BUSY PERIOD ANALYSIS OF THE LEVEL DEPENDENT PH/PH/1/K QUEUE

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Abstract. In this paper, we study the transient behavior of a level dependent PH/PH/1/K queue during the busy period. We derive in closed-form the joint transform of the length of the busy period, the number of customers served during the busy period, and the number of losses during the busy period. We differentiate between two types of losses: the overflow losses that are due to a full queue and the losses due to an admission controller. For the M/PH/1/K, M/PH/1/K under a threshold policy, and PH/M/1/K queues we determine simple expressions for their joint transform.

Keywords: PH/PH/1/K queue; Phase-type distributions; Level dependent queues; Busy period; Absorbing Markov chains; Matrix Analytical Approach;

1. Introduction

In practice, it is often the case that arrivals and their service times depend on the system state. For example, in telecommunication systems this happens at the packet switch (router): when its buffer size increases, a controller drops the arriving packets with an increasing probability. In human based service systems, it is known that there is a strong correlation between the volume of work demanded from a human and her/his productivity. Moreover, the transient performance measures of a system are important for understanding the system evolution. All these facts motivate us to study the transient measures of a state dependent queueing system.

The transient regime of queueing systems is much more difficult to analyze than the steady state regime. This explains the scarcity of transient research results in this field compared to the steady state regime. A good exception is the M/M/1 queue which has been well studied in both transient and steady state regimes. This paper is devoted to the study of the more general case of the transient behavior of the state dependent PH/PH/1/K, i.e., the state dependent PH/PH/1 queue with finite waiting room of size $K-1$. In particular, we shall analyze the transient measures related to the busy period.

Takács in [14, Chap 1] was among the first to derive the transient probabilities of the M/M/1/K, referred to as $P_{ij}(t)$. Basically, these are the probabilities that at time t the queue length is j given it was i at time zero. Building on these probabilities

Takács also determined the transient probabilities of the M/M/1 queue by taking the limit of $P_{ij}(t)$ for $K \to \infty$. For the M/G/1/K, Cohen [6, Chap III.6] computed the Laplace transform of $P_{ij}(t)$ and the bivariate transform of the number of customers served and number of losses due to overflow during the busy period. This is done using complex analysis. Specifically, the joint transform is presented as a fraction of two contour integrals that involve K and the Laplace-Stieltjes transform (LST) of the customers' service time. Rosenlund in [12] extended Cohen's result by deriving the joint transform of the busy period length, the number of customers served and the number of losses during the busy period. In a similar way to [12], Rosenlund in [13] analyzed the G/M/1/K and gave the trivariate transform. The approach of Rosenlund is more probabilistic than Cohen's analysis. However, Rosenlund's final results for the trivariate transform for M/G/1/K and G/M/1/K are represented as a fraction of two contour integrals. For more recent works on the busy period analysis of M/G/1/K we refer to [7, 15]. Recently, there was an increased interest in the expected number of losses during the busy period in the M/G/1/K queue with equal arrival and service rate; see, e.g., [1, 11, 16]. In this case, the interesting phenomenon is that the expected number of losses during the busy period in the M/G/1/K equals one for all values of $K \geq 1$.

In this paper, we shall assume that the distribution of the inter-arrival times and service times is phase-type. For this reason, the embedding of the queue length process at the instants of departures or arrivals becomes unnecessary in order to analyze its steady state distribution. We emphasize that is a key difference between our approach and those used in [6, 12, 13]. For an algorithmic method of the LST of the busy period in the PH/PH/1 queue see, e.g., [9, 10]. Bertsimas et al. in [4] derived in closed form the LST of the busy period in the PH/PH/1 queue as a function of the roots of a specific function that involves the LST of the inter-arrival and service times.

In [2], we extended the results of Rosenlund in [12] for the M/M/1/K in several ways. First, we studied a state dependent M/M/1/K with admission control. Second, we considered the residual busy period that is initiated with $n \geq 1$ customers. Moreover, we derived the distribution of the maximum number of customers during the busy period and other related performance measures. In this paper, we shall extend these results by considering the level dependent PH/PH/1/K queue. In a similar way to [2], this shall be done using the theory of absorbing Markov chains. The key point is to model the event that the system becomes empty as absorbing. Contrary to the analysis in [2], the derivation of the joint transform shall not use the explicit inverse of some Toeplitz matrices, however, we shall here proceed with a different approach that is based on the analyticity of probability generating functions.

