Polynomial matrices and feedback

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EINDHOVEN UNIVERSITY OF TECHNOLOGY

Department of Mathematics and Computing Science

Memorandum COSOR 84 - 01

Polynomial Matrices and Feedback

by

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January, 1984

Introduction

In this paper we describe the use of feedback with respect tot some poly-
nomial matrix constructions.

Consider the following problem

(1)   Given a polynomial matrix $P(\lambda) \in \mathbb{R}[\lambda]^{p \times q}$ (the set of $p \times q$-matrices

with entries in $\mathbb{R}[\lambda]$ (the set of real polynomials in λ)) such that

$q > p$ and that $P(\ )$ has full rank for all $\lambda \in \mathbb{C}$ (the set of complex

numbers).

Construct a matrix $Q(\lambda) \in \mathbb{R}[\lambda]^{(q-p) \times q}$ such that

(2)
$$\begin{bmatrix} P(\lambda) \\ Q(\lambda) \end{bmatrix}$$

is unimodular.

Of course this problem has a well-known solution. Algorithms providing us
with $Q(\lambda)$ are mostly based on elementary row (column) operations, reducing
$P(\lambda)$ to some simple form. Having obtained such a simple form (for instance
lower triangular, Hermite form(like), Smith form(like)) the construction
of $Q(\lambda)$ is straightforward.

Our construction is not based on elementary row (column) operations. We
work on real matrices directly.

One of the main problems concerning the methods based on elementary opera-
tions, which, in turn, are based on the Euclidean algorithm, is their bad
numerical behaviour.

Our method is based on a numerically reliable method for the construction of a feedback matrix solving the deadbeat control problem for a generalized state space system $Ex_{k+1} = Ax_k + Bu_k$. This method is closely related to [2]. The method we use, also gives the inverse of (2) in a straightforward way.

A number of applications of $Q(\lambda)$ in (1), (2) can be found in [1].

Preliminaries

In this section we describe the generalized deadbeat control problem. The solution of this problem will be needed for the construction of $Q(\lambda)$ in (1), (2).

This problem is the following

(3) Given (E,A,B) , $E \in \mathbb{R}^{n \times n}$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$.

Construct a matrix $F \in \mathbb{R}^{m \times n}$ such that all generalized eigenvalues of the pencil $[\lambda E - (A + BF)]$ are zero.

Observe that for the solvability of (3) we must have that E is regular because generalized eigenvalues at infinity are not allowed.

This problem is called the generalized deadbeat control problem because it is a deadbeat control problem for the generalized state space system

$$Ex_{k+1} = Ax_k + Bu_k \; ; \; x_k \in \mathbb{R}^n \; , \; u_k \in \mathbb{R}^m \; , \; k = 0,1,2,\ldots \; .$$

Problem (3) is equivalent to the usual deadbeat control problem for a system $(E^{-1}A, E^{-1}B)$. However, we will consider (3) because of the possible ill conditioning of E with respect to inversion.

An equivalent statement for all generalized eigenvalues of $[\lambda E - (A+BF)]$ being zero is

$$[E - \lambda(A + BF)] \text{ is a unimodular matrix .}$$

This can be seen as follows.

If $[E - \lambda(A+BF)]$ is unimodular we must have that E is regular. Therefore $[\lambda E - (A+BF)]$ does not have generalized eigenvalues at infinity. Now suppose that $[\lambda E - (A+BF)]$ has a nonzero eigenvalue λ_1. Then $\lambda_1 Ex - (A+BF)x = 0$ for some nonzero $x \in \mathbb{C}^n$. Therefore $Ex - \frac{1}{\lambda}(A+BF)x = 0$, contradicting the unimodularity of $[E - \lambda(A+BF)]$. The other implication can be proved similarly.

Next we consider the solvability of problem (3).

It is well-known that there exists a matrix F such that $[E^{-1}A + E^{-1}BF]$ is nilpotent if and only if the noncontrollable eigenvalues of $E^{-1}A$ (for the system $(E^{-1}A, E^{-1}B)$ are zero. Therefore (3) is solvable of and only if all generalized eigenvalues λ_i of $[\lambda E - A]$, such that $[\lambda_i E - A, B]$ does not have full rank, are zero.

