

Finite deformation theory of hierarchically arranged porous solids - I. Balance of mass and momentum.

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FINITE DEFORMATION THEORY OF HIERARCHICALLY ARRANGED POROUS SOLIDS—I. BALANCE OF MASS AND MOMENTUM

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Abstract—A finite deformation theory of porous solids, including an hierarchical arrangement of the pores is presented. The momentum balance equation and the mass balance equation are derived by means of a formal averaging procedure. The procedure transforms the discrete network of pores into a continuum, without sacrificing essential information about orderly intercommunication of the pores. The distinction between different hierarchical levels of pores is achieved by means of a hierarchical parameter. The theory has applications, particularly in the field of the mechanics of blood perfused soft tissues, where the distinction between arterioles, capillaries and venules is essential for a correct quantification of regional blood perfusion of the tissue. Deformations are assumed quasi-static. Both solid and fluid are assumed incompressible.

INTRODUCTION

The experiments of Darcy and Ritter [1] are generally considered as the starting point of the theory of flow through porous media. Terzaghi introduced the principle of effective stress which combined with Darcy's law opened the way to modelling of consolidation phenomena in porous media. Biot [2] has generalized the consolidation theory to three-dimensional deformation of elastic porous media and later to viscoelastic media and media exhibiting finite deformation. Conservation of momentum in a biphasic porous medium is of the form:

$$\nabla \cdot \sigma^{\text{eff}} - \nabla \cdot p = 0 \quad (1)$$

with σ^{eff} the effective stress tensor and p the hydrodynamic pressure. The pressure and stress as used in equation (1) are not measured at the level of the individual pore, grain or fiber but rather as averages over a number of pores, grains or fibers [3]. The widespread application of basic equations of porous media mechanics to areas far beyond those they were designed for call for thorough experimental and theoretical verification. The need for theoretical verification has led to the setting up of a mathematical theory which allows the derivation of macroscopic laws—such as Darcy's law—from a law valid on the microscopic level of the individual pore. In this context, the averaging procedure developed by Slattery [4] and Whitaker [5] plays an important role. The development of mixture theory by Truesdell [6] and others has embedded porous media theories in a broader scope. Bowen [7, 8] derived finite deformation equations for compressible and incompressible porous solids saturated with N immiscible fluids from mixture theory. An overview of developments in porous media theories up to 1983 is found in Bedford and Drumheller [9]. Wilson and Aifantis [10] analysed fractured porous media in which two intercommunicating fluids saturate the solid. Recently, Bai and Roegiers [11] extended this two porosity model to include thermal effects.

In this paper the mathematical micro-macro transformation theory or formal averaging procedure, developed by Whitaker [5] and Slattery [4], is applied to the specific situation where the pores of the medium are arranged in an hierarchical sequence. An example of such an

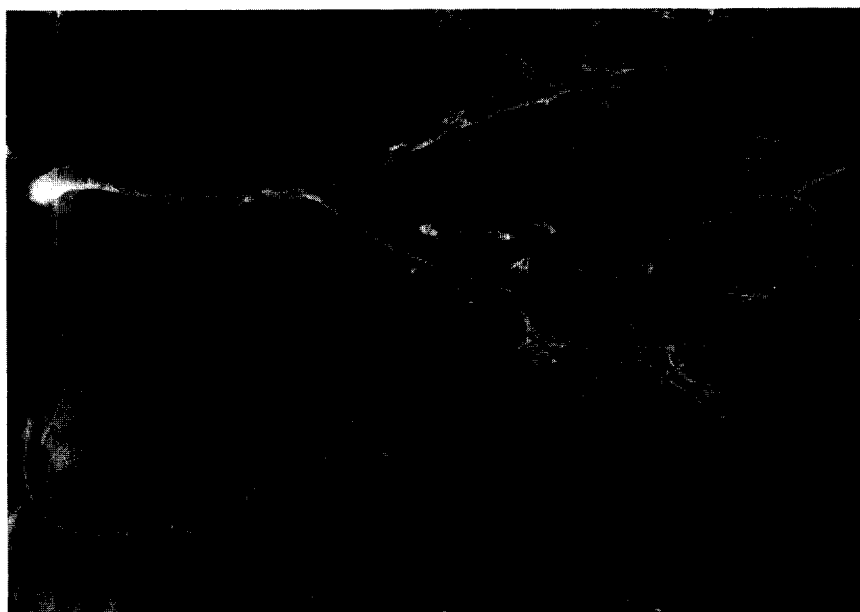


Fig. 1. Scanning electron micrograph of coronary artery branches penetrating into the heart wall of a dog.

