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EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computer Science

CASA-Report 09-27
August 2009

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ISSN: 0926-4507

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Preprint of an article submitted for consideration in

August 12, 2009

Abstract

This paper deals with the analysis of Hamiltonian Hopf as well as saddle-centre bifurcations in 4-DOF systems defined by perturbed isotropic oscillators (1:1:1:1 resonance), in the presence of two quadratic symmetries Ξ and L_1. When we normalize the system with respect to the quadratic part of the energy and carry out a reduction with respect to a 3-torus group we end up with a 1-DOF system with several parameters on the thrice reduced phase space. Then, we focus our analysis on the evolution of relative equilibria around singular points of this reduced phase space. In particular, dealing with the Hamiltonian Hopf bifurcation the 'geometric approach' is used, following the steps set up by one of the authors in the context of 3-DOF systems. In order to see the interplay between integrals and physical parameters in the analysis of bifurcations, we consider as perturbation a one-parameter family, which in particular includes one of the classical Stark-Zeeman models (parallel case) in 3 dimensions.

Keywords: *Hamiltonian system, fourfold 1:1 resonance, bifurcation, normal form, reduction, Hamiltonian Hopf bifurcation*

1 Introduction

T The analysis of bifurcations of relative equilibria of perturbed Hamiltonian systems with symmetries covers a vaste literature. Among them, recently progress has been made, in particular, in the understanding of bifurcations emanating from singular points of the algebraic varieties defining the reduced phase spaces. More precisely we are interested here in enlarging the studies done in relation to the 1:1:1 resonance [7, 14, 15, 16, 17, 18, 19, 20, 21], considering now one more resonance with the same frequency; a preliminary report communicating part of the full study presented here was presented recently [12](see also [11]). In [28] it is shown that the reduced phase space of the n-dimensional harmonic oscillator is $\mathbb{CP}^{n-1}$. Since then, the cases $n = 2, 3$ have been considered by many authors. To our knowledge, the case $n = 4$ has not been treated systematically. Only when perturbed Keplerian systems have been immersed in 4-D through the KS or Moser regularization transformation [4, 24, 25, 5, 9, 10], authors have encountered the need of studying the 1:1:1:1 resonance, but with the constraint imposed by the KS transformation, which in our notation corresponds to one of our integrals (see (3), (4)) being zero: $\Xi = 0$.

To be more precise, on $\mathbb{R}^8$ with the standard symplectic form consider a Hamiltonian system with Hamiltonian

$$H = H_2 + H_3 + H_4 + \text{h.o.t.}\ , \tag{1}$$

with

$$H_2(q, Q) = \tfrac{1}{2}\,(Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2) + \tfrac{1}{2}\,\omega^2(q_1^2 + q_2^2 + q_3^2 + q_4^2)\ , \tag{2}$$

and H_i are polynomials in the variables (q_j, Q_j). Moreover we will assume that H has two rotationals integrals

$$\Xi(q, Q) = q_1 Q_2 - Q_1 q_2 + q_3 Q_4 - Q_3 q_4\ , \tag{3}$$

$$L_1(q, Q) = -q_1 Q_2 + Q_1 q_2 + q_3 Q_4 - Q_3 q_4\ . \tag{4}$$

In this paper, without loss of generality, in what follows we take $\omega = 1$ to simplify our study.

Two types of situations may be modeled by the Hamiltonian function (1). The first one corresponds to the Taylor expansion around elliptic equilibria of dynamical systems in the resonance 1:1:1:1; the second type appears when, making use of the KS transformation and rescaling of the independent variable, perturbed Keplerian-Coulomb systems are transformed into perturbed isotropic oscillators. In the first case the equilibrium is taken as the new origin, thus the value of the Hamiltonian is zero at it (and also $H_2 = n = 0$), and will take small values when we study the effects of H_i. In the second case, as it is well known, the value of the Hamiltonian is (see eq (3.2) p. 59 [10]) not zero; then, our perturbation analysis will be around that value.

Our interest is to deal with the general scenario corresponding to the 4-D harmonic oscillator and its toral reductions, to the point where we can identify thrice reduced phase spaces of different type, with one or two singular points. As we said before the regular reduction related to the energy normalization $H_2 = n$ lead us to $\mathbb{CP}^3$. The second singular reduction defined by the action associated to $\Xi = \xi$ has a reduced space diffeomorphic to $\mathbb{S}^2_{n-\xi} \times \mathbb{S}^2_{n+\zeta}$. Later, dealing with applications when $\Xi = 0$, we connect our problem with perturbed Keplerian systems; it is well known that the reduced space for normalized perturbed Keplerian systems can be identified with $\mathbb{S}^2_n \times \mathbb{S}^2_n$ [4]. It is for this reason that the reductions are performed in this order. Finally, if the integral $L_1 = \ell$ is also considered, the associated third (singular) reduction gives as reduced phase space a surface, the nature of which depends on the values of the integrals H_2, Ξ and L_1. That surface has has one or two singular points depending on the relative values of ξ and ℓ.

In [19, 20, 21] it was shown that Hamiltonian Hopf bifurcations (HHB) can only arise at cone-like singular points of the reduced phase spaces. Here, after having identified the reduced phase spaces, we focus on these points for systems originally defined in 4-D. Although the methods already developed: 'algebraic', 'geometric', etc. are, in principle, directly applicable in this case, we still have to perform a careful analysis of the role played by and relative influence of the integrals as well as its interrelation with the physical parameters of the system studied.

In order to illustrate our analysis, in this paper we consider the perturbed harmonic oscillator in resonance 1:1:1:1 given by

$$\begin{aligned} H(q,Q) \;=\; & \frac{1}{2}(Q,Q) + \frac{1}{2}(q,q) \\ & + \varepsilon \left[\lambda \,(q_1^2+q_2^2+q_3^2+q_4^2)(-q_1^2-q_2^2+q_3^2+q_4^2)\right. \\ & \quad \left. + \, 4\,(q_1^2+q_2^2+q_3^2+q_4^2)\{(q_1q_4-q_2q_3)^2+(q_1q_3+q_2q_4)^2\}\right], \end{aligned} \tag{5}$$

where $(\,,\,)$ is the standard inner product, λ is an external parameter and ε an small parameter.

Note that the application has a secondary importance in this paper. Nonetheless, apart from its intrinsic physical interest, this model is generalizing the perturbed Keplerian systems, due to its simple definition as a quadratic 1-parameter perturbation in 3-D Cartesian co-ordinates. As we see it as a particular case of a perturbed 4-D oscillator, we ask ourselves: Does, in order to have a HHB in this oscillator, the integral Ξ have to be zero?

In this respect, in order to make the presentation clear, the physical model we consider (see below), is made of two terms, which only introduce one parameter λ, defined by the ratio of the quantities defining the electric and magnetic forces, both considered to be small with respect to the unperturbed part.

The paper is organized as follows. In Sec. 2 we carry out the $\mathbb{T}^3$ normalization and reduction of problem identifying the invariants defining those spaces. We apply this process to the parallel Stark-Zeeman problem. The structure of the thrice reduced phase spaces, regular and singular, is presented in detail. In Sec. 3 the energy-momentum mapping is introduced and the analysis of is critical values lead to identify the bifurcations lines associated to Hamiltonian Hopf and saddle-center bifurcations. In Sec. 4 we present the collection of the different flows. In Sec. 5 we study in detail the Hamiltonian Hopf bifurcation connected to the singular points of our thrice reduced phase space. Finally in Sec. 6 we related our work with the parallel Stark-Zeeman case as a perturbed Keplerian system, whose normalized dynamics is connected with the $\Xi = 0$ manifold through the KS transformation.

2 Normalization and Toral Reduction

2.1 Reduction to $\mathbb{CP}^3$

There are 16 invariants for the action corresponding to H_2 (see [6]):

$$\begin{array}{lll}
\pi_1 = Q_1^2 + q_1^2 \ , & \pi_2 = Q_2^2 + q_2^2 \ , & \pi_3 = Q_3^2 + q_3^2 \ , \\
\pi_4 = Q_4^2 + q_4^2 \ , & \pi_5 = Q_1Q_2 + q_1q_2 \ , & \pi_6 = Q_1Q_3 + q_1q_3 \ , \\
\pi_7 = Q_1Q_4 + q_1q_4 \ , & \pi_8 = Q_2Q_3 + q_2q_3 \ , & \pi_9 = Q_2Q_4 + q_2q_3 \ , \\
\pi_{10} = Q_3Q_4 + q_3q_4 \ , & \pi_{11} = -Q_1q_2 + q_1Q_2 \ , & \pi_{12} = -Q_1q_3 + q_1Q_3 \ , \\
\pi_{13} = -Q_1q_4 + q_1Q_4 \ , & \pi_{14} = -Q_2q_3 + q_2Q_3 \ , & \pi_{15} = -Q_2q_4 + q_2Q_4 \ , \\
\pi_{16} = -Q_3q_4 + q_3Q_4 \ . & &
\end{array} \tag{6}$$

These invariants can be easily derived using complex conjugate co-ordinates and are subjected to the relations given in Table (1) , defining the first reduced phase space.

Theorem 2.1 *The invariants given in (6) verify the relations given in Table 1. These relations define $\mathbb{CP}^3$ as a submanifold of $\mathbb{R}^{16}$.*

Proof : One can verify the relations Table (1) given by the theorem by substituting at both sides the definitions of π_i given in (6). In order to show that they define $\mathbb{CP}^3$ we introduce complex conjugate coordinates (z_1, z_2, z_3, z_4), by means of

$$z_1 = Q_1 + iq_1, \quad z_2 = Q_2 + iq_2, \quad z_3 = Q_3 + iq_3, \quad z_4 = Q_4 + iq_4$$

and we introduce the invariants of the harmonic oscillator over $\mathbb{C}^4$

$$\begin{array}{lll}
\sigma_1 = z_1\overline{z}_1 = \pi_1, & \sigma_2 = z_2\overline{z}_2 = \pi_2 \ , & \\
\sigma_3 = z_3\overline{z}_3 = \pi_3, & \sigma_4 = z_4\overline{z}_4 = \pi_4 \ , & \\
\sigma_5 = z_1\overline{z}_2 = \pi_5 + i\pi_{11}, & \sigma_6 = z_1\overline{z}_3 = \pi_6 + i\pi_{12}, & \sigma_7 = z_1\overline{z}_4 = \pi_7 + i\pi_{13} \ , \\
\sigma_8 = z_2\overline{z}_3 = \pi_8 + i\pi_{14}, & \sigma_9 = z_2\overline{z}_4 = \pi_9 + i\pi_{15}, & \sigma_{10} = z_3\overline{z}_4 = \pi_{10} + i\pi_{16} \ .
\end{array} \tag{7}$$

$\pi_1\pi_2 = \pi_5^2 + \pi_{11}^2$ $\pi_1\pi_4 = \pi_7^2 + \pi_{13}^2$ $\pi_2\pi_4 = \pi_9^2 + \pi_{15}^2$	$\pi_1\pi_3 = \pi_6^2 + \pi_{12}^2$ $\pi_2\pi_3 = \pi_8^2 + \pi_{14}^2$ $\pi_3\pi_4 = \pi_{10}^2 + \pi_{16}^2$
$\pi_1\pi_8 = \pi_5\pi_6 + \pi_{11}\pi_{12}$ $\pi_1\pi_9 = \pi_5\pi_7 + \pi_{11}\pi_{13}$ $\pi_1\pi_{10} = \pi_6\pi_7 + \pi_{12}\pi_{13}$ $\pi_2\pi_6 = \pi_5\pi_8 - \pi_{11}\pi_{14}$ $\pi_2\pi_7 = \pi_5\pi_9 - \pi_{11}\pi_{15}$ $\pi_2\pi_{10} = \pi_8\pi_9 + \pi_{14}\pi_{15}$ $\pi_3\pi_5 = \pi_6\pi_8 + \pi_{12}\pi_{14}$ $\pi_3\pi_7 = \pi_6\pi_{10} - \pi_{12}\pi_{16}$ $\pi_3\pi_9 = \pi_8\pi_{10} - \pi_{14}\pi_{16}$ $\pi_4\pi_5 = \pi_7\pi_9 + \pi_{13}\pi_{15}$ $\pi_4\pi_6 = \pi_7\pi_{10} + \pi_{13}\pi_{16}$ $\pi_4\pi_8 = \pi_9\pi_{10} + \pi_{15}\pi_{16}$	$\pi_1\pi_{14} = \pi_5\pi_{12} - \pi_6\pi_{11}$ $\pi_1\pi_{15} = \pi_5\pi_{13} - \pi_7\pi_{11}$ $\pi_1\pi_{16} = \pi_6\pi_{13} - \pi_7\pi_{12}$ $\pi_2\pi_{12} = \pi_5\pi_{14} + \pi_8\pi_{11}$ $\pi_2\pi_{13} = \pi_5\pi_{15} + \pi_9\pi_{11}$ $\pi_2\pi_{16} = \pi_8\pi_{15} - \pi_9\pi_{14}$ $\pi_3\pi_{11} = \pi_8\pi_{12} - \pi_6\pi_{14}$ $\pi_3\pi_{13} = \pi_{10}\pi_{12} + \pi_6\pi_{16}$ $\pi_3\pi_{15} = \pi_{10}\pi_{14} + \pi_8\pi_{16}$ $\pi_4\pi_{11} = \pi_9\pi_{13} - \pi_7\pi_{15}$ $\pi_4\pi_{12} = \pi_{10}\pi_{13} - \pi_7\pi_{16}$ $\pi_4\pi_{14} = \pi_{10}\pi_{15} - \pi_9\pi_{16}$
$\pi_5\pi_{10} - \pi_7\pi_8 = \pi_{13}\pi_{14} + \pi_{11}\pi_{16}$ $\pi_7\pi_8 - \pi_6\pi_9 = \pi_{13}\pi_{14} - \pi_{12}\pi_{15}$ $\pi_5\pi_{10} + \pi_{11}\pi_{16} = \pi_6\pi_9 + \pi_{12}\pi_{15}$	$\pi_{10}\pi_{11} + \pi_5\pi_{16} = \pi_8\pi_{13} - \pi_7\pi_{14}$ $\pi_9\pi_{12} + \pi_6\pi_{15} = \pi_7\pi_{14} + \pi_8\pi_{13}$ $\pi_{10}\pi_{11} - \pi_5\pi_{16} = \pi_9\pi_{12} - \pi_6\pi_{15}$
$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 2n$	

Table 1: Relations among the invariants given in (6)

Then, relations given by Table (1) read

$$
\begin{array}{lll}
\sigma_1\sigma_2 = \sigma_5\overline{\sigma}_5 & \sigma_1\sigma_3 = \sigma_6\overline{\sigma}_6 & \sigma_1\sigma_4 = \sigma_7\overline{\sigma}_7 \\
\sigma_2\sigma_3 = \sigma_8\overline{\sigma}_8 & \sigma_2\sigma_4 = \sigma_9\overline{\sigma}_9 & \sigma_3\sigma_4 = \sigma_{10}\overline{\sigma}_{10} \\
\sigma_1\sigma_8 = \overline{\sigma}_5\sigma_6 & \sigma_1\sigma_9 = \overline{\sigma}_5\sigma_7 & \sigma_1\sigma_{10} = \overline{\sigma}_6\sigma_7 \\
\sigma_2\sigma_6 = \sigma_5\sigma_8 & \sigma_2\sigma_7 = \sigma_5\sigma_9 & \sigma_2\sigma_{10} = \overline{\sigma}_8\sigma_9 \\
\sigma_3\sigma_5 = \overline{\sigma}_8\sigma_6 & \sigma_3\sigma_7 = \sigma_6\sigma_{10} & \sigma_3\sigma_9 = \sigma_8\sigma_{10} \\
\sigma_4\sigma_5 = \overline{\sigma}_9\sigma_7 & \sigma_4\sigma_6 = \overline{\sigma}_{10}\sigma_7 & \sigma_4\sigma_8 = \overline{\sigma}_{10}\sigma_9 \\
\sigma_5\sigma_{10} = \overline{\sigma}_8\sigma_7 & \sigma_6\sigma_9 = \sigma_7\sigma_8 & \sigma_5\overline{\sigma}_{10} = \sigma_6\overline{\sigma}_9 \\
\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 = 2n & &
\end{array}
\tag{8}
$$

Note that from the third element of the first column of (8) to the last in the third column we have the complex version of the two columns which, given in terms of π_i, are appearing in (1). If we replace now in (8) each σ by its corresponding complex expressions $z \in \mathbb{C}$ we obtain the definition of $\mathbb{CP}^3$ given in for instance [6]. **q.e.d.**

This invariants provide an orbit mapping

$$\rho_1 : \mathbb{R}^8 \to \mathbb{R}^{16}; (q_1, q_2, q_3, q_4, Q_1, Q_2, Q_3, Q_4) \to (\pi_1, \cdots, \pi_{16}) \, .$$

We may now obtain the reduced phase space by putting $H_2 = n$ in the image of ρ_1. This is a regular reduction, whence the reduced phase space is a smooth symplectic manifold. As was shown the relations given in Table (1) define a 6 dimensional reduced phase space isomorphic to $\mathbb{CP}^3$. The brackets for the invariants π_i defining the Poisson structure on the reduced phase space are given in Table 2.