An equivalent condition for this is

$$(4) \qquad [E - \lambda A, \lambda B]$$

is right invertible for $\lambda \in \mathbb{C}$ (or, equivalently, right invertible over $\mathbb{R}[\lambda]$) . This can easily be seen (remember that E is regular).

We will need a solution for (1) for the case $P(\lambda) = [E - \lambda A, \lambda B]$. Therefore, let F be such that $[E^{-1}A + E^{-1}BF]$ is nilpotent. Such an F exists by the pole placement theorem because all nonzero eigenvalues of $E^{-1}A$ are controllable for the system $(E^{-1}A, E^{-1}B)$.

Now we have that

$$\begin{bmatrix} E - \lambda A & \lambda B \\ F & I \end{bmatrix}$$

is unimodular because

(5)
$$\begin{bmatrix} E - \lambda A & \lambda B \\ F & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -F & I \end{bmatrix} = \begin{bmatrix} E - \lambda(A+BF) & \lambda B \\ 0 & I \end{bmatrix} .$$

Here $[E - \lambda(A+BF)]$ is unimodular because $[I - (E^{-1}A + E^{-1}BF)]$ is unimodular. An explicit formula for $[E - (A+BF)]^{-1}$ is

$$[E - \lambda(A + BF)]^{-1} = [I + \lambda(E^{-1}A + E^{-1}BF) + \ldots + \lambda^{n-1}(E^{-1} + E^{-1}BF)^{n-1}]E^{-1} .$$

Therefore a solution $Q(\lambda)$ to problem (1) for $P(\lambda) = [E - \lambda A, \lambda B]$ is

$$Q(\lambda) = [F \quad I] .$$

In the next section we will show that problem (1) for general $P(\lambda)$, may be solved by solving (1) for a pencil $[E - \lambda A, \lambda B]$ where (E, A, B) is derived from $P(\lambda)$ in a straightforward way.

Results

Let $P(\lambda) \in \mathbb{R}[\lambda]^{p \times q}$ be such that $P(\lambda)$ has rank p for all $\lambda \in \mathbb{C}$ $(q > p)$. $P(\lambda)$ can be written as

$$P(\lambda) = P_0 + P_1\lambda + \ldots + P_n\lambda^n ,$$

where $P_i \in \mathbb{R}^{p \times q}$, $i = 0, \ldots, n$.

Because $P(0)$ has rank p we have that P_0 has rank p.
We may assume that

(6) $\qquad P_0 = [P_e, 0]$

where P_e is a regular $p \times p$-matrix because we can solve problem (1) for $P(\lambda)$
if we can solve (1) for $P(\lambda)U$ where U is a regular matrix. Of course U may
be taken to be unitary in order to obtain this particular form for P_0.

Next we partition $P(\lambda)$ as

$$P(\lambda) = [P_e, 0] + [P_{a1}, P_{b1}]\lambda + \ldots + [P_{an}, P_{bn}]\lambda^n$$

where

$$P_{ai} \in \mathbb{R}^{p \times p}, \ P_{bi} \in \mathbb{R}^{p \times (q-p)} \quad \text{for } i = i, \ldots, n .$$

Consider matrices E, A, B defined by

(7) $\qquad E = \begin{bmatrix} I & & & \\ & \cdot & & \\ & & \cdot & \\ & & & I \\ & & & & P_e \end{bmatrix}, A = \begin{bmatrix} 0 & & & & -P_{an} \\ I & \cdot & & & \cdot \\ & \cdot & \cdot & & \cdot \\ & & & 0 & \cdot \\ & & & I & -P_{a1} \end{bmatrix}, B = \begin{bmatrix} P_{bn} \\ \cdot \\ \cdot \\ \cdot \\ P_{b1} \end{bmatrix}$

It can easily be seen that $[E - \lambda A, \lambda B]$ has full rank for all $\lambda \in \mathbb{C}$.
Let $F \in \mathbb{R}^{(q-p) \times np}$ be a feedback matrix such that

$$\begin{bmatrix} E - \lambda A & B \\ F & I \end{bmatrix}$$

is unimodular. (See previous section for the existence of F)

