$A_n = \begin{cases} 0 & \text{if } p < 1 \\ \gamma n & \text{if } 1 < p < 2 \\ \gamma n + \frac{2n}{\pi}(H_1(B_n) - H_2(B_2)) & \text{if } p = 1. \end{cases}$$

$$\log E e^{itX_p} = \begin{cases} t\gamma i - c|t|^p(1 - i\mathrm{sgn}(t)\beta \tan \frac{p\pi}{2}) & \text{if } p \neq 1 \\ t\gamma i - c|t|(1 - i\mathrm{sgn}(t)\frac{2\beta}{\pi} \log \frac{1}{|t|}) & \text{if } p = 1. \end{cases}$$

If ϕ is aperiodic and $\mathbb{Z}$-valued, then $||B_n P_{T_n}(1_{\{S_n\phi = k_n\}}) - f_{X_p}(\kappa)||_\infty \to 0$ as $n \to \infty$ and $\frac{k_n - A_n}{B_n} \to \kappa$, f_{X_p} denotes the density of X_p

ϕ is called *aperiodic* if

$$e^{it\phi} = \frac{\lambda g}{g \circ T} \Rightarrow t = 0, \lambda = 1 \text{ and } g = 1$$

In particular: $B_n \mu(S_n\phi = k_n) \to f_{X_p}(\kappa)$.

Let $I \subset \mathbb{R}$ be an interval, $k_n \in \mathbb{Z}$, $\frac{k_n - A_n}{B_n} \to \kappa$, ϕ $\mathbb{R}$-valued and aperiodic.

Local Limit Theorem.

$$B_n P_{T^n}(1_{\{S_n\phi \in k_n + I\}}) \to |I| f_{X_p}(\kappa)$$

In particular

$$B_n \mu(S_n\phi \in k_n + I) \to |I| f_{X_p}(\kappa).$$

Let $(X, \mathcal{A}, m, T, \alpha)$ be a mixing, probability preserving Gibbs-Markov map, and let G be a subgroup of $\mathbb{R}^d$ of form $G = A(\mathbb{R}^k \times \mathbb{Z}^\ell)$ where $k + \ell = d$ and $A \in GL(d, \mathbb{R})$. Suppose that

$$\phi : X \to G$$

is aperiodic, Lipschitz continuous on each $a \in \alpha$ and $D_\alpha \phi := \sup_{a \in \alpha} C_{\phi|a} < \infty$.

The *skew product* is $T_\phi : X \times G \to X \times G$ defined by

$$T_\phi(x, g) = (Tx, g + \phi(x)).$$

Theorem:
1) Either T_ϕ is totally dissipative, or T_ϕ is pointwise dual ergodic.
2) If G is discrete and T_ϕ is conservative, then T_ϕ is exact.

Suppose that

$$\frac{\phi_n}{B_n} \to X_p \quad \text{in distribution}$$

where $B_n > 0$ and X_p is nondegenerate p-stable.
Theorem: T_ϕ is conservative iff

$$\sum_{n=1}^{\infty} \frac{1}{B_n^d} = \infty,$$

In this case, T_ϕ is pointwise dual ergodic with return sequence

$$a_n(T_\phi) \sim f_{X_p}(0) \sum_{k=0}^{n} \frac{1}{B_k^d}.$$

CONVERGENCE IN σ-FINITE MEASURE SPACES

Let $(\Omega, \mathcal{A}, T, \mu)$ be pointwise dual ergodic.
$a(n)$ return sequence
$\phi(N, \epsilon) = \sup\left\{ \frac{a(n)}{a(nN^\epsilon)} : 1 \le n \le N^{1-\epsilon} \right\}$

Theorem: $\forall \epsilon > 0,\ \lim_N \phi(N, \epsilon) = 0 \Rightarrow \forall f \in L_1(\mu)$

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{na(n)} S_n f = \int f d\mu \qquad \text{(log-averages)}$$

in measure on every subset of finite measure.

