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# Asymptotic normality of the deconvolution kernel density estimator under the vanishing error variance

Bert van Es

Korteweg-de Vries Institute for Mathematics  
Universiteit van Amsterdam  
Plantage Muidergracht 24  
1018 TV Amsterdam  
The Netherlands  
a.j.vanes@uva.nl

Shota Gugushvili

EURANDOM

Technische Universiteit Eindhoven  
P.O. Box 513  
5600 MB Eindhoven  
The Netherlands  
gugushvili@eurandom.tue.nl  
Phone: +31 40 2478113  
Fax: +31 40 2478190

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## Abstract

Let  $X_1, \dots, X_n$  be i.i.d. observations, where  $X_i = Y_i + \sigma_n Z_i$  and the  $Y$ 's and  $Z$ 's are independent. Assume that the  $Y$ 's are unobservable and that they have the density  $f$  and also that the  $Z$ 's have a known density  $k$ . Furthermore, let  $\sigma_n$  depend on  $n$  and let  $\sigma_n \rightarrow 0$  as  $n \rightarrow \infty$ . We consider the deconvolution problem, i.e. the problem of estimation of the density  $f$  based on the sample  $X_1, \dots, X_n$ . A popular estimator of  $f$  in this setting is the deconvolution kernel density estimator. We derive its asymptotic normality under two different assumptions on the relation between the sequence  $\sigma_n$  and the sequence of bandwidths  $h_n$ . We also consider several simulation examples which illustrate different types of asymptotics corresponding to the derived theoretical results and which show that there exist situations where models with  $\sigma_n \rightarrow 0$  have to be preferred to the models with fixed  $\sigma$ .

*Keywords:* Asymptotic normality, deconvolution, Fourier inversion, kernel type density estimator.

*AMS subject classification:* Primary 62G07; Secondary 62G20  
*Running title:* Deconvolution kernel density estimator

# 1 Introduction

The classical deconvolution problem consists of estimation of the density  $f$  of a random variable  $Y$  based on the i.i.d. copies  $Y_1, \dots, Y_n$  of  $Y$ , which are corrupted by an additive measurement error. More precisely, let  $X_1, \dots, X_n$  be i.i.d. observations, where  $X_i = Y_i + Z_i$  and the  $Y$ 's and  $Z$ 's are independent. Assume that the  $Y$ 's are unobservable and that they have the density  $f$  and also that the  $Z$ 's have a known density  $k$ . Such a model of measurements contaminated by an additive measurement error has numerous applications in practice and arises in a variety of fields, see for instance Carroll et al. (2006). Notice that the  $X$ 's have a density  $g$  which is equal to the convolution of  $f$  and  $k$ . The deconvolution problem consists in estimation of the density  $f$  based on the sample  $X_1, \dots, X_n$ .

A popular estimator of  $f$  is the deconvolution kernel density estimator, which was proposed in Carroll and Hall (1988) and Stefanski and Carroll (1990), see also pp. 231–233 in Wasserman (2007) for an introduction. Additional recent references can be found e.g. in van Es et al. (2008). Let  $w$  be a kernel and  $h_n > 0$  a bandwidth. The deconvolution kernel density estimator  $f_{nh_n}$  is constructed as

$$f_{nh_n}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \frac{\phi_w(h_n t) \phi_{emp}(t)}{\phi_k(t)} dt = \frac{1}{n} \sum_{j=1}^n \frac{1}{h_n} w_{h_n} \left( \frac{x - X_j}{h_n} \right), \quad (1)$$

where  $\phi_{emp}$  denotes the empirical characteristic function, i.e.  $\phi_{emp}(t) = n^{-1} \sum_{j=1}^n \exp(itX_j)$ ,  $\phi_w$  and  $\phi_k$  are Fourier transforms of functions  $w$  and  $k$ , respectively, and

$$w_{h_n}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \frac{\phi_w(t)}{\phi_k(t/h_n)} dt.$$

Depending on the rate of decay of the characteristic function  $\phi_k$  at plus and minus infinity, deconvolution problems are usually divided into two groups, ordinary smooth deconvolution problems and supersmooth deconvolution problems. In the first case it is assumed that  $\phi_k$  decays to zero at plus and minus infinity algebraically (an example of such  $k$  is the Laplace density) and in the second case the decay is essentially exponential (in this case  $k$  can be e.g. a standard normal density). In general, the faster  $\phi_k$  decays at plus and minus infinity (and consequently smoother the density  $k$  is), the more difficult the deconvolution problem becomes, see e.g. Fan (1991a). The usual smoothness condition imposed on the target density  $f$  is that it belongs to the class  $\mathcal{C}_{\alpha,L} = \{f : |f^{(\ell)}(x) - f^{(\ell)}(x+t)| \leq L|t|^{\alpha-\ell} \text{ for all } x \text{ and } t\}$ , where  $\alpha > 0$ ,  $\ell = \lfloor \alpha \rfloor$  (the integer part of  $\alpha$ ) and  $L > 0$  are known constants, cf. Fan (1991a). Then, if  $k$  is ordinary smooth of order  $\beta$  (see e.g. Assumption C (ii) below for a definition), the optimal rate of convergence for the estimator  $f_{nh_n}(x)$  with the mean square error used as the performance criterion

is  $n^{-\alpha/(2\alpha+2\beta+1)}$ , while if  $k$  is supersmooth of order  $\lambda$  (see Assumption B (ii)), the optimal rate of convergence is  $(\log n)^{-\alpha/\lambda}$ , see Fan (1991a). The latter convergence rate is rather slow and it suggests that the deconvolution problem is not practically feasible in the supersmooth case, since it seems samples of very large size are required to obtain reasonable estimates. Hence at first sight it appears that the nonparametric deconvolution with e.g. the Gaussian error distribution (a popular choice in practice) cannot lead to meaningful results for moderate sample sizes and is practically irrelevant. However, it was demonstrated by exact MISE (mean integrated square error) computations in Wand (1998) that, despite the slow convergence rate in the supersmooth case, the deconvolution kernel density estimator performs well for reasonable sample sizes, if the noise level measured by the noise-to-signal ratio  $\text{NSR} = \text{Var}[Z](\text{Var}[Y])^{-1}100\%$ , cf. Wand (1998), is not too high. Clearly, an ‘ideal case’ in a deconvolution problem would be that not only the sample size  $n$  is large, but also that the error term variance is small. This leads one to an idealised model  $X = Y + \sigma_n Z$ , where now  $\text{Var}[Z] = 1$  and  $\sigma_n$  depends on  $n$  and tends to zero as  $n \rightarrow \infty$ . The idea to consider  $\sigma_n \rightarrow 0$  was already proposed in Fan (1992) and was further developed in Delaigle (2008). We refer to these works for additional motivation. These papers deal mainly with the mean integrated square error of the estimator of  $f$ . Here we will study its asymptotic normality. Asymptotic normality of the deconvolution kernel density estimator in the deconvolution problem with fixed error term variance was derived in Fan (1991b) and van Es and Uh (2004, 2005). For a practical situation where  $\sigma_n \rightarrow 0$  can arise, see e.g. Section 4.2 of Delaigle (2008), where an example of measurement of sucrase in intestinal tissues is considered and inference is drawn on the density of the sucrase content. Sucrase is a name of several enzymes that catalyse the hydrolysis of sucrose to fructose and glucose.

It trivially follows from (1) that the deconvolution kernel density estimator for the model that we consider, i.e.  $X_i = Y_i + \sigma_n Z_i$  with  $\sigma_n \rightarrow 0$  as  $n \rightarrow \infty$ , is defined as

$$f_{nh_n}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \frac{\phi_w(h_n t) \phi_{emp}(t)}{\phi_k(\sigma_n t)} dt = \frac{1}{n} \sum_{j=1}^n \frac{1}{h_n} w_{r_n} \left( \frac{x - X_j}{h_n} \right), \quad (2)$$

where

$$w_{r_n}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \frac{\phi_w(t)}{\phi_k(r_n t)} dt, \quad (3)$$

$r_n = \sigma_n/h_n$  and  $\phi_k$  now denotes the characteristic function of the random variable  $Z$  with a density  $k$ . We will also use  $\rho_n = r_n^{-1} = h_n/\sigma_n$  and in this case we will denote the function  $w_{r_n}$  by  $w_{\rho_n}$ . Observe that if  $w$  is symmetric, (2) will be real-valued.