arrangement is the microcirculatory bed of biological tissues (Fig. 1). In a microcirculatory bed, blood flows from arteries to arterioles, to capillaries, venules and veins. Darcy's law as such is not able to describe microcirculatory flow. The very definition of pressure and flow as averages over a number of pores, makes it impossible to distinguish between arterial, capillary and venous pressures and flows. This is the reason why a different set of macroscopic law is developed in which pressure and flow are selectively averaged according to the prevailing hierarchical pore structure. This leads to averaged equations of balance of momentum and balance of mass. The equations are valid for an incompressible porous solid saturated with a continuous spectrum of intercommunicating incompressible fluids. In the companion paper an extended Darcy equation and constitutive relationships are derived. The present papers are restricted to the theoretical derivation of the governing equations. The first steps towards experimental verification of the present theory in the limiting case of a flow through a rigid porous medium are presented elsewhere [12, 13].

DEFINITIONS

The averaging procedure

Let r be a representative elementary volume of the deformed medium at time t . The volume r is centered around a point P . The point P defines a position vector $\underline{x} = O\vec{P}$ in which O is a fixed origin. The volume r is shared by solid (r^S) and fluid (r^F). The characteristic length of r is chosen sufficiently large so as to provide a fair continuum representation of all (discontinuous) properties of the solid and the fluid phase, and sufficiently small so as to avoid that macroscopic variations of these properties throughout the domain are smoothed out. The current volume fraction n^X of phase r^X in r is defined as:

$$n^X = \frac{r^X}{r}. \quad (2)$$

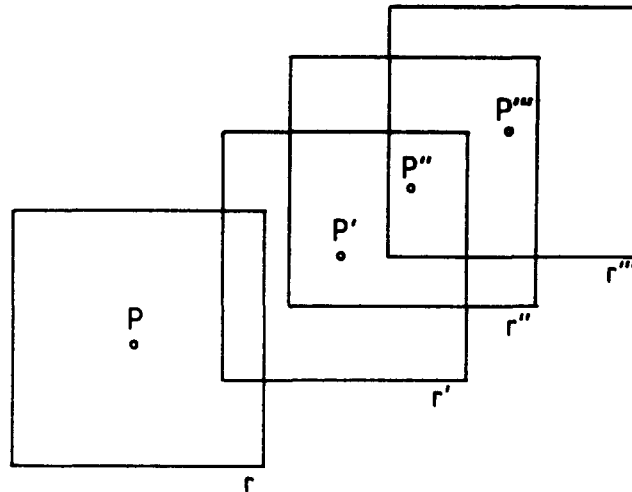


Fig. 2. The averaging procedure transforms the porous medium into a continuum. P, P', P'' and P''' are different points of the domain. With each of these points corresponds a different elementary volume.

Let f be some property, pertaining only to phase r^X of r . Then the real-volume average of f is defined as:

$$\langle f \rangle_{\underline{x}}^* = \frac{1}{r^X} \int_{r^X} f \, dr \tag{3}$$

and the bulk-volume average according to:

$$\langle f \rangle_X = \frac{1}{r} \int_{r^X} f \, dr = n^X \langle f \rangle_{\underline{x}}^* \tag{4}$$

These averages can be viewed as point macroscopic quantities associated with the centroid of r , which may lie in any phase of r . $\langle f \rangle$ and $\langle f \rangle^*$ are defined at each point in a fictitious continuum, and their values may thus change from point to point even within a given volume r . This can be easily understood if we consider that each point of the continuum is the centroid of a different elementary volume r (Fig. 2). All real-volume averages and bulk-volume averages are assumed continuous functions of the position vector \underline{x} of the centroid of r . If we chose the position vector of the material particles of the phase r^X as property f in equation (2), we obtain the average position vector of the phase r^X in r . In this theory it is assumed that at any time t , and at any position \underline{x} of the centroid of r , the average position vector of the phase r^X equals the position vector \underline{x} of the centroid of r :

$$\langle \underline{x} \rangle_{\underline{x}}^* = \underline{x} \tag{5}$$

This assumption implies homogeneous distribution of the different constituents in the volume r and thus restricts the applicability of the theory to the case where the volume fractions n^X do not change significantly along the characteristic length of r . Provided that a similar assumption holds for the averaged quantities $\langle f \rangle_{\underline{x}}^*$ we can show that the real-volume average of a real volume average equals the real-volume average itself:

$$\langle \langle f \rangle_{\underline{x}}^* \rangle_{\underline{x}}^* = \langle f \rangle_{\underline{x}}^* \tag{6}$$

