Vertex ranking of asteroidal triple-free graphs

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Abstract. We present an efficient algorithm for computing the vertex ranking number of an asteroidal triple-free graph. Its running time is bounded by a polynomial in the number of vertices and the number of minimal separators of the input graph.

1 Introduction

A vertex ranking of a graph is a vertex coloring by a linear ordered set of colors such that for every path in the graph with end vertices of the same color there is a vertex on this path with a higher color. The vertex ranking problem askes for a vertex ranking with a minimum number of colors.

Asteroidal triples in graphs were introduced in [15], where the interval graphs are characterized as those chordal graphs without asteroidal triples. In the meantime asteroidal triples (or ATs, for short) turned out to be an important concept. Their absence in a graph forces several nice structures [6], however, a characterization of AT-free graphs as intersection graphs is not known.

The AT-free graphs do not form a class of perfect graphs. Nevertheless efficient algorithms are known for some domination problems restricted to this class [7]. The complements of comparability graphs are an important subclass of AT-free graphs, which shows that the class of AT-free graphs is a relatively large class of graphs. While in general the number of minimal separators of an AT-free graph cannot be bounded by any polynomial of the number of vertices,

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this is the case for the complements of comparability graphs of partial orders of bounded dimension [2].

The dimension of a partial order is one of the most carefully studied parameters of a partial order [19]. Yannakakis [20] showed that determining whether a partial order has dimension at most d is NP-complete for any fixed $d \geq 3$. Furthermore, while many problems have been shown to be efficiently solvable on partial orders of dimension 2, no NP-complete partial order problem was known to be solvable by a polynomial time algorithm for partially ordered sets of some fixed dimension greater than 2. This changed when a polynomial time algorithm was found for the treewidth of a cocomparability graph of fixed dimension [14]. In this paper we show that for the vertex ranking problem a similar result can be obtained.

If the intersection model of a cocomparability graph is part of the input, it was already shown in [8] that the vertex ranking problem can be solved. However, if only the graph is given as input, this does not yield a polynomial time algorithm, since we cannot find the representation efficiently. Thus, until now, it was unclear whether the problem was easy because the class of graphs is well behaved, or because having the representation (which is the solution to an NP-complete problem) is such a powerful tool that it gave the solution.

Much work has been done in finding optimal rankings of trees. For trees there is now a linear time algorithm finding an optimal vertex ranking [16]. For the closely related edge ranking problem on trees an $O(n^3 \log n)$ algorithm was given in [18]. Efficient vertex ranking algorithms were known for very few other classes of graphs. The vertex ranking problem is trivial on split graphs and it is solvable in linear time on cographs [17]. Recently a $O(n^4)$ algorithm for vertex ranking of interval graphs was presented in [1]. The approach presented in [8] can be used to design a $O(n^3)$ algorithm computing an optimal vertex ranking for interval graphs. In [8] also and $O(n^6)$ algorithm was presented computing the vertex ranking of permutation graphs.

The decision problem 'Given a graph G and a positive integer k, has G a vertex ranking with at most k colors' is NP-complete, even when restricted to cobipartite or bipartite graphs [3]. In view of this it is interesting to notice that for each *constant* t, the class of graphs with vertex ranking number at most t is recognizable in linear time [3]. This follows from the fact that for each t, the class of graphs with vertex ranking number at most t is minor closed and from the recent results of Robertson and Seymour.

In [11], among other things, an $O(\sqrt{n})$ bound is given for the vertex ranking number of a planar graph and the authors describe a polynomial time algorithm which finds a ranking using only $O(\sqrt{n})$ colors. For graphs in general there is an approximation algorithm known with factor $O(\log^2 n)$ [4, 12]. In [4] it is also shown that one plus the *pathwidth* of a graph is a lower bound for the vertex ranking number of the graph (hence a planar graph has pathwidth $O(\sqrt{n})$, (which was also shown in [12] using different methods).

