# Bell's inequality and the coincidence-time loophole

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This paper analyzes the effects of time-dependence in the Bell inequality. A generalized inequality is derived for the case when coincidence and non-coincidence is controlled by timing that depends on the detector settings. Needless to say, this inequality is violated by quantum mechanics and by experimental data provided that the loss of measurement pairs through failure of coincidence is small enough, but the quantitative bound is more restrictive in this case than in the previously analyzed "efficiency loophole."

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The Bell inequality [1] and its descendants (see e.g., Ref. [2]) have been the main argument on the EPR-paradox [3, 4] for the last forty years. A new research field of 'experimental metaphysics' has formed, where the goal is to show that the concept of local realism is inconsistent with quantum mechanics, and ultimately with the real world. The experiments which have been performed to verify this have not been completely conclusive, but they point in a certain direction: Nature cannot be described by a local realist model (see Ref. [5–7] for instance). The reason for saying "not been completely conclusive" is the existence of certain "loopholes" in these experiments. There has been considerable discussion in the literature on this (see Refs. [8–16] among others), and this paper is motivated by recent claims (e.g., Ref. [14]) that time-dependence has been omitted in the Bell inequality, so that the Bell inequality is totally invalid. The present analysis shows that this is not the case, although one may find that certain bounds are higher than one would naïvely expect. Furthermore, the loophole can be closed with relative ease in experiments, and indeed *is* in some modern experiments.

The situation is as follows: in the standard Bell setup (see Fig. 1), we want to take into account that the time-correlation is not perfect between results at one site and results at the other. Usually, there is a "coincidence window" in which events are counted as being simultaneous, even when there is a finite (i.e., nonzero) time between them. There is a possibility that, in this type of setup, the local setting may change the time at which the local event happens. And this would have implications; a certain local event may be simultaneous with a remote event, or not, depending on the local detector setting. The result will be that the simultaneity of two detection events will depend on *both settings*, even though the underlying physical processes that control this are completely local. We will examine this situation in detail and derive precise bounds for violation of the appropriate Bell inequality.

To perform the intended formal examination of this, we need to put the hidden-variable model into formal language. We arrive at a probabilistic model [20]. Here, the hidden variable is a point $\lambda$ in a "sample space" $\Lambda$, the space of all possible values of the hidden variable. The measurement results are

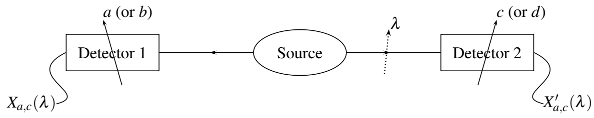


FIG. 1: The Bell setup. There are two local parameters ("detector settings"), one at each site, having one of have two values, denoted $a$ and $b$ for one site, and $c$ and $d$ for the other. The RVs describing the results are denoted, e.g., $X_{a,c}$ and $X'_{a,c}$, for the two sites.

described by random variables (RVs) $X(\lambda)$ which take their values in the value space $V$, usually taken as $\{-1,+1\}$ in the spin-$\frac{1}{2}$ case (see Fig. 1). There is a probability measure $P$ on the space $\Lambda$, used to calculate the probabilities of the different outcomes and the expectation value $E$, where

$$E(X)=\int_\Lambda X(\lambda)dP(\lambda)=\int_\Lambda X\,dP, \tag{1}$$

suppressing the parantheses. We then obtain the expectation of the product of the results as $E(XX')$, usually denoted "correlation" in this context [21]. Finally, after using *locality* only four RVs remain; $A$, $B$, $C'$ and $D'$, see below. Now, we have

*Theorem 1 (The Clauser-Horne-Shimony-Holt (CHSH) inequality)* The following three prerequisites are assumed to hold except at a null set:

***(i)*** *Realism.* Measurement results can be described by probability theory, using two families of RVs, e.g.,

$$\begin{aligned} X_{a,c} &: \Lambda \to V \\ &\lambda \mapsto X_{a,c}(\lambda) \\ X'_{a,c} &: \Lambda \to V \\ &\lambda \mapsto X'_{a,c}(\lambda). \end{aligned}$$

