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for Nonlinear Systems

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Abstract

A general setting is developed which describes controlled invariance for nonlinear control systems and which incorporates the previous approaches dealing with controlled invariant (co -) distributions. A special class of controlled invariant subspaces, called controllability cospaces, is introduced. These geometric notions are shown to be useful for deriving a (geometric) solution to the dynamic disturbance decoupling problem and for characterizing the so-called fixed dynamics for the general input-output noninteracting control problem via dynamic compensation. These fixed dynamics are a major issue for studying noninteracting control with stability. The class of quasi-static state feedbacks is used.

1 Introduction

During the last two decades, nonlinear control theory was developed thanks to the increasing number of researchers involved in this area. A main stream of the research in the 80's was the generalization, at least partially, of the so-called geometric approach which proved to be particularly efficient for linear time-invariant systems (see [32] for an overview). In this linear theory, controlled invariance is a fundamental notion.

The study of controlled invariance for nonlinear systems of the form

$$\dot{x} = f(x) + g(x)u \tag{1}$$

where $x \in I\!\!R^n$, $u \in I\!\!R^m$, was initiated in [6]. In this paper invariants were sought under feedback transformations of the form

$$u = \alpha(x) + v \tag{2}$$

Later on, controlled invariance was tackled by various authors ([20],[15],[23],[24]). The group of feedback transformations acting on (1) was enlarged to transformations of the form

$$u = \alpha(x) + \beta(x)v \tag{3}$$

where $\beta(x)$ is square and locally invertible. These works yielded the definition of a controlled invariant distribution. The key was found for the solution of synthesis problems, such as the disturbance decoupling problem and the noninteracting control problem, via regular (or invertible) static state feedback (see the textbooks [18],[25] for an overview). The study of controlledm invariance under the class of feedbacks (3) remains active - see [8],[14] for recent contributions.

Some limits of this by now well established theory appeared at the end of the 80's in the characterization of left- or right-invertibility for nonlinear systems or for synthesis problems involving dynamic feedback. A nice understanding of these problems is provided by a differential algebraic theory ([13]).

In linear theory, it has been shown that controllability subspaces play an important role in applications. These controllability subspaces are a special class of controlled invariant subspaces. An analogous notion of controllability distribution was defined for nonlinear systems ([26]). Recently, dynamic controllability distributions were considered ([30]). It has been shown that these distributions may be used to characterize the invertibility of a system. In this paper, a dual notion called "controllability cospace", is defined. These controllability cospaces incorporate the annihilators of the dynamic controllability distributions introduced in ([30]).

The goal of this paper is to introduce a generalized notion of controlled invariance by allowing an enlarged class of feedback transformations acting on (1). The motivation is to clarify the geometric structure of nonlinear systems and to develop an (algebro -) geometric framework to tackle synthesis problems via dynamic feedback. Relations exist with both the differential geometric and the differential algebraic approach, but these will not be outlined in this paper.

We can summarize the existing results related to the study of controlled invariance for nonlinear systems in the following table:

Feedback	References
$u = \alpha(x) + v$	Brockett
$u = \alpha(x) + \beta(x)v$	Isidori *et al.* Hirschorn Nijmeijer *et al.*,...
$u = \alpha(x, v, \dot{v}, \cdots, v^{(k)})$	

To complete the table, we investigate in this paper the controlled invariance for nonlinear systems under feedback transformations called quasi-static state feedback.

In the sequel we consider a nonlinear control system (1), where the entries of $f(x)$ and $g(x)$ are meromorphic functions of x. It is assumed that rank $g(x) = m$ and that $n \geq 1$.

1

The organization of this paper is as follows. In Section 2 we define the generalized notion of invariance with respect to the dynamics (1). Section 3 is devoted to controlled invariance and related properties. Controllability cospaces and their applications are treated in Section 4.

2 Invariant subspaces

We follow the notations and setting of [12]. Let $\mathcal{K}$ denote the field of meromorphic functions of $x, u, \dot{u}, \cdots, u^{(n-1)}$. $\mathcal{E}$ is the formal vector space spanned by $\{dx, du, d\dot{u}, \cdots, du^{(n-1)}\}$ over $\mathcal{K}$. The notation dx stands for $\{dx_1, \cdots, dx_n\}$ and $du^{(k)}$ for $\{du_1^{(k)}, \cdots, du_m^{(k)}\}$. Let $\mathcal{X} := \mathrm{span}_{\mathcal{K}}\{dx\}$ and $\mathcal{U} := \mathrm{span}_{\mathcal{K}}\{du, d\dot{u}, \cdots, du^{(n-1)}\}$.

Consider a subspace $\Omega \subset \mathcal{X}$. Define

$$\dot{\Omega} = \mathrm{span}_{\mathcal{K}}\{\dot{\omega} \mid \omega \in \Omega\} \tag{4}$$

where $\omega = \sum_{i=1}^{n} \omega_i(x, u, \dot{u}, \cdots, u^{(n-1)})dx_i$ and time-derivation is defined by $\dot{\omega} = \sum_{i=1}^{n}(\omega_i d\dot{x}_i + \dot{\omega}_i dx_i)$. Thus $\dot{\omega} \in \mathrm{span}_{\mathcal{K}}\{dx, d\dot{x}\}$.

Definition 2.1 A subspace $\Omega \subset \mathcal{X}$ is said to be invariant with respect to (1) if

$$\dot{\Omega} \subset \Omega + \mathrm{span}_{\mathcal{K}}\{du\} \tag{5}$$

∎

Remark 2.2 Let $\mathcal{K}_k$ be the field of meromorphic functions of $x, u, \cdots, u^{(k)}$ and define

$$\mathcal{K}' = \bigcup_{k \in I\!N} \mathcal{K}_k$$

Then (5) is equivalent to the statement that $(\Omega + \mathrm{span}_{\mathcal{K}'}\{du^{(k)} \mid k \geq 0\})$ is a differential vector space, with the derivation defined above.

Example 2.3 Let Ω be an involutive invariant codistribution for (1) and let (x_1, x_2) be a local system of coordinates such that $\Omega = \mathrm{span}\{dx_1\}$. Then in the coordinates (x_1, x_2), (1) takes the form (cf. [18],[25])

$$\begin{array}{rcl}
\dot{x}_1 & = & f_1(x_1) + g_1(x_1)u \\
\dot{x}_2 & = & f_2(x_1, x_2) + g_2(x_1, x_2)u
\end{array} \tag{6}$$

Then

$$\dot{\Omega} = \mathrm{span}_{\mathcal{K}}\{d\dot{x}_1\} = \mathrm{span}_{\mathcal{K}}\{d(f_1(x_1) + g_1(x_1)u)\} \subset \Omega + \mathrm{span}_{\mathcal{K}}\{du\} \tag{7}$$

Hence Ω is invariant in the sense of Definition 2.1.

When a given subspace is not invariant, it is interesting to know whether or not there exists a feedback transformation that renders it invariant. This is the topic of the next section.

3 Controlled invariant subspaces

In this section we define and characterize the controlled invariance of subspaces $\Omega \subset \mathcal{X}$ under quasi-static state feedback. In Subsection 3.1 we first define quasi-static state feedback, based on ([9],[10],[11]). In Subsection 3.2 we give a definition of controlled invariance under quasi-static state feedback. We make some remarks about the smallest controlled invariant subspace containing some given subspace in Subsection 3.3. As shown in [28], this subspace allows to characterize the solvability conditions of the dynamic disturbance decoupling problem. In Subsection 3.4 some properties of controlled invariance under regular static state feedback (3) are given. Finally,

conditions for controlled invariance of subspace $\Omega \subset \mathcal{X}$ under quasi-static state feedback are investigated in Subsection 3.5.

Throughout this paper we employ the following terminology. A vector $\omega \subset \mathcal{E}$ is called *exact* if there exists a $\phi \in \mathcal{K}$ such that $\omega = d\phi$. A subspace $\Omega \subset \mathcal{E}$ of dimension r is called *exact* if there exist functions $\phi_1, \cdots, \phi_r \in \mathcal{K}$ such that $\Omega = \mathrm{span}_{\mathcal{K}}\{d\phi_1, \cdots, d\phi_r\}$. Given subspaces $\Omega_1 \subset \Omega_2 \subset \mathcal{E}$, (Ω_2/Ω_1) is said to be *exact* if there exist functions $\phi_1, \cdots, \phi_d \in \mathcal{E}$, with $d = \dim(\Omega_2) - \dim(\Omega_1)$, such that $\Omega_2 = \Omega_1 \oplus \mathrm{span}_{\mathcal{K}}\{d\phi_1, \cdots, d\phi_d\}$, or, in other words, (Ω_2/Ω_1) is isomorphic to an exact subspace of $\mathcal{E}$. Consider a subspace $\Omega \subset \mathcal{E}$. Then clearly $0 \subset \Omega$ is exact. Furthermore, if $\Omega_1 \subset \Omega$, $\Omega_2 \subset \Omega$ are exact, then also $\Omega_1 + \Omega_2 \subset \Omega$ is exact. Hence there exists a unique maximal exact subspace in Ω.

3.1 Quasi-static state feedback

Consider the nonlinear system (1). A *generalized static state feedback* for (1) is a feedback of the form

$$u = \phi(x, v, \cdots, v^{(r)}) \tag{8}$$

where $v \in I\!\!R^m$ denotes the new controls. Let $\mathcal{K}_v$ denote the field of meromorphic functions of $\{x, \{v^{(k)} \mid k \geq 0\}\}$ and define the formal vector space $\mathcal{E}_v := \mathrm{span}_{\mathcal{K}_v}\{d\xi \mid \xi \in \mathcal{K}_v\}$. As in [9],[10], we define the following *filtrations* ([2]) of $\mathcal{E}_v$:

$$\begin{aligned} \mathcal{V}_{-1} &:= \mathrm{span}_{\mathcal{K}_v}\{dx\} \\ \mathcal{V}_k &:= \mathrm{span}_{\mathcal{K}_v}\{dx, dv, \cdots, dv^{(k)}\} \quad (k \geq 0) \end{aligned} \tag{9}$$

$$\begin{aligned} \mathcal{U}_{-1} &:= \mathrm{span}_{\mathcal{K}_v}\{dx\} \\ \mathcal{U}_k &:= \mathrm{span}_{\mathcal{K}_v}\{dx, d\phi, \cdots, d\phi^{(k)}\} \quad (k \geq 0) \end{aligned} \tag{10}$$

The filtrations $\mathcal{U}_k$ and $\mathcal{V}_k$ are said to have *bounded difference* ([2]) if there exists an $s \in I\!\!N$ such that for all $k \geq -1$

$$\begin{aligned} \mathcal{U}_k &\subset \mathcal{V}_{k+s} \\ \mathcal{V}_k &\subset \mathcal{U}_{k+s} \end{aligned} \tag{11}$$

Definition 3.4 ([9],[10],[11]) u given by (8) is said to be a *quasi-static state feedback* for (1) if the filtrations $\mathcal{U}_k$ and $\mathcal{V}_k$ have bounded difference.