If two properties f and g are statistically uncorrelated within the phase r^X , we can write:

$$\langle fg \rangle_{\underline{x}}^* = \langle f \rangle_{\underline{x}}^* \langle g \rangle_{\underline{x}}^* \tag{7}$$

$$\langle fg \rangle_X = \langle f \rangle_X \langle g \rangle_X = \langle f \rangle_{\underline{x}}^* \langle g \rangle_X \tag{8}$$

An averaged quantity, which is of particular interest in this study is the average initial position

vector \underline{X} of the solid. Consider all the solid particles in the volume r at time t . At time $t = 0$, each of these particles had an initial position $\underline{x}|_{t=0}$. These initial position vectors can be averaged over the solid phase of r :

$$\underline{X} = \langle \underline{x}|_{t=0} \rangle_s^* \quad (9)$$

It is assumed that there exists a one-to-one correspondence between the current position vector \underline{x} of the centroid of r and the average initial position vector \underline{X} :

$$\underline{x} = \chi(\underline{X}). \quad (10)$$

From now on, the vector \underline{X} will simply be named initial position vector, and the vector \underline{x} current position vector.

Displacement, velocity and strain

The displacement vector is defined as:

$$\underline{u} = \underline{x} - \underline{X} \quad (11)$$

and the deformation tensor is:

$$\underline{F} = ({}^\circ\nabla\underline{x})^C = ({}^\circ\nabla\underline{u})^C + \underline{I} \quad (12)$$

in which:

$${}^\circ\nabla = \frac{\partial}{\partial \underline{X}} \quad (13)$$

are the gradient operators with respect to the initial configuration. The relative volume change of the mixture is given by:

$$J = \frac{r}{\chi^{-1}(r)} = \det \underline{F}. \quad (14)$$

Three different types of time derivatives are used:

—the partial time derivative $\frac{\partial}{\partial t}$, i.e. the time derivative for an observer fixed in space ($\underline{x} = \text{constant}$).

—the local material time derivative $\frac{d}{dt} := \dots$, i.e. the time derivative for an observer fixed to the local material. This material can be fluid or solid according to the phase in which we reside.

—the average time derivative $\frac{D}{Dt} := \dots$, i.e. the time derivative for an observer fixed to the average position of the solid.

The local material time derivative is linked to the partial time derivative according to:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{\underline{x}} \cdot \nabla \quad (15)$$

in which $\dot{\underline{x}}$ represents the velocity of the local material particle. The local material time derivative and the average time derivative are related as follows:

$$\frac{D}{Dt} = \left\langle \frac{d}{dt} \right\rangle_s^* \quad (16)$$

The physical meaning of the averaging signs in equations (16) is the averaging of the time

derivatives with respect to different observers, each of them following individual solid particles. The average velocity of the solid can thus be written as:

$$\dot{\underline{x}} = \frac{D}{Dt} \underline{x} = \frac{D}{Dt} \underline{x} = \langle \dot{\underline{x}} \rangle_S^* \tag{17}$$

The hierarchic parameter, pressure and flow

If averaging is to be performed without loss of the hierarchy between pores, a parameter should be defined, quantifiable for each pore, and specifying the hierarchical level to which the pore belongs. In case of soft tissue, we would call such a parameter an arteriovenous parameter as it differentiates between arterial, arteriolar, capillary, venular and venous compartments. In Huyghe *et al.* [12] an hierarchic parameter x_0 is defined in terms of vessel diameters. In case of a deformable medium, these diameters would be measured in some definite reference state. Different pore compartments are defined as “hierarchic parameter intervals”. In particular, we can define an infinitesimal compartment $r^f(x_0) dx_0$ as the volume of all fluid with an hierarchic parameter between x_0 and $x_0 + dx_0$ in the deformed elementary volume r . We know that:

$$\int_{\alpha}^{\beta} r^f(x_0) dx_0 = \text{total fluid volume in } r = r^F \tag{18}$$

where r^f is the fluid volume per unit hierarchic parameter, α is the lower bound of x_0 in r and β is the upper bound of x_0 in r .