***(ii)*** *Locality.* A measurement result should be independent of the remote setting, e.g.,

$$\begin{aligned} A(\lambda) &\stackrel{\text{def}}{=} X_{a,c}(\lambda)=X_{a,d}(\lambda) \\ C'(\lambda) &\stackrel{\text{def}}{=} X'_{a,c}(\lambda)=X'_{b,c}(\lambda). \end{aligned}$$

***(iii)*** *Measurement result restriction.* The results may only range from $-1$ to $+1$,

$$V = \{x \in \mathbb{R}; -1 \leq x \leq +1\}.$$

Then

$$\left|E(AC') + E(AD')\right| + \left|E(BC') - E(BD')\right| \leq 2. \tag{2}$$

The proof consists of simple algebraic manipulations inside each of the two expressions on the right hand side, followed by application of the triangle inequality on each expression.

Previous treatments have discussed several loopholes in this inequality, but the most similar issue to the present is the "detector efficiency" problem. A simple formalism to use is that of Ref. [12], where inefficient detectors, or in more general terms, inefficient *measurement setups* are modeled by having the measurement-result RVs *undefined* at points in $\Lambda$ where no detection occurs. This means that, e.g., the RVs $A$ and $C'$ will only be defined at subsets of $\Lambda$ denoted $\Lambda_A$ and $\Lambda_{C'}$, resp.. The averaging must now be restricted to the set where the RV in question is defined, and the probability measure adjusted accordingly. In the language of probability theory we need the *conditional* expectation value

$$E(A|\Lambda_A) = \int_{\Lambda_A} X_A dP_A, \tag{3}$$

using the conditional probability measure

$$P_A(S) = P(S|\Lambda_A) \text{ for all events } S. \tag{4}$$

In the detector efficiency case, the first correlation in the CHSH inequality then is $E(AC'|\Lambda_A \cap \Lambda_{C'})$, the expectation of $AC'$ conditioned on both factors in the product being defined (that both results are observed). This is the correlation that would be obtained from an experimental setup where the coincidence counters are told to ignore single particle events.

In the present case, there is a slight difference. In a hidden-variable model, the detection times $T$ and $T'$ at the two sites can be described as RVs that depend on the settings, e.g.,

$$\begin{aligned} T_{a,c} : \Lambda &\to \mathbb{R} \\ \lambda &\mapsto T_{a,c}(\lambda) \\ T'_{a,c} : \Lambda &\to \mathbb{R} \\ \lambda &\mapsto T'_{a,c}(\lambda). \end{aligned} \tag{5}$$

A "coincidence" then occurs when the two times differ by less than some predetermined time interval $\Delta T$. In mathematical language this corresponds to saying that coincidences occur for certain values of the hidden variable $\lambda$, e.g., at the settings $a$ and $c$, values in the set

$$\Lambda_{AC'} \stackrel{\text{def}}{=} \left\{\lambda : \left|T_{a,c}(\lambda) - T'_{a,c}(\lambda)\right| < \Delta T\right\}. \tag{6}$$

In the following, we will concentrate on the set $\Lambda_{AC'}$ (but it does help to remember its origin), and this set can vary depending on both detector settings. Note that no assumption has to be made on the locality of the detection times $T$ and $T'$ at the two sites; they may depend on both settings. While unphysical, remember that we are concentrating on the resulting statistics of $A$, $B$, $C'$, and $D'$ here, and only use the times to find coincidences. If it makes the reader feel better, he/she may use an implicit locality assumption ($T_a = T_{a,c}$ and $T'_c = T'_{a,c}$), but that is of no consequence below.

The first correlation in the CHSH inequality then is $E(AC'|\Lambda_{AC'})$, the expectation of $AC'$ conditioned on coincidences for the settings $a$ and $c$. The original CHSH inequality is no longer valid, and the reason can be seen in the start of the proof where one wants to add

$$\begin{aligned} &\left|E(AC'|\Lambda_{AC'}) + E(AD'|\Lambda_{AD'})\right| \\ &\quad = \left|\int_{\Lambda_{AC'}} AC' dP_{AC'} + \int_{\Lambda_{AD'}} AD' dP_{AD'}\right|. \end{aligned} \tag{7}$$

The integrals on the right-hand side cannot easily be added when $\Lambda_{AC'} \neq \Lambda_{AD'}$, since we are taking expectations over different ensembles $\Lambda_{AC'}$ and $\Lambda_{AD'}$, with respect to different probability measures.

