Hitting times and the running maximum of Markovian growth collapse processes

Hitting Times and the Running Maximum of Markovian Growth Collapse Processes

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May 11, 2009

Abstract

We consider a Markovian growth collapse process on the state space $\mathscr{E} = [0, \infty)$ which evolves as follows. Between random downward jumps the process increases with slope one. Both the jump intensity and the jump sizes depend on the current state of the process. We are interested in the behavior of the first hitting time $\tau_y = \inf\{t \geq 0 | X_t = y\}$ as y becomes large and the growth of the maximum process $M_t = \sup\{X_s | 0 \leq s \leq t\}$ as $t \to \infty$. We consider the recursive sequence of equations $\mathcal{A}m_n = m_{n-1}$, $m_0 \equiv 1$, where $\mathcal{A}$ is the extended generator of the MGCP, and show that the solution sequence (which is essentially unique and can be given in integral form) is related to the moments of τ_y. The Laplace transform of τ_y can be expressed in closed form (in terms of an integral involving a certain kernel) in a similar way. We derive asymptotic results for the running maximum: (i) if $m_1(y)$ is of rapid variation, we have $M_t/m^{-1}(t) \xrightarrow{d} 1$; (ii) if $m_1(y)$ is of regular variation with index $a \in (0, \infty)$ and the MGCP is ergodic, then $M_t/m^{-1}(t) \xrightarrow{d} Z_a$, where Z_α has a Frechet distribution. We present several examples.

1 Introduction and known results

A Markovian growth collapse process (MGCP) is a Markov process $(X_t)_{t \geq 0}$ on the state space $\mathscr{E} = [0, \infty)$ with no upward jumps and piecewise deterministic right-continuous paths. The process X_t increases linearly with slope one between the jumps. Hence it

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can be written in the form

$$X_t = X_0 + t - \sum_{k=1}^{N_t} B_k, \quad t \geq 0$$

where $(N_t)_{t\geq 0}$ is a state-dependent counting process and the downward jump sizes $B_k > 0$ also depend on the current state. MGCPs can be encountered in a large variety of applications, of which we mention growth population models, risk processes, neuron firing, and window sizes in transmission control protocols, and have been studied in [14, 8, 7, 28]. They form a special class of piecewise deterministic Markov processes [11, 10, 6].

We are interested in the behavior of the first hitting time

$$\tau_y = \inf\{t \geq 0 | X_t = y\}$$

of the level $y \geq 0$ and in the running maximum process

$$M_t = \sup\{X_s | 0 \leq s \leq t\}.$$

Note that $X_{\tau_y} = X_{\tau_y -} = y$ almost surely. The main objective of this paper is to evaluate the Laplace transform $\mathbb{E}e^{-s\tau_y}$ and the moments of τ_y for the introduced growth collapse model, in particular the function $m(y) = \mathbb{E}\tau_y$, and to derive a quite general result for the convergence of M_t.

More formally, let $T_1, T_2, \ldots$ denote the times of the successive collapses (jumps) of the MGCP and let $\lambda(\cdot)$ be the jump intensity of the process, so that the probability of a jump during $[t, t+h]$ given that $X_t = x$ is $\lambda(x)h + o(h)$ as $h \to 0$. The probability of a jump from x into the set $[0, y]$ is given by $\mu_x(y)$ where μ_x is the distribution function of a probability measure on $[0, x)$ for each $x \in \mathscr{E}$. We assume that

- $\lambda : \mathscr{E} \to [0, \infty)$ is locally integrable and $\int_0^\infty \lambda(u)\, du = \infty$ so that $\mathbb{P}(T_1 = \infty) = 0$.
- $\mathbb{E}(N_t) < \infty$ for all $t \geq 0$.

Let $\mathbb{P}_x$ and $\mathbb{E}_x$ denote conditional probability and expectation given that $X_0 = x$. It is easy to see that under $\mathbb{P}_x$ the first hitting time of level $y > x$ has the same distribution as $\tau_y - \tau'_x$ under $\mathbb{P}_0$, where τ'_x is independent of τ_y and has the same distribution as τ_x. The process X_t can also be viewed as a regenerative process if we define cycles as the times between successive visits to some fixed recurrent state $z \in [0, \infty)$. Let C_k denote the length of the kth cycle, where the first cycle starts at time $C_0 = \tau_z$. Let $S_k = \sum_{i=0}^k C_i$ and let $K(t)$ denote the current cycle at time t. Then $S_{K(\tau_y)-1} \leq \tau_y = S_{K(\tau_y)-1} + \overline{\tau}_y$, where $\overline{\tau}_y$ is distributed as the first hitting time of level y, starting from z and given that the process stays above z. Renyi's theorem states that if $\mu_C = \mathbb{E}(C_1) < \infty$ then

$$\frac{\mathbb{P}(K(\tau_y)=1)}{\mu_C} S_{K(\tau_y)-1} \xrightarrow{d} Z \ , \quad y \to \infty,$$

where $\xrightarrow{d}$ denotes weak convergence and Z is an exponential random variable with unit mean (see the extended version given as Theorem 2.4 in [17]). Let $\xi_i = \max\{X_t | t \in [S_i, S_{i+1}]\}$ denote the ith cycle maximum and $G(y) = \mathbb{P}(\xi_1 \le y) = 1 - \mathbb{P}(K(\tau_y) = 1)$ the common distribution function of the ξ_i. If $\overline{\tau}_y$ is small compared to $S_{K(\tau_y)-1}$, then we can expect that

$$\frac{1-G(y)}{\mu_C} \tau_y \xrightarrow{d} Z \ , \quad y \to \infty. \tag{1}$$

The fact that this is indeed true if X_t is ergodic is known as Keilson's theorem [18]. Propositions 2 and 3 in [8] imply that for an MGCP $\mathbb{E}\tau_y^n < \infty$ for all $y \ge 0$ and all $n \in \mathbb{N}$ and that X_t is ergodic if $\limsup_{x\to\infty} \lambda(x) \int_0^x \mu_x(y)\, dy > 1$. Moreover, it can be shown (see e.g. [3], Proposition 4.1) that the convergence in (1) also holds in expectation so that

$$m(y) \sim \frac{\mu_C}{1-G(y)} \ , \quad y \to \infty. \tag{2}$$

Consequently, any asymptotic result for the function m is at the same time a result for the tail of G. It then follows from (1) and (2) that in the ergodic case

