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A Method for Searching the Limit Cycles of High Order Sigma-Delta Modulators

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Abstract— **In this paper an approach for description and validation of potential limit cycles of high order Sigma-Delta modulators is presented. The approach is based on a parallel decomposition of the modulator. In this representation, the general N-th order modulator is transformed into a decomposition of low order, generally complex modulators, which interact only through the quantizer function. The decomposition considered helps to describe easily the time domain behavior of the modulator. Based on this, the conditions for the existence of limit cycles in the high order modulator for constant inputs, are obtained. They are determined by the periodicity conditions for the states of the first order modulators. In this case, the state variables are uncoupled and the obtained conditions are very easy to be checked. Limit cycles correspond to periodic output sequences and the proposed method includes description and validation of possible sequences.**

I. INTRODUCTION

Sigma-Delta ($\Sigma\Delta$) modulation has become in recent years an increasingly popular choice for robust and inexpensive analog-to-digital (AD) and digital-to-analog (DA) conversion [l], [2]. As a result of this, AD and DA converters based on 1-bit $\Sigma\Delta$ modulators are widely used in different applications. Despite the widespread use of $\Sigma\Delta$ modulators theoretical understanding of $\Sigma\Delta$ concept is still very limited, because these systems are nonlinear, due to the presence of a discontinuous nonlinearity - the quantizer. Limit cycles are well known phenomena that often appear in practical $\Sigma\Delta$ modulators. In frequency domain they correspond to discrete peaks in the frequency spectrum of the modulator. If these peaks are inside the signal base band, the total harmonic distortion increases. Because of this, for data processing applications it is very important to predict and describe possible limit cycles.

Since the pioneering work of Gray and his co-workers beginning with [3], a number of researchers have contributed to the development of a theory of $\Sigma\Delta$ modulation. Main results concerning the limit cycles for low order $\Sigma\Delta$ modulators are presented in [4], [5], [6]. In [7] a procedure for characterizing and validating potential limit cycles is presented. This enables to carry out an exhaustive search for cyclic sequences up to a length of 40 clock periods. In [8], [9] authors use state space approach and present a mathematical framework for the description of limit cycles in 1-bit $\Sigma\Delta$ modulators for constant inputs.

Here we present a method for description and validation of potential limit cycles for high order modulators. Especially we focus our attention on constant input signals. The method is based on the decomposition of the general *N-th* order modulator presented in [10], [11], [12]. Using this presentation the modulator could be considered as made up of *N* first order modulators (generally complex modulators), which interact only through the quantizer function. The decomposition considered helps to easily describe the time domain behavior of the modulator. Based on this the conditions for the existence of limit cycles in the high order modulators are presented. These conditions are to be easily checked and they are the basis of the searching procedure. All cases of poles of the loop filter transfer function are considered.

The paper is organized as follows. In the next section we give a brief overview of the decomposition technique. In Section III we consider the time domain behavior of high order modulators and describe their limit cycle performance. Then, in Section IV we present the conditions for validation of limit cycles. In Section V we give several examples and we end up with some conclusions in Section VI.

II. PARALLEL DECOMPOSITION OF HIGH ORDER SIGMA DELTA MODULATORS

The structure of a basic Sigma-Delta modulator is shown in Fig. 1, and it consists of a filter with transfer function $G(z)$ followed by a one-bit quantizer in a feedback loop. The system operates in discrete time.

The input to the loop is a discrete-time sequence $u(n)$ from *[-1, 1]*. The discrete-time sequence $\mathbf{x}(n)$ is the output of

the filter and the input to the quantizer. Let us consider a *N-th* order modulator with a loop filter with a transfer function $G(z)$. Suppose the transfer function has N real distinct roots of the denominator.
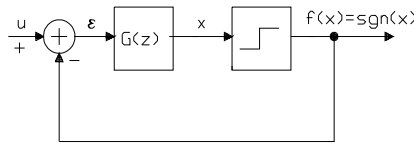

Figure 1.   Basic structure of Sigma Delta modulator.