Theorem: $\exists \epsilon, \gamma > 0$ with $\phi(N, (\log N)^{-\gamma}) = 0(\frac{1}{(\log N)^\epsilon})$
$\exists A \in B,\ \mu(A) = 1\ \exists \overline{a}(n) = (1 + \beta_n)a(n)$ such that

$$\beta_n = 0(\frac{1}{(\log n)^\gamma})$$

$$\sum_{k=1}^{n} \hat{T}^k 1_A \leq \overline{a}(n)$$

$$\Rightarrow$$

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{na(n)} S_n f = \int f d\mu \qquad \text{a.s.}$$

Remark: $a(n) = n^\alpha L(n)(\alpha > 0) \Rightarrow \forall f \in L_1^+(\mu)$ the following is equivalent:
$\frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{na(n)} S_n f \to \int f d\mu$ a.s.
$\frac{1}{a(N)} \sum_{n=1}^{N} \frac{f \circ T^n}{a(n)} \to \int f d\mu$ a.s. (Chung Erdös averages)

Theorem: Let T have a Darling-Kac set with α-mixing return time process, $a(n) = n^\alpha L(n),\ 0 < \alpha < 1$
Let $\phi \uparrow$ and $\phi(n)/n \downarrow$. Then for $K_\alpha = \frac{\Gamma(1+\alpha)}{\alpha^2(1-\alpha)^{1-\alpha}}$
(a) $\sum_{n=1}^{\infty} \frac{1}{n} e^{-\beta\phi(n)} < \infty \forall \beta > 1 \quad \Rightarrow$

$$\limsup_{n\to\infty} \frac{1}{n^\alpha L(\frac{n}{\phi(n)})\phi(n)^{a-\alpha}} S_n f \leq K_\alpha \int f d\mu; \forall f \in L_1^+(\mu)$$

(b) $\sum_{n=1}^{\infty} \frac{1}{n} e^{-r\phi(n)} = \infty\ \forall r < 1 \quad \Rightarrow$

$$\limsup_{n\to\infty} \frac{1}{n^\alpha L(\frac{n}{\phi(n)})\phi(n)^{1-\alpha}} S_n f \geq K_\alpha \int f d\mu;\ \forall f \in L_1^+(\mu)$$

(c)

$$\limsup_{n\to\infty} \frac{1}{n^\alpha L(\frac{n}{LLn})(LLn)^{1-\alpha}} S_n f = K_\alpha \int f d\mu;\ \forall f \in L_1^+(\mu)$$

Theorem: Under the same assumptions as before, but $\alpha = 1$, there exists a constant K_T such that

$$\limsup_{n\to\infty} \frac{1}{nL(\frac{n}{LLn})} S_n f = K_T \int f d\mu$$

and, moreover, if $L(\frac{n}{LLn}) \sim L(n)$

$$\limsup_{n\to\infty} \frac{1}{nL(n)} S_n f = \int f d\mu \; \forall f \in L_1^+(\mu)$$

Let $(X_k)_{k=1}^\infty$ be a stationary c.f.m. process on $(A, \mathcal{A}, m)$. W.l.o.g $A = \Sigma$ with the m–preserving transformation $S(y_k)_{k=1}^\infty = (y_{k+1})_{k=1}^\infty$ so that $X_1(y) = y_1$. It is well known that S is ergodic .
Now we let

$$X = \{x = (y,n) : 1 \le n \le X_1(y), y \in A\}$$

$$\mathcal{B} = \bigvee_{n=1}^{\infty} \mathcal{A} \cap \{X_1 \ge n\} \times \{n\}$$

$$\mu(B \times \{n\}) = m(B) \qquad (B \in \mathcal{A} \cap \{X_1 \ge n\})$$

$$T(y,n) = \begin{cases} (y, n+1), & \text{if } X_1(y) \ge n+1 \\ (Sy, 1), & \text{if } X_1(y) = n. \end{cases}$$

By Kakutani's theorem T is a conservative, ergodic, measure preserving transformation on $(X, \mathcal{B}, \mu)$.
By Kac's formula $\mu(X) = EX_1$.
X_1 is the first return time function φ to A, and S is the induced transformation T_A. A is a D–K set for T.