$$\frac{\tau_y}{m(y)} \xrightarrow{d} Z \ , \quad y \to \infty. \tag{3}$$

Clearly $M_t \ge y$ if and only if $\tau_y \le t$, so that various probabilistic properties of τ_y can be expressed in terms of properties of M_t, in particular via the relation $\mathbb{P}(\tau_y \le t) = \mathbb{P}(M_t \ge y)$. Clearly $\max_{i \le K(t)-1} \xi_i \le M_t \le \max_{i \le K(t)} \xi_i$, and since $K(t) \approx t/\mu_C$ it is to be expected that $\mathbb{P}(M_t \le y)$ is close to $G(y)^{t/\mu_C}$. Indeed, it is shown in [26] that $\sup_{y\ge 0} \left|\mathbb{P}(M_t \le y) - G(y)^{t/\mu_C}\right| \to 0$ as $t \to \infty$. Hence classical extreme value theory for i.i.d. variables can be applied to find possible limits of the (properly normalized) process M_t. The following results are known for general regenerative processes (see [26, 2]). Let $\vec{G}(t) = \inf\{x : 1 - G(x) \le 1/t\}$. A function $f : [0, \infty) \to [0, \infty)$ is called regularly varying if

$$\frac{f(\lambda y)}{f(y)} \to \lambda^a \ , \quad y \to \infty,$$

for all $\lambda > 0$ and we then write $f \in \mathcal{R}_a$. Suppose that X_t is ergodic. Then, if $1-G \in \mathcal{R}_{-a}$ for some $a > 0$, we have

$$\frac{M_t}{\vec{G}(t/\mu_C)} \xrightarrow{d} Z_a \ , \quad t \to \infty, \tag{4}$$

where $\mathbb{P}(Z_a \le x) = e^{-x^{-a}}$ (Frechet distribution). If $1 - G(y) = \exp\big(- \int_0^y [1/\delta(u)]\big)\ du$ for some absolutely continuous function $\delta > 0$ having density $\delta'(y) \to 0$ as $y \to \infty$, then

$$\frac{M_t - \vec{G}(t/\mu_C)}{\delta(\vec{G}(t/\mu_C))} \xrightarrow{d} Z_G \ , \quad t \to \infty,$$

where $\mathbb{P}(Z_G \le x) = e^{-e^{-x}}$ (Gumbel distribution).

In this paper we supplement the above known results by the following contributions. In Section 2 we consider the recursive sequence of equations $\mathcal{A}m_n = m_{n-1}$, $m_0 \equiv 1$, where $\mathcal{A}$ is the extended generator of the MGCP, and show that the solution sequence (which is essentially unique and can be given in integral form) is related to the moments of τ_y: we have for example $\mathbb{E}\tau_y = m_1(y)$ and $\mathbb{E}\tau_y^2 = 2(m_1(y)^2 - m_2(y))$. The Laplace transform of τ_y can be expressed in closed form (in terms of an integral involving a certain kernel) in a similar way. We also prove the alternative series expansion

$$1/\mathbb{E}e^{-s\tau_y} = \sum_{n=0}^{\infty} m_n(x)s^n, \quad s, x \ge 0. \tag{5}$$

Without assuming ergodicity of the MGCP it can be shown (using (5)) that the relation $m_2(y) = o(m_1(y))$ as $y \to \infty$ implies (3). In Section 3 we derive asymptotic results for the running maximum: (i) if $m_1(y)$ is of rapid variation, we have $M_t/m^{-1}(t) \xrightarrow{d} 1$; (ii) if $m_1(y)$ is of regular variation with index $a \in (0, \infty)$ and the MGCP is ergodic, then $M_t/m^{-1}(t) \xrightarrow{d} Z_a$. In Section 4 we present several examples. In the case of separable jump measures (i.e., $\mu_x(y) = \nu(y)/\nu(x)$ for some function $\nu(x)$) we give various explicit results on τ_y. Moreover, we prove that if ν is regularly varying with index b and $x\lambda(x)$ tends to some limit $a \in (b + 1, \infty]$ as $x \to \infty$, then $M_t/m^{-1}(t) \xrightarrow{d} Z_a$ if $a < \infty$ and $M_t/m^{-1}(t) \xrightarrow{d} Z_a$ if $a = \infty$. In applications $\lambda(x)$ is usually nondecreasing, leading to $a = \infty$; a typical case is $\lambda(x) = \lambda x^\beta$ for some $\beta > 0$. If $\nu(x) = x$, a collapse causes the cut-off of a uniform fraction of the current value, which can be modeled by the multiplication by a random variable that is uniform on $(0, 1)$. We also present several closed-form expressions in the general case of jumps generated by multiplication by $(0, 1)$-valued random variables. Finally, the results on regularly and rapidly varying functions that are used throughout are collected in an appendix.

We note that instead of studying models with linear increase, we could also study MGCPs Y_t with a more general deterministic inter-jump behavior, say $dY_t = r(Y_t)\ dt$, where $r(x)$ is a Lipschitz continuous function. It turns out that we can easily transform X_t into Y_t and vice versa by means of the transformation $X_t = \theta(Y_t)$, where $\theta(x) = \int_z^x 1/r(u)\ du$ measures the time the process Y_t needs to increase from 0 to x. It then follows indeed that $dX_t = (d\theta(Y_t)/dt)(dY_t/dt) = 1$ in between jumps. If $\widehat{\tau}_y$ and $\widehat{M}_t$ denote the first hitting time and the maximum process of Y_t, then it is easy to see that $\tau_y = \widehat{\tau}_{\theta^{-1}(y)}$ and $M_t = \theta(\widehat{M_t})$.

2 Integral equations and series representations

Our derivations require the notion of the extended generator of the Markov process X_t. A measurable function $f : [0, \infty) \to [0, \infty)$ belongs to the domain of the extended generator if the process

$$M_t^f = f(X_t) - \int_0^t g(X_s)\, ds, \quad t \geq 0 \tag{6}$$

is a martingale for some measurable function $g : [0, \infty) \to [0, \infty)$. In this case we write $\mathcal{A}f(x) = g(x)$ and call $\mathcal{A}$ the extended generator. Note that $\mathcal{A}$ can be multi-valued.

[11] gives broad sufficient conditions for a function to be a member of the domain. Let $\mathscr{M}_{abs}$ denote the set of absolutely continuous functions $f : [0, \infty) \to [0, \infty)$ with locally bounded non-negative Lebesgue density $f'(x)$. If $f \in \mathscr{M}_{abs}$, then f is non-decreasing and since $X_t \leq t$ a.s. we have $f(X_t) \leq f(t)$ a.s., yielding the bound

$$\mathbb{E} \sum_{n=0}^{N_t} |f(X_{T_i-}) - f(X_{T_i})| \leq 2f(t)\mathbb{E}N_t < \infty$$

for all $t \geq 0$. It follows from [11], Theorem (26.14), that the functions in $\mathscr{M}_{abs}$ belong to the domain of the extended generator and that $\mathcal{A}f(x)$ is given by

$$\mathcal{A}f(x) \quad = \quad f'(x) - \lambda(x) \int_0^x (f(x) - f(y))\; d\mu_x(y)$$

which, after applying Fubini's theorem, can be written as

$$\mathcal{A}f(x) \quad = \quad f'(x) - \lambda(x) \int_0^x f'(y)\mu_x(y)\, dy. \tag{7}$$

Note that the actual domain of the extended generator may be much larger than $\mathscr{M}_{abs}$, but $\mathscr{M}_{abs}$ suffices here, since the relevant functions that appear throughout this paper belong to $\mathscr{M}_{abs}$.

