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EINDHOVEN UNIVERSITY OF TECHNOLOGY
Faculty of Mathematics and Computing Science

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LINEAR HOMOGENEOUS RELATIONS FOR THE HYPERGEOMETRIC FUNCTION I. MIXED DIFFERENCE-DIFFERENTIAL RELATIONS

by

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INTRODUCTION.

There exist a surprisingly large number of relations for the hypergeometric functions. In the first section a structure theorem is derived for relations of a special kind, the so-called *linear homogeneous partial mixed difference-differential equations*. This structure theorem hinges on a generalization of STAFFORD's theorem on the structure of ideals in the WEYL-algebra's, which is proved in section 2 using noncommutative localization. In section 3 some other consequences of this localization technique are proved.

Let me end this introduction with some notational remarks.

$\mathbb{N}_0 = \{0,1,2,\ldots\}$. For $x_1,\ldots,x_m$ and $\alpha \in \mathbb{Z}^m$, $x_1^{\alpha_1},\ldots,x_m^{\alpha_m}$ will be abbreviated to x^α, and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_m$.

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§ 1. LINEAR RELATIONS FOR THE HYPERGEOMETRIC FUNCTION.

Let $F(a,b;c;z)$ be the hypergeometric function :

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!\ (c)_n} z^n ,$$

where $(a)_n = a(a+1)\ldots(a+n-1)$, etc. ... .
As is wellknown the hypergeometric function satisfies a score of homogeneous equations involving differentiation with respect to z, or a shift over an integer in a, b or c. For instance (SLATER [7], ch. 1) :

$$\left[z(1-z) \frac{\partial^2}{\partial z^2} + (c-(1+a-b)z) \frac{\partial}{\partial z} - ab\right] F(a,b;c;z) = 0 ;$$

$$a(1-z)\ F(a+1,b;c;z) + (c-2a+(a-b)z)\ F(a,b;c;z) = (c-a)\ F(a-1;b;c;z) ;$$

$$aF(a+1,b;c;z) - bF(a,b+1;c;z) = (a-b)\ F(a,b;c;z) ;$$

$$c(1-z)\ F(a,b;c;z) + (c-a)\ zF(a,b;c+1;z) = cF(a,b+1;c;z) ;$$

$$c \frac{\partial}{\partial z} F(a,b;c;z) = abF(a+1,b+1;c+1;z) .$$

Let $\partial = \frac{\partial}{\partial z}$, φ_a the shift operator in a: $\varphi_a\ f(a,b,c,z) = f(a+1,b,c,z)$, and φ_b and φ_c likewise in b and c, then all these relations have the form :

$$LF(a,b;c;z) = 0$$

with $L \in \mathbb{Q}[a,b,c,z,\varphi_a,\varphi_a^{-1},\varphi_b,\varphi_b^{-1},\varphi_c,\varphi_c^{-1},\partial]$.

So studying this kind of relations for the hypergeometric function, which I shall call linear homogeneous partial mixed difference-differential equations, leads to the study of the set of linear partial mixed difference-differential operators. It is not difficult to see that this set forms an associative $\mathbb{Q}$-algebra, and that $[\varphi_a,a] = \varphi_a$, $[\varphi_b,b] = \varphi_b$, $[\varphi_c,c] = \varphi_c$, $[\delta,z] = 1$ and that all other basic commutators vanish.

In fact I shall study more in general the k-algebra's $G_{m,n}(k) = k[y_1,\dots,y_m,z_1,\dots,z_n,\varphi_1,\dots,\varphi_m,\delta_1,\dots,\delta_n]$, where k is a field of characteristic zero, and the only nonvanishing commutators are

$$[\varphi_i,y_i] = \varphi_i \quad ; \quad [\delta_i,z_i] = 1 .$$

In the special case that m=0 one recovers the WEYL-algebra $A_n(k)$, and the case m=3, n=1, k=$\mathbb{Q}$ yields the algebra introduced in connection with the hypergeometric function. Recall that for the WEYL-algebra's STAFFORD [8] proved the following theorem.

THEOREM 1. *Every left ideal of* $A_n(k)$ *may be generated by two elements.*

In section 2 I shall prove the following generalization:

THEOREM 2. *Every left ideal of* $G_{m,n}(k)$ *may be generated by two elements.*

This theorem can be proven using the same arguments as STAFFORD uses for the proof of THEOREM 1, but a much faster way consists in considering $G_{m,n}(k)$ as a localization of $A_{m+n}(k)$, as is done in section 2. Moreover, this yields a whole set of generalizations of theorems valid for the WEYL-algebra's (see for instance BJÖRK [2], ch. 1).

Applying this theorem to $G_{3,1}(\mathbb{Q})$ yields now the structure theorem for the linear homogeneous partial mixed difference-differential relations of the hypergeometric functions:

THEOREM 3. *There exist two homogeneous linear partial mixed difference-differential relations for the hypergeometric function, which imply all relations of this kind.*

Proof. Let $L(F) = \{L \in G_{3,1}(\mathbb{Q}) \mid LF = 0\}$. $L(F)$ forms a left ideal in $G_{3,1}(\mathbb{Q})$, and is hence generated by two elements.