Next we observe that

$$
\begin{bmatrix}
I & 0 & . & . & . & . & 0 \\
\lambda I & I & . & & & & . \\
. & \lambda I & . & . & & & . \\
. & & . & . & . & & . \\
. & & & . & . & . & . \\
\lambda^{n-1}I & . & & . & \lambda I & I & 0 \\
0 & . & . & . & . & 0 & I
\end{bmatrix}
\begin{bmatrix}
0 & 0 & . & . & 0 & \lambda P_{an} & \vdots & \lambda P_{bn} \\
-\lambda I & . & . & & & . & \vdots & . \\
0 & . & . & & & . & \vdots & . \\
. & . & . & & 0 & & \vdots & . \\
. & & .. & & I & \lambda P_{a2} & \vdots & . \\
0 & . & . & 0 & -\lambda I & P_e+\lambda P_{a1} & \vdots & \lambda P_{b1} \\
\hline
 & & & F & & & \vdots & I
\end{bmatrix}
=
\begin{bmatrix}
I & 0 & . & . & 0 & \vdots & X_n(\lambda) & \vdots & Y_n(\lambda) \\
0 & . & . & & . & \vdots & . & \vdots & . \\
. & . & . & . & . & \vdots & . & \vdots & . \\
. & & . & . & 0 & \vdots & . & \vdots & . \\
0 & . & . & 0 & I & \vdots & . & \vdots & . \\
0 & . & . & . & 0 & \vdots & X_1(\lambda) & \vdots & Y_1(\lambda) \\
\hline
 & & F_1 & & & \vdots & F_2 & \vdots & I
\end{bmatrix}
= \overline{P}(\lambda) .
$$

Here $X_i(\lambda) = P_{ai} + P_{a,i+1}\lambda^2 + \ldots + P_{a,n}\lambda^{n-i+1}$ for $1 = 2,\ldots,n$.

The matrices $Y_i(\lambda)$ are defined analogously for $i = 2,\ldots,n$.

Futhermore

$$[X_1(\lambda),Y_1(\lambda)] = P(\lambda) .$$

Observe that $\overline{P}(\lambda)$ is unimodular (because both factors in the left hand side are unimodular).

Let $U = [U_1,U_2]$ be a unitary matrix such that

$$
[U_1,U_2]
\begin{bmatrix}
I & & & \\
 & . & & \\
 & & . & \\
 & & & I \\
\hline
 & F_1 & &
\end{bmatrix}
=
\begin{bmatrix}
 & & \\
 & R & \\
 & & \\
\hline
 & 0 &
\end{bmatrix}
$$

where R is regular $\left(U_1 \in \mathbb{R}^{((n-1)p+q-p)\times(n-1)p}, U_2 \in \mathbb{R}^{((n-1)p+(q-p)\times(q-p)} \right)$.

Then

$$(8) \qquad \begin{bmatrix} U_1 & 0 & U_2 \\ 0 & I & 0 \end{bmatrix} \overline{P}(\lambda) = \begin{bmatrix} R & \vline & \star \\ \hline 0 & \vline & P(\lambda) \\ \hline 0 & \vline & Q(\lambda) \end{bmatrix}$$

for some polynomial matrix $Q(\lambda)$.

The matrix

$$\begin{bmatrix} P(\lambda) \\ Q(\lambda) \end{bmatrix}$$

is unimodular because $\overline{P}(\lambda)$ is unimodular, R is regular and U is unitary.

This shows that $Q(\lambda)$ is a solution to problem (1).

Observe that the degree of $Q(\lambda)$ is less than the degree of $P(\lambda)$

The inverse of (2) (and also a right inverse of $P(\lambda)$) can easily be constructed using the polynomial matrix

$$\begin{bmatrix} E - \lambda A & \lambda B \\ F & I \end{bmatrix}^{-1} .$$

Up to now we have used two unitary tranformations (6), (8) in order to obtain $Q(\lambda)$ where we assumed that F can be computed. If we show that a feedback matrix for the generalized deadbeat control problem can be constructed in a numerically reliable way we will have obtained a reliable method for the construction of a matrix $Q(\lambda)$ solving problem (1).

Algorithms

In this section we present a collection of numerically reliable algorithms for the construction of a feedback matrix F such that $[E - \lambda(A+BF)]$ is unimodular (F is a solution to problem (3)).