Remark 3.5 It is easily verified that a regular static state feedback (3) is a quasi-static state feedback.

The following result is also easily proved.

Proposition 3.6 *Let u given by (8) be a quasi-static state feedback. Then there exist an integer $s \in I\!\!N$ and a function $\psi(x, u, \cdots, u^{(s)})$ such that, locally,*

$$v = \psi(x, u, \cdots, u^{(s)}) \tag{12}$$

∎

3.2 Controlled invariance

Consider the control system (1) together with a quasi-static state feedback (8) and define $\mathcal{V} := \mathrm{span}_{\mathcal{K}_v}\{dv^{(k)} \mid k \geq 0\}$. Let us denote $\Theta^{(k)}$ as the time derivative of order k of Θ along the trajectories of the system (1), and $\Theta^{[k]}$ as the time derivative of order k of Θ along the trajectories of the closed loop system (1) fed back with (8). We will write simply $\dot{\Theta}$ for $\Theta^{(1)}$.

Definition 3.7 A subspace $\Omega \subset \mathcal{X}$ is said to be controlled invariant for (1) if there exists a quasi-static state feedback (8) such that for (1)(8) one has

$$\Omega^{[1]} \subset \Omega + \mathcal{V} \tag{13}$$

The definition of controlled invariance given in Definition 3.7 is in accordance with the well known definition of a controlled invariant codistribution. Recall from e.g. [18],[25] that a codistribution Ω is controlled invariant if there exists a regular static state feedback (3) such that

$$\begin{aligned}
\mathcal{L}_{f+g\alpha}\Omega &\subset \Omega \\
\mathcal{L}_{(g\beta)_{*i}}\Omega &\subset \Omega \quad (i = 1, \cdots, m)
\end{aligned} \tag{14}$$

Let $\omega \in \Omega$. Then for (1,3) we have

$$\omega^{[1]} = \mathcal{L}_{f+g\alpha}\omega + \sum_{i=1}^{m}(v_i \mathcal{L}_{(g\beta)_{*i}}\omega + \langle \omega, (g\beta)_{*i}\rangle dv_i) \in \Omega + \mathcal{V} \tag{15}$$

when we interpret Ω as a subspace of $\mathrm{span}_{\mathcal{K}}\{dx\}$.

The following theorem gives a necessary condition for controlled invariance. For (1), let $\mathcal{G}$ denote the distribution spanned by the input vector fields. Define the subspace $\mathcal{G}^{\perp} \subset \mathcal{X}$ by

$$\mathcal{G}^{\perp} = \{\omega \in \mathcal{X} \mid \langle \omega, g\rangle \equiv 0, \ \forall g \in \mathcal{G}\} \tag{16}$$

Theorem 3.8 *Let $\Omega \subset \mathcal{X}$. Then Ω is controlled invariant only if*

$$(\widehat{\Omega \cap \mathcal{G}^{\perp}}) \subset \Omega \tag{17}$$

Proof By definition of $\mathcal{G}^{\perp}$, $(\widehat{\Omega \cap \mathcal{G}^{\perp}}) \subset \mathcal{X}$. Controlled invariance of Ω then implies (17). ∎

Remark 3.9 In fact, using (15), it may be shown that (17) is equivalent to the well known conditions $\mathcal{L}_f(\Omega \cap \mathcal{G}^{\perp}) \subset \Omega$, $\mathcal{L}_{g_i}(\Omega \cap \mathcal{G}^{\perp}) \subset \Omega (i = 1, \cdots, m)$ for controlled invariance of involutive codistributions (cf. [18], [25]).

3.3 The smallest controlled-invariant subspace containing a given subspace

Given a subspace $\Pi \subset \mathcal{X}$, it is unclear whether (or under what conditions) there exists a smallest controlled invariant subspace containing Π. This is due to the fact that for two controlled invariant subspaces $\Omega_1, \Omega_2 \subset \mathcal{X}$, we do not necessarily have that $\Omega_1 \cap \Omega_2$ is controlled invariant, so that we cannot use the "standard" arguments (as in e.g. [32],[18],[25]). In this subsection we will give some comments on this question.

We will use the following notation. Given a subspace $\Pi \subset \mathcal{X}$, we define

$$\Pi_* := \mathcal{X} \cap (\Pi + \Pi^{(1)} + \cdots + \Pi^{(n-1)}) \tag{18}$$

In what follows, we will need the following lemma.

Lemma 3.10 *Consider a subspace $\Omega \subset \mathcal{X}$ satisfying $(\Omega \cap \mathcal{G}^{\perp}) = \{0\}$. Then we have for all $k \in \mathbb{N}$:*

$$\mathcal{X} \cap (\Omega^{(1)} + \cdots + \Omega^{(k)}) = \{0\} \tag{19}$$

4

Proof Let $d := \dim(\Omega)$, and let $\omega_1, \cdots, \omega_d$ be a basis of Ω, with

$$\omega_i = \sum_{j=1}^{n} \omega_{ij}(x, u, \cdots, u^{(r)}) dx_j \quad (i = 1, \cdots, d) \tag{20}$$

Let $A(x, u, \cdots, u^{(r)})$ be the (d, n)-matrix with entries ω_{ij} $(i = 1, \cdots, d; j = 1, \cdots, n)$. Since $\omega_1, \cdots, \omega_d$ forms a basis of Ω, the matrix A has full row rank over $\mathcal{K}_u$. We may now characterize Ω by

$$\Omega = \text{rowspan}_{\mathcal{K}_u}(A(x, u, \cdots, u^{(r)}) \; 0 \; \cdots \; 0) \tag{21}$$

while $\Omega^{(k)}$ $(k = 1, 2, \cdots)$ may be characterized by

$$\Omega^{(k)} = \text{rowspan}_{\mathcal{K}_u}(X_{k0} \; X_{k1} \; \cdots \; X_{kk-1} \; (Ag) \; 0 \; \cdots \; 0) \tag{22}$$

for certain matrices $X_{k0}, \cdots, X_{kk-1}$. Now assume that (Ag) is not right-invertible over $\mathcal{K}_u$. This implies that there exists a non-zero row-vector $\eta^T := (\eta_1 \cdots \eta_d)$ such that

$$\eta^T(Ag) = 0 \tag{23}$$

However, this would imply that $\omega := \sum_{j=1}^{d} \eta_j \omega_j$ satisfies

$$\langle \omega, \tau \rangle = 0 \quad (\forall \tau \in \mathcal{G}) \tag{24}$$

which contradicts the fact that $(\Omega \cap \mathcal{G}^\perp) = \{0\}$. Hence we have that (Ag) is right-invertible over $\mathcal{K}_u$. Next, let $\omega \in \mathcal{X} \cap (\Omega^{(1)} + \cdots + \Omega^{(k)})$ $(k \in \{1, 2, \cdots\})$. Since $\omega \in (\Omega^{(1)} + \cdots + \Omega^{(k)})$, we may represent ω by a row-vector

$$(\eta_1^T \; \cdots \; \eta_k^T) \begin{pmatrix} X_{10} & (Ag) & 0 & \cdots & \cdots & 0 \\ X_{20} & X_{21} & (Ag) & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ X_{k0} & X_{k1} & X_{k2} & \cdots & X_{kk-1} & (Ag) \end{pmatrix}$$

The fact that $\omega \in \mathcal{X}$ implies that necessarily

$$(\eta_1^T \; \cdots \; \eta_k^T) \begin{pmatrix} (Ag) & 0 & 0 & \cdots & \cdots & 0 \\ X_{21} & (Ag) & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ X_{k1} & X_{k2} & X_{k3} & \cdots & X_{kk-1} & (Ag) \end{pmatrix} = 0$$

and thus

$$\eta_i^T(Ag) = 0$$

which give $\eta_i^T = 0$, since (Ag) is right-invertible. Thus, $\omega = 0$, which establishes our claim. ∎

Proposition 3.11 *Let $\Omega \subset \mathcal{X}$ be a subspace satisfying $\widehat{\Omega \cap \mathcal{G}^\perp} \subset \Omega$. Then*

$$\Omega_* = \Omega$$

Proof Let $\tilde{\Omega}$ be such that

$$\Omega = (\Omega \cap \mathcal{G}^\perp) \oplus \tilde{\Omega} \tag{25}$$

By hypothesis we have

$$(\widehat{\Omega \cap \mathcal{G}^\perp}) \subset \Omega \tag{26}$$

We now prove by induction that we have

$$(\Omega \cap \mathcal{G}^\perp)^{(k)} \subset \Omega + \tilde{\Omega}^{(1)} + \cdots + \tilde{\Omega}^{(k-1)} \quad (k = 1, 2, \cdots) \tag{27}$$

By (26), we have that (27) holds for $k = 1$. Next assume that (27) holds for $k = 1, \cdots, \ell - 1$. Then

$$(\Omega \cap \mathcal{G}^\perp)^{(\ell)} = ((\Omega \cap \mathcal{G}^\perp)^{(\ell-1)})^{(1)} \overset{\text{IH}}{\subset} (\Omega + \tilde{\Omega}^{(1)} + \cdots + \tilde{\Omega}^{(\ell-2)})^{(1)} =$$

$$(\Omega^{(1)} + \tilde{\Omega}^{(2)} + \cdots + \tilde{\Omega}^{(\ell-1)}) \overset{(25)}{=} ((\widehat{\Omega \cap \mathcal{G}^\perp}) + \tilde{\Omega}^{(1)} + \cdots + \tilde{\Omega}^{(\ell-1)}) \overset{(26)}{\subset}$$

$$(\Omega + \tilde{\Omega}^{(1)} + \cdots + \tilde{\Omega}^{(\ell-1)})$$

which establishes (27). Using (27) and the modular distributive rule (see e.g. [32, Section 0.3]) we obtain

$$\Omega_* = \mathcal{X} \cap (\Omega + \Omega^{(1)} + \cdots + \Omega^{(n-1)}) =$$

$$\mathcal{X} \cap (\Omega + (\Omega \cap \mathcal{G}^\perp)^{(1)} + \tilde{\Omega}^{(1)} + \cdots + (\Omega \cap \mathcal{G}^\perp)^{(n-1)} + \tilde{\Omega}^{(n)}) \subset \tag{28}$$

$$\mathcal{X} \cap (\Omega + \tilde{\Omega}^{(1)} + \cdots + \tilde{\Omega}^{(n)}) = \Omega + \mathcal{X} \cap (\tilde{\Omega}^{(1)} + \cdots + \tilde{\Omega}^{(n-1)})$$

Since by definition of $\tilde{\Omega}$ we have that $(\tilde{\Omega} \cap \mathcal{G}^\perp) = \{0\}$, we obtain from (28) and Lemma 3.10 that

$$\Omega_* \subset \Omega \tag{29}$$

Furthermore, we have by definition of Ω_* that

$$\Omega \subset \Omega_* \tag{30}$$

Hence we have that $\Omega_* = \Omega$, which establishes our claim. ∎

Corollary 3.12 *Consider a subspace* $\Pi \subset \mathcal{X}$ *and let* $\Omega \subset \mathcal{X}$ *be a controlled invariant subspace containing* Π. *Then* $\Pi_* \subset \Omega$.