The corresponding block diagram of modulator model is given in Fig. 2 [10], [11].
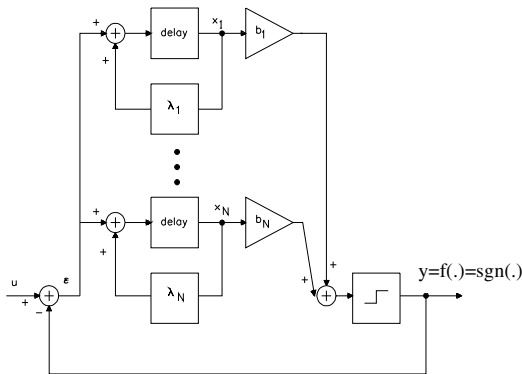

Figure 2.   Block diagram of the modulator model using detailed parallel form of the loop filter.

Based on this presentation the state equations of the Sigma Delta modulator are

$$x_k(n+1) = \lambda_k x_k(n) + \left[u(n) - f\left(\sum_{i=1}^{N} b_i x_i(n) \right) \right] =$$
$$= \lambda_k x_k(n) + \left[u(n) - f\left(\mathbf{b}^T \mathbf{x}(n) \right) \right] = \quad (1)$$
$$= \lambda_k x_k(n) + \left[u(n) - y(n) \right], k = 1, 2, ..., N,$$

where λ_1, λ_2 ,...,λ_N are poles (or modes) of the loop filter, $\mathbf{b}=(b_1, b_2, ..., b_N)^T$ is the vector of fractional components coefficients and $\mathbf{x}=(x_1, x_2, ..., x_N)^T$ is the state vector. The quantizer function is a sign function

$$y(n) = f\left(\sum_{i=1}^{N} b_i x_i(n) \right) = f\left(\mathbf{b}^T \mathbf{x}(n) \right) = \begin{cases} 1, & \mathbf{b}^T \mathbf{x}(n) \geq 0 \\ -1, & \mathbf{b}^T \mathbf{x}(n) < 0 \end{cases} \quad (2)$$

It should be stressed that the original approach in [10] is extended in [12] for complex pairs of poles. In this case, the block diagram is the same as in Figure 2, but for every complex conjugated pair of poles λ_i, λ_{i+1} the corresponding coefficients b_i, b_{i+1} are also complex conjugated. Thus the sigma delta modulator could be considered as made up of N first order complex modulators. However, the contribution $b_i\lambda_i + b_{i+1}\lambda_{i+1}$ of every complex conjugated pair of poles λ_i, λ_{i+1} to the weighted sum of the input of the quantizer is real [12]. Since the model is based on the parallel presentation of the loop filter it can be used for calculations only and the results in the paper are easily obtained.

Without loosing the generality, the case of repeated poles will be considered when the pole λ_1 is repeated with order two i.e. $\lambda_1 = \lambda_2$. Thus the state equations of the Sigma Delta modulator become [11]

$$x_1(n+1) = \lambda_1 x_1(n) + \left[u(n) - y(n) \right]$$
$$x_2(n+1) = x_1(n) + \lambda_2 x_2(n)$$
$$x_k(n+1) = \lambda_k x_k(n) + \left[u(n) - y(n) \right], \quad (3)$$
$$k = 3, ..., N \ .$$

III.   TIME DOMAIN BEHAVIOR AND DESCRIPTION OF LIMIT CYCLES OF SIGMA DELTA MODULATORS

Without loss of generality we will consider the case with real distinct poles. Then the discrete time sequence for state variables x_1, x_2, ..., x_N is given by:

$$x_k(1) = \lambda_k x_k(0) + \left[u(0) - y(0) \right],$$
$$x_k(2) = \lambda_k x_k(1) + \left[u(1) - y(1) \right] = \lambda_k^2 x_k(0) +$$
$$+ \left[u(0) - y(0) \right] \lambda_k^1 + \left[u(1) - y(1) \right]$$
$$.............$$
$$x_k(n) = \lambda_k x_k(n-1) + \left[u(n-1) - y(n-1) \right] = \quad (4)$$
$$= \lambda_k^n x_k(0) + \left[u(0) - y(0) \right] \lambda_k^{n-1} + \left[u(1) - y(1) \right] \lambda_k^{n-2} +$$
$$+ ... \left[u(n-2) - y(n-2) \right] \lambda_k^1 + \left[u(n-1) - y(n-1) \right] =$$
$$= \lambda_k^n x_k(0) + \sum_{i=0}^{n-1} \left[u(i) - y(i) \right] \lambda_k^{n-i-1}$$
$$k = 1, 2, ..., N \ .$$

