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**Derivative Formula and Applications
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Derivative Formula and Applications for Hyperdissipative Stochastic Navier-Stokes/Burgers Equations*

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Abstract

By using coupling method, a Bismut type derivative formula is established for the Markov semigroup associated to a class of hyperdissipative stochastic Navier-Stokes/Burgers equations. As applications, gradient estimates, dimension-free Harnack inequality, strong Feller property, heat kernel estimates and some properties of the invariant probability measure are derived.

AMS subject Classification: 60J75, 60J45.

Keywords: Bismut formula, coupling, strong Feller, stochastic Navier-Stokes equation.

1 Introduction

Let H be the divergence free sub-space of $L^2(\mathbb{T}^d; \mathbb{R}^d)$, where $\mathbb{T}^d := (\mathbb{R}/[0, 2\pi])^d$ is the d -dimensional torus. The d -dimensional Navier-Stokes equation (for $d \geq 2$) reads

$$dX_t = \{\nu \Delta X_t - B(X_t, X_t)\}dt,$$

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where $\nu > 0$ is the viscosity constant and $B(u, v) := \mathbf{P}(u \cdot \nabla)v$ for $\mathbf{P} : L^2(\mathbb{T}^d; \mathbb{R}^d) \rightarrow H$ the orthogonal projection (see e.g. [13]). When $d = 1$ and $H = L^2(\mathbb{T}^d; \mathbb{R}^d)$, this equation reduces to the Burgers equation. In recent years, the stochastic Navier-Stokes equations have been investigated intensively, see e.g. [6] for the ergodicity of 2D Navier-Stokes equations with degenerate noise, and see [3, 5, 12] for the study of 3D stochastic Navier-Stokes equations. The main purpose of this paper is to establish the Bismut type derivative formula for the Markov semigroup associated to stochastic Navier-Stokes type equations, and as applications, to derive gradient estimates, Harnack inequality, and strong Feller property for the semigroup.

We shall work with a more general framework as in [8], which will be reduced to a class of hyperdissipative (i.e. the Laplacian has a power larger than 1) stochastic Navier-Stokes/Burgers equations in Section 2.

Let $(H, \langle \cdot, \cdot \rangle, \|\cdot\|_H)$ be a separable real Hilbert space, and $(L, \mathcal{D}(L))$ a positively definite self-adjoint operator on H with $\lambda_0 := \inf \sigma(L) > 0$, where $\sigma(L)$ is the spectrum of L . Let $V = \mathcal{D}(L^{1/2})$, which is a Banach space with norm $\|\cdot\|_V := \|L^{1/2} \cdot\|$. Let Q be a Hilbert-Schmidt linear operator on H with $\text{Ker } Q = \{0\}$. Then $\mathcal{D}(Q^{-1}) := Q(H)$ is a Banach space with norm $\|x\|_Q := \|Q^{-1}x\|_H$. In general, for $\theta > 0$, let $V_\theta = \mathcal{D}(L^{\theta/2})$ with norm $\|L^{\theta/2} \cdot\|_H$. We assume that there exist two constants $\theta \in (0, 1]$ and $K_1 > 0$ such that $V_\theta \subset \mathcal{D}(Q^{-1})$ and

$$\mathbf{(A0)} \quad \|u\|_Q^2 \leq K_1 \|u\|_{V_\theta}^2, \quad u \in V_\theta.$$

Moreover, let

$$B : V \times V \rightarrow H$$

be a bilinear map such that

$$\mathbf{(A1)} \quad \langle v, B(v, v) \rangle = 0, \quad v \in V;$$

$$\mathbf{(A2)} \quad \text{There exists a constant } C > 0 \text{ such that } \|B(u, v)\|_H^2 \leq C \|u\|_H^2 \|v\|_V^2, \quad u, v \in V;$$

$$\mathbf{(A3)} \quad \text{There exists a constant } K_2 > 0 \text{ such that } \|B(u, v)\|_Q^2 \leq K_2 \|u\|_{V_\theta}^2 \|v\|_{V_\theta}^2, \quad u, v \in V.$$

Finally, let W_t be the cylindrical Brownian motion on H . We consider the following stochastic differential equation on H :

$$(1.1) \quad dX_t = QdW_t - \{LX_t + B(X_t)\}dt,$$

where $B(X_t) := B(X_t, X_t)$. According to [8], for any initial value $X_0 \in H$ the equation (1.1) has a unique strong solution, which gives rise to a Markov process on H (see Appendix for details). For any $x \in H$, let X_t^x be the solution starting at x . Let $\mathcal{B}_b(H)$ be the set of all bounded measurable functions on H . Then

$$P_t f(x) := \mathbb{E}f(X_t^x), \quad x \in H, t \geq 0, f \in \mathcal{B}_b(H)$$

defines a Markov semigroup $(P_t)_{t \geq 0}$.

We shall adopt a coupling argument to establish a Bismut type derivative formula for P_t , which will imply explicit gradient estimates and the dimension-free Harnack inequality in the sense of [14]. This type of Harnack inequality has been applied to the study of several models of SDEs and SPDEs, see e.g. [4, 7, 9, 11, 10, 15] and references within.

For $f \in \mathcal{B}_b(H)$, $h \in V_\theta$, $x \in H$ and $t > 0$, let

$$D_h P_t f(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \{P_t f(x + \varepsilon h) - P_t f(x)\}$$

provided the limit in the right-hand side exists. Let $\tilde{B}(u, v) = B(u, v) + B(v, u)$.

Theorem 1.1. *Assume that **(A0)**-**(A3)** hold for some constants $\theta \in (0, 1]$, $K_1, K_2, C > 0$. Then for any $t > 0$, $h \in V_\theta$ and $f \in \mathcal{B}_b(H)$, $D_h P_t f$ exists on H and satisfies*

$$(1.2) \quad D_h P_t f(x) = \mathbb{E} \left\{ f(X_t^x) \int_0^t \left\langle Q^{-1} \left(\frac{1}{t} e^{-sL} h - \frac{t-s}{t} \tilde{B}(X_s^x, e^{-sL} h) \right), dW_s \right\rangle \right\}, \quad x \in H.$$

Let V_θ^* be the dual space of V_θ . According to Theorem 1.1, under assumptions **(A0)**-**(A3)** we may define the gradient $DP_t f : H \rightarrow V_\theta^*$ by letting

$$v_\theta^* \langle DP_t f(x), h \rangle_{V_\theta} = D_h P_t f(x), \quad x \in H, h \in V_\theta.$$

We shall estimate

$$\|DP_t f(x)\|_{V_\theta^*} := \sup_{\|h\|_{V_\theta} \leq 1} |D_h P_t f(x)|, \quad x \in H.$$

To this end, let $\|Q\|$ and $\|Q\|_{HS}$ be the operator norm and the Hilbert-Schmidt norm of $Q : H \rightarrow H$ respectively.