For definitions and properties of classes of well-structured graphs not given here we refer to [5, 9, 10].

In this paper we show that the vertex ranking problem can be solved efficiently for AT-free graphs with a polynomial number of minimal separators.

2 Preliminaries

2.1 Preliminaries on AT-free graphs

An independent set of three vertices is called an *asteroidal triple* if between each pair in the triple there exists a path that avoids the neighborhood of the third. A graph is *AT-free* if it does not contain an asteroidal triple. Recently the structure of AT-free graphs has been studied extensively (see [6]). The class of AT-free graphs contains various well-known graph classes, as e.g., interval, permutation, trapezoid and cocomparability graphs. A good reference for more information on all these subclasses is [10].

2.2 Preliminaries on separators

All graphs in this paper are simple and undirected. $G[W]$ will be the subgraph of G *induced* by the vertices of W.

Our main tool will be the set of minimal separators.

Definition 1. A vertex set S is an a,b*-separator* if the removal of S separates a and b in distinct connected components. If no proper subset of S is an a,b-separator then S is a *minimal* a,b*-separator.* S is a *minimal separator* if there exist non adjacent vertices a and b such that S is a minimal a,b-separator. An *inclusion minimal separator* is a separator such that no other separator is properly contained in it.

Definition 2. Let $G = (V, E)$ be a graph and let S and C be vertex sets of G. Then C is S*-full* in G if every vertex in S has a neighbor in C. We say that $G[C]$ is a *full component* of $G[V \setminus S]$ if C is S-full in G and $G[C]$ is a connected component of $G[V \setminus S]$.

Obviously a vertex set S is a minimal a,b-separator of G iff a and b are vertices in different full components of $G[V \setminus S]$. Furthermore, S is an inclusion minimal separator of G iff $G[V \setminus S]$ is disconnected and all connected components of $G[V \setminus S]$ are S-full in G.

Lemma 3. *Let $G = (V, E)$ be a graph with separator S_1. Let a and b be non-adjacent vertices in a connected component $G[C]$ of $G[V \setminus S]$. Let $S \subset C$ be a minimal a,b-separator in $G[C]$. Then there exists a minimal a,b-separator S_2 of G with the following properties:*

1. *$S \subseteq S_2 \subseteq S \cup S_1$*
2. *The connected component $G[A]$ of $G[C \setminus S]$ containing vertex a is a full component of $G[V \setminus S_2]$.*

Proof. Let $G[A]$ be the connected component of $G[C \setminus S]$ and let $G[B]$ be the connected component of $G[C \setminus S]$ containing b. Clearly, $S^* := S_1 \cup S$ separates a and b in G and the connected components containing a and b are $G[A]$ and $G[B]$ respectively.

Remove all vertices from S^* which do not have a neighbor in A. Let the resulting set be S_2. Notice that no vertex of S is removed in the process, hence $S \subseteq S_2 \subseteq S \cup S_1$. Since only vertices of S^* are removed which do not have a neighbor in $G[A]$, $G[A]$ is a full component of $G[V \setminus S_2]$. (Notice that when we remove vertices of S^*, the component of $G[V \setminus S^*]$ containing the vertex b may grow, but not the component containing a.) □

In [13] the following result is shown.

Theorem 4. *For a positive number R, let $\mathcal{G}(R)$ be the class of graphs with at most R minimal separators. There exists an algorithm running in time $O(n^5 R)$, which, for all R, given a graph G with n vertices, either detects that $G \notin \mathcal{G}(R)$ or lists all minimal separators in G.*

2.3 Preliminaries on rankings

Definition 5. Let $G = (V, E)$ be a graph and let t be some integer. A *(vertex) t-ranking* is a coloring $c : V \to \{1, \ldots, t\}$ such that for every pair of vertices x and y with $c(x) = c(y)$ and for every path between x and y there is a vertex z on this path with $c(z) > c(x)$. The *vertex ranking number* of G, $\chi_r(G)$, is the smallest value t for which the graph admits a t-ranking.