The normal form of H with respect to H_2 can now be expressed in these invariants. We get $\bar{H} = H_2 + \varepsilon \bar{H}^\lambda + \mathcal{O}(\varepsilon^2)$ with

$$
\begin{aligned}
\bar{H}^\lambda(\pi) \;=\; & \frac{1}{16}\lambda\left[6\,(\pi_1+\pi_2+\pi_3+\pi_4)\,(-\pi_1-\pi_2+\pi_3+\pi_4)+8\,(\pi_{11}-\pi_{16})\,(\pi_{11}+\pi_{16})\right] \\
& +\frac{1}{32}(\pi_1+\pi_2+\pi_3+\pi_4)\,\big[-6\,(\pi_1+\pi_2-\pi_3-\pi_4)^2+16\,(\pi_7-\pi_8)^2+16\,(\pi_6+\pi_9)^2 \\
& \qquad -16\,(\pi_{13}-\pi_{14})^2-16\,(\pi_{12}+\pi_{15})^2-8\,(\pi_{11}-\pi_{16})^2-8\,(\pi_{11}+\pi_{16})^2\big] \\
& +\frac{1}{2}(\pi_1+\pi_2-\pi_3-\pi_4)\,(\pi_{11}-\pi_{16})\,(\pi_{11}+\pi_{16}).
\end{aligned} \tag{9}
$$

The corresponding Poisson system is $\dot{\pi}_i = \{\pi_i, \bar{H}^\lambda\}$.

2.2 Reduction to $\mathbb{S}^2_{n+\xi} \times \mathbb{S}^2_{n-\xi}$

We assume from now on that the families of systems we are interested in have as a first integral the quadratic function Ξ; this is the case of our model given by Eq. (5) as can be checked immediately. Then, to further reduce from $\mathbb{CP}^3$ to a new space, which turns out to be diffeomorphic to $\mathbb{S}^2_{n+\xi} \times \mathbb{S}^2_{n-\xi}$, one would have to fix $\Xi = \xi$ and divide out the S^1-action generated by Ξ. In order to carry out the reduction induced by the rotational symmetry Ξ on $\mathbb{CP}^3$, we will make use of the following proposition, which determines the algebra of the corresponding invariant functions. Those functions, as well as the relations among them, give the structure of this reduced space

Proposition 2.2 *Let φ_Ξ be the S^1-action generated by the Poisson flow of Ξ over $\mathbb{CP}^3$. The functions*

$$
\begin{array}{ll}
H_2(\pi) = \frac{1}{2}\,(\pi_1+\pi_2+\pi_3+\pi_4)\,, & \Xi(\pi) = \pi_{11}+\pi_{16}\,, \\
L_1(\pi) = -\pi_{11}+\pi_{16}\,, & L_2(\pi) = \pi_{12}+\pi_{15}\,, \\
L_3(\pi) = -\pi_{13}+\pi_{14}\,, & K_1(\pi) = \frac{1}{2}\,(-\pi_1-\pi_2+\pi_3+\pi_4)\,, \\
K_2(\pi) = -\pi_7+\pi_8\,, & K_3(\pi) = -\pi_6-\pi_9\,,
\end{array} \tag{10}
$$

$\{,\}$	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8
π_1	0	0	0	0	$2\pi_{11}$	$2\pi_{12}$	$2\pi_{13}$	0
π_2	0	0	0	0	$-2\pi_{11}$	0	0	$2\pi_{14}$
π_3	0	0	0	0	0	$-2\pi_{12}$	0	$-2\pi_{14}$
π_4	0	0	0	0	0	0	$-2\pi_{13}$	0
π_5	$-2\pi_{11}$	$2\pi_{11}$	0	0	0	π_{14}	π_{15}	π_{12}
π_6	$-2\pi_{12}$	0	$2\pi_{12}$	0	$-\pi_{14}$	0	π_{16}	π_{11}
π_7	$-2\pi_{13}$	0	0	$2\pi_{13}$	$-\pi_{15}$	$-\pi_{16}$	0	0
π_8	0	$-2\pi_{14}$	$2\pi_{14}$	0	$-\pi_{12}$	$-\pi_{11}$	0	0
π_9	0	$-2\pi_{15}$	0	$2\pi_{15}$	$-\pi_{13}$	0	$-\pi_{11}$	$-\pi_{16}$
π_{10}	0	0	$-2\pi_{16}$	$2\pi_{16}$	0	$-\pi_{13}$	$-\pi_{12}$	$-\pi_{15}$
π_{11}	$2\pi_5$	$-2\pi_5$	0	0	$\pi_2 - \pi_1$	π_8	π_9	$-\pi_6$
π_{12}	$2\pi_6$	0	$-2\pi_6$	0	π_8	$\pi_3 - \pi_1$	π_{10}	$-\pi_5$
π_{13}	$2\pi_7$	0	0	$-2\pi_7$	π_9	π_{10}	$\pi_4 - \pi_1$	0
π_{14}	0	$2\pi_8$	$-2\pi_8$	0	π_6	$-\pi_5$	0	$\pi_3 - \pi_2$
π_{15}	0	$2\pi_9$	0	$-2\pi_9$	π_7	0	$-\pi_5$	π_{10}
π_{16}	0	0	$2\pi_{10}$	$-2\pi_{10}$	0	π_7	$-\pi_6$	π_9

$\{,\}$	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}	π_{16}
π_1	0	0	$-2\pi_5$	$-2\pi_6$	$-2\pi_7$	0	0	0
π_2	$2\pi_{15}$	0	$2\pi_5$	0	0	$-2\pi_8$	$-2\pi_9$	0
π_3	0	$2\pi_{16}$	0	$2\pi_6$	0	$2\pi_8$	0	$-2\pi_{10}$
π_4	$-2\pi_{15}$	$-2\pi_{16}$	0	0	$2\pi_7$	0	$2\pi_9$	$2\pi_{10}$
π_5	π_{13}	0	$\pi_1 - \pi_2$	$-\pi_8$	$-\pi_9$	$-\pi_6$	$-\pi_7$	0
π_6	0	π_{13}	$-\pi_8$	$\pi_1 - \pi_3$	$-\pi_{10}$	π_5	0	$-\pi_7$
π_7	π_{11}	π_{12}	$-\pi_9$	$-\pi_{10}$	$\pi_1 - \pi_4$	0	π_5	π_6
π_8	π_{16}	π_{15}	π_6	π_5	0	$\pi_2 - \pi_3$	$-\pi_{10}$	$-\pi_9$
π_9	0	π_{14}	π_7	0	π_5	$-\pi_{10}$	$\pi_2 - \pi_4$	π_8
π_{10}	$-\pi_{14}$	0	0	π_7	π_6	π_9	π_8	$\pi_3 - \pi_4$
π_{11}	$-\pi_7$	0	0	π_{14}	π_{15}	$-\pi_{12}$	$-\pi_{13}$	0
π_{12}	0	$-\pi_7$	$-\pi_{14}$	0	π_{16}	π_{11}	0	$-\pi_{13}$
π_{13}	$-\pi_5$	$-\pi_6$	$-\pi_{15}$	$-\pi_{16}$	0	0	π_{11}	π_{12}
π_{14}	π_{10}	$-\pi_9$	π_{12}	$-\pi_{11}$	0	0	π_{16}	$-\pi_{15}$
π_{15}	$\pi_4 - \pi_2$	$-\pi_8$	π_{13}	0	$-\pi_{11}$	$-\pi_{16}$	0	π_{14}
π_{16}	$-\pi_8$	$\pi_4 - \pi_3$	0	π_{13}	$-\pi_{12}$	π_{15}	$-\pi_{14}$	0

Table 2: The bracket relations $\{\pi_i, \pi_j\}$.

are φ_Ξ-invariants functions. Those functions are bounded to the following constraints

$$\begin{aligned} K_1^2+K_2^2+K_3^2+L_1^2+L_2^2+L_3^2 &= H_2^2+\Xi^2\,, \\ K_1L_1+K_2L_2+K_3L_3 &= H_2\,\Xi\,. \end{aligned} \tag{11}$$

Moreover, these functions define the orbit mapping

$$\rho_2:\mathbb{R}^{16}\to\mathbb{R}^8; (\pi_1,\cdots,\pi_{16})\to(K_1,K_2,K_3,L_1,L_2,L_3,H_2,\Xi).$$

As we are making the reduction to $\mathbb{CP}^3$ we have that $H_2=n$. Moreover, if we fix $\Xi=\xi$, then it results that relations (11) define what we call the double reduced space, which is diffeomorphic to $\mathbb{S}^2_{n+\xi}\times\mathbb{S}^2_{n-\xi}$.

Proof : The Poisson flow generated by Ξ in $\mathbb{CP}^3$ is given by $\dot\pi_i=\{\pi_i,\Xi\}$, that is,

$$\begin{array}{llll} \dot\pi_1=-2\pi_5, & \dot\pi_2=2\pi_5, & \dot\pi_3=-2\pi_{10}, & \dot\pi_4=2\pi_{10}, \\ \dot\pi_5=\pi_1-\pi_2, & \dot\pi_6=-(\pi_7+\pi_8), & \dot\pi_7=\pi_6-\pi_9, & \dot\pi_8=\pi_6-\pi_9, \\ \dot\pi_9=\pi_7+\pi_8, & \dot\pi_{10}=\pi_3-\pi_4, & \dot\pi_{11}=0, & \dot\pi_{12}=-(\pi_{13}+\pi_{14}), \\ \dot\pi_{13}=\pi_{12}-\pi_{15}, & \dot\pi_{14}=\pi_{12}-\pi_{15}, & \dot\pi_{15}=\pi_{13}+\pi_{14}, & \dot\pi_{16}=0, \end{array}$$

from which we obtain that the invariant functions are the ones given by (10), i.e. we build up invariants which are zero along the flow. There is an alternative way for obtaining the invariants, which is considering the Poisson structure given by Table 2.

The relations among the invariants are given by the Casimirs of the Poisson structure defined by Table 3. The Casimirs of the structure will be H_2, Ξ joint with the ones given in (11). There should be four Casimirs because the Poisson structure on the image of the orbit map is an 8×8 matrix of rank four.

$\{,\}$	K_1	K_2	K_3	L_1	L_2	L_3
K_1	0	$-2L_3$	$2L_2$	0	$-2K_3$	$2K_2$
K_2	$2L_3$	0	$-2L_1$	$2K_3$	0	$-2K_1$
K_3	$-2L_2$	$2L_1$	0	$-2K_2$	$2K_1$	0
L_1	0	$-2K_3$	$2K_2$	0	$-2L_3$	$2L_2$
L_2	$2K_3$	0	$-2K_1$	$2L_3$	0	$-2L_1$
L_3	$-2K_2$	$2K_1$	0	$-2L_2$	$2L_1$	0

Table 3: Poisson structure for $(K_1,K_2,K_3,L_1,L_2,L_3)$.

The Casimirs, as we know, have to commute with all the elements of the algebra, in such a way that the linear Casimirs will be H_2 and Ξ. We move now to build quadratic Casimirs which generically are given by

$$\tau=a_1K_1^2+a_2K_2^2+a_3K_3^2+b_1L_1^2+b_2L_2^2+b_3L_3^2+aH_2^2+b\Xi^2,$$

and

$$
\begin{aligned}
\theta \;=\; & a_1 H_2 K_1 + a_2 H_2 K_2 + a_3 H_2 K_3 + a_4 H_2 \Xi + a_5 H_2 L_1 + a_6 H_2 L_2 + a_7 H_2 L_3 \\
& + a_8 K_1 K_2 + a_9 K_1 K_3 + a_{10} K_1 \Xi + a_{11} K_1 L_1 + a_{12} K_1 L_2 + a_{13} K_1 L_3 \\
& + a_{14} K_2 K_3 + a_{15} K_2 \Xi + a_{16} K_2 L_1 + a_{17} K_2 L_2 + a_{18} K_2 L_3 \\
& + a_{19} K_3 \Xi + a_{20} K_3 L_1 + a_{21} K_3 L_2 + a_{22} K_3 L_3 \\
& + a_{23} \Xi L_1 + a_{24} \Xi L_2 + a_{25} \Xi L_3 \\
& + a_{26} L_1 L_2 + a_{27} L_1 L_3 \\
& + a_{28} L_2 L_3.
\end{aligned}
$$

We start imposing that all the Poisson brackets of τ with K_i and L_i are zero (this trivially happens with H_2 and Ξ)

$$
\begin{aligned}
\{\tau, K_1\} = 0 \;\; &\Rightarrow \;\; a_2 = b_3 \text{ and } a_3 = b_2, \quad \{\tau, K_2\} = 0 \Rightarrow a_1 = b_3 \text{ and } a_3 = b_1, \\
\{\tau, K_3\} = 0 \;\; &\Rightarrow \;\; a_2 = b_1 \text{ and } {}_1 = b_2, \quad \{\tau, L_1\} = 0 \Rightarrow b_2 = b_3 \text{ and } a_3 = a_2, \\
\{\tau, L_2\} = 0 \;\; &\Rightarrow \;\; b_1 = b_3 \text{ and } a_3 = a_1, \quad \{\tau, L_3\} = 0 \Rightarrow b_2 = b_1 \text{ and } a_1 = a_2,
\end{aligned}
$$

from which we obtain $a_1 = a_2 = a_3 = a_4 = b_1 = b_2 = b_3$.

We repeat the process with θ, and we obtain $a_{11} = a_{17} = a_{22} \neq 0$ and $a_4 \neq 0$, while all the other coefficients are zero. Thus, the Casimirs we were searching are:

$$
\begin{aligned}
c_1(K_1^2 + K_2^2 + K_3^2 + L_1^2 + L_2^2 + L_3^2) \;&=\; c_2 H_2^2 + c_3 \Xi^2, \\
c_4(K_1 L_1 + K_2 L_2 + K_3 L_3) \;&=\; c_5 H_2 \Xi,
\end{aligned}
$$

By substituting (10) one obtains (11).

$\{,\}$	σ_1	σ_2	σ_3	δ_1	δ_2	δ_3
σ_1	0	$-2\sigma_3$	$2\sigma_2$	0	0	0
σ_2	$2\sigma_3$	0	$-2\sigma_1$	0	0	0
σ_3	$-2\sigma_2$	$2\sigma_1$	0	0	0	0
δ_1	0	0	0	0	$-2\delta_3$	$2\delta_2$
δ_2	0	0	0	$2\delta_3$	0	$-2\delta_1$
δ_3	0	0	0	$-2\delta_2$	$2\delta_1$	0

Table 4: Poisson structure of the twice reduced space in (σ, δ) coordinates.