To each infinitesimal compartment $r^f(x_0) dx_0$ corresponds a volume fraction n^f per unit x_0 :

$$n^f = \frac{r^f(x_0) dx_0}{r dx_0} = \frac{r^f}{r} \tag{19}$$

It follows that:

$$\int_{\alpha}^{\beta} n^f dx_0 = n^F = \frac{r^F}{r} \tag{20}$$

Any property f of the fluid can be averaged over the infinitesimal fluid compartments $r^f(x_0) dx_0$:

$$\langle f^f \rangle_t^* = \frac{1}{r^f(x_0) dx_0} \int_{r^f(x_0) dx_0} f dr \tag{21}$$

$$\langle f \rangle_t = \frac{1}{r} \int_{r^f(x_0) dx_0} f dr \tag{22}$$

These different averages interrelate according to:

$$\langle f \rangle_t = n^f dx_0 \langle f \rangle_t^* \tag{23}$$

e.g., we can average the local fluid pressure p^f :

$$\langle p^f \rangle_t^* = \frac{1}{r^f(x_0) dx_0} \int_{r^f(x_0) dx_0} p^f dr \tag{24}$$

There is a definite fluid flow corresponding to each compartment $r^f(x_0) dx_0$. Fluid flow is a vectorial variable. At point \underline{x} of the fluid phase, and at time t , the local velocity of fluid with respect to an observer fixed in space is $\dot{\underline{x}}(\underline{x}, t)$. The local velocity of fluid with respect to an observer fixed to the solid surrounding \underline{x} is:

$$\underline{v} = \dot{\underline{x}} - \langle \dot{\underline{x}} \rangle_S^* \tag{25}$$

The spatial fluid flow \underline{q} is defined according to:

$$\underline{q}(x_0, \underline{x}, t) = \frac{n^f(x_0, \underline{x}, t)}{r^f(x_0)} \int_{r^f(x_0)} \underline{v}(\underline{x}, t) \, d\mathbf{r}$$

or

$$\underline{q} = n^f \langle \underline{v} \rangle_t^* \quad (26)$$

Fluid particles move from one compartment to the other. As the hierarchic parameter x_0 is different for different compartments, it is clear that the x_0 -value belonging to the compartment in which the fluid particle currently resides, changes in time. We can thus define a material time derivative \dot{x}_0^f for an observer fixed to the fluid particle. This derivative quantifies the motion of a fluid particle from one compartment to another. In analogy to equation (26), the hierarchic flow q_0 is defined according to

$$q_0 = n^f \langle \dot{x}_0 \rangle_t^* \quad (27)$$

From equations (19), (21) and (27) we know that:

$$q_0 = \frac{1}{r} \int_{r^f(x_0)} \dot{x}_0 \, d\mathbf{r} \quad (28)$$

It is easy to show that q_0 is the flow through the surface characterized by an arteriovenous parameter x_0 in the volume r . Indeed, this flow is:

$$\frac{1}{r} \int_{\text{AVP}=\underline{x}_0} \underline{v} \cdot d\mathbf{a} \quad (29)$$

The flux through an elementary surface element $d\mathbf{a}$ is rewritten in terms of an elementary volume $d\mathbf{r}$ built on top of it:

$$d\mathbf{a} \, dx_0 = \nabla x_0 \, d\mathbf{r} \quad (30)$$

reducing expression (29) to:

$$\frac{1}{r} \int_{\text{AVP}=\underline{x}_0} \underline{v} \cdot d\mathbf{a} = \frac{1}{r} \int_{r^f(x_0)} \underline{v} \cdot \nabla x_0 \, d\mathbf{r} = q_0 \quad (31)$$

which was to be shown. It will be useful to write the fluid flows q_0 and \underline{q} as one four-dimensional flow vector:

$$\mathbf{q}^4 = \begin{pmatrix} q_0 \\ \underline{q} \end{pmatrix} \quad (32)$$

THE SLATTERY-WHITAKER AVERAGING THEOREM

An essential property of averages defined in equations (3, 4) is the Slattery-Whitaker theorem. Proved simultaneously and independently by Slattery [4] and Whitaker [5], it relates the average of the gradient to the gradient of the average.

THEOREM. If

I f is a scalar-valued function defined in phase X of volume v

II $r(\underline{x}) \subset v$ is an averaging volume obtained by translating a fixed volume r to a position such that $\underline{x} \in v$ is the centroid of $r(\underline{x})$ then it holds that:

$$\langle \nabla f \rangle = \langle f \rangle + \frac{1}{r} \int_{\partial r^X \cap (-dr)} f \, d\mathbf{a} \quad (33)$$

in which $\partial r^X \cap (\sim dr)$ is the interface between phase r^X and other phases of the volume r , and $d\mathbf{a}$ is a vector of size da , perpendicular to the surface da of the interface, pointing away from the phase r^X . It is obvious that similar expressions hold for the divergence of a vector-valued function \underline{f} . Equation (33) will be the key step in the formal averaging procedures described in the next sections. The derivation of this averaging theorem is found in Slattery [14, pp. 192–196]. An essential condition for the validity of equation (33) is that the averaging volume is kept constant in size and shape, and is not rotated when translated from one point of the domain to another.