Corollary 1.2. *Under assumptions of Theorem 1.1.*

(1) *For any $t > 0$, $x \in H$ and $f \in \mathcal{B}_b(H)$,*

$$\|DP_t f(x)\|_{V_\theta^*}^2 \leq (P_t f^2(x)) \left\{ \frac{2K_1}{t} + \frac{4K_2}{\lambda_0^{2-\theta}} (\|x\|_H^2 + \|Q\|_{HS}^2 t) \right\}.$$

(2) *Let $f \in \mathcal{B}_b(H)$ be positive. For any $x \in H$, $t > 0$ and $\delta \geq 4\sqrt{K_2} \|Q\| \lambda_0^{(\theta-3)/2}$,*

$$\begin{aligned} \|DP_t f(x)\|_{V_\theta^*} &\leq \delta \{P_t(f \log f) - (P_t f) \log P_t f\}(x) \\ &\quad + \frac{2}{\delta} \left\{ \frac{K_1}{t} + \frac{2K_2}{\lambda_0^{1-\theta}} (\|x\|_H^2 + \|Q\|_{HS}^2 t) \right\} P_t f(x). \end{aligned}$$

(3) Let $\alpha > 1, t > 0$ and $f \geq 0$. The Harnack inequality

$$(P_t f(x))^\alpha \leq (P_t f^\alpha(y)) \exp \left[\frac{2\alpha \|x - y\|_{V_\theta}^2}{\alpha - 1} \left\{ \frac{K_1}{t} + \frac{2K_2}{\lambda_0^{1-\theta}} (\|x\|_H^2 \vee \|y\|_H^2 + \|Q\|_{HS}^2 t) \right\} \right]$$

holds for $x, y \in H$ such that

$$\|x - y\|_{V_\theta} \leq \frac{(\alpha - 1)\lambda_0^{(3-\theta)/2}}{4\alpha\|Q\|\sqrt{K_2}}.$$

In particular, P_t is V_θ -strong Feller, i.e. $\lim_{\|y-x\|_{V_\theta} \rightarrow 0} P_t f(y) = P_t f(x)$ holds for $f \in \mathcal{B}_b(H), t > 0, x \in H$.

As applications of the Harnack inequality derived above, we have the following result.

Corollary 1.3. Under assumptions of Theorem 1.1. P_t has an invariant probability measure μ such that $\mu(V) = 1$ and hence, $\mu(V_\theta) = 1$. If moreover $\theta \in (0, 1)$, then:

- (1) P_t has a unique invariant probability measure μ , and the measure has full support on V_θ .
- (2) P_t has a density $p_t(x, y)$ on V_θ w.r.t. μ . Moreover, let $r_0 = \frac{(\alpha-1)\lambda_0^{(3-\theta)/2}}{4\alpha\|Q\|\sqrt{K_2}}$ and $B_\theta(x, r_0) = \{y : \|y - x\|_{V_\theta} \leq r_0\}$,

$$\begin{aligned} & \left(\int_{V_\theta} p_t(x, y)^{(\alpha+1)/\alpha} \mu(dy) \right)^\alpha \\ & \leq \frac{1}{\int_{B_\theta(x, r_0)} \exp \left[- \frac{2\alpha \|x-y\|_{V_\theta}^2}{\alpha-1} \left\{ \frac{K_1}{t} + \frac{2K_2}{\lambda_0^{1-\theta}} (\|x\|_H^2 \vee \|y\|_H^2 + \|Q\|_{HS}^2 t) \right\} \right] \mu(dy)} < \infty \end{aligned}$$

holds for any $t > 0, \alpha > 1$ and $x \in V_\theta$.

Note that the Harnack inequality presented in Corollary 1.2 is local in the sense that $\|x - y\|_{V_\theta}$ has to be bounded above by a constant. To derive a global Harnack inequality, we need to extend the gradient-entropy inequality in Corollary 1.2 (2) to all $\delta > 0$. In this spirit, we have the following result.

Theorem 1.4. Under assumptions of Theorem 1.1.

- (1) For any $\delta > 0$ and any positive $f \in \mathcal{B}_b(H)$,

$$\begin{aligned} \|DP_t f(x)\|_{V_\theta^*} & \leq \delta \{P_t(f \log f) - (P_t f) \log P_t f\}(x) \\ & \quad + \frac{2}{\delta} \left\{ \frac{K_1}{t \wedge t_\delta} + \frac{2K_2 e}{\lambda_0^{1-\theta}} (\|x\|_H^2 + \|Q\|_{HS}^2 t) \right\} P_t f(x), \quad x \in H, t > 0 \end{aligned}$$

holds for $t_\delta := \frac{\delta^2 \lambda_0^{3-\theta}}{4\|Q\|^2 e K_2}$.

(2) Let $\alpha > 1, t > 0$ and $f \geq 0$. Then

$$(P_t f(x))^\alpha \leq (P_t f^\alpha(y)) \exp \left[\frac{2\alpha \|x - y\|_{V_\theta}^2}{\alpha - 1} \left\{ K_1 \left(\frac{1}{t} \vee \frac{4\alpha^2 \|Q\|^2 e K_2 \|x - y\|_\theta^2}{(\alpha - 1)^2 \lambda_0^{3-\theta}} \right) + \frac{2K_2 e}{\lambda_0^{1-\theta}} (\|x\|_H^2 \vee \|y\|_H^2 + \|Q\|_{HS}^2 t) \right\} \right]$$

holds for all $x, y \in \mathbb{H}$.

The remainder of the paper is organized as follows. We first consider in Section 2 a class of stochastic Navier-Stokes type equations to illustrate our results, then prove these results in Section 3.

2 Stochastic hyperdissipative Navier-Stokes/Burgers equations

Let $\mathbb{T}^d = (\mathbb{R}/[0, 2\pi])^d$ for $d \geq 1$. Let Δ be the Laplace operator on \mathbb{T}^d . To formulate Δ using spectral representation, we first consider the complex L^2 space $L^2(\mathbb{T}^d; \mathbb{C}^d)$. Recall that for $a = (a_1, \dots, a_d), b = (b_1, \dots, b_d) \in \mathbb{C}^d$, we have $a \cdot b = \sum_{i=1}^d a_i \bar{b}_i$. Let

$$e_k(x) = (2\pi)^{-d/2} e^{i(k \cdot x)}, \quad k \in \mathbb{Z}^d, x \in \mathbb{T}^d.$$

Then $\{e_k : k \in \mathbb{Z}^d\}$ is an ONB of $L^2(\mathbb{T}^d; \mathbb{C})$. Obviously, for a sequence $\{u_k\}_{k \in \mathbb{Z}^d} \subset \mathbb{C}^d$,

$$u := \sum_{k \in \mathbb{Z}^d} u_k e_k \in L^2(\mathbb{T}^d; \mathbb{R}^d)$$

if and only if $\bar{u}_k = u_{-k}$ holds for any $k \in \mathbb{Z}^d$ and $\sum_{k \in \mathbb{Z}^d} |u_k|^2 < \infty$. By spectral representation, we may characterize $(\Delta, \mathcal{D}(\Delta))$ on $L^2(\mathbb{T}^d; \mathbb{R}^d)$ as follows:

$$\Delta u = - \sum_{k \in \mathbb{Z}^d} |k|^2 u_k e_k, \quad u := \sum_{k \in \mathbb{Z}^d} u_k e_k \in \mathcal{D}(\Delta),$$

$$\mathcal{D}(\Delta) := \left\{ \sum_{k \in \mathbb{Z}^d} u_k e_k : u_k \in \mathbb{C}^d, \bar{u}_k = u_{-k}, \sum_{k \in \mathbb{Z}^d} |u_k|^2 |k|^4 < \infty \right\}.$$