Now, in order to show that this orbit space is diffeomorphic to $\mathbb{S}^2_{n+\xi} \times \mathbb{S}^2_{n-\xi}$, keeping in mind that we are in $\mathbb{CP}^3$ we fix values for $H_2 = n$ and $\Xi = \xi$ and introduce new coordinates

$$
\begin{aligned}
\sigma_1 = K_1 + L_1, \quad \sigma_2 = K_2 + L_2, \quad \sigma_3 = K_3 + L_3, \\
\delta_1 = L_1 - K_1, \quad \delta_2 = L_2 - K_2, \quad \delta_3 = L_3 - K_3,
\end{aligned}
\tag{12}
$$

which applied to (11) leads to

$$\begin{aligned} \sigma_1^2 + \sigma_2^2 + \sigma_3^2 &= (n+\xi)^2, \\ \delta_1^2 + \delta_2^2 + \delta_3^2 &= (n-\xi)^2, \end{aligned} \tag{13}$$

in other words $\mathbb{S}^2_{n+\xi} \times \mathbb{S}^2_{n-\xi}$. Moreover, if $\xi = 0$, we have $\mathbb{S}^2_n \times \mathbb{S}^2_n$, which together with its Poisson structure given by Table 4, is precisely the reduced phase space associated to perturbed Keplerian systems normalized by the energy (see for instance [10]).

In Fig. (1) a representation of the twice reduced phase spaces for different values of the integral ξ is given.

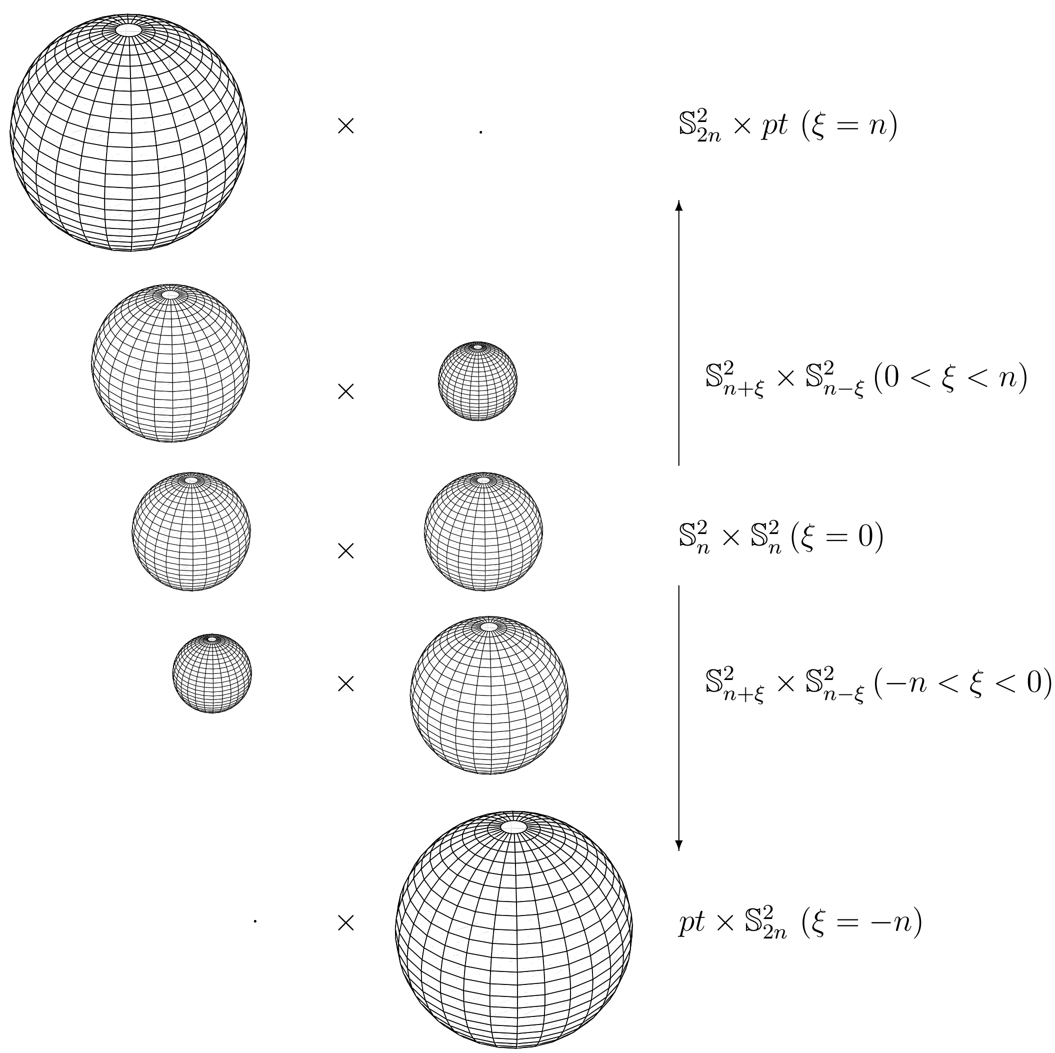

Figure 1: Twice reduced orbit space $\mathbb{S}^2_{n+\xi} \times \mathbb{S}^2_{n-\xi}$ for several values of the integral ξ

Note that the values n and ξ satisfy $n \geq 0$ and $n \geq |\xi|$. The latter being an immediate consequence of

$$H_2(q,Q) \pm \Xi(q,Q) = \tfrac{1}{2}\left((q_1 - Q_2)^2 + (q_2 + Q_1)^2 + (q_3 - Q_4)^2 + (q_4 + Q_3)^2\right) \geq 0. \quad (14)$$

q.e.d.

Applying the previous result to our family, from (9) the twice reduced Hamiltonian, modulo constants, is given by

$$\bar{\bar{H}}^\lambda(K,L) = \lambda(\frac{3}{2}nK_1 - \frac{1}{2}\xi\, L_1) + n(-\frac{3}{2}K_1^2 + K_2^2 + K_3^2 - \frac{1}{2}L_1^2 - L_2^2 - L_3^2) + \xi K_1 L_1 + \mathcal{O}(\varepsilon). \quad (15)$$

Thus, the system has two degrees of freedom because the relations, including $\Xi = \xi$, define a four-dimensional reduced phase space. The corresponding Poisson system is

$$\dot{K}_i = \{K_i, \bar{\bar{H}}^\lambda\}, \quad \dot{L}_i = \{L_i, \bar{\bar{H}}^\lambda\}\ ,$$

which follows from the relations given in Table 3.

In addition the system has a remaining integral L_1. This allows us to reduce our system to one degree of freedom.

2.3 Reduction to one degree of freedom

To further reduce from $\mathbb{S}^2_{n+\xi} \times \mathbb{S}^2_{n-\xi}$ to $\mathcal{V}_{n\,\xi\,\ell}$ one would have to fix $L_1 = \ell$ and divide out the S^1-action generated by L_1. This can be done as a consequence of the following Proposition, based on the study of the flow generated by $L_1 = \pi_{16} - \pi_{11}$. With the Poisson structure given by Table 3., we will be able to analyse the thrice reduced flow of our system.

Proposition 2.3 *Let φ_{L_1} be the S^1-action generated by the Poisson flow defined by L_1 over $\mathbb{S}^2_{n+\xi} \times \mathbb{S}^2_{n-\xi}$. Together with the invariants H_2, Ξ and $L-1$, the functions*

$$\begin{aligned}
M(K,L) &= \tfrac{1}{2}\left(K_2^2 + K_3^2 + L_2^2 + L_3^2\right), & N(K,L) &= \tfrac{1}{2}\left(K_2^2 + K_3^2 - L_2^2 - L_3^2\right), \\
Z(K,L) &= K_2L_2 + K_3L_3\ , & S(K,L) &= K_2L_3 - K_3L_2\ , \\
K(K,L) &= K_1\ .
\end{aligned} \quad (16)$$

generate the algebra of the φ_{L_1}-invariant functions on $\mathbb{S}^2_{n+\xi} \times \mathbb{S}^2_{n-\xi}$, which are constraint by the relations

$$\begin{aligned}
&K^2 + 2M = n^2 + \xi^2 - \ell^2\ , \\
&\ell\, K + Z = n\,\xi\ , \\
&M^2 - N^2 = Z^2 + S^2\ ,
\end{aligned} \quad (17)$$

this provides an orbit mapping

$$\rho_3 : \mathbb{R}^8 \to \mathbb{R}^8; (K_1, K_2, K_3, L_1, L_2, L_3, H_2, \Xi) \to (M, N, Z, S, K, L_1, H_2, \Xi).$$

When we fix a $H_2 = n$, $\Xi = \xi$ and $L_1 = \ell$, we have that (17) gives the surface

$$\mathcal{V}_{n\,\xi\,\ell} = \{(S, K, N) \mid 4(N^2 + S^2) = [(n + \xi)^2 - (K + \ell)^2]\,[(n - \xi)^2 - (K - \ell)^2]\} \quad (18)$$

defining the thrice reduced orbit space in the (S, K, N)-space.

From now on it will be convenient denote the right hand in (18) as

$$f(K) = [(n + \xi)^2 - (K + \ell)^2]\,[(n - \xi)^2 - (K - \ell)^2]. \quad (19)$$

Proof : Over the space $(K_1, K_2, K_3, L_2, L_3)$ the S^1-action of L_1 is determined by the vector field

$$X_{L_1} : \quad \begin{pmatrix} \dot{K}_1 \\ \dot{K}_2 \\ \dot{K}_3 \\ \dot{L}_2 \\ \dot{L}_3 \end{pmatrix} = 2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ L_2 \\ L_3 \end{pmatrix}$$

The S^1-action of this vector field is given

$$\varphi_{L_1} \begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ L_2 \\ L_3 \end{pmatrix} = 2 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos 2t & \sin 2t & 0 & 0 \\ 0 & -\sin 2t & \cos 2t & 0 & 0 \\ 0 & 0 & 0 & \cos 2t & \sin 2t \\ 0 & 0 & 0 & -\sin 2t & \cos 2t \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ L_2 \\ L_3 \end{pmatrix}$$

Note that this action is not free, because K_1 is an invariant for this action, in other words the reduced space has singular points, connected with the axis K_1. The invariant associated to this action φ_{L_1} are

$$\begin{aligned} &X = \tfrac{1}{2}\,(K_2^2 + K_3^2), \quad Y = \tfrac{1}{2}\,(L_2^2 + L_3^2), \quad Z = K_2 L_2 + K_3 L_3, \\ &S = K_2 L_3 - K_3 L_2, \quad K = K_1 \end{aligned}$$

bound by the constraint

$$4XY = Z^2 + S^2. \quad (20)$$

This result may be found in, for instance, [20]. As it is more convenient, we will use

$\{,\}$	X	Y	Z	S	K	L_1
X	0	$-2KS$	$-2SL_1$	$-4KX+2ZL_1$	$2S$	0
Y	$2KS$	0	$2SL_1$	$4KY-2ZL_1$	$-2S$	0
Z	$2SL_1$	$-2SL_1$	0	$-4(X-Y)L_1$	0	0
S	$4KX-2ZL_1$	$-4KY+2ZL_1$	$4(X-Y)L_1$	0	$-4X+4Y$	0
K	$-2S$	$2S$	0	$4(X-Y)$	0	0
L_1	0	0	0	0	0	0

Table 5: Poisson structure in (X, Y, Z, S, K, L_1) invariants

$\{,\}$	M	N	Z	S	K	L_1
M	0	$4KS$	0	$-4KN$	0	0
N	$-4KS$	0	$-4L_1S$	$-4(KM-L_1Z)$	$4S$	0
Z	0	$4L_1S$	0	$-4L_1N$	0	0
S	$4KN$	$4(KM-L_1Z)$	$4L_1N$	0	$-4N$	0
K	0	$-4S$	0	$4N$	0	0
L_1	0	0	0	0	0	0

Table 6: Poisson structure in (M, N, Z, S, K, L_1) invariants

$M = X+Y$ and $N = X-Y$ instead of X and Y. Then, considering the relations defining the twice reduced orbit space (11), joint with (20) and fixing values for the symmetries, we obtain the thrice reduced space (18). This space is the surface of revolution obtained by rotation of $\sqrt{f(K)}$ along the K axis.

The relation (20) and L_1 are Casimirs of the Poisson structure on the space of C^∞ functions in the invariants (X, Y, Z, S, K, L_1), given by the Table 5. When using variables M and N the Poisson structure is given by Table 6, where Casimirs are L_1 and the third relation in (17). **q.e.d.**

The Hamiltonian (15) on the third reduced phase space (modulo constants) is reduced to:

$$\mathcal{H}(-, K, N) = 2nN + \left(\frac{3}{2}n\lambda + \ell\xi\right) K - \frac{3}{2}nK^2 + \mathcal{O}(\varepsilon) \tag{21}$$

(We use $\mathcal{H}$ instead of $\bar{\bar{\bar{H}}}$ for the sake of simplicity). In (K, N, S)-space the energy surfaces are parabolic cylinders. The intersection with the reduced phase space give the trajectories of the thrice reduced system. Tangency with the reduced phase spaces gives relative equilibria that correspond to three dimensional tori on the original phase space.

Note that the reduced phase spaces as well as the Hamiltonian are invariant under the discrete symmetry $S \to -S$. Furthermore the reduced phase space is invariant under the discrete symmetry $N \to -N$. We choose not to further reduce our reduced phase space with respect to these discrete symmetries like in [22] because the three dimensional picture makes it easy to access information about the reduced orbits and this way one does not introduce additional critical points (fixed points) which need special attention.

2.4 Structure of the thrice reduced phase space $\mathcal{V}_{n,\xi,\ell}$

Before we begin searching for possible relative equilibria and their bifurcations in our system, we need to understand the geometric nature of all the possible reduced spaces in (n, ξ, ℓ) parameter space.

To this end we study $f(K)$ which can be written as

$$f(K) = (K + n + \xi + \ell)(K - n - \xi + \ell)(K - n + \xi - \ell)(K + n - \xi - \ell),$$

thus, the four zeroes of $f(K)$ are given by

$$K_1 = -\ell - n - \xi\ ,\ K_2 = \ell + n - \xi\ ,\ K_3 = \ell - n + \xi\ ,\ K_4 = -\ell + n + \xi\ .$$

So $f(K)$ is positive (or zero) when

$$\max(K_1, K_3) \leqslant K \leqslant \min(K_2, K_4),$$

i.e. in the subsequent intervals of K:

$$\begin{array}{lll} \ell < \xi\,,\ -\ell < \xi & K_1 < K_3 < K_2 < K_4 & K \in [K_3, K_2] \\ \ell > \xi\,,\ -\ell < \xi & K_1 < K_3 < K_4 < K_2 & K \in [K_3, K_4] \\ \ell < \xi\,,\ -\ell > \xi & K_3 < K_1 < K_2 < K_4 & K \in [K_1, K_2] \\ \ell > \xi\,,\ -\ell > \xi & K_3 < K_1 < K_4 < K_2 & K \in [K_1, K_4] \end{array} \tag{22}$$

To find the double zeroes of $f(K)$ we compute the discriminant of $f(K) = 0$. It is

$$(\ell - n)^2(\ell + n)^2(\ell - \xi)^2(\ell + \xi)^2(n - \xi)^2(n + \xi)^2\ .$$

Thus there are double zeroes at $\ell = \pm n$, $\ell = \pm\xi$ and $\xi = \pm n$. Triple zeroes occur when $|\ell| = |\xi| = n$. And quadruple zeroes only occur when $\ell = n = \xi = 0$.