CONSERVATION OF MOMENTUM

Bulk volume equilibrium

Equilibrium of an elementary volume r of the deformed mixture is provided if and only if:

$$\int_{\partial r} \underline{\sigma} \cdot d\mathbf{a} = \int_r \rho \underline{\ddot{x}} dr \tag{34}$$

where $\underline{\sigma}$ is the Cauchy stress tensor, ρ is the local density and ∂r is the boundary of r .

It should be understood that in equation (34) each infinitesimal volume dr is occupied only by one phase. The stress $\underline{\sigma}$ in the volume dr is the regular Cauchy stress tensor as we define it in a single phase material. Neglecting inertial forces and subdividing the r -space, equation (34) transforms into:

$$\int_{\partial r^S \cap \partial r} \underline{\sigma} \cdot d\mathbf{a} + \int_{\partial r^F \cap \partial r} \underline{\sigma} \cdot d\mathbf{a} = 0. \tag{35}$$

I II

Term II transforms into:

$$\int_{\partial r^F \cap \partial r} \underline{\sigma} \cdot d\mathbf{a} = - \int_{\partial r^F \cap \partial r} p^f d\mathbf{a} + \int_{\partial r^F \cap \partial r} \underline{\tau}^f \cdot d\mathbf{a} \tag{36}$$

where p^f is the fluid pressure and $\underline{\tau}^f$ is the fluid shear stress.

We assume that the solid phase is incompressible. Therefore, it is natural to split the stress in the solid phase into a hydrodynamic pressure p^s , independent from deformation and an effective stress $\underline{\sigma}^s$ resulting from deformation of the solid:

$$\underline{\sigma} = -p^s \underline{I} + \underline{\sigma}^s. \tag{37}$$

Hence:

$$\int_{\partial r^S \cap \partial r} \underline{\sigma} \cdot d\mathbf{a} = - \int_{\partial r^S \cap \partial r} p^s d\mathbf{a} + \int_{\partial r^S \cap \partial r} \underline{\sigma}^s \cdot d\mathbf{a}. \tag{38}$$

Reordering, assembling and dividing all the terms of the equilibrium by the volume r results in:

$$- \frac{1}{r} \int_{\partial r^S \cap \partial r} p^s d\mathbf{a} + \frac{1}{r} \int_{\partial r^S \cap \partial r} \underline{\sigma}^s \cdot d\mathbf{a} + \frac{1}{r} \int_{\partial r^F \cap \partial r} \underline{\tau}^f \cdot d\mathbf{a} - \frac{1}{r} \int_{\partial r^F \cap \partial r} p^f d\mathbf{a} = 0. \tag{39}$$

Applying subsequently the divergence theorem and the Slattery–Whitaker theorem to the second term of equation (39), yields:

$$\begin{aligned} \frac{1}{r} \int_{\partial r^S \cap \partial r} \underline{\sigma}^s \cdot d\mathbf{a} &= \frac{1}{r} \int_{\partial r^S} \underline{\sigma}^s \cdot d\mathbf{a} - \frac{1}{r} \int_{\partial r^S \cap (\sim \partial r)} \underline{\sigma}^s \cdot d\mathbf{a} = \frac{1}{r} \int_{r^S} \nabla \cdot \underline{\sigma}^s dr - \frac{1}{r} \int_{\partial r^S \cap (\sim \partial r)} \underline{\sigma}^s \cdot d\mathbf{a} \\ &= \langle \nabla \cdot \underline{\sigma}^s \rangle_S - \frac{1}{r} \int_{\partial r^S \cap (\sim \partial r)} \underline{\sigma}^s \cdot d\mathbf{a} = \nabla \cdot \langle \underline{\sigma}^s \rangle_S. \end{aligned} \tag{40}$$

In a similar way the first and third terms of equation (39) are transformed:

$$-\frac{1}{r} \int_{\partial r^s \cap \partial r} p^s \, d\mathbf{a} = -\nabla \langle p^s \rangle_s \tag{41}$$

$$\frac{1}{r} \int_{\partial r^f \cap \partial r} \mathbf{t}^f \cdot d\mathbf{a} = \nabla \cdot \langle \mathbf{t}^f \rangle_F. \tag{42}$$