To formulate the Navier-Stokes/Burgers type equation, when $d \geq 2$ we consider the sub-space divergence free elements of $L^2(\mathbb{T}^d; \mathbb{R}^d)$. It is easy to see that a smooth vector field

$$u = \sum_{k \in \mathbb{Z}^d} u_k e_k$$

is divergence free if and only if $u_k \cdot k = 0$ holds for all $k \in \mathbb{Z}^d$. Moreover, to make the spectrum of $-\Delta$ strictly positive, we shall not consider non-zero constant vector fields. Therefore, the Hilbert space we are working on becomes

$$H := \left\{ \sum_{k \in \hat{\mathbb{Z}}^d} u_k e_k : u_k \in \mathbb{C}^d, (d-1)(u_k \cdot k) = 0, \bar{u}_k = u_{-k}, \sum_{k \in \hat{\mathbb{Z}}^d} |u_k|^2 < \infty \right\},$$

where $\hat{\mathbb{Z}}^d = \mathbb{Z}^d \setminus \{0\}$. Since when $d = 1$ the condition $(d-1)(u_k \cdot k) = 0$ is trivial, the divergence free restriction does not apply for the one-dimensional case.

Let $(A, \mathcal{D}(A)) = (-\Delta, \mathcal{D}(\Delta))|_H$, the restriction of $(\Delta, \mathcal{D}(\Delta))$ on H , and let $\mathbf{P} : L^2(\mathbb{T}^d; \mathbb{R}^d) \rightarrow H$ be the orthogonal projection. Let

$$L = \lambda_0 A^{\delta+1}$$

for some constants $\lambda_0, \delta > 0$. As in Section 1, define $V = \mathcal{D}(L^{1/2})$ and $V_\theta = \mathcal{D}(L^{\theta/2})$. Then

$$B : V \times V \rightarrow H; \quad B(u, v) = \mathbf{P}(u \cdot \nabla)v$$

is a continuous bilinear (see the (b) in the proof of Theorem 2.1 below). Let $Q = A^{-\sigma}$ for some $\sigma > 0$, and let W_t be the cylindrical Brownian motion on H . Obviously, $\|Q\| \leq 1$ and when $\sigma > \frac{d}{4}$,

$$\|Q\|_{HS}^2 \leq \sum_{k \in \hat{\mathbb{Z}}^d} |k|^{-4\sigma} < \infty.$$

We consider the stochastic differential equation

$$(2.1) \quad dX_t = QdW_t - (LX_t + B(X_t))dt,$$

where $B(u) := B(u, u)$ for $u \in V$. Thus, we are working on the stochastic hyperdissipative Navier-Stokes (for $d \geq 2$) and Burgers (for $d = 1$) equations.

Theorem 2.1. *Let $\delta > \frac{d}{2}$, $\sigma \in (\frac{d}{4}, \frac{\delta}{2}]$ and $\theta \in [\frac{2\sigma+1}{\delta+1}, 1]$. Then all assertions in Section 1 hold for $K_1 = \frac{1}{\lambda_0^\theta}$ and*

$$K_2 = \frac{4^{2\delta\theta+1}}{\lambda_0^{2\theta}} \sum_{k \in \hat{\mathbb{Z}}^d} |k|^{-2(\delta+1)\theta} < \infty.$$

Proof. Since $\sigma > \frac{d}{4}$, $Q : H \rightarrow H$ is Hilbert-Schmidt. By Theorem 1.1 and its consequences, it suffices to verify assumptions **(A0)**-**(A3)**. Since **(A1)** is trivial for $d = 1$ and follows from the divergence free property for $d \geq 2$, we only have to prove **(A0)**, **(A2)** and **(A3)**. Let

$$u = \sum_{k \in \hat{\mathbb{Z}}^d} u_k e_k, \quad v = \sum_{k \in \hat{\mathbb{Z}}^d} v_k e_k$$

be two elements in V_θ .

(a) Since $\theta \in [\frac{2\sigma+1}{\delta+1}, 1]$ implies $4\sigma \leq 2\theta(\delta+1)$, we have

$$\|u\|_Q^2 = \sum_{k \in \hat{\mathbb{Z}}^d} |u_k|^2 |k|^{4\sigma} \leq \frac{1}{\lambda_0^\theta} \sum_{k \in \hat{\mathbb{Z}}^d} \lambda_0^\theta |u_k|^2 |k|^{2\theta(\delta+1)} = \frac{1}{\lambda_0^\theta} \|u\|_{V_\theta}^2.$$

Thus, **(A0)** holds for $K_1 = \frac{1}{\lambda_0^\theta}$.

(b) It is easy to see that

$$(2.2) \quad B(u, v) = \mathbf{P} \sum_{l, m \in \hat{\mathbb{Z}}^d, m \neq l} i(u_{l-m} \cdot m) v_m e_l.$$

By Hölder inequality,

$$\begin{aligned} \|B(u, v)\|_H^2 &\leq \sum_{l \in \hat{\mathbb{Z}}^d} \left(\sum_{m \in \hat{\mathbb{Z}}^d \setminus \{l\}} |u_{l-m}| \cdot |m| \cdot |v_m| \right)^2 \\ &\leq \sum_{l \in \hat{\mathbb{Z}}^d} \left(\sum_{m \in \hat{\mathbb{Z}}^d \setminus \{l\}} |u_{l-m}|^2 |m|^{-2\delta} \right) \sum_{m \in \hat{\mathbb{Z}}^d} |v_m|^2 |m|^{2(\delta+1)} \\ &\leq \frac{1}{\lambda_0} \left(\sum_{m \in \hat{\mathbb{Z}}^d} |m|^{-2\delta} \right) \|u\|_H^2 \|v\|_V^2. \end{aligned}$$

Since $\delta > \frac{d}{2}$, we have $\sum_{m \in \hat{\mathbb{Z}}^d} |m|^{-2\delta} < \infty$. Thus, **(A2)** holds for some constant C .

(c) By (2.2), we have

$$\begin{aligned} \|B(u, v)\|_Q^2 &:= \|A^\sigma B(u, v)\|_H^2 \leq \sum_{l \in \hat{\mathbb{Z}}^d} |l|^{4\sigma} \left(\sum_{m \in \hat{\mathbb{Z}}^d} |u_{l-m}| \cdot |m| \cdot |v_m| \right)^2 \\ (2.3) \quad &\leq 2 \sum_{l \in \hat{\mathbb{Z}}^d} |l|^{4\sigma} \left(\sum_{|m| > \frac{|l|}{2}, m \neq l} |u_{l-m}| \cdot |m| \cdot |v_m| \right)^2 \\ &\quad + 2 \sum_{l \in \hat{\mathbb{Z}}^d} |l|^{4\sigma} \left(\sum_{|m| \leq \frac{|l|}{2}, m \in \hat{\mathbb{Z}}^d} |u_{l-m}| \cdot |m| \cdot |v_m| \right)^2 := 2I_1 + 2I_2. \end{aligned}$$