To get a consistent estimator, we need to control the bandwidth  $h_n$ . The usual condition to get consistency in kernel density estimation is that the

bandwidth  $h_n$  depends on  $n$  and is such that  $h_n \rightarrow 0, nh_n \rightarrow \infty$ , see e.g. Theorem 6.27 in Wasserman (2007). Since in our model we assume  $\sigma_n \rightarrow 0$ , additional assumptions on  $h_n$ , which relate it to  $\sigma_n$ , are needed. In essence we distinguish two cases:  $\sigma_n/h_n \rightarrow r$  with  $0 \leq r < \infty$ , or  $\sigma_n/h_n \rightarrow \infty$ . Conditions on the target density  $f$ , the density  $k$  of  $Z$  and kernel  $w$  will be tailored to these two cases.

The remaining part of the paper is organised as follows: in Section 2 we will present the obtained results. Section 3 contains several simulation examples illustrating the results from Section 2. All the proofs are given in Section 4.

## 2 Results

### 2.1 The case $0 \leq r < \infty$

We first consider the case when  $0 \leq r < \infty$ . We will need the following conditions on  $f$ ,  $w$ ,  $k$  and  $h_n$ .

**Assumption A.**

- (i) The density  $f$  is such that  $\phi_f$  is integrable.
- (ii)  $\phi_k(t) \neq 0$  for all  $t \in \mathbb{R}$  and  $\phi_k$  has a bounded derivative.
- (iii) The kernel  $w$  is symmetric, bounded and continuous. Furthermore,  $\phi_w$  has support  $[-1, 1]$ ,  $\phi_w(0) = 1$ ,  $\phi_w$  is differentiable and  $|\phi_w(t)| \leq 1$ .
- (iv) The bandwidth  $h_n$  depends on  $n$  and we have  $h_n \rightarrow 0, nh_n \rightarrow \infty$ .
- (v)  $\sigma_n \rightarrow 0$  and  $r_n = \sigma_n/h_n \rightarrow r$ , where  $0 \leq r < \infty$ .

Notice that Assumption A (i) implies that  $f$  is continuous and bounded. Assumption  $\phi_k(t) \neq 0$  for all  $t \in \mathbb{R}$  is standard in kernel deconvolution and is unavoidable when using the Fourier inversion approach to deconvolution. Furthermore, a variety of kernels satisfy Assumption A (iii), see e.g. examples in van Es and Uh (2005). Also notice that  $w$  is not necessarily a density, since it may take on negative values. Observe that in Assumption A (v) we do not exclude the case  $r = 0$ .

The following theorem establishes asymptotic normality in this case.

**Theorem 1.** *Let Assumption A hold and let the estimator  $f_{nh_n}$  be defined by (2). Then*

$$\sqrt{nh_n}(f_{nh_n}(x) - \mathbb{E}[f_{nh_n}(x)]) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, f(x) \int_{-\infty}^{\infty} |w_r(u)|^2 du\right) \quad (4)$$

as  $n \rightarrow \infty$ .

Notice that unlike the asymptotic normality theorem for the deconvolution kernel density estimator in the supersmooth deconvolution problem

with fixed  $\sigma$ , that was obtained in van Es and Uh (2004, 2005), the asymptotic variance in (4) now depends on  $f$ . When  $r_n = 0$  for all  $n$ , we recover the asymptotic normality theorem for an ordinary kernel density estimator, see Parzen (1962).

## 2.2 The case $r = \infty$

We turn to the case  $r = \infty$ . In this case we have to make the distinction between the ordinary smooth and supersmooth deconvolution problems. We first consider the supersmooth case. We will need the following condition.

### Assumption B.

- (i) The density  $f$  is such that  $\phi_f$  is integrable.
- (ii)  $\phi_k(t) \neq 0$  for all  $t \in \mathbb{R}$  and  $\phi_k(t) \sim C|t|^{\lambda_0} \exp(-|t|^\lambda/\mu)$  for some constants  $\lambda > 1, \mu > 0$  and real constants  $\lambda_0$  and  $C$ .
- (iii)  $w$  is a bounded, symmetric and continuous function. Furthermore,  $\phi_w$  is supported on  $[-1, 1]$ ,  $\phi_w(0) = 1$  and  $|\phi_w(t)| \leq 1$ . Moreover,

$$\phi_w(1-t) = At^\alpha + o(t^\alpha)$$

as  $t \downarrow 0$ , where  $A \in \mathbb{R}$  and  $\alpha \geq 0$  are some numbers.

- (iv) The bandwidth  $h_n$  depends on  $n$  and we have  $h_n \rightarrow 0, nh_n \rightarrow \infty$ .
- (v)  $\sigma_n \rightarrow 0$  and  $\sigma_n^\lambda/h_n^{\lambda-1} \rightarrow \infty$ .

Assumption B (i)-(iv) correspond to those in van Es and Uh (2005). Assumption B (v) is stronger than  $\sigma_n/h_n \rightarrow \infty$ , but it is essential in the proof of Theorem 2. Denote  $\zeta(\rho_n) = \exp(1/(\mu\rho_n^\lambda))$ . The following theorem holds true.

**Theorem 2.** *Let Assumption B hold and let the estimator  $f_{nh_n}$  be defined by (2). Furthermore, assume that  $\mathbb{E}[Y_j^2] < \infty$  and  $\mathbb{E}[Z_j^2] < \infty$ . Then*

$$\frac{\sqrt{n}\sigma_n}{\rho_n^{\lambda(1+\alpha)+\lambda_0-1}\zeta(\rho_n)}(f_{nh_n}(x) - \mathbb{E}[f_{nh_n}(x)]) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \frac{A^2}{2\pi^2C^2} \left(\frac{\mu}{\lambda}\right)^{2+2\alpha} (\Gamma(\alpha+1))^2\right) \quad (5)$$

as  $n \rightarrow \infty$ .

When  $\sigma_n = 1$  for all  $n$ , the arguments given in the proof of this theorem are still valid, and hence we can also recover the asymptotic normality theorem of van Es and Uh (2005) for the deconvolution kernel density estimator in the supersmooth deconvolution problem.

Finally, we consider the ordinary smooth case.

### Assumption C.

- (i) The density  $f$  is such that  $\phi_f$  is integrable.

(ii)  $\phi_k(t) \neq 0$  for all  $t \in \mathbb{R}$  and  $\phi_k(t)t^\beta \rightarrow C, \phi'_k(t)t^{\beta+1} \rightarrow -\beta C$  as  $t \rightarrow \infty$ , where  $\beta \geq 0$  and  $C \neq 0$  are some constants.

(iii)  $\phi_w$  is symmetric and continuously differentiable. Furthermore,  $\phi_w$  is supported on  $[-1, 1]$ ,  $|\phi_w(t)| \leq 1$  and  $\phi_w(0) = 1$ .

(iv) The bandwidth  $h_n$  depends on  $n$  and we have  $h_n \rightarrow 0, nh_n \rightarrow \infty$ .

(v)  $\sigma_n \rightarrow 0$  and  $\sigma_n/h_n \rightarrow \infty$ .

For the discussion on Assumption C (i)–(iv) see Fan (1991b).

**Theorem 3.** *Let Assumption C hold and let the estimator  $f_{nh_n}$  be defined by (2). Then*

$$\sqrt{nh_n\rho_n^{2\beta}}(f_{nh_n}(x) - \mathbb{E}[f_{nh_n}(x)]) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \frac{f(x)}{2\pi C^2} \int_{-\infty}^{\infty} |t|^{2\beta} |\phi_w(t)|^2 dt\right) \quad (6)$$

as  $n \rightarrow \infty$ .