Or as another consequence, recall that for any distinct triples $(\ell_i, m_i, n_i) \in \mathbb{Z}^3$ there exist $\lambda_i \in \mathbb{Q}[a,b,c,z] \setminus \{0\}$ such that

$$\sum_{i=1}^{3} \lambda_i \, F(a+\ell_i, b+m_i; c+n_i, z) = 0 \qquad \text{(SLATER [7], ch. 1)}$$

Now from theorem 2 one derives that there exist two relations of the type $\sum_{i=1}^{h} \lambda_i \, F(a+\ell_i; b+m_i; c+n_i; z) = 0$ generating the others, considering the $G_{3,0}(\mathbb{Q}(z))$- module generated by F.

As a final remark notice that the EULER-identity (SLATER [7], ch. 1) :

$$F(a,b;c;z) = (1-z)^{c-a-b} \, F(c-a, c-b; c; z)$$

is of a completely different nature, and hence does not fall in the range of theorem 3. To include this type of relations the ring of operators to be considered becomes far more involved.

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§ 2. LOCALIZATIONS OF THE WEYL-ALGEBRA.

To get an idea about the sequel consider the algebra $G_{1,0}(\mathbb{Q}) = \mathbb{Q}[y,\varphi,\varphi^{-1}]$. Untill now this algebra was viewed as a set of operators acting upon functions in the argument y.
Assume that these functions allow a *Mellintransform* :

$$\bar{f}(u) = \int_\Gamma u^{-y} f(y)\,dy ,$$

where Γ is a contour which is invariant under φ. Then

$$\overline{\varphi f}(u) = \int_\Gamma u^{-y} f(y+1)\,dy = \int_\Gamma u^{-y+1} f(y)\,dy = u\bar{f}(u)$$

$$\overline{yf}(u) = \int_\Gamma y u^{-y} f(y)\,dy = -u \frac{d}{du} \int_\Gamma u^{-y} f(y)\,dy = -u \frac{d}{du}\, f(u).$$

Hence $\mathbb{Q}[y,\varphi]$ can also be considered as a set of operators on functions in the argument u, and is isomorphic to $\mathbb{Q}[u,u\frac{d}{du}]$. Localizing to φ then yields an isomorphism between $G_{1,0}(\mathbb{Q})$ and to localization of the WEYL-algebra $A_1(\mathbb{Q})$ to z.

This isomorphism is of course completely independent of the existence of the Mellintransform, and can be described formally by means of localization techniques, which hold in general for $A_n(k)$.
Therefore let $S_m \subset A_n(k)$ be the multiplicative subset generated by $1,z_1,\ldots,z_m$. Then S_m satisfies :

LEMMA 1. *For all* $s \in S_m$ *and* $a \in A_n(k)$ *there exist* $s' \in S_m$ *and* $a' \in A_n(k)$ *such that* $s'a = a's$.

Proof. Any $s' = z^\alpha$ with α very large will do.

According to STENSTRÖM [9], ch. II, prop 1.4 this property allows one to talk about localizing $A_n(k)$ to S_m. $S_m^{-1} A_n(k)$ has as elements expressions of the type $s^{-1}a$, $s \in S_m$, $a \in A_n(k)$, where $s_1^{-1}a_1$ and $s_2^{-1}a_2$ are identified if $s_2a_1 = s_1a_2$. Then $S_m^{-1}A_n(k)$ is a ringextension of $A_n(k)$ due to LEMMA 1.

Before I proceed, let me introduce some notation that will come in handy later on. In the algebra $A_{m+n}(k)$ denote the variables that are going to be localized by $x_1,\dots,x_m$, and the remaining by $z_1,\dots,z_n$. S_m is generated by $1,x_1,\dots,x_m$; $\partial_i = \frac{\partial}{\partial z_i}$, and $d_i = \frac{\partial}{\partial x_i}$.

LEMMA 2. *The set* $\{x^\alpha z^\beta d^\lambda \partial^\mu \mid \alpha \in \mathbb{Z}^m, \lambda \in \mathbb{N}_0^m, \beta,\mu \in \mathbb{N}_0^n\}$ *forms a Hamel basis for the* k *linear space* $S_m^{-1} A_{m+n}(k)$.

Proof. This is an easy consequence of BJÖRK [2], ch. 1, prop 1.2.

A similar lemma holds for $G_{m,n}(k)$ as is immediate from the definition :

LEMMA 3. *The set* $\{\varphi^\alpha z^\beta y^\lambda \partial^u \mid \alpha \in \mathbb{Z}^m, \lambda \in \mathbb{N}_0^m, \beta,\mu \in \mathbb{N}_0^n\}$ *forms a Hamel basis for the* k *linear space* $G_{m,n}(k)$.