The last term of equation (39) needs more attention. The contribution of each compartment $r^f(x_0) \, dx_0$ to the last term of equation (39) is $-1/r \int_{(\partial r^f \, dx_0) \cap \partial r} p^f \, d\mathbf{a}$. Applying subsequently the divergence theorem and the Slattery–Whitaker theorem, yields:

$$\begin{aligned} -\frac{1}{r} \int_{(\partial r^f \, dx_0) \cap \partial r} p^f \, d\mathbf{a} &= -\frac{1}{r} \int_{\partial r^f \, dx_0} p^f \, d\mathbf{a} + \frac{1}{r} \int_{(\partial r^f \, dx_0) \cap (\sim \partial r)} p^f \, d\mathbf{a} \\ &= -\langle \nabla p^f \rangle + \frac{1}{r} \int_{(\partial r^f \, dx_0) \cap (\sim \partial r)} p^f \, d\mathbf{a} \\ &= -\nabla \langle p^f \rangle_f = -\nabla (n^f \langle p^f \rangle_f^*) \, dx_0. \end{aligned} \tag{43}$$

Integration of (43) throughout the x_0 -range yields for the last term of equations (39):

$$\begin{aligned} \frac{1}{r} \int_{\partial r^f \cap \partial r} p^f \, d\mathbf{a} &= -\int_{\alpha}^{\beta} \nabla (n^f \langle p^f \rangle_f^*) \, dx_0 \\ &= -\nabla \int_{\alpha}^{\beta} n^f \langle p^f \rangle_f^* \, dx_0. \end{aligned} \tag{44}$$

Equations (40)–(42) and (44) are fitted into equations (39), using equation (23):

$$\nabla \cdot [\langle \sigma^s \rangle_s + \langle \mathbf{t}^f \rangle_F - I \int_{\alpha}^{\beta} n^f (\langle p^f \rangle_f^* - \langle p^s \rangle_s^*) \, dx_0] - \nabla \langle p^s \rangle_s^* = 0. \tag{45}$$

We define the effective stress of the mixture as:

$$\sigma^{\text{eff}} = \langle \sigma^s \rangle_s + \langle \mathbf{t}^f \rangle_F - I \int_{\alpha}^{\beta} n^f (\langle p^f \rangle_f^* - \langle p^s \rangle_s^*) \, dx_0. \tag{46}$$

Therefore equation (45) transforms into:

$$\nabla \cdot \sigma^{\text{eff}} - \nabla \langle p^s \rangle_s^* = 0. \tag{47}$$

The momentum balance equation (47) is identical to the equation derived for biphasic fluid–solid mixtures (1). The effective stress of biphasic mixtures usually only depends on the history of the strain tensor while in the case of a hierarchical fluid–solid mixture one should expect the effective stress as defined in equation (46) to depend on the distribution of fluid volume over the hierarchical range.

CONSERVATION OF MASS

Mass balance of the fluid phase

As the fluid is assumed incompressible, conservation of mass of fluid is equivalent to conservation of fluid volume. The change in volume of an infinitesimal fluid compartment $r^f(x_0) \, dx_0$ is therefore equal to the fluid volume which flows through the boundary $\partial r^f(x_0) \, dx_0$:

$$\frac{\partial}{\partial t} r^f(x_0) \, dx_0 + \int_{\partial r^f(x_0) \, dx_0} \dot{\mathbf{x}} \cdot d\mathbf{a} = 0. \tag{48}$$

Dividing by the volume r and using Green's theorem, we find

$$\frac{\partial n^f}{\partial t} dx_0 + \langle \nabla \cdot \dot{x} \rangle_f = 0. \tag{49}$$

The Slattery–Whitaker theorem is applied to the last term of equation (49):

$$\begin{aligned} \langle \nabla \cdot \dot{x} \rangle_f &= \nabla \cdot \langle \dot{x} \rangle_f + \frac{1}{r} \int_{\partial(r^f dx_0) \cap (\sim \partial r)} \dot{x} \cdot d\mathbf{a} \\ &= \nabla \cdot (n^f \langle \dot{x} \rangle_f^*) dx_0 + \frac{1}{r} \int_{\partial(r^f dx_0) \cap (\sim \partial r)} \mathbf{v} \cdot d\mathbf{a} \end{aligned} \tag{50}$$

with

$$* \mathbf{v} = \dot{x} - \langle \dot{x} \rangle_s^*, \text{ the fluid velocity relative to the surrounding solid.}$$

$$* \int_{\partial(r^f dx_0) \cap (\sim \partial r)} \langle \dot{x} \rangle_s^* \cdot d\mathbf{a} = 0 \text{ due to the substantial constancy of } \langle \dot{x} \rangle_s^* \text{ along the characteristic length of } r.$$