In the sequel we need the kernel $K_s(x, y) = \lambda(x)\mu_x(y) + s$, where $x \geq y \geq 0, s \geq 0$, and its iterates $K_s^1(x, y) = K_s(x, y)$ and

$$K_s^n(x, y) = \int_y^x K_s(x, u) K_s^{n-1}(u, y)\, du, \quad n \geq 2.$$

It is straightforward to show that

$$K_s^n(x,y) \leq (\lambda(x)+s)\frac{\left(\int_y^x (\lambda(u)+s)\, du\right)^{n-1}}{(n-1)!} \tag{8}$$

(cf. Lemma 1 in [27]). Hence, the resolvent kernel

$$R_s(x,y) = 1 + \sum_{k=1}^{\infty} K_s^k(x,y)$$

is well-defined and converges for all $s \geq 0$ and all $x \geq y \geq 0$. Moreover, it follows from (8) that

$$R_s(x,y) \leq 1 + (\lambda(x)+s) \cdot \exp\Big(\int_y^x (\lambda(u)+s)\, du\Big).$$

Theorem 1. *1. Let $m_0(x) = 1$. For all $n \in \mathbb{N}$ there exists a unique solution $m_n \in \mathscr{M}_{abs}$ of the equation $\mathcal{A}m_n(x) = m_{n-1}(x)$ with initial condition $m_n(0) = 0$. Moreover,*

$$m_n(y) = \int_0^y \Big(m_{n-1}(x) + \int_0^x R_0(x,u) m_{n-1}(u)\, du\Big)\, dx, \quad n \geq 2. \tag{9}$$

2. *We have $\mathbb{E}\tau_y = m_1(y)$, so that $m(y) = m_1(y)$, and $\operatorname{Var}\tau_y = m_1(y)^2 - 2m_2(y)$.*

3. *For all $s \geq 0$ there is a unique solution $\psi(s,\cdot)$ in $\mathscr{M}_{abs}$ of the equation $\mathcal{A}\psi(s,x) = s\psi(s,x)$ with initial condition $\psi(s,0) = 1$. Moreover,*

$$\psi(s,y) = 1 + s\int_0^y \Big(1 + \int_0^x R_s(x,u)\, du\Big)\, dx. \tag{10}$$

4. *The Laplace transform of τ_y is given by $\mathbb{E}e^{-s\tau_y} = 1/\psi(s,y)$.*

Proof. A generator equation $\mathcal{A}f(x) = z(x)$ with $z \in \mathscr{M}_{abs}$ can be written as an integral equation for the density f', namely

$$f'(x) = z(x) + \int_0^x K(x,y) f'(y)\, dy, \tag{11}$$

where $K(x,y) =_{def} K_0(x,y) = \lambda(x)\mu_x(y)$. Similarly, the equation

$$\mathcal{A}f(x) = sf(x), \quad f(0) = 1,$$

is equivalent to

$$f'(x) = sf(x) + \int_0^x K(x,y) f'(y)\, dy = s + \int_0^x K_s(x,y) f'(y)\, dy. \tag{12}$$

It is well-known that a solution of (11) is given by

$$f'(x) = z(x) + \int_0^x R_0(x,y)z(y)\,dy. \tag{13}$$

and that (12) is solved by

$$f'(x) = s\Big(1 + \int_0^x R_s(x,y)\,dy\Big). \tag{14}$$

Note that certainly $f \in \mathscr{M}_{abs}$, since f is absolutely continuous and f' is locally bounded and non-negative. The homogeneous equation

$$h'(x) \quad = \quad \int_0^x K_s(x,y)h'(y)\,dy,$$

is solved in the set of absolutely continuous functions only by constant functions h. This is immediate from the fact that iteration yields

$$\begin{aligned} |h'(x)| &= \Big| \int_0^x K_s^n(x,y)h'(y)\,dy \Big| \\ &\le (\lambda(x)+s) \int_0^x \frac{\big(\int_y^x (\lambda(u)+s)\,du\big)^{n-1}}{(n-1)!} |h'(y)|\,dy \end{aligned}$$

for all $n \in \mathbb{N}$ and hence $h'(x) = 0$. Consequently, solutions of (11) and (12) are unique in $\mathscr{M}_{abs}$, once we specify $f(0)$.

Since $m_1(x) = 1$, it follows that the process $U_{1,t} = m_1(X_t) - t$ is a martingale. Now, since $\mathbb{E}\tau_y < \infty$ then on $\{\tau_y > t\}$ we have $|U_{1,t}| \le t + m_1(X_{\tau_y}) \le t + m_1(y) = O(t)$, so that

$$\mathbb{E}\,(U_{1,t}; \tau_y > t) = \mathbb{E}\,(U_{1,t} | \tau_y > t)\, o(1/t) \to 0 \tag{15}$$

as $t \to \infty$. This justifies optional stopping for the martingale $U_{1,t}$ at time τ_y (see [15]) and it follows from $m_1(0) = 0$ that $m_1(y) = m(y) = \mathbb{E}\tau_y$.

The integrated process $I_t = \int_0^t s\,d\,(m(X_s) - s)$ is also a martingale (see again [15]) and it follows by partial integration that

$$I_t = tm(X_t) - \tfrac{1}{2}t^2 - m_2(X_t) + D_t,$$

where $D_t = m_2(X_t) - \int_0^t m(X_s)\,ds$ is the Dynkin martingale of the function m_2. Hence the difference

$$U_{2,t} = tm(X_t) - \tfrac{1}{2}t^2 - m_2(X_t)$$

of I_t and D_t is a martingale, too. Optional stopping, which can be justified as in (15), leads to

$$\operatorname{Var} \tau_y = \mathbb{E}\tau_y^2 - (\mathbb{E}\tau_y)^2 = m(y)^2 - 2m_2(y),$$

showing part 2. We now turn to the function ψ. Since $\mathcal{A}\psi(s,x) = s\psi(s,x)$ with $\psi(0,x) = 1$, we have

$$\psi'(s,x) = s + \int_0^x K_s(x,y)\psi'(s,y)\, dy,$$

which is tantamount to (12). Following the discussion above we conclude that a unique solution in $\mathscr{M}_{abs}$ exists and $\psi(s,\cdot)$ is given in terms of the associated resolvent kernel as in (10), so that part 3 is proved.