The paper is organized as follows. In Section 1.1, we give a detailed description of the model and the assumptions made. Section 2 reports our results that shall be presented in a number of different Theorems, Propositions, and Corollaries. More precisely, Theorem 1 gives our main result for the four variate transform as function of the inverse of a specific matrix. Proposition 1 presents a numerical recursion to invert this matrix. In Propositions 2, 3, and 4, we derive the closed form expressions for the four variate transform for the M/PH/1/K, the level dependent M/PH/1/K, and PH/M/1/K queues.

1.1. **Model.** We consider a level dependent PH/PH/1/K queueing system, i.e., a level dependent PH/PH/1 queue with finite waiting room of size $K-1$ customers. The arrival process is a renewal process with phase-type inter-arrival times distribution and with Laplace-Stieltjes transform (LST) $\phi_i(w)$, $Re(w) \geq 0$, in the case where the queue length is $i \in \{0, 1, \ldots, K\}$. The service times distribution is phase-type with LST $\xi_i(w)$, in the case where the queue length is $i \in \{0, 1, \ldots, K\}$. A phase-type distribution can be represented by an initial distribution vector π, a transient generator $\mathbf{F}$, and an absorption rate vector F^o, i.e., $\mathbf{F}^{-1}F^0 = -e^T$, where e^T is a column vector with all entries equal to one. For more details we refer, e.g., to [9, p. 44]. Then, it is well-known that the LST of the inter-arrival times can be written as follows

$$\phi_i(w) = f_i(w\mathbf{I} - \mathbf{F}_i)^{-1}F_i^o, \quad \mathrm{Re}(w) \geq 0, \tag{1}$$

where the initial probability distribution f_i is a row vector of dimension M_a, the transient generator $\mathbf{F}_i$ is an M_a-by-M_a matrix, and the absorption rate vector F_i^o is a column vector of dimension M_a. Similarly, the LST of the service times reads

$$\xi_i(w) = s_i(w\mathbf{I} - \mathbf{S}_i)^{-1}S_i^o, \quad \mathrm{Re}(w) \geq 0, \tag{2}$$

where s_i is a row vector of dimension M_s, $\mathbf{S}_i$ is an M_s-by-M_s matrix, and S_i^o is a column vector of dimension M_s.

We assume that an admission controller is installed at the entry of the queue that has the duty of dropping the arriving customers with probability p_i when the queue length is $i \in \{0, 1, \ldots, K\}$. In other words, the customers are admitted in the queue with probability $q_i = 1 - p_i$ when its queue length is i. The arrivals to the queue of size K are all lost. It should be clear that in this case $p_K = 1$ and $q_K = 0$.

We are interested in the queue behavior during the *busy period* which is defined as: the time interval that starts with an arrival that joins an empty queue and ends at the first time the queue becomes empty again. We note that an arrival to an empty queue is admitted in the system with probability q_0, $0 < q_0 \leq 1$. Similarly, we define the residual busy period as the busy period initiated with $n \geq 1$ customers. Note that for $n = 1$ the residual busy period and the busy period are equal. In the following, we shall assume that, unless otherwise stated, at the beginning of the residual busy period the distribution vector of the phase of the inter-arrival times and service times is distributed according to f_n and s_n.

Consider an arbitrary residual busy period. Let B_n denote its length. Let S_n denote the total number of served customers during B_n. Let L_n denote the total number of losses, i.e. arrivals that are not admitted in the queue either due to the admission control or to the full queue, during B_n. We shall differentiate between the two types of losses. Let L_n^c denote the total number of losses that are not admitted in the queue due to the admission control, during B_n. Let L_n^o denote the total number of the overflow losses that are not admitted in the queue because it is full, i.e. due to $p_K = 1$, during B_n. In this paper, we determine the joint transform $\mathbb{E}\big[e^{-wB_n} z_1^{S_n} z_2^{L_n^c} z_3^{L_n^o}\big]$, $Re(w) \geq 0$, $|z_1| \leq 1$, $|z_2| \leq 1$, and $|z_3| \leq 1$. We will use the theory of absorbing Markov chains. This is done by modeling the event that "the queue jumps to the empty state" as an absorbing event. Tracking the number of customers served and losses before the absorption occurs gives the desired result.