Theorem: Suppose that $m(\{X_1 \ge t\} = \big(\Gamma(1-\alpha)\Gamma(1+\alpha)a(t)\big)^{-1}$, where $a(t)$ is regularly varying with index $\alpha \in (0,1)$. Let b be the inverse of a. Then for $\phi(n) \uparrow$ and $\phi(n)/n \downarrow$ as $n \uparrow \infty$, we have:
(a) If $\sum_{n=1}^\infty \frac{1}{n} \exp\big[-\beta\phi(n)\big] < \infty$ for all $\beta > 1$ then

$$\liminf_{n\to\infty} \frac{1}{b(n/\phi(n))\phi(n)} \sum_{k=1}^n X_k \ge K_\alpha^{-1/\alpha} \qquad \text{a.e.}$$

(b) If $\sum_{n=1}^\infty \frac{1}{n} \exp\big[-r\phi(n)\big] = \infty$ for all $r < 1$ then

$$\liminf_{n\to\infty} \frac{1}{b(n/\phi(n))\phi(n)} \sum_{k=1}^n X_k \le K_\alpha^{-1/\alpha} \qquad \text{a.e.}$$

(c)

$$\liminf_{n\to\infty} \frac{1}{b(n/L_2(n))L_2(n)} \sum_{k=1}^n X_k = K_\alpha^{-1/\alpha} \qquad \text{a.e.}$$

Corollary: Suppose that

$$\sup_{t \le s \le tL_2(t)} \left| \frac{h(s)}{h(t)} - 1 \right| \to 0 \quad \text{as } t \to \infty.$$

If $\phi(n)\uparrow$ and $\phi(n)/n\downarrow$ as $n\uparrow\infty$, then:
(a) If $\sum_{n=1}^{\infty}\frac{1}{n}\exp\left[-\beta\phi(n)\right]<\infty$ for all $\beta>1$ then

$$\liminf_{n\to\infty}\frac{1}{b(n)\phi(n)^{1-1/\alpha}}\sum_{k=1}^{n}X_k\geq K_\alpha^{-1/\alpha}\qquad\text{a.e.}$$

(b) If $\sum_{n=1}^{\infty}\frac{1}{n}\exp\left[-r\phi(n)\right]=\infty$ for all $r<1$ then

$$\liminf_{n\to\infty}\frac{1}{b(n)\phi(n)^{1-1/\alpha}}\sum_{k=1}^{n}X_k\leq K_\alpha^{-1/\alpha}\qquad\text{a.e.}$$

(c)

$$\liminf_{n\to\infty}\frac{1}{b(n)L_2(n)^{1-1/\alpha}}\sum_{k=1}^{n}X_k=K_\alpha^{-1/\alpha}\qquad\text{a.e.}$$

Theorem: Set

$$B(n)=b(n/L_2(n))L_2(n),\quad H_n(t)=\frac{1}{B(n)}\sum_{k=1}^{[nt]}X_k.$$

(i) $\{H_n\}_{n\in\mathbb{N}}$ is precompact in

$$\Delta_\uparrow=\{x:\mathbb{R}_+\to[0,\infty],\ x(0)=0,\ x\uparrow\}\ \text{a.e.}$$

(ii)

$$\{H_n\}'=K(\alpha)=\{x\in\Delta_\uparrow:\int_0^b(x'(t))^{-\frac{\alpha}{1-\alpha}}dt\leq K_\alpha^{\frac{1}{1-\alpha}}\}\ \text{a.e.}$$

where $b=\inf\{t:x(t)=\infty\}$.