The limit cycles correspond to periodic solutions in time domain. The periodic solutions can be observed at the output of the modulator as repetitive sequences of 1's and -1's. Let's consider a periodic sequence $y(0), y(1), ..., y(M-1)$ with length M at the output of the modulator. In this case $y(M)=y(0), y(M+1)=y(1),..., y(2M-1)=y(M-1)$, etc. Every periodic output sequence corresponds to a periodic sequence in the states, i.e. every state variable x_k is periodic. This can be observed easily if we write the state variable x_k after L periods.

$$x_k(L.M) = \lambda_k^{L.M} x_k(0) + \sum_{i=0}^{LM-1} \left[u(i) - y(i) \right] \lambda_k^{LM-i-1} \quad (5)$$
$$k = 1, 2, ..., N \ .$$

Taking into account that every $[u(i)-y(i)]$ is the same after each M samples, (5) can be rewritten as

$$x_k(L.M) = \lambda_k^{L.M} x_k(0) + \sum_{i=0}^{LM-1} \left[u(i) - y(i) \right] \lambda_k^{LM-i-1} =$$
$$= \lambda_k^{L.M} x_k(0) + \sum_{p=0}^{L-1} \lambda_k^{p.M} \left(\sum_{i=0}^{M-1} \left[u(i) - y(i) \right] \lambda_k^{M-i-1} \right) \quad (6)$$
$$= \lambda_k^{L.M} x_k(0) + \frac{1 - \lambda_k^{LM}}{1 - \lambda_k^M} \left(\sum_{i=0}^{M-1} \left[u(i) - y(i) \right] \lambda_k^{M-i-1} \right)$$
$$k = 1, 2, ..., N \ .$$

The above is correct, because the geometric series

$\sum_{p=0}^{L-1} \lambda_k^{p.M}$ has a value $\dfrac{1-\lambda_k^{LM}}{1-\lambda_k^M}$.

If $|\lambda_k|<1$, for every L that is large enough (after enough time)

$$x_k(L.M) = \frac{1}{1-\lambda_k^M}\left(\sum_{i=0}^{M-1}[u(i)-y(i)]\lambda_k^{M-i-1}\right),$$

i.e. $x_k(L.M)$ does not depend on L. This means repetition of the value of state x_k after every M instances, i.e. the states are periodic.

If $|\lambda_k|>1$, it follows from (6) that the boundness of the states is ensured iff

$$x_k(0) = \frac{1}{1-\lambda_k^M}\left(\sum_{i=0}^{M-1}[u(i)-y(i)]\lambda_k^{M-i-1}\right) \qquad (7)$$

and thus $x_k(L.M) = x_k(0)$. This means that the initial condition with respect to x_k should be taken in accordance with (7) in order to ensure stability of the solution. This fits with the results in [11], [12] concerning the stability of high order modulators when only one pole of the loop filter is larger than 1.

If $\lambda_k=1$, $x_k(L.M) = x_k(0)$ for every L and every $x_k(0)$, because at the periodic orbit $\sum_{i=0}^{M-1}[u-y(i)]=0$ for constant input signal u. This actually means that periodicity with respect to x_k is ensured.

In the case of complex pair of poles the results are similar, but the initial conditions connected with the complex conjugated pair of poles are also complex conjugated. We should stress again that the contribution of the state variables corresponding to these poles, to the input of the quantizer is real [12].

In the case of repeated poles, based on Eq. (3) we obtain

$$x_1(M) = \lambda_1^M x_1(0) + \sum_{i=0}^{M-1}[u(i)-y(i)]\lambda_1^{M-i-1}$$

$$x_2(M) = \lambda_1^M x_2(0) + M.\lambda_1^{M-1}x_1(0) +$$

$$+\sum_{i=0}^{M-2}(M-i-1).[u(i)-y(i)]\lambda_1^{M-i-2} \qquad (8)$$

$$.............$$

$$x_k(M) = \lambda_k^M x_k(0) + \sum_{i=0}^{M-1}[u(i)-y(i)]\lambda_k^{M-i-1}$$

$$k = 3,4,...,N \ .$$

The solution of $x_k(0) = x_k(M)$, $k = 1,3,...,N$ with respect to $x_1(0), x_3(0), \dots$ and $x_N(0)$ is given by (7). From (8) the solution of $x_2(0) = x_2(M)$ with respect to $x_2(0)$ is

$$x_2(0) = \frac{M.\lambda_1^{n-1}x_1(0) + \sum_{i=0}^{M-2}(M-i-1).[u(i)-y(i)]\lambda_1^{M-i-2}}{1-\lambda_1^M} \ . \qquad (9)$$

