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# Diffusion constants and martingales for senile random walks

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## Abstract

We derive diffusion constants and martingales for senile random walks with the help of a time-change. We provide direct computations of the diffusion constants for the time-changed walks. Alternatively, the values of these constants can be derived from martingales associated with the time-changed walks. Using an inverse time-change, the diffusion constants for senile random walks are then obtained via these martingales. When the walks are diffusive, weak convergence to Brownian motion can be shown using a martingale central limit theorem.

## 1 Introduction and general framework

In this paper we study random walks on  $\mathbb{Z}^d$  for dimensions  $d \geq 1$ , which can be viewed as time-changes of random walks that were named *senile reinforced* and *senile persistent* random walks in [4]. We will use this terminology also in this paper, although senile persistent random walks were originally introduced and studied under the name of *directionally reinforced* random walks in [6, 7]. We would like to note that the reinforcement of senile random walks is of a different kind than that of more traditional edge reinforced random walks, as introduced by Coppersmith and Diaconis [3]. For more details and discussion, we refer to the introductions and references in [4, 6, 7], and to the recent survey paper [8] on reinforced random processes.

Recurrence and transience properties of senile random walks were studied in the papers [4, 7], and scaling limits are identified in [5, 6]. In this paper, rather than taking the senile random walks themselves as our starting point, we start by studying other random walks that are later interpreted as time-changes of senile random walks. The idea of looking at these time-changed walks is not new, and has also been used in the mentioned references. However, this paper presents a different approach to identify the diffusion constants and weak limits of the walks under study, using mainly martingale techniques.

Indeed, below we will provide new, “direct” calculations of the diffusion constants for the time-changed random walks, and we show that these random walks are close to martingales (for the persistent case, this has also been observed

in [6]). Using martingale theory, we can then derive the diffusion constants for the senile random walks by an inverse time-change. This confirms that Theorem 2.5 in [4] holds under a slightly weaker moment condition, as conjectured by the authors. Finally, we will show that under appropriate conditions for which the walks are diffusive, weak convergence of senile random walks to Brownian motion follows from a martingale central limit theorem.

We will now introduce a general framework for the time-changed walks we want to study below. Generally, the walks are described by a sequence  $W = (W_1, W_2, \dots)$  of random variables taking values in  $\mathbb{Z}^d$ . For each  $n \in \mathbb{N} := \{1, 2, \dots\}$ , we will write  $W_n$  (the position at time  $n$ ) as the sum of  $n$  random steps, where the  $m$ th step ( $m \in \mathbb{N}$ ) has a direction  $D_m$  taking values in  $\{e_1, e_2, \dots, e_{2d}\}$ , the unit vectors of  $\mathbb{Z}^d$ , and a length  $L_m \in \{0, 1, 2, \dots\}$ .

Actually, for the single purpose of relating our walks to senile random walks later on, we will write each step length  $L_m$  as a function of a random variable  $T_m$  taking values in  $\mathbb{N} = \{1, 2, \dots\}$ . These variables  $T_m$  are i.i.d. (hence, so are the step lengths) and define the random time-change linking our random walks to senile random walks. Below, we will use the notation  $T$  for a generic variable distributed as any one of the  $T_m$ . The distribution of the random times  $T_m$  is specified in terms of a function  $f : \mathbb{N} \rightarrow [-1, \infty)$  (the reinforcement function) by  $\mathbb{P}(T \geq 1) = 1$  and

$$\mathbb{P}(T \geq k) = \prod_{l=1}^{k-1} \frac{1 + f(l)}{2d + f(l)} \quad \text{for } k = 2, 3, \dots \quad (1.1)$$

This specific form of the distribution of the  $T_m$  is introduced only to make the link with senile random walk. For now, we do not put any restrictions on the function  $f$ , but later on, we will require that either  $\mathbb{E}(T)$  is finite or both  $\mathbb{E}(T^2)$  and  $\mathbb{E}(T)$  are finite, depending on whether we consider the persistent or the reinforced case.

Thus, following the description above, we can write

$$W_n := \sum_{m=1}^n D_m L_m \quad \text{for all } n \in \mathbb{N}, \quad (1.2)$$

where the laws of the  $D_m$  and  $L_m$  are yet to be specified. In sections 2 and 3 we consider two specific instances of this general class of random walks, related to senile persistent and senile reinforced random walks, respectively. Our first aim will be to compute the *diffusion constants* for these walks, which for a general walk  $X = (X_1, X_2, \dots)$  is defined by

$$C_X := \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}(|X_n|^2), \quad (1.3)$$

provided the limit exists and is finite. To find the diffusion constants for the senile random walks, we will then make use of martingales associated with the time-changed walks, and these martingales will also be used to prove weak convergence to Brownian motion when the senile walks are diffusive.

## 2 The persistent case

We start with the persistent case, for which the definition of the walk is somewhat easier than in the reinforced case, but the analysis is harder. In this case, we take

$$L_m = T_m \quad \text{for all } m \in \mathbb{N}, \quad (2.1)$$

and the directions of different steps obey the rule that the direction at each step has to be different from the direction at the previous step, but all remaining choices of direction are equally likely. Formally, this means that the directions  $D_m$  satisfy

$$\mathbb{P}(D_1 = e_i) = \frac{1}{2d} \quad \text{for each } i = 1, 2, \dots, 2d, \quad (2.2)$$

and for all  $m \in \mathbb{N}$ ,

$$\mathbb{P}(D_{m+1} = e_i \mid D_m) = \frac{1}{2d-1} \mathbb{1}(D_m \neq e_i) \quad \text{for each } i = 1, 2, \dots, 2d, \quad (2.3)$$

where  $\mathbb{1}(A)$  is the indicator of the event  $A$ . Equations (2.1)–(2.3) completely specify the law of the random walk defined by (1.2). For the remainder of this section we will write  $W^p = (W_1^p, W_2^p, \dots)$  for this walk, where the superscript  $p$  is used to single out the *persistent* case studied here.

### 2.1 Direct calculation of the diffusion constant

We will now provide a direct calculation of the diffusion constant for the random walk  $W^p$  defined above. It will be clear from the computation that we have to require that  $\mathbb{E}(T^2) < \infty$  (which implies  $\mathbb{E}(T) < \infty$ ). The diffusion constant is then given by the following proposition.