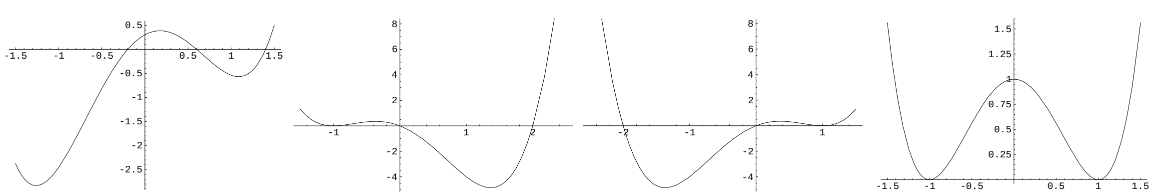

Figure 2: Snapshots of $f(K)$. From left to right: $\ell \neq \xi \neq n$, $\ell = -\xi$, $\ell = \xi$, $\ell = \xi = 0$

By (14) we have $H_2 \pm \Xi \geqslant 0$, similarly $H_2 \pm L_1 \geqslant 0$, and $H_2 \pm K_1 \geqslant 0$. Consequently we may represent the admissible K-values in the following way. In (ξ, ℓ, K)-space consider the cube $-n \leqslant \xi \leqslant n$, $-n \leqslant \ell \leqslant n$, $-n \leqslant K \leqslant n$. Consider the tetrahedron circumscribed by this cube (see Fig. 3), such that the diagonals of the top face of the cube, connecting $(-n,-n,n)$ and (n,n,n), and of the bottom face of the cube, connecting $(n,-n,-n)$ and $(-n,n,-n)$, are part of this tetrahedron. Below we will show that the vertices of the tetrahedron correspond to invariant circles, and the edges to invariant two-tori. The lines in the top face and bottom face of the cube correspond to the two-tori at the cone-like singular points of the reduced phase spaces..

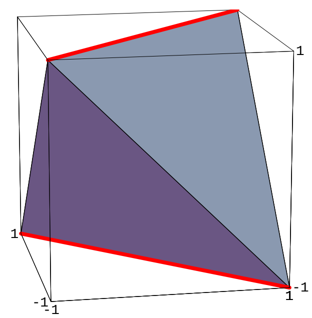

Figure 3: Tetrahedron

One expects three-tori to bifurcate of this tetrahedron. In (ξ,ℓ,n)-space this tetrahedron corresponds to the square section in Fig. 4.

Next we study the possible roots of $f(K)$ and the related reduced spaces and, for certain cases, the corresponding configurations in original phase space.

Four different roots. When we have four different roots we have two possible cases:

(i) $\ell \neq \xi, \qquad \xi,\ell \neq 0.$

(ii) $\ell \neq \xi, \qquad \xi = 0$ or $\ell = 0.$

For the first case, suppose that $\xi > \ell$ (remember that $|\xi| < n, |\ell| < n$; see (14)), we have that $f(K)$ is positive if $\ell - n + \xi < K < \ell + n - \xi$. Similarly, for the other configurations of ξ and ℓ the limits of K are given by two roots of multiplicity one. The reduced phase spaces are diffeomorphic to $\mathbb{S}^2$. In the second case the reduced phase spaces are also diffeomorphic to $\mathbb{S}^2$ but symmetric with repect to reflection in the plane $K = 0$. When ℓ goes to ξ the two last roots of the left figure in Fig. (2) form a double root and in the reduced spaces appears a singularity as we can see below.

One double root $\ell = \pm\xi \neq 0$. If $\ell = \xi$ and $\ell = -\xi$ we have double roots at $K = \pm n$ and the reduced phase space is a turnip, having a cone-like singularity. This case will be a Hamiltonian Hopf candidate.

Two double roots $\ell = \xi = 0$. Two double roots appear if $\ell = \xi = 0$ and we have lemon shaped reduced phase spaces having two cone-like singular points.

One double root $\ell = \pm n$. When $\ell = \pm n$ we have also a double root, but in this case the reduced space is a point and $f(K) = 0$ when $k = \xi$. All possible configurations are gathered in Fig. (4).

When the reduced phase space is a point, this point lifts to four points $\{\pm\sigma_1^{(0)}, 0, 0, \pm\delta_1^{(0)}, 0, 0\}$ in $S^2 \times S^2$, because

$$K_1^2 + L_1^2 = n^2 + \xi^2 \ , \ K_1 \, L_1 = n \, \xi \ ,$$

which implies,

$$\sigma_1^2 = (n + \xi)^2 \ , \ \delta_1^2 = (n - \xi)^2 \ .$$

If $\ell = n$ we have $\pi_1 + \pi_2 + \pi_3 + \pi_4 - 2(\pi_{16} - \pi_{11}) = 0$, and $(q_2 - Q_1)^2 + (q_1 + Q_2)^2 + (q_3 - Q_4)^2 + (q_4 + Q_3)^2 = 0$, which holds if

$$q_2 = Q_1, \ q_1 = -Q_2, \ q_4 = -Q_3, \ q_3 = Q_4,$$

and we have

$$\begin{aligned} n &= L_1 = q_1^2 + q_2^2 + q_3^2 + q_4^2, \\ \xi &= K_1 = -q_1^2 - q_2^2 + q_3^2 + q_4^2, \\ K_2 &= K_3 = L_2 = L_3 = 0. \end{aligned}$$

and $f(K) = 0$.

In original phase space space this lifts to a 2-torus given by

$$\tfrac{1}{2}\,(n + \xi) = q_3^2 + q_4^2 \ , \ \tfrac{1}{2}\,(n - \xi) = q_1^2 + q_2^2 \ , \ q_2 = Q_1 \ , \ q_1 = -Q_2 \ , \ q_4 = -Q_3 \ , \ q_3 = Q_4 \ .$$

When $\ell = -n$ we get the 2-torus given by

$$\tfrac{1}{2}\,(n + \xi) = q_3^2 + q_4^2 \ , \ \tfrac{1}{2}\,(n - \xi) = q_1^2 + q_2^2 \ , \ q_2 = -Q_1 \ , \ q_1 = Q_2 \ , \ q_4 = Q_3 \ , \ q_3 = -Q_4 \ .$$

When $\xi = n$ we get the 2-torus given by

$$\tfrac{1}{2}\,(n + \ell) = q_3^2 + q_4^2 \ , \ \tfrac{1}{2}\,(n - \ell) = q_1^2 + q_2^2 \ , \ q_2 = -Q_1 \ , \ q_1 = Q_2 \ , \ q_4 = -Q_3 \ , \ q_3 = Q_4 \ .$$

When $\xi = -n$ we get the 2-torus given by

$$\tfrac{1}{2}\,(n + \ell) = q_3^2 + q_4^2 \ , \ \tfrac{1}{2}\,(n - \ell) = q_1^2 + q_2^2 \ , \ q_2 = Q_1 \ , \ q_1 = -Q_2 \ , \ q_4 = Q_3 \ , \ q_3 = -Q_4 \ .$$

Triple roots $|\ell| = |\xi| = n$. In this case a triple zero appears when $K = |\ell| = |\xi| = n$. From the above computations we see that in this case the 2-tori become 1-tori, i.e.

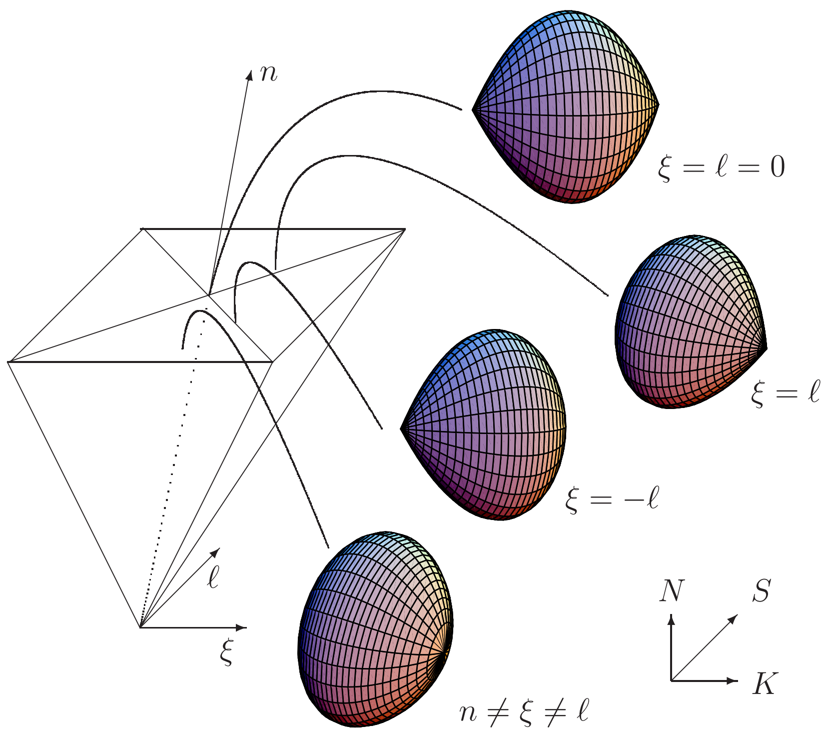

Figure 4: The thrice reduced phase space over the parameter space. K is the symmetry axis of each surface.

topological circles. On the second reduced phase space $\mathbb{S}^2 \times \mathbb{S}^2$ this circle reduces to to $\sigma_1^2 = 2n^2$, that is $(\pm 2n, 0, 0; 0, 0, 0)$, and the second $\mathbb{S}^2$ reduces to a point.

Quadruple roots. Finally, the subcase $\ell = \xi = n = 0$ corresponds with a quadruple root at the origin of the $\{n, \xi, \ell\}$-space.

Note that the composition of the three orbit maps gives an orbit map from $\mathbb{R}^8 \to \mathbb{R}^8$, which is an orbit map for the three torus action. This action is generated by the rotational flows of the three integrals H_2, Ξ, L_1, which are independent and commute. In [**?**]sjamaar91 it is shown that reduction in stages gives the same result as applying the overall orbit map and that the order of the reductions does not matter. Due to the shape of the reduced phase spaces the intersection of the reduced Hamiltonian and these spaces will generically be a circle or a disjoint union of two circles. Thus the generic fibre of the energy momentum map

$$\mathcal{EM} : \mathbb{R}^8 \to \mathbb{R}^4; (q, Q) \to (\bar{H}_\lambda, H_2, \Xi, L_2)$$

will be a $\mathbb{T}^4$ or a disjoint union of two $\mathbb{T}^4$. A point on the regular part of a reduced phase space will correspond to a $\mathbb{T}^3$. Consequently the generic relative equilibrium, i.e.

stationary point for the reduced Hamiltonian system on the final reduced phase space, will correspond to a $\mathbb{T}^3$. Singular points on reduced phase spaces will correspond to orbit types of the $\mathbb{T}^3$-action different from $\mathbb{T}^3$. Thus singular points will correspond to lower dimensional tori. There will be fibres that are a point (the origin which is a stationary point of the original system and a fixed point for all circle symmetries), a circle (two of the reductions will have a stationary point) or a $\mathbb{T}^2$ (one reduction will have a stationary point). The rank of the energy momentum map $\mathbb{R}^8 \to (H, H_2, \Xi, L_2)$ will correspond to the dimension of the fibre. In the following the relative equilibria for the reduced Hamiltonian system corresponding to the singular points of the reduced phase space will be studied in more detail.

Note that the above computed relative equilibria that are a point, $\mathbb{S}^1$ or $\mathbb{T}^2$ correspond to the lower dimensional strata of the symplectic leaf stratification of the $\mathbb{T}^3$-orbit space.

3 Energy Momentum Mapping and Relative Equilibria

In the search for equilibria in a Hamiltonian system we may follow two different paths, depending on the analytical or geometrical approach taken for that study. We may deal directly with the Poisson flow over the thrice reduced orbit space (analytical approach), obtaining finally an equation in the variable K, whose coefficients are functions of (n, ξ, ℓ, λ), the distinguished and physical parameters of the reduced system. After we have found the equilibria we still need to compute the corresponding values of the energy. In other words, we have to identify the manifolds where those equilibria live (for details see Egea [11]). In what follows we take the second approach, the geometric one, where the normalized and truncated energy function $\mathcal{H}$ plays a main role because the equilibria are given by the tangencies of the surface $\mathcal{H} = h$ with the reduced orbit space. In fact, as the Hamiltonian does not depend on S, it is possible to implement our study considering tangencies in the plane (K, N). More precisely, here we will take $(K, 2N)$ which will allow to work with $f(K)$ directly.

3.1 Critical values of the energy momentum mapping.

Consider the Hamiltonian (21) at a level of energy $\mathcal{H} = h$, from which we have

$$N = \frac{3}{4}K^2 - (\frac{\xi\ell}{2n} + \frac{3\lambda}{4})K + \frac{h}{2n}. \tag{23}$$

Substituting this in the equation for the reduced phase space given by $4N^2 - f(K) = 0$, $(S = 0)$, we obtain the following polynomial of degree four

$$P^{\lambda}_{h,n,\ell,\xi}(K) = b_1K^4 + a_2K^3 + b_3K^2 + b_4K + b_5, \tag{24}$$

where

$$
\begin{aligned}
b_1 &= 5\,n^2\\
b_2 &= -6\,n\,(2\,\ell\,\xi+3\,n\,\lambda)\\
b_3 &= 12\,n\,h+4\,\ell^2\,\left(2\,n^2+\xi^2\right)+12\,\ell\,n\,\xi\,\lambda+n^2\,\left(8\,\left(n^2+\xi^2\right)+9\,\lambda^2\right)\\
b_4 &= -\,4\left(2\,\ell\,\left(h+4\,n^3\right)\,\xi+3\,n\,h\,\lambda\right)\\
b_5 &= 4(\,h^2-\,n^2\,(-\ell+n-\xi)\,(\ell+n-\xi)\,(-\ell+n+\xi)\,(\ell+n+\xi)).
\end{aligned}
$$

Double roots of this polynomial, solutions of the system

$$
P^{\lambda}_{h,n,\xi,a}(K)=0, \qquad \frac{d}{dK}P^{\lambda}_{h,n,\xi,a}(K)=0, \tag{25}
$$

determine the critical values of the energy momentum mapping $\mathcal{EM}$,

$$
\mathcal{EM}:\mathbb{R}^8\to\mathbb{R}^4:(q,Q)\mapsto(H,H_2,\Xi,L_1). \tag{26}
$$

For a fixed value of λ, the locus defined by the set of values (h,n,ξ,ℓ), denoted by Δ_λ, for which the polynomial (24) has double roots, is the discriminant of this polynomial. This discriminant corresponds to the singularity of the the energy momentum mapping. Critical values (h,n,ξ,ℓ) of $\mathcal{EM}$ are in correspondence with values (h,n,ξ,ℓ) for which the level curve $\mathcal{H}=h$ is tangent to the reduced space $V_{n,\xi,\ell}$, or to those values where the curve passes through the singular point $(0,n,0)$ of the reduced space (for short we refer to it just with $K=n$). As we know, such singular points exist for the critical values $(h,n,\xi,\pm\xi)$.

The full analysis of all the relative equilibria of our system will not be tackled here. We will mainly concentrate on the case $\xi=\ell$, when the reduced space has singular points and also consider the case $\xi=0$, $\lambda=0$: The Zeeman model.

3.1.1 Searching Hamiltonian Hopf bifurcations

From this point on let us focus on the case where the two first integrals are equal $\ell=\xi\neq 0$, *i.e.* when the thrice reduced space has a singularity at $K=n$. The quartic polynomial (24) given by

$$
\begin{aligned}
P^{\lambda}_{h,n,\xi}(K) &= 5n^2K^4-6n\left(2\xi^2+3n\lambda\right)K^3\\
&\quad+\left(12nh+8n^4+4\xi^4+12n\xi^2\lambda+n^2\left(16\xi^2+9\lambda^2\right)\right)K^2\\
&\quad+\left(-8\left(h+4n^3\right)\xi^2-12nh\lambda\right)K+\left(2h+n\left(-2\xi^2+3n\left(n-\lambda\right)\right)\right)^2,
\end{aligned} \tag{27}
$$

has, in general, four roots whose analytical expressions may be obtained using Cardan formulas. Here, we satisfy ourselves only with the search for double roots with the goal of building the overall view on how they evolve over the space of parameters.