When  $\sigma_n = 1$ , we recover the asymptotic normality theorem of Fan (1991b) for a deconvolution kernel density estimator in the ordinary smooth deconvolution problem.

As a general conclusion, we notice that Theorems 1–3 demonstrate that the asymptotics of  $f_{nh_n}(x)$  depend in an essential way on the relationship between the sequences  $\sigma_n$  and  $h_n$ . In case  $r_n \rightarrow r < \infty$ , the asymptotics are similar to those in the direct density estimation, while when  $r = \infty$ , they resemble those in the classical deconvolution problem.

### 3 Simulation examples

In this section we consider several simulation examples for the supersmooth deconvolution case covered by Theorems 1 and 2. We do not pretend to produce an exhaustive simulation study. Our examples serve as a mere illustration of the asymptotic results from the previous section.

It follows from Theorems 1–3 that for a fixed point  $x$  and a large enough  $n$ , a suitably centred and normalised estimator  $f_{nh_n}(x)$  is approximately normally distributed with mean and standard deviation given in these three theorems. Suppose we have fixed the sample size  $n$  and the bandwidth  $h_n$ , generated a sample of size  $n$ , evaluated the estimate  $f_{nh_n}(x)$  and have repeated this procedure  $N$  times, where  $N$  is sufficiently large. This will give us  $N$  values of  $f_{nh_n}(x)$ . We then can evaluate the sample mean and the sample standard deviation of this set of values  $f_{nh_n}(x)$ . Under appropriate conditions these should be close to the ones predicted by Theorems 1 and 2. In particular, in the setting of Theorem 1, the mean  $M$  and the standard deviation  $SD$  must be approximately given by

$$M = f * w_{h_n}(x), \quad SD = \frac{1}{\sqrt{nh_n}} f(x) \int_{-\infty}^{\infty} |w_{\sigma_n/h_n}(u)|^2 du, \quad (7)$$



while in the setting of Theorem 2 they are approximately equal to

$$M = f * w_{h_n}(x), \quad SD = \frac{A}{\sqrt{2}\pi C} \left(\frac{\mu}{\lambda}\right)^{1+\alpha} \Gamma(\alpha + 1) \frac{\rho_n^{\lambda(1+\alpha)+\lambda_0-1} \zeta(\rho_n)}{\sqrt{n}\sigma_n}. \quad (8)$$

We first concentrate on Theorem 1. Let  $f$  and  $k$  be standard normal densities, let  $n = 1000$  and suppose  $\sigma_n = 0.1$ . The noise level measured by the noise-to-signal ratio is thus rather low and equals  $\text{NSR} = 1\%$ . Suppose that a kernel  $w$  is given by

$$w(x) = \frac{48 \cos x}{\pi x^4} \left(1 - \frac{15}{x^2}\right) - \frac{144 \sin x}{\pi x^5} \left(2 - \frac{5}{x^2}\right). \quad (9)$$

Its corresponding Fourier transform is given by  $\phi_w(t) = (1 - t^2)^3 1_{[-1,1]}(t)$ . Here  $A = 8$  and  $\alpha = 3$ . A good performance of this kernel in deconvolution context was established in Delaigle and Hall (2006). Assume that the number of replications  $N = 500$ . Before we proceed any further, we need to fix the bandwidth. We opted for a theoretically optimal bandwidth, i.e. the bandwidth that minimises

$$\text{MISE}[f_{nh_n}] = \text{E} \left[ \int_{-\infty}^{\infty} (f_{nh_n}(x) - f(x))^2 dx \right], \quad (10)$$

the mean-squared error of the estimator  $f_{nh}$ . To find this optimal bandwidth, we considered a sequence of bandwidths  $h = 0.01 * k, k = 1, 2, \dots, K$ , where  $K$  is a large enough integer, passed to the Fourier transforms in (10) via Parseval's identity, cf. Wand (1998), and then used the numerical integration. This procedure resulted in  $h_n = 0.1$ . For real data the above method does not work, because (10) depends on the unknown  $f$ , and we refer to Delaigle (2008) for data-dependent bandwidth selection methods. However, once again we stress the fact that in order to reach a specific goal of these simulation examples, the bandwidth  $h_n$  must be the same for all  $N$  replications. This excludes the use of a data-dependent procedure. To speed up the computation of the estimates, binning of observations was used, see e.g. Silverman (1982) and Jones and Lotwick (1984) for related ideas in kernel density estimation.

Under these assumptions we evaluated the sample means and standard deviations of  $f_{nh_n}(x)$  for  $x$  from a grid on the interval  $[-3, 3]$  with mesh size  $\Delta = 0.1$ . These then were plotted in Figure 1 together with the theoretical values from (7). We notice that the sample means match the theoretical values very well. This can be also explained by the fact that the bandwidth  $h_n$  is quite small. The match between the sample standard deviations and the theoretical standard deviations is slightly less satisfactory. It also turns out that Theorem 2 is clearly not applicable in this case: an evaluation of the theoretical standard deviation  $SD$  in (8) yields a very large value 3.41646, which grossly overestimates the sample standard deviation for any point  $x$ .

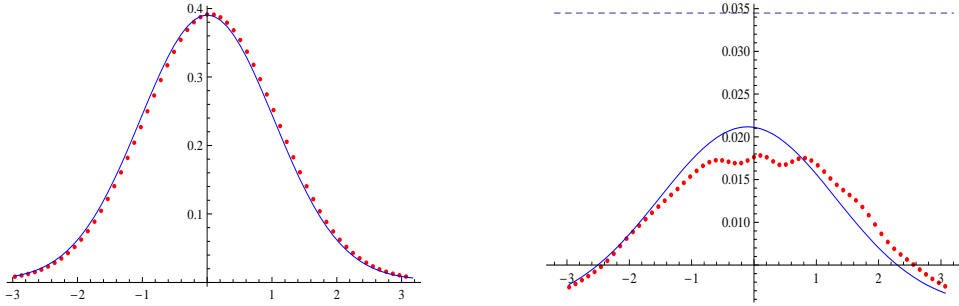


Figure 1: The sample means and the theoretical means (left display, a dotted and a solid line, respectively) together with the sample standard deviations and the two theoretical standard deviations corresponding to Theorems 1 and 2 (right display, a dotted, a solid and a dashed line, respectively). Here the target density  $f$  and the density  $k$  of a random variable  $Z$  are standard normal densities, the noise variance  $\sigma_n^2 = 0.01$ , the sample size  $n = 1000$ , the bandwidth  $h_n = 0.1$  and the kernel  $w$  is given by (9). The number of replications equals  $N = 500$ . The integral in (11) and not its asymptotic expansion was used to evaluate the standard deviation in Theorem 2.

The reason for this seems to be that both the sample size  $n$  and the error variance  $\sigma_n^2$  appear to be too small for the setting of Theorem 2.

At this point the following remark is in order. Reviewing the proof of Theorem 2, one sees that the following asymptotic equivalence is used:

$$\int_0^1 \phi_w(s) \exp[s^\lambda / (\mu h^\lambda)] ds \sim A\Gamma(\alpha + 1) \left(\frac{\mu}{\lambda} h^\lambda\right)^{1+\alpha} e^{1/(\mu h^\lambda)} \quad (11)$$

as  $h \rightarrow 0$ . This explains the shape of the normalising constant in Theorem 2. However, the direct numerical evaluation of the integral in (11) (with the same parameters and the kernel as in our example above) shows that the approximation in (11) is good only for very small values of  $h$  and that it is quite inaccurate for larger values of  $h$ , see a discussion in van Es and Gugushvili (2008). Obviously, one can correct for the poor approximation of the sample standard deviation by the theoretical standard deviation by using the left-hand side of (11) instead of its approximation. Nevertheless, this still leads to a very large (compared to the sample standard deviation) value of the theoretical standard deviation for our particular example, namely 0.034477.