**Proposition 2.1.** *Assume  $\mathbb{E}(T^2) < \infty$ . Then the diffusion constant of the random walk  $W^p$  is given by*

$$C^p := \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}(|W_n^p|^2) = \frac{d \mathbb{E}(T^2) - \mathbb{E}(T)^2}{d}.$$

*Proof.* It is easy to see that

$$\mathbb{E}(|W_n^p|^2) = n \mathbb{E}(T^2) + 2 \sum_{k=1}^{n-1} \sum_{m=1}^{n-k} \mathbb{E}(D_m \cdot D_{m+k} L_m L_{m+k}), \quad (2.4)$$

where for all  $m, k \geq 1$ , by independence of the step lengths,

$$\mathbb{E}(D_m \cdot D_{m+k} L_m L_{m+k}) = \mathbb{E}(T)^2 \mathbb{E}(D_m \cdot D_{m+k}). \quad (2.5)$$

Now note that on the event  $E_{mk} := \{D_{m+k-1} \cdot D_{m+k} = 0\}$ ,  $D_m \cdot D_{m+k}$  takes on the values  $\pm 1$  with equal probabilities, by (2.3). On the other hand, on the

complementary event  $E_{mk}^c$ , we have that  $D_{m+k} = -D_{m+k-1}$ . Therefore, using independence again,

$$\begin{aligned}\mathbb{E}(D_m \cdot D_{m+k}) &= \mathbb{E}(D_m \cdot D_{m+k} \mathbb{1}(E_{mk})) - \mathbb{E}(D_m \cdot D_{m+k-1} \mathbb{1}(E_{mk}^c)) \\ &= -\frac{1}{2d-1} \mathbb{E}(D_m \cdot D_{m+k-1}).\end{aligned}\tag{2.6}$$

Iterating this recursion relation, it follows that

$$\mathbb{E}(D_m \cdot D_{m+k}) = \left(\frac{-1}{2d-1}\right)^k.\tag{2.7}$$

Plugging this expression into (2.4), we obtain

$$\begin{aligned}\mathbb{E}(|W_n^p|^2) &= n \mathbb{E}(T^2) + 2 \mathbb{E}(T)^2 \sum_{k=1}^{n-1} (n-k) \left(\frac{-1}{2d-1}\right)^k \\ &= n \mathbb{E}(T^2) - \mathbb{E}(T)^2 \left(\frac{n}{d} + \frac{2d-1}{2d^2} \left[\left(\frac{-1}{2d-1}\right)^n - 1\right]\right).\end{aligned}\tag{2.8}$$

By (1.3), this equation identifies the value of the diffusion constant if we take the limit  $n \rightarrow \infty$ .  $\square$

## 2.2 Martingales for the persistent random walk

The purpose of this subsection is to show that the walk  $W^p$  is within bounded distance from a martingale at each step. More precisely, we will see that adding a correction of constant length to each position  $W_n^p$  gives us a martingale. In fact, the proof of Proposition 2.2 below identifies a second martingale by direct calculation, which can be used to provide an alternative derivation of the diffusion constant for the walk  $W^p$ .

To state our result, we introduce the filtration  $\{\mathcal{F}_n : n \in \mathbb{N}\}$ , where

$$\mathcal{F}_n := \sigma(D_1, T_1, D_2, T_2, \dots, D_n, T_n) \quad \text{for all } n \in \mathbb{N}.\tag{2.9}$$

Now define a new random walk  $M^p$  by

$$M_n^p := W_n^p - \frac{\mathbb{E}(T)}{2d} D_n \quad \text{for all } n \in \mathbb{N}.\tag{2.10}$$

As before, we assume that  $\mathbb{E}(T^2) < \infty$ . Then the following proposition identifies two martingales associated with the walk  $W^p$ .

**Proposition 2.2.** *Let  $C^p$  be the diffusion constant appearing in Proposition 2.1. Then  $\{(M_n^p, \mathcal{F}_n) : n \in \mathbb{N}\}$  and  $\{(|M_n^p|^2 - n C^p, \mathcal{F}_n) : n \in \mathbb{N}\}$  are martingales.*

*Proof.* The essential ingredients for the proof are: (i) that the events  $E_{n1} := \{D_n \cdot D_{n+1} = 0\}$  and its complement  $E_{n1}^c$  are independent of the events in  $\mathcal{F}_n$ ,

(ii) that on the event  $E_{n1}^c$ ,  $D_{n+1} = -D_n$ , and (iii) that on the event  $E_{n1}$ ,  $D_{n+1}$  is distributed symmetrically (orthogonal to  $D_n$ ). Observing that

$$M_{n+1}^p = M_n^p + \frac{\mathbb{E}(T)}{2d} D_n + D_{n+1} \left( L_{n+1} - \frac{\mathbb{E}(T)}{2d} \right), \quad (2.11)$$

it is then not difficult to verify that

$$\begin{aligned} \mathbb{E}(M_{n+1}^p \mid \mathcal{F}_n) &= \mathbb{E}(M_{n+1}^p \mathbb{1}(E_{n1}) \mid \mathcal{F}_n) + \mathbb{E}(M_{n+1}^p \mathbb{1}(E_{n1}^c) \mid \mathcal{F}_n) \\ &= M_n^p. \end{aligned} \quad (2.12)$$

Next we use (2.11) again, as well as  $|D_n|^2 = 1$ , to compute

$$\begin{aligned} |M_{n+1}^p|^2 &= |M_n^p|^2 + \frac{\mathbb{E}(T)^2}{2d^2} + L_{n+1}^2 - \frac{\mathbb{E}(T)}{d} L_{n+1} \\ &\quad + D_n \cdot M_n^p \frac{\mathbb{E}(T)}{d} + 2D_{n+1} \cdot M_n^p \left( L_{n+1} - \frac{\mathbb{E}(T)}{2d} \right) \\ &\quad + \frac{\mathbb{E}(T)}{d} D_n \cdot D_{n+1} \left( L_{n+1} - \frac{\mathbb{E}(T)}{2d} \right). \end{aligned} \quad (2.13)$$

In the same way as before, a straightforward calculation now leads to

$$\begin{aligned} \mathbb{E}(|M_{n+1}^p|^2 \mid \mathcal{F}_n) &= \mathbb{E}(|M_{n+1}^p|^2 \mathbb{1}(E_{n1}) \mid \mathcal{F}_n) + \mathbb{E}(|M_{n+1}^p|^2 \mathbb{1}(E_{n1}^c) \mid \mathcal{F}_n) \\ &= |M_n^p|^2 + \frac{d \mathbb{E}(T^2) - \mathbb{E}(T)^2}{d}, \end{aligned} \quad (2.14)$$

confirming the proposition.  $\square$

### 2.3 Connection with senile persistent random walk

As alluded to in the introduction, the random walk  $W^p$  studied above can be seen as a time-change of another random walk  $S^p$ , called *senile persistent random walk*, sampled at the random times

$$\tau_n := \sum_{k=1}^n T_k \quad \text{for all } n \in \mathbb{N}. \quad (2.15)$$

The connection between the two walks is best established through the inverse of this time-change. That is, we introduce the random map  $\tau^{-1} : \mathbb{N} \rightarrow \mathbb{N}$  by setting

$$\tau_n^{-1} := \inf\{m \in \mathbb{N} : \tau_m \geq n\} \quad \text{for each } n \in \mathbb{N}. \quad (2.16)$$

Thus, for any point  $\omega$  of the sample space,  $\tau_n^{-1}(\omega)$  is the time  $m$  such that  $\tau_{m-1}(\omega)$  is less than  $n$  and  $\tau_m(\omega)$  is at least  $n$ . Note that for every  $n \in \mathbb{N}$ ,  $\tau_n^{-1}$  is a stopping time with respect to the filtration  $\{\mathcal{F}_n : n \in \mathbb{N}\}$ , since (setting  $\tau_0 := 0$ )

$$\{\tau_n^{-1} \leq k\} = \bigcup_{m=1}^k \{\tau_{m-1} < n \leq \tau_m\} = \{n \leq \tau_k\} \in \mathcal{F}_k. \quad (2.17)$$

We also remark that  $\tau_n^{-1} \leq n$  a.s., since  $\tau_n$  is necessarily at least equal to  $n$ .