The problem here is that the ensemble on which the correlations are evaluated changes with the settings, while the original Bell inequality requires that they stay the same. In effect, the Bell inequality only holds on the common part of the four different ensembles $\Lambda_{AC'}$, $\Lambda_{AD'}$, $\Lambda_{BC'}$, and $\Lambda_{BD'}$, i.e., for correlations of the form

$$E(AC'|\Lambda_{AC'} \cap \Lambda_{AD'} \cap \Lambda_{BC'} \cap \Lambda_{BD'}). \tag{8}$$

Unfortunately our experimental data comes in the form

$$E(AC'|\Lambda_{AC'}), \tag{9}$$

so we need an estimate of the relation of the common part to its constituents:

$$\begin{aligned} \delta &= \inf_{\text{settings}} \frac{P(\Lambda_{AC'} \cap \Lambda_{AD'} \cap \Lambda_{BC'} \cap \Lambda_{BD'})}{P(\Lambda_{AC'})} \\ &= \inf_{\text{settings}} P(\Lambda_{AD'} \cap \Lambda_{BC'} \cap \Lambda_{BD'}|\Lambda_{AC'}). \end{aligned} \tag{10}$$

This is a purely theoretical construct, not available in experimental data, but we will relate it to experimental data below. Anyhow, fixing this relative size, we can prove

*Theorem 2 (The CHSH inequality with coincidence restriction)* The prerequisites (i–iii) of Theorem 1 are assumed to hold except at a null set, as is

***(iv)*** *Coincident events.* Correlations are obtained on subsets of $\Lambda$, namely on

$$\Lambda_{AC'},\ \Lambda_{AD'},\ \Lambda_{BC'},\ \text{or } \Lambda_{BD'}.$$

Then

$$\begin{aligned} &\left|E(AC'|\Lambda_{AC'}) + E(AD'|\Lambda_{AD'})\right| \\ &\quad + \left|E(BC'|\Lambda_{BC'}) - E(BD'|\Lambda_{BD'})\right| \leq 4 - 2\delta. \end{aligned} \tag{11}$$

*Proof.* The proof consists of two steps; the first part is similar to the proof of Theorem 1, using the intersection

$$\Lambda_{\mathrm{I}} = \Lambda_{AC'} \cap \Lambda_{AD'} \cap \Lambda_{BC'} \cap \Lambda_{BD'}, \tag{12}$$

on which coincidences occur for all relevant settings. This ensemble may be empty, but only when $\delta = 0$ and then the inequality is trivial, so $\delta > 0$ can be assumed in the rest of the proof. Now (i–iii) yields

$$\Big|E(AC'|\Lambda_{\mathrm{I}}) + E(AD'|\Lambda_{\mathrm{I}})\Big| + \Big|E(BC'|\Lambda_{\mathrm{I}}) - E(BD'|\Lambda_{\mathrm{I}})\Big| \le 2. \tag{13}$$

The second step is to translate this into an expression with $E(AC'|\Lambda_{AC'})$ and so on. For brevity, let $\Lambda_{\mathrm{O}} = \Lambda_{AD'} \cap \Lambda_{BC'} \cap \Lambda_{BD'}$ and denote "set complement" by $\complement$. Then, $\Lambda_{\mathrm{I}} = \Lambda_{\mathrm{O}} \cap \Lambda_{AC'}$ and

$$\begin{aligned} E(AC'|\Lambda_{AC'}) =& P(\Lambda_{\mathrm{O}}|\Lambda_{AC'})E(AC'|\Lambda_{\mathrm{O}} \cap \Lambda_{AC'}) \\ &+ P(\Lambda_{\mathrm{O}}^{\complement}|\Lambda_{AC'})E(AC'|\Lambda_{\mathrm{O}}^{\complement} \cap \Lambda_{AC'}). \end{aligned} \tag{14}$$