In our second example we left  $\sigma_n, n$  and  $k$  the same as above, but as  $f$  we took a mixture of two normal densities with means  $-1$  and  $1$  and equal variance  $0.375$ . The mixing probability was taken to be equal to  $0.5$ . The density  $f$  is bimodal and is plotted in Figure 2. The simulation results for this density are reported in Figure 3. The conclusions are the same as for the first example. One can easily recognise a bimodal shape of the target density  $f$  by looking at the sample standard deviation.

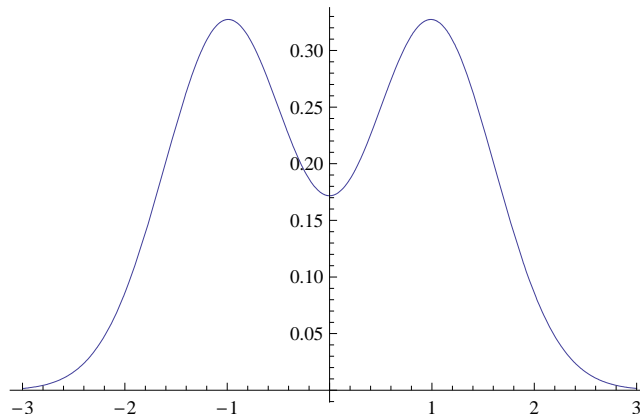


Figure 2: The density  $f$ : a mixture of two normal densities with means  $-1$  and  $1$  and equal variance  $0.375$ . The mixing probability is taken to be equal to  $0.5$ .

In our third example we again considered the standard normal density, but we increased the sample size to  $n = 10000$ . The results are reported in Figure 4. As can be seen, the match between the sample standard deviations and the theoretical standard deviations as computed using Theorem 1 is less satisfactory than in the previous example. The explanation lies in the fact that, even though the noise level is low when judged by itself, it is still a bit large compared to the sample size that we have in this case. Also Theorem 2 remains unapplicable, as it still produces considerably larger values of the theoretical standard deviation compared to the sample standard deviation ( $0.0166319$  after the necessary correction using (11)).

In the next three examples we kept the standard normal densities  $f$  and  $k$ , but increased the sample size  $n$  to  $100000$ . The error variance  $\sigma_n^2$  was consecutively taken to be  $0.01, 1$  and  $4$ , i.e. we considered three different noise levels,  $1\%$ ,  $100\%$  and  $400\%$ . A transition from the asymptotics described by Theorem 1 to those described by Theorem 2 is clearly visible in the resulting plots, see Figures 5–7. Figure 5 also indicates that there exist intermediate situations not immediately covered by either of the two theorems. Notice that Figure 7 seems to confirm a general, albeit not intuitive message of Theorem 2, which says that the asymptotic standard deviation does not depend on a point  $x$ , but only on the error density  $k$ : there is a large neighbourhood around zero for which the sample standard deviation is almost constant.

In our final example we considered the case when the density  $f$  is again a mixture of two normal densities (see above for details). The simulation results for this density are reported in Figure 8. In this last example the bandwidth  $h_n = 0.44$  was on purpose not selected as a minimiser of  $\text{MISE}[f_{nh_n}]$ , but was taken to be the same as when estimating a standard normal density (see Figure 7 above). Notice that the sample standard deviation is almost constant in the neighbourhood of the origin and is of the same magnitude

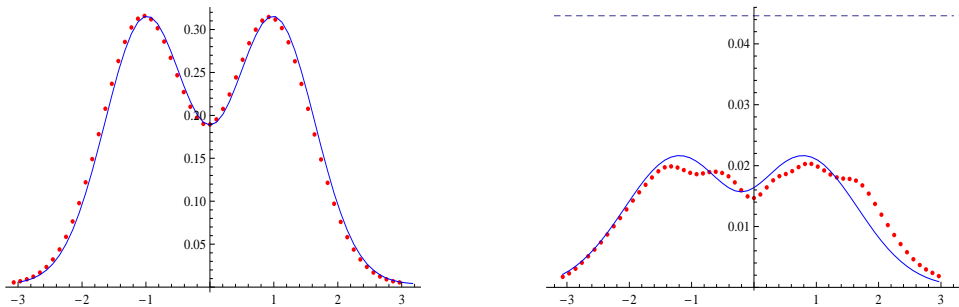


Figure 3: The sample means and the theoretical means (left display, a dotted and a solid line, respectively) together with the sample standard deviations and the two theoretical standard deviations corresponding to Theorems 1 and 2 (right display, a dotted, a solid and a dashed line, respectively). Here the target density  $f$  is a mixture of two normal densities with means equal to  $-1$  and  $1$  and the same variance  $0.375$ , the mixing probability is  $0.5$ , the density  $k$  of a random variable  $Z$  is a standard normal density, the noise variance  $\sigma_n^2 = 0.01$ , the sample size  $n = 1000$ , the bandwidth  $h_n = 0.08$  and the kernel  $w$  is given by (9). The number of replications equals  $N = 500$ . The integral in (11) and not its asymptotic expansion was used to evaluate the standard deviation in Theorem 2.

as the one depicted in Figure 7. This seems to provide an additional confirmation of the statement of Theorem 2, which says that the limit variance of the estimator  $f_{nh_n}$  does not depend on the target density  $f$ . Also notice that because of the fact that  $h_n$  is relatively large, the smoothed version of  $f$ , i.e.  $f * w_{h_n}$ , is unimodal instead of being bimodal.

As a preliminary conclusion (we also considered some other examples not reported here), our simulation examples seem to suggest that the asymptotics given by Theorem 2 correspond to the less realistic scenarios of high noise level and very large sample size. This provides further motivation for the study of deconvolution problems under the assumption  $\sigma_n \rightarrow 0$  as  $n \rightarrow \infty$ .

## 4 Proofs

To prove Theorem 1, we will need the following modification of Bochner's lemma, see Parzen (1962) for the latter.

**Lemma 1.** *Suppose that for all  $y$  we have  $K_n(y) \rightarrow K(y)$  as  $n \rightarrow \infty$  and that  $\sup_n |K_n(y)| \leq K^*(y)$ , where the function  $K^*$  is such that  $\int_{-\infty}^{\infty} K^*(y) dy < \infty$  and  $\lim_{y \rightarrow \infty} y K^*(y) = 0$ . Furthermore, suppose that  $g_n$  is a sequence of densities, such that*

$$\lim_{n \rightarrow \infty} \sup_{|u| \leq \epsilon_n} |g_n(x - u) - f(x)| \rightarrow 0 \quad (12)$$

for some sequence  $\epsilon_n \downarrow 0$ , such that  $\epsilon_n/h_n \rightarrow \infty$  as  $n \rightarrow \infty$  for a sequence

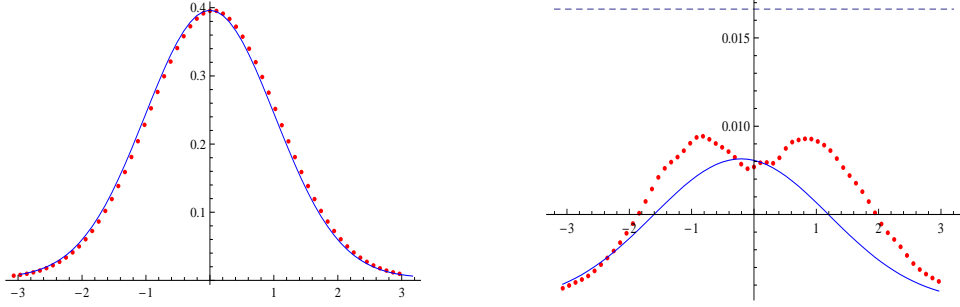


Figure 4: The sample means and the theoretical means (left display, a dotted and a solid line, respectively) together with the sample standard deviations and the two theoretical standard deviations corresponding to Theorems 1 and 2 (right display, a dotted, a solid and a dashed line, respectively). Here the target density  $f$  and the density  $k$  of a random variable  $Z$  are standard normal densities, the noise variance  $\sigma_n^2 = 0.01$ , the sample size  $n = 10000$ , the bandwidth  $h_n = 0.07$  and the kernel  $w$  is given by (9). The number of replications equals  $N = 500$ . The integral in (11) and not its asymptotic expansion was used to evaluate the standard deviation in Theorem 2.