Let (E,A,B) be a matrix triple such that $[E - \lambda A, \lambda B]$ is right invertible over $\mathbb{R}[\lambda]$. Here $E \in \mathbb{R}^{n \times n}$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$.

Algorithm

$$i := 1; \quad n_i := n; \quad s_i := 1; \quad E_i := E; \quad A_i := A; \quad B_i := B;$$

while $n_i > 0$ and $s_i > 0$ and $B_i \neq 0$ do

begin

Step 1: Using a minor modification of the singular value decomposition we

have (some of the zero-matrices may be empty)

$$B_i = U_i \begin{bmatrix} 0 & 0 & 0 \\ 0 & D_{ib} & 0 \\ 0 & 0 & D_{ig} \end{bmatrix} V_i^T$$

where U_i, V_i are unitary matrices and D_{ib}, D_{ig} are diagonal matrices

together containing the singular values of B_i. D_{ib} contains only "bad"

singular values (think of "bad" as "too small") and D_{ig} contains g_i

"good" singular values of B_i (think of "good" as "large enough").

We will assume that "good" implies positive.

If $g_i = 0$ then

begin

Test whether all generalized eigenvalues of $\lambda E_i - A_i$ are zero.

(this can be done by means of the QZ algorithm) If not all gene-

ralized eigenvalues of $\lambda E_i - A_i$ are zero we have to define "good" and

"bad" differently in order to obtain at least one "good" singular

value. $(g_i > 0)$.

end

Step 2: If $g_i > 0$ the following partioned matrices are computed

$$\left(\begin{bmatrix} E_{ia} & E_{ib} \\ E_{if} & E_{ig} \end{bmatrix}, \begin{bmatrix} A_{ia} & A_{ib} \\ A_{if} & A_{ig} \end{bmatrix}, \begin{bmatrix} B_{ib} \\ B_{ig} \end{bmatrix} \right) := \left(U_i^T E_i, U_i^T A_i, U_i^T B_i \right).$$

Here

$$B_{ib} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & D_{ib} & 0 \end{bmatrix} V_i^T, \quad B_{ig} = \begin{bmatrix} 0 & 0 & D_{ig} \end{bmatrix} V_i^T$$

$$E_{ig} \in \mathbb{R}^{g_i \times g_i}, \quad A_{ig} \in \mathbb{R}^{g_i \times g_i}.$$

The dimensions of the other matrices involved are chosen accordingly.

Step 3: If $g_i > 0$ a unitary matrix W_i is computed such that

$$\begin{bmatrix} A_{ia} & A_{ib} \\ A_{if} & A_{ig} \end{bmatrix} W_i = \begin{bmatrix} \overline{A}_{ia} & 0 \\ \overline{A}_{if} & \overline{A}_{ig} \end{bmatrix}, \quad \overline{A}_{ig} \in \mathbb{R}^{g_i \times g_i}.$$

$$F_i := [0 , F_{ig}] := V_i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -D_{ig}^{-1} \overline{A}_{ig} \end{bmatrix}.$$

Now it is clear that

$$\begin{bmatrix} \overline{A}_{ia} & 0 \\ \overline{A}_{if} & \overline{A}_{ig} \end{bmatrix} + \begin{bmatrix} B_{ib} \\ B_{ig} \end{bmatrix} F_i = \begin{bmatrix} \overline{A}_{ia} & 0 \\ \overline{A}_{if} & 0 \end{bmatrix}.$$

Step 4: If $g_i > 0$ a unitary matrix Y_i is computed such that

$$Y_i^T \begin{bmatrix} E_{ia} & E_{ib} \\ E_{if} & E_{ig} \end{bmatrix} W_i = \begin{bmatrix} \overline{E}_{ia} & 0 \\ \overline{E}_{if} & \overline{E}_{ig} \end{bmatrix} \quad , \quad \overline{E}_{ig} \in \mathbb{R}^{g_i \times g_i} \quad .$$