Proof Using the definition of Π_*, the fact that $\Pi \subset \Omega$, and combining the results of Theorem 3.8 and Propositio 3.11, we obtain

$$\Pi_* = \mathcal{X} \cap (\Pi + \Pi^{(1)} + \cdots + \Pi^{(n)}) \subset \mathcal{X} \cap (\Omega + \Omega^{(1)} + \cdots + \Omega^{(n-1)}) = \Omega_* = \Omega$$

which establishes our claim. ∎

The subspace Π_* defined in (18) is, by Corollary 3.12, a candidate for being the smallest controlled invariant subspace containing Π. If Π is exact, it can be shown that indeed it *is*. This may be shown in the following way. Let $r = \dim \Pi$ and choose meromorphic functions $h_1(x), \cdots, h_r(x)$ such that $\Pi = \text{span}_{\mathcal{K}}\{dh_1, \cdots, dh_r\}$. Next consider the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{31}$$

Then for this system, $\Pi_* = \mathcal{X} \cap \mathcal{Y}$, where $\mathcal{Y} = \text{span}_{\mathcal{K}}\{dy, \cdots, dy^{(n-1)}\}$. (The subspace $\mathcal{X} \cap \mathcal{Y}$ was introduced in [7] for the study of the minimal order input-output decoupling problem.) If the system (31) is right-invertible, one can construct a quasi-static state feedback which renders Π_* invariant by using the construction in [28]. If (31) is not right-invertible, the same construction, together with Lemma 1 from [22] may be used to show that Π_* is controlled invariant. Summarizing, we have the following result:

6

Theorem 3.13 *Consider a subspace* $\Pi \subset \mathcal{X}$ *which is exact. Then* $\Pi_* := \mathcal{X} \cap (\Pi + \cdots + \Pi^{(n-1)})$ *is the smallest controlled invariant subspace containing* Π. ∎

A nice application of the subspace $\Pi_* = \mathcal{X} \cap \mathcal{Y}$ (the smallest controlled invariant subspace containing the differential of the output) was shown in ([28]). This subspace allows to characterize the solvability conditions of disturbance decoupling problem by means of quasi-static state feedback, and then by dynamic state feedback. This condition is in accordance with the one used in case of the static disturbance decoupling problem where the concept of supremal controlled invariant subspace or supremal controlled invariant distribution contained in kernel of the output is applied (see [18],[32]).

3.4 Characterization of controlled invariant subspaces under regular static state feedback

In this subsection we investigate under what conditions a subspace $\Omega \subset \mathcal{X}$ is controlled invariant under regular static state feedback. Recall from Subsection 3.2 that a regular static state feedback is a special sort of quasi-static state feedback. A first result is the following.

Proposition 3.14 *Consider a d-dimensional subspace* $\Omega \subset \mathcal{X}$. *Assume that* Ω *is controlled invariant under a quasi-static state feedback of the form* $u = \phi(x, v)$. *Then* Ω *admits a basis* $\omega_1, \cdots, \omega_d$ *with*

$$\omega_i = \sum_{j=1}^{n} \omega_{ij}(x) dx_j \tag{32}$$

Proof Assume that $\Omega = \mathrm{span}_{\mathcal{K}}\{\tilde{\omega}_1, \cdots, \tilde{\omega}_d\}$, with

$$\tilde{\omega}_i = \sum_{j=1}^{n} \tilde{\omega}_{ij}(x, u) dx_j \tag{33}$$

Let $A(x, u)$ be the matrix with entries $\tilde{\omega}_{ij}$ $(i = 1, \cdots, d; j = 1, \cdots, n)$. Viewing Ω as a linear subspace (over $\mathcal{K}$) of $\mathcal{X} \oplus \mathrm{span}_{\mathcal{K}}\{du\}$, it may be characterized by

$$\Omega = \mathrm{rowspan}_{\mathcal{K}} (A(x, u) \quad 0) \tag{34}$$

Similarly, $\Omega + \dot{\Omega}$ is characterized by

$$\dot{\Omega} = \mathrm{rowspan}_{\mathcal{K}} \begin{pmatrix} A(x, u) & 0 \\ B(x, u, \dot{u}) & (Ag)(x, u) \end{pmatrix} \tag{35}$$

where

$$B(x, u, \dot{u}) = \sum_{i=1}^{n} \frac{\partial A}{\partial x_i}(x, u)\dot{x}_i(x, u) + \sum_{j=1}^{m} \frac{\partial A}{\partial u_j}\dot{u}_j + A(x, u)\left(f_x(x) + \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}u\right) \tag{36}$$

with f_x the Jacobian of f. Since Ω is rendered invariant via $u = \phi(x, v)$ there exist matrices $P(x, v, \dot{v})$ and $Q(x, v)$ such that

$$B(x, \phi, \dot{\phi})dx + (Ag)(x, \phi)d\phi = P(x, v, \dot{v})A(x, \phi)dx + Q(x, v)dv \tag{37}$$

or

$$\begin{aligned} B(x, \phi, \dot{\phi}) &= P(x, v, \dot{v})A(x, \phi) - (Ag)(x, \phi)\phi_x(x, v) \\ (Ag)(x, \phi)\phi_v(x, v) &= Q(x, v) \end{aligned} \tag{38}$$

7

Since $\phi_v(x,v)$ is invertible, this yields

$$B(x,\phi,\dot\phi) = P(x,v,\dot v)A(x,\phi) - Q(x,v)\phi_v(x,v)^{-1}\phi_x(x,v) \tag{39}$$

Furthermore, by the Inverse Function Theorem there locally exists a function $\psi(x,u)$ such that $u = \phi(x,v)$ is equivalent to $v = \psi(x,u)$ and $\psi_x(x,u) = -\phi_v(x,\psi(x,u))^{-1}\phi_x(x,\psi(x,u))$. Hence (39) yields

$$B(x,u,\dot u) = \tilde P(x,u,\dot u)A(x,u) + \tilde Q(x,u)\psi_x(x,u) \tag{40}$$

where $\tilde P(x,u,\dot u) = P(x,\psi(x,u),\dot\psi(x,u,\dot u))$ and $\tilde Q(x,u) = Q(x,\psi(x,u))$. Taking partial derivatives with respect to $\dot u_i$, we obtain

$$\frac{\partial A}{\partial u_i} = \frac{\partial \tilde P}{\partial \dot u_i}A(x,u) \quad (i = 1,\cdots,m) \tag{41}$$

Obviously,

$$\frac{\partial^2 \tilde P}{\partial \dot u_i \partial \dot u_j} = 0 \quad (i,j = 1,\cdots,m)$$

Hence there exist matrices $R_i(x,u)$ ($i = 1,\cdots,m$) such that

$$\frac{\partial A}{\partial u_i} = R_i(x,u)A(x,u) \tag{42}$$

Using arguments from the theory of linear time-varying ordinary differential equations this yields that $A(x,u)$ is of the form

$$A(x,u) = \Phi(x,u)\Psi(x)$$

with $\Phi(x,u)$ square invertible. Hence

$$\Omega = \mathrm{rowspan}_{\mathcal{K}}\left(A(x,u) \ \ 0\right) = \mathrm{rowspan}_{\mathcal{K}}\left(\Psi(x) \ \ 0\right) \tag{43}$$

which establishes our claim. If $\Omega = \mathrm{rowspan}_{\mathcal{K}}\left(A(x,u,\cdots,u^{(\ell)}) \ \ 0\right)$ with $\ell > 1$, the claim is established by using the same arguments together with an induction argument. ∎

From the above proposition it follows that the set of subspaces $\Omega \subset \mathcal{X}$ that are controlled invariant under a quasi-static state feedback $u = \phi(x,v)$ may be identified with the set of "classical" controlled invariant codistributions. The following theorem gives a characterization of controlled invariance in our generalized framework.

Theorem 3.15 *Let $\Omega \subset \mathcal{X}$ be a subspace such that*

$$(\Omega + \dot\Omega)/\Omega \ \ is \ exact \tag{44}$$

and which admits a basis satisfying (32). Then Ω is controlled invariant under a quasi-static state feedback $u = \phi(x,v)$ if and only if

$$\widehat{(\Omega \cap \mathcal{G}^\perp)} \subset \Omega \tag{45}$$

Moreover, if the conditions above are satisfied, then $\phi(x,v)$ rendering Ω invariant may be chosen of the form (3).