It is known that the process $e^{-st}\psi(s,X_t)$ is a martingale (see e.g. [15], p.175, or [25]). Optional stopping at τ_y, which can be justified as in [19], leads to $\mathbb{E}e^{-s\tau_y}\psi(s,y) = \psi(s,0) = 1$, so that part 4 is proved. $\square$

Remark. For $n = 1$ one can prove the equation $\mathcal{A}m(x) = 1$ by an alternative probabilistic reasoning, avoiding the use of martingales. The equation to be solved becomes

$$m'(y) \quad = \quad 1 + \lambda(y) \int_0^y m'(u)\mu_y(u)\, du. \tag{16}$$

Consider the first jump time T_1. If $T_1 \geq y$ then $\tau_y = y$, while if $T_1 < y$ then τ_y is equal to T_1 plus the hitting time of y, starting at X_{T_1}. Hence,

$$\tau_y \overset{d}{=} y\mathbb{1}_{\{T_1 \geq y\}} + \big(T_1 + \tau'_y - \tau''_{X_{T_1}}\big)\mathbb{1}_{\{T_1 < y\}},$$

where the families $(\tau'_y)_{y\geq 0}$ and $(\tau''_y)_{y\geq 0}$ are independent of each other, both are independent of $(X_t)_{t\geq 0}$, and $\tau''_y \overset{d}{=} \tau'_y \overset{d}{=} \tau_y$ for all $y \geq 0$. It follows that

$$m(y) = y + \frac{\mathbb{E}(T_1 - \tau''_{X_{T_1}}; T_1 < y)}{\mathbb{P}(T_1 \geq y)}.$$

Conditioning on T_1 yields

$$m(y) = y + \frac{\int_0^y \Big(t - \int_0^t m(u) d\mu_t(u)\Big)\mathbb{P}(T_1 \in dt)}{\mathbb{P}(T_1 \geq y)}.$$

Using $\dfrac{d}{dy}\mathbb{P}(T_1 \leq y) = \lambda(y)\mathbb{P}(T_1 \geq y)$, we obtain (16) after a short calculation.

Theorem 2. *We have the power series representation*

$$\psi(s,x) = \sum_{n=0}^{\infty} m_n(x) s^n \tag{17}$$

for all $s \geq 0$ and all $x \geq 0$.

Proof. To show (17), we first prove by induction that $m_n(y) \leq m(y)^n/n!$, which is certainly true for $n = 1$. Moreover, if the assumption holds for $n-1$, then

$$\begin{aligned} m_n(y) &= \int_0^y \left(m_{n-1}(x) + \int_0^x R_0(x,u) m_{n-1}(u)\, du \right) dx \\ &\leq \int_0^y m_{n-1}(x) m'(x)\, dx \leq \int_0^y \frac{m(x)^{n-1}}{(n-1)!} m'(x)\, dx = \frac{m(x)^n}{n!}. \end{aligned}$$

It follows that the series in (17) converges for all $s \geq 0$ and all $x \geq 0$. The function

$$h(x) =_{def} \sum_{n=0}^{\infty} s^n m_n(x) \leq e^{sm(x)}$$

is in $\mathscr{M}_{abs}$ and $h_k(x) =_{def} \sum_{n=0}^{k} s^n m_n(x)$ converges to $h(x)$ pointwise as $k \to \infty$. Since

$$\mathcal{A} h_k(x) = \sum_{n=1}^{k} s^n m_{n-1}(x) = s \sum_{n=0}^{k-1} s^n m_n(x) = s h_k(x) - s^k m_k(x),$$

it follows that $|\mathcal{A} h_k(x) - s h_k(x)| \leq (sm(x))^k/k!$. In particular, $\mathcal{A} h_k(x) - s h_k(x)$ tends to zero as $k \to \infty$. Hence, $\mathcal{A} h(x) = s h(x)$ and, by the uniqueness property, $\psi(s,x) = h(x)$. □

The following corollary is needed in the next section.

Corollary 1. *The function*

$$s \mapsto \frac{\psi(s,y) - 1}{s}$$

is increasing for all $y \in [0,\infty)$ and in particular $\psi(s,y) \geq 1 + sm(y)$.

Proof. $\dfrac{\psi(s,y)-1}{s} = \sum_{n=1}^{\infty} m_n(y) s^{n-1}$. □

The next theorem gives a sufficient criterion for $\tau_y/m(y)$ to be asymptotically exponential without the assumption of ergodicity.

Theorem 3. *If $m_2(y) = o(m(y)^2)$, then $\dfrac{\tau_y}{m(y)} \xrightarrow{d} Z$ as $y \to \infty$.*

Proof. We carry out an induction proof to show that $m_n(y) = o(m(y)^n)$ for all $n \geq 2$. We have $m_2(y) = o(m(y)^2)$ by assumption. If the assertion is true for $n-1$, we obtain, using the representation (9) and monotonicity of the functions $m_n(y)$,

$$\begin{aligned} m_n(y) &= \int_0^y \big(m_{n-1}(x) + \int_0^x R_0(x,u) m_{n-1}(u)\, du\big)\, dx \\ &\leq m_{n-1}(y) \int_0^y \big(1 + \int_0^x R_0(x,u)\, du\big)\, dx \\ &= o(m^{n-1}(y)) m(y) = o(m(y)^n). \end{aligned}$$

It now follows from Theorem 1 that

$$\lim_{y\to\infty} \mathbb{E} e^{-s\tau_y/m(y)} = \lim_{y\to\infty} \psi(s/m(y), y)^{-1} = \lim_{y\to\infty} \big(\sum_{n=0}^{\infty} s^n \frac{m_n(y)}{m(y)^n}\big)^{-1}. \tag{18}$$

Since $\sup_y m_n(y)/m(y)^n \leq 1/n!$ and $\lim_{y\to\infty} m_n(y)/m(y)^n = 0$ for all $n \geq 2$, we can use Lebesgue's convergence theorem and conclude that the right-hand side of (18) tends to $1/(1+s)$, as $y \to \infty$, i.e., to the Laplace transform of Z. This completes the proof. $\square$

3 Asymptotics of the running maximum

We now consider the asymptotic behavior of M_t in two cases: (i) $m(x)$ is regularly varying and (ii) $m(x)$ is rapidly varying. Assuming ergodicity, case (i) is a straightforward consequence of known results. Case (ii) is more complicated. Let $m^{-1}(t)$ be the inverse function of the monotone increasing function $m(x)$.