Then we have

$$\left(Y_i^T U_i^T E_i W_i \; , \; Y_i^T U_i^T A_i W_i + Y_i^T U_i^T B F_i \; , \; Y_i^T U_i^T B \right) =$$

$$\left(\begin{bmatrix} E_{i+1} & 0 \\ \star & \overline{E}_{ig} \end{bmatrix} \; , \; \begin{bmatrix} A_{i+1} & 0 \\ \star & 0 \end{bmatrix} \; , \; \begin{bmatrix} B_{i+1} \\ \star \end{bmatrix} \right)$$

for some

$$E_{i+1} \in \mathbb{R}^{(n_i - g_i) \times (n_i - g_i)} \; , \; A_{i+1} \in \mathbb{R}^{(n_i - g_i) \times (n_i - g_i)} \; , \; B_{i+1} \in \mathbb{R}^{(n_i - g_i) \times m} \quad .$$

Step 5: $n_{i+1} := n_i - g_i; \; s_{i+1} := g_i; \; i := i + 1$.

end (of while-loop)

Observe that

- $(E_{i+1}, A_{i+1}, B_{i+1})$ does not depend on F_i

- $(E_{i+1}, A_{i+1}, B_{i+1})$ satisfies the solvability condition for (1) with

$$(E, A, B) = (E_{i+1}, A_{i+1}, B_{i+1}) \quad .$$

for each cycle i of this part of the algorithm.

It will be clear that termination of this while-loop will be obtained after

n cycles at most.

We have termination because $n_i = 0$ or $s_i = 0$ or $B_i = 0$ for some i.

If $n_{i_0+1} = 0$ we have obtained (after i_0 cycles)

$$n_1, \ldots, n_{i_0}; \ g_1, \ldots, g_{i_0}; \ W_1, \ldots, W_{i_0}; \ U_1, \ldots, U_{i_0}; \ Y_1, \ldots, Y_{i_0}; \ F_{1g}, \ldots, F_{i_0 g}$$

(W_{i_0} and Y_{i_0} may be taken to be identity matrices).

If $s_{i_1+1} = 0$ we have obtained (after i_1 cycles)

$$n_1, \ldots, n_{i_1}; \ g_1, \ldots, g_{i_1-1}; \ W_1, \ldots, W_{i_1-1}; \ U_1, \ldots, U_{i_1-1}; \ Y_1, \ldots, Y_{i_1-1};$$

$$F_{1g}, \ldots, F_{i_1,g} \ , \ E_{i_1} \in \mathbb{R}^{n_{i_1} \times n_{i_1}}, \ A_{i_1} \in \mathbb{R}^{n_{i_1} \times n_{i_1}}, \ B_{i_1} \in \mathbb{R}^{n_{i} \times m}$$

such that $\lambda E_{i_1} - A_{i_1}$ has generalized eigenvalues equal to zero.

If $B_{i_2} = 0$ we have obtained (after $i_2 - 1$ cycles)

$$n_1, \ldots, n_{i_2}; \ g_1, \ldots, g_{i_2-1}; \ W_1, \ldots, W_{i_2-1}; \ U_1, \ldots, U_{i_2-1}; \ Y_1, \ldots, Y_{i_2-1};$$

$$F_{1g}, \ldots, F_{i_2-1,g} \ , \ E_{i_2} \in \mathbb{R}^{n_{i_2} \times n_{i_2}}, \ A_{i_2} \in \mathbb{R}^{n_{i_2} \times n_{i_2}}$$

such that $\lambda E_{i_2} - A_{i_2}$ has generalized eigenvalues equal to zero because
(3) is solvable.

Next we compute unitary matrices X_ℓ, $Z_\ell \in \mathbb{R}^{n \times n}$

$$X_\ell := U_1 Y_1 \begin{bmatrix} U_2 & 0 \\ 0 & I_{m_2} \end{bmatrix} \begin{bmatrix} Y_2 & 0 \\ 0 & I_{m_2} \end{bmatrix} \cdots \begin{bmatrix} U_\ell & 0 \\ 0 & I_{m_\ell} \end{bmatrix} \begin{bmatrix} Y_\ell & 0 \\ 0 & I_{m_\ell} \end{bmatrix}$$

$$Z_\ell := W_1 \begin{bmatrix} W_2 & 0 \\ 0 & I_{m_2} \end{bmatrix} \cdots \begin{bmatrix} W_\ell & 0 \\ 0 & I_{m_\ell} \end{bmatrix}$$

for $\ell = i_0$, i_1-1, i_2-1 if $n_{i_0+1} = 0$, $s_{i_1+1} = 0$, $B_{i_2} = 0$ respectively.