If $\lambda_k=1$, to ensure $x_2(0) = x_2(M)$ to be satisfied

$$x_1(0) = -\frac{\sum_{i=0}^{M-2}(M-i-1).[u(i)-y(i)]\lambda_1^{M-i-2}}{M} \ . \qquad (10)$$

Thus if we choose $x_1(0)$ in accordance to (10), conditions (8) are satisfied for every $x_2(0)$.

IV. VALIDATIONS OF LIMIT CYCLES OF SIGMA DELTA MODULATORS

The results in the previous section have been derived without matching the time sequence of the states $x_k(0), x_k(1), \dots, x_k(M-1)$, $k=1,2,\dots N$ with the time sequence of the output signal $y(0), y(1), \dots, y(M-1)$ in the framework of one period. In fact, to have a valid output sequence $y(0), y(1), \dots, y(M-1)$, (2) should be satisfied. Thus,

$$\left(\sum_{k=1}^{N}b_k x_k(n)\right) = \left(\mathbf{b}^T\mathbf{x}(n)\right) \geq 0, \ if \ y(n)=1$$

$$\left(\sum_{k=1}^{N}b_k x_k(n)\right) = \left(\mathbf{b}^T\mathbf{x}(n)\right) < 0, \ if \ y(n)=-1 \qquad (11)$$

$$n=1,2,...,M$$

or

$$\sum_{k=1}^{N}b_k \lambda_k^n x_k(0) \geq -\sum_{k=1}^{N}b_k\left(\sum_{i=0}^{n-1}[u(i)-y(i)]\lambda_k^{n-i-1}\right), \ if \ y(n)=1$$

$$\sum_{k=1}^{N}b_k \lambda_k^n x_k(0) < -\sum_{k=1}^{N}b_k\left(\sum_{i=0}^{n-1}[u(i)-y(i)]\lambda_k^{n-i-1}\right), \ if \ y(n)=-1$$

$$n=1,2,...,M \ . \qquad (12)$$

Hence

$$\sum_{k=1}^{N}b_k \lambda_k^n x_k(0) \geq -\sum_{i=0}^{n-1}\left([u(i)-y(i)]\sum_{k=1}^{N}b_k\lambda_k^{n-i-1}\right), \ if \ y(n)=1$$

$$\sum_{k=1}^{N}b_k \lambda_k^n x_k(0) < -\sum_{i=0}^{n-1}\left([u(i)-y(i)]\sum_{k=1}^{N}b_k\lambda_k^{n-i-1}\right), \ if \ y(n)=-1$$

$$n=1,2,...,M \ . \qquad (13)$$

Conditions (13) can be rewritten in the following form:

$$\sum_{k=1}^{N}b_k \lambda_k^n x_k(0) \geq A_n, \ if \ y(n)=1$$

$$\sum_{k=1}^{N}b_k \lambda_k^n x_k(0) < A_n, \ if \ y(n)=-1 \qquad (14)$$

$$n=1,2,...,M \ ,$$

where

$$A_n = -\sum_{i=0}^{n-1}\left([u(i)-y(i)]\sum_{k=1}^{N}b_k\lambda_k^{n-i-1}\right). \qquad (15)$$

In the case of a complex pair of poles λ_i, λ_{i+1} the result has the same form. It should be noted that the left and right parts of inequalities (14) are real.