Proof The necessity was proven in Theorem 3.8.
Assume that (45) holds. Note that $\Omega + \dot\Omega \subset \mathrm{span}_{\mathcal{K}}\{dx,du\}$. Let $\tilde\Omega \subset \mathcal{X}$ be such that $\Omega = (\Omega \cap \mathcal{G}^\perp) \oplus \tilde\Omega$. Assume that $\dot{\tilde\Omega} \cap \mathcal{X} \neq \{0\}$. This implies that there is an $\tilde\omega \in \dot{\tilde\Omega}$, $\tilde\omega \neq 0$, such that $\tilde\omega \in \mathcal{X}$ and hence $\tilde\omega \in (\Omega \cap \mathcal{G}^\perp)$, which gives a contradiction. Thus

$$\dot{\tilde\Omega} \cap \mathcal{X} = \{0\} \tag{46}$$

By (44), there exists a $v_1(x, u)$ such that

$$\Omega + \dot\Omega = \Omega \oplus \text{span}_{\mathcal{K}}\{dv_1\} \tag{47}$$

Since (45) and (46) hold, we must have that $(\partial v_1/\partial u)$ has full row rank. Then there exists a function $v_2(u)$ such that $(\partial v/\partial u)$ is square and invertible, where $v = (v_1^T\ v_2^T)^T$. By (47) we now have that

$$\Omega^{[1]} \subset \Omega + \mathcal{V} \tag{48}$$

Moreover, since $(\partial v/\partial u)$ is invertible, there exists a $\psi(x, v)$ such that $u = \psi(x, v)$. Hence ψ defines a quasi-static state feedback and thus Ω can be rendered invariant via quasi-static state feedback. Since we are dealing with a control system (1) that is affine in u, it is easily seen that v can be taken affine in u and thus ψ can be taken affine in v. This implies that Ω can be rendered invariant via a static state feedback (3). ∎

Remark 3.16

(i) If Ω is exact, then clearly also $(\Omega + \dot\Omega)/\Omega$ is exact. Hence the set of subspace $\Omega \subset \mathcal{X}$ such that $(\Omega + \dot\Omega)/\Omega$ is exact, incorporates the "classical" involutive codistributions.

(ii) The condition $(\Omega + \dot\Omega)/\Omega$ is exact is not necessary for controlled invariance. This can be seen from the following counter example. Take the system $\dot x_1 = u_1$, $\dot x_2 = u_2$, $\dot x_3 = 0$ and $\Omega = \text{span}_{\mathcal{K}}\{dx_1 + x_2 dx_3\}$. It is straightforwardly to check that $(\widehat{\Omega \cap \mathcal{G}^{\perp}}) \subset \Omega$ and that $(\Omega + \dot\Omega)/\Omega$ is not exact. However, with the regular static state feedback $u_1 = v_1 - x_3 v_2$, $u_2 = v_2$ we obtain

$$\dot\Omega = \text{span}_{\mathcal{K}}\{dv_1 - x_3 dv_2\} \subset \Omega + \mathcal{V}$$

and hence Ω is controlled invariant.

3.5 Some characterizations of controlled invariance

Conditions are derived for controlled invariance of a subspace under a quasi-static state feedback.

3.5.1 The general case : a sufficient condition

Let us consider a general subspace $\Omega \subset \mathcal{X}$. Define by induction:

$$\hat\Omega_0 \ := \ 0$$

$$\Omega_0 \ := \ \Omega$$

$$\hat\Omega_{k+1} \ := \ \text{maximal exact subspace in } \frac{\Omega_k + \dot\Omega_k}{\Omega_k}$$

$$\Omega_{k+1} \ := \ \Omega_k + \hat\Omega_{k+1}$$

Furthermore, define

$$k^* := \max\{k \geq 1 \mid \dim(\hat\Omega_k) > \dim(\hat\Omega_{k-1})\}$$

Theorem 3.17 *Let $\Omega \subset \mathcal{X}$. If*

(i) $(\widehat{\Omega \cap \mathcal{G}^{\perp}}) \subset \Omega$

(ii) $\dfrac{\Omega_{k^*-1} + \dot\Omega_{k^*-1}}{\Omega_{k^*-1}}$ *is exact.*

9

then Ω is controlled invariant for (1).

Proof From the definition of k^*, there exist vector valued $dv_1, \cdots, dv_{k^*}$ in $\mathcal{E}$, where each dv_i is non-empty, such that

$$
\begin{aligned}
\hat{\Omega}_1 &= \text{span}_{\mathcal{K}}\{dv_1\} \subset \frac{\Omega_0 + \dot{\Omega}_0}{\Omega_0} \\[2mm]
\hat{\Omega}_2 &= \text{span}_{\mathcal{K}}\{d\dot{v}_1, dv_2\} \subset \frac{\Omega_1 + \dot{\Omega}_1}{\Omega_1} \\[2mm]
&\quad\vdots \\[2mm]
\hat{\Omega}_{k^*} &= \text{span}_{\mathcal{K}}\{dv_1^{(k^*-1)}, dv_2^{(k^*-2)}, \cdots, dv_{k^*}\} \subset \frac{\Omega_{k^*-1} + \dot{\Omega}_{k^*-1}}{\Omega_{k^*-1}}
\end{aligned}
\tag{49}
$$

Note that from *(ii)* the last inclusion in (49) is in fact an equality. We now have

$$
\begin{aligned}
\dot{\Omega} &\subset \quad \Omega_0 + \hat{\Omega}_1 + \dot{\Omega}_0 + \dot{\hat{\Omega}}_1 = \Omega_1 + \dot{\Omega}_1 \subset \cdots \subset \\[2mm]
&\quad \Omega_{k^*-1} + \dot{\Omega}_{k^*-1} = \Omega_{k^*-1} + \text{span}_{\mathcal{K}}\{dv_1^{(k^*-1)}, \cdots, dv_{k^*}\} \subset \\[2mm]
&\quad \Omega + \text{span}_{\mathcal{K}}\{dv^{(k)} \mid k \geq 0\}
\end{aligned}
\tag{50}
$$

It remains to be shown that v defines a quasi-static state feedback. From the above construction, one has

$$
\begin{aligned}
v_1 &= \phi_1(x, u) \\
v_2 &= \phi_2(x, v_1, \dot{v}_1, u) \\
&\quad\vdots \\
v_{k^*} &= \phi_{k^*}(x, \{v_i^{(j)} \mid 1 \leq i \leq k^* - 1, 0 \leq j \leq k^* - i\}, u)
\end{aligned}
\tag{51}
$$

¿From *(i)*, $(\partial(\phi_1, \cdots, \phi_{k^*})/\partial u)$ has full row rank. Thus there exists $v_{k^*+1} = \phi_{k^*+1}(u)$ such that $(\partial(\phi_1, \cdots, \phi_{k^*+1})/\partial u)$ is square invertible. From the Inverse Function Theorem, there exists a function ψ such that $u = \psi(x, v, \dot{v}, \cdots, v^{(k^*)})$. By applying this feedback, one has

$$
\Omega^{[1]} \subset \Omega + \text{span}_{\mathcal{K}}\{dv^{(k)} \mid k \geq 0\}
$$

$\blacksquare$

Remark 3.18 The above theorem only gives sufficient conditions for the controlled invariance of a subspace $\Omega \subset \mathcal{X}$. In Theorem 3.8 it was shown that *(i)* is also a necessary condition. But the condition *(ii)* is not. This is shown by the following example.

Example 3.19 ([18])We consider a nonlinear system on $\mathbb{R}^4$ with three inputs u_1, u_2, u_3 given by:

$$
\dot{x}_1 = u_1, \quad \dot{x}_2 = x_4 + u_2, \quad \dot{x}_3 = x_3 u_1 + u_2, \quad \dot{x}_4 = u_3
$$

Let $\Omega = \text{span}_{\mathcal{K}}\{dx_1 - u_1 dx_3, dx_4\}$. Then Ω is not exact, and $\dot{\Omega}$ is given by

$$
\dot{\Omega} = \text{span}_{\mathcal{K}}\{(1 - u_1 x_1)du_1 - u_1 du_2 - \dot{u}_1 dx_3 - u_1^2 dx_1, du_3\}.
$$

Ω is invariant by $u_1 = v_1$, $u_2 = -\dfrac{\dot{v}_1}{v_1^2}x_1 - v_1 x_1 + v_2$ and $u_3 = v_3$. One obtains $k^* = 1$, but $\dfrac{\Omega + \dot{\Omega}}{\Omega}$ is not exact.

10

3.5.2 A special case

Let us consider a subspace $\Omega \subset \mathcal{X}$ such that

$$\Omega = \Omega \cap \mathcal{G}^\perp + \Phi_* \tag{52}$$

where Φ is an any exact subspace in $\mathcal{X}$.

Proposition 3.20 *Let $\Omega \subset \mathcal{X}$ satisfy (52), then Ω is controlled invariant if and only if*

$$\widehat{(\Omega \cap \mathcal{G}^\perp)} \subset \Omega \tag{53}$$

Proof By Theorem 3.8 we only need to show the sufficiency. Clearly Φ_* is controlled invariant (see Theorem 3.13). Hence there exists a quasi-static feedback (8) such that

$$\Phi_*^{[1]} \subset \Phi_* + \mathcal{V}$$

Now (53) implies that

$$\Omega^{[1]} \subset \Omega + \mathcal{V}$$

and hence Ω is controlled invariant. ∎

An effective way to compute Φ_* satisfying (52) is given by the following proposition.

Proposition 3.21 *Let $\Omega \subset \mathcal{X}$ be a subspace such that (53) holds. Then there exists an exact subspace $\Phi \subset \Omega$ satisfying (52) if and only if*

$$\Omega = \Omega \cap \mathcal{G}^\perp + \hat{\Phi}_* \tag{54}$$

where $\hat{\Phi}$ is the largest exact subspace in Ω

Proof
Assume that (54) holds. Taking $\Phi = \hat{\Phi}$, we then have (52). Conversely, assume that there exists an exact subspace $\Phi \subset \mathcal{X}$ such that (52) holds. Clearly $\Phi_* \subset \hat{\Phi}_*$. Now $\hat{\Phi} \subset \Omega$ implies by Proposition 3.11 that $\hat{\Phi}_* \subset \Omega$. Thus

$$\Omega = \Omega \cap \mathcal{G}^\perp + \Phi_* \subset \Omega \cap \mathcal{G}^\perp + \hat{\Phi}_* \subset \Omega$$

Hence (54) is verified. ∎

4 Controllability cospaces

In this section, controllability cospaces are studied under quasi-static state feedbacks as a special class of controlled invariant subspaces previously defined. These controllability cospaces are related to the dual of dynamic controllability distributions (see [30]). In Subsection 4.1 we first define controllability cospaces. An algorithm which characterizes these cospaces is then given in Subsection 4.2 and some properties are discussed. In Subsection 4.3 we derive an algorithm computing the smallest controllability cospace containing a given exact subspace, while its applications are treated in Subsection 4.4 and Subsection 4.5.