Here I_{m_i} is the $m_i \times m_i$ identity matrix; $m_i = n - n_i$, $i = 2, \ldots, \ell$.

The matrix $F_g \in \mathbb{R}^{m \times n}$ is formed as follows

$$F_g := [\; 0 \quad F_{\ell g}, \ldots, F_{1g} \;]$$

where the zero matrix is empty if $n_{i_0+1} = 0$, for some i_0. Otherwise this

matrix is a $m \times n_{i_1}$ or $m \times n_{i_2}$ zero matrix depending on the termination condition

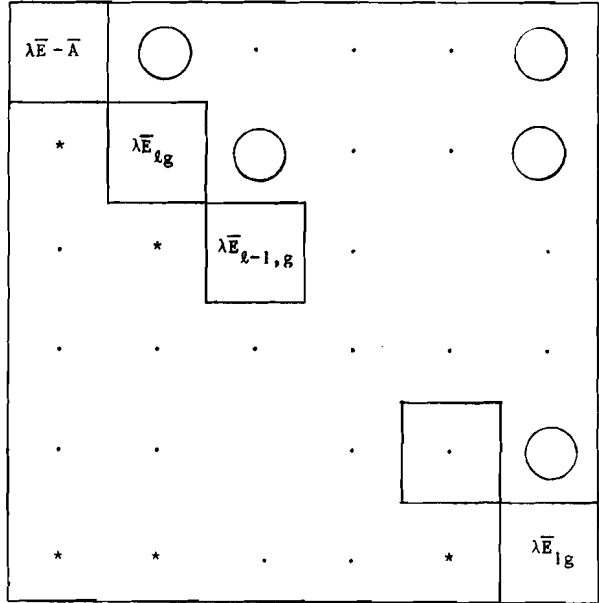$s_{i_1+1} = 0$ or $B_{i_2} = 0$ respectively.

In the final step of the algorithm we compute $F \in \mathbb{R}^{m \times n}$

$$F := F_g Z_\ell^T$$

end of the algorithm.

In order to prove that the matrix F is a solution to problem (3) we observe

that

$$(2) \quad \lambda x_\ell^T E z_\ell - \left(x_\ell^T A z_\ell + x_\ell^T B F \right) =$$

(matrix figure with blocks labeled $\lambda\bar{E} - \bar{A}$, $\lambda\bar{E}_{\ell g}$, $\lambda\bar{E}_{\ell-1,g}$, $\lambda\bar{E}_{1g}$ along the diagonal)

Here $\lambda\bar{E} - \bar{A}$ = "empty", $\lambda E_{i_1} - A_{i_1}$, $\lambda E_{i_2} - A_{i_2}$ for the termination condition $n_{i_0+1} = 0$, $s_{i_1+1} = 0$, $B_{i_2} = 0$ respectively.

This shows that $[\lambda E - (A+BF)]$ only has generalized eigenvalues equal to zero.

Therefore $[E - \lambda(A+BF)]$ is unimodular.

Discussion of the algorithms

The algorithm described in the previous section in fact represents a collection of algorithms because any selection policy between "good" and "bad" with respect to singular values, generally results in a different algorithm.

Concerning termination of the algorithm we observe the following.

If the matrix triple (E,A,B), where E is regular, represents a controllable generalized system (this means that $(E^{-1}A, E^{-1}B)$ is controllable) then the condition $B_i = 0$ will not terminate the while loop because in each cycle we

have that (E_i, A_i, B_i) is controllable.

Controllability of (E,A,B) is not really a restriction because any generalized system (E,A,B) may be transformed (using unitary transformations) into a generalized system of the form

$$(10) \qquad \left(\begin{bmatrix} E_b & 0 \\ * & E_g \end{bmatrix}, \begin{bmatrix} A_b & 0 \\ * & A_g \end{bmatrix}, \begin{bmatrix} 0 \\ B_g \end{bmatrix} \right)$$

where (E_g, A_g, B_g) is controllable (E_g is regular because E is regular). Observe that solvability of (3) for (E,A,B) menas that (3) is solvable for (E_g, A_g, B_g) and that $[E_b - \lambda A_b]$ is unimodular.