By definition a vertex ranking is a proper coloring. Hence $\chi_r(G) \geq \chi(G)$ for every graph G. We call a $\chi_r(G)$-ranking of G an optimal ranking. Clearly, $\chi_r(K_n) = n$, where K_n is a complete graph on n vertices. Furthermore, the vertex ranking number of a disconnected graph is equal to the maximum vertex ranking number of its components.

Lemma 6. *Let $G = (V, E)$ be connected, and let c be a t-ranking of G. Then there is at most* one *vertex x with $c(x) = t$.*

Proof. Assume there are two vertices with color t. Since G is connected, there is a path between these two vertices. By definition this path must contain a vertex with color at least $t + 1$. This is a contradiction. □

Remark. Notice that if c is a t-ranking of a graph G and H is a subgraph of G, then the restriction c' of c to the vertices of H is a t-ranking for H.

This observation together with Lemma 6 leads to the following lemma which appeared in [11].

Lemma 7. *A coloring $c : V \to \{1, \ldots, t\}$ is a t-ranking for a graph $G = (V, E)$ if and only if for each $1 \leq i \leq t$, each connected component of the subgraph $G[\{x \mid c(x) \leq i\}]$ of G has at most one vertex y with $c(y) = i$.*

The following theorem, presented in [8], is our main tool for designing efficient vertex ranking algorithms on special classes of graphs.

Theorem 8. *Let $G = (V, E)$ be a a graph and $\mathcal{S}$ a nonempty collection of subsets of V containing all inclusion minimal separators of G. Then*

$$\chi_r(G) = \min_{S \in \mathcal{S}} |S| + \max_{C} \chi_r(G[C])$$

where C ranges over the vertex sets of all connected components of $G[V \setminus S]$.

3 Preliminaries on blocks

We introduce blocks, which are a basic concept for our algorithm.

Definition 9. A k-block of G consists of k minimal separators $S_1, \ldots, S_k$ of G with $S_i \not\subseteq S_j$ whenever $i \neq j$ and of a vertex set C such that $G[C]$ is a connected component of $G[V \setminus (S_1 \cup \cdots \cup S_k)]$ and such that C is contained in a full component of $G[V \setminus S_i]$ for each $i = 1, \ldots, k$.

In case of AT-free graphs we only need 1-blocks and 2-blocks. For reasons of clarity we give the definitions explicitly.

Definition 10. A *1-block* is a pair (S, C), where S is a minimal separator and $G[C]$ is a S-full component of $G[V \setminus S]$.

We want to determine the vertex ranking number $\chi_r(G[C])$ for all 1-blocks (S, C), by decomposing it into smaller 1-blocks and 2-blocks. The vertex ranking for the graph G then follows from Theorem 8. We introduce 2-blocks.

Definition 11. A triple (S_1, C, S_3) is called *2-block* of a graph $G = (V, E)$ if S_1 and S_3 are minimal separators of G with $S_1 \not\subseteq S_3$ and $S_3 \not\subseteq S_1$, and $C \subset V$ induces a connected component $G[C]$ of $G[V \setminus (S_1 \cup S_3)]$ such that for $i = 1, 3$ there exist S_i-full components $G[C_i]$ of $G[V \setminus S_i]$ with $C \subseteq C_i$ and $S_{4-i} \setminus S_i \subseteq C_i$.

4 Decomposing 1-blocks

Consider a 1-block (S, C). If C is a clique then $\chi_r(G[C]) = |C|$. Henceforth, assume this is not the case.

The following theorem gives the decomposition of 1-blocks.