$h_n \rightarrow 0$ . Then

$$\lim_{n \rightarrow \infty} \frac{1}{h_n} \int_{-\infty}^{\infty} K_n \left( \frac{x-y}{h_n} \right) g_n(y) dy = f(x) \int_{-\infty}^{\infty} K(y) dy. \quad (13)$$

*Proof.* The proof follows the same lines as the proof of Lemma 2.1 in Fan (1991b). We have

$$\begin{aligned} & \left| \frac{1}{h_n} \int_{-\infty}^{\infty} K_n \left( \frac{x-y}{h_n} \right) g_n(y) dy - f(x) \int_{-\infty}^{\infty} K(y) dy \right| \\ & \leq \left| \frac{1}{h_n} \int_{-\infty}^{\infty} K_n \left( \frac{x-y}{h_n} \right) g_n(y) dy - f(x) \frac{1}{h_n} \int_{-\infty}^{\infty} K_n \left( \frac{y}{h_n} \right) dy \right| \\ & \quad + f(x) \left| \int_{-\infty}^{\infty} K_n(y) dy - \int_{-\infty}^{\infty} K(y) dy \right| = I + II. \end{aligned}$$

Notice that  $II$  converges to zero by the dominated convergence theorem. We turn to  $I$ . Splitting the integration region into the sets  $\{|u| \leq \epsilon_n\}$  and  $\{|u| > \epsilon_n\}$  for some  $\epsilon_n > 0$ , we obtain that

$$\begin{aligned} I & \leq \left| \int_{\{|u| \leq \epsilon_n\}} (g_n(x-u) - f(x)) \frac{1}{h_n} K_n \left( \frac{u}{h_n} \right) du \right| \\ & \quad + \left| \int_{\{|u| > \epsilon_n\}} (g_n(x-u) - f(x)) \frac{1}{h_n} K_n \left( \frac{u}{h_n} \right) du \right| \\ & = III + IV. \end{aligned}$$

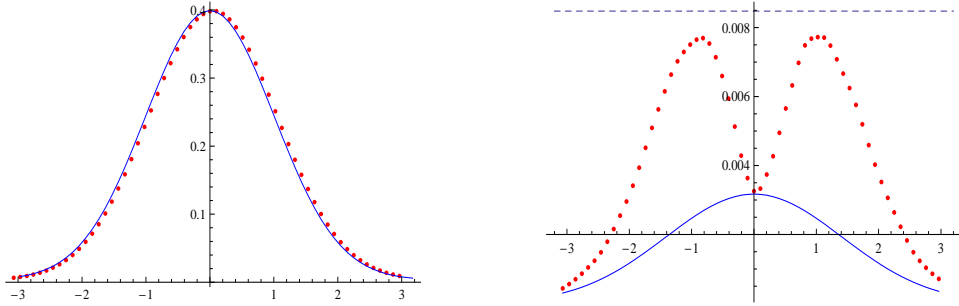


Figure 5: The sample means and the theoretical means (left display, a dotted and a solid line, respectively) together with the sample standard deviations and the two theoretical standard deviations corresponding to Theorems 1 and 2 (right display, a dotted, a solid and a dashed line, respectively). Here the target density  $f$  and the density  $k$  of a random variable  $Z$  are standard normal densities, the noise variance  $\sigma_n^2 = 0.01$ , the sample size  $n = 100000$ , the bandwidth  $h_n = 0.05$  and the kernel  $w$  is given by (9). The number of replications equals  $N = 500$ . The integral in (11) and not its asymptotic expansion was used to evaluate the standard deviation in Theorem 2.

For *III* we have

$$III \leq \sup_{|u| \leq \epsilon_n} |g_n(x-u) - f(x)| \int_{-\infty}^{\infty} K^*(u) du.$$

By (12) the right-hand side of the above expression vanishes as  $n \rightarrow \infty$ . Now we consider *IV*. Using the fact that  $g_n$  is a density (and hence that it is positive and integrates to one), we have

$$\begin{aligned} IV &\leq \int_{|u| > \epsilon_n} g_n(x-u) \frac{1}{h_n} \left| K^* \left( \frac{u}{h_n} \right) \right| du + f(x) \int_{|u| > \epsilon_n} \frac{1}{h_n} K^* \left( \frac{u}{h_n} \right) du \\ &\leq \frac{1}{\epsilon} \sup_{|y| > \epsilon_n/h_n} |y K^*(y)| + f(x) \int_{|y| > \epsilon_n/h_n} K^*(y) dy. \end{aligned}$$

Notice that the right-hand side in the last inequality vanishes as  $n \rightarrow \infty$ , because we assumed that  $\epsilon_n/h_n \rightarrow \infty$ . Combination of these results yields the statement of the lemma.  $\square$

*Proof of Theorem 1.* The main steps of the proof are similar to those on pp. 1069–1070 of Parzen (1962). Let  $\delta$  be an arbitrary positive number. Denote

$$V_{nj} = \frac{1}{h_n} w_{r_n} \left( \frac{x - X_j}{h_n} \right),$$

where  $w_{r_n}$  is defined by (3) and notice that (2) is an average of the i.i.d. random variables  $V_{n1}, \dots, V_{nn}$ . We have

$$\text{Var}[V_{nj}] = \text{E}[V_{nj}^2] - (\text{E}[V_{nj}])^2. \quad (14)$$

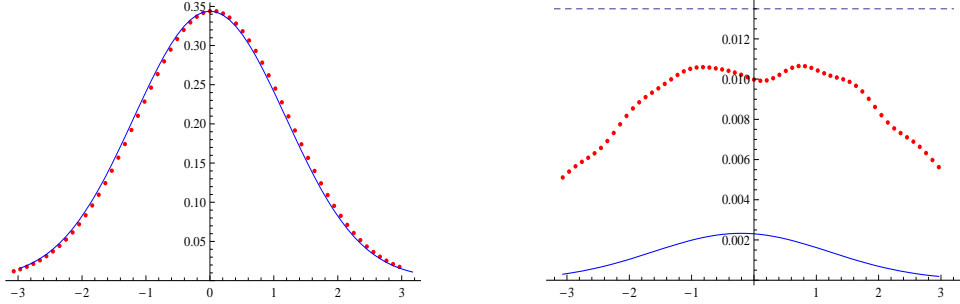


Figure 6: The sample means and the theoretical means (left display, a dotted and a solid line, respectively) together with the sample standard deviations and the two theoretical standard deviations corresponding to Theorems 1 and 2 (right display, a dotted, a solid and a dashed line, respectively). Here the target density  $f$  and the density  $k$  of a random variable  $Z$  are standard normal densities, the noise variance  $\sigma_n^2 = 1$ , the sample size  $n = 100000$ , the bandwidth  $h_n = 0.24$  and the kernel  $w$  is given by (9). The number of replications equals  $N = 500$ . The integral in (11) and not its asymptotic expansion was used to evaluate the standard deviation in Theorem 2.