By the Schwartz inequality,

$$I_1 \leq \sum_{l \in \hat{\mathbb{Z}}^d} |l|^{4\sigma} \left(\sum_{|m| > \frac{|l|}{2}, m \neq l} |u_{l-m}|^2 |l-m|^{2(\delta+1)\theta} |m|^{2-2(\delta+1)\theta} \right) \sum_{|m| > \frac{|l|}{2}, m \neq l} |v_m|^2 |m|^{2(\delta+1)\theta} |l-m|^{-2(\delta+1)\theta}.$$

Since $\theta \geq \frac{2\sigma+1}{\delta+1}$ implies that $4\sigma - 2(\delta+1)\theta + 2 \leq 0$, if $|m| > \frac{|l|}{2}$ and $|l| \geq 1$ we have

$$|l|^{4\sigma}|m|^{-2(\delta+1)\theta+2} \leq 4^{(\delta+1)\theta-1}|l|^{4\sigma-2(\delta+1)\theta+2} \leq 4^{(\delta+1)\theta-1}.$$

Therefore,

$$(2.4) \quad \begin{aligned} I_1 &\leq \frac{1}{\lambda_0^\theta} 4^{(\delta+1)\theta-1} \|u\|_{V_\theta}^2 \sum_{l \in \hat{\mathbb{Z}}^d} \sum_{|m| > \frac{|l|}{2}, m \neq l} |v_m|^2 |m|^{2(\delta+1)\theta} |l-m|^{-2(\delta+1)\theta} \\ &\leq \frac{1}{\lambda_0^{2\theta}} 4^{(\delta+1)\theta-1} \left(\sum_{m \in \hat{\mathbb{Z}}^d} |m|^{-2(\delta+1)\theta} \right) \|u\|_{V_\theta}^2 \|v\|_{V_\theta}^2. \end{aligned}$$

Similarly, when $|m| \leq \frac{|l|}{2}$ we have $|l-m| \geq \frac{|l|}{2}$ and thus, due to $4\sigma - 2(\delta+1)\theta \leq 0$,

$$|l|^{4\sigma}|l-m|^{-2(\delta+1)\theta} \leq 4^{(\delta+1)\theta}|l|^{4\sigma-2(\delta+1)\theta} \leq 4^{(\delta+1)\theta}|m|^{4\sigma-2(\delta+1)\theta}.$$

Therefore,

$$\begin{aligned} I_2 &\leq \sum_{l \in \hat{\mathbb{Z}}^d} |l|^{4\sigma} \left(\sum_{1 \leq |m| \leq \frac{|l|}{2}} |u_{l-m}|^2 |l-m|^{2(\delta+1)\theta} |m|^{2-2(\delta+1)\theta} \right) \sum_{1 \leq |m| \leq \frac{|l|}{2}} |v_m|^2 |m|^{2(\delta+1)\theta} |l-m|^{-2(\delta+1)\theta} \\ &\leq \frac{4^{(\delta+1)\theta}}{\lambda_0^{2\theta}} \left(\sum_{m \in \hat{\mathbb{Z}}^d} |m|^{4\sigma-4(\delta+1)\theta+2} \right) \|u\|_{V_\theta}^2 \|v\|_{V_\theta}^2 \leq \frac{4^{(\delta+1)\theta}}{\lambda_0^{2\theta}} \left(\sum_{m \in \hat{\mathbb{Z}}^d} |m|^{-2(\delta+1)\theta} \right) \|u\|_{V_\theta}^2 \|v\|_{V_\theta}^2, \end{aligned}$$

where the last step is due to $4\sigma - 2(\delta+1)\theta + 2 \leq 0$ mentioned above. Combining this with (2.3) and (2.4), we prove **(A3)** for the desired K_2 which is finite since $\theta \geq \frac{2\sigma+1}{\delta+1}$ and $\sigma > \frac{d}{4}$ imply that $2(\delta+1)\theta \geq 4\sigma + 1 > d$. \square

3 Proofs of Theorem 1.1 and consequences

We first present an exponential estimate of the solution, which will be used in the proof of Theorem 1.1.

Lemma 3.1. *In the situation of Theorem 1.1, we have*

$$\mathbb{E} \exp \left[\frac{\lambda_0^2}{2\|Q\|^2} \int_0^t \|X_s^x\|_V^2 ds \right] \leq \exp \left[\frac{\lambda_0^2}{2\|Q\|^2} (\|x\|_H^2 + \|Q\|_{HSt}^2) \right], \quad x \in H, t \geq 0.$$

Moreover, for any $t > 0$ and $x \in H$,

$$\mathbb{E} \exp \left[\frac{2}{\|Q\|^2 e t} \int_0^t \|X_s^x\|_V^2 ds \right] \leq \exp \left[\frac{2}{\|Q\|^2 t} (\|x\|_H^2 + \|Q\|_{HSt}^2) \right].$$

Proof. (a) Since $\langle B(u, v), v \rangle = 0$, by the Itô formula we have

$$(3.1) \quad d\|X_t^x\|_H^2 \leq -2\|X_t^x\|_V^2 dt + \|Q\|_{HS}^2 dt + 2\langle X_t^x, QdW_t \rangle.$$

Let

$$\tau_n := \inf\{t \geq 0 : \|X_t^x\|_H \geq n\}.$$

By Theorem 4.1 below we have $\tau_n \rightarrow \infty$ as $n \rightarrow \infty$. So, for any $\lambda > 0$ and $n \geq 1$,

$$\begin{aligned} \mathbb{E} \exp \left[\lambda \int_0^{t \wedge \tau_n} \|X_s^x\|_V^2 ds \right] &\leq \mathbb{E} \exp \left[\frac{\lambda}{2} (\|x\|_H^2 + \|Q\|_{HS}^2 t) + \lambda \int_0^{t \wedge \tau_n} \langle X_s^x, QdW_s \rangle \right] \\ &\leq \exp \left[\frac{\lambda}{2} (\|x\|_H^2 + \|Q\|_{HS}^2 t) \right] \left(\mathbb{E} \exp \left[2\lambda^2 \|Q\|^2 \int_0^{t \wedge \tau_n} \|X_s^x\|_H^2 ds \right] \right)^{1/2} < \infty. \end{aligned}$$

Since $\|\cdot\|_H^2 \leq \frac{1}{\lambda_0} \|\cdot\|_V^2$, this implies that

$$\mathbb{E} \exp \left[\lambda \int_0^{t \wedge \tau_n} \|X_s^x\|_V^2 ds \right] \leq e^{\frac{\lambda}{2} (\|x\|_H^2 + \|Q\|_{HS}^2 t)} \left(\mathbb{E} \exp \left[\frac{2\lambda^2 \|Q\|^2}{\lambda_0} \int_0^{t \wedge \tau_n} \|X_s^x\|_V^2 ds \right] \right)^{1/2}.$$

Letting $\lambda = \frac{\lambda_0^2}{2\|Q\|^2}$, we obtain

$$\mathbb{E} \exp \left[\frac{\lambda_0^2}{2\|Q\|^2} \int_0^{t \wedge \tau_n} \|X_s^x\|_V^2 ds \right] \leq \exp \left[\frac{\lambda_0^2}{2\|Q\|^2} (\|x\|_H^2 + \|Q\|_{HS}^2 t) \right].$$

This proves the first inequality by letting $n \rightarrow \infty$.