4.1 Definition of controllability cospaces

Controllability cospaces consist of vectors which are autonomous after applying certain quasi-static state feedback $u = \psi(x, v, \cdots, v^{(r)})$ and zeroing certain input channels v_j, where $j \in \mathcal{J} \subset \{1, \cdots, m\}$. Such nonregular transformations are not defined for every element in $\mathcal{K}_v$. One possibility to circumvent this problem is to consider the module $\text{span}_{\mathcal{A}}\{dx\}$ over the ring of analytic functions rather than the linear space over the field of meromorphic functions. Another way is chosen here; it consists in taking a particular basis of a given subspace of $\text{span}_{\mathcal{K}}\{dx\}$ so that its time derivative is well defined when applying nonregular feedback. Such a basis always exists. More precisely, let $\Theta \subset \mathcal{X}$ be a subspace which admits a basis $\theta_1, \cdots, \theta_d$ with

$$\theta_i = \sum_{k=1}^{n} \frac{\alpha_{ik}(x, v, \cdots, v^{(\nu)})}{\beta_{ik}(x, v, \cdots, v^{(\nu)})} dx_i,$$

where α_{ik} and β_{ik} are in $\mathcal{A}$, the ring of analytic functions of $\{x, v^{(k)} \mid k \geq 0\}$. Obviously, we can choose another basis for Θ, $\tilde{\theta}_1, \cdots, \tilde{\theta}_d$, in the module $\text{span}_{\mathcal{A}}\{dx\}$ over the ring $\mathcal{A}$ by taking

$$\tilde{\theta}_i = (\prod_{k=1}^{n} \beta_{ik})\theta_i$$

Definition 4.22 *A subspace $\mathcal{C} \subset \mathcal{X}$ is said to be a controllability cospace for (1) if there exists a quasi-static state feedback (8) and a set of integers $\mathcal{J} \subset \{1, \cdots, m\}$ such that for (1),(8) one has*

$$\mathcal{C}^{[1]} \subset \mathcal{C} + \mathcal{V} \tag{55}$$

and

$$\mathcal{C} = \max\{\Theta \subset \mathcal{X} \mid \text{span}_{\mathcal{K}}\{\tilde{\theta}_i^{[1]} \mid_{v_j=0, j \in \mathcal{J}}\} \subset \Theta\} \tag{56}$$

where $\tilde{\theta}_i$ is defined as above.

This means that $\mathcal{C}$ is the largest autonomous subspace in $\mathcal{X}$ of the closed loop system. Moreover, by this definition, it is clear that a controllability cospace is controlled invariant. The following example illustrates the above definition.

Example 4.23 Consider again the nonlinear system given in Example 3.19. Let $\mathcal{C} = \text{span}_{\mathcal{K}} \{dx_1, d(x_2 - x_3), dx_4 - u_1 dx_3\}$, and suppose that $u_1 = v_1 + c$ where c is a non zero constant, $u_2 = v_2$ and $u_3 = v_3 + \dot{v}_1 x_3 + (v_1 + c)^2 x_3 + (v_1 + c)v_2$, which is quasi-static since $v_1 = u_1 - c$ and $v_2 = u_2$ and $v_3 = u_3 - \dot{u}_1 x_3 - u_1^2 x_3 - u_1 u_2$. From this, it is easy to show that

$$\mathcal{C}^{[1]} = \text{span}_{\mathcal{K}}\{dx_4 - u_1 dx_3, dv_3 + (x_3(v_1 + c) + v_2)dv_1 + x_3 d\dot{v}_1\} \subset \mathcal{C} + \mathcal{V}$$

and

$$\mathcal{C}^{[1]} \mid_{v_1=0, v_3=0} = \text{span}_{\mathcal{K}}\{dx_4 - u_1 dx_3\} \subset \mathcal{C}$$

Furthermore

$$\mathcal{C} = \max\{\Theta \subset \mathcal{X} \mid \Theta^{[1]} \mid_{v_1=0, v_3=0} \subset \Theta\}$$

Hence $\mathcal{C}$ is a controllability cospace in the sense of Definition 4.22.

4.2 Controllability cospace algorithm

First of all, we give an algorithm characterizing the controllability cospaces, called *the controllability cospace algorithm*. Some properties of a general controllability cospace are then derived. Let $\mathcal{C}$ be a given subspace and define a sequence $\mathcal{S}_\mu$ according to

$$\mathcal{S}_0 \quad := \quad \mathcal{X}$$

$$\mathcal{S}_{\mu+1} \quad := \quad \mathrm{span}_{\mathcal{K}}\{\omega \in \mathcal{S}_\mu \mid \dot{\omega} \in \mathcal{S}_\mu + \dot{\mathcal{C}}\} \quad (\mu \in I\!N)$$

(57)

The $\mathcal{S}_\mu$ sequence is decreasing. Thus, there exists a $\mu^* \in I\!N$ such that $\mathcal{S}_{\mu^*} = \mathcal{S}_{\mu^*+1} = \ldots = \mathcal{S}^*$. The algorithm (57) yields a necessary condition for a subspace $\mathcal{C}$ of $\mathcal{X}$ to be a controllability cospace, which is shown in the following lemma.

Lemma 4.24 *Let* $\mathcal{C} \subset \mathcal{X}$. *If* $\mathcal{C}$ *is a controllability cospace, then* $\mathcal{C} = \mathcal{S}^*$

Proof Assume that $\mathcal{C}$ is a controllability cospace. Let $\{\tilde{\omega}_i\}$ be a basis for $\mathcal{C}$ in the module $\mathrm{span}_{\mathcal{A}}\{dx\}$ over the ring $\mathcal{A}$. Then by definition, there exists a quasi-static state feedback (8) and a set of integers $\mathcal{J} \subset \{1, \cdots, m\}$ such that $\mathcal{C}^{[1]} \subset \mathcal{C} + \mathcal{V}$ and $\mathcal{C}^{[1]} = \mathrm{span}_{\mathcal{K}}\{\tilde{\omega}_i^{[1]} \mid_{v_j=0, j \in \mathcal{J}}\} \subset \mathcal{C}$. According to (57), write

$$\mathcal{S}^* = \mathrm{span}_{\mathcal{K}}\{\omega \in \mathcal{X} \mid \dot{\omega} \in \mathcal{S}^* + \dot{\mathcal{C}}\}$$

(58)

Let $\omega \in \mathcal{C}$. We have $\dot{\omega} \in \dot{\mathcal{C}}$ and hence $\omega \in \mathcal{S}^*$. This implies that $\mathcal{C} \subset \mathcal{S}^*$. Now, $\dot{\mathcal{S}}^* \subset \mathcal{S}^* + \dot{\mathcal{C}}$. By the feedback which renders $\mathcal{C}^{[1]} \subset \mathcal{C}$, one has $\mathcal{S}^{*[1]} \subset \mathcal{S}^*$. Since $\mathcal{C}$ is the largest subspace in $\mathcal{X}$ such that $\mathcal{C}^{[1]} \subset \mathcal{C}$, one has $\mathcal{S}^* \subset \mathcal{C}$. ∎

In the next section, we give an algorithm computing the smallest controllability cospace containing a given subspace, based on algorithm (57).

4.3 The smallest controllability cospace containing a given subspace

In general, the intersection of two controllability cospaces is not a controllability cospace. Thus it is unclear if there exists a smallest controllability cospace containing some given subspace. However, if an exact subspace $\Pi \subset \mathcal{X}$ is given, then there exists a smallest one containing Π.

Consider a nonlinear system given by (1). By Theorem 3.13 Π_* is the smallest controlled invariant subspace containing Π. The next theorem will relate Π_* to the smallest controllability cospace containing Π.

Theorem 4.25 *Define the sequence* $\mathcal{D}_\mu$ *according to*

$$\mathcal{D}_0 \quad = \quad \mathcal{X}$$

$$\mathcal{D}_{\mu+1} \quad = \quad \mathrm{span}_{\mathcal{K}}\{\omega \in \mathcal{D}_\mu \mid \dot{\omega} \in \mathcal{D}_\mu + \dot{\Pi}_*\} \quad (\mu \in I\!N)$$

(59)

Then $\mathcal{D}_* = \lim_{\mu \to \infty} \mathcal{D}_\mu$ *is the smallest controllability cospace containing* Π.

Proof Note that

$$\mathcal{D}_* = \mathrm{span}_{\mathcal{K}}\{\omega \in \mathcal{X} \mid \dot{\omega} \in \mathcal{D}_* + \dot{\Pi}_*\}$$

(60)

Let $r = \dim \Pi$. Now, Π is exact implies that there exist meromorphic functions $\varphi_1(x), \cdots, \varphi_r(x)$ such that $\Pi = \mathrm{span}_{\mathcal{K}}\{d\varphi_1, \cdots, d\varphi_r\}$. Consider the system (1) with a fictitious output $\varphi = (\varphi_1, \cdots, \varphi_r)^T$. We decompose the output φ as $\varphi = (\tilde{\varphi}, \hat{\varphi})^T$ so that the system (1) with the output $\tilde{\varphi}$ is right-invertible. Define $\rho := \dim(\tilde{\varphi})$.

13

Construct a quasi-static state feedback $u = \phi(x, v, \ldots, v^{(r)})$, by taking $v_i = \tilde{\varphi}_i^{(n'_i)}$ where $\{n'_i\}$ is the set of orders of zeros at infinity, for $i = 1, \cdots, \rho$ and $v_i = w_i$ for $i = \rho+1, \ldots, m$. This feedback always renders Π_* invariant. Thus, $\mathcal{D}_*$ is rendered invariant too, *i.e.* $\mathcal{D}_*^{[1]} \subset \mathcal{D}_* + \mathcal{V}$. Let now $\{\tilde{\omega}_i\}$ be a basis for $\mathcal{D}_*$ in the module $\text{span}_{\mathcal{A}}\{dx\}$ over the ring $\mathcal{A}$. If we set $v_i = 0$ for $i = 1, \ldots, \rho$ one obtains

$$\mathcal{D}_*^{[1]} = \text{span}_{\mathcal{K}}\{\tilde{\omega}_i^{[i]} \mid_{v_j=0, j=1, \cdots, \rho}\} \subset \mathcal{D}_*$$

and $\mathcal{D}_*$ is then a controllability cospace. In order to prove that $\mathcal{D}_*$ is the smallest controllability cospace containing Π, we consider another controllability cospace $\mathcal{D}$ such that $\mathcal{D} \supset \Pi$. By definition $\mathcal{D}$ is controlled invariant and according to Lemma 4.24, $\mathcal{D}$ satisfies

$$\mathcal{D} = \text{span}_{\mathcal{K}}\{\omega \in \mathcal{X} \mid \dot{\omega} \in \mathcal{D} + \dot{\mathcal{D}}\} \tag{61}$$

Since Π_* is the smallest controlled invariant subspace containing Π, this implies that $\mathcal{D} \supset \Pi_*$. From (60) and (61), one has $\mathcal{D}_* \subset \mathcal{D}$. ∎

Now we consider a nonlinear system given by (31). Clearly $\Omega_* = \mathcal{X} \cap \mathcal{Y}$ is the smallest controlled invariant subspace containing the differential of the output. Therefore the smallest controllability cospace containing the differential of the output is given by the next corollary.