Observe that

$$\mathbb{E}[V_{nj}^2] = \int_{-\infty}^{\infty} \frac{1}{h_n^2} \left| w_{r_n} \left( \frac{x-y}{h_n} \right) \right|^2 g_n(y) dy, \quad (15)$$

where  $g_n$  denotes the density of  $X_j$ . Integration by parts gives

$$w_{r_n}(u) = \frac{1}{iu} \int_{-1}^1 e^{-itu} \left( \frac{\phi_w(t)}{\phi_k(r_nt)} \right)' dt,$$

and hence

$$|w_{r_n}(u)| \leq \frac{1}{|u|} \int_{-1}^1 \left| \frac{\phi_w'(t)\phi_k(r_nt) - r_n\phi_w(t)\phi_k'(r_nt)}{(\phi_k(r_nt))^2} \right| dt$$

Furthermore,  $\lim_{n \rightarrow \infty} r_n = r < \infty$  implies that there exists a positive number  $a$ , such that  $\sup r_n \leq a < \infty$ . Notice that

$$\inf_{t \in [-1,1]} |\phi_k(r_nt)| = \inf_{s \in [-r_n, r_n]} |\phi_k(s)| \geq \inf_{s \in [-a, a]} |\phi_k(s)|.$$

Therefore

$$|w_{r_n}(u)| \leq c_{ak} \frac{1}{|u|} \int_{-1}^1 (|\phi_w'(t)| + |\phi_w(t)|) dt, \quad (16)$$

where the constant  $c_{ak}$  does not depend on  $n$ , but only on the density  $k$  and the number  $a$ . On the other hand

$$|w_{r_n}(u)| \leq \frac{1}{2\pi} \int_{-1}^1 \frac{|\phi_w(t)|}{\inf_{s \in [-a, a]} |\phi_k(s)|} dt < \infty. \quad (17)$$

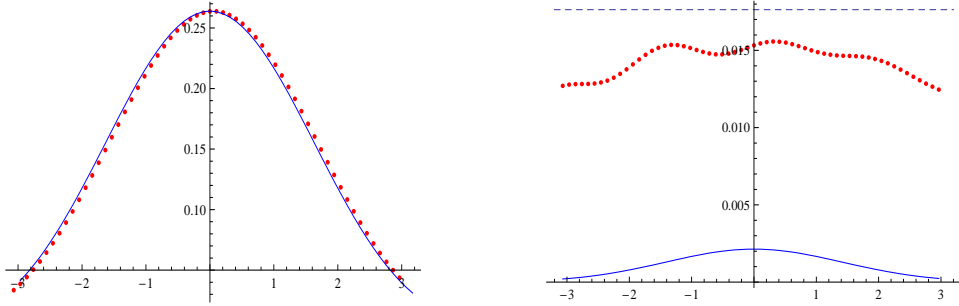


Figure 7: The sample means and the theoretical means (left display, a dotted and a solid line, respectively) together with the sample standard deviations and the two theoretical standard deviations corresponding to Theorems 1 and 2 (right display, a dotted, a solid and a dashed line, respectively). Here the target density  $f$  and the density  $k$  of a random variable  $Z$  are standard normal densities, the noise variance  $\sigma_n^2 = 4$ , the sample size  $n = 100000$ , the bandwidth  $h_n = 0.44$  and the kernel  $w$  is given by (9). The number of replications equals  $N = 500$ . The integral in (11) and not its asymptotic expansion was used to evaluate the standard deviation in Theorem 2.

Combining (16) and (17), we obtain that

$$|w_{r_n}(u)| \leq \min \left( C_1, \frac{C_2}{|u|} \right), \quad (18)$$

where the constants  $C_1$  and  $C_2$  do not depend on  $n$ . Observe that the function on the right-hand side of (18) is square integrable. Next, we have

$$\sup_{|u| \leq \epsilon_n} |g_n(x-u) - f(x)| \leq \sup_{|u| \leq \epsilon_n} |g_n(x-u) - g_n(x)| + |g_n(x) - f(x)| = I + II.$$

for an arbitrary  $\epsilon_n > 0$ . By the Fourier inversion argument for  $I$  we obtain

$$\begin{aligned} |I| &\leq \left| \sup_{|u| \leq \epsilon_n} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi_f(t) \phi_k(r_n t) (e^{itu} - 1) dt \right| \\ &\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |\phi_f(t)| \sup_{|u| \leq \epsilon_n} |e^{itu} - 1| dt. \end{aligned}$$

Notice that  $\sup_{|u| \leq \epsilon_n} |e^{itu} - 1| \leq \epsilon_n |t| \rightarrow 0$  for every fixed  $t$ . Furthermore,  $\sup_{|u| \leq \epsilon_n} |e^{itu} - 1| \leq 2$  and  $\phi_f$  is integrable. Let  $\epsilon_n \downarrow 0$  as  $n \rightarrow \infty$ . Then by the dominated convergence theorem  $I$  will vanish as  $n \rightarrow \infty$ . A similar Fourier inversion argument and another application of the dominated convergence theorem shows that  $II$  also vanishes as  $n \rightarrow \infty$ . Thus (12) is satisfied. Now (15), (18) and Lemma 1 imply that

$$\mathbb{E}[V_{n_j}^2] \sim \frac{1}{h_n} f(x) \int_{-\infty}^{\infty} |w_r(u)|^2 du. \quad (19)$$



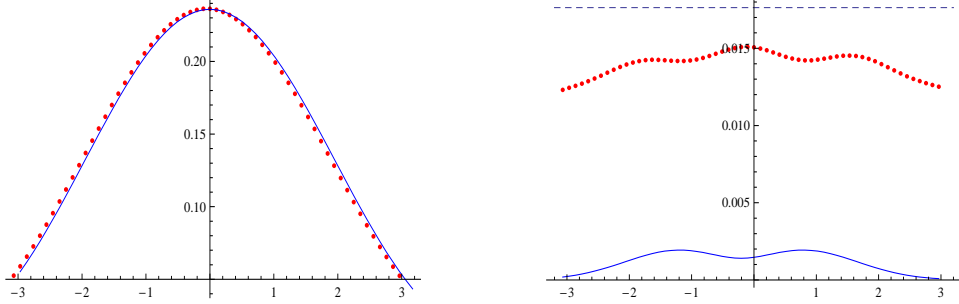


Figure 8: The sample means and the theoretical means (left display, a dotted and a solid line, respectively) together with the sample standard deviations and the two theoretical standard deviations corresponding to Theorems 1 and 2 (right display, a dotted, a solid and a dashed line, respectively). Here the target density  $f$  is a mixture of two normal densities with means equal to  $-1$  and  $1$  and the same variance  $0.375$ , the mixing probability is  $0.5$ , the density  $k$  of a random variable  $Z$  is a standard normal density, the noise variance  $\sigma_n^2 = 4$ , the sample size  $n = 100000$ , the bandwidth  $h_n = 0.44$  and the kernel  $w$  is given by (9). The number of replications equals  $N = 500$ . The integral in (11) and not its asymptotic expansion was used to evaluate the standard deviation in Theorem 2.

Furthermore, by Fubini's theorem

$$\begin{aligned}
\mathbb{E}[V_{nj}] &= \frac{1}{h_n} \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(\frac{-itx}{h_n}\right) \mathbb{E}\left[\exp\left(\frac{itX_j}{h_n}\right)\right] \frac{\phi_w(t)}{\phi_k(r_nt)} dt \\
&= \int_{-\infty}^{\infty} \exp\left(-\frac{itx}{h_n}\right) \mathbb{E}\left[\exp\left(\frac{itY_j}{h_n}\right)\right] \mathbb{E}\left[\exp\left(\frac{it\sigma_n Z_j}{h_n}\right)\right] \frac{\phi_w(t)}{\phi_k(r_nt)} dt \\
&= \frac{1}{h_n} \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-\frac{itx}{h_n}\right) \phi_f\left(\frac{t}{h_n}\right) \phi_w(t) dt.
\end{aligned} \tag{20}$$

The last expression is bounded uniformly in  $h_n$  due to Assumption A (i) and (iii), which can be seen by a change of the integration variable  $t/h_n = s$ . Moreover, using (15), (17) and (19), we have that

$$\mathbb{E}[|V_{nj}^{2+\delta}|] = \int_{-\infty}^{\infty} \frac{1}{h^{2+\delta}} \left|w_{r_n}\left(\frac{x-y}{h}\right)\right|^{2+\delta} g_n(y) dy \tag{21}$$

is of order  $h_n^{-1-\delta}$ . Combination of the above results now yields

$$\frac{\mathbb{E}[|V_{nj} - \mathbb{E}[V_{nj}]|^{2+\delta}]}{n^{\delta/2} (\text{Var}[V_{nj}])^{1+\delta/2}} \rightarrow 0 \tag{22}$$

as  $h_n \rightarrow 0$ ,  $nh_n \rightarrow \infty$ . Therefore  $f_{nh_n}(x)$  satisfies Lyapunov's condition for asymptotic normality in the triangular array scheme, see Theorem 7.3 in Billingsley (1968), and hence it is asymptotically normal, i.e.