The senile persistent random walk  $S^p$  can now be defined by

$$S_n^p := W_{\tau_n^{-1}}^p + D_{\tau_n^{-1}}(n - \tau_{\tau_n^{-1}}) \quad \text{for all } n = 1, 2, \dots, \quad (2.18)$$

where  $\tau_{\tau_n^{-1}} = \sum_{m=1}^{\tau_n^{-1}} T_m$ . It is not obvious from this formal definition how the walk  $S^p$  behaves, so let us discuss this in more detail. First observe that  $S_{\tau_n}^p = W_n^p$ , so that we can indeed interpret  $W^p$  as the senile random walk  $S^p$  sampled at the times  $\tau_n$ . Next we note that by (2.18), in between times  $\tau_{n-1}$  and  $\tau_n$ , the walk moves in a straight line from the position  $W_{\tau_{n-1}}$  to  $W_n$ , taking steps of unit length. Therefore, we see that the random walk  $S^p$  is a walk which persists to move in a given direction for a random time distributed like  $T$ , then chooses a new direction at random, moves in that direction for a random time distributed again like  $T$ , and so on.

It is now instructive to interpret the role of the function  $f$  appearing in the distribution (1.1) of the random times  $T_n$  from the behaviour of the walk  $S^p$ . Looking at equation (1.1), we see that the walk  $S^p$ , after having moved in the same direction for  $n$  steps, chooses to make the next step again in the same direction with a probability given by  $(1 + f(n))/(2d + f(n))$ . Furthermore, all other choices of direction for the next step are equally likely. This description of the walk  $S^p$  corresponds to how the model was originally defined in [7].

Our next objective is to find the diffusion constant for the senile persistent random walk  $S^p$ . It is given by the following result.

**Proposition 2.3.** *Suppose that  $\mathbb{E}(T^2) < \infty$ . Then the diffusion constant of the senile persistent random walk  $S^p$  is given by*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}(|S_n^p|^2) = \frac{d \mathbb{E}(T^2) - \mathbb{E}(T)^2}{d \mathbb{E}(T)} = \frac{C^p}{\mathbb{E}(T)}.$$

*Proof.* The key observation is that at time  $n$ ,  $S^p$  is not far from  $M^p$  at the stopping time  $\tau_n^{-1}$ . To be precise, from the definitions (2.10) and (2.18) we see that for all  $n \in \mathbb{N}$ ,

$$S_n^p = M_{\tau_n^{-1}}^p + D_{\tau_n^{-1}} \left( \frac{\mathbb{E}(T)}{2d} + n - \tau_{\tau_n^{-1}} \right) =: M_{\tau_n^{-1}}^p + X_n^p, \quad (2.19)$$

where we have introduced  $X_n^p$  to denote the difference between  $S_n^p$  and  $M_{\tau_n^{-1}}^p$ . By the triangle inequality and Hölder's inequality, we then have

$$\left| \mathbb{E}(|S_n^p|^2) - |M_{\tau_n^{-1}}^p|^2 - |X_n^p|^2 \right| \leq 2 \sqrt{\mathbb{E}(|M_{\tau_n^{-1}}^p|^2) \mathbb{E}(|X_n^p|^2)}. \quad (2.20)$$

Observe that  $|X_n^p|$  is bounded by the sum of a constant and the term  $\tau_{\tau_n^{-1}} - n$ , which takes values between 0 and  $T_{\tau_n^{-1}}$ . Therefore, to prove Proposition 2.3, it suffices to show that, on the one hand,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}(|M_{\tau_n^{-1}}^p|^2) = \frac{d \mathbb{E}(T^2) - \mathbb{E}(T)^2}{d \mathbb{E}(T)}, \quad (2.21)$$

and on the other hand,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}(T_{\tau_n^{-1}}^2) = 0. \quad (2.22)$$

We will now show (2.21) and (2.22) by appealing to Proposition 2.2 and the law of large numbers for the random times  $\tau_n$ .

First we will prove (2.21). To ease the notation, we shall write  $C^p$  for the diffusion constant of the walk  $W^p$  appearing in Proposition 2.1. Now we observe that (by standard martingale theory, see e.g. [1, Theorem 6.7.3]) Proposition 2.2 and the fact that  $\tau_1^{-1}, \tau_2^{-1}, \dots$  is an increasing sequence of stopping times imply that the process

$$\{|M_{\tau_n^{-1}}^p|^2 - \tau_n^{-1} C^p : n \in \mathbb{N}\} \quad (2.23)$$

is a martingale with respect to the filtration  $\{\mathcal{F}_{\tau_n^{-1}} : n \in \mathbb{N}\}$  defined by

$$\mathcal{F}_{\tau_n^{-1}} := \{A \in \mathcal{F} : A \cap \{\tau_n^{-1} \leq k\} \in \mathcal{F}_k \text{ for all } k = 1, 2, \dots, n\}. \quad (2.24)$$

Therefore,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}(|M_{\tau_n^{-1}}^p|^2 - \tau_n^{-1} C^p) = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}(|M_1^p|^2 - C^p) = 0. \quad (2.25)$$

But the strong law of large numbers dictates that  $n^{-1}\tau_n \xrightarrow{\text{a.s.}} \mathbb{E}(T)$ , from which it follows that  $n^{-1}\tau_n^{-1} \xrightarrow{\text{a.s.}} \mathbb{E}(T)^{-1}$ . Since  $\tau_n^{-1} \leq n$  a.s., we therefore have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}(\tau_n^{-1}) = \frac{1}{\mathbb{E}(T)} \quad (2.26)$$

by bounded convergence. Together with (2.25), this implies (2.21).

