Solution to Problem 65-7*: Solution to an integral equation

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Problem 65-6, *A Ratio of Two Definite Multiple Integrals*,* by A. J. STRECOK (Argonne National Laboratory).

Determine

$$I(-1, x)/I(1, x),$$

where

$$I(s, x) = \int_0^x \int_0^t t(x^2 - t^2)^{-1/2} e^{s(t^2-u^2)}\, du\, dt$$

for nonzero s and x.

Solution by P. J. SHORT (White Sands Missile Range).

The solution here can be obtained from the solution to *Problem* 64-8 (October, 1965). Denoting the integral in *Problem* 64-8 by $I^*(x)$, it follows by making the substitutions $v = u\sqrt{s}$ and $w = t\sqrt{s}$ that

$$I(s, x) = \frac{\sqrt{\pi}}{2s} I^*(x\sqrt{s}) = \frac{\pi}{4s}\{e^{sx^2} - 1\}.$$

Whence,

$$\frac{I(-1, x)}{I(1, x)} = e^{-x^2}.$$

Also solved by R. D. ADAMS (University of Kansas), D. BEIN (Fairleigh Dickinson University), C. J. BOUWKAMP (Technological University, Eindhoven, Netherlands), M. P. FRIEDMAN (Smithsonian Institute), D. L. LANSING (NASA—Langley Research Center), C. B. A. PECK (two solutions) (Pennsylvania State College), H. B. ROSENSTOCK (U. S. Naval Research Laboratory), L. RUBENFELD (Courant Institute of Mathematical Sciences), P. A. SCHEINOK (Hahnemann Medical College), C. B. SHAW, JR. (Electro-Optical Systems, Inc.), S. SPITAL (California State Polytechnic College), F. W. STEUTEL (Technische Hogeschool Twente, Enschede, Netherlands) and the proposer.

Problem 65-7*, *Solution to an Integral Equation*,† by A. J. STRECOK (Argonne National Laboratory).

Show that a solution of the equation

$$(1) \qquad 1 = e^{-x^2} + \int_0^x 2te^{-t^2}\psi(x, t)\, dt$$

is given by

$$(2) \qquad \psi(x, t) = \frac{2}{\pi}(x^2 - t^2)^{-1/2} \int_0^t e^{u^2}\, du,$$

and give conditions on $\psi(x, t)$ which make this a unique solution.

* Work performed under the auspices of the United States Atomic Energy Commission.

† Work performed under the auspices of the United States Atomic Energy Commission.

Solution by C. J. Bouwkamp (Technological University, Eindhoven, Netherlands).

The first part of the problem is simple in view of the solution given of *Problem* 65-6 if applied to $s = -1$, viz.,

$$\int_0^x 2te^{-t^2}\psi(x, t)\, dt = \frac{4}{\pi} I(-1, x) = 1 - e^{-x^2}, \qquad x > 0.$$

The second part of the problem is not well posed. Equation (1) is not a proper integral equation. The functional equation (1) has an abundance of solutions, and conditions on $\psi(x, t)$ that will make the particular solution (2) the only solution of (1) seem to be quite artificial. This is substantiated by noting it is easy to construct solutions of the types $f(t)g(x)$ and $f(t) + g(x)$, where $f(t)$ is wholly arbitrary. Thus,

$$\psi_1(x, t) = f(t)(1 - e^{-x^2}) \left(\int_0^x 2te^{-t^2}f(t)\, dt\right)^{-1}$$

and

$$\psi_2(x, t) = 1 + f(t) - (1 - e^{-x^2})^{-1} \int_0^x 2te^{-t^2}f(t)\, dt$$

are solutions of (1) for any $f(t)$ making the right-hand side meaningful. Apparently, the proposer is interested in solutions that are singular at $x = t$. However, there is a whole class of such solutions. This is shown as follows.

Assume that (1) is meant to hold for $x > 0$, and that $\psi(x, t)$ is defined and continuous for $0 \leqq t < x$. It is somewhat easier to discuss the problem in transformed variables. To that end, set

$$x^2 = \xi, \quad t^2 = \tau, \quad \chi(\xi, \tau) = \psi(x, t).$$

Then (1) becomes

$$1 = e^{-\xi} + \int_0^\xi e^{-\tau}\chi(\xi, \tau)\, d\tau, \tag{3}$$

and the function (2) corresponds to

$$\chi(\xi, \tau) = \frac{1}{\pi} (\xi - \tau)^{-1/2} \int_0^\tau e^u u^{-1/2}\, du, \tag{4}$$

this being a particular slution of (3). In fact, by invoking the theory of Abel's integral equation, this can be proved independently of the solution to *Problem* 65-6.

Now assume that

$$\chi(\xi, \tau) = (\xi - \tau)^{-\alpha} e^\tau g(\tau),$$

where $g(t)$ is an unknown function and $0 < \alpha < 1$. Then (4) transforms into the generalized Abel equation

$$\int_0^\xi \frac{g(\tau)}{(\xi - \tau)^\alpha}\, d\tau = 1 - e^{-\xi},$$

which is solved by

$$g(\tau) = \frac{e^{-\tau} \sin (\alpha\pi)}{\pi} \int_0^\tau e^u u^{\alpha-1} \, du.$$

That is to say, in terms of the original variables, the function

$$\psi_3(x, t) = \frac{2 \sin (\alpha\pi)}{\pi(x^2 - t^2)^\alpha} \int_0^t e^{u^2} u^{2\alpha-1} \, du \tag{5}$$

is a particular solution of (1) for any α with $0 < \alpha < 1$. The proposer's solution (2) is obtained for $\alpha = 1/2$.

Of course, there exist still other solutions of (1). Thus, if the proposer writes his unknown function $\psi(x, t)$ as a product of the kernel function $(x^2 - t^2)^{-1/2}$ and an unknown function $h(t)$, the solution indicated for $h(t)$ is unique, by invoking the existing theory of Abel's integral equation. But then the kernel function should have been prescribed.

Also solved by R. D. ADAMS (University of Kansas), M. P. FRIEDMAN (Smithsonian Institution), F. W. STEUTEL (Technische Hogeschool Twente, Enschede, Netherlands) and P. P. WANG (Bell Telephone Laboratories).