Theorem: $a(n)=n^\alpha L(n),\alpha\in[0,1]\Rightarrow\ \forall f\in L_1^+(\mu)$

$$\frac{S_nf}{a(n)}\to Y_\alpha\int f\,d\mu\quad\text{"weakly"}$$

where Y_α is a stable subordinator

Remark: *"weak" convergence* is

$$g:\overline{\mathbb{R}}_+\to\mathbb{R},q\text{ invariant }\ q\ll\mu$$

$$\int_\Omega g(\frac{S_nf}{a(n)})dq\to E(g(Y_\alpha\int f\,d\mu))$$

Y_α is determined by its Laplace transform

$$E\exp(zY_\alpha)=\sum_{n=0}^{\infty}z^n\frac{\Gamma(1+\alpha)^n}{\Gamma(1+n\alpha)}$$

DIMENSION THEORY

Let (Ω, d) be a metric space. For a subset $F \subset \Omega$ define the outer s-dimensional Hausdorff measure $m_H^s(F)$ of F by

$$m_H^s(F) = \lim_{\delta - 0} \inf\left\{ \sum_{U \in \mathcal{U}} (\mathrm{diam}(U))^s : \mathcal{U} \text{ a cover; } \sup_{U \in \mathcal{U}} \mathrm{diam}(U) < \delta \right\}.$$

Definition: The Hausdorff dimension $\dim_H(F)$ of F is defined by

$$\dim_H(F) = \inf\{s : m_H^s(F) < \infty\} = \sup\{s : m_H^s(F) = \infty\}.$$

Note: Instead of taking the diameter as a measurement for U one may take other functions on U. For example

$$m_?^s(F) = \lim_{\delta \to 0} \inf\left\{ \sum_{U \in \mathcal{U}} \psi(U)(\phi(U))^s : \mathcal{U} \text{ a cover; } \sup_{U \in \mathcal{U}} \mathrm{diam}(U) < \delta;\ \mathcal{U} \subset ? \right\}.$$

This leads to the Carathéodory-Pesin dimension (in particular Billingsley dimension). If $\phi = 1$ and ψ is a function of $\mathrm{diam}(U)$ denote the corresponding outer measure by m_ψ.

Definition: The lower (upper) pointwise dimension in $\omega \in \Omega$ of a probability measure μ on Ω is defined as

$$\underline{\dim}_\mu(\omega) = \liminf_{\epsilon \to 0} \frac{\log \mu(B(\omega, \epsilon))}{\log \epsilon};$$
$$\overline{\dim}_\mu(\omega) = \limsup_{\epsilon \to 0} \frac{\log \mu(B(\omega, \epsilon))}{\log \epsilon}.$$

Frostman's Type Lemma: If $\Omega \subset \mathbb{R}^n$ is compact, then there exists a constant $b(n)$ such that for any $F \subset \Omega$ and any finite measure μ on F:

If $\limsup_{r - 0} \frac{\mu(B(\omega,r))}{r^d} \geq C$ for all $\omega \in F$, then $m_H^d(F) \leq b(n)C^{-1}\mu(F)$.

If $\limsup_{r - 0} \frac{\mu(B(\omega,r))}{r^d} \leq C < \infty$ for all $\omega \in F$, then $m_H^d(F) \geq b(n)^{-1}C\mu(F)$.

Note that this is the same in spirit as Young's result: If $\underline{\dim}_\mu \geq d$, then $\dim_H(F) \geq d$. If $\overline{\dim}_\mu \leq d$, then $\dim_H(F) \leq d$.

It also uses the same ideas as for Frostman's lemma: If $d_1 \leq \underline{\dim}_\mu \leq d_2$ then the Hausdorff dimension $\dim(\mu) := \inf\{\dim_H(Y) : \mu(Y) = 1\}$ of μ satisfies $d_1 \leq \dim(\mu) \leq d_1$.