It remains to show (2.22). To this end, we write

$$\frac{1}{n} \mathbb{E}(T_{\tau_n^{-1}}^2) = \frac{1}{n} \sum_{k=1}^n \mathbb{E}(T_k^2 \mathbb{1}(\tau_n^{-1} = k)). \quad (2.27)$$

Fix  $\epsilon > 0$  small and set  $k_{\pm} = \lfloor n(\mathbb{E}(T)^{-1} \pm \epsilon) \rfloor$ . We will split the sum over  $k$  in (2.27) into three contributions with  $k_- \leq k \leq k_+$ ,  $k > k_+$  and  $k < k_-$ , respectively, and show that each contribution is either of order  $\epsilon$  or vanishes as  $n \rightarrow \infty$ . For the first contribution, we have

$$\frac{1}{n} \sum_{k=k_-}^{k_+} \mathbb{E}(T_k^2 \mathbb{1}(\tau_n^{-1} = k)) \leq \frac{1}{n} \sum_{k=k_-}^{k_+} \mathbb{E}(T_k^2) \rightarrow 2\epsilon \mathbb{E}(T^2) \quad \text{as } n \rightarrow \infty. \quad (2.28)$$

Next we recall that  $\{\tau_n^{-1} = k\} = \{\tau_{k-1} < n, \tau_k \geq n\}$ , from which it follows that  $\mathbb{1}(\tau_n^{-1} = k) \leq \mathbb{1}(\tau_{k-1} < n)$ . Since for all  $k, m \in \mathbb{N}$ ,  $\mathbb{P}(\tau_{k+m} < n) \leq \mathbb{P}(\tau_k < n)$ , we therefore have

$$\begin{aligned} \frac{1}{n} \sum_{k=k_++1}^n \mathbb{E}(T_k^2 \mathbb{1}(\tau_n^{-1} = k)) &\leq \frac{1}{n} \sum_{k=k_++1}^n \mathbb{E}(T_k^2) \mathbb{P}(\tau_{k-1} < n) \\ &\leq \mathbb{P}(\tau_{k_+} < n) \mathbb{E}(T^2). \end{aligned} \quad (2.29)$$



This contribution to (2.27) vanishes in the limit  $n \rightarrow \infty$  by the law of large numbers for  $\tau_n$ .

To deal with the last contribution to (2.27), we set  $\delta := \frac{1}{2}\mathbb{E}(T)\epsilon$ , and observe that  $\mathbb{1}(\tau_n^{-1} = k, T_k \leq \delta n) \leq \mathbb{1}(\tau_{k-1} \geq (1-\delta)n)$ . Since for all  $k, m \in \mathbb{N}$ ,  $\mathbb{P}(\tau_{k-m} \geq (1-\delta)n) \leq \mathbb{P}(\tau_k \geq (1-\delta)n)$ , we therefore have

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^{k_- - 1} \left[ \mathbb{E}(T_k^2 \mathbb{1}(\tau_n^{-1} = k, T_k \leq \delta n)) + \mathbb{E}(T_k^2 \mathbb{1}(\tau_n^{-1} = k, T_k > \delta n)) \right] \\ \leq \mathbb{P}(\tau_{k_-} \geq (1-\delta)n) \mathbb{E}(T^2) + \mathbb{E}(T^2 \mathbb{1}(T > \delta n)) \end{aligned} \quad (2.30)$$

in the same way as before. By the law of large numbers and the assumption that  $\mathbb{E}(T^2) < \infty$ , this contribution also vanishes as  $n \rightarrow \infty$ . We have shown that the expression (2.27) can be made arbitrarily small for  $n \rightarrow \infty$ , which proves (2.22). This completes the proof of Proposition 2.3.  $\square$

## 2.4 Weak convergence to Brownian motion

We will now show weak convergence of  $S^p$  to Brownian motion, by applying a martingale central limit theorem to the martingale  $\{M_{\tau_n}^p : n \in \mathbb{N}\}$  studied above. We follow Billingsley [2, Section 18]. Let  $D[0, \infty)$  be the metric space of right-continuous real functions on  $[0, \infty)$  with left-hand limits which has the Skorohod topology, as in [2, Section 16]. Generally, we will denote by  $W$  standard Brownian motion on any functional space under consideration, and we write  $\Rightarrow_n$  to denote weak convergence with  $n$ . Setting  $S_0 := 0$ , for the senile persistent random walk, the following holds:

**Theorem 2.4.** *Assume  $\mathbb{E}(T^2) < \infty$ . For every  $t \geq 0$  and  $n \in \mathbb{N}$ , define*

$$Z_t^n := \sqrt{\frac{d \mathbb{E}(T)}{n C^p}} S_{[nt]}^p, \quad (2.31)$$

where  $C^p$  is the diffusion constant of  $W^p$  appearing in Proposition 2.1. Then  $Z^n \Rightarrow_n W$  in the sense of  $D[0, \infty)^d$ .

*Proof.* First we recall that  $S_n^p$  is close to  $M_{\tau_n}^p$ , as expressed by (2.19) in the proof of Proposition 2.3. In fact, the proof of Proposition 2.3 shows that the contribution from the correction term  $X^p$  vanishes, that is,  $X_{[nt]}^p / \sqrt{n} \Rightarrow_n 0$  for every  $t > 0$ . Therefore, it suffices to prove weak convergence to Brownian motion for  $M_{\tau_n}^p$ .

We recall that  $\{M_{\tau_n}^p : n \in \mathbb{N}\}$  is a martingale with respect to the filtration  $\mathcal{F} := \{\mathcal{F}_{\tau_n} : n \in \mathbb{N}\}$  defined by (2.24). Let us now write  $M_{\tau_n}^i$ ,  $i = 1, 2, \dots, d$ ,

for the one-dimensional marginals of  $M_{\tau_n}^P$ . For each  $n \in \mathbb{N}$ , define

$$\xi_{n1}^i := \sqrt{\frac{d \mathbb{E}(T)}{n C^p}} M_{\tau_1}^i \quad i = 1, 2, \dots, d; \quad (2.32)$$

$$\xi_{nk}^i := \sqrt{\frac{d \mathbb{E}(T)}{n C^p}} \left( M_{\tau_k}^i - M_{\tau_{k-1}}^i \right) \quad i = 1, 2, \dots, d; \quad k = 2, 3, \dots \quad (2.33)$$

Then for each  $i$ , the  $\xi_{nk}^i$  form a triangular array of martingale differences with respect to the filtration  $\mathcal{F} := \{\mathcal{F}_{\tau_k} : k \in \mathbb{N}\}$ .

By (2.11) we have for  $k \geq 2$

$$\begin{aligned} |\xi_{nk}^i| &= \sqrt{\frac{d \mathbb{E}(T)}{n C^p}} \mathbb{1}(\tau_k^{-1} \neq \tau_{k-1}^{-1}) \left| D_{\tau_k}^i L_{\tau_k} + \frac{\mathbb{E}(T)}{2d} \left( D_{\tau_{k-1}}^i - D_{\tau_k}^i \right) \right| \\ &\leq \sqrt{\frac{d \mathbb{E}(T)}{n C^p}} \sum_{l \leq k} \mathbb{1}(\tau_k^{-1} = l, \tau_{k-1}^{-1} = l-1) \left( L_l + \frac{\mathbb{E}(T)}{d} \right) \\ &= \sqrt{\frac{d \mathbb{E}(T)}{n C^p}} \sum_{l \leq k} \mathbb{1}(\tau_{l-1} = k-1) \left( T_l + \frac{\mathbb{E}(T)}{d} \right). \end{aligned} \quad (2.34)$$

Setting  $\tau_0 := 0$ , it is clear that this bound also holds for  $k = 1$ .