Corollary 4.26 *Define the sequence $\mathcal{C}_\mu$ according to*

$$\mathcal{C}_0 = \mathcal{X}$$
$$\mathcal{C}_{\mu+1} = \text{span}_{\mathcal{K}}\{\omega \in \mathcal{C}_\mu \mid \dot{\omega} \in \mathcal{C}_\mu + \dot{\Omega}_*\} \quad (\mu \in \mathbb{N}) \tag{62}$$

Then $\mathcal{C}_* = \lim_{\mu \to \infty} \mathcal{C}_\mu$ is the smallest controllability cospace containing $\text{span}_{\mathcal{K}}\{dh(x)\}$.

Remark 4.27 When specialized to linear systems, the sequence $\mathcal{C}_\mu$ (62) turns out to be equal to the dual of the sequence $\mathcal{R}_\mu$ (the sequence computing the maximal controllability subspace in kernel of the output mapping). A proof of this can be found in the appendix.

4.4 The block input-output decoupling problem

Now, we apply the smallest controllability cospaces $\mathcal{C}_*$ previously defined to solve a quasi-static state feedback input-output decoupling problem. For this, we consider the system (1) together with the partitioned output blocks y_i for $i = 1, \ldots, k$, given by:

$$y_i = h_i(x) \tag{63}$$

The problem can be stated as follows: find a quasi-static state feedback and a partition of the new control $v = (v_1^T, \ldots, v_k^T)^T$ such that the new input v_i^T only affectsy the output y_i.

Define $\mathcal{C}_{i*}$ and Ω_{i*} to be the smallest controllability cospace and the smallest controlled invariant subspace respectively, both containing $\text{span}_{\mathcal{K}}\{dh_i(x)\}$.

First, let us give the following property which is derived from Theorem 5.1 in ([29]).

Property 4.28 *Consider system (31), and assume that $\dim(\mathcal{G}^\perp) = n - m$. Let ρ be its differential output rank. Then*

$$\dim(\mathcal{G}^\perp + \Omega_*) = \dim(\mathcal{G}^\perp + \mathcal{C}_*) = (n\text{-}m\text{+}\rho). \tag{64}$$

Moreover, if the system (31) is right-invertible, then

$$\dim(\mathcal{G}^\perp + \Omega_*) = \dim(\mathcal{G}^\perp + \mathcal{C}_*) = (n\text{-}m\text{+}\rho). \tag{65}$$

14

This property is a generalization of known result on linear system. It gives a geometric interpretation for the rank of a system. That was also derived by Respondek in [30] using dynamic controllability distributions.

Corollary 4.29 *The block input-output decoupling problem via quasi-static state feedback for the system (1),(63) is solvable if and only if*

$$\dim\left(\frac{\mathcal{G}^\perp + \mathcal{C}_*}{\mathcal{G}^\perp}\right) = \sum_{i=1}^{k}\dim\left(\frac{\mathcal{G}^\perp + \mathcal{C}_{i*}}{\mathcal{G}^\perp}\right) \tag{66}$$

Proof If ρ denotes the rank of the system (1),(63) and ρ_i denotes the rank of the subsystem (1) with the output y_i, then by Property 4.28, (66) is equivalent to

$$\rho = \sum_{i=1}^{k}\rho_i \tag{67}$$

So, (66) coincides with the condition given by Di Benedetto *et al.* ([12]), in case of the dynamic block decoupling problem. Thus, the proof in [12] also proves this result. ∎

Furthermore, the controllability cospaces also allow to characterize the fixed dynamics with respect to any quasi-static feedback. This will be the topic of the next section.

4.5 Dimension of fixed modes by quasi-static state feedback

The problem of noninteraction and stability of nonlinear systems by means of static feedback has been considered by Isidori and Grizzle [21]. They have shown that there exists a fixed internal dynamics, called P^* *dynamics* whose stability is a necessary condition to solve the noninteracting control problem with stability. In the case where the P^* *dynamics* is unstable, Wagner in [31] has shown that there exists a well-defined dynamics, called Δ_{mix} *dynamics*, which cannot be eliminated by any regular dynamic feedback that renders the considered system noninteractive. The Δ_{mix} *dynamics* must then be asymptotically stable if noninteracting control and stability is to be achieved by means of dynamic state feedback. A sufficient condition to solve the problem of noninteracting control with stability by means of dynamic state feedback was given in ([4],[5]). In these references, the problem of dynamic feedback noninteracting control with stability is solved if some regularity assumptions are satisfied, the Δ_{mix} *dynamics* is asymptotically stable and each decoupled subsystem is asymptotically stabilizable.

All the results above are valid under the assumption that the decoupling matrix $A(x)$ is nonsingular. In the case where $A(x)$ is singular and the system is square and invertible, Zhan *et al.* [33] introduced the so-called Canonical Dynamic Decoupling Algorithm to construct a canonical dynamic extension. They have shown that the dynamically decoupled system is stable only if the Δ_{mix} *dynamics* of the canonical dynamic extension is stable, which is an intrinsic property of the given system.

In this section, we investigate the case where the decoupling matrix is not necessarily invertible and study the noninteracting control problem with stability by means of quasi-static state feedback. The goal is to show that the controllability cospaces introduced before are able to describe intrinsic geometric conditions generalizing the above ones.

Let us consider a square invertible nonlinear affine system (Σ) of the form

$$\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x)u_i, \quad x \in I\!R^n, \quad u_i \in I\!R \\ y_i &= h_i(x), \quad i = 1, \cdots, m, \quad y_i \in I\!R \end{aligned} \tag{68}$$

Let $\{n_i'\}$ be the set of orders of zeros at infinity, where $n_1' > n_2' > \cdots > n_m'$. Permute if necessary y_i such that the corresponding order of zero at infinity is n_i'. Let $\mathcal{C}_{i*}$ be the smallest controllability cospace containing $\text{span}_\mathcal{K}\{dh_i(x)\}$. A first result is the following.

15

Lemma 4.30 *Suppose that the system (68) is decouplable by a quasi-static state feedback $u = \psi(x, v, \cdots, v^{(s)})$. Then, there always exist coordinates $\xi = (\xi_0, \xi_1, \cdots, \xi_m, \hat{\xi})$ such that the system (68) is presented in the following form:*

$$
\begin{aligned}
\dot{\xi}_0 &= f_0(\xi_0) \\
\dot{\xi}_1 &= f_1(\xi_0, \xi_1, v_1) \\
&\vdots \\
\dot{\xi}_m &= f_m(\xi_0, \xi_m, v_m) \\
\dot{\hat{\xi}} &= \hat{f}(\xi, v, \dot{v}, \cdots, v^{(s)}) \\
\\
y_i &= h_i(\xi_0, \xi_i)
\end{aligned}
\tag{69}
$$

To prove Lemma 4.30, we first need the following property of $\mathcal{C}_{i*}$.

Lemma 4.31 *For a scalar output $y_i = h_i(x)$, $\mathcal{C}_{i*}$ is an exact subspace.*

Proof Let Ω_{i*} be the smallest controlled invariant subspace containing $\mathrm{span}_\mathcal{K}\{dh_i(x)\}$. If Δ_i^* is the maximal controlled invariant distribution in $\ker\{dh_i(x)\}$, we have $\Omega_{i*} = {\Delta_i^*}^\perp$. Let now $\mathcal{R}_i^*$ be the maximal controllability distribution in $\ker\{dh_i(x)\}$. Clearly ${\mathcal{R}_i^*}^\perp$ is a controllability cospace containing $\mathrm{span}_\mathcal{K}\{dh_i(x)\}$, and thus $\mathcal{C}_{i*} \subset {\mathcal{R}_i^*}^\perp$. From [18], we have

$$
\mathcal{R}_i^* = \Delta_i^* \cap \left([f, \mathcal{R}_i^*] + \sum_{j=1}^m [g_j, \mathcal{R}_i^*] + \mathcal{G}\right)
\tag{70}
$$

and thus

$$
\begin{aligned}
{\mathcal{R}_i^*}^\perp &= \Omega_{i*} + [f, \mathcal{R}_i^*]^\perp \cap \left(\bigcap_{j=1}^m [g_j, \mathcal{R}_i^*]^\perp\right) \cap \mathcal{G}^\perp \\
\\
&= \{\omega \in \mathcal{X} \mid \exists \omega_1 \in \Omega_{i*}, \exists \omega_2 \in \mathcal{G}^\perp \text{ such that } \omega = \omega_1 + \omega_2 \\
\\
&\quad \text{and } (\forall \tau \in \mathcal{R}_i^*)(\forall \sigma \in \{f, g_1, \cdots, g_m\})(\langle[\sigma, \tau], \omega_2\rangle = 0)\}
\end{aligned}
\tag{71}
$$

Let $\omega \in {\mathcal{R}_i^*}^\perp$. Hence there exist $\omega_1 \in \Omega_{i*}$ and $\omega_2 \in \mathcal{G}^\perp$ such that $\omega = \omega_1 + \omega_2$, and $\forall \tau \in \mathcal{R}_i^*, \forall \sigma \in \{f, g_1, \cdots, g_m\}$, one has $\langle[\sigma, \tau], \omega_2\rangle = 0$. Compute

$$\dot{\omega} = \dot{\omega}_1 + \dot{\omega}_2$$

Clearly $\dot{\omega}_1 \in \dot{\Omega}_{i*}$. Furthermore

$$
\begin{aligned}
\dot{\omega}_2 &= \mathcal{L}_f \omega_2 + \sum_{j=1}^m (u_j \mathcal{L}_{g_j} \omega_2 + \langle\omega_2, g_j\rangle du_j) \\
\\
&= \mathcal{L}_f \omega_2 + \sum_{j=1}^m u_j \mathcal{L}_{g_j} \omega_2
\end{aligned}
\tag{72}
$$