(b) Next, due to the first inequality and the Jensen inequality, we only have to prove the second one for $t \leq \lambda_0^{-2}$. In this case, let

$$\beta(s) = e^{(\lambda_0^2 - t^{-1})s}, \quad s \in [0, t].$$

By the Itô formula, we have

$$d\|X_s^x\|_H^2 \beta(s) = \left\{ -2\|X_s^x\|_V^2 \beta(s) + \beta'(s) \|X_s^x\|_H^2 + \beta(s) \|Q\|_{HS}^2 \right\} ds + 2\beta(s) \langle X_s^x, QdW_s \rangle.$$

Thus, for any $\lambda > 0$,

$$(3.2) \quad \begin{aligned} &\mathbb{E} \exp \left[2\lambda \int_0^{t \wedge \tau_n} \|X_s^x\|_V^2 \beta(s) ds - \lambda \|x\|_H^2 - \lambda \|Q\|_{HS}^2 t \right] \\ &\leq \mathbb{E} \exp \left[2\lambda \int_0^{t \wedge \tau_n} \beta(s) \langle X_s^x, QdW_s \rangle + \lambda \int_0^{t \wedge \tau_n} \beta'(s) \|X_s^x\|_H^2 ds \right] \\ &\leq \left(\mathbb{E} \exp \left[2\lambda \int_0^{t \wedge \tau_n} \|X_s^x\|_V^2 \beta(s) ds \right] \right)^{1/2} \left(\mathbb{E} \exp \left[4\lambda \int_0^{t \wedge \tau_n} \beta(s) \langle X_s^x, QdW_s \rangle \right. \right. \\ &\quad \left. \left. - 2\lambda \int_0^{t \wedge \tau_n} \|X_s^x\|_H^2 (\lambda_0^2 \beta(s) - \beta'(s)) ds \right] \right)^{1/2}. \end{aligned}$$

Note that the first inequality in the above display implies that

$$\mathbb{E} \exp \left[2\lambda \int_0^{t \wedge \tau_n} \|X_s^x\|_V^2 \beta(s) ds \right] < \infty, \quad n \geq 1.$$

Let

$$\lambda = \frac{1}{t \|Q\|^2}.$$

By our choice of $\beta(s)$ and noting that $t \leq \lambda_0^{-2}$ so that $\beta(s) \leq 1$, we have

$$\frac{1}{2} (4\lambda)^2 \beta(s)^2 \|Q\|^2 \leq 2\lambda^2 \beta(s) \|Q\|^2 \leq 2\lambda (\lambda_0^2 \beta(s) - \beta'(s)).$$

Therefore,

$$\mathbb{E} \exp \left[4\lambda \int_0^{t \wedge \tau_n} \beta(s) \langle X_s^x, Q dW_s \rangle - 2\lambda \int_0^{t \wedge \tau_n} \|X_s^x\|_H^2 (\lambda_0^2 \beta(s) - \beta'(s)) ds \right] \leq 1.$$

Combining this with (3.2) for $\lambda = (t \|Q\|^2)^{-1}$, we obtain

$$\mathbb{E} \exp \left[\frac{2}{\|Q\|^2 et} \int_0^{t \wedge \tau_n} \|X_s^x\|_V^2 ds \right] \leq \exp \left[\frac{2}{\|Q\|^2 2t} (\|x\|_H^2 + \|Q\|_{HSt}^2) \right].$$

This completes the proof by letting $n \rightarrow \infty$. \square

Proof of Theorem 1.1. Simply denote $X_s = X_s^x$, which solves (2.1) for $X_0 = x$. For given $h \in V_\theta$ and $\varepsilon > 0$, by Theorem 4.1 below the equation

$$(3.3) \quad dY_s = Q dW_s - \left\{ LY_s + B(X_s) + \frac{\varepsilon}{t} e^{-Ls} h \right\} ds, \quad Y_0 = x + \varepsilon h$$

has a unique solution. So,

$$d(X_s - Y_s) = -L(X_s - Y_s) ds + \frac{\varepsilon}{t} e^{-Ls} h ds.$$

This implies that

$$(3.4) \quad \begin{aligned} X_s - Y_s &= e^{-Ls} (X_0 - Y_0) + \frac{\varepsilon}{t} \int_0^s e^{-L(s-r)} e^{-Lr} h dr \\ &= \frac{\varepsilon(t-s)}{t} e^{-Ls} h =: Z_s, \quad s \in [0, t]. \end{aligned}$$

Let

$$\eta_s = B(X_s + Z_s) - B(X_s) - \frac{\varepsilon}{t} e^{-Ls} h,$$

which is well-defined since according to Lemma 3.1, $X \in V$ holds $\mathbb{P} \times ds$ -a.e. Then, by (3.4) the equation (3.3) reduces to

$$(3.5) \quad dY_s = QdW_s - \{LY_s + B(Y_s)\}ds + \eta_s ds = Qd\tilde{W}_s - \{LY_s + B(Y_s)\}ds,$$

where

$$\tilde{W}_s := W_s + \int_0^s Q^{-1}\eta_r dr, \quad s \in [0, t].$$

By **(A0)** and **(A3)** we have

$$(3.6) \quad \begin{aligned} \|Q^{-1}\eta_s\|_H^2 &\leq \frac{2\varepsilon^2 K_1^2}{t^2} \|h\|_{V_\theta}^2 + 2\|\tilde{B}(X_s, Z_s) + B(z_s, z_s)\|_Q^2 \\ &\leq \varepsilon^2 C(t) (\|h\|_{V_\theta}^2 + \varepsilon^2 \|h\|_{V_\theta}^4 + \|h\|_{V_\theta}^2 \|X_s\|_{V_\theta}^2). \end{aligned}$$

Since $\theta \leq 1$ so that $\|\cdot\|_{V_\theta} \leq c\|\cdot\|_V$ holds for some constant $c > 0$, combining (3.6) with Lemma 3.1 we concluded that

$$\mathbb{E}e^{\int_0^t \|\eta_s\|_Q^2 ds} < \infty$$

holds for small enough $\varepsilon > 0$. By the Girsanov theorem, in this case

$$R_s := \exp \left[- \int_0^s \langle Q^{-1}\eta_r, dW_r \rangle - \frac{1}{2} \int_0^s \|\eta_r\|_Q^2 dr \right], \quad s \in [0, t]$$

is a martingale and $\{\tilde{W}_s\}_{s \in [0, t]}$ is the cylindrical Brownian motion on H under the probability measure $\mathbb{R}_t \mathbb{P}$. Combining this with (3.5) and the fact that $Y_t = X_t$ due to (3.4), for small $\varepsilon > 0$ we have

$$P_t f(x + \varepsilon h) = \mathbb{E}[R_t f(Y_t)] = \mathbb{E}[R_t f(X_t)].$$

Therefore, by the dominated convergence theorem due to Lemma 3.1 and (3.6), we conclude that

$$\begin{aligned} D_h P_t f(x) &:= \lim_{\varepsilon \rightarrow 0} \frac{P_t f(x + \varepsilon h) - P_t f(x)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\frac{R_t - 1}{\varepsilon} f(X_t) \right] = -\mathbb{E} \left\{ f(X_t) \lim_{\varepsilon \rightarrow 0} \int_0^t \left\langle Q^{-1} \frac{\eta_s}{\varepsilon}, dW_s \right\rangle \right\} \\ &= -\mathbb{E} \left\{ f(X_t) \int_0^t \left\langle Q^{-1} \left(\frac{t-s}{t} \tilde{B}(e^{-Ls}h, X_s) - \frac{1}{t} e^{-Ls}h \right), dW_s \right\rangle \right\}, \end{aligned}$$

where the last step is due to the bilinear property of B , which implies that

$$\begin{aligned}
\frac{\eta_s}{\varepsilon} &= \frac{1}{\varepsilon} \tilde{B}(X_s, z_s) + \frac{1}{\varepsilon} B(Z_\varepsilon) - \frac{1}{t} e^{-Ls} h \\
&= \frac{t-s}{t} \tilde{B}(X_s, e^{-Ls} h) - \frac{1}{t} e^{-Ls} h + \frac{\varepsilon(t-s)}{t} B(e^{-Ls} h, e^{-Ls} h).
\end{aligned}$$