A word on the notation: throughout $x := y$ will designate that by definition x is equal to y, $1_{\{E\}}$ the indicator function of any event E ($1_{\{E\}}$ is equal to one if E is true and zero otherwise), x^T the transpose vector of x, e_i the unit row vector of appropriate dimension with all entries equal to zero except the i-th entry that is one, and $\mathbf{I}$ the identity matrix of appropriate dimension. We use $\otimes$ as the Kronecker product operator defined as follows. Let $\mathbf{X}$ and $\mathbf{Y}$ be two matrices and $x(i,j)$ and $y(i,j)$ denote the (i,j)-entries of $\mathbf{X}$ and $\mathbf{Y}$ respectively then $\mathbf{X} \otimes \mathbf{Y}$ is a block matrix where the (i,j)-block is equal to $x(i,j)\mathbf{Y}$.

2. Results

Before reporting our main result we shall first introduce a set of matrices, then we define our key absorbing Markov chain (AMC), and finally we order the AMC states in a proper way that yields a nice structure. The event that the queue becomes empty, i.e. the end of the busy period, is modeled as an absorbing event which justifies the need of the theory of absorbing Markov chains.

Let us define the following K-by-K block matrices: the matrix $\mathbf{A}$ that is an upper bidiagonal block matrix with i-th upper element equal to $q_i(F_i^o f_i) \otimes \mathbf{I}$ and i-th diagonal element equal to $\mathbf{F}_i \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{S}_i$, the matrix $\mathbf{B}$ that is a lower diagonal matrix with i-th lower diagonal element equal to $\mathbf{I} \otimes (S_i^o s_i)$, and the matrix $\mathbf{C}$ that is a diagonal matrix with i-th diagonal element, $i = 1, \ldots, K-1$, equal to $p_i(F_i^o f_i) \otimes \mathbf{I}$ and K-th element equal to $\mathbf{0}$, and the matrix $\mathbf{D}$ that is a zero block matrix with (K,K)-block element equal to $(F_K^o f_K) \otimes \mathbf{I}$. Note that T_i^o is a column vector and f_i is a row vector thus $F_i^o f_i$ is a matrix. Similarly, $S_i^o s_i$ is a matrix. Moreover, note that $\mathbf{A} + \mathbf{B}$ represents the generator of a level dependent PH/PH/1/K queue restricted to strictly positive queue length, see, e.g., [9, Chap. 3]. Let us denote $\mathbf{Q}_K(w, z_1, z_2, z_3) = w\mathbf{I} - \mathbf{A} - z_1\mathbf{B} - z_2\mathbf{C} - z_3\mathbf{D}$. For ease of presentation, we shall refer to $\mathbf{Q}_K(w, z_1, z_2, z_3)$ as $\mathbf{Q}_K$.

Let $Q(t) := \big(Ph_s(t), Ph_a(t), N(t), S(t), L^c(t), L^o(t)\big)$ denote the continuous-time Markov process with discrete state-space $\xi := \{1, \cdots, M_s\} \times \{1, \cdots, M_a\} \times \{0, 1, \cdots, K\} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$, where $Ph_s(t)$ represents the phase of the (if any) customer in service at time t, $Ph_a(t)$ the phase of the inter-arrival time at time t, $N(t)$ represents the number of customers in the queue at time t, $S(t)$ the number of served customers from the queue until t, $L^c(t)$ the number of losses due to admission control in the queue until t, $L^o(t)$ the number of overflow losses in the queue until t, and $\mathbb{N}$ the set of non-negative integers. States with $N(t) = 0$ are absorbing. We refer to this absorbing Markov process by AMC. The absorption of AMC occurs when the queue becomes empty, i.e., $N(t) = 0$. By setting the initial state of AMC at $t = 0$ to $(p_s, p_a, n, 0, 0, 0)$, $n \geq 1$, $p_s \in \{1, \cdots, M_s\}$ with distribution vector equal to s_n and $p_a \in \{1, \cdots, M_a\}$ with distribution vector equal to f_n, the time until absorption is equal to B_n, the residual busy period length. Moreover, it is clear that S_n (resp. L^o_n and L^c_n), the total number of departures (resp. losses) during the residual busy period, is equal to $S(B_n + \epsilon) = S_n$ (resp. $L^c(B_n + \epsilon) = L^c_n$ and $L^o(B_n + \epsilon) = L^o_n$), $\epsilon > 0$.