Now fix  $\epsilon > 0$  and set  $\delta := \frac{1}{2} \epsilon \sqrt{C^p/d \mathbb{E}(T)}$ . Then from the bound on  $|\xi_{nk}^i|$ , it follows that for  $n$  sufficiently large we have that

$$\begin{aligned} &\sum_{k \leq nt} \mathbb{E}((\xi_{nk}^i)^2 \mathbb{1}(|\xi_{nk}^i| \geq \epsilon)) \\ &\leq \frac{2d \mathbb{E}(T)}{n C^p} \mathbb{E} \left[ \sum_{k \leq nt} \sum_{l \leq k} \mathbb{1}(\tau_{l-1} = k-1) T_l^2 \mathbb{1}(T_l \geq \delta \sqrt{n}) \right]. \end{aligned} \quad (2.35)$$

Interchanging the order of summation and using that  $\sum_{k=i}^{\lfloor nt \rfloor} \mathbb{1}(\tau_{k-1} = k-1) \leq 1$ , we arrive at

$$\sum_{k \leq nt} \mathbb{E}((\xi_{nk}^i)^2 \mathbb{1}(|\xi_{nk}^i| \geq \epsilon)) \leq \frac{2d \mathbb{E}(T)}{n C^p} \sum_{l \leq nt} \mathbb{E} [T_l^2 \mathbb{1}(T_l \geq \delta \sqrt{n})]. \quad (2.36)$$

Since the  $T_l$  are i.i.d. and  $\mathbb{E}(T^2) < \infty$ , we conclude that for every  $t \geq 0$ ,

$$\sum_{k \leq nt} \mathbb{E}((\xi_{nk}^i)^2 \mathbb{1}(|\xi_{nk}^i| \geq \epsilon)) \rightarrow_n 0. \quad (2.37)$$

Now put

$$(\sigma_{nk}^i)^2 := \mathbb{E}((\xi_{nk}^i)^2 \mid \mathcal{F}_{\tau_{k-1}}), \quad (2.38)$$

where we define  $\mathcal{F}_{\tau_0}$  to be the trivial  $\sigma$ -field  $\{\emptyset, \Omega\}$ . By Proposition 2.2 and the symmetry of our random walks, for every  $i = 1, 2, \dots, d$ ,

$$\left\{ \left( (M_{\tau_n}^i)^2 - \tau_n^{-1} C^p / d, \mathcal{F}_{\tau_n} \right) : n \in \mathbb{N} \right\} \quad (2.39)$$

is a martingale. Therefore, for  $k \geq 2$ ,

$$\begin{aligned} (\sigma_{nk}^i)^2 &= \frac{d \mathbb{E}(T)}{n C^p} \mathbb{E} \left( (M_{\tau_k^{-1}}^i)^2 + (M_{\tau_{k-1}^{-1}}^i)^2 - 2M_{\tau_k^{-1}}^i M_{\tau_{k-1}^{-1}}^i \mid \mathcal{F}_{\tau_{k-1}^{-1}} \right) \\ &= \frac{\mathbb{E}(T)}{n} \left( \mathbb{E}(\tau_k^{-1} \mid \mathcal{F}_{\tau_{k-1}^{-1}}) - \tau_{k-1}^{-1} \right). \end{aligned} \quad (2.40)$$

Next we observe that for all  $l, m \in \mathbb{N}$  (considering  $l > m$  and  $l \leq m$  in turn),

$$\{\tau_k^{-1} \leq l\} \cap \{\tau_{k-1}^{-1} \leq m\} = \{\tau_l \geq k\} \cap \{\tau_m \geq k-1\} \in \mathcal{F}_m. \quad (2.41)$$

It follows by (2.24) that  $\{\tau_k^{-1} \leq l\} \in \mathcal{F}_{\tau_{k-1}^{-1}}$  for all  $l$ , and hence, that the random variable  $\tau_k^{-1}$  is in fact  $\mathcal{F}_{\tau_{k-1}^{-1}}$ -measurable. Therefore, for  $k \geq 2$ ,

$$(\sigma_{nk}^i)^2 = \frac{\mathbb{E}(T)}{n} (\tau_k^{-1} - \tau_{k-1}^{-1}). \quad (2.42)$$

By the strong law of large numbers, it immediately follows that for every  $t \geq 0$ ,

$$\sum_{k \leq nt} (\sigma_{nk}^i)^2 = \sum_{k \leq nt} \mathbb{E}((\xi_{nk}^i)^2 \mid \mathcal{F}_{\tau_{k-1}^{-1}}) \xrightarrow{\text{a.s.}}_n t. \quad (2.43)$$

Now write (taking  $M_0^i := 0$ )

$$Y_t^{ni} := \sum_{k \leq nt} \xi_{nk}^i = \sqrt{\frac{d \mathbb{E}(T)}{n C^p}} M_{\tau_{[nt]}^{-1}}^i. \quad (2.44)$$

Then Theorem 18.2 in [2] states that because (2.43) and (2.37) both hold,  $Y^{ni} \Rightarrow_n W$  in the sense of  $D[0, \infty)$ . In other words, we have shown that the one-dimensional marginals  $Y^{ni}$  converge weakly to Brownian motion. We now want to extend this to weak convergence of  $Y^n = (Y^{n1}, \dots, Y^{nd})$ . The proof of Theorem 18.2 in [2] shows that for each  $i$  the laws of the one-dimensional marginals  $Y^{ni}$  form a tight family. But since the product of compact sets in  $D[0, \infty)$  is a compact set, this implies tightness of the family of laws of the  $Y^n$ . It remains to show that all finite-dimensional distributions of  $Y^n$  converge to those of  $d$ -dimensional Brownian motion.

To show this, we need the additional result that for  $i \neq j$  and all  $n \in \mathbb{N}$ ,

$$\mathbb{E}((M_{n+1}^i - M_n^i)(M_{n+1}^j - M_n^j) \mid \mathcal{F}_n) = 0. \quad (2.45)$$

This can be seen by using (2.11) and noting that on the event  $\{D_n \cdot D_{n+1} \neq 0\}$ ,  $D_{n+1} = -D_n$  whereas on the event  $\{D_n \cdot D_{n+1} = 0\}$ ,  $D_{n+1}^i D_n^j$  takes on each of the values  $\pm 1$  with equal probabilities. It follows that for  $i \neq j$  and fixed  $n$ , the  $\xi_{nk}^i \xi_{nk}^j$  are martingale differences with respect to the filtration  $\mathcal{F}$ .