Calculation principle:

Let m be conformal w.r.t. to $\exp f$. Then

$$m(T^n(B(x,r))) = \int_{B(x,r)} \exp[-S_n f]dm$$

which is in some cases $\sim m(B(x,r))\exp[-S_n f(x)]$.

Let T be conformal, then

$$\mathrm{diam} T^n(B(x,r)) \sim r|(T^n)'|.$$

Then

$$\frac{m(B(x,r))}{r^\kappa \psi(r)} \sim \exp[-S_n f(x) + \kappa S_n \log|T'|(x) - \log\psi(r)].$$

To apply the LIL, we need:

$$\int f dm - \kappa \int \log|T'|dm = 0.$$

One can show (Young)

$$\dim(m) = \inf\{\dim_H(Y) : m(Y) = 1\} = \kappa = \frac{\int f dm}{\int \log|T'|dm}.$$

For $\kappa \geq 0$ let

$$\psi_\kappa(t) = t\exp\left(\kappa\sqrt{\log 1/t \ \log\log\log 1/t}\right).$$

Makarov's results: $\exists C > 0$ (independent of B) such that, if B is a Jordan domain, then the harmonic measure ν on ∂B satisfies $\nu \ll m_{\psi_C}$ and $\nu \perp m_H^\alpha$ for every $\alpha > 1$.

In particular, the Hausdorff dimension of the harmonic measure equals 1. Moreover, there exists a Jordan domain for which there is some κ so that $\nu \perp m_{\psi_\kappa}$.

Result of Pryztycki, Urbański, Zdunik: Let B be an RB–domain, i.e. the boundary ∂B is conformally self–similar and repelling (for example all simply connected basins of immediate attraction to an attractive periodic point of a rational map are RB–domains).

Then there exists a number $c(B) \geq 0$ such that

(a) $\nu \perp m_{\psi_c}$ for every $0 < c < c(B)$,

(b) $\nu \ll m_{\psi_c}$ for every $c > c(B)$,

(c) If $c(B) = 0$ then $\nu \ll m_H^1$ and ∂B is a real-analytic Jordan curve.

Parabolic rational functions (cf. section Prob. Theory for MFS): (results also hold for parabolic MFS??) Let m be the equilibrium measure for the Hölder continuous potential f with pressure $P(T, f) > \sup_z f(z)$. In this case, for a Hölder continuous function ϕ, the function ϕ^* satisfies an a.s. invariance principle with respect to T^* and m^*. Hence we obtain the following results from the properties of the stopping time N_C and the fact that m and m^* are equivalent.

Theorem: Let ϕ be Hölder continuous. If $c_\phi^2 > 0$, then

$$m\Big(\{x \in J(T) : \sum_{j=0}^{n-1} [\phi(T^j(x)) - m(\phi)] > \frac{c_\phi}{\sqrt{m^*(N_C)}} \psi(n)\sqrt{n} \text{ for } \infty \text{ many } n \in \mathbb{N}\}\Big)$$
$$= \begin{cases} 0 & \text{if } \psi \text{ belongs to the lower class,} \\ 1 & \text{if } \psi \text{ belongs to the upper class.} \end{cases}$$

Moreover,

$$\limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} \phi \circ T^k - nm(\phi)}{\sqrt{2c_\phi^2 (m^*(N_C))^{-1} n \log\log n}} = 1 \qquad m \text{ a.e.}$$

Remark: Recall that $\psi : [1, \infty) \to \mathbb{R}_+$ belongs to the lower (upper) class if it is non-decreasing and if the integral

$$\int_1^\infty \frac{\psi(t)}{t} \exp\left[-\frac{1}{2}\psi(t)^2\right] dt$$

converges (diverges).