During a residual busy period, the processes $S(t)$, $L^c(t)$, and $L^o(t)$ are counting processes. To take advantage of this property, we order the transient states of the AMC, i.e. $(i, j, k, l, m, o) \in \xi \setminus \{(\cdot, \cdot, 0, \cdot, \cdot, \cdot)\}$, increasingly first according to o, then m, l, k, j, and finally according to i. In the following, we shall express the generator of AMC as a function of the aforementioned matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$. The proposed ordering induces that the generator matrix of the transitions between the transient states of AMC, denoted by $\mathbf{G}$, is an infinite upper-diagonal block matrix with diagonal blocks equal to $\mathbf{G}_0$ and upper-diagonal blocks equal to $\mathbf{U}_0$, i.e.,

$$\mathbf{G} = \begin{pmatrix} \mathbf{G}_0 & \mathbf{U}_0 & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{G}_0 & \mathbf{U}_0 & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{3}$$

We note that $\mathbf{G}_0$ denotes the generator matrix of the transitions which do not induce any modification in the number of overflow losses, i.e., $L^o_n(t)$. Moreover, $\mathbf{U}_0$ denotes the transition rate matrix of the transitions that represent an arrival to a full queue (an overflow), i.e., transitions between the transient states (i, j, K, l, m, o) and $(i, j', K, l, m, o + 1)$, where j' is the initial phase of the next inter-arrival time after an overflow loss. For this reason, $\mathbf{U}_0$ is a block diagonal matrix with diagonal blocks equal to $\mathbf{U}_{00}$. The blocks $\mathbf{U}_{00}$ are in turn diagonal block matrices with entries equal to $\mathbf{D}$. The block matrix $\mathbf{G}_0$ is also an infinite upper-diagonal block matrix with diagonal blocks equal to $\mathbf{G}_1$, and upper-diagonal blocks equal to $\mathbf{U}_1$. Therefore, $\mathbf{G}_0$ has the following canonical form:

$$\mathbf{G}_0 = \begin{pmatrix} \mathbf{G}_1 & \mathbf{U}_1 & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{G}_1 & \mathbf{U}_1 & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{4}$$

where $\mathbf{U}_1$ denotes the transition rate matrix of the transitions that represent a dropped arriving customer by the admission controller, i.e., transitions between the transient states (i, j, k, l, m, o) and $(i, j', k, l, m+1, o)$. For this reason, $\mathbf{U}_1$ is a block matrix of diagonal entries equal to $\mathbf{C}$. The matrix $\mathbf{G}_1$ is the generator matrix of the transition between the transient states (i, j, k, l, m, o) and (i', j', k', l', m, o), i.e. the transitions that do not induce any modification in the number of overflow losses and of losses due to the admission controller. Observe that $\mathbf{G}_1$ has the following canonical form:

$$\mathbf{G}_1 = \begin{pmatrix} \mathbf{G}_2 & \mathbf{B} & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{G}_2 & \mathbf{B} & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{5}$$