Theorem 4. *If $m \in \mathcal{R}_a$ for some $a \in (0,\infty)$ and X_t is ergodic, then*

$$M_t/m^{-1}(t) \xrightarrow{d} Z_a \ , \quad t \to \infty. \tag{19}$$

Proof. Recall that $\vec{G}(t)$ denotes the generalized inverse function of $t \mapsto 1 - G(1/t)$. According to relation (2) we have $[1-G(\vec{G}(y))]m(\vec{G}(y)) \to \mu_C$ and hence $m(\vec{G}(y)) \sim \mu_c y$ or

$$m(y) \sim \mu_c \vec{G}^{-1}(y) \ , \quad y \to \infty.$$

Consequently, as $m \in \mathcal{R}_a$, we have $\vec{G}^{-1} \in \mathcal{R}_a$. It follows that $\vec{G} \in \mathcal{R}_{1/a}$ and, from the definition of $\vec{G}(t)$, that $1 - G \in \mathcal{R}_{-a}$. Hence the conditions for convergence in (4) are fulfilled. Since $\vec{G}^{-1}$ is increasing and unbounded, a result in [12] implies that $\vec{G}(t) \sim m^{-1}(\mu_c t)$, yielding $M_t/m^{-1}(t) \xrightarrow{d} Z_a$ as $t \to \infty$. $\square$

Theorem 5. *If $m \in \mathcal{R}_\infty$, then*

$$\mathbb{E}\Big(\big(\frac{M_t}{m^{-1}(t)}\big)^n\Big) \to 1 \tag{20}$$

for all $n \geq 0$. In particular

$$M_t/m^{-1}(t) \xrightarrow{d} 1 \ , \quad t \to \infty. \tag{21}$$

Proof. We define $\mu_n(t) = \mathbb{E}M_t^n$ and show that

$$\big(m^{-1}(\frac{1}{s})\big)^{-n} \int_0^\infty e^{-st} \, d\mu_n(t) \to 1 \ , \quad s \to 0 \tag{22}$$

for every $n \in \mathbb{N}$. By Karamata's Tauberian theorem, (22) implies that $\mathbb{E}(M_t^n) \sim (m^{-1}(t))^n$ as $t \to \infty$, and hence we have proved (20). Since the constant moment sequence obviously satisfies Carleman's criterion, (19) follows immediately.

To prove (22), let $y = m^{-1}(1/s)$. Then

$$\begin{aligned}
y^{-n} \int_0^\infty e^{-st} \, d\mu_n(t) &= y^{-n} \int_0^\infty e^{-st} \frac{d}{dt}\big(\mathbb{E}M_t^n\big) \, dt \\
&= y^{-n} \int_0^\infty e^{-st} \frac{d}{dt}\Big(\int_0^\infty nu^{n-1}\mathbb{P}(M_t > u) \, du\Big) \, dt \\
&= y^{-n} \int_0^\infty nu^{n-1} \int_0^\infty e^{-st}\mathbb{P}(\tau_u \in dt) \, du \\
&= y^{-n} \int_0^\infty \frac{nu^{n-1}}{\psi(s,u)} \, du \\
&= J_0^\infty(y),
\end{aligned}$$

where we define

$$J_a^b(y) =_{def} \int_a^b \frac{nu^{n-1}}{\psi(1/m(y), yu)} \, du.$$

We show that $J_0^\infty(y) \to 1$ by dividing the range of integration in three parts:

(A) $J_w^\infty(y) \to 0$ for any $w > 1$:
According to Corollary 1 we have $\psi(s, y) \geq 1 + sm(y)$. Hence,

$$\psi(\frac{1}{m(y)}, uy) \geq 1 + \frac{m(uy)}{m(y)} = 1 + \frac{r(uy)}{r(y)}u^{n+1},$$

where $r(x) =_{def} x^{n+1}m(x)$ is again rapidly varying. The convergence $r(uy)/r(y) \to \infty$ is uniform for $u \geq w > 1$ (see (34) in the Appendix). In particular, for all $K > 0$ we ultimately have $\inf_{u \geq w} r(uy)/r(y) \geq K$ for large y, yielding

$$J_w^\infty(y) \ \leq \ \int_w^\infty \frac{nu^{n-1}}{1 + [r(uy)/r(y)]u^{n+1}} \, du \leq \int_w^\infty \frac{nu^{n-1}}{1 + Ku^{n+1}} \, du$$

for w sufficiently large, so that $J_w^\infty(y)$ tends to zero as $y \to \infty$.

(B) $J_1^w(y) \to 0$ for any $w > 1$:
This is clear, since the integrand tends to zero and is uniformly bounded by nw^{n-1} on the bounded interval $[1, w]$.

(C) $J_0^1(y) \to 1$:
Let $u \in (0,1)$ and choose an $\varepsilon > 0$; then it follows from (18) that

$$\lim_{y\to\infty} \psi(\frac{s}{m(y)}, yu) = \lim_{y\to\infty} \psi(\frac{s}{m(yu)} \frac{m(yu)}{m(y)}, yu) \le \lim_{y\to\infty} \psi(\frac{\varepsilon}{m(yu)}, yu) = 1 + \varepsilon.$$

On the other hand, again using Corollary 1,

$$\lim_{y\to\infty} \psi(\frac{s}{m(y)}, yu) \ge \lim_{y\to\infty} \Big(1 + s\frac{m(yu)}{m(y)}\Big) = 1,$$

and hence $J_0^1(y) \to \int_0^1 nu^{n-1}\, du = 1$. □

4 Applications to special cases

We have seen that $m(y) = \mathbb{E}(\tau_y)$ serves as a normalizing function in (3) (in the ergodic case) and in Theorem 3, while its inverse $m^{-1}(t)$ plays a similar role in Theorems 4 and 5. Therefore explicit formulas for these functions are of special interest. In several examples one can compute $m(y)$ via the unique solution in $\mathscr{M}_{abs}$ of the integral equation given in Theorem 1 for $n = 1$, which reads as

$$m'(y) \quad = \quad 1 + \lambda(y) \int_0^y m'(u)\mu_y(u)\, du. \tag{23}$$

Similarly we can solve the equations

$$\psi'(s,y) \quad = \quad s\psi(s,y) + \lambda(y) \int_0^y \psi'(s,u)\mu_y(u)\, du \tag{24}$$

$$m_2'(y) \quad = \quad m(y) + \lambda(y) \int_0^y m_2'(u)\mu_y(u)\, du \tag{25}$$

in $\mathscr{M}_{abs}$ to find $\mathbb{E}e^{-s\tau_y} = 1/\psi(s,y)$ and $\operatorname{Var}\tau_y = m_1(y)^2 - 2m_2(y)$. Let us consider a few examples.

4.1 Separable jump measures

Suppose that the jump measures μ_x are given in the form

$$\mu_x(y) = \frac{\nu(y)}{\nu(x)}, \quad x \ge y \ge 0$$

for some non-decreasing function $\nu : [0,\infty) \to [0,\infty)$ (defining 0/0 as 0). We give some examples at the end of this subsection.