Theorem 12. *Let (S_1, C) be a 1-block of an AT-free graph $G = (V, E)$, let S be an inclusion minimal separator of $G[C]$, and let $G[A]$ be a connected component of $G[C \setminus S]$. Then there is a set S_2 such that (S_2, A) is a 1-block of G or (S_1, A, S_2) is a 2-block of G.*

Proof. We choose a vertex $a \in A$ and $b \in C \setminus (A \cup S)$. Then $S_1 \cup S$ is an a, b-separator of G. Let S_2 be a minimal a, b-separator with $S_2 \subseteq S_1 \cup S$. Then we have $S \subseteq S_2$ and A is contained in one full component of $G[V \setminus S_2]$.

If $S_2 = N(A) \setminus A$ then (S_2, A) is a 1-block of G.

Assume $N(A) \setminus (A \cup S_2) \neq \emptyset$. We show that (S_1, A, S_2) is a 2-block. Clearly, $S_2 \not\subseteq S_1$, since $S \not\subseteq S_1$. But also, $S_1 \not\subseteq S_2$, since there is a vertex of A with a neighbor in $S_1 \setminus S_2$.

Clearly, $S_2 \setminus S_1 = S$ is contained in the full component $G[C]$ of $G[V \setminus S_1]$.

Consider a full component $G[C^*]$ other than $G[C]$ of $G[V \setminus S_1]$. Hence every vertex of $S_1 \setminus S_2$ has a neighbor in C^*. It follows that there is connected component $G[D]$ of $G[V \setminus S_2]$ containing all vertices of $S_1 \setminus S_2$. Since there is a vertex of A with a neighbor in $S_1 \setminus S_2$, this component D contains all vertices of A. It follows that $G[D]$ is a full component of $G[V \setminus S_2]$, since every vertex of S_2 has a neighbor in A. This proves that (S_1, A, S_2) is a 2-block. □

5 Decomposing 2-blocks

We need the following lemma.

Lemma 13. *Let (S_1, C, S_3) be a 2-block of an AT-free graph $G = (V, E)$ and let S be an inclusion minimal separator of $G[C]$. Then for all minimal separators S_2 of G with $S \subseteq S_2 \subseteq S \cup S_1 \cup S_3$ holds $S_1 \subseteq S_2 \cup S_3$ or $S_3 \subseteq S_2 \cup S_1$ or the vertices of $S_1 \setminus S_2$ and $S_3 \setminus S_2$ are in different connected components of $G[V \setminus S_2]$.*

Proof. Suppose $S_1 \not\subseteq (S_2 \cup S_3)$ and $S_3 \not\subseteq (S_2 \cup S_1)$. Choose vertices s_1 and s_3 in $S_1 \setminus (S_2 \cup S_3)$ and in $S_3 \setminus (S_2 \cup S_1)$ respectively. Clearly, there also exists a vertex $s_2 \in S_2 \setminus (S_1 \cup S_3)$.

Assume $S_1 \setminus S_2$ and $S_3 \setminus S_2$ are contained in one connected component of $G[V \setminus S_2]$. Then there is another full component of $G[V \setminus S_2]$ without vertices of $S_1 \cup S_3$. Choose a vertex v_2 in this component. Choose a vertex vertices v_1 in a full component of $G[V \setminus S_1]$ that contains no vertices of $S_2 \cup S_3$ and a vertex v_3 in a full component of $G[V \setminus S_3]$ that contains no vertex of $S_1 \cup S_2$. Notice that there is a path from v_1 to v_3, via s_1 and s_3, avoiding the neighborhood of v_2. Similar paths exist from v_1 to v_2 and from v_3 to v_2. It follows that v_1, v_2 and v_3 form an AT. □

This section shows how 2-blocks are decomposed. Let (S_1, C, S_3) be a 2-block of an AT-free graph G. Clearly, if C is a clique, then the vertex ranking number of $G[C]$ is $|C|$. Assume henceforth this is not the case.