Now fix  $s, t \geq 0$ . Define  $\eta_{nk}$  as  $a \xi_{nk}^i + b \xi_{nk}^j$  for  $k \leq [ns]$  and as  $b \xi_{nk}^j$  for  $[ns] < k \leq [n(s+t)]$ . Then, by (2.43) and because the  $\xi_{nk}^i \xi_{nk}^j$  are martingale differences,

$$\sum_{k \leq nt} \mathbb{E}(\eta_{nk}^2 \mid \mathcal{F}_{\tau_{k-1}^{-1}}) \xrightarrow{\text{a.s.}}_n a^2 s + b^2 (s+t). \quad (2.46)$$

Therefore, by Theorem 18.1 in [2],  $aY_s^{ni} + bY_{s+t}^{nj} \Rightarrow_n aW_s^i + bW_{s+t}^j$ , where the  $W^i$  are the one-dimensional marginals of  $d$ -dimensional Brownian motion. Because this holds for all  $a$  and  $b$ , by the Cramér-Wold argument  $(Y_s^{ni}, Y_{s+t}^{nj}) \Rightarrow_n (W_s^i, W_{s+t}^j)$ . It is easy to see that this argument can be generalized to show that all finite-dimensional distributions of  $Y^n$  converge to those of  $d$ -dimensional Brownian motion. This completes the proof.  $\square$

### 3 The reinforced case

We now turn our attention to the reinforced case, where we set

$$L_m = \mathbb{1}(T_m \text{ is odd}) \quad \text{for all } m \in \mathbb{N}. \quad (3.1)$$

Thus, the lengths of the steps of the random walk are i.i.d. variables taking values in  $\{0, 1\}$ . However, the directions of different steps are not independent. Namely, the step following a step of length 0 may not have the same direction as the previous step, and the step following a step of length 1 may not be in the opposite direction of the previous step. All other choices of direction are equally likely. Formally, the directions  $D_m$  satisfy

$$\mathbb{P}(D_1 = e_i) = \frac{1}{2d} \quad \text{for each } i = 1, 2, \dots, 2d, \quad (3.2)$$

and for all  $m \in \mathbb{N}$  and each  $i = 1, 2, \dots, 2d$ ,

$$\begin{aligned} & \mathbb{P}(D_{m+1} = e_i \mid D_m, L_m) \\ &= \frac{1}{2d-1} \mathbb{1}(D_m \neq e_i, L_m = 0) + \frac{1}{2d-1} \mathbb{1}(D_m \neq -e_i, L_m = 1). \end{aligned} \quad (3.3)$$

Equations (3.1)–(3.3) completely specify the law of the random walk defined by (1.2). For the remainder of this section, we will denote this walk by  $W^r = (W_1^r, W_2^r, \dots)$ , where the superscript  $r$  is used to identify the *reinforced* case.

From (3.1), it may not come as a surprise that the quantity

$$p := \mathbb{P}(T \text{ is odd}) \quad (3.4)$$

plays an important role in the analysis of the reinforced case. In fact, if  $d = 1$  we see from (3.3) that the walk has a trivial behaviour if  $p = 1$ , since then it keeps moving in the same direction. Let us therefore take the opportunity to exclude this special case from the analysis for the remainder of this section, so that we don't have to repeat the condition that  $p < 1$  if  $d = 1$  all the time. Note, however, that the case  $p = 1$  is perfectly fine and nontrivial in higher dimensions. Also, when we consider the time-changed walk  $W^r$  there will be no problem in allowing  $\mathbb{P}(T = \infty) > 0$ , where we may assume either that “ $T_n$  is odd” is false, or that “ $T_n$  is odd” is true if  $T_n = \infty$ , whichever one prefers.

### 3.1 Direct calculation of the diffusion constant

We will now compute the diffusion constant for the random walk  $W^r$  defined above. In terms of the parameter  $p = \mathbb{P}(T \text{ is odd})$ , the diffusion constant is identified by the following proposition.

**Proposition 3.1.** *The diffusion constant of the random walk  $W^r$  is given by*

$$C^r := \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}(|W_n^r|^2) = \frac{dp}{d-p}.$$

*Proof.* We start from the observation that

$$\mathbb{E}(|W_n^r|^2) = np + 2 \sum_{k=1}^{n-1} \sum_{m=1}^{n-k} \mathbb{E}(D_m \cdot D_{m+k} L_m L_{m+k}), \quad (3.5)$$

where, since  $D_m \cdot D_{m+k} L_m$  is independent of  $L_{m+k}$ ,

$$\mathbb{E}(D_m \cdot D_{m+k} L_m L_{m+k}) = p \mathbb{E}(D_m \cdot D_{m+k} L_m). \quad (3.6)$$

Now note that on the event  $E_{mk} := \{D_{m+k-1} \cdot D_{m+k} = 0\}$ ,  $D_m \cdot D_{m+k}$  takes on the values  $\pm 1$  with equal probabilities by (3.3). On the other hand, on the complementary event  $E_{mk}^c$  we have that  $D_{m+k} = D_{m+k-1} (2L_{m+k-1} - 1)$ . Therefore, again using independence,

$$\begin{aligned} \mathbb{E}(D_m \cdot D_{m+k} L_m) &= \mathbb{E}(D_m \cdot D_{m+k-1} L_m (2L_{m+k-1} - 1) \mathbb{1}(E_{mk}^c)) \\ &= \frac{2p-1}{2d-1} \mathbb{E}(D_m \cdot D_{m+k-1} L_m). \end{aligned} \quad (3.7)$$

Iterating this recursion and using (3.6), we obtain

$$\begin{aligned} \mathbb{E}(D_m \cdot D_{m+k} L_m L_{m+k}) &= p \left( \frac{2p-1}{2d-1} \right)^{k-1} \mathbb{E}(D_m \cdot D_{m+1} L_m) \\ &= \frac{p^2}{2d-1} \left( \frac{2p-1}{2d-1} \right)^{k-1}. \end{aligned} \quad (3.8)$$

Substituting this result into (3.5) yields

$$\begin{aligned} \mathbb{E}(|W_n^r|^2) &= np + \frac{2p^2}{2d-1} \sum_{k=1}^{n-1} (n-k) \left( \frac{2p-1}{2d-1} \right)^{k-1} \\ &= n \frac{dp}{d-p} + \frac{p^2(2d-1)}{2(d-p)^2} \left[ \left( \frac{2p-1}{2d-1} \right)^n - 1 \right]. \end{aligned} \quad (3.9)$$

The value of the diffusion constant for the random walk  $W^r$  follows.  $\square$

### 3.2 Martingales for the reinforced random walk

The purpose of this subsection is to identify martingales associated with the random walk  $W^r$  introduced above. Our main observation is that if we add a correction of constant length (but random direction) to the positions  $W_n^r$ , then we obtain a martingale. To be precise, define

$$M_n^r := W_n^r + \frac{p}{2(d-p)} D_n (2L_n - 1) \quad \text{for all } n \in \mathbb{N}, \quad (3.10)$$

and let  $\{\mathcal{F}_n : n \in \mathbb{N}\}$  be the filtration defined by

$$\mathcal{F}_n := \sigma(D_1, T_1, D_2, T_2, \dots, D_n, T_n) \quad \text{for all } n \in \mathbb{N}. \quad (3.11)$$

The following proposition identifies two martingales associated with  $W^r$ .