We now have

$$\begin{aligned} &\Big|E(AC'|\Lambda_{AC'}) - \delta E(AC'|\Lambda_{\mathrm{I}})\Big| \\ &\quad \le \Big|P(\Lambda_{\mathrm{O}}^{\complement}|\Lambda_{AC'})E(AC'|\Lambda_{\mathrm{O}}^{\complement} \cap \Lambda_{AC'})\Big| \\ &\qquad + \Big|P(\Lambda_{\mathrm{O}}|\Lambda_{AC'})E(AC'|\Lambda_{\mathrm{O}} \cap \Lambda_{AC'}) - \delta E(AC'|\Lambda_{\mathrm{I}})\Big| \\ &\quad \le P(\Lambda_{\mathrm{O}}^{\complement}|\Lambda_{AC'})E\big(|AC'|\big|\Lambda_{\mathrm{O}}^{\complement} \cap \Lambda_{AC'}\big) \\ &\qquad + \Big(P(\Lambda_{\mathrm{O}}|\Lambda_{AC'}) - \delta\Big)E\big(|AC'|\big|\Lambda_{\mathrm{I}}\big) \\ &\quad \le P(\Lambda_{\mathrm{O}}^{\complement}|\Lambda_{AC'}) + P(\Lambda_{\mathrm{O}}|\Lambda_{AC'}) - \delta = 1 - \delta, \end{aligned} \tag{15}$$

which, together with Ineq. (13) and the triangle inequality, yields the desired result after some simple manipulations. □

Let us now relate this to experimental quantities. In this context, the quantity of greatest interest is the probability of coincidence:

$$\gamma \overset{\text{def}}{=} \inf_{\text{settings}} P(\Lambda_{AC'}). \tag{16}$$

We now have

$$\delta \ge 4 - \frac{3}{\gamma} \tag{17}$$

because (Bonferroni)

$$\begin{aligned} &P(\Lambda_{AD'} \cap \Lambda_{BC'} \cap \Lambda_{BD'}|\Lambda_{AC'}) \\ &\quad \ge P(\Lambda_{AD'}|\Lambda_{AC'}) + P(\Lambda_{BC'}|\Lambda_{AC'}) + P(\Lambda_{BD'}|\Lambda_{AC'}) - 2, \end{aligned} \tag{18}$$

and

$$\begin{aligned} P(\Lambda_{BD'}|\Lambda_{AC'}) &= \frac{P(\Lambda_{AC'} \cap \Lambda_{BD'})}{P(\Lambda_{AC'})} \\ &= \frac{P(\Lambda_{AC'}) + P(\Lambda_{BD'}) - P(\Lambda_{AC'} \cup \Lambda_{BD'})}{P(\Lambda_{AC'})} \\ &\ge 1 + \frac{P(\Lambda_{BD'}) - 1}{P(\Lambda_{AC'})} \ge 1 + \frac{\gamma - 1}{\gamma} = 2 - \frac{1}{\gamma}. \end{aligned} \tag{19}$$

Putting this into our modified CHSH inequality we arrive at

$$\begin{aligned} &\Big|E(AC'|\Lambda_{AC'}) + E(AD'|\Lambda_{AD'})\Big| \\ &\quad + \Big|E(BC'|\Lambda_{BC'}) - E(BD'|\Lambda_{BD'})\Big| \le \frac{6}{\gamma} - 4. \end{aligned} \tag{20}$$

The bound for violation by quantum mechanics here is $\gamma > 3 - \frac{3}{\sqrt{2}} \approx 0.8787$, which is considerably higher than the corresponding value for the detector-efficiency case, $\frac{1}{\sqrt{2}} \approx 0.7071$.