Theorem 14. *Let (S_1, C, S_3) be a 2-block of an AT-free graph $G = (V, E)$, let S be an inclusion minimal separator of $G[C]$, and let $G[A]$ be a connected component of $G[C \setminus S]$. Then there is a set S_2 such that (S_2, A) is a 1-block of G or one of (S_1, A, S_2) and (S_3, A, S_2) is a 2-block of G.*

Proof. We choose a vertex $a \in A$ and $b \in C \setminus (S \cup A)$. Then $S_1 \cup S \cup S_3$ is an a, b-separator of G. Let S_2 be a minimal a, b-separator with $S_2 \subseteq S_1 \cup S \cup S_3$. Then we have $S \subseteq S_2$ and A is in one full component of $G[V \setminus S_2]$.

First assume that there exist vertices s_1 and s_3 in $S_1 \cap N(A) \setminus (S_2 \cup S_3)$ and in in $S_3 \cap N(A) \setminus (S_2 \cup S_1)$ respectively. Then by lemma 13 s_1 and s_3 would belong to different connected components of $G[V \setminus S_2]$ contradicting the fact that s_1 and s_3 are both in $N(A)$.

If $S_2 = N(A) \setminus A$ then (S_2, A) is a 1-block of G.

Finally assume that $S_2 \subset N(A) \setminus A$. Without loss of generality we can now assume that that $N(A) \setminus (A \cup S_2) \subseteq S_1$. Then (S_1, A, S_2) is a 2-block of G which follows in the same manner as in the proof of Theorem 12. □

6 The algorithm

Assume our input is an AT-free graph $G = (V, E)$ on n vertices with R minimal separators. Using the algorithm of [13] we first compute the set Δ of all minimal separators of G. This takes time $O(n^5 R)$. Next we compute a list $\mathcal{B}$ of all vertex sets $C \subseteq V$ such that $C = V$ or there is a separator $S \in \Delta$ such that (S, C) is a 1-block of G or there exist separators $S_1, S_2 \in \Delta$ such that (S_1, C, S_2) is a 2-block of G. Creating list $\mathcal{B}$ is possible in time $O(n^2 R^2)$ since we have to run $R(R+1)/2$ times a subroutine computing connected components. We sort the elements of $\mathcal{B}$ by the number of vertices in time $O(n^2 R^2)$. Now the set $\mathcal{B}$ contains at most nR^2 different vertex sets $C \subseteq V$. For each $C \in \mathcal{B}$ we compute the vertex ranking number $\chi_r(G[C])$ in the following way.

If C is a clique of G then $\chi_r(G[C]) = |C|$. Otherwise C is representable as block of G and decomposes into smaller blocks with ranking numbers computed before. By theorem 8 we have

$$\chi_r(G[C]) = \min_{S \in \Delta} |S \cap C| + \max_{C'} \chi_r(G[C'])$$

where $G[C']$ ranges over all connected components of $G[C \setminus S]$. The last ranking number computed this way is $\chi_r(G)$ for $C = V$. By lemma 3 the set $\{S \cap C : S \in \Delta\}$ is non-empty and contains all minimal separators of $G[C]$. By Theorems 12 and 14 for the inclusion minimal separators of $G[C]$ the ranking numbers of the connected components $G[C']$ are computed before. For components $G[C']$ with $C' \notin \mathcal{B}$ we assume ranking number $|C'|$. These components cannot realize the minimum. Per component $G[C]$ we need time $O(n^2 R)$ for this step. Hence the total running time of our algorithm is $O(n^5 R + n^3 R^3)$.

7 Conclusion

We have given a efficient algorithm computing the ranking number of AT-free graphs. This generalizes in a non trivial way the result of [8]. Notice that in this paper we do not assume any intersection model. As a matter of fact, no intersection model for AT-free graphs is known.

To us it is unknown whether similar results as obtained in this paper hold for graphs in general. To be more precise, we do not know if there is an algorithm computing the vertex ranking number for graphs in general, with a running time polynomial in the number of vertices and the number of minimal separators.

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