$$\frac{f_{nh_n}(x) - \mathbb{E}[f_{nh_n}(x)]}{\sqrt{\text{Var}[f_{nh_n}(x)]}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1).$$

Formula (4) is then immediate from this fact, formulae (14), (19), (20) and Slutsky's lemma, see Corollary 2 on p. 31 of Billingsley (1968).  $\square$

*Proof of Theorem 2.* The proof follows the same line of thought as the proof of Theorem 1 in van Es and Uh (2005). For an arbitrary  $0 < \epsilon < 1$  we have

$$f_{nh_n}(x) = \frac{1}{2\pi nh_n} \sum_{j=1}^n \int_{-\epsilon}^{\epsilon} \exp\left(is \left(\frac{X_j - x}{h_n}\right)\right) \frac{\phi_w(s)}{\phi_k(s/\rho_n)} ds \quad (23)$$

$$+ \frac{1}{2\pi nh_n} \sum_{j=1}^n \left( \int_{-1}^{-\epsilon} + \int_{\epsilon}^1 \right) \exp\left(is \left(\frac{X_j - x}{h_n}\right)\right) \frac{\phi_w(s)}{\phi_k(s/\rho_n)} ds. \quad (24)$$

The integral in (23) is real-valued, which can be seen by taking its complex conjugate. Using Assumption B (i), the variance of (23) can be bounded as follows:

$$\begin{aligned} \text{Var} & \left[ \frac{1}{2\pi nh_n} \sum_{j=1}^n \int_{-\epsilon}^{\epsilon} \exp\left(is \left(\frac{X_j - x}{h_n}\right)\right) \frac{\phi_w(s)}{\phi_k(s/\rho_n)} ds \right] \\ & \leq \frac{1}{4\pi^2 nh_n^2} \mathbb{E} \left[ \left( \int_{-\epsilon}^{\epsilon} \exp\left(is \left(\frac{X_j - x}{h_n}\right)\right) \frac{\phi_w(s)}{\phi_k(s/\rho_n)} ds \right)^2 \right] \\ & \leq \frac{1}{4\pi^2 nh_n^2} \left( \int_{-\epsilon}^{\epsilon} \frac{1}{|\phi_k(s/\rho_n)|} ds \right)^2 \\ & \leq \frac{1}{4\pi^2 nh_n^2} (2\epsilon)^2 \left( \frac{1}{\inf_{-\epsilon \leq s \leq \epsilon} |\phi_k(s/\rho_n)|} \right)^2 \\ & = O \left( \frac{1}{\pi^2} \frac{1}{n} \frac{1}{\sigma_n^2} \left( \frac{\epsilon}{\rho_n} \right)^{2-2\lambda_0} \exp \left( \frac{2\epsilon^\lambda}{\mu \rho_n^\lambda} \right) \right). \end{aligned}$$

Hence the contribution of (23) minus its expectation is of order

$$O_P \left( \frac{1}{\sigma_n} \frac{1}{\sqrt{n}} \left( \frac{\epsilon}{\rho_n} \right)^{1-\lambda_0} \exp \left( \frac{\epsilon^\lambda}{\mu \rho_n^\lambda} \right) \right).$$

By comparing this to the normalising constant in (5), by Slutsky's lemma we see that (23) can be neglected when considering the asymptotic normality of  $f_{nh_n}(x)$ .

The term (24) can be written as

$$\frac{1}{2\pi n h_n C} \sum_{j=1}^n \left( \int_{-1}^{-\epsilon} + \int_{\epsilon}^1 \right) \exp \left( i s \left( \frac{X_j - x}{h_n} \right) \right) \phi_w(s) \left( \frac{|s|}{\rho_n} \right)^{-\lambda_0} \exp \left( \frac{|s|^\lambda}{\mu \rho_n^\lambda} \right) ds \quad (25)$$

$$\begin{aligned} &+ \frac{1}{2\pi n h_n} \sum_{j=1}^n \left( \int_{-1}^{-\epsilon} + \int_{\epsilon}^1 \right) \exp \left( i s \left( \frac{X_j - x}{h_n} \right) \right) \phi_w(s) \\ &\times \left( \frac{1}{\phi_k(s/\rho_n)} - \frac{1}{C} \left( \frac{|s|}{\rho_n} \right)^{-\lambda_0} \exp \left( \frac{|s|^\lambda}{\mu \rho_n^\lambda} \right) \right) ds. \end{aligned} \quad (26)$$

Observe that both (25) and (26) are real. Expression (25) equals

$$\frac{1}{\pi n \sigma_n C} \rho_n^{\lambda_0 - 1} \sum_{j=1}^n \int_{\epsilon}^1 \cos \left( s \left( \frac{X_j - x}{h_n} \right) \right) \phi_w(s) s^{-\lambda_0} \exp \left( \frac{s^\lambda}{\mu \rho_n^\lambda} \right) ds. \quad (27)$$

By formula (21) of van Es and Uh (2005)

$$\cos \left( s \left( \frac{X_j - x}{h_n} \right) \right) = \cos \left( \frac{X_j - x}{h_n} \right) + R_{n,j}(s), \quad (28)$$

where  $R_{n,j}(s)$  is a remainder term satisfying

$$|R_{n,j}| \leq (|x| + |X_j|) \left( \frac{1-s}{h_n} \right), \quad (29)$$

whence by Lemma 5 of van Es and Uh (2005) the expression (27) equals

$$\begin{aligned} &\frac{1}{\pi \sigma_n C} \rho_n^{\lambda_0 - 1} \int_{\epsilon}^1 \phi_w(s) s^{-\lambda_0} \exp \left( \frac{s^\lambda}{\mu \rho_n^\lambda} \right) ds \frac{1}{n} \sum_{j=1}^n \cos \left( \frac{X_j - x}{h_n} \right) + \frac{1}{n} \sum_{j=1}^n \tilde{R}_{n,j} \\ &= \frac{1}{\pi \sigma_n C} A(\Gamma(\alpha + 1) + o(1)) \left( \frac{\mu}{\lambda} \right)^{1+\alpha} \rho_n^{\lambda(1+\alpha) + \lambda_0 - 1} \zeta(\rho_n) \frac{1}{n} \sum_{j=1}^n \cos \left( \frac{X_j - x}{h_n} \right) \\ &+ \frac{1}{n} \sum_{j=1}^n \tilde{R}_{n,j}, \end{aligned}$$

where

$$\tilde{R}_{n,j} = \frac{1}{\pi \sigma_n C} \rho_n^{\lambda_0 - 1} \int_{\epsilon}^1 R_{n,j}(s) \phi_w(s) s^{-\lambda_0} \exp \left( \frac{s^\lambda}{\mu \rho_n^\lambda} \right) ds.$$

By (29) and Lemma 5 of van Es and Uh (2005) the latter expression can be bounded as

$$\begin{aligned} |\tilde{R}_{n,j}| &\leq \frac{1}{\pi \sigma_n C} (|x| + |X_j|) \rho_n^{\lambda_0 - 1} \int_{\epsilon}^1 \left( \frac{1-s}{h_n} \right) \phi_w(s) s^{-\lambda_0} \exp \left( \frac{s^\lambda}{\mu \rho_n^\lambda} \right) ds \\ &= \frac{1}{\pi \sigma_n h_n C} A \left( \frac{\mu}{\lambda} \right)^{\alpha+2} (\Gamma(\alpha + 2) + o(1)) \rho_n^{\lambda(2+\alpha) + \lambda_0 - 1} \zeta(\rho_n) (|x| + |X_j|). \end{aligned}$$

Hence

$$\text{Var}[\tilde{R}_{n,j}] \leq \text{E}[\tilde{R}_{n,j}^2] = O\left(\frac{1}{\sigma_n^2 h_n^2} \rho_n^{2(\lambda(2+\alpha)+\lambda_0-1)} (\zeta(\rho_n))^2\right).$$

Here we used the fact that  $\text{E}[Y_j^2] + \text{E}[Z_j^2] < \infty$  together with the fact that being convergent, the sequence  $\sigma_n$  is bounded, which implies that  $\text{E}[X_j^2]$  is bounded uniformly in  $n$ . By Chebyshev's inequality it follows that

$$\frac{1}{n} \sum_{j=1}^n (\tilde{R}_{n,j} - \text{E} \tilde{R}_{n,j}) = O_P\left(\frac{1}{\sigma_n h_n} \frac{\rho_n^{\lambda(2+\alpha)+\lambda_0-1} \zeta(\rho_n)}{\sqrt{n}}\right). \quad (30)$$

After multiplication of this term by the normalising factor from (5) we obtain that the resulting expression is of order  $\rho_n^\lambda (\sigma_n h_n)^{-1} = h_n^{\lambda-1} \sigma_n^\lambda$ . Assumption B (v) and Slutsky's lemma then imply that the remainder term (30) can be neglected when considering the asymptotic normality of  $f_{nh_n}(x)$ .