Let us see whether this bound is necessary and sufficient. At the same time, we answer the question if it would be possible to lower the bound by putting further natural constraints on the model. This will be done by construction of an *ad hoc* model that will give the quantum predictions at the settings $a = 0$, $b = \pi/2$, $c = \pi/4$, and $d = -\pi/4$, and the additional natural constraints are: it will only use local data, even for timing; the marginal distributions are correct; there is full correlation if the settings are equal at the two sites; and the coincidence probability is the same at our specified pairs of settings.

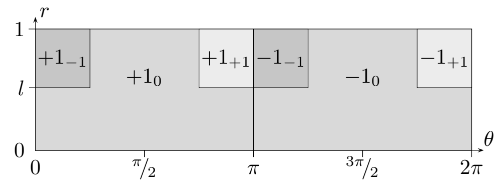


FIG. 2: Outcome pattern for the detectors. The subscripts are the detection times. Thus $+1_0$ means outcome $+1$ at time 0.

The model is as follows: the hidden variable $\lambda$ is a pair $(\theta, r)$ of coordinates, uniformly distributed over the rectangle indicated in Fig. 2. The local detector setting corresponds to a shift in the $\theta$-direction of the pattern, with wrap-around when necessary. The result is obtained according to the diagram (the subscript is the detection time which can be $\pm 1$ or 0). To make the behaviour interesting we choose $\Delta T$ to be 3/2, so that a time-difference of zero or one time unit(s) is a coincidence while a time-difference of two time units will not be a coincidence.

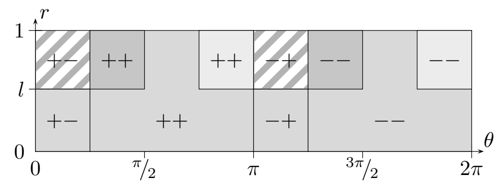


FIG. 3: Coincidences occur as follows: the events are truly coincident in the middle-gray areas, and since $\Delta T > 1$ events are "coincident" in the other grey areas, but since $\Delta T < 2$ events are "non-coincident" in the hatchmarked areas.

For example, for the settings $a = 0$ and $c = \pi/4$ at the two sites, there will be coincidences at the $\lambda$s indicated in Fig. 3, so that the probability of coincidence is $3/4+l/4$, while the probability of getting $++$ or $--$ is $3/4$. For the settings $b=\pi/2$ and $d=-\pi/4$ at the two sites, the probability of coincidence would again be $3/4+l/4$, while the probability of getting $++$ or $--$ would only be $l/4$, so that

$$\begin{aligned}E(AC'|\Lambda_{AC'}) &= E(AD'|\Lambda_{AD'})\\ &= E(BC'|\Lambda_{BC'}) = -E(BD'|\Lambda_{BD'}) = \frac{3-l}{3+l}.\end{aligned} \tag{21}$$

Setting $(3-l)/(3+l)=1/\sqrt{2}$, i.e., $l=3(3-2\sqrt{2})\approx 0.5147$ we obtain

$$\gamma=\frac{3+l}{4}=\frac{3+3(3-2\sqrt{2})}{4}=3-\frac{3}{\sqrt{2}}, \tag{22}$$

which saturates the derived coincidence probability bound. This model does what we have asked of it so far, especially, it violates the Bell inequality maximally. The model does not have constant coincidence probability for *all* angular settings but can easily be modified so that it does [22]. Furthermore, the interference pattern is not sinusoidal, and delays are discrete, but this will be a subject of further research (see e.g., Ref. [13]).

We have seen that a useful Bell inequality does hold if sufficiently many events are simultaneous, in stark contrast to claims to the opposite in, e.g., Ref. [14, 16]. That it needs to be modified when events are allowed to drop from the statistics is not surprising but to be expected, cf. previous analysis in the low-efficiency case. It is perhaps more surprising that the bound on the amount of coincidences is higher in this case than in the efficiency case. The reason for this is that the set of coincidences ("counted" events) $\Lambda_{AC'}$ factors in the efficiency case, i.e., $\Lambda_{AC'}=\Lambda_A\cap\Lambda_{C'}$ (see Ref. [12]), while here the set cannot be factorized. Thus, the present treatment is a proper generalization of the previous results. A major remaining challenge is to extend the analysis to the situation when coincidence, detection *and* memory loopholes (see Refs. [17, 18]) are all present.