□

Proof of Corollary 1.2. (1) By (1.2) and the Schwartz inequality, for any h with $\|h\|_{V_\theta} \leq 1$, we have

$$\begin{aligned}
(3.7) \quad |D_h P_t f(x)|^2 &\leq (P_t f(x))^2 \mathbb{E} \int_0^t \left\| \frac{1}{t} e^{-Ls} h - \frac{t-s}{t} \tilde{B}(X_s^x, h) \right\|_Q^2 ds \\
&\leq 2(P_t f^2(x)) \left\{ \frac{K_1}{t} + \mathbb{E} \int_0^t \|\tilde{B}(X_s^x, h)\|_Q^2 ds \right\},
\end{aligned}$$

where the last step is due to the fact that **(A0)** implies

$$(3.8) \quad \|e^{-Ls} h\|_Q^2 \leq K_1 \|e^{-Ls} h\|_{V_\theta}^2 \leq K_1 \|h\|_{V_\theta}^2.$$

Next, by **(A3)** and $\theta \leq 1$ we have

$$(3.9) \quad \|\tilde{B}(X_s^x, h)\|_Q^2 \leq 4K_2 \|h\|_{V_\theta}^2 \|X_s^x\|_{V_\theta}^2 \leq \frac{4K_2}{\lambda_0^{1-\theta}} \|X_s^x\|_V^2.$$

Combining this with (3.1) we obtain

$$\mathbb{E} \int_0^t \|\tilde{B}(X_s^x, h)\|_Q^2 ds \leq \frac{2K_2}{\lambda_0^{1-\theta}} (\|x\|_H^2 + \|Q\|_{HS}^2 t).$$

The proof of (1) is completed by this and (3.7).

(2) Let $f \geq 0$ and h be such that $\|h\|_{V_\theta} \leq 1$. Let

$$M_t = \int_0^t \left\langle Q^{-1} \left(\frac{t-s}{t} \tilde{B}(e^{-Ls} h, X_s) - \frac{1}{t} e^{-Ls} h \right), dW_s \right\rangle.$$

By (1.2) and the Young inequality (see e.g. [2, Lemma 2.4]),

$$(3.10) \quad |D_h P_t f(x)| \leq \delta \{P_t(f \log f) - (P_t f) \log P_t f\}(x) + \{\delta \log \mathbb{E} e^{\frac{1}{\delta} M_t}\} P_t f(x), \quad \delta > 0.$$

Since by (3.8) and (3.9) we have

$$\begin{aligned}
\langle M \rangle_t &= \int_0^t \left\| \frac{1}{t} e^{-Ls} h - \frac{t-s}{t} \tilde{B}(X_s^x, h) \right\|_Q^2 ds \\
&\leq \frac{2K_1}{t} + \frac{4K_2}{\lambda_0^{1-\theta}} \int_0^t \|X_s^x\|_V^2 ds,
\end{aligned}$$

it follows from Lemma 3.1 that for any $\delta \geq \delta_0 := 4\sqrt{K_2} \|Q\| \lambda_0^{(\theta-3)/2}$,

$$\begin{aligned}
\mathbb{E} \exp \left[\frac{1}{\delta} M_t \right] &\leq \left(\mathbb{E} \exp \left[\frac{2}{\delta^2} \langle M \rangle_t \right] \right)^{1/2} \leq \left(\mathbb{E} \exp \left[\frac{2}{\delta_0^2} \langle M \rangle_t \right] \right)^{\delta_0^2/(2\delta^2)} \\
&\leq \exp \left[\frac{2K_1}{\delta^2 t} \right] \left(\mathbb{E} \exp \left[\frac{8K_2}{\delta_0^2 \lambda_0^{1-\theta}} \int_0^t \|X_s^x\|_V^2 ds \right] \right)^{\delta_0^2/(2\delta^2)} \\
&= \exp \left[\frac{2K_1}{\delta^2 t} \right] \left(\mathbb{E} \exp \left[\frac{\lambda_0^2}{2\|Q\|^2} \int_0^t \|X_s^x\|_V^2 ds \right] \right)^{\delta_0^2/(2\delta^2)} \\
&\leq \exp \left\{ \frac{2K_1}{\delta^2 t} + \frac{\lambda_0^2 \delta_0^2}{4\delta^2 \|Q\|^2} (\|x\|_H^2 + \|Q\|_{HS}^2 t) \right\} \\
&= \exp \left\{ \frac{2}{\delta^2} \left(\frac{K_1}{t} + \frac{2K_2}{\lambda_0^{1-\theta}} (\|x\|_H^2 + \|Q\|_{HS}^2 t) \right) \right\}.
\end{aligned}$$

Combining this with (3.10) we prove (2).

(3) According to e.g. [4, proof of Proposition 4.1]), the V_θ -strong Feller property of P_t follows from the claimed Harnack inequality, which we prove below by using an argument in [2, Proof of Theorem 1.2]. Let $x \neq y$ be such that

$$(3.11) \quad \|x - y\|_{V_\theta} \leq \frac{\alpha - 1}{\alpha \delta_0} \text{ for } \delta_0 := \frac{4\|Q\|\sqrt{K_2}}{\lambda_0^{(3-\theta)/2}}.$$

Let

$$\beta_s = 1 + s(\alpha - 1), \quad \gamma_s = x + s(y - x), \quad s \in [0, 1].$$

We have

$$\begin{aligned}
&\frac{d}{ds} \log(P_t f^{\beta(s)})^{\alpha/\beta(s)}(\gamma_s) \\
&= \frac{\alpha(\alpha - 1)}{\beta(s)^2} \cdot \frac{P_t(f^{\beta(s)} \log f^{\beta(s)}) - (P_t f^{\beta(s)}) \log P_t f^{\beta(s)}}{P_t f^{\beta(s)}}(\gamma_s) + \frac{\alpha D_{y-x} P_t f^{\beta(s)}}{\beta(s) P_t f^{\beta(s)}}(\gamma_s) \\
&\geq \frac{\alpha \|x - y\|_{V_\theta}}{\beta(s) P_t f^{\beta(s)}(\gamma_s)} \left\{ \frac{\alpha - 1}{\beta(s) \|x - y\|_{V_\theta}} \left(P_t(f^{\beta(s)} \log f^{\beta(s)}) - (P_t f^{\beta(s)}) \log P_t f^{\beta(s)} \right)(\gamma_s) \right. \\
&\quad \left. - \|DP_t f^{\beta(s)}(\gamma_s)\|_{V_\theta}^* \right\}.
\end{aligned}$$