The upper-diagonal blocks of $\mathbf{G}_1$ represent the transition between the transient states $(i, j, k, l\ , m, o)$ and $(i', j, k-1, l+1, m, o)$, i.e. a transition that models a departure from the queue. For this reason, the upper-diagonal blocks are equal to the aforementioned matrix $\mathbf{B}$, which is a lower diagonal matrix of i-th element equal to $S_i^o s_i \otimes \mathbf{I}$. The matrix $\mathbf{G}_2$ represents the transitions due to a modification in the inter-arrival phase, service phase, or an arrival that is admitted in the queue. Therefore, $\mathbf{G}_2$ is equal to the previously mentioned matrix $\mathbf{A}$, which is an upper-diagonal matrix of the following form:

$$\mathbf{G}_2 = \begin{pmatrix} \mathbf{F}_1 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{S}_1 & q_1 F_1^o f_1 \otimes \mathbf{I} & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{F}_2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{S}_2 & q_2 F_2^o f_2 \otimes \mathbf{I} & \cdots & \cdots \\ \vdots \ddots & \ddots & \ddots & \ddots & \ddots \\ \cdots & \cdots & \cdots & \cdots & \mathbf{F}_K \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{S}_K \end{pmatrix}. \tag{6}$$

In the following we model the event that the queue becomes empty, i.e. the end of the busy period, as an absorbing event. The joint transform is deduced by determining the last state visited before absorption.

We are now ready to formulate our main result.

Theorem 1 (Level dependent queue)**.** *Assume that the residual busy period starts with n customers at time zero, and at time zero the phases of the inter-arrival time and the service time are distributed according to f_n and s_n. The joint transform of B_n, S_n, and L_n is then given by*

$$\mathbb{E}_d\big[e^{-wB_n} z_1^{S_n} z_2^{L_n^c} z_3^{L_n^o}\big] = z_1 e_n \otimes f_n \otimes s_n \mathbf{Q}_K^{-1} (e_1 \otimes e)^T \otimes S_1^o.$$

Proof: Let us denote

$$\pi_{i,j,k,l,m,o}(t) := \mathbb{P}\big(Q(t) = (i, j, k, l, m, o) \mid (p_s, p_a, n, 0, 0, 0)\big).$$

The Laplace transform of $\pi_{i,j,k,l,m,o}(t)$ denotes

$$\tilde{\pi}_{i,j,k,l,m,o}(w) = \int_{t=0}^{\infty} e^{-wt} \pi_{i,j,k,l,m,o}(t) dt, \quad Re(w) \geq 0.$$

Moreover, let us define the following row vectors:

$$\begin{aligned} \tilde{\Pi}_{j,k,l,m,o}(w) &= \big(\tilde{\pi}_{1,j,k,l,m,o}(w), \cdots, \tilde{\pi}_{M_s,j,k,l,m,o}(w)\big), \\ \tilde{\Pi}_{k,l,m,o}(w) &= \big(\tilde{\Pi}_{1,k,l,m,o}(w), \cdots, \tilde{\Pi}_{M_a,k,l,m,o}(w)\big), \\ \tilde{\Pi}_{l,m,o}(w) &= \big(\tilde{\Pi}_{1,l,m,o}(w), \cdots, \tilde{\Pi}_{K,l,m,o}(w)\big). \end{aligned}$$

The Kolmogorov backward equation of the absorbing state $(i, j, 0, l, m, o)$ reads

$$\frac{d}{dt} \pi_{i,j,0,l,m,o}(t) = \pi_{i,j,1,l-1,m,o}(t) S_1^o(i), \tag{7}$$

where $S_1^o(i)$ is the i-th entry of S_1^o. Since $(i,j,0,l,m,o)$ is an absorbing state it is easily seen that

$$\begin{aligned}\pi_{i,j,0,l,m,o}(t) &= \mathbb{P}\Big(B_n < t, Ph_s(B_n) = i, Ph_a(B_n) = j, S_n = l, L_n^c = m,\\ &\qquad L_n^o = o \mid (p_s, p_a, n, 0, 0, 0)\Big).\end{aligned}$$