If the matrix triple (E,A,B), where E is regular, represents a controllable generalized system such that all generalized eigenvalues are non zero ($E^{-1}A$ is regular) then the condition $n_i = 0$ for some i will terminate the while loop.

We may restrict our algorithm to this case because a controllable generalized system (E,A,B) having some zero generalized eigenvalues can be transformed into the following form (using the QZ algorithm)

$$(11) \qquad \left(\begin{bmatrix} E_z & * \\ 0 & E_n \end{bmatrix}, \begin{bmatrix} A_z & * \\ 0 & A_n \end{bmatrix}, \begin{bmatrix} B_z \\ B_n \end{bmatrix} \right)$$

where the generalized eigenvalues of $\lambda E_z - A_z$ all are zero and $\lambda E_n - A_n$ only has non zero generalized eigenvalues. Furthermore it will be clear that (E_n, A_n, B_n) is controllable.

If we apply our algorithm to (E_n, A_n, B_n) we will obtain a matrix F_n such that $[0, F_n]$ solves (3) for the controllable system (11). A solution to problem (3) for the system (10) can be obtained straightforwardly in this case.

We have chosen to describe the algorithm for the general case of a system (E, A, B), where E is regular such that (3) is solvable, and not only for a controllable system or even a controllable system having only non zero generalized eigenvalues because we can deal with a larger class of cases in this way.

If we take the special version of the algorithm:

"good" = "non zero"; "bad" = "zero"

we obtain a feedback matrix F such that $(E^{-1}A + E^{-1}BF)^k = 0$ where k satisfies

$$k = \min_F \left\{ \ell_F \mid (E^{-1}A + E^{-1}BF)^{\ell_F} = 0 \right\} .$$

Furthermore F has minimum Frobenius norm (see [3]).

With respect to the numerical properties of the algorithm we observe that the construction of the matrix F_g may be postponed (it is not necessary to calculate F_{ig} within the while-loop) until the matrices X_ℓ and Z_ℓ have been computed. This can be seen as follows.
In (9) it can be seen that

$$X_\ell^T A Z_\ell + X_\ell^T B F_g = \begin{bmatrix} \boxed{\bar{A}} & & & & \bigcirc \\ & \boxed{0} & & & \\ & & \boxed{0} & & \\ & & & \boxed{0} & \\ \star & & & & \boxed{0} \end{bmatrix} .$$

Let

$$X_\ell^T A Z_\ell = \begin{bmatrix} \overline{A} & A_{\ell+1,\ell} & \cdots & A_{\ell+1,1} \\ A_{\ell,\ell+1} & \cdots\cdots\cdots\cdots & A_{\ell,1} \\ \vdots & & \vdots \\ A_{1,\ell+1} & \cdots\cdots\cdots\cdots & A_{11} \end{bmatrix} \quad , \quad X_\ell^T B = \begin{bmatrix} B_{\ell+1} \\ B_\ell \\ \vdots \\ B_1 \end{bmatrix} .$$

Here

$$A_{ij} \in \mathbb{R}^{g_i \times g_j} \qquad i = 1,\ldots,\ell; \ j = 1,\ldots,\ell$$

$$A_{i,\ell+1} \in \mathbb{R}^{g_i \times r} \qquad i = 1,\ldots,\ell$$

$$A_{\ell+1,j} \in \mathbb{R}^{r \times g_j} \qquad j = 1,\ldots,\ell$$

$$\overline{A} \in \mathbb{R}^{r \times r}$$

where

$$r = n - \sum_{i=1}^{\ell} g_i$$

and

$$B_i \in \mathbb{R}^{g_i \times m} \qquad i = 1,\ldots,\ell$$

$$B_{\ell+1} \in \mathbb{R}^{r \times m} .$$

Then we have

$$\begin{bmatrix} B_{\ell+1} \\ B_\ell \\ \vdots \\ B_i \end{bmatrix} F_{ig} = - \begin{bmatrix} A_{\ell+1,i} \\ A_{\ell,i} \\ \vdots \\ A_{i,i} \end{bmatrix} \quad , \quad i = 1,\ldots,\ell$$

which shows that the construction of $F_g = [\,0\, , F_{\ell,g}\, ,\ldots, F_{1,g}]$ merely consists of solving a set of linear equations.