We know from the start that $K = n$ is an equilibrium of the system. Thus, it will be a double root of (27) under the constraint

$$h = \frac{-3\,n^3 + 2\,n\,\xi^2 + 3\,n^2\,\lambda}{2}. \tag{28}$$

Indeed, if the energy verifies (28) we have that $(K - n)^2$ factorizes (27), in other words $K = n$ is a double root. The relation between λ and ξ is readely obtained

$$\lambda_\pm = 2n - \frac{2\xi^2}{3n} \pm \frac{4}{3}\sqrt{n^2 - \xi^2}, \tag{29}$$

which defines the two leaves of a bifurcation surface in (n, ξ, λ). Later on we come back to these branches of the bifurcation curve.

The other two roots are

$$K_{3,4} = \frac{5\,n^3 - 6\,n\,\xi^2 - 9\,n^2\,\lambda \pm \sqrt{\Psi}}{5n^2}$$

where $\Psi = n^2\,(4\,\xi^4 + 12\,n\,\xi^2\,\lambda + n^2\,(-20\,\xi^2 + 9\,\lambda^2))$. Then, we will have a second equilibrium if $\Psi = 0$, and that occurs when

$$\lambda = \frac{2\,\left(\pm\sqrt{5}\,n\,\xi - \xi^2\right)}{3\,n}, \tag{30}$$

and the equilibrium takes de value

$$K = -n \pm \frac{6}{\sqrt{5}}\,\xi\,,$$

from which we conclude that the condition for an equilibrium of multiplicity three will be $\xi = \pm\frac{\sqrt{5}}{3}n$. Besides, as $|K| < n$ we have that

$$K = -n + \frac{6}{\sqrt{5}}\,\xi, \quad \xi \in (0, \sqrt{5}/3), \qquad K = -n - \frac{6}{\sqrt{5}}\,\xi, \quad \xi \in (-\sqrt{5}/3, 0). \tag{31}$$

In Fig. (5) we see tangencies and their evolution along the curve which is defined by (30). Note also how the size of the reduced space changes with ξ. Thus, we have obtained that, as $K = n$ is an equilibrium of the system, it is possible to identify another equilibrium in the same manifold of energy, satisfying $N < 0$, *i. e.*, that is located on the lower branch, if we keep in mind that for equilibria $S = 0$. It also occurs that if $K = n$ is of multiplicity one, any other equilibrium can not be of higher multiplicity. Moreover when $K = n$ is an equilibrium of multiplicity two there are not more equilibria on the same energy manifold, because $K = n$ is a root of (27) with multiplicity three, which prevent the presence of any other double root needed for another equilibrium.

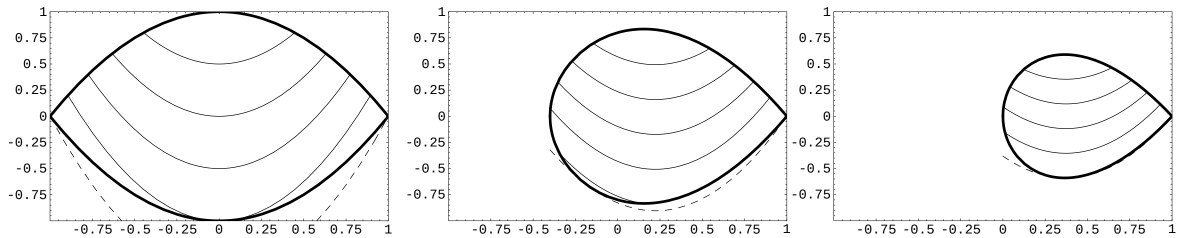

Figure 5: Equilibria at $K = n$ and $K = -n + \frac{6}{\sqrt{5}}\,\xi$ for $n = 1$. From left to right we have the cases $\xi = 0$, $\xi = 0.3$ and $\xi = \frac{1}{2}$.

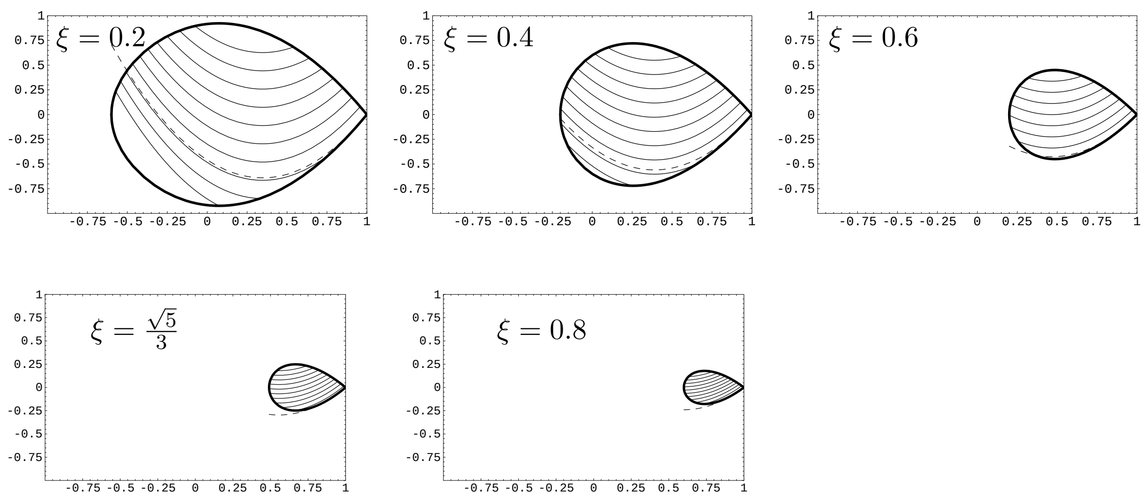

Figure 6: Levels of energy over the reduced space for several values of ξ, where $n = 1$ and $\lambda = \lambda\_$ in all the cases. the dashed line represents the level of energy for which we have the double (or triple in the first figure of the second raw) contact between the Hamiltonian and the reduced space at $K = n$.

Note that the triple roots of (27) at $K = n$, corresponding to relative equilibria with algebraic multiplicity two, are the natural candidates to exibits a Hamiltonian Hopf bifurcation. Indeed, if we obtain the roots of the system

$$P^{\lambda}_{h,n,\xi}(K) = 0, \quad \frac{d}{dK} P^{\lambda}_{h,n,\xi}(K) = 0, \quad \frac{d^2}{dK^2} P^{\lambda}_{h,n,\xi}(K) = 0, \tag{32}$$

if we impose $K = n$ to be a solution, the relation among the integrals to be satisfied is

$$\lambda_{\pm} = 2n - \frac{2\xi^2}{3n} \pm \frac{4}{3}\sqrt{n^2 - \xi^2}, \tag{33}$$

which is precisely one of the bifurcation lines we are after. If we replace it in (28) we obtain the energy manifold on which $K = n$ is solution of the system (32):

$$h = \frac{3\,n^3 \pm 4\,n^2\,\sqrt{n^2 - \xi^2}}{2}. \tag{34}$$

Now we are going to represent the tangencies of he Hamiltonian with the reduced space over the bifurcation curve defined by the two branches $\lambda_{\pm}$. In Fig. (6) appears a section $S = 0$ of the reduced space on which we have drawn curves for of the energy for several values of the Hamiltonian in $(K, 2N)$ when we are over $\lambda_{(-)}$. In the first three figures we observe the existence of two simple equilibria joint with an equilibrium of multiplicity two at $K = n$. The first figure of the second row corresponds to $\xi = \sqrt{5}n/3$, and we identify a triple equilibrium at $K = n$ and a simple equilibrium. In the next figure $K = n$ is a double equilibrium, in other words we pass from four to three equilibria. It is worth noticing how the evolution of ξ modifies the shape of the reduced space reaching a limit where the reduced space collapses to a point at $K = n$.

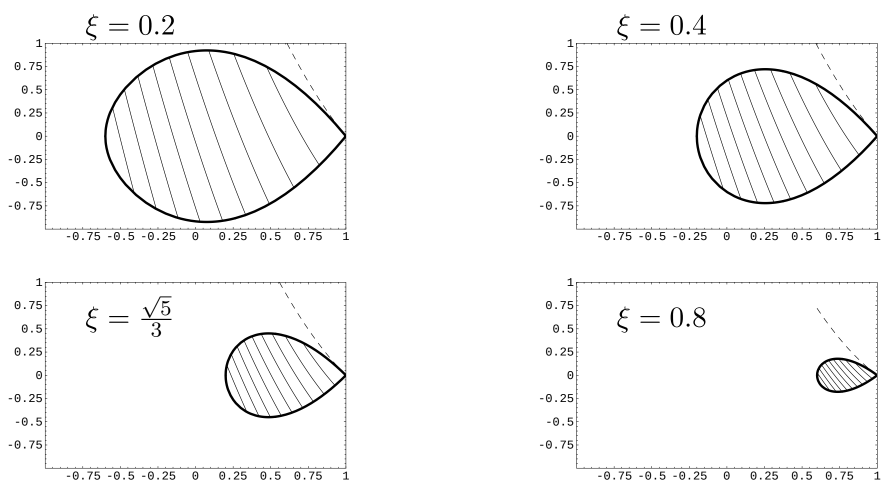

Figure 7: Levels of energy over the reduced space for several values of ξ, and $\lambda = \lambda_{-}$ in all the cases. The dashed line represents the level of energy for which the double contact (or triple in the first figure of the second row) occurs between the Hamiltonian and the reduced space at $K = n$.

In Fig. (7) we repeat the same scenario but now on the branch $\lambda_{(+)}$ which, corresponds to tangencies with the upper branch of the reduced space. In the first three cases the topology is the same: an equilibrium of multiplicity two at $K = n$ and a second equilibrium over the lower branch of the reduced space.

The particular case of triple roots at $K = n$ of the polynomial (27) has to be studied in detail. Indeed, for

$$\xi = \pm n, \quad \lambda = \frac{4}{3}\,n \tag{35}$$

we also have triple roots for $K = n$ with energy

$$h = \frac{3}{2}n^3, \tag{36}$$

but the reduced space is a point for those values. Thus an analysis in $\mathbb{S}^2_{n+\xi} \times \mathbb{S}^2_{n-\xi}$ which reduces to $\mathbb{S}^2_{2n} \times (0,0,0)$, isomorphic to $\mathbb{S}^2$, is nedeed. We will see that the transit from $\xi = 0$ to $\xi \neq 0$ presents different types of bifurcations as function of the parameter λ.

3.1.2 Saddle-node bifurcations

Still within the case $\ell = \xi$ we look here for double roots such that $K \neq n$. To do that we analyze conditions for triple roots of the polynomial $P^{\lambda}_{h,n,\xi}(K)$ given by (27), which requires computing first and second derivatives: $\frac{d}{dK}P^{\lambda}_{h,n,\xi}(K)$ and $\frac{d^2}{dK^2}P^{\lambda}_{h,n,\xi}(K)$. Then, we built the resultant R_1 of $P^{\lambda}_{h,n,\xi}(K)$ and $\frac{d}{dK}P^{\lambda}_{h,n,\xi}(K)$ with respect to K, and the resultant R_2 of $P^{\lambda}_{h,n,\xi}(K)$ and $\frac{d^2}{dK^2}P^{\lambda}_{h,n,\xi}(K)$ with resepct to K, where R_1 and R_2 are polynomials in (n, ξ, λ, h). It remains to obtain the resultant R_3 of R_1 and R_2 with respect to h, and finally we obtain R_3 which is a polynomial depending on (n, ξ, λ, h), one of whose factors R_{3a}, given by

$$\begin{aligned} R_{3a} \;=\; & 64\xi^{12} + 192(3\lambda - 50)\xi^{10} + 144(\lambda(15\lambda - 388) + 268)\xi^8 \\ & + 32(9\lambda(3\lambda(5\lambda - 138) + 244) - 4168)\xi^6 \\ & + 36\left(\lambda\left(3\lambda\left(45\lambda^2 - 984\lambda - 296\right) + 13408\right) + 5360\right)\xi^4 \\ & + 12\left(3\lambda\left(3\lambda\left(3\lambda\left(9\lambda^2 - 78\lambda - 152\right) - 656\right) - 6320\right) - 20000\right)\xi^2 \\ & + (3\lambda + 2)^3(3\lambda + 10)^3, \end{aligned} \tag{37}$$

allows to obtain the bifurcation curve, by means of an implicit representation, which determines the double roots represented in Fig. (8).

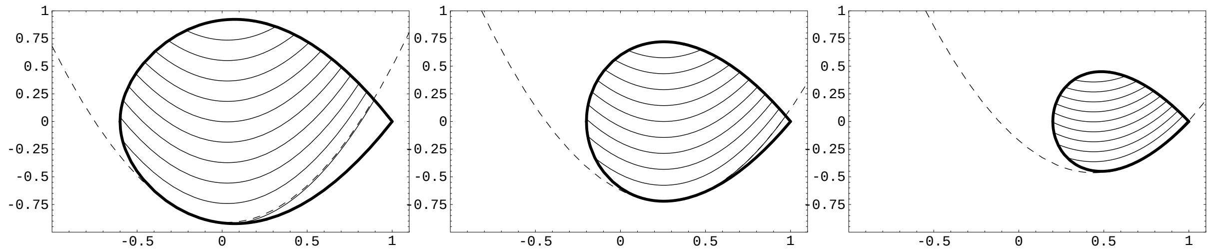

Figure 8: Levels of energy over the reduced space for several values of ξ, where $n = 1$ and $\lambda = \lambda_s$ in all cases. The dashed line represents the level of energy for which the double contact occurs between the Hamiltonian and the reduced space at $K = n$.

4 Bifurcation Panorama

4.1 Hamiltonian Hopf and saddle-center bifurcation lines

In the previous section, when $\xi = \ell$, we have obtained equilibria of multiplicity two, joint with the corresponding bifurcation lines. For a fixed value of n, double equilibria define a bifurcation line in the plane (λ, ξ). In Fig. (9) appear bifurcation lines defined by (29) and (37), joint with a curve (dashed line) which defines the presence of equilibria over the same level of energy.

In the case when $K = n$ is an equilibrium of multiplicity two the system is defined along the curves **D**, **E** and **F**. The curve **F** exists when $\lambda \in (4/3, 10/3)$ and it is symmetric with respect to the $\xi = 0$ axis, i.e. it has two branches in ξ one positive and the other negative; this curve corresponds to the branch λ_+ given by (29). The branch λ_- in (29) relates to **D** and **E** which, as we will see, define two different types of bifurcation. The curve **D** lives in the interval $\xi \in (-\sqrt{5}n/3, \sqrt{5}n/3)$, meanwhile the curve **E** has two branches in the intervals $\xi(-1, -\sqrt{5}n/3) \cup (\sqrt{5}n/3, 1)$.

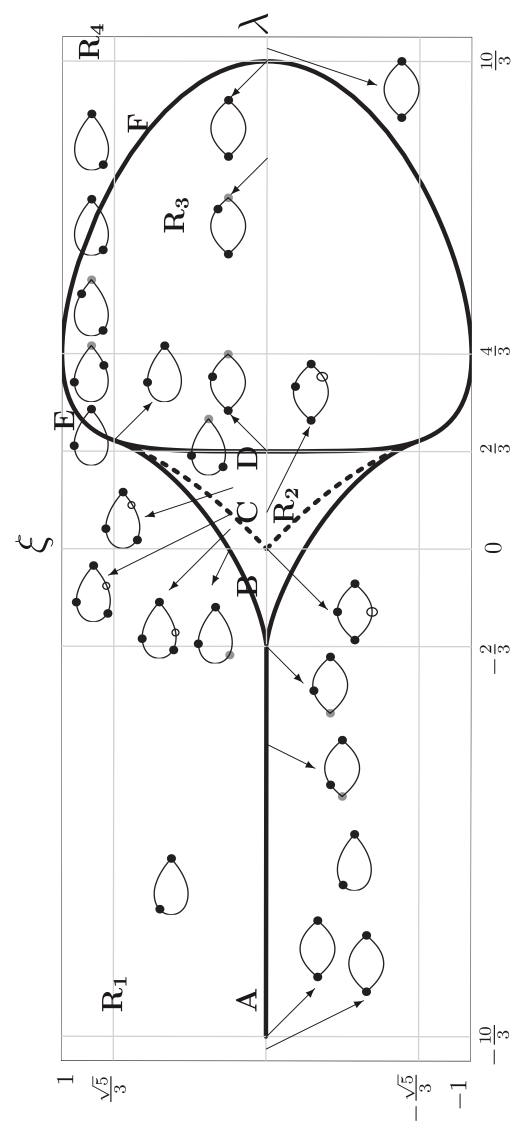

Figure 9: Bifurcation lines in the (λ, ξ) plane defined by multiplicity two equilibria.