Now, let $\tau \in \mathcal{R}_i^*$ and $\sigma \in \{f, g_1, \cdots, g_m\}$. Then

$$
\begin{aligned}
\langle\tau, \mathcal{L}_\sigma \omega_2\rangle &= \mathcal{L}_\sigma\langle\tau, \omega_2\rangle - \langle[\sigma, \tau], \omega_2\rangle \\
\\
&= \mathcal{L}_\sigma\langle\tau, \omega_2\rangle = \mathcal{L}_\sigma\langle\tau, (\omega - \omega_1)\rangle = 0
\end{aligned}
\tag{73}
$$

where the last equality follows from the fact that $\omega \in {\mathcal{R}_i^*}^\perp$ and $\omega_1 \in \Omega_{i*} \subset {\mathcal{R}_i^*}^\perp$. By (72),(73), we then have $\dot{\omega}_2 \in {\mathcal{R}_i^*}^\perp$, and hence

$$\dot{\widehat{{\mathcal{R}_i^*}^\perp}} \subset \dot{\Omega}_{i*} + {\mathcal{R}_i^*}^\perp$$

16

By construction, C_{i*} is the largest subspace in $\mathcal{X}$ which verifies $\dot{C}_{i*} \subset C_{i*} + \dot{\Omega}_{i*}$. This implies $\mathcal{R}_i^{*\perp} \subset C_{i*}$. So C_{i*} is the annihilator of $\mathcal{R}_i^*$, which is known to be involutive ([26],[18]). Hence C_{i*} is exact, which establishes our claim. ∎

Proof of Lemma 4.30 By Lemma 4.31, C_{i*} is an exact subspace. Thus, $\dot{C}_{i*}$ as well as $\sum_{j=0}^{n_i'-1} C_{i*}^{(j)}$ are also exact. Let us define C_0 as the uncontrollable subspace of (Σ) which is the subspace $\mathcal{H}_\infty$ introduced in [1]. It is obvious that for each $i = 1, \cdots, m$

$$C_0 = \sum_{j=0}^{n_i'-1} C_{i*}^{(j)} \cap \sum_{k \neq i} \sum_{j=0}^{n_k'-1} C_{k*}^{(j)}$$

Let $d\xi_0$ to be a basis of C_0, thus $\dot{\xi}_0 = f_0(\xi_0)$. For an invertible system, we can construct a quasi-static state feedback which decouples system (Σ) by taking $v_i = y_i^{(n_i')}$. For $i = 1, \cdots, m$, then choose $d\xi_i$ such that $\{d\xi_0, d\xi_i\}$ is a basis of $\sum_{j=0}^{n_i'-1} C_{i*}^{(j)}$. Then one has

$$\dot{\xi}_i = f_i(\xi_0, \xi_i, v_i)$$

Complete the new coordinates by choosing $\hat{\xi}$, such that $\{d\xi_0, d\xi_1, \cdots, d\xi_m, d\hat{\xi}\}$ is linearly independent. Thus, one has

$$\dot{\hat{\xi}} = \hat{f}(\xi, v, \dot{v}, \cdots, v^{(s)}),$$

and (69) is established. ∎

Using this Lemma, the following corollary is obtained

Corollary 4.32 *The dimension of the fixed dynamics with respect to any quasi-static feedback which decouples the system, is*

$$n - \dim \left(\sum_{i=1}^{m} \sum_{j=0}^{n_i'-1} C_{i*}^{(j)} \right) \tag{74}$$

Remark 4.33 From the definition of the structure at infinity, one gets

$$\dim \left(\sum_{i=1}^{m} \sum_{j=0}^{n_i'-1} C_{i*}^{(j)} \right) = \dim \left(\mathcal{X} \cap \sum_{i=1}^{m} \sum_{j \geq 0} C_{i*}^{(j)} \right) \tag{75}$$

From Remark 4.33 and Corollary 4.32, the following theorem is derived.

Theorem 4.34 *For a square invertible nonlinear system, the dimension of the fixed dynamics with respect to any quasi-static state feedback is*

$$n - \dim \left(\mathcal{X} \cap \sum_{i=1}^{m} \sum_{j \geq 0} C_{i*}^{(j)} \right)$$

∎

The above theorem allows to characterize the dimension of the fixed dynamics by computing C_{i*} only. Under additional technical conditions as in [21], one describes the fixed dynamics. Suppose that the origin is an equilibrium point of f and the quasi-static state feedback rendering (68) noninteractive preserves this equilibrium point, then the induced fixed dynamics are

$$\dot{\hat{\xi}} = \hat{f}(0, \cdots, 0, \hat{\xi}).$$

And the asymptotic stability of these dynamics is a necessary condition for noninteracting control with internal stability.

The next example illustrates theorem 4.34.

Example 4.35 Let us consider a nonlinear system given by:

$$\dot{x}_1 = u_1, \ \dot{x}_2 = x_4 + x_3 u_1, \ \dot{x}_3 = x_3 + x_4, \ \dot{x}_4 = u_2, \ \dot{x}_5 = x_1 + x_2$$

$$y_1 = x_1, \ y_2 = x_2$$

We have $\{n_i'\} = \{2, 1\}$. Permute then y_i, and thus $C_{1*} = \{dx_2\}$ and $C_{2*} = \{dx_1\}$. The quasi-static feedback which decouples the system is $u_1 = v_1$ and $u_2 = v_2 - (x_3 + x_4)v_1 - x_3\dot{v}_1$, where (v_1, v_2) is a new input vector.

It is clear that $C_0 = 0$. We choose $d\xi_1 = \{dx_2, d(x_4 + x_3 u_1)\}$ as a basis of $\{C_{1*} + \dot{C}_{1*}\}$, and thus

$$\dot{\xi}_1 = \begin{pmatrix} \dot{\xi}_{11} \\ \dot{\xi}_{12} \end{pmatrix} = \begin{pmatrix} \xi_{12} \\ v_2 \end{pmatrix} \tag{76}$$

Choose now $\{d\xi_2\} = \{dx_1\}$ as a basis of C_{2*}, and one has

$$\dot{\xi}_2 = v_1 \tag{77}$$

We complete our coordinate transformation by taking $\hat{\xi} = \begin{pmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \end{pmatrix} = \begin{pmatrix} x_4 \\ x_5 \end{pmatrix}$. So in the new coordinates $(\xi_1, \xi_2, \hat{\xi})$, the considered system becomes

$$\begin{aligned}
\dot{\xi}_{11} &= \xi_{12} \\
\dot{\xi}_{12} &= v_2 \\
\dot{\xi}_2 &= v_1 \\
\dot{\hat{\xi}}_1 &= v_2 - (\xi_{12} - \hat{\xi}_1) - \hat{\xi}_1 v_1 + (\xi_{12} - \hat{\xi}_1)\dot{v}_1/v_1 \\
\dot{\hat{\xi}}_2 &= \xi_2 + \xi_{11}
\end{aligned} \tag{78}$$

$$\begin{aligned}
y_1 &= \xi_2 \\
y_2 &= \xi_{11}
\end{aligned}$$

Clearly $\dim(\hat{\xi}) = 2$. It equals $n - \dim(\mathcal{X} \cap (\sum\limits_{i=1}^{m}\sum\limits_{j \geq 0} C_{i*}^{(j)})) = n - \dim(dx_1, dx_2, dx_4 + u_1 dx_3)$. Thus, the dimension of the fixed dynamics is two. Since the origin is an equilibrium point, the fixed dynamics are then

$$\begin{aligned}
\dot{\hat{\xi}}_1 &= \hat{\xi} \\
\dot{\hat{\xi}}_2 &= 0
\end{aligned} \tag{79}$$

Similarly to Wagner's and Battilotti's results, in the case where no quasi-static state feedback can render the system simultaneously noninteractive and stable, a suitable dynamic feedback may still solve the problem. This reduces to the results in Zhan *et al.* [33].

Finally, we can then summarize the existing results related to the dimension of the fixed dynamics of a nonlinear square decoupled system in the following table:

Feedback	$A(x)$ invertible	$A(x)$ non-invertible
(Quasi) Static	dim $(\mathcal{P}^*)$ (Isidori & Grizzle)	$n - \dim(\mathcal{X} \cap (\sum\limits_{i=1}^{m}\sum_{j \geq 0} C_i^{*^{(j)}}))$
Dynamic	dim (Δ_{mix}) (Wagner, Battilotti)	dim $(\Delta_{mix}(\Sigma_p))$ (Zhan *et al.*)

5 Conclusions

A generalized notion of controlled invariance under quasi-static state feedback for nonlinear systems was introduced. It was shown that this notion coincides with the "classical" notion of a controlled invariant distribution under regular static state feedback. Using the generalized notion of controlled invariance, a condition for the controlled invariance of not necessarily involutive distributions was derived. For a subspace $\Omega \subset \mathcal{X}$, we gave sufficient conditions for controlled invariance under quasi-static state feedback. Furthermore, a necessary and sufficient condition was also given, but it was only made for a special class of Ω.

For a controllability cospace $\mathcal{C} \subset \mathcal{X}$, some properties were derived by means of the controllability cospace algorithm. Moreover the smallest controllability cospace containing the differential of the output allowed to solve the block input-output decoupling problem. It also characterized the dimension of the fixed dynamics with respect to any quasi-static state feedback in the case of one to one decoupling.

This paper leaves some interesting open questions, which are the topic for further research. A first question is related to necessary and sufficient conditions for controlled invariance for a general class of subspaces. A second question is whether (or under what conditions) there exists a smallest controlled invariant subspace containing some given subspace. It seems that for the answer to both questions a better understanding of quasi-static state feedback is needed.

Finally, let us remark that throughout the paper we have restricted ourselves to "Kalmanian" systems and to subspaces $\Omega \subset \mathcal{X}$. However, the definition of controlled invariance and the characterizations of controlled invariance in this paper can, *mutatis mutandis*, be translated to non-Kalmanian systems and subspaces $\Omega \subset \mathcal{X} \times \mathcal{U}$.

References

[1] Aranda-Bricaire, E., C.H. Moog and J.B. Pomet, *A linear algebraic framework for dynamic feedback linearization*, to appear in IEEE Trans. on Automatic control, March 1995.

[2] Atiyah, M.F., and I.G. MacDonald, *Introduction to commutative algebra*, Addison-Wesley, Reading Mass., 1969.

[3] Basile, G. and G. Marro, *Controlled and conditionned invariants in linear system theory*, New Jersey, Prentice Hall, 1992.

[4] Battilotti, S., *A sufficient condition for nonlinear noninteracting control with stability via dynamic state feedback*, IEEE Trans. Automat. Control, Vol. 36, No. 9, (1991), pp. 1033-1045.

[5] Battilotti, S. and W.P. Dayawansa, *Noninteracting control with stability for a class of nonlinear system*, Systems & Control Letters, Vol. 17, (1991), pp. 327-338.