Theorem 6. *For an MGCP with $\mu_x(y) = \nu(x)/\nu(y)$ as above and general intensity function $\lambda(x)$, the mean of the first hitting time τ_y is given in closed form by*

$$m(y) \quad = \quad y + \int_0^y \frac{\lambda(x)}{\nu(x)} \int_0^x \nu(w) \exp\Big(\int_w^x \lambda(v)\, dv \Big)\, dw\, dx. \tag{26}$$

The variance of τ_y can be computed from (26) and

$$m_2(y) = \int_0^y \Big[m(x) + \frac{\lambda(x)}{\nu(x)} \int_0^x m(w)\nu(w) \exp\Big(\int_w^x \lambda(v)\, dv \Big)\, dw \Big]\, dx. \tag{27}$$

Proof. Let $A \in \mathscr{M}_{abs}$ be arbitrary and define $z(x) = \int_0^x z'(u)\, du$ by setting

$$z'(y) =_{def} A(y) + \frac{\lambda(y)}{\nu(y)} \int_0^y A(w)\nu(w) \exp\Big(\int_w^y \lambda(v)\, dv \Big)\, dw.$$

A straightforward calculation yields

$$\begin{aligned}
&\int_0^y \nu(u)(z'(u) - A(u))\, du = \int_0^y \lambda(u) \int_0^u A(w)\nu(w) \exp\Big(\int_w^u \lambda(v)\, dv \Big)\, dw\, du \\
= \;& \int_0^y A(w)\nu(w) \int_w^y \lambda(u) \exp\Big(\int_w^u \lambda(v)\, dv \Big)\, du\, dw \\
= \;& \int_0^y A(w)\nu(w) \Big(\exp\Big(\int_w^y \lambda(v)\, dv \Big) - 1 \Big)\, dw \\
= \;& \int_0^y A(w)\nu(w) \exp\Big(\int_w^y \lambda(v)\, dv \Big)\, dw - \int_0^y A(u)\nu(u)\, du \\
= \;& \frac{\nu(y)}{\lambda(y)}(z'(y) - A(y)) - \int_0^y A(u)\nu(u)\, du.
\end{aligned}$$

Hence,

$$A(y) = z'(y) - \lambda(y) \int_0^y z'(u) \frac{\nu(u)}{\nu(y)}\, du.$$

Letting $A(y) = 1$ and $A(y) = m(y)$ we obtain equations (26) and (27), respectively. □

Regarding the Laplace transform of τ_y, the required solution to $\mathcal{A}f(x) = sf(x)$ does not seem easy to find. If all functions involved are smooth enough we can transform the generator equation into

$$\frac{\partial^2}{\partial x^2}\psi(s,x) - (s + \xi(x) + \lambda(x))\, \psi'(s,x) + s\xi(x)\psi(s,x) = 0, \quad \psi(0,x) = 1, \tag{28}$$

where

$$\xi(x) = \frac{\lambda'(x)}{\lambda(x)} - \frac{\nu'(x)}{\nu(x)}.$$

Fixing s and defining $h(x)$ by $\psi(s,x) = e^{h(x)}$ we arrive at the Riccati equation

$$h'^2(x) + h''(x) - (s + \xi(x) + \lambda(x))\, h'(x) + s\xi(x) = 0, \quad h'(0) = s,$$

which is difficult to solve in general.

Now we turn to the running maximum. For regularly varying $\nu(x)$ we have the following result.

Theorem 7. *Suppose that*

$$\lim_{x\to\infty} x\lambda(x) = a \in (1,\infty]$$

and that $\nu \in \mathcal{R}_b$ for some $b < a-1$.

(a) If $a < \infty$, then

$$M_t/m^{-1}(t) \xrightarrow{d} Z_{a-b}\ , \quad t \to \infty.$$

(b) If $a = \infty$, then $M_t/m^{-1}(t) \xrightarrow{d} 1$ as $t \to \infty$.

Proof. We have $b \geq 0$ because $\nu(x)$ is nondecreasing. By Proposition 3 in [8], X_t is ergodic if $\limsup_{x\to\infty} \lambda(x) \int_0^x \mu_x(y)\, dy > 1$. In our case, for $a < \infty$ this lim sup is given by

$$\begin{aligned}\limsup_{x\to\infty} \lambda(x) \int_0^x \frac{\nu(y)}{\nu(x)}\, dy &= \limsup_{x\to\infty} \frac{a}{x\nu(x)} \int_0^x \nu(y)\, dy \\ &= \limsup_{x\to\infty} \frac{a}{x\nu(x)} \frac{x\nu(x)}{b+1} \\ &= \frac{a}{b+1} > 1\end{aligned}$$

(where in we have used Theorem 8, part 1, in the Appendix for the second equality) and for $a = \infty$ it is infinite. By Theorems 4 and 5, it remains to show that $m \in \mathcal{R}_{a-b}$. This is done in the following lemma (in which no inequality between a and b is assumed). $\square$

Lemma 1. *Suppose that $\nu \in \mathcal{R}_b$ for some $b < \infty$ and that $x\lambda(x) \to a \in (0,\infty]$. Then $m \in \mathcal{R}_{1\vee(a-b)}$.*

Proof. If $x\lambda(x) \to a$, then $\lambda \in \mathcal{R}_{-1}$ and it follows that

$$x \mapsto \frac{\lambda(x)}{\nu(x)} \exp\Big(\int_0^x \lambda(s)\, ds\Big) \in \mathcal{R}_{a-1-b}.$$

If $b-a \geq -1$, then by Theorem 9 in the Appendix

$$x \mapsto \frac{\lambda(x)\exp\big(\int_0^x \lambda(s)\,ds\big)}{\nu(x)} \int_0^x \nu(u)\exp\Big(-\int_0^u \lambda(s)\,ds\Big)\,du \in \mathcal{R}_0,$$

yielding $m \in \mathcal{R}_1$. If $b-a < -1$, then $\int_0^\infty \nu(u)\exp\big(-\int_0^u \lambda(s)\,ds\big)\,du < \infty$ and

$$x \mapsto \frac{\lambda(x)\exp\big(\int_0^x \lambda(s)\,ds\big)}{\nu(x)} \int_0^x \nu(u)\exp\Big(-\int_0^u \lambda(s)\,ds\Big)\,du \in \mathcal{R}_{-1-b+a}$$

and $m \in \mathcal{R}_{a-b}$. Note that $a-b>1$ implies that $y/m(y) \to 0$. If $a=\infty$, then $x \mapsto \nu(x)\exp\big(-\int_0^x \lambda(s)\,ds\big)$ is slowly varying and $x \mapsto \int_0^x \nu(u)\exp\big(-\int_0^u \lambda(s)\,ds\big)\,du$ is in $\mathcal{R}_1$. Thus $m \in \mathcal{R}_\infty$. $\square$