The bifurcation line defined in implicit form by (37) corresponds with an equilibrium of multiplicity two whose evolution with ξ is represented in Fig. (8). Over the Fig. (9)it appears represented by curve **B**. The segment **A** lives over $\xi = 0$ and $\lambda \in (-10/3, -2/3)$, meanwhile the curve **B**, composed by the two symmetric branches with respect to $\xi = 0$, exists for $\xi \in (-\sqrt{5}n/3, \sqrt{5}n/3)$. The extrema of segment **A** define bifurcation points and are in correspondence with the values of λ which makes zero (37) when $\xi = 0$ as we will see later on.

Finally, the discontinous line which appears in Fig. (9) corresponds the geometric locus of the parameter plane where, the on the reduced space we have two equilibria over the same level of energy, whose analytical expression is given by (30). Note that this curve is not a bifurcation line.

Bifurcation curves divide the parameter plane in several regions characterize d for having and invariant number of points pof equilibria. These regions are $\mathbf{R_1}$, $\mathbf{R_2}$, $\mathbf{R_3}$ and $\mathbf{R_4}$. Passing from one region to another requires to cross any of those lines, and a change in the number of equilibria as well as their stability occurs. A particular case in parameter plane corresponds to two points from where we have access to three different regions. We refer to the points defined by $\lambda = 20n/27$ and $\xi = \pm\sqrt{5}n/3$. From them we may reach $\mathbf{R_1}$, $\mathbf{R_2}$ and $\mathbf{R_3}$. As we will see later they are degenerate.

Apart from the bifurcation lines, Fig. (9) also contains several representation of the $S = 0$ section of the reduced phase space; all the relative equilibria are located in that plane. They have been painted black, grey or white depending if they are stable with index $+1$, or unstable with indices 0 or -1 respectively.

In Fig. (10) we see a zoom around ($\xi = \pm\sqrt{5}n/3$, $\lambda = 20n/27$), one of the points in parameter plane from which we may access three different regions; more precisely $\mathbf{R_1}$, $\mathbf{R_2}$ and $\mathbf{R_3}$. Later we show what makes these points to be special: from the Hamiltonian Hopf bifurcation analysis, we will see that those bifurcations are degenerate.

4.1.1 The segment A.

The segment **A** is the region in parameter space (λ, ξ) defined by the points $(-10/3, 0)$ and $(-2/3, 0)$. These are bifurcation points of the system and verify the Eq. (37) for $\xi = 0$. Indeed, in that case we have

$$(3\lambda + 2)^3(3\lambda + 10)^3 = 0. \tag{38}$$

The nature of the flow evolution around this line is different to the bifurcation lines $\mathbf{B}, \mathbf{D}, \mathbf{E}$ and **F**. We say that **A** is a bifurcation line because crossing it there is a change 2-3-2 from in the number of equilibria, when we move from the positive $\mathbf{R_1}$ to the negative crossing **A**.

We might have also taken as definition of the bifurcation curve the ones over which the

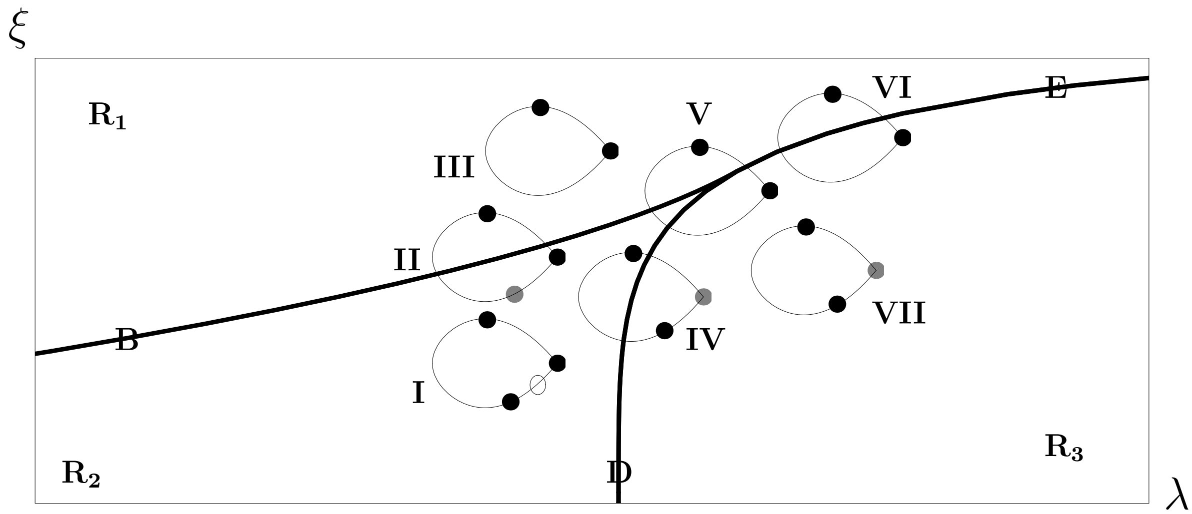

Figure 10: Detail of the bifurcations given in Fig. (9) around $(\frac{20}{27}, \frac{\sqrt{5}}{3})$.

system has an equilibrium of multiplicity two which bifurcate when we slightly move the values in the parameter plane. With this definition **A** would not be a bifurcation line except for the extreme points of that segment, where the system has an equilibrium of multiplicity two at the extremum $K = -n$ of the reduced space. The special character of this segment **A** relies on the fact that, when we leave that segment, the reduced space changes drastically loosing one of their singular points and, thus, an equilibrium. None of the two equilibria of the system in $\mathbf{R_1}$ bifurcates the third equilibrium that we have over **A**; this one is related to the singular point associated to the reduced space. But not always occurs the same pattern, i.e. the appearance-disappearance of singular and equilibria of the system. For instance in Fig. 9 we may observe that the system has three equilibria both when $\xi = 0$ or if $\xi \neq 0$, situation which is easy to understand from then geometry of the system, i.e., from the tangencies of the reduced space and energy surfaces.

4.2 Bifurcations lines and stability

Completing Fig. (9), we include the Fig. (11) which shows the evolution of the phase flow over the parameter plane $(\xi = \ell, \lambda)$ for $n = 1$. The curves on that plane are bifurcation curves dividing it in four regions in each of which the flow is topologicaly equivalent. Moreover, along the segment A there are also structural changes in the phase flow. In order to illustrate de bifurcations when crossing bifurcation lines, we include several flows for each region. We have chosen to present 'north' and 'south' views of the reduced space (for the case $\xi = 0$ we present 'north'-'south' and lateral views), which allows to show in a global way the flow over the reduced space. It should be noticed that all the figures are scaled; in other words we consider the case $\xi = \pm n$ when the reduced space shrinks to a point at the end of these paragraphs.

Figures are labeled by a number and a letter. The number represents the region of the parameter plane, meanwhile the letter allows to distinguish between two flows topologicaly equivalent. Thus, for example 2a and 2b are qualitatively equivalent, although the flow is different in both cases.

Let us describe the flow and bifurcations reading the figure from right to left. When passing from 1 to 2a it corresponds to a supercritical Hopf bifurcation where the singular point $K = n$ changes from stable to unstable, and a stable point bifurcates from it. We observe that the transit 2a to 2b in the north view (from $K > 0$) manifests by the motion of this equilibrium, changing as a function of λ, as this value decreases.

When we cross from 2b to 3a we have a supercritical Hopf bifurcation at the singular point which becomes unstable as we may see in the south view (negative K axis). If we observe the transit from 2b to 2c we have the same structure in the flow, although we see that when ξ decreases the shape of the reduced space changes from turnip to a lemon shape. From 2c to 4a, taking the south view, we see a subcritical Hopf bifurcation: the singular point changes stability and a unstable equilibria bifurcates from it. From 4a to 4b we see how the unstable point is moving until a saddle-center bifurcation occurs when we mobe from 4b to 3b. Again we observe the equivalence between 3b and 3a, and the same comment we made for 2b and 2c applies here.

The bifurcations for the case $\xi = 0$ occurs over segment A. Let consider first the transit from 4(a/b) to 5, where we have that the number of equilibria (4) do not changes, but only one of the stable points becomes singular; when we pass from 3(a/b) to 7 something similar occurs for the two stable equlibria. In the transit from 3(a/b) to segment A, 6(a/b), an unstable equilibrium at the singular point $K = -n$ appears.

Referring now to the bifurcations in the $\xi = 0$ case we have that in the transit from 7 to 6b we identify a supercritical Hopf bifurcation at the point $K = -n$ which changes its stability and a stable equilibrium bifurcates. This equilibrium evolves reaching the highest point at the north when $\lambda = 0$, the rest of the equilibria being the singular points, both stable, and a unstable point at the south, just at the minimum. This unstable equilibrium occurs when the point $K = -n$ changes its stability in the transit from 6a to 5. If we follows the evolution with λ over $\xi = 0$ we have again a subcritical bifurcation when we cross D and a supercritical in F over $K = n$. Note that the phenomenology at $K = -n$ in the evolution when $\lambda < 0$ goes to 0 that we just have mentioned, it is equivalent to the one that occurs at $K = n$ when $\lambda > 0$ goes to 0.

Finally, we have to add the description of the bifurcation related with the border of the domain in the parameter plane (λ, ξ): the lines $(\lambda, \pm n)$. As we have said above, due to the scaling of the figures, we do not present the evolution of those spaces growing from a point. When we pass from the lines $\xi = \pm n$ to $|\xi| < n$ a bifurcation is taken place. If

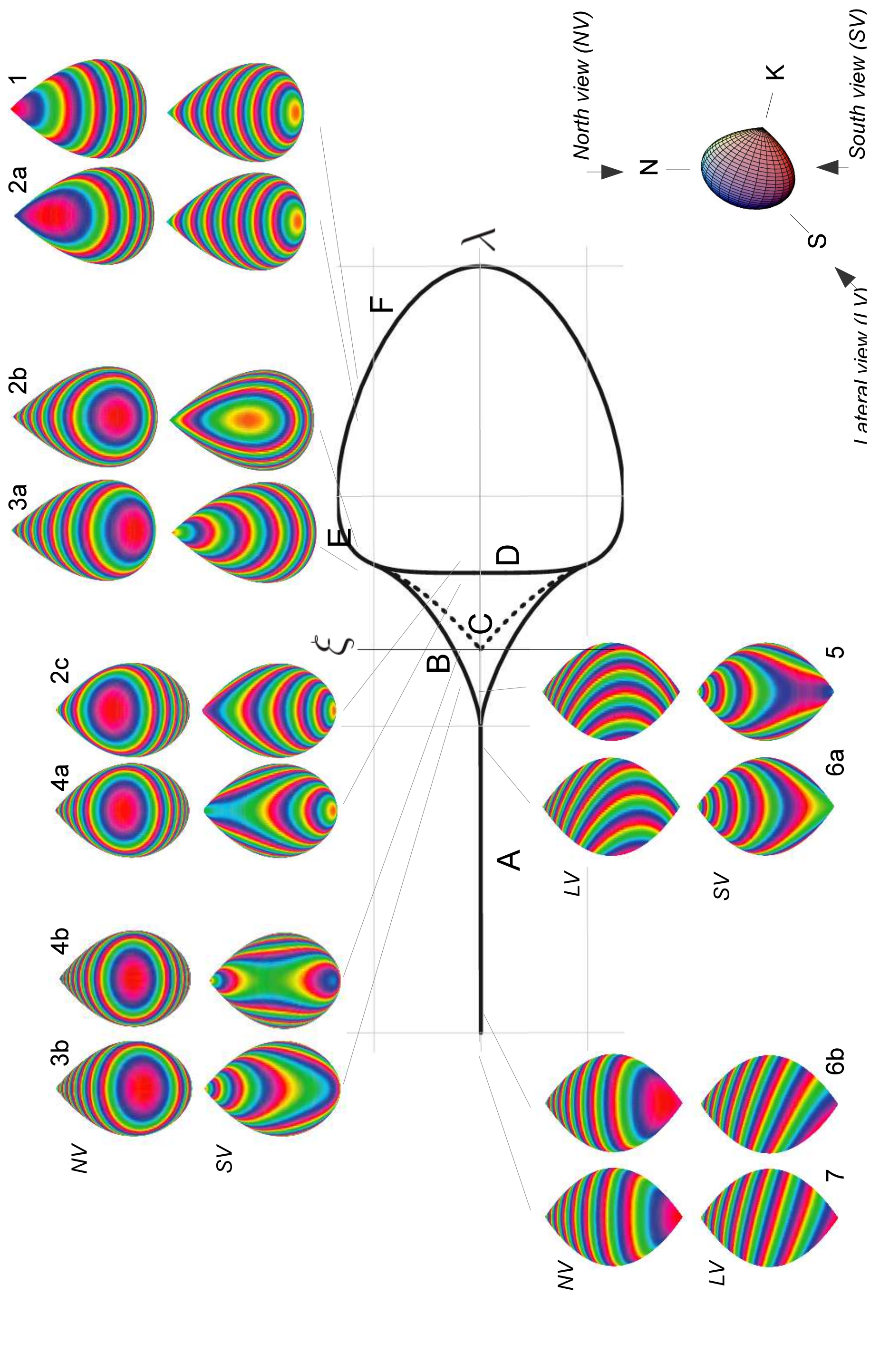

Figure 11: Evolution of the phase flows in thrice reduced over the parameter plane and the bifurcations

$\lambda \neq 4/3$ this point bifurcates into a singular and a regular point both stable. Only when $\lambda = 4/3$ the situation is different. We have that three points bifurcate from the singular reduced space moving transversaly entering region R_3. We may also move to regions R_1 or R_4 leaving from $(4/3, \pm n)$ tangencially.

5 Hamiltonian Hopf Bifurcations (HHB)

In this section we analyze the presence of Hamiltonian Hopf bifurcations in the thrice reduced space using the geometric method, because it is more adecuate for systems of one degree of freedom. For more details on this bifurcation see [23].

The previous results allows to give the following theorem, which we will prove along this Section. Recall the function (29) on which we base our study

$$\lambda_{\pm} = 2n - \frac{2\xi^2}{3n} \pm \frac{4}{3}\sqrt{n^2 - \xi^2}.$$

Theorem 5.1 *Let us consider the normalized truncated 4D Hamiltonian system with Hamiltonian $\bar{H} = H_2 + \varepsilon \bar{H}^{\lambda}$ as given by (9). This system has integrals $\Xi = \xi$, $L_1 = \ell$, and $H_2 = n$. We have that:*

(i) for $\xi = \ell$ and $\lambda = \lambda_+(n, \xi)$ with $-n < \xi < n$ given by (29) the system has a supercritical HHB. In the particular case $\xi = 0$ we have that $\lambda = 10n/3$.

(ii) For $\xi = \ell$ and $\lambda = \lambda_-(n, \xi)$ with $-\frac{\sqrt{5}n}{3} < \xi < \frac{\sqrt{5}n}{3}$ given by (29) the system has a subcritical HHB. In the particular case $\xi = 0$ we have that $\lambda = 2n/3$.

(iii) For $\xi = \ell$ and $\lambda = \lambda_-(n, \xi)$ with $-n < \xi < -\frac{\sqrt{5}n}{3} \cup \frac{\sqrt{5}n}{3} < \xi < n$ given by (29) the system has a supercritical HHB. In the limit cases $\xi = \pm\sqrt{5}n/3$ we have a HHB degenerated.