[6] Brockett, R.W., *Feedback invariants for nonlinear systems*, Proceedings IFAC Congress, Helsinki, 1978, pp. 1115-1120.

[7] Cao, L. and Y.F. Zheng, *On minimal compensators for decoupling control*, Systems & Control Letters, Vol. 18, (1992), pp. 121-128.

[8] Cheng, D. and T.J. Tarn, *New result on (f,g)-invariance*, Systems & Control letters, Vol. 12, No. 4, (1989), pp. 319-326.

[9] Delaleau, E., and M. Fliess, *An algebraic interpretation of the structure algorithm with an application to feedback decoupling*, Proceedings NOLCOS 1992, Bordeaux, France, pp. 489-494.

[10] Delaleau, E., and M. Fliess, *Algorithme de structure, filtrations et découplage*, C.R. Acad. Sci. Paris, **t.315, Série I**, (1992), pp. 101-106.

[11] Delaleau, E., *Sur les derivées de l'entrée en représentation et commande des systèmes non linéaires*, Thèse de docteur en sciences, Université Paris XI Orsay, 1993.

[12] Di Benedetto, M.D., J.W. Grizzle and C.H. Moog, *Rank invariants of nonlinear systems*, SIAM J. Control Optimiz., **27**, (1989), pp. 658-672.

[13] Fliess, M., *Automatique et corps différentiels*, Forum Math., **1**, (1989), pp. 227-238.

[14] Grasse, K.A., *On controlled invariance for fully nonlinear systems*, International Journal of Control, Vol. 56, No. 5, (1992), pp. 1121-1137.

[15] Hirschorn, R.M., *(A, B)-invariant distributions and disturbance decoupling of nonlinear systems*, SIAM J. Control Optimiz., **19**, (1981), pp. 1-19.

[16] Huijberts, H.J.C., H. Nijmeijer and L.L.M. van der Wegen, *Dynamic disturbance decoupling for nonlinear systems*, SIAM J. Control Optimiz., **30**, (1992), pp. 336-349.

[17] Huijberts, H.J.C. and C.H. Moog, *Controlled invariance of nonlinear systems: nonexact forms speak louder than exact forms*, in Systems and Networks: Mathematical Theory and Application (Vol. II), U. Helmke, R. Mennicken and J. Saurer, eds., Akademie Verlag, Berlin, 1994, pp. 245-248.

[18] Isidori, A., *Nonlinear control systems* (Second Edition), Springer Verlag, Berlin, 1989.

[19] Isidori, A., *Control of nonlinear systems via dynamic state feedback*, in Algebraic and geometric methods in nonlinear control theory, M. Fliess and M. Hazewinkel, eds., Reidel, Dordrecht, 1986.

[20] Isidori, A., A.J. Krener, C. Gori-Giorgi and S. Monaco, *Nonlinear decoupling via feedback: a differential geometric approach*, IEEE Trans. Automat. Control, **AC-26**, 1981, pp. 331-345.

[21] Isidori, A. and J.W. Grizzle, *Fixed modes and nonlinear noninteracting control with stability*, IEEE Trans. Automat. Control, Vol. 33, No. 10, (1988), pp. 907-914.

[22] Moog, C.H., *Nonlinear decoupling and structure at infinity*, MCSS, **1**, (1988), pp. 257-268.

[23] Nijmeijer, H., *Controlled invariance for affine control systems*, Int. J. Control, **34**, (1981), pp. 824-833.

[24] Nijmeijer, H., and A.J. van der Schaft, *Controlled invariance for nonlinear systems*, IEEE Trans. Automat. Control, **AC-27**, (1982), pp. 904-914.

[25] Nijmeijer, H., and A.J. van der Schaft, *Nonlinear dynamical control systems*, Springer Verlag, New York, 1990.

[26] Nijmeijer, H., *Controllability distributions for nonlinear control systems*, Systems & Control letters, Vol. 2, No. 2, (1982), pp. 122-129.

[27] Nijmeijer, H. and J.M. Schumacher, *The regular local noninteracting control problem for nonlinear control systems*, SIAM J. Control Optimiz., 24, (1986), pp. 1232-1245.

[28] Perdon, A.M., Y.F. Zheng, C.H. Moog and G. Conte, *Disturbance decoupling for nonlinear systems: a unified approach*, KYBERNETIKA, Vol. 29, No. 5, (1993), pp. 479-489.

[29] Pereira da Silva, P.S., *Disturbance decoupling by regular dynamic feedback for affine nonlinear systems : a linear algebraic approach*, Proceding of IFAC Control Conference, Sydney, 1993, pp. VI-387-VI-390.

[30] Respondek, W., *Dynamic controllability distributions in nonlinear systems*, Proceedings 2nd IFAC Workshop on Systems Structure and Control, Prague, 1992, pp. 256-258.

[31] Wagner, K.G., *Nonlinear noninteracting with stability by dynamic state feedback*, SIAM J. Control and Optimiz., Vol. 29, No. 3, (1991), pp. 609-622.

[32] Wonham, W.M., *Linear multivariable control: a geometric approach* (3rd Edition), Springer Verlag, Berlin, 1985.

[33] Zhan, W., T.J. Tarn and A. Isidori, *A canonical dynamic extension for noninteraction with stability for affine nonlinear square systems*, Systems & Control Letters, Vol. 17, (1991), pp. 177-184.

Appendix

According to Remark 4.27, we will prove that the sequence (62) computing C_* is the same as the one computing $\mathcal{R}^{*\perp}$ (the dual of $\mathcal{R}^*$, the maximal controllability subspace in kernel of the output) for linear time invariant systems. We proceed by induction. First, we recall some basic operations that we need.

Consider a linear system given by :

$$\begin{aligned}
\dot{x} &= Ax + Bu, \ x \in I\!R^n, u \in I\!R^m \\
y &= Cx
\end{aligned} \tag{80}$$

Identify elements of $I\!R^n$ with column vectors, while elements of $I\!R^{n\perp}$, its dual, are identified with row-vectors. Thus, $\omega = \sum_{i=1}^{n} \alpha_i dx_i \in I\!R^{n\perp}$ is identified with the row-vector $\alpha := (\alpha_1, \ldots, \alpha_n) \in I\!R^n$. With this notation,

$$\dot{\omega} = \alpha d\dot{x} = \alpha A dx + \alpha B du \in (I\!R^n \times I\!R^m)^\perp \tag{81}$$

is identified with the row-vector $(\alpha A \ \ \alpha B)$.

Let a subspace $V \subset I\!R^n$ be given. Then

$$\begin{aligned}
(AV)^\perp &= \{\omega \in \text{span}\{dx\} \,|< \omega, Av >= 0, \forall v \in V\} \\
&= \{\alpha \in I\!R^n \mid \alpha Av = 0, \forall v \in V\} = \{\alpha \in I\!R^n \mid \alpha A \in V^\perp\} \tag{82} \\
&=: \ ^{-1}AV^\perp
\end{aligned}$$

if $\omega = \alpha dx \in (AV)^\perp \cap B^\perp$, then

$$\dot{\omega} = \alpha A dx + \alpha B du = \alpha A dx \simeq \alpha A \in V^\perp \tag{83}$$

The two sequences to be compared are:

$$\left\{ \begin{aligned}
\mathcal{R}_0^\perp &:= \mathcal{X} \\
\mathcal{R}_{\mu+1}^\perp &:= \mathcal{V}^{*\perp} + \ ^{-1}A\mathcal{R}_\mu^\perp \cap \text{Im}B^\perp \quad (\mu \in I\!N)
\end{aligned} \right. \tag{84}$$

and

$$\left\{ \begin{aligned}
\mathcal{C}_0 &:= \mathcal{X} \\
\mathcal{C}_{\mu+1} &:= \{\omega \in \mathcal{C}_\mu \mid \dot{\omega} \in \mathcal{C}_\mu + \dot{V}^{*\perp}\} \quad (\mu \in I\!N)
\end{aligned} \right. \tag{85}$$

where $\mathcal{V}^*$ is the maximal controlled invariant subspace in $\text{Ker}C$ for the system (80). For step 0, it is obvious that $\mathcal{R}_0^\perp = \mathcal{C}_0$. Suppose that $\mathcal{R}_\mu^\perp = \mathcal{C}_\mu$ for $\mu = 0, \cdots, \ell$. Let $\omega \in \mathcal{R}_{\ell+1}^\perp$, thus there exist $\omega_1 \in \mathcal{V}^{*\perp}$ and $\omega_2 \in \ ^{-1}A\mathcal{R}_\ell^\perp \cap B^\perp$ such that $\omega = \omega_1 + \omega_2$. By (83) , $\dot{\omega}_2 \in \mathcal{R}_\ell^\perp = \mathcal{C}_\ell$ and hence $\mathcal{R}_{\ell+1}^\perp \subset \mathcal{C}_{\ell+1}$. To show the other inclusion, let $\omega \in \mathcal{C}_{\ell+1}$, then

$$\dot{\omega} \in \mathcal{C}_\ell + \dot{V}^{*\perp} = \mathcal{R}_\ell^\perp + \dot{V}^{*\perp} \tag{86}$$

Thus, there exists $\omega_1 \in \mathcal{V}^{*\perp}$ and $\omega_2 \in \mathcal{R}_\ell^\perp$ such that $\dot{\omega} = \dot{\omega}_1 + \omega_2$. Let now $\dot{\omega}_0 = \overbrace{\dot{\omega} - \dot{\omega}_1} = \omega_2$. So, $\dot{\omega}_0 \in \mathcal{R}_\ell^\perp$. This implies that

$$\begin{aligned}
\omega_0 &\in \{\omega = \alpha dx \mid \dot{\omega} \in \mathcal{R}_\ell^\perp\} = \{\alpha dx \mid \alpha A dx + \alpha B du \in \mathcal{R}_\ell^\perp\} \\
&= \{\alpha \mid \alpha A \in \mathcal{R}_\ell^\perp\} \cap B^\perp = \ ^{-1}A\mathcal{R}_\ell^\perp \cap B^\perp \tag{87}
\end{aligned}$$

So, $\omega = \omega_1 + \omega_0 \in \mathcal{R}_{\ell+1}^\perp$, which yields that $\mathcal{C}_{\ell+1} \subset \mathcal{R}_{\ell+1}^\perp$. Thus, we have that $\mathcal{C}_\mu = \mathcal{R}_\mu^\perp$ for all $\mu \in I\!N$, which establishes our claim. ∎