Examples. **(A)** *Renewal age processes.* If $\nu(x) \equiv 1$, then $\mu_x(y) \equiv 1$, i.e., the process restarts at zero after each jump. This is the age process from renewal theory, where the renewal epochs have a distribution with density $x \mapsto \lambda(x)\exp\big(-\int_0^x \lambda(u)\,du\big)$, $x \geq 0$. Note that τ_y is the first time at which the current lifetime reaches y. Equations (26) and (27) yield after some further calculations

$$\begin{aligned} m(y) &= \int_0^y \exp\Big(\int_w^y \lambda(v)\,dv\Big)\,dw \\ m_2(y) &= \int_0^y (y-w)\exp\Big(\int_w^y \lambda(v)\,dv\Big)\,dw. \end{aligned}$$

The Laplace transform of τ_y is given by

$$\mathbb{E}e^{-s\tau_y} = \Big(1 + s\int_0^y \exp\Big(\int_u^y (\lambda(w)+s)\,dw\Big)\,du\Big)^{-1}.$$

This follows immediately from (24) which reads as

$$\psi'(s,y) \;=\; s\psi(s,y) + \lambda(y)(\psi(s,y)-1).$$

The case where $\lambda(x) \equiv \lambda$ is constant has been discussed in [20].

(B) *Coupled intensity rate and jump measure.* Let $\nu(x) = \lambda(x)$ for all $x \geq 0$. Then we obtain from (26)

$$m(y) \;=\; \int_0^y \exp\Big(\int_0^x \lambda(w)\,dw\Big)\,dx.$$

Moreover, it follows from (28) that

$$\frac{\partial^2}{\partial x^2}\psi(s,x) = (s+\lambda(x))\,\psi'(s,x)$$

and thus $\psi'(s,x) = s\,\exp\big(sx + \int_0^x \lambda(w)\,dw\big)$. Hence

$$\psi(s,y) = 1 + s\int_0^y \exp\big(sx + \int_0^x \lambda(s)\,ds\big)\,dx$$

and thus

$$\mathbb{E}e^{-s\tau_y} = \left(1 + s\int_0^y \exp\big(sx + \int_0^x \lambda(s)\,ds\big)\,dx\right)^{-1}.$$

This generalizes the result for the particular case $\lambda(x) = \lambda x$ and $\mu_x(y) = y/x$ in [8].

4.2 MGCPs with multiplicative jumps

Consider the case where at each jump time the current level of the process is multiplied by an independent random variable Q having a distribution function F whose support is contained in $[0,1)$ (i.e., $F(1-) = 1$). Due to their importance in applications these MGCPs have been frequently studied [23, 24, 21, 9, 16, 22, 4, 13, 1]. Clearly $\mu_y(u) = F(u/y)$ and if we assume that $\lambda(x) \equiv \lambda$, then

$$m'(y) = 1 + \lambda\int_0^y m'(u)F(u/y)\,du, \tag{29}$$

$$\psi'(s,y) = s\psi(s,y) + \lambda\int_0^y \psi'(s,u)F(u/y)\,du. \tag{30}$$

Suppose that $m(\cdot)$ and $\psi(s,\cdot)$ can be expanded into power series: $m(x) = \sum_{k=1}^\infty a_k x^k$ and $\psi(s,x) = \sum_{k=1}^\infty b_k x^k$. Then

$$1 = \sum_{k=1}^\infty a_k k x^{k-1} - \lambda\sum_{k=1}^\infty \theta_k x^k = \sum_{k=1}^\infty \left(a_{k+1}(k+1) - \lambda\theta_k a_k\right)x^k,$$

where $\theta_k = 1 - \mathbb{E}Q^k$. Hence,

$$a_{k+1} = \lambda\frac{\theta_k}{k+1}a_k\ , \quad a_1 = m'(0) = 1$$

yielding

$$m(x) = \frac{1}{\lambda}\sum_{k=1}^\infty \frac{\prod_{i=1}^{k-1}\theta_i}{k!}(\lambda x)^k. \tag{31}$$

Similarly, the power series of $\psi(s,x)$ satisfies

$$s\sum_{k=1}^\infty b_k x^k = \sum_{k=1}^\infty \left(b_{k+1}(k+1) - \lambda\theta_k b_k\right)x^k$$

which means that

$$b_{k+1} = \frac{\lambda\theta_k + s}{k+1} b_k \ , \quad b_0 = \psi(s,0) = 1$$

and therefore leads to

$$\psi(s,x) = 1 + s\sum_{k=1}^{\infty} \frac{\prod_{i=1}^{k-1}(\lambda\theta_i + s)}{k!} x^k. \tag{32}$$

The two power series in (31) and (32) obviously have infinite radius of convergence and satisfy the defining equations in Theorem 1, so that they are indeed the desired solutions.

A more comprehensive treatment of multiplicative MGCPs, including the asymptotic behavior of m, is given in [21].

Two special cases. (a) The collapse consists of a multiplication by a deterministic constant $q \in [0,1)$, i.e., $F(x) = \mathbb{1}_{\{x \geq q\}}$. Then $\theta_a = 1 - q^a$ and, using the q-series symbols $(q)_k = \prod_{i=1}^{k}(1-q^i)$ and $(c;q)_k = \prod_{i=0}^{k-1}(1-cq^i)$, we obtain

$$m(x) = \frac{1}{\lambda}\sum_{k=1}^{\infty} \frac{(q)_{k-1}}{k!}(\lambda x)^k$$

and

$$\psi(s,x) = 1 + \sum_{k=1}^{\infty} \frac{(\lambda+s)^k(\frac{\lambda}{\lambda+s};q)_k}{k!} x^k.$$

(b) $Q = U^{1/\alpha}$ for some $\alpha > 0$ and a uniform random variable U on $(0,1)$. Then

$$m(x) = \alpha\lambda^{-1}\int_0^{\lambda x} u^{-\alpha}e^u \int_0^u t^{\alpha-1}e^{-t}\,dt\,du$$

and

$$\psi(s,x) = H(\alpha\frac{s}{s+\lambda}, \alpha; (\lambda+s)x).$$

where $H(a,b;x) = \sum_{k=0}^{\infty} \dfrac{(a)_k}{(b)_k}\dfrac{x^k}{k!}$ is a standard hypergeometric function.