**Proposition 3.2.** *Let  $C^r$  be the diffusion constant appearing in Proposition 3.1. Then  $\{(M_n^r, \mathcal{F}_n) : n \in \mathbb{N}\}$  and  $\{(|M_n^r|^2 - n C^r, \mathcal{F}_n) : n \in \mathbb{N}\}$  are martingales.*

*Proof.* The key observation is that even though the direction  $D_{n+1}$  itself depends on  $D_n$  and  $L_n$ , the events  $E_{n1} := \{D_n \cdot D_{n+1} = 0\}$  and its complement  $E_{n1}^c$  are independent of the events in  $\mathcal{F}_n$ , and have the probabilities  $(2d-2)/(2d-1)$  and  $1/(2d-1)$ , respectively. Moreover, on the event  $E_{n1}$ ,  $D_{n+1}$  is distributed symmetrically (orthogonal to  $D_n$ ), and on the event  $E_{n1}^c$ , we have  $D_{n+1} = D_n(2L_n - 1)$ . Now observe that for all  $n \in \mathbb{N}$ ,

$$\begin{aligned} M_{n+1}^r &= M_n^r - \frac{p}{2(d-p)} D_n (2L_n - 1) \\ &\quad + \frac{p}{2(d-p)} D_{n+1} (2L_{n+1} - 1) + D_{n+1} L_{n+1}. \end{aligned} \quad (3.12)$$

A simple computation then yields

$$\begin{aligned} \mathbb{E}(M_{n+1}^r \mid \mathcal{F}_n) &= \mathbb{E}(M_{n+1}^r \mathbb{1}(E_{n1}) \mid \mathcal{F}_n) + \mathbb{E}(M_{n+1}^r \mathbb{1}(E_{n1}^c) \mid \mathcal{F}_n) \\ &= M_n^r. \end{aligned} \quad (3.13)$$

Likewise, for all  $n \in \mathbb{N}$  we can write

$$\begin{aligned} |M_{n+1}^r|^2 &= |M_n^r|^2 + \frac{p^2}{2(d-p)^2} + \frac{d}{d-p} \mathbb{1}(L_{n+1} = 1) \\ &\quad - D_n \cdot M_n^r \frac{p}{d-p} (2L_n - 1) + D_{n+1} \cdot M_n^r \left[ \frac{p}{d-p} (2L_{n+1} - 1) + 2L_{n+1} \right] \\ &\quad - \frac{p}{2(d-p)} D_n \cdot D_{n+1} (2L_n - 1) \left[ \frac{p}{d-p} (2L_{n+1} - 1) + 2L_{n+1} \right]. \end{aligned} \quad (3.14)$$

A straightforward computation gives

$$\begin{aligned} \mathbb{E}(|M_{n+1}^r|^2 \mid \mathcal{F}_n) &= \mathbb{E}(|M_{n+1}^r|^2 \mathbb{1}(E_{n1}) \mid \mathcal{F}_n) + \mathbb{E}(|M_{n+1}^r|^2 \mathbb{1}(E_{n1}^c) \mid \mathcal{F}_n) \\ &= |M_n^r|^2 + \frac{dp}{d-p}. \end{aligned} \quad (3.15)$$

This confirms that the two processes of the proposition are martingales with respect to the filtration  $\{\mathcal{F}_n : n \in \mathbb{N}\}$ .  $\square$

### 3.3 Connection with senile reinforced random walk

Like in the persistent case, the walk  $W^r$  can be interpreted as a *senile reinforced random walk* sampled at the random times

$$\tau_n := \sum_{k=1}^n T_k \quad \text{for all } n \in \mathbb{N}. \quad (3.16)$$

As before, we concentrate on the inverse time-change defined by

$$\tau_n^{-1} := \inf\{m \in \mathbb{N} : \tau_m \geq n\} \quad \text{for each } n \in \mathbb{N}. \quad (3.17)$$

We recall that the random times  $\tau_n^{-1}$  are stopping times with respect to the filtration  $\{\mathcal{F}_n : n \in \mathbb{N}\}$ , and that  $\tau_n^{-1} \leq n$  almost surely.

We may define the senile reinforced random walk  $S^r$  on  $\mathbb{Z}^d$  by

$$S_n^r := W_{\tau_n^{-1}}^r - D_{\tau_n^{-1}} \mathbb{1}(\tau_{\tau_n^{-1}} - n \text{ is odd}) \quad \text{for all } n \in \mathbb{N}, \quad (3.18)$$

where  $\tau_{\tau_n^{-1}} = \sum_{m=1}^{\tau_n^{-1}} T_m$ . Observe that indeed  $S_{\tau_n}^r = W_n^r$  for all  $n \in \mathbb{N}$ , so that we may interpret  $W^r$  as the senile random walk  $S^r$  sampled at the times  $\tau_n$ . Furthermore, by the definition (3.18), in between times  $\tau_{n-1}^{-1}$  and  $\tau_n^{-1}$ , the walk  $S^r$  jumps back and forth between the positions  $W_{n-1}^r$  and  $W_n^r$ . Thus, the senile reinforced random walk  $S^r$  is a walk that traverses an edge back and forth for a random time distributed like  $T$ , then selects a new edge at random and traverses that edge for a random time again distributed like  $T$ , and so on.

As in the persistent case, this description gives us an interpretation of the reinforcement function  $f$  defining the distribution of the random times  $T_n$  in (1.1). Namely, the walk  $S^r$  has the property that after it has been traversing the same edge back and forth for the last  $n$  steps, it will choose to traverse that edge again in the next step with probability  $(1 + f(n))/(2d + f(n))$ . Furthermore, all other choices for the next edge are equally likely. This corresponds to the original definition of the model in [4].

At this stage, we should note that the walk gets stuck on an edge in case  $T_n = \infty$  for some  $n \in \mathbb{N}$ . For the remainder of this section, we will rule out this possibility by assuming  $\mathbb{P}(T = \infty) = 0$ . However, we note that the following proposition and proof hold perfectly well if  $\mathbb{E}(T) = \infty$ , when we take division by  $\infty$  to yield 0.