Grigull's result: A parabolic rational map is expansive and open, hence admits a Markov partition. Denote this partition by $\mathcal{P}$, and let

$$\mathcal{P}^n = \mathcal{P} \vee ... \vee T^{-n+1}\mathcal{P}.$$

For $z \in J(T)$ denote $p^n(z)$ the atom of $\mathcal{P}^n$ containig z. The invariance principle for the jump transformation gives the law of iterated logarithm for the information function.

Theorem: Let f be Lipschitz continuous. Then for a.e. z

$$\limsup_{n\to\infty} \frac{\log m(p^n(z)) + nh_m(T)}{\sqrt{2c_f^2 (m^*(N_C))^{-1} n \log\log n}} = 1,$$

provided c_f^2 defined above is non-zero.

Makarov's result for equilibrium measures for parabolic rational maps: Denote the (modified) Lyapunov exponent by $\chi = m^*(N_C)m(\log|T'|)$ and the Hausdorff dimension of m by τ. For a function $\psi : [1, \infty) \longrightarrow \mathbb{R}_+$ define for sufficiently small $t > 0$

$$\tilde{\psi}(t) = t^\tau \exp\left(c_{f+\tau\log|T'|} \chi^{-1/2} \psi(-\log t)\sqrt{-\log t} \right)$$

Theorem:
(1) If ψ belongs to the lower class, then

$$m \ll m_{\tilde{\psi}}.$$

(2) If ψ belongs to the upper class, then

$$m \perp m_{\tilde{\psi}}.$$

Makarov's result for SBR measures for parabolic rational maps: No a.s. results for parabolic maps and the potential $f = -h \log |T'|$ known (??) when the invariant measure is finite, (except Birkhoff's theorem).

The jump transformation still satisfies an a.s. invariance principle, thus allowing upper and lower class results.

We restrict to the case of the boundary ∂B of a petal of a rationally indifferent fixed point of a parabolic rational map T (the result generalizes in fact to so-called parabolic Jordan domains). Let

$$\phi = \log |(T^*)' \circ R| - \log |(F^*)'|,$$

where $R : \{|z| < 1\} \to B$ denotes the Riemann map, F the extension of $R^{-1} \circ T \circ R$ to an open set containing the closed unit disc, and where * denote the respective jump transformations. For a function $\psi : [1, \infty) \to (0, \infty)$ define (for sufficiently small $t > 0$)

$$\tilde{\psi}(t) = t \exp\left(\frac{c_\phi}{\sqrt{\chi}} \psi(-\log t) \sqrt{-\log t} \right)$$

(χ denotes the Lyapunov exponent of $T^* \circ R$ with repsect to Lebesgue measure).

Theorem: Suppose that $c_\phi^2 > 0$. Then:
a) If ψ belongs to the lower class, then $\nu \ll m_{\tilde{\psi}}$.
b) If ψ belongs to the upper class, then $\nu \perp m_{\tilde{\psi}}$.
c) $c_\phi^2 = 0$ if and only if ∂B is real-analytic.

Infinite SBR measure for parabolic rational maps: Let Ω denote the set of parabolic points ω (i.e. $\exists p \ni \ T^p(\omega) = \omega, (T^p)'(\omega) = 1$) (which are always contained in J). Let T_ω^{-p} denote the analytic inverse branch of T^p which fixes ω. Then define $p(\omega)$ by

$$T_\omega^{-p}(z) = z - a(z - \omega)^{p(\omega)+1} + \ldots$$

Next define

$$\alpha(\omega) = \frac{p(\omega) + 1}{p(\omega)} h,$$

and

$$\alpha = \min_{\omega \in \Omega} \alpha(\omega).$$

The SBR-measure of T is infinite iff $\alpha > 2$ ($\dim_H(J) \leq 1$ implies $\alpha > 2$).

Theorem: Let $h = \dim_H(J) < 1$, m be Sullivan's h-conformal measure and

$$f(t) = t^h \left(\log \frac{1}{t}\right)^{\gamma(1-h)}.$$

Then

a) $m \ll m_f$ if $\gamma > (\alpha - 1)^{-1}$.

b) $m \perp m_f$ if $\gamma \leq (\alpha - 1)^{-1}$.