For the case of repeated poles (Eq.(3)), (14) becomes

$$\sum_{k=1}^{N} b_k \lambda_k^n x_k(0) + (M-1)b_2 \lambda_1^{n-1} x_1(0) \geq A_n \text{, if } y(n)=1$$

$$\sum_{k=1}^{N} b_k \lambda_k^n x_k(0) + (M-1)b_2 \lambda_1^{n-1} x_1(0) < A_n \text{, if } y(n)=-1 \quad (16)$$

$$n = 1,2,...,M ,$$

where

$$A_n = -\sum_{i=0}^{n-1} \left([u(i)-y(i)] \sum_{\substack{k=1 \\ k \neq 2}}^{N} \left(b_k \lambda_k^{n-i-1} + (n-i-1)b_2 \lambda_1^{n-i-2} \right) \right) . \quad (17)$$

Conditions (14) or (16) validate the limit cycles connected with the corresponding vector of initial conditions $x_k(0)$, $k=1,2,...,N$ that are obtained in the previous section. It should be stressed that these conditions can be easily checked and very simply implemented.

The strategy for searching the limit cycles consists in the following. For every periodic output sequence of 1's and -1's with arbitrary length M, we check whether initial conditions that ensure periodicity after the first period exist. Afterward the validity of the corresponding output sequences has to be checked.

V. EXAMPLES

To demonstrate the applicability of the approach we consider four third order modulators with the following parameters $\lambda = [\lambda_1 \ \lambda_2 \ \lambda_3]^T$, $\mathbf{b}=[b_1 \ b_2 \ b_3]^T$ and u: $\lambda=[0.7 \ 0.8 \ 0.9]^T$, $\mathbf{b}=[1 \ 1 \ 1]^T$, $u=0.4=const$; $\lambda=[0.8 \ 0.9 \ 1]^T$, $\mathbf{b}=[1 \ 1 \ 1]^T$, $u=0.5=const$; $\lambda=[0.9 \ 0.9 \ 1]^T$, $\mathbf{b}=[1 \ 1 \ 1]^T$, $u=0.5=const$; $\lambda=[1 \ 1 \ 0.9]^T$, $\mathbf{b}=[1 \ 1 \ 1]^T$, $u=0.2=const$.

Let's check the output sequence 1, 1, -1 with length $M=3$ for the first modulator. The initial conditions that lead to this periodic output sequence without transient are $x_1(0)=1.0441$, $x_2(0)=1.0984$, $x_3(0)=1.3801$. For the second modulator $x_1(0)=0.8875$, $x_2(0)=0.8156$, and every $x_3(0)$ from [1.624, 3.026] leads to a periodic output sequence 1, 1, 1, -1 with length $M=4$. For the third modulator $x_1(0)=0.8156$, $x_2(0)=-0.6879$ and every $x_3(0)$ from [0.8784, 2.1799] leads to the periodic output sequence with length $M=4$, $y(0)=1$, $y(1)=1$, $y(2)=1$, $y(3)=-1$. For the forth modulator, the initial conditions that lead to a periodic output sequence 1, 1, -1, 1, -1 without transient are $x_1(0)=0.8$, $x_3(0)=0.8398$ and every $x_2(0)$ from interval [-0.7558, 0.4002].

The last example is a second order $\Sigma\Delta$ modulator with the following loop filter transfer function

$$G(z) = \frac{2r\cos\theta z^{-1} - r^2 z^{-2}}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}} .$$

Here $\lambda_1 = \alpha+j\beta$, $\lambda_2 = \alpha-j\beta$, $b_1 = \delta-j\gamma$, $b_2 = \delta+j\gamma$ and $\alpha=r.\cos\theta$, $\beta=r.\sin\theta$, $\delta=r.\cos\theta$, $\gamma=r.(\cos2\theta)/(2\sin\theta)$. If $r=0.9$ and $\theta=30$ and we try the output sequence 1, 1, -1, the initial conditions that lead to this periodic output sequence without transient are $x_1(0)=0.779-j0.123$, $x_2(0)=0.779+j0.123$.

Simulations confirm these results.

VI. CONCLUSIONS

In this paper we present an approach for characterization and validation of potential limit cycles of one bit high order Sigma-Delta modulators with constant input. The approach is general because it uses the general form of a Sigma-Delta modulator. It is based on a parallel decomposition of the modulator. In this representation, the general N-th order modulator is transformed into a decomposition of low order, generally complex modulators, which interact only through the quantizer function. The results are given for all possible cases of poles of the loop filter transfer function. The advantage of the approach is that because of the decomposition, the state variables are uncoupled and obtained conditions are very easy to be checked. The formulas are very easy to be implemented and straightforward calculations allow an analytical check and validation of possible limit cycles with arbitrarily length.

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