The variance of (26) can be bounded by

$$\frac{1}{4\pi^2 n h_n^2 C^2} \left( \left( \int_{-1}^{-\epsilon} + \int_{\epsilon}^1 \right) |\phi_w(s)| \left( \frac{|s|}{\rho_n} \right)^{-\lambda_0} \exp\left(\frac{|s|^\lambda}{\mu \rho_n^\lambda}\right) |u(s/\rho_n)| ds \right)^2,$$

where the function  $u$  is given by

$$u(y) = \frac{C|y|^{\lambda_0} \exp(-|y|^\lambda \mu^{-1})}{\phi_k(y)} - 1. \quad (31)$$

This function is bounded on  $\mathbb{R} \setminus (-\delta, \delta)$ , where  $\delta$  is an arbitrary positive number. It follows that  $u(s/\rho_n)$  is also bounded and tends to zero for all fixed  $s$  with  $|s| \geq \epsilon$  as  $\rho_n \rightarrow 0$ . Hence the variance of (26) is of smaller order compared to the variance of (25), which can be shown by the dominated convergence theorem via an argument similar to the one in the proof of Lemma 5 of van Es and Uh (2005). Therefore by Slutsky's lemma (26) can be neglected when considering asymptotic normality of (5).

Combination of the above observations yields that it suffices to study

$$\frac{A}{\pi C} \left(\frac{\mu}{\lambda}\right)^{1+\alpha} (\Gamma(\alpha+1) + o(1)) U_{nh_n}(x), \quad (32)$$

where

$$U_{nh_n}(x) = \frac{1}{\sqrt{n}} \sum_{j=1}^n \left( \cos\left(\frac{X_j - x}{h_n}\right) - \text{E} \left[ \cos\left(\frac{X_j - x}{h_n}\right) \right] \right).$$

Observe that

$$\frac{X_j - x}{h_n} = \frac{Y_j - x}{h_n} + \frac{\sigma_n}{h_n} Z_j = \frac{Y_j - x}{h_n} + \frac{Z_j}{\rho_n}$$

and that by the same arguments as in the proof of Lemma 6 in van Es and Uh (2005), both  $(Y_j - x)/h_n \bmod 2\pi$  and  $Z/\rho_n \bmod 2\pi$  converge in distribution to a random variable with a uniform distribution on  $[0, 2\pi]$ . Furthermore, these two random variables are independent. Now notice that for two independent random variables  $W_1$  and  $W_2$  the sum  $W_1 + W_2 \bmod 2\pi$  equals in distribution  $(W_1 \bmod 2\pi + W_2 \bmod 2\pi) \bmod 2\pi$ . Moreover, if  $W_1$  and  $W_2$  are uniformly distributed on  $[0, 2\pi]$ , then also  $W_1 + W_2 \bmod 2\pi$  is uniformly distributed on  $[0, 2\pi]$ , see Scheinok (1965). Using these two facts, by exactly the same arguments as in the proof of Lemma 6 of van Es and Uh (2005) we finally obtain that  $U_{nh_n}(x) \xrightarrow{D} \mathcal{N}(0, 1/2)$ . The latter in conjunction with (32) entails (5).  $\square$

*Proof of Theorem 3.* The proof employs an approach similar to the proof of Theorem 2.1 of Fan (1991b). We have

$$\mathbb{E}[V_{nj}^2] = \int_{-\infty}^{\infty} \frac{1}{h_n^2} \left| w_{\rho_n} \left( \frac{x-y}{h_n} \right) \right|^2 g_n(y) dy.$$

By equation (3.1) of Fan (1991b) (with  $h_n$  replaced by  $\rho_n$ ) we have

$$\left| \frac{\rho_n^\beta \phi_w(t)}{\phi_k(t/\rho_n)} \right| \leq w_0(t),$$

where  $w_0$  is a positive integrable function. Hence by the dominated convergence theorem

$$\rho_n^\beta w_{\rho_n}(y) \rightarrow \frac{1}{2\pi C} \int_{-1}^1 e^{-itx} t^\beta \phi_w(t) dt.$$

Furthermore, again by equation (3.1) of Fan (1991b) we have  $|\rho_n^\beta w_{\rho_n}(y)| \leq C_2$  for some constant  $C_2$  independent of  $n$  and  $y$ , while equation (2.7) of Fan (1991b) implies that  $|\rho_n^\beta w_{\rho_n}(y)| \leq C_1/|y|$ . Combination of these two bounds gives

$$|\rho_n^\beta w_{\rho_n}(y)| \leq \min \left( \frac{C_1}{|y|}, C_2 \right). \quad (33)$$

Since the fact that  $g_n$  satisfies (12) can be shown exactly as in the proof of Theorem 1, by Lemma 1 we then obtain that

$$\begin{aligned} \mathbb{E}[V_{nj}^2] &\sim \frac{f(x)}{h_n \rho_n^{2\beta}} \int_{-\infty}^{\infty} \left[ \frac{1}{2\pi C} \int_{-1}^1 e^{-ity} t^\beta \phi_w(t) dt \right]^2 dy \\ &= \frac{1}{h_n \rho_n^{2\beta}} \frac{f(x)}{2\pi C^2} \int_{-1}^1 |t|^{2\beta} |\phi_w(t)|^2 dt, \end{aligned} \quad (34)$$

where the last equality follows from Parseval's identity. Furthermore, by Fubini's theorem and the dominated convergence theorem we have

$$\begin{aligned}
\mathbb{E}[V_{nj}] &= \frac{1}{h_n} \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-\frac{itx}{h_n}\right) \mathbb{E}\left[\exp\left(\frac{itX_j}{h_n}\right)\right] \frac{\phi_w(t)}{\phi_k(t/\rho_n)} dt \\
&= \frac{1}{h_n} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx/h} \phi_f\left(\frac{t}{h_n}\right) \phi_w(t) dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi_f(t) \phi_w(h_n t) dt \\
&\rightarrow f(x).
\end{aligned} \tag{35}$$

The dominated convergence theorem is applicable because of Assumption B (i) and (iii). Finally, let us consider  $\mathbb{E}[|V_{nj}^{2+\delta}|]$ . Writing

$$\mathbb{E}[|V_{nj}|^{2+\delta}] = \int_{-\infty}^{\infty} \frac{1}{h_n^{2+\delta}} \left| w_{\rho_n}\left(\frac{x-y}{h_n}\right) \right|^{2+\delta} g_n(y) dy, \tag{36}$$

and using (33) and Lemma 1, we obtain that

$$\mathbb{E}[|V_{nj}|^{2+\delta}] = O(h_n^{-1-\delta} \rho_n^{-\beta(2+\delta)}).$$

Combination of (34), (35) and (36) yields that Lyapunov's condition is fulfilled and hence that  $f_{nh_n}(x)$  is asymptotically normal. Formula (6) then follows from (34) and (35). This completes the proof.  $\square$

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