More generally, let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a homeomorphism with $\varphi(0) = 0$ such that $\varphi(2x)/\varphi(x)$ is bounded above and away from 1 for large x, and

$$\widetilde{\varphi}(t) = t^h \left(\varphi\big(\log(\frac{1}{t})\big)\right)^{1-h}.$$

Then m is absolutely continuous or orthogonal with respect $m_{\widetilde{\varphi}}$ depending on whether

$$\int_1^\infty \frac{\varphi^{-1}(x)}{x^\alpha} dx$$

converges or diverges.

MULTIFRACTAL FORMALISM

The theory of multifractals has its origin in Kolmogorov's work 1941 for completely developed turbulence. His third hypothesis that the energy dissipation is lognormal distributed was questioned by Mandelbrot in 1972/4. Based on these ideas Frisch, Parisi and later Halsey et al. developed a first simple formalism for multifractals. The connection with thermodynamics was pointed out by Fujisaka in 1987. Since then this connection is one of the basic object of research in fractal geometry. More generally one may consider the connection between large deviation theory and multifractal formalism.

Multifractal Principle: Let X be a set and $h : X \to \mathbb{R}$. Let D be a real valued function defined on all (or part of) subsets of X. Then we define the spectrum w.r. to h and D by

$$f(\alpha) = D\Big(\{x \in X : \ h(x) = \alpha\}\Big).$$

Calculate $f(\alpha)$, e.g. if D is the dimension function.

Thermodynamic Formalism: Let (Σ, S) be a subshift of finite type (e.g. obtained from a finite Markov partition). Let $\mathcal{H}$ denote the class of Hölder-continuous functions on Σ. Let $\phi, \psi \in \mathcal{H}$ and define $f : \Sigma \to \mathbb{R}$ by

$$f(\alpha) = \dim_H(\{x \in \Sigma : \lim_{n \to \infty} \exp[S_n(\phi - \psi)(x)] = \alpha\}).$$

In case μ is the Gibbs measure for ϕ and $\psi = \log 2$ (2 is the expanding constant),

$$X_\alpha = \{x \in \Sigma : \lim_{n\to\infty} \frac{\log \mu(B(x, 2^{-n})}{\log 2^{-n}} = \alpha\},$$

provided the pressure $P(\phi - \psi) = 0$ (cf. the definition of a Gibbs measure).
Proposition: Let $\phi, \psi \in \mathcal{H}$ so that $\psi < 0$ and $P(\phi) = 0$. Then there exists a unique function

$$\beta : \mathbb{R} \to \mathbb{R}$$

satisfying

$$P(t\phi + \beta(t)\psi) = 0.$$

β is real-analytic with $\beta' < 0$ and $\beta'' \geq 0$. These derivatives vanish only in isolated points or $\beta'' = 0$. In the first case one obtains that $\alpha = \beta'$ is invertible and the domain of definition of its Legendre transform $f := \beta^*$ is the intervall $\Gamma :=$ image of α. Hence

$$\beta^*(\alpha(t)) = t\alpha(t) + \beta(t).$$

Local Large Deviation Theorem.

$z(R) \sim z$ means that $\sup_{R' \geq R} \left| \frac{z(R')}{z} - 1 \right| \to 0$ as $R \to \infty$.