Hence, the Laplace transform of the l.h.s. of (7) is equal to the joint transform $\mathbb{E}_d\big[e^{-wB_n}\mathbf{1}_{\{Ph_s(B_n)=i\}}\cdot\mathbf{1}_{\{Ph_a(B_n)=j\}}\cdot\mathbf{1}_{\{S_n=l\}}\cdot\mathbf{1}_{\{L_n^c=m\}}\cdot\mathbf{1}_{\{L_n^o=o\}}\big]$. Taking the Laplace transform on both sides in (7) and summing over all values of i and j gives that

$$\begin{aligned}\mathbb{E}_d\big[e^{-wB_n}\cdot\mathbf{1}_{\{S_n=l\}}\cdot\mathbf{1}_{\{L_n^c=m\}}\cdot\mathbf{1}_{\{L_n^o=o\}}\big] &= \sum_{j=1}^{M_a}\tilde{\Pi}_{j,1,l-1,m,o}(w)S_1^o\\ &= \tilde{\Pi}_{1,l-1,m,o}(w)e^T\otimes S_1^o\\ &= \tilde{\Pi}_{l-1,m,o}(w)(e_1\otimes e)^T\otimes S_1^o.\end{aligned}$$

Removing the condition on S_n, L_n^c, and L_n^o we deduce that

$$\begin{aligned}\mathbb{E}_d\big[e^{-wB_n}z_1^{S_n}z_2^{L_n^c}z_3^{L_n^o}\big] &= \sum_{l=1}^{\infty}\sum_{m=0}^{\infty}\sum_{o=0}^{\infty}z_1^l z_2^m z_3^o\tilde{\Pi}_{l-1,m,o}(w)(e_1\otimes e)^T\otimes S_1^o\\ &= z_1\sum_{l=0}^{\infty}z_1^l\sum_{m=0}^{\infty}z_2^m\sum_{o=0}^{\infty}z_3^o\tilde{\Pi}_{l,m,o}(w)(e_1\otimes e)^T\otimes S_1^o. \quad (8)\end{aligned}$$

We now derive the r.h.s. of $\mathbb{E}_d\big[e^{-wB_n}z_1^{S_n}z_2^{L_n^c}z_3^{L_n^o}\big]$. Taking the Laplace transforms of the Kolmogorov backward equations of AMC we find that

$$\begin{aligned}\tilde{\Pi}_{l,m,o}(w)(w\mathbf{I}-\mathbf{A}) &= \mathbf{1}_{\{l,m,o=0\}}e_n\otimes f_n\otimes s_n+\mathbf{1}_{\{l\geq 1\}}\tilde{\Pi}_{l-1,m,o}(w)\mathbf{B}\\ &\quad +\mathbf{1}_{\{m\geq 1\}}\tilde{\Pi}_{l,m-1,o}(w)\mathbf{C}+\mathbf{1}_{\{o\geq 1\}}\tilde{\Pi}_{l,m,o-1}(w)\mathbf{D}, \quad (9)\end{aligned}$$

where $e_n\otimes f_n\otimes s_n$ represents the initial state vector of AMC. Multiplying (9) by $z_1^l z_2^m z_3^o$ and summing the result first over all o, then m, and finally l yields that

$$\sum_{l=0}^{\infty}z_1^l\sum_{m=0}^{\infty}z_2^m\sum_{o=0}^{\infty}z_3^o\tilde{\Pi}_{l,m,o}(w)(w\mathbf{I}-\mathbf{A}-z_1\mathbf{B}-z_2\mathbf{C}-z_3\mathbf{D}) = e_n\otimes f_n\otimes s_n. \quad (10)$$

Note that $(w\mathbf{I}-\mathbf{A}-z_1\mathbf{B}-z_2\mathbf{C}-z_3\mathbf{D})$, $Re(w)>0$, is invertible since it has a dominant main diagonal. Inserting (10) into (8) completes the proof. □

Remark 1. *Assume that the residual busy period starts with n customers at time zero, and at time zero the phases of the inter-arrival time and the service time are distributed according to some distribution vectors f_{n0} and s_{n0}. The joint transform of B_n, S_n, and L_n is then given by*

$$\mathbb{E}_d\big[e^{-wB_n}z_1^{S_n}z_2^{L_n^c}z_3^{L_n^o}\big] = z_1 e_n\otimes f_{n0}\otimes s_{n0}\mathbf{Q}_K^{-1}(e_1\otimes e)^T\otimes S_1^o.$$