(iv) For $\xi = \ell = 0$ and $\lambda = -\frac{2n}{3}$ the system has a subcritical HHB.

(v) For $\xi = \ell = 0$ and $\lambda = -\frac{10n}{3}$ the systems has a supercritical HHB.

5.1 Linear Analysis around $K = \pm n$

The standard scenario for a Hamiltonian Hopf bifurcation occurs in smooth 2-DOF Hamiltonian systems with an equilibrium point, which depends on a parameter, such that the linear flow around this point exhibits a Krein collision. In particular, over this bifurcation point the linear vector field has pure imaginary eigenvalues and can not be diagonalized. Thus, it has sense to study in $\mathbb{S}^2_{n+\xi} \times \mathbb{S}^2_{n-\xi}$, such behavior of our system.

The vector field generated by the twice reduced Hamiltonian system $\bar{\bar{H}}^\lambda$ over $\mathbb{S}^2_{n+\xi}\times\mathbb{S}^2_{n-\xi}$ is given by

$$
\begin{aligned}
\dot{n} &= 0,\\
\dot{K}_1 &= 8\,(K_3\,L_2 - K_2\,L_3)\;n,\\
\dot{K}_2 &= L_3\,(-2\,K_1\,n + 2\,L_1\,\xi + 3\,n\,\lambda) - K_3\,(6\,L_1\,n + \xi\,(-2\,K_1+\lambda)),\\
\dot{K}_3 &= L_2\,(2\,K_1\,n - 2\,L_1\,\xi - 3\,n\,\lambda) + K_2\,(6\,L_1\,n + \xi\,(-2\,K_1+\lambda)),\\
\dot{\Xi} &= 0,\\
\dot{L}_1 &= 0,\\
\dot{L}_2 &= -10\,K_1\,K_3\,n + 2\,L_1\,L_3\,n + 2\,K_3\,L_1\,\xi + 2\,K_1\,L_3\,\xi + 3\,K_3\,n\,\lambda - L_3\,\xi\,\lambda,\\
\dot{L}_3 &= -2\,L_1\,(L_2\,n + K_2\,\xi) + 2\,K_1\,(5\,K_2\,n - L_2\,\xi) - 3\,K_2\,n\,\lambda + L_2\,\xi\,\lambda\}.
\end{aligned}
\tag{39}
$$

Remember that $\xi = \ell$ is the condition for the presence of a singular point at $K = n$, $S = N = 0$ (we focus on this case because the other case $K = -n$ is similar). In the twice reduced space with coordinates $\{K_1, K_2, K_3, L_1, L_2, L_3\}$, this point corresponds to $\{n, 0, 0, \xi, 0, 0\}$. Then, the linearized vector field is given by

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -\xi\,(4n+\lambda) & 0 & 0 & 0 & -\Delta_1\\
0 & 0 & \xi\,(4n+\lambda) & 0 & 0 & 0 & \Delta_1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -\Delta_2 & 0 & 0 & 0 & \xi\,(4n-\lambda)\\
0 & 0 & \Delta_2 & 0 & 0 & 0 & \xi\,(-4n+\lambda) & 0
\end{pmatrix},
$$

with $\Delta_1 = 2n^2 - 2\xi^2 - 3n\lambda$ and $\Delta_2 = 10n^2 - 2\xi^2 - 3n\lambda$. Eigenvalues are given by

$$\pm\sqrt{-\Theta \pm 2\xi\sqrt{\Theta}\lambda - \xi^2\lambda^2}$$

with

$$\Theta = 20\,n^4 - 8\,n^2\,\xi^2 + 4\,\xi^4 - 36\,n^3\,\lambda + 12\,n\,\xi^2\,\lambda + 9\,n^2\,\lambda^2.$$

Along the bifurcation curve (29) we have that $\Theta = 0$, in which case we have two pure imaginary eigenvalues

$$\pm\, i\lambda\xi$$

If $\Theta < 0$ we have two pairs of complex eigenvalues

$$\pm\sqrt{-(\Theta+\lambda^2\xi^2) \pm i\lambda\xi\sqrt{|\Theta|}},$$

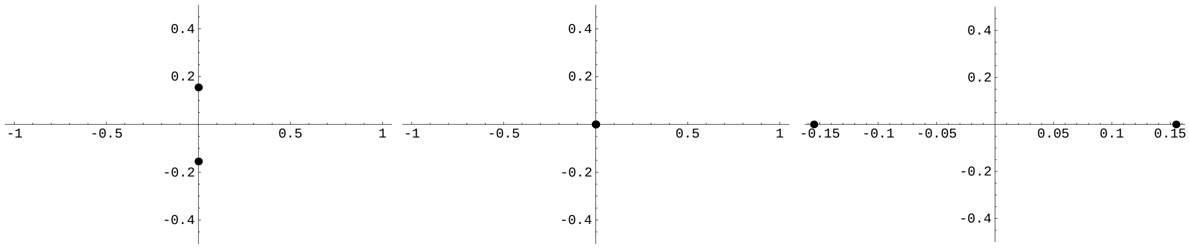

Figure 12: Eigenvalues of the linearized vector field around $K = n$ for the case $\xi = 0$, $n = 1$. At the center the system is over the curve **D** of Fig. 9, which corresponds with $\lambda = 2/3$. At the left we have eigenvalues for a value of λ less than $\lambda = 2/3$ and to the right we represent the situation after the bifurcation.

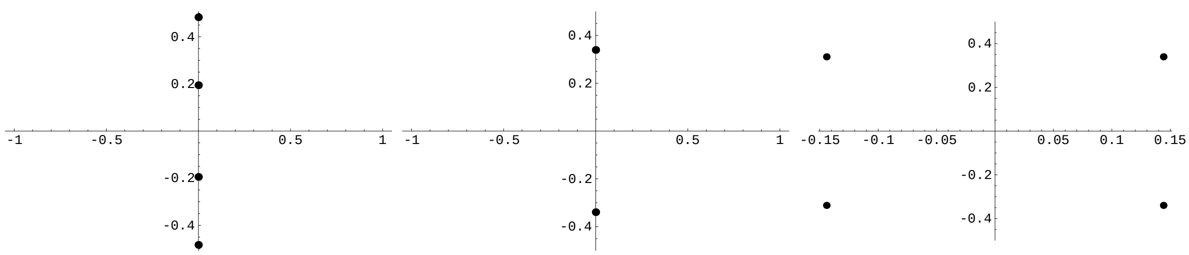

Figure 13: Eigenvalues of the linearized vector field around $K = n$ for the case $\xi = 0$, $n = 1$. At the center the system is over the curve **D** of Fig. 9. At the left eigenvalues correspond to $\mathbf{R_2}$ before crossing the curve, and to the right we are in $\mathbf{R_3}$ after crossing the bifurcation curve.

and if $\Theta > 0$ we have two pairs of pure imaginary eigenvalues

$$\pm\sqrt{-(\sqrt{\Theta} \pm \lambda\xi)^2}$$

Thus, when we cross the bifurcation curve, (see curves **D**, **E** and **F** in Fig. 9), we are in the scenario of the Hamiltonian Hopf bifurcation. This curve goes to the origin when n tends to zero.

We may consider several ways of crossing the bifurcation curve, depending on the variation of λ, ξ or both at the same time. For instance, the passage through $(\lambda, \xi) = (2/3, 0)$ might give rise to different scenarios for the behavior of the eigenvalues. In Fig. (12) we cross the line D along the line $\xi = 0$, in Fig. (13)we cross the line D away from the point $(\lambda, \xi) = (2/3, 0)$, and in Fig. (14) the point $(\lambda, \xi) = (2/3, 0)$ is approached along the line $\xi = 0$ and then the path moves off this line.

From the special cases, apart from the tangencies to the bifurcation curves for which do not have bifurcations, we may consider the case $\xi = 0$. Then, the behavior of the eigenvalues become degenerate. For values $\lambda = \frac{2}{3}n$ and $\lambda = \frac{10}{3}n$ the linearized system is nilpotent. Eigenvalues go from a pair of double pure imaginary to another pair which is

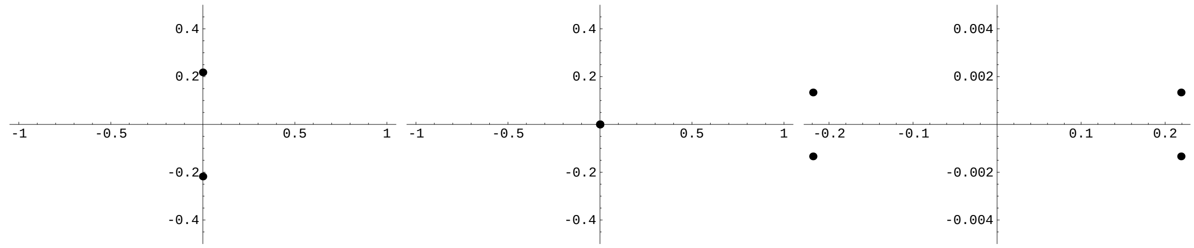

Figure 14: Eigenvalues of the linearized vector field around $K=n$ for the case $\xi=0$, $n=1$. At the left we have eigenvalues in $\mathbf{R_2}$ with $\xi \neq 0$, and to the right in $\mathbf{R_3}$. Thus, we have changed both λ and ξ, versus what we do in Figs. 12 and 13 where we only change λ.

now double real value through a quadruple zero eigenvalues. Due to the presence of the S^1 symmetry associated to L_1 this gives rise to a Hamiltonian Hopf bifurcation because the nilpotent part of the quadratic Hamiltonian can be embedded in a Lie algebra isomorphic to $sl(2,\mathbb{R})$ which commutes with the semisimple Hamiltonian that generates the symmetry S^1. In this case singularity theory allows also to obtain the standard form for the Hopf bfurcation. Note that the more reduced standard form which encapsulates a Hamiltonian Hopf bifurcation obtained in [23] only has a quadratic nilpotent part.

Indeed, when $\xi=0$ the linearization around the equilibrium $K=\pm n$ translates in $\mathbb{S}^2_{n+\xi}\times\mathbb{S}^2_{n-\xi}$ to the linearization around $(K_1,K_2,K_3,L_1,L_2,L_3)=(\pm n,0,0,0,0,0)$ which is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & n(3\lambda\mp 2n) \\ 0 & 0 & 0 & 0 & 0 & 0 & -n(3\lambda\mp 2n) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & n(3\lambda\mp 10n) & 0 & 0 & 0 & 0 \\ 0 & 0 & -n(3\lambda\mp 10n) & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{40}$$

from which we have the following eigenvalues

$$\pm\sqrt{n^2\left(\lambda\mp\frac{2}{3}n\right)\left(\lambda\mp\frac{10}{3}n\right)}, \tag{41}$$

which are zero when

$$\lambda=\pm\frac{2}{3}n,\quad \lambda=\pm\frac{10}{3}n,$$

as we have announced. In he following section we prove the existence of HHB applying the geometric criterium.

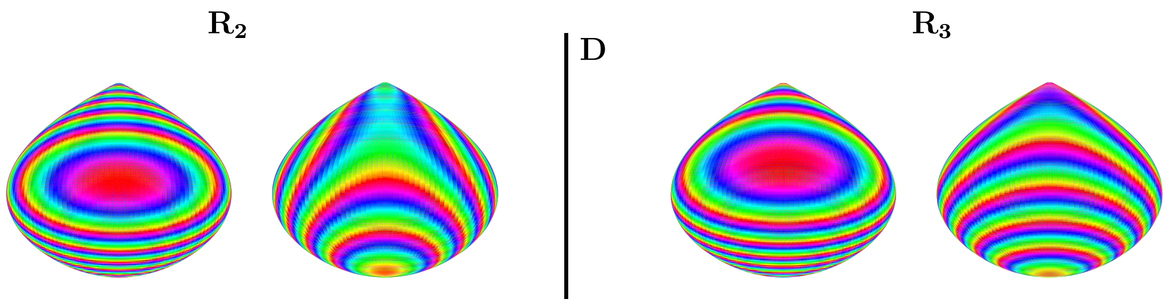

Figure 15: 'North' and 'south' views. From $\mathbf{R_2}$ to $\mathbf{R_3}$. Subcritic Hopf. The point $K = n$ changes its stability and an unstable equilibrium bifurcates from it, as we can see at the south region.

5.2 Non degenerate Hopf Bifurcations at $K = \pm n$

Hopf bifurcations ought to appear at cone type singularities in the reduced space (see [19], [20],[21]). The following Lemma helps in determining such singularities which, on the other hand, we have already detected in the analysis of the reduced spaces.

Lemma 5.2 *If a polynomial $p(x)$ has a zero at $x = c$ with algebraic multiplicity two, then*

$$\left(\frac{d}{dx}\sqrt{p}\right)(c) \neq 0 \ .$$

As a consequence, the cone type singularities appear at the double zeros of $f(K)$, because $f(K)$ is positive in a neigborhood of this zero.

What limit do we have for the presence of HHB? Due to the nature of the thrice reduced phase space, if $\ell = n$ then the space reduces to a point, thus the scenario for a Hamiltonian Hopf bifurcation disappears. We have not analyzed what happens when ℓ is near n, i.e. the 'explosion' of the reduced space from a point. What type of bifurcation is this? The same situation occurs when $\xi = \pm n$. In what follows we will focus on the analysis of the cone like singularity present at $K = n$ when $\xi = \ell$, and the cone like singularity at $K = \pm n$ when $\xi = \ell = 0$.

5.3 Tangencies

The equilibria of the system are defined by the points of tangency between the Hamiltonian and the reduced space. In particular, we will apply the geometric criterium given in [21] in order to determine a HHB. Solving $\mathcal{H} = h$ for $2N$, we may build the energy manifold in the plane $(K, 2N)$

$$g(K) = 2N = \frac{3}{2}\left(K^2 - \lambda K\right) - \frac{\xi^2}{n}K + \frac{h}{n}.$$

We consider also

$$\phi_+(K) = \sqrt{f(K)} = (n-K)\sqrt{(n+K+2\xi)(n+K-2\xi)}$$

and

$$\phi_-(K) = \sqrt{f(K)} = -(n-K)\sqrt{(n+K+2\xi)(n+K-2\xi)}.$$

which represent upper and lower arcs of the reduced space, where $2\xi - n \leq K \leq n$. Thus, as $4N^2 + 4S^2 = f(K)$, they are the section $S = 0$ of the reduced space. Imposing the tangency of both curves at $K = n$ (with $\ell = \xi$) we obtain

$$\begin{aligned} g'(n) = \left.\frac{dg}{dK}\right|_{K=n} &= 3\,n - \frac{\xi^2}{n} - \frac{3\,\lambda}{2}, \\ \phi'_+ = \left.\frac{d\phi_+}{dK}\right|_{K=n} &= -2\sqrt{n^2-\xi^2}, \\ \phi'_- = \left.\frac{d\phi_-}{dK}\right|_{K=n} &= 2\sqrt{n^2-\xi^2}, \end{aligned}$$

where the slope of the upper arc is negative and the lower arc positive. Equaling these expressions we obtain

$$h = n\left(\frac{-3\,n^2}{2} + \xi^2 + \frac{3\,n\,\lambda}{2}\right), \quad \lambda_\pm = \frac{2}{3}\left(3n - \frac{\xi^2}{n} \pm 2\sqrt{n^2-\xi^2}\right),$$

for upper arc $(+)$ and lower arc $(-)$ respectively. Indeed, the tangency of the Hamiltonian with the upper arc at $(n, 0)$ is obtained making

$$g'(K) - \phi'_+(K) = 0$$

and is given by

$$h = \frac{3\,n^3}{2} - 2\,n^2\sqrt{n^2-\xi^2}, \quad \lambda_+ = \frac{2}{3}\left(3n - \frac{\xi^2}{n} + 2\sqrt{n^2-\xi^2}\right)$$

where λ defines the upper arc of the bifurcation curve $\Theta = 0$. For the tangency at the lower arc we have

$$h = \frac{3\,n^3}{2} + 2\,n^2\sqrt{n^2-\xi^2}, \quad \lambda_- = \frac{2}{3}\left(3n - \frac{\xi^2}{n} - 2\sqrt{n^2-\xi^2}\right),$$

where λ defines the lower arc of the bifurcation curve. Note that these branches define the curves **D**, **E** and **F** obtained in (29).