4.3 The Cramér-Lundberg model in risk theory

The classical risk-reserve process in the Cramér-Lundberg model is given by

$$R_t = t - \sum_{k=1}^{N_t} U_k$$

where the claims U_k are independent, have a common distribution function B, and N_t is a Poisson process with intensity λ which is independent of the U_k. Let $\underline{R}_t = \inf_{s \le t} R_s$ and consider the reflected process

$$X_t = R_t - \underline{R}_t.$$

X_t can be interpreted as a risk-reserve process, where successive ruins are ignored. It is easy to see that X_t is an MGCP with $\lambda(x) = \lambda$ and $\mu_y(u) = 1 - B(y - u)$ and

$$m'(y) \quad = \quad 1 + \lambda \int_0^y m'(u)(1 - B(y - u))\, du = 1 + (m' * B_\lambda)(y),$$

where $B_\lambda(x) = \lambda \int_0^x (1 - B(u))\, du$ and $*$ denotes convolution. In what follows we write $\mu_n = \int_0^\infty u^n\, dB(u)$.

The above renewal equation has the unique solution

$$m'(x) = \sum_{k=0}^{\infty} B_\lambda^{k*}(x).$$

Therefore the asymptotic behavior of $m(y)$ can be deduced from known results in renewal theory.

Theorem 8. *1. If $\lambda\mu_1 = 1$ and $\mu_2 < \infty$, then $m(y) \sim \dfrac{1}{\lambda\mu_2} \cdot y^2$.*

2. If $1 < \lambda\mu_1 < \infty$ or $\lambda\mu_1 = \infty$ and $\int_0^\infty e^{-\delta x}\, dB_\lambda(x) = 1$ for some real δ, then

$$m(y) \sim \frac{e^{\gamma y}}{\gamma^2 \int_0^\infty u e^{-\gamma u}\, dB_\lambda(u)},$$

where $\gamma > 0$ is the solution of $\int_0^\infty e^{-\gamma x}\, dB_\lambda(x) = 1$. In this case we have

$$M_t / m^{-1}(t) \xrightarrow{d} 1.$$

3. If $\lambda\mu_1 < 1$, then $m(y) \sim \dfrac{y}{1 - \lambda\mu_1}$. In addition, if there is a solution β of the equation $\int_0^\infty e^{\beta x}\, dB_\lambda(x) = 1$ with $\int_0^\infty u e^{\beta u}\, dB_\lambda(u) < \infty$, then $|\dfrac{y}{1 - \lambda\mu_1} - m(y)|$ tends to a constant as $y \to \infty$.

Proof. The three cases follow from Propositions 6.1 and 7.2 and Theorem 7.1 in [3]. Note that

$$\int_0^\infty u^n\, dB_\lambda(u) = \lambda \int_0^\infty u^n (1 - B(u))\, du = \frac{\lambda}{n+1} \int_0^\infty u^{n+1}\, dB(u) = \frac{\lambda\mu_{n+1}}{n+1}. \quad \square$$

Regarding the Laplace transform of τ_y, the equation for ψ is given by

$$\psi'(s,y) \quad = \quad s\psi(s,y) + \lambda \int_0^y \psi'(s,y)(u)(1-B(y-u))\,du = s\psi(s,y) + (\psi(s,\cdot)' * B_\lambda)(y)$$

and does not seem to be solvable in general. However, for the transform

$$\Psi_s(t) = \int_0^\infty e^{-tx}\psi(s,dx)$$

there is a nice explicit formula in terms of the Laplace transform $\beta(t) = \int_0^\infty e^{-tx}\,dB(x)$ of B. We obtain

$$\Psi_s(t) \quad = \quad s\frac{\Psi_s(t)+1}{t} + \lambda\Psi_s(t)\frac{1-\beta(t)}{t}.$$

Hence,

$$\Psi_s(t) = \frac{s}{t-\lambda(1-\beta(t))-s}.$$

5 Appendix: regular and rapid variation

We state here some useful results about regular variation, which are needed in this paper. For further information regarding regular variation the reader is referred to the comprehensive monograph [5].

A function $f:[0,\infty)\to[0,\infty)$ is *r*egularly varying with index $a\in[0,\infty]$ if for all $\lambda>0$

$$\frac{f(\lambda y)}{f(y)} \to \lambda^a \quad , \quad y\to\infty. \tag{33}$$

In this case we write $f\in\mathcal{R}_a$. f is called rapidly varying if $a=\infty$ and slowly varying if $a=0$. The convergence in (33) is uniform for

$$\lambda \quad \in \quad \begin{cases} [c_1,\infty),\ c_1>0 & ,\ a<0 \\ [c_1,c_2],\ c_1>0 & ,\ a=0 \\ (0,c_2],\ c_2>0 & ,\ a>0 \\ (0,c_1)\cup(c_2,\infty),\ c_1<1<c_2 & ,\ a=\infty \end{cases} \tag{34}$$

f is regularly varying with index $a<\infty$ if and only if f is of the form

$$f(x) \sim c\cdot\exp\Big(\int_1^x U(w)\,dw\Big) \tag{35}$$

where $c > 0$ and $wU(w) \to a$. On the other hand, if

$$f(x) \sim c(x) \cdot \exp\Big(\int_1^x V(w)\, dw \Big), \tag{36}$$

where $c(x)$ is non-decreasing and $wV(w) \to \infty$ as $w \to \infty$, then f is rapidly varying. For f to be rapidly varying it is sufficient to show that $f(\lambda y)/f(y) \to \infty$ for all $\lambda > 1$, or that

$$\frac{xf'(x)}{f(x)} \to \infty \ , \quad x \to \infty.$$

If $f \in \mathcal{R}_a$ with index $a > 0$ and is increasing, then its inverse, denoted by f^{-1}, is regularly varying with index $1/a$ and vice versa, where we agree to understand $1/\infty = 0$ (see [5], Theorems 1.5.12 and 2.4.7). Finally, Karamata's theorem (see [5], Section 1.6) clarifies the behavior of the integral of a function in $\mathcal{R}_a$.

Theorem 9. *Let $f \in \mathcal{R}_a$ and $F(x) = \int_0^x f(w)\, dw$.*

1. *If f is locally bounded and $a > -1$, then $F(x) \sim \dfrac{x}{a+1} f(x)$.*
2. *If $a = -1$ and $xf(x)$ is locally integrable, then $x \mapsto \int_0^x f(u)\, du$ is slowly varying and $\int_0^x f(u)\, du/(xf(x)) \to \infty$. If additionally $\int_0^\infty f(u)\, du < \infty$, then $x \mapsto \int_x^\infty f(u)\, du < \infty$ is slowly varying.*
3. *If $a < -1$, then $\int_x^\infty f(u)\, du < \infty$ for large x and $\int_x^\infty f(u)\, du \sim \dfrac{xf(x)}{|a+1|}$.*
4. *If $a = \infty$, then $F \in \mathcal{R}_\infty$.*

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