Therefore, applying (2) to

$$\delta := \frac{\alpha - 1}{\beta(s)\|x - y\|_{V_\theta}}$$

which is larger than δ_0 according to (3.11), we obtain

$$\begin{aligned} \frac{d}{ds} \log(P_t f^{\beta(s)})^{\alpha/\beta(s)}(\gamma_s) &\geq -\frac{2\alpha\|x - y\|_{V_\theta}}{\delta\beta(s)} \left\{ \frac{K_1}{t} + \frac{2K_2}{\lambda_0^{1-\theta}} (\|\gamma_s\|_H^2 + \|Q\|_{HSt}^2) \right\} \\ &\geq -\frac{2\alpha\|x - y\|_{V_\theta}^2}{\alpha - 1} \left\{ \frac{K_1}{t} + \frac{2K_2}{\lambda_0^{1-\theta}} (\|x\|_H^2 \vee \|y\|_H^2 + \|Q\|_{HSt}^2) \right\}. \end{aligned}$$

Integrating over $[0, 1]$ w.r.t. ds , we derive the desired Harnack inequality. \square

Proof of Corollary 1.3. Since $u \mapsto \|u\|_V^2$ is a compact function on H , i.e. for any $r > 0$ the set $\{u \in H : \|u\|_V \leq r\}$ is relatively compact in H , (3.1) implies the existence of the invariant probability measure satisfying (1) by a standard argument (see e.g. [15, Proof of Theorem 1.2]). Moreover, any invariant probability measure μ satisfies $\mu(\|\cdot\|_V^2) < \infty$, hence, $\mu(V) = 1$. Below, we assume $\theta \in (0, 1)$ and prove (1) and (2) respectively.

(1) Let μ be an invariant probability measure, we first prove it has full support on μ .

$$r_0 = \frac{\lambda_0^{(3-\theta)/2}}{8\|Q\|\sqrt{K_2}}.$$

By Corollary 1.2(3) for $\alpha = 2$, for any fixed $t > 0$ there exists a constant $C(t) > 0$ such that

$$(P_t f(x))^2 \leq (P_t f^2(y)) e^{C(t)(\|x\|_H^2 + \|y\|_H^2)}, \quad \|x - y\|_{V_\theta} \leq r_0.$$

Applying this inequality n times, we may find a constant $c(t, n) > 0$ such that

$$(3.12) \quad (P_t f(x))^{2n} \leq (P_t f^{2n}(y)) e^{C(t, n)(\|x\|_H^2 + \|y\|_H^2)}, \quad \|x - y\|_{V_\theta} \leq nr_0.$$

Since V is dense in V_θ , to prove that μ has full support on V_θ , it suffices to show that

$$(3.13) \quad \mu(B_\theta(x, \varepsilon)) > 0, \quad x \in V, \varepsilon > 0$$

holds for $B_\theta(x, \varepsilon) := \{y : \|y - x\|_{V_\theta} < \varepsilon\}$. Since $\mu(V_\theta) = 1$, there exists $n \geq 1$ such that $\mu(B_\theta(x, nr_0)) > 0$. Applying (3.12) to $f = 1_{B_\theta(x, \varepsilon)}$ we obtain

$$\mathbb{P}(\|X_t^x - x\|_{V_\theta} < \varepsilon)^{2n} \int_{B_\theta(x, nr_0)} e^{-C(t, n)(\|x\|_H^2 + \|y\|_H^2)} \mu(dy) \leq \mu(B_\theta(x, \varepsilon)).$$

So, if $\mu(B_\theta(x, \varepsilon)) = 0$ then

$$(3.14) \quad \mathbb{P}(\|X_t^x - x\|_{V_\theta} \geq \varepsilon) = 1, \quad t > 0.$$

To see that this is impossible, let us observe that for any $m \geq 1$ there exists a constant $c(m) > 0$ such that

$$(3.15) \quad \|\cdot\|_{V_\theta}^2 \leq c(m)\|\cdot\|_H^2 + \frac{1}{(\lambda_0 m)^{1-\theta}}\|\cdot\|_V^2$$

holds. Moreover, using $\langle \cdot, \cdot \rangle$ to denote the duality w.r.t H , we have

$$\begin{aligned} 2\langle X_t^x - x, LX_t^x \rangle &= 2\|X_t^x - x\|_V^2 + 2\langle X_t^x - x, Lx \rangle \\ &\geq 2\|X_t^x - x\|_V^2 - 2\|X_t^x - x\|_V\|x\|_V \geq \|X_t^x - x\|_V^2 - \|x\|_V^2 \end{aligned}$$

and due to **(A1)** and **(A2)**,

$$2\langle X_t^x - x, B(X_t^x) \rangle = -2\langle x, B(X_t^x) \rangle \leq 2C\|x\|_H\|X_t^x\|_V\|X_t^x\|_H \leq \frac{1}{2}\|X_t^x - x\|_V^2 + c_1 + c_2\|X_t^x\|_H^2$$

holds for some constants c_1, c_2 depending on x . Combining theses with the Itô formula for $\|X_t^x - x\|_H^2$, we arrive at

$$d\|X_t^x - x\|_H^2 \leq -\frac{1}{2}\|X_t^x - x\|_V^2 dt + (c_3 + c_2\|X_t^x\|_H^2)dt + 2\langle X_t^x - x, QdW_t \rangle$$

for some constant $c_3 > 0$. Since $\mathbb{E}\|X_t\|_H^2$ is bounded for $t \in [0, 1]$, this implies that

$$\mathbb{E} \int_0^t \|X_s^x - x\|_V^2 ds \leq c_0 t, \quad t \in [0, 1]$$

holds for some constant $c_0 > 0$. Combining this with (3.15) and noting that $t \mapsto X_t^x$ is continuous in H , we conclude that

$$\limsup_{t \rightarrow 0} \frac{1}{t} \int_0^t \mathbb{E}\|X_s^x - x\|_{V_\theta}^2 ds \leq \frac{c_0}{(\lambda_0 m)^{1-\theta}}, \quad m \geq 1.$$

Letting $m \rightarrow \infty$ we obtain

$$\lim_{t \rightarrow 0} \frac{1}{t} \int_0^t \mathbb{E}\|X_s^x - x\|_{V_\theta}^2 ds = 0.$$

this is contractive to (3.14).

Next, if the invariant probability measure is not unique, we may take two different extreme elements μ_1, μ_2 of the set of all invariant probability measures. It is well-known that μ_1 and μ_2 are singular with each other. Let D be a μ_1 -null set, since μ_1 has full support on V_θ and $P_t 1_D$ is continuous and $\mu_1(P_t 1_D) = \mu_1(D) = 0$, we have $P_t 1_D \equiv 0$.

Thus, $\mu_2(D) = \mu_2(P_t 1_D) = 0$. This means that μ_2 has to be absolutely continuous w.r.t. μ_1 , which is contradictive to the singularity of μ_1 and μ_2 .