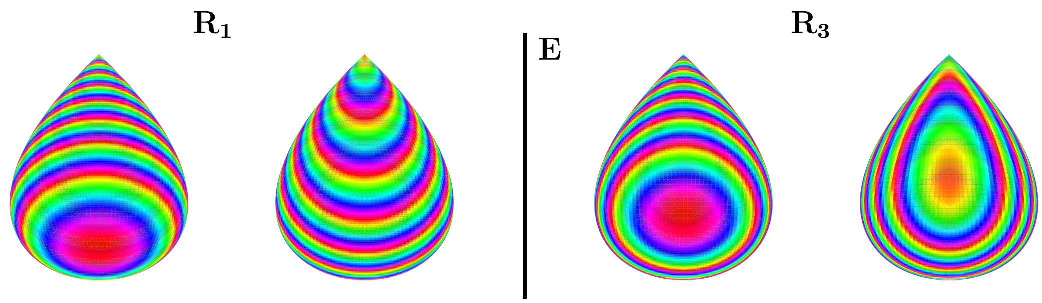

Figure 16: 'North' and 'south' views. Transit from $\mathbf{R_1}$ to $\mathbf{R_3}$. Supercritical Hopf. The singular point $K = n$ switch from stable to unstable and a stable point bifurcates from it, as it can be seen in the south region.

After we have studied tangency conditions we move now to analyze transversality conditions, require for the presence of HHB. The variation of the reduced Hamiltonian with λ through the cone singularity is given by

$$\frac{d}{d\lambda} g'_\lambda(n) = -\frac{3}{2},$$

and, as this can not be zero, the transversality condition is verified.

The second condition we have to check now is the non degeneracy of the higher order terms of the Hamiltonian. We have to analyze when the Hamiltonian has a second order contact at the vertex of the cone and when is tangent to the cone from inside and from outside. Let be

$$u_+(K) = g(K) - \phi_+(K) \quad \text{y} \quad u_-(K) = g(K) - \phi_-(K).$$

For checking the non degeneracy condition we have

$$u''_+ = \left.\frac{d^2 u_+}{dK^2}\right|_{k=n} = 3 + \frac{2n}{\sqrt{n^2 - \xi^2}}, \quad u''_- = \left.\frac{d^2 u_-}{dK^2}\right|_{k=n} = 3 - \frac{2n}{\sqrt{n^2 - \xi^2}},$$

from which we conclude that, with respect to the upper arc, we always have tangency from the outside of the cone and, thus, a supercritical HHB, which also exists when $\xi = 0$. Then we have proven part (i) of Theorem 5.1.

With respect to the lower arc of the reduced space we have tangency from inside when $-\frac{\sqrt{5}}{3}n < \xi < \frac{\sqrt{5}}{3}n$, in which case we have a subcritical HHB. For the range $-n < \xi - \frac{\sqrt{5}}{3}n$ and $\frac{\sqrt{5}}{3}n < \xi < n$ the tangency occurs from outside the cone and, thus, we have a supercritical HHB, which also occurs when $\xi = 0$. For the cases $\xi = \pm\frac{\sqrt{5}}{3}n$ the HHB is degenerate. Thus, we have proven statements (ii) and (iii) of Theorem 5.1 respectively.

In relation to the degenerate HHB, it is important to note that at this point a transition occurs between a supercritical and a subcritical bifurcation. Moreover we have there an equilibrium of multiplicity three. More on degenerate Hamiltonian Hopf bifurcations can be found in [26].

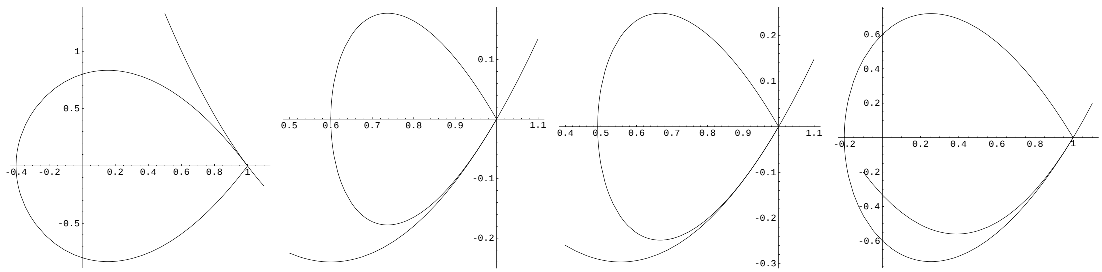

Figure 17: From left to right we have the tangency with the upper branch of the reduced space for $\xi = 0.3$ (supercritical). Moreover we see three tangencies with the lower branch for $\xi = 0.8$, $\sqrt{5}/3$ and $\xi = 0.4$ representing supercritical, degenerate and subcritical respectively. In all cases $n = 1$.

We conclude this section considering the HHB at $K = -n$ when $\xi = 0$, and whose geometric analysis follows the same path we have followed around $K = n$. Verifying transversality conditions we find that for $\lambda = -10n/3$ we have a supercritical HHB and for $\lambda = -2n/3$ a subcritical HHB. Indeed, the transversality condition is satisfied because

$$\frac{d}{d\lambda} g'_\lambda(-n) = -\frac{3}{2},$$

is not zero. The non degeneracy comes from

$$u''_+ = \left.\frac{d^2 u_+}{dK^2}\right|_{k=-n} = 5, \quad u''_- = \left.\frac{d^2 u_-}{dK^2}\right|_{k=-n} = 1,$$

from which we obtain that in the positive branch there is a tangency from outside which gives rise to a supercritical HHB, meanwhile for the lower branch there is a tangency from inside to outside which determines a subcritical HHB. As we know both are present for $\lambda = -10n/3$ and $\lambda = -2n/3$ respectively. This proves statements (iv) and (v) of Theorem 5.1.

6 The Stark-Zeeman Hamiltonian System $\xi = 0$. The Zeeman Case $\xi = \lambda = 0$.

As we have mentioned in the Introduction, the study of the particular case $\xi = 0$ is connected with applications to perturbed Keplerian (Coulomb) Systems in three degrees of freedom through the KS transformation. In that respect, perhaps one of the more studied problem under this approach is the Stark-Zeeman Hamiltonian system, *i.e.* the hydrogen atom under the perturbation of electric and magnetic fields. There is a vast list of papers on these systems; we refer the reader simply to two of them [13], [27] and all the references therein.

Briefly, let us mention there are two models under the name Stark-Zeeman Hamiltonian system according to the relative position of both fields: (a) parallel, or (b) crossed electric magnetic fields (some authors consider the generic case with any relative angle). Their corresponding Hamiltonian are

$$(a) \qquad \mathcal{H} = \tfrac{1}{2}\,\mathbf{X}^2 - \frac{1}{\mathbf{x}} + c_3\, x_3 + c_4\,(x_1^2 + x_2^2), \tag{42}$$

where c_3 and c_4 represent the strength of the electric and the magnetic field, and the second case is

$$(b) \qquad \mathcal{H} = \tfrac{1}{2}\,\mathbf{X}^2 - \frac{1}{\mathbf{x}} + F x_2 + \tfrac{1}{2}\,G(x_2 X_3 - x_3 X_2) + \frac{1}{8} G^2 (x_2^2 + x_3^2), \tag{43}$$

with F and G again the strength of the fields. The difference between both models manifests itself in the fact that the parallel case has an axial symmetry which reads as a second integral: the third component of the angular momentum. Thus, (a) is a 2-DOF system, meanwhile (b) is a 3-DOF system. For recent work on the system defined by (b) see Efstathiou *et al.* [9].

When we consider the regularized and normalized Hamiltonian system for the parallel case then this corresponds to the sytem considered in this paper when $\xi = 0$. Coefficients of the quartic (24) are now

$$\begin{aligned}
b_1 &= 5n^2, \\
b_2 &= -18n^2\lambda, \\
b_3 &= 8n^2\ell^2 + \left(8n^2 + 9\lambda^2\right) n^2 + 12nh, \\
b_4 &= -12nh\lambda, \\
b_5 &= -4n^6 + 8n^4\ell^2 - 4n^2\ell^4 + 4h^2.
\end{aligned}$$

Moreover, if we limit ourselves to the Zeeman case $\lambda = 0$, the quartic takes the form

$$5n^2K^4 + \left(12\,n\,h + 8\,n^2\,\ell^2 + 8\,n^4\right) K^2 + 4\,\left(h^2 - (-n\,\ell^2 + n^3)^2\right) = 0 \tag{44}$$

whose roots are

$$K = \pm\sqrt{\frac{2(-3h - 2n(n^2 + \ell^2))}{5n} \pm \frac{2\,\chi}{5n})} \tag{45}$$

with

$$\chi = \sqrt{4\,h^2 + 9\,n^2\,\ell^4 - 2\,n^4\,\ell^2 + 9\,n^6 + 12\,n\,h\,\,(n^2 + \ell^2)}.$$

Proceeding as explained in (25), we obtain the equilibria which are given by

$$K = 0, \quad K = \pm\sqrt{\frac{2(-3h - 2n(n^2 + \ell^2))}{5n}}. \tag{46}$$

Indeed, in order to obtain the first solution of (46) the equality $h = \pm n\,(\ell^2 - n^2)$ has to be verified, meanwhile we obtain the second when $\chi = 0$ or, in other words, we have $h = \frac{1}{2}\,(-3\,\ell^2\,n \pm 2\,\sqrt{5}\,\ell\,n^2 - 3\,n^3)$, from which finally the possible roots are

$$K = 0, \quad K = \pm\sqrt{\ell^2 \pm \frac{6\,\ell\,n}{\sqrt{5}} + n^2},$$

Looking for double equilibria we find that they only appear in $K = 0$ as triple roots, as it can be seen imposing that (44) defines a double solution $K = 0$, which ends being triple. This occurs when $h = -\frac{2}{3}n(\ell^2 + n^2)$. This can be directly computed using he Poisson flow. In conclusion we have a triple root at $K = 0$ for $\ell = \pm\frac{n}{\sqrt{5}}$ which evolves into two double roots when $|\ell| < \frac{n}{\sqrt{5}}$. These two moving points become the singular points of the reduced space when $\ell = 0$.

In Fig. 18 we may see the geometry of the problem and the tangencies between the reduced space and the Hamiltonian surfaces which corresponds to the Zeeman case. For more information see Cushman [5], van der Meer [25], etc

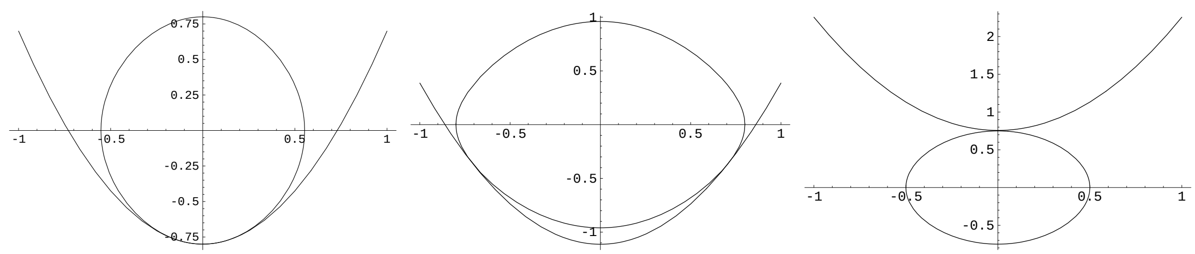

Figure 18: From left to right we have an equilibrium of multiplicity three at $K = 0$ for $\ell = -\frac{1}{\sqrt{5}}$, two equilibria for $\ell = 0.2$ and one equilirium for $\ell = 0.5$. In all the cases we have taken $n = 1$ and $\xi = \lambda = 0$. As $\xi \neq \ell$ the reduced space is not singular.

It is easy to see that we obtain Hamiltonian function (5) when we use the KS type transformation with case (a). Besides, as we have assumed both parameters small and of the same order, we only need to take $c_3 = \varepsilon\,\lambda$ and $c_4 = \varepsilon$. One might ask what is the interest of the study done in this paper in relation to the system defined by Hamiltonian (42)? A complete answer requires to take the integral $\Xi = 0$ and discuss L_1 (the axial symmetry) both in the generic case $L_1 \neq 0$ and in the particular case $L_1 = 0$. In other words, this paper only covers $\Xi = L_1 = 0$. Special as it might appear, this is precisely the scenario which still needed understanding, at least in mathematical terms, because the system has bifurcations of rectilinear orbits which correspond to the singular points of the thrice reduced space. Making use of invariant theory approach only the particular case $\lambda = 0$, the pure Zeeman effect, had been already studied by Cushman [5].

As we said before the system defined by (42) has been subject of research since the 80's. In particular Cacciani *et al.* [1, 2, 3] published that they were able to isolate at laboratory level three different states when dealing with lithium. Again, dealing with Hamiltonian (42) and assuming the azimuthal quantum number to be zero ($\Xi = L_1 = 0$ in our present notation) and using normalization techniques which brought the reduced system to a 2-sphere, Salas *et al.* [29] (see also [30]), found the way of explaining this situation. They reported two pitchfork bifurcations in their reduced phase space, both related to rectilinear orbits in the Oz axis. Just to complete the description of what was behind those bifurcations of the reduced system, the analysis using the Poincaré surfaces of section was made by those authors confirming bifurcation scenarios. The numerical evidence of bifurcations in the original system gave grounds for the publications of those results. We hope the present paper will serve to fix definitively these previous misconceptions: we report now that the 'two pitchfork bifurcations in the reduced phase space' are actually the two Hamiltonian Hopf bifurcations that we see along the line $\Xi = L_1 = 0$ in Fig. (9).

Conclusions

Completing a preliminary report [12], the Hamiltonian fourfold 1:1 resonance in the presence of a toroidal symmetry and is full reduction (regular and singular) is studied in detail. The tori foliation of these perturbed superintegrable systems creates several scenarios depending on the relative values of the integrals. In particular the invariants defining the thrice reduced orbit space are given, as well as the relations among integrals labeling those spaces with one or two singular points. The second aspect on which we focus is the bifurcations of uniparametric Hamiltonian families within is toroidal symmetry context.

Enlarging previous results on the 1:1 and 1:1:1 resonances, attention is put in the interplay between integrals and physical parameters, considering as a benchmark the generalized Stark-Zeeman model in 4 dimensions. We identify a continuos evolution of Hamiltonian Hopf bifurcations (HHB) as well as saddle-center bifurcations, but only in well defined intervals of the physical parameter. In particular, for the physical 3 dimension case $\Xi = 0$, we set up a misleading explanation ([29]) on the bifurcation analysis corresponding to the polar case $L_1 = 0$. What is portrayed there as pitchfork bifurcations is, in fact, a HHB scenario.

Although the setting for the analysis of other similar systems is set up, to refer to generic patterns about these systems, say reconstruction for instance, is still open; our work has to be considered only as a step to that goal. More precisely, we tackle how the well established geometric approach to the HHB extends to 4 dimensions. In that respect we have been working also with the generalized van der Waals problem in 4 dimensions, another uniparametric family which, versus the Stark-Zeeman, presents several integrable cases. This fact introduces another ingredient in the analysis which we are exploring [8].

As in the model considered here the normalization process was truncated at first order, a number of situations appear whose analysis leads to a degenerecy condition, thus demanding more analysis including higher order terms. This work is now in progress.

Acknowledgements

We are in debt to Prof. Abad for the software Esferas that we have used for painting the thrice reduced space. The authors acknowledge support from Ministerio de Educación of Spain, grants BFM2003-02137 and MTM2006-06961 and support from a grant from the Gobierno Regional of Murcia (Fundación Séneca).

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