Proposition 1. *The joint transform B_1, S_1, L_1^c, and L_1^o is given by*

$$\mathbb{E}_d\big[e^{-wB_1}z_1^{S_1}z_2^{L_1^c}z_3^{L_1^o}\big] \quad = \quad z_1f_1\otimes s_1\big(\mathbf{X}_1\big)^{-1}e^T\otimes S_1^o,$$

where $\mathbf{X}_i$, $i=1,\dots,K-1$, satisfies the following (backward) recursion

$$\mathbf{X}_i \quad = \quad w\mathbf{I}-\mathbf{F}_i\otimes\mathbf{I}-\mathbf{I}\otimes\mathbf{S}_i-z_2p_iF_i^of_i\otimes\mathbf{I}-z_1q_iF_i^of_i\otimes\mathbf{I}\big(\mathbf{X}_{i+1}\big)^{-1}\mathbf{I}\otimes S_{i+1}^os_{i+1},$$

with

$$\mathbf{X}_K=w\mathbf{I}-\mathbf{F}_K\otimes\mathbf{I}-\mathbf{I}\otimes\mathbf{S}_K-z_3F_K^of_K\otimes\mathbf{I}.$$

Proof: According to Theorem 1 the joint transform of B_1, S_1, L_1^c, and L_1^o can be written as

$$\mathbb{E}_d\big[e^{-wB_1}z_1^{S_1}z_2^{L_1^c}z_3^{L_1^o}\big]=z_1f_1\otimes s_1\mathbf{Q}_K(1,1)e^T\otimes S_1^o,$$

where $\mathbf{Q}_K(1,1)$ is the $(1,1)$-block entry of $\mathbf{Q}_K^{-1}$. Let us partition the matrix $\mathbf{Q}_K$ as follows

$$\mathbf{Q}_K=\left(\begin{array}{c|c}\mathbf{F}_{11} & \mathbf{F}_{12}\\ \hline \mathbf{F}_{21} & \mathbf{Q}_{K-1}\end{array}\right), \tag{11}$$

where $\mathbf{F}_{11}:=w\mathbf{I}-\mathbf{F}_1\otimes\mathbf{I}-\mathbf{I}\otimes\mathbf{S}_1-z_2p_1F_1^of_1\otimes\mathbf{I}$, $\mathbf{F}_{12}:=-e_1\otimes q_1F_1^of_1\otimes\mathbf{I}$, $\mathbf{F}_{21}:=-z_1(e_1)^T\otimes\mathbf{I}\otimes S_2^os_2$, $\mathbf{Q}_{K-1}$ is obtained from the matrix $\mathbf{Q}_K$ by removing its first blocks row and first blocks column. A simple linear algebra gives that the inverse of $\mathbf{Q}_K$ reads

$$\mathbf{Q}_K^{-1}=\left(\begin{array}{c|c}(\mathbf{F}_{11}^*)^{-1} & -\mathbf{F}_{11}^{-1}\mathbf{F}_{12}(\mathbf{F}_{22}^*)^{-1}\\ \hline -\mathbf{F}_{22}^{-1}\mathbf{F}_{21}(\mathbf{F}_{11}^*)^{-1} & (\mathbf{F}_{22}^*)^{-1}\end{array}\right), \tag{12}$$

where $\mathbf{F}_{11}^*:=\mathbf{F}_{11}-\mathbf{F}_{12}\mathbf{Q}_{K-1}^{-1}\mathbf{F}_{21}$ and $\mathbf{F}_{22}^*:=\mathbf{Q}_{K-1}-\mathbf{F}_{21}\mathbf{F}_{11}^{-1}\mathbf{F}_{12}$. It is readily seen that

$$\begin{aligned}\mathbb{E}_d\big[e^{-wB_1}z_1^{S_1}z_2^{L_1^c}z_3^{L_1^o}\big] &= z_1f_1\otimes s_1(\mathbf{F}_{11}^*)^{-1}e^T\otimes S_1^o\\ &= z_1f_1\otimes s_1(\mathbf{F}_{11}-\mathbf{F}_{12}(\mathbf{Q}_{K-1})^{-1}\mathbf{F}_{21})^{-1}e^T\otimes S_1^o\\ &= z_1f_1\otimes s_1\Big(w\mathbf{I}-\mathbf{F}_1\otimes\mathbf{I}-\mathbf{I}\otimes\mathbf{S}_1-z_2p_1F_1^of_1\otimes\mathbf{I}\\ &\qquad -q_1F_1^of_1\otimes\mathbf{I}\mathbf{Q}_{K-1}(1,1)\mathbf{I}\otimes S_2^os_2\Big)^{-1}e^T\otimes S_1^o,\end{aligned} \tag{13}$$