Therefore this algorithm has the same numerical behaviour as the algorithms in [2] and [3]. This shows that the algorithm may be considered to be a numerically reliable algorithm.

Formally this generalized deadbeat control algorithm is not numerically stable (backward stability). It can be proved, as in [3], that the feedback matrix F is an exact solution to the generalized deadbeat control problem for

$$(E + \delta E, \ A + \delta A, \ B + \delta B)$$

where

$$\| \delta E, \delta A, \delta B \| = \psi(\varepsilon) \ \cdot \ \varphi(\| E \|, \| A \|, \| B \|, \| F \|) \ .$$

Here $\psi(\varepsilon)$ is of the order of the relative machine precision ε and φ is a bilinear function in $\| E \|, \| A \|, \| B \|$ and $\| F \|$. ($\| \ \|$ is the Frobenius norm).

It can also be proved that, analogously to [4], that there exists a perturbation $\delta P(\lambda)$ such that F is an exact solution to the generalized deadbeat control problem for $(E + \delta E, \ A + \delta A, \ B + \delta B)$ where this latter generalized system is obtained from $P(\lambda) + \delta P(\lambda)$ in the same way as (E,A,B) is obtained from $P(\lambda)$ (this perturbation has the same structure as (E,A,B)).

In (8) we obtain a polynomial matrix $Q(\lambda)$ such that there exists a perturbation $\delta Q(\lambda)$ having the property that

$$\begin{bmatrix} P(\lambda) + \delta P(\lambda) \\ Q(\lambda) + \delta Q(\lambda) \end{bmatrix}$$

is exactly unimodular. The perturbations $\delta P(\lambda)$, $\delta Q(\lambda)$ consist of polynomials whose coefficients are of the order of ε. However, these coefficients are also

functions of $\| F \|$.

The existence of $\delta Q(\lambda)$ is proved using forward stability analysis and $\delta P(\lambda)$
is obtained using a backward stability argument.

Details of the proof will be omitted.

Observe that the choice of the identity matrix I in

$$
\begin{bmatrix}
E - \lambda A & \lambda B \\
F & I
\end{bmatrix}
$$

is an arbitrary one. We could have chosen any regular matrix D instead because

$$
\begin{bmatrix}
E - \lambda A & \lambda B \\
DF & D
\end{bmatrix}
$$

also is unimodular if F is a solution to the generalized deadbeat control
problem.

The choise of a "good" D and the exploitation of the (sometimes existing) free-
dom for F in order to obtain a "good" unimodular matrix

$$
\begin{bmatrix}
P(\lambda) \\
Q(\lambda)
\end{bmatrix}
$$

is a point of current research.

Examples

The algorithm has been used to compute $Q(\lambda)$ for various matrices $P(\lambda)$. Here
the coefficients of $P(\lambda)$ have been chosen uniformly distributed in $[-1,1]$.
After having computed $Q(\lambda)$ the determinant of $\begin{bmatrix} P(\lambda) \\ Q(\lambda) \end{bmatrix}$ has been computed for

$\lambda = 0, 0.1, 0.2, \ldots, 0.9$. Let d denote the number of digits in the determinant

of the computed unimodular matrix that does not depend on λ. The number of rows

of $P(\lambda)$ is p, the number of columns of $P(\lambda)$ is q and the degree of $P(\lambda)$ is n.

The following table shows the "unimodularity" of the matrix $\begin{bmatrix} P(\lambda) \\ Q(\lambda) \end{bmatrix}$ for various

choices of p,q,n. If d is large then the matrix is highly unimodular.

p	q	n	d
1	7	5	11
3	7	2	9
6	7	5	8
3	7	8	9
1	15	25	9
10	25	5	9
1	2	25	8
1	2	35	7
20	30	2	8

unimodularity of $\begin{bmatrix} P(\lambda) \\ Q(\lambda) \end{bmatrix}$.

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