**Proposition 3.3.** *For the senile reinforced random walk  $S^r$ , the diffusion constant is given by*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}(|S_n^r|^2) = \frac{1}{\mathbb{E}(T)} \frac{dp}{d-p} = \frac{C^r}{\mathbb{E}(T)}.$$

*Proof.* The proof proceeds in the same way as the proof of Proposition 2.3, and is in fact somewhat simpler. First we want to express  $S^r$  in terms of the martingale  $M^r$ . By the definitions (3.10) and (3.18) we have for all  $n \in \mathbb{N}$ ,

$$\begin{aligned} S_n^r &= M_{\tau_n^{-1}}^r - D_{\tau_n^{-1}} \left( \mathbb{1}(\tau_{\tau_n^{-1}} - n \text{ is odd}) + \frac{p}{2(d-p)} (2L_{\tau_n^{-1}} - 1) \right) \\ &=: M_{\tau_n^{-1}}^r + X_n^r, \end{aligned} \quad (3.19)$$

where we have introduced  $X_n^r$  to denote the difference between  $S^r$  at time  $n$  and  $M^r$  at time  $\tau_n^{-1}$ . By the triangle inequality and Hölder's inequality,

$$\left| \mathbb{E}(|S_n^r|^2 - |M_{\tau_n^{-1}}^r|^2 - |X_n^r|^2) \right| \leq 2 \sqrt{\mathbb{E}(|M_{\tau_n^{-1}}^r|^2) \mathbb{E}(|X_n^r|^2)}. \quad (3.20)$$

Since  $|X_n^r|$  is bounded by a constant, it therefore suffices to show that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}(|M_{\tau_n^{-1}}^r|^2) = \frac{C^r}{\mathbb{E}(T)} \quad (3.21)$$

in order to prove Proposition 3.3.

But Proposition 3.2 and the fact that the  $\tau_n^{-1}$  are stopping times with respect to the filtration  $\{\mathcal{F}_n : n \in \mathbb{N}\}$  imply that  $\{|M_{\tau_n^{-1}}^r|^2 - \tau_n^{-1} C^r : n \in \mathbb{N}\}$  is a martingale with respect to the filtration  $\{\mathcal{F}_{\tau_n^{-1}} : n \in \mathbb{N}\}$  defined by (2.24). Therefore,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}(|M_{\tau_n^{-1}}^r|^2 - \tau_n^{-1} C^r) = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}(|M_1^r|^2 - C^r) = 0. \quad (3.22)$$

The strong law of large numbers dictates that  $n^{-1} \tau_n^{-1} \xrightarrow{\text{a.s.}} \mathbb{E}(T)^{-1}$ , from which we get by bounded convergence that  $n^{-1} \mathbb{E}(\tau_n^{-1}) \rightarrow \mathbb{E}(T)^{-1}$ . Together with the previous result this implies (3.21), proving the proposition.  $\square$

### 3.4 Weak convergence to Brownian motion

Weak convergence to Brownian motion for the senile reinforced random walk can be shown in the same way as for the persistent case, studied in Theorem 2.4. As before, we let  $D[0, \infty)$  be the metric space of right-continuous real functions on  $[0, \infty)$  with left-hand limits, and we write  $W$  for Brownian motion and  $\Rightarrow_n$  for weak convergence with  $n$ . Of course, to get Brownian motion we need the random walk to be diffusive, so by Proposition 3.3 we have to assume  $\mathbb{E}(T) < \infty$ . Then, setting  $S_0^r := 0$ , the following holds:

**Theorem 3.4.** *Assume  $\mathbb{E}(T) < \infty$ . For every  $t \geq 0$  and  $n \in \mathbb{N}$ , define*

$$Z_t^n := \sqrt{\frac{d \mathbb{E}(T)}{n C^r}} S_{[nt]}^r, \quad (3.23)$$

where  $C^r$  is the diffusion constant of  $W^r$  appearing in Proposition 3.1. Then  $Z^n \Rightarrow_n W$  in the sense of  $D[0, \infty)^d$ .



*Proof.* For the senile reinforced random walk, the difference  $X_n^r$  between  $S_n^r$  and  $M_{\tau_n}^r$ , as defined by (3.21), is uniformly bounded by a constant. Therefore, it suffices to prove weak convergence to Brownian motion for  $M_{\tau_n}^r$  instead of  $S_n^r$ .

As in the proof of Theorem 2.4, let us write  $M_{\tau_n}^i$ ,  $i = 1, 2, \dots, d$ , for the one-dimensional marginals of  $M_{\tau_n}^r$ . For each  $n \in \mathbb{N}$ , define

$$\xi_{n1}^i := \sqrt{\frac{d \mathbb{E}(T)}{n C^r}} M_{\tau_1}^i \quad i = 1, 2, \dots, d; \quad (3.24)$$

$$\xi_{nk}^i := \sqrt{\frac{d \mathbb{E}(T)}{n C^r}} (M_{\tau_k}^i - M_{\tau_{k-1}}^i) \quad i = 1, 2, \dots, d; \quad k = 2, 3, \dots \quad (3.25)$$

Then for each  $i$ , the  $\xi_{nk}^i$  form a triangular array of martingale differences with respect to the filtration  $\mathcal{F} := \{\mathcal{F}_{\tau_k} : k \in \mathbb{N}\}$ .

By (3.12) it is clear that all random variables  $\sqrt{n}|\xi_{nk}^i|$  are uniformly bounded by a constant. We conclude that for every  $\epsilon > 0$  and  $t \geq 0$ ,

$$\sum_{k \leq nt} \mathbb{E}((\xi_{nk}^i)^2 \mathbb{1}(|\xi_{nk}^i| \geq \epsilon)) \rightarrow_n 0. \quad (3.26)$$

Moreover, we can follow the steps (2.38)–(2.43) in the proof of Theorem 2.4 to see that for every  $t \geq 0$ ,

$$\sum_{k \leq nt} \mathbb{E}((\xi_{nk}^i)^2 \mid \mathcal{F}_{\tau_{k-1}}) \xrightarrow{\text{a.s.}}_n t. \quad (3.27)$$

If we now write, setting  $M_0^r := 0$ ,

$$Y_t^{ni} := \sum_{k \leq nt} \xi_{nk}^i = \sqrt{\frac{d \mathbb{E}(T)}{n C^r}} M_{\tau_{[nt]}}^i, \quad (3.28)$$

then Theorem 18.2 in [2] states that  $Y^{ni} \Rightarrow_n W$  in the sense of  $D[0, \infty)$ . In particular, for each  $i$  the laws of the one-dimensional marginals  $Y^{ni}$  form a tight family, which implies that the family of laws of  $Y^n = (Y^{n1}, \dots, Y^{nd})$  is tight as well. It remains to show that all finite-dimensional distributions of  $Y^n$  converge to those of  $d$ -dimensional Brownian motion.

As in the persistent case, the result will follow if we can show that for  $i \neq j$  and all  $n \in \mathbb{N}$ ,

$$\mathbb{E}((M_{n+1}^i - M_n^i)(M_{n+1}^j - M_n^j) \mid \mathcal{F}_n) = 0. \quad (3.29)$$

This can be shown by using (3.12) and noting that on the event  $\{D_n \cdot D_{n+1} \neq 0\}$ ,  $D_{n+1} = D_n(2L_n - 1)$  whereas on the event  $\{D_n \cdot D_{n+1} = 0\}$ ,  $D_{n+1}^i D_n^j$  takes the values  $\pm 1$  with equal probabilities. It follows that for  $i \neq j$  and fixed  $n$ , the  $\xi_{nk}^i \xi_{nk}^j$  are martingale differences with respect to the filtration  $\mathcal{F}$ . The proof can now be completed as in the persistent case.  $\square$

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