Theorem: (Kesseböhmer) Let ϕ and ψ be Hölder-continuous functions on a subshift of finite type such that $\psi < 0$ and $P(\phi) = 0$, and let μ denote the Gibbs measure for the potential $\beta(0)\psi$. Assume that no non-trivial linear combination of ϕ and ψ is cohomologous to a $2\pi\mathbb{Z}$-periodic function. Let $n(R)$ be defined by

$$-S_{n(R)}\psi \leq R < -S_{n(R)+1}\psi.$$

Then for all compact sets K, $\kappa \in K$ and all $a < b$

$$\begin{aligned} &\mu(S_{n(R)}\phi + \kappa R \in [a, b]) \\ &\sim \frac{C(\kappa) \int_a^b e^{-\alpha^{-1}(\kappa)s} ds}{\sqrt{2\pi\beta''(\alpha^{-1}(\kappa))}} R^{-1/2} \exp[(f(\kappa) - \beta(0))R]. \end{aligned}$$

The convergence is uniform in $\kappa \in K$ and $C(\kappa)$ is bounded away from 0 and infinty.

Corollary: (Large deviation)

$$\mu(\{x : S_{n(R)(x)}\phi(x) \geq -\alpha(t)R\}) \sim \frac{C(\alpha(t))}{\sqrt{2\pi\beta''(t)}} R^{-1/2} \exp[(f(\alpha(t)) - \beta(0))R] \quad t > 0,$$

$$\mu(\{x : S_{n(R)(x)}\phi(x) \leq -\alpha(t)R\}) \sim \frac{C(\alpha(t))}{\sqrt{2\pi\beta''(t)}} R^{-1/2} \exp[(f(\alpha(t)) - \beta(0))R] \quad t < 0.$$

Corollary: Let m denote the Gibbs measure for ϕ and $C_n(x)$ the cylinder of length n containing x. Then

$$\begin{aligned}&\mu(\{x:\ \log m(C_{n(-\log r)(x)}(x)) \geq \alpha(t)\log r\})\\ &\qquad\sim \frac{C(\alpha(t))}{\sqrt{2\pi\beta''(t)}}\sqrt{-\log r}\exp[-(f(\alpha(t))-\beta(0))\log r] \qquad t>0,\end{aligned}$$

$$\begin{aligned}&\mu(\{x:\ \log m(C_{n(-\log r)(x)}(x)) \leq \alpha(t)\log r\})\\ &\qquad\sim \frac{C(\alpha(t))}{\sqrt{2\pi\beta''(t)}}\sqrt{-\log r}\exp[-(f(\alpha(t))-\beta(0))\log r] \qquad t<0.\end{aligned}$$

Corollary: (Local central limit theorem)

$$\sqrt{R}\mu(S_{n(R)}\phi+\alpha(0)R-u\sqrt{R}\in[a,b])\sim\frac{(b-a)e^{-u^2/2\beta''(0)}}{\sqrt{2\pi\beta''(0)}}.$$

Corollary: (Central limit theorem)

$$\begin{aligned}&\mu\left(\{x\in\Sigma:\frac{S_{n(R)}\phi+\alpha(0)R}{\sqrt{\beta''(0)R}}\leq u\}\right)\\ &\qquad\to\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u}\exp[-t^2/2]dt.\end{aligned}$$

Corollary:

$$\begin{aligned}&\mu\left(\{x\in\Sigma:\frac{\log[m(C_{n(-\log r)(x)}(x))]-\alpha(0)\log r}{\sqrt{-\beta''(0)\log r}}\leq u\}\right)\\ &\qquad\to\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u}\exp[-t^2/2]dt\end{aligned}$$

as $r\to 0$.

Corollary: For $t\geq 0$

$$\begin{aligned}&\lim_{r\to 0}-\frac{1}{\log r}\log\mu(\log[m(C_{n(-\log r)(x)}(x))]\geq-\beta'(t)\log r)\\ &\qquad=-t\beta'(t)+\beta(t)-\beta(0)=f(\alpha(t))-\beta(0),\end{aligned}$$

and for $t\leq 0$

$$\begin{aligned}&\lim_{r\to 0}-\frac{1}{\log r}\log\mu(\log[m(C_{n(-\log r)(x)}(x))]\leq-\beta'(t)\log r)\\ &\qquad=-t\beta'(t)+\beta(t)-\beta(0)=f(\alpha(t))-\beta(0).\end{aligned}$$

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