Note that several modern experiments are not affected by this loophole, such as the ion trap experiment by Rowe et al [15], because there, *all* experimental runs produce coincidences (although there, *locality* is not strictly enforced). In the case of optical experiments that use a pulsed variant of the common parametric down-conversion source of Ref. [19], one can use the natural assumption that the setting-dependent delays described by $T_{a,c}$ and $T'_{a,c}$ do not depend on the relation between pulse length and pulse spacing. Then, if the pulse (i.e., the driving pulse of the parametric downconverter) is short in comparison to the pulse spacing, one can assume that any delays that occur will not delay photons from the time-window of one pulse to the next, or at least that this will happen only with very low probability. Now, the driving pulse will provide a well-defined, *pre-determined* coincidence window and this will, in effect, remove this loophole. What remains is a lowered efficiency, because photons may be delayed enough to arrive outside this coincidence window. Proper selection of a somewhat longer window length may be needed. In any case, using a pulsed source and making these two natural assumptions will enable use of the previous lower bound (e.g., from Ref. [12]).

In conclusion, we have shown that the coincidence loophole is significantly more damaging than the well-studied detection problem. Fortunately, the damage can be quantified. The results underline the importance of eliminating coincidence post-selection in future pulsed experiments.

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[1] J. S. Bell, Physics **1**, 195 (1964).
[2] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969).
[3] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[4] D. Bohm and Y. Aharonov, Phys. Rev. **108**, 1070 (1957).
[5] A. Aspect, P. Grangier, and G. Roger, Phys. Rev. Lett. **47**, 460 (1981).
[6] A. Aspect, P. Grangier, and G. Roger, Phys. Rev. Lett. **49**, 91 (1982).
[7] A. Aspect, J. Dalibard, and G. Roger, Phys. Rev. Lett. **49**, 1804 (1982).
[8] P. Pearle, Phys. Rev. D **2**, 1418 (1970).
[9] J. F. Clauser and M. A. Horne, Phys. Rev. D **10**, 526 (1974).
[10] A. Garg and N. D. Mermin, Phys. Rev. D **35**, 3831 (1987).
[11] A. Shimony, *Search for a naturalistic world view* (Cambridge Univ. Press, Cambridge, 1993), Vol. II.
[12] J.-Å. Larsson, Phys. Rev. A **57**, 3304 (1998).
[13] J.-Å. Larsson, Phys. Lett. A **256**, 245 (1999).
[14] K. Hess and W. Philipp, Proc. Nat. Acad. Sci. USA **98**, 14224 (2001).
[15] M. A. Rowe *et al.*, Nature (London) **409**, 791 (2001).
[16] L. Accardi, K. Imafuku., and M. Regoli, Infinite Dimensional Analysis, Quantum Probability and Related Fields **5**, 1 (2002).
[17] R. D. Gill, in *Mathematical Statistics and Applications*, Vol. 42 of *IMS Lecture Notes – Monograph Series*, edited by M. Moore, S. Froda, and C. Léger (Institute of Mathematical Statistics, Beachwood, Ohio, 2003), pp. 133–154, quant-ph/0110137.
[18] R. D. Gill, in *Foundations of Probability and Physics - 2*, Vol. 5 of *Math. Modelling in Phys., Engin., and Cogn. Sc.* (Växjö Univ. Press, Växjö, Sweden, 2003), pp. 179–206, quant-ph/0301059.
[19] P. G. Kwiat *et al.*, Phys. Rev. Lett. **75**, 4337 (1995).
[20] It may seem that we only discuss the "deterministic" case here, but a generalization to the "stochastic" case is straightforward and will not be done here.
[21] The reason for this is that in the simplest case $|X|=|X'|=1$ and $E(X)=E(X')=0$, and then the correlation is precisely $E(XX')$. Here, this terminology will be retained even for cases when this ceases to be valid, as it has become standard in this context
[22] The relative delays in our ad hoc model vary in probability, but this can also be fixed, introducing some additional complexity.