where $\mathbf{Q}_{K-1}(1,1)$ is the (1,1)-block entry of $\mathbf{Q}_{K-1}^{-1}$. $\mathbf{Q}_{K-1}$ is a tridiagonal block matrix. Repeating the same way of partitioning the matrix $\mathbf{Q}_K$ to $\mathbf{Q}_{K-1}$ one can show that

$$\mathbf{Q}_{K-1}(1,1)=w\mathbf{I}-\mathbf{F}_2\otimes\mathbf{I}-\mathbf{I}\otimes\mathbf{S}_2-z_2p_2F_2^of_2\otimes\mathbf{I}-q_2F_2^of_2\otimes\mathbf{I}\mathbf{Q}_{K-2}(1,1)\mathbf{I}\otimes S_3^os_3.$$

$\mathbf{Q}_{K-2}(1,1)$ is the (1,1)-block entry of $\mathbf{Q}_{K-2}^{-1}$ and $\mathbf{Q}_{K-2}$ is obtained from the matrix $\mathbf{Q}_{K-1}$ by removing its first row and first column. For this reason, we deduce by induction that $\mathbb{E}_d\big[e^{-wB_1}z_1^{S_1}z_2^{L_1^c}z_3^{L_1^o}\big]$ satisfies the recursion defined in Proposition 1. □

2.1. **M/PH/1/K Queue.** For the M/PH/1/K we have that $-\mathbf{F}_i = F_i^o f_i = \lambda$, $i = 1, \cdots, K$, $\mathbf{S}_i = \mathbf{S}$ and $S_i^o s_i = S^o s$, $i = 1, \cdots, K$. Let $\xi(w) = s(w\mathbf{I} - \mathbf{S})^{-1} S^o$ denote the LST of the service times. Moreover, we assume that $q_i = q$, $i = 1, \cdots, K-1$.

Lemma 1. *The function* $x - z_1\xi\big(w + \lambda(1 - qx - pz_2)\big)$ *has* $M_s + 1$ *distinct non-null roots* $r_1, \cdots, r_{M_s+1}$*, such that* $0 < |r_1| < |r_2| < \cdots < |r_{M_s+1}|$.

Proof. It is well known that $\xi(w)$, the LST of the service times which has a phase-type distribution of M_s phases, is a rational function. Therefore, the denominator of $\xi(w)$ is a polynomial in w of degree M_s and the numerator is a polynomial of degree $< M_s$. For this reason, the numerator of $x - z_1\xi(w + \lambda(1 - qx - pz_2))$ is a polynomial in x of degree $M_s + 1$. Therefore, the function $x - z_1\xi\big(w + \lambda(1 - qx - pz_2)\big)$ has $M_s + 1$ roots. It is easily checked that zero is not a root of this function.

For the sake of clarity of the presentation, we will assume that these roots are distinct. In Section 3 we shall relax this assumption by considering that $r_{i+l} = r_i + l\epsilon$, $\epsilon > 0$, $i \in \{1, \ldots, M_s + 1\}$ and $l = 0, \ldots, L-1$, and taking the limit in our final result for $\epsilon \to 0$. This means, we have that r_i is a root of multiplicity L. ☐

Proposition 2 (M/PH/1/K Queue)**.** *The joint transform of* B_n*,* S_n*,* L_n^o*, and* L_n^c *for the M/PH/1/K queue is given by*

$$\mathbb{E}\big[e^{-wB_n} z_1^{S_n} z_2^{L_n^c} z_3^{L_n^o}\big] = \frac{\frac{1}{2\pi\mathbf{i}} \int_{D_\alpha} \frac{1}{x^{K-1-n}} \frac{1}{qx + pz_2 - z_3} \frac{dx}{x - z_1\xi(w + \lambda(1 - qx - pz_2))}}{\frac{1}{2\pi\mathbf{i}} \int_{D_\alpha} \frac