(2) As observe above that $P_t 1_D \equiv 0$ for any μ -null set D . So, P_t has a transition density $p_t(x, y)$ w.r.t. μ on V_θ . Next, let $f \geq 0$ such that $\mu(f^\alpha) \leq 1$. By the Harnack inequality in Corollary 1.2(3), we have

$$(P_t f(x))^\alpha \int_{B_\theta(x, r_0)} \exp \left[-\frac{2\alpha \|x - y\|_{V_\theta}^2}{\alpha - 1} \left\{ \frac{K_1}{t} + \frac{2K_2}{\lambda_0^{1-\theta}} (\|x\|_H^2 \vee \|y\|_H^2 + \|Q\|_{HS}^2 t) \right\} \right] \mu(dy) \leq 1.$$

Then the desired estimate on $\int p_t(x, z)^{(\alpha+1)/\alpha} \mu(dz)$ follows by taking

$$f(\cdot) = p_t(x, \cdot).$$

Proof of Theorem 1.4. (1) Let M_t be in the proof of Corollary 1.2 (2). By (??), for $\delta > 0$ we have

$$\begin{aligned} \mathbb{E} \exp \left[\frac{M_t}{\delta} \right] &\leq \left(\mathbb{E} \exp \left[\frac{2\langle M \rangle_t}{\delta^2} \right] \right)^{1/2} \\ &\leq \exp \left[\frac{2K_1}{\delta^2 t} \right] \left(\exp \left[\frac{8K_2}{\lambda_0^{1-\theta} \delta^2} \int_0^t \|X_s^x\|_V^2 ds \right] \right)^{1/2}. \end{aligned}$$

If $t \leq t_\delta$ then

$$\frac{8K_2}{\lambda_0^{1-\theta} \delta^2} \leq \frac{2\lambda_0^2}{\|Q\|^2 e t},$$

so that by the Jensen inequality and the second inequality in Lemma 3.1,

$$\begin{aligned} \mathbb{E} \exp \left[\frac{M_t}{\delta} \right] &\leq \exp \left[\frac{2K_1}{\delta^2 t} \right] \left(\exp \left[\frac{2\lambda_0^2}{\|Q\|^2 e t} \int_0^t \|X_s^x\|_V^2 ds \right] \right)^{\frac{2K_2 \|Q\|^2 e t}{\delta^2 \lambda_0^{3-\theta}}} \\ &\leq \exp \left[\frac{2K_1}{\delta^2 t} + \frac{4K_2 e}{\delta^2 \lambda_0^{1-\theta}} \right], \quad t \leq t_\delta. \end{aligned}$$

Combining this with (3.10) we prove the desired gradient estimate for $t \leq t_\delta$. By the gradient estimate for $t = t_\delta$ and the semigroup property, when $t > t_\delta$ we have

$$\begin{aligned} \|DP_t f(x)\|_{V_\theta^*} &= \|DP_{t_\delta}(P_{t-t_\delta} f)(x)\|_{V_\theta^*} \leq \delta \{ P_{t_\delta}((P_{t-t_\delta} f) \log P_{t-t_\delta} f) \\ &\quad - (P_t f) \log P_t f \}(x) + \frac{2}{\delta} \left\{ \frac{K_1}{t_\delta} + \frac{2K_2 e}{\lambda_0^{1-\theta}} (\|x\|_H^2 + \|Q\|_{HS}^2 t) \right\} P_t f(x). \end{aligned}$$

This implies the desired gradient estimate for $t > t_\delta$ since due to the Jensen inequality

$$P_{t_\delta}((P_{t-t_\delta} f) \log P_{t-t_\delta} f) \leq P_t f \log f.$$

(2) Repeating the proof of Corollary 1.3 (3) using the inequality in Theorem 1.4 (1) instead of Corollary 1.2 (2) for $\delta = \frac{\alpha-1}{\beta(s)\|x-y\|_{V_\theta}}$, we obtain

$$\frac{d}{ds} (\log P_t f^{\beta(s)})^{\alpha/\beta(s)} \geq -\frac{2\alpha\|x-y\|_{V_\theta}^2}{\alpha-1} \left\{ \frac{K_1}{t \wedge t_\delta} + \frac{2K_2 e}{\lambda_0^{1-\theta}} (\|x\|_H^2 \vee \|y\|_H^2 + \|Q\|_{HS}^2 t) \right\}.$$

This completes the proof by integrating over $[0, 1]$ w.r.t. ds and noting that

$$t_\delta = \frac{\delta^2 \lambda_0^{3-\theta}}{4\|Q\|^2 e K_2} \geq \frac{(\alpha-1)^2 \lambda_0^{3-\theta}}{4\alpha^2 \|Q\|^2 K_2 e \|x-y\|_{V_\theta}^2}$$

since

$$\delta = \frac{\alpha-1}{\beta(s)\|x-y\|_{V_\theta}} \geq \frac{\alpha-1}{\alpha\|x-y\|_{V_\theta}}.$$

□

4 Appendix

We aim to verify the existence and uniqueness of the solution to (1.1) by using the main result of [8].

Theorem 4.1. *Assume (A1) and (A2). For any $X_0 \in H$ the equation (1.1) has a unique solution X_t , which is a continuous Markov process on H such that*

$$\mathbb{E} \left(\sup_{t \in [0, T]} \|X_t\|_H^p + \int_0^T \|X_t\|_V^2 dt \right) < \infty$$

holds for any $p > 1$ and \mathbb{P} -a.s.

$$X_t = X_0 - \int_0^t (LX_s + B(X_s)) ds + QW_t, \quad t \geq 0$$

holds on H .

Proof. Let V^* be the dual space of V w.r.t. H . Then for any $v \in V$,

$$A(v) := -(Lv + B(v)) \in V^*.$$

It suffices to verify assumptions (H1)-(H4) in [8, Theorem 1.1] for the functional A . The hemicontinuity assumption H1) follows immediately from the bilinear property of B . Next, by (A2) and the bilinear property of B , we have

$$\begin{aligned} V^* \langle A(v_1) - A(v_2), v_1 - v_2 \rangle_V &= -\|v_1 - v_2\|_V^2 + \|B(v_2 - v_1, v_1) - B(v_2 - v_1, v_2)\|_V \\ &\leq -\|v_1 - v_2\|_V^2 + C\|v_1 - v_2\|_H^2 \|v_1\|_V^2. \end{aligned}$$

So, the assumption (H2) in [8] holds for $\rho(v) := c\|v\|_V^2$. Moreover, by **(A1)** we have

$$v^* \langle A(v), v \rangle_V \leq -\|v\|_V^2.$$

Thus, the coercivity assumption (H3) in [8] holds for $\theta = 1, \alpha = 2, K = 0$ and $f = \text{constant}$. Finally, **(A2)** implies that

$$\|A(v)\|_{V^*}^2 \leq 2\|v\|_V^2 + 2\|L^{-1/2}B(v)\|_H^2 \leq 2\|v\|_V^2 + \frac{2c}{\lambda_0}\|v\|_H^2\|v\|_V^2.$$

Therefore, the growth condition (H4) in [8] holds for some constant $f, K > 0$ and $\alpha = \beta = 2$. \square

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