The boundary surface $\partial(r^f dx_0) \cap (\sim \partial r)$ of the infinitesimal fluid compartment is subdivided into a solid–fluid interface, a fluid–fluid interface corresponding to $HP = x_0$ and a fluid–fluid interface corresponding to $HP = x_0 + dx_0$ ($HP =$ hierarchic parameter). Hence, the second term of equation (50) can be split into:

$$\frac{1}{r} \int_{\partial(r^f dx_0) \cap (\sim \partial r)} \mathbf{v} \cdot d\mathbf{a} = \frac{1}{r} \int_{S-F \text{ interface}} \mathbf{v} \cdot d\mathbf{a} + \frac{1}{r} \int_{HP=x_0} \mathbf{v} \cdot d\mathbf{a} + \frac{1}{r} \int_{HP=x_0+dx_0} \mathbf{v} \cdot d\mathbf{a} \tag{51}$$

Mass exchange between fluid and solid phase is neglected:

$$\frac{1}{r} \int_{S-F \text{ interface}} \mathbf{v} \cdot d\mathbf{a} = 0 \tag{52}$$

The last two terms of equations (51) need more attention. A fluid particle crosses the surface $HP = x_0$. Its velocity relative to the surface is the velocity \mathbf{v} defined in equation (25), because the surface is fixed to the solid. The rate of exchange of the hierarchical parameter x_0 for an observer fixed to the fluid particle equals:

$$\dot{x}_0 = \mathbf{v} \cdot \nabla x_0. \tag{53}$$

Considering the elementary volume δr ($\ll dr$) obtained by translating the elementary surface $d\mathbf{a}$ from $HP = x_0$ to $HP = x_0 + \delta x_0$, we write:

$$d\mathbf{a} = -\frac{\delta r}{\delta x_0} \nabla x_0 \tag{54}$$

and similarly for the surface $HP = x_0 + dx_0$:

$$d\mathbf{a} = \frac{\delta r}{\delta x_0} \nabla x_0. \tag{55}$$

Introducing equations (52), (53), (54) and (55) into equation (51), yields

$$\begin{aligned}
 \frac{1}{r} \int_{\partial(r^f dx_0) \cap (\sim \partial r)} \underline{v} \cdot d\underline{q} &= -\frac{1}{r \delta x_0} \int_{r^f(x_0) \delta x_0} \dot{x}_0 \delta r + \frac{1}{r \delta x_0} \int_{r^f(x_0 + dx_0) \delta x_0} \dot{x}_0 \delta r \\
 &= -\frac{1}{\delta x_0} \langle \dot{x}_0(x_0) \rangle_t + \frac{1}{\delta x_0} \langle \dot{x}_0(x_0 + dx_0) \rangle_t \\
 &= -\frac{1}{\delta x_0} n^f(x_0) \delta x_0 \langle \dot{x}_0(x_0) \rangle_t^* + \frac{1}{\delta x_0} n^f(x_0 + dx_0) \delta x_0 \langle \dot{x}_0(x_0 + dx_0) \rangle_t^* \\
 &= \frac{\partial}{\partial x_0} (n^f \langle \dot{x}_0 \rangle_t^*) dx_0.
 \end{aligned} \tag{56}$$

Substituting equation (56) into equation (57) and equation (50) into equation (49), yields:

$$\frac{\partial n^f}{\partial t} dx_0 + \underline{\nabla} \cdot (n^f \langle \dot{\underline{x}} \rangle_t^*) dx_0 + \frac{\partial}{\partial x_0} (n^f \langle \dot{x}_0 \rangle_t^*) dx_0 = 0$$

or

$$\frac{\partial n^f}{\partial t} + \underline{\nabla}^4 \cdot (n^f \langle \dot{\underline{x}}^4 \rangle_t^*) = 0 \tag{57}$$

with

$$\underline{\nabla}^4 = \begin{pmatrix} \frac{\partial}{\partial x_0} \\ \frac{\partial}{\partial \underline{x}} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x_0} \\ \underline{\nabla} \end{pmatrix} \tag{58}$$

$$\underline{\underline{x}}^4 = \begin{pmatrix} x_0 \\ \underline{x} \end{pmatrix}. \tag{59}$$

Mass balance of the solid phase

In a similar way, it can be shown for the solid that:

$$\frac{\partial}{\partial t} (1 - n^F) + \underline{\nabla} \cdot ((1 - n^F) \langle \dot{\underline{x}} \rangle_t^*) = 0$$

or:

$$-\frac{\partial n^F}{\partial t} + \underline{\nabla} \cdot ((1 - n^F) \langle \dot{\underline{x}} \rangle_t^*) = 0. \tag{60}$$

Assumptions, similar to those mentioned for the fluid mass equation, are required here:

- incompressibility of the solid
- no mass exchange between solid and fluid phase.