These two lemma's imply the existence of an isomorphism σ of k-linear spaces between $G_{m,n}(k)$ and $S_m^{-1}A_{m+n}(k)$:

$$\sigma(\varphi^\alpha z^\beta y^\lambda \partial^u) = (-1)^{|\lambda|} x^\alpha z^\beta d^\lambda \partial^u .$$

The point is of course :

LEMMA 4. σ *is an isomorphism of* k-*algebra's*.

Proof. The only nonobvious part is to prove whether $\sigma(y_i\varphi_i) = \sigma(y_i)G(\varphi_i)$.

Since $y_i \varphi_i = \varphi_i y_i - \varphi_i$ one has

$$\sigma(y_i \varphi_i) = \sigma(\varphi_i, y_i) - \sigma(\varphi_i) = -x_i d_i - x_i = -d_i x_i = \sigma(y_i)\ \sigma(\varphi_i).$$

Having proved that $G_{m,n}(k)$ is isomorphic to a localization of $A_{m+n}(k)$ the only point left to verify is that theorem 1 stays valid under localization. To this end recall the definition of being saturated. An ideal $J \subset A_{m+n}(k)$ is called S_m-*saturated* if for all $a \in A_{m+n}(k)$, for which $sa \in J$ for some $s \in S_m$, $a \in J$, or stated differently

$$J = S_m^{-1}\ J \cap A_{m+n}(k).$$

If J is an ideal of $S_m^{-1}A_{m+n}(k)$, then $J \cap A_{m+n}(k)$ is obviously an S_m-saturated ideal of $A_{m+n}(k)$. In fact (STENSTRÖM [9], ch. 1 exc 11) :

LEMMA 5. *There is a one-one correspondence between the* S_m-*saturated ideals of* $A_{m+n}(k)$ *and the ideals of* $S_m^{-1}A_{m+n}(k)$, *and if* $J \triangleleft S_m^{-1}A_{m+n}(k)$ *then* $J = S_m^{-1}(J \cap A_{m+n}(k))$.

Corollary. THEOREM 2.

Proof. Let J be an ideal of $G_{m,n}(k)$, then $\sigma(J)$ is an ideal of $S_m^{-1}A_{m+n}(k)$. Let a_1 and a_2 generate $\sigma(J) \cap A_{m+n}(k)$, then a_1 and a_2 generate $\sigma(J)$ and hence J is generated by $\sigma^{-1}(a_1)$ and $\sigma^{-1}(a_2)$.

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§ 3. OTHER COROLLARIES.

1. *Generalized hypergeometric functions.* Note that a similar theorem holds for all generalized hypergeometric functions. For some cases where the parameters are integers this seems to be known.

2. *Module structure.* From LEMMA 4 and STAFFORD [8], TH 3.3/3.9 immediately follows :

THEOREM 4. *Let* M *be a finitely generated* $G_{m,n}(k)$*-module. Then* $M = M' \oplus G^s_{m,n}(k)$ *where* $s \in \mathbb{N}_0$, rank $M' \leqslant 1$ and M' *can be generated by two elements.*

3. *Global homological dimension.* Since $G_{m,n}(k)$ is flat over $A_{m+n}(k)$ (STENSTRÖM [9], ch. II, prop 3.5) gl.dim $G_{m,n}(k) \leqslant A_{m+n}(k) = m+n$ (ROOS [6]). Since k, considered as a $G_{m,n}(k)$ module has a minimal projective resolution of length m+n as one easily verifies this proves :

THEOREM 5. (*See also* BJÖRK [2] ch. 3, th 2.5) gl.dim $G_{m,n}(k) = m+n$.

4. *Krulldimension.* Using RENTSCHLER-GABRIEL's definition of Krulldimension, one proves :

THEOREM 6. K.dim $G_{m,n}(k) = m+n$.

Proof. From LEMMA 5 and prop. a of RENTSCHLER-GABRIEL [5] one has K.dim $G_{m,n}(k) \leqslant$ K.dim $A_{m+n}(k) = m+n$ (NOUAZY-GABRIEL [4]). Now proceed with induction to m.

The case m=0 being clear, let J be an ideal of $G_{m-1,n}(k)$, then $J\,[y_m,\varphi_m,\varphi_m^{-1}]/(y_m)$ is a submodule of $G_{m,n}(k)/(y_m)$ and hence again by prop. a of [5] one concludes that $K.\dim G_{m,n}(k)/(y_m) \geqslant m+n-1$ and now the infinite sequence of ideals in $G_{m,n}(k)$:

$$G_{m,n}(k) \supset (y_m) \supset (y_m)^2 \supset \dots$$

has factors of Krulldimension not less than m+n-1. Hence

$$m+n \leqslant \mathrm{Kdim}\ G_{m,n}(k) \leqslant \mathrm{Kdim}\ A_{m+n}(k) = m+n.$$

5. *Bernstein dimension.* For any $G_{m,n}(k)$ module M, one may define its Bernstein dimension d(M) (see GABBER [3] or BJÖRK [2]).
As was proved there one has $d(M) \geqslant m+n$ for any finitely generated non zero $G_{m,n}(k)$ module M.

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