Total mass balance

Equation (57) can be integrated over the x_0 -range:

$$\frac{\partial}{\partial t} \int_{\alpha}^{\beta} n^f dx_0 + \int_{\alpha}^{\beta} \underline{\nabla} \cdot (n^f \langle \dot{\underline{x}} \rangle_t^*) dx_0 + \int_{\alpha}^{\beta} \frac{\partial}{\partial x_0} (n^f \langle \dot{x}_0 \rangle_t^*) dx_0 = 0$$

or:

$$\frac{\partial n^F}{\partial t} + \int_{\alpha}^{\beta} \underline{\nabla} \cdot (n^f \langle \dot{\underline{x}} \rangle_t^*) dx_0 = 0. \tag{61}$$

Adding equation (61) to equation (60), yields the total mass balance equation:

$$\int_{\alpha}^{\beta} \nabla \cdot [n^f(\dot{x})^* - \langle \dot{x} \rangle^*] dx_0 + \nabla \cdot \langle \dot{x} \rangle^* = 0 \tag{62}$$

or, according to equations (25), (26) and (17),

$$\int_{\alpha}^{\beta} \nabla \cdot \underline{q} dx_0 + \nabla \cdot \underline{\hat{u}} = 0. \tag{63}$$

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NOMENCLATURE

Tensor notation

\underline{a}	vector in 3D space
$\underline{\hat{a}}^4$	vector in 4D space
$\underline{\hat{a}}$	second order tensor in 3D space
$\underline{\hat{a}}^4$	second order tensor in 4D space
\underline{ab}	dyadic product of the vectors \underline{a} and \underline{b}
$\underline{\hat{a}} \cdot \underline{\hat{b}}$	dot product of the vectors $\underline{\hat{a}}$ and $\underline{\hat{b}}$
$\underline{\hat{a}} \cdot \underline{\hat{b}}$	dot product of a second order tensor and a vector
$\underline{\hat{a}} \cdot \underline{\hat{b}}$	dot product of two second order tensors, such that
	$\forall \underline{c} (\underline{\hat{a}} \cdot \underline{\hat{b}}) \cdot \underline{c} = \underline{\hat{a}} \cdot (\underline{\hat{b}} \cdot \underline{c})$
$\ \underline{a}\ $	length of vector \underline{a}
$\underline{\hat{a}}^C$	conjugate of $\underline{\hat{a}}$
$\underline{\hat{a}}^{-1}$	inverse of $\underline{\hat{a}}$
$\det(\underline{\hat{a}})$	determinant of $\underline{\hat{a}}$
\underline{I}	unit second order tensor

Set notation

$A \cap B$	intersection of set A and set B
$A \cup B$	union of set A and set B
$\sim A$	complementary set of set A
$\forall a$	for all a

Specific symbols

da	elementary surface in current configuration
\underline{da}	vector of size a perpendicular to da
$\frac{d}{dt}$ or \cdot	local material time derivative
$\frac{D}{Dt}$ or \circ	average time derivative

$\frac{\partial}{\partial t}$	partial time derivative
∂V	boundary surface of volume V
\underline{F}	deformation tensor
\underline{J}	Jacobian
n^f	current fluid volume fraction per unit hierarchical parameter
n^F	current total fluid volume fraction (current porosity)
n^X	current volume fraction of phase X
p^f	local fluid pressure
p^s	local solid pressure
\underline{q}	spatial fluid flow vector
$\underline{\hat{q}}^4$	fluid flow vector
$\underline{\hat{q}}$	integrated fluid flow vector
$\underline{\hat{r}}$	representative volume in current configuration
r^f	fluid volume in r per unit hierarchic parameter
r^X	volume of phase X in r
t	time
\underline{u}	displacement vector
$\underline{\hat{v}}$	relative fluid velocity
$\underline{\hat{x}}_0$	hierarchic parameter
\underline{x}	current position vector
$\underline{\hat{X}}$	average initial position vector of the solid
$\underline{\hat{x}}^4$	$\begin{pmatrix} x_0 \\ x \end{pmatrix}$
$\underline{\hat{X}}^4$	$\begin{pmatrix} x_0 \\ X \end{pmatrix}$
χ	transformation from initial to current configuration
$\underline{\sigma}^s$	part of local Cauchy stress tensor due to deformation of the solid
$\underline{\sigma}$	local Cauchy stress tensor
$\underline{\sigma}^{eff}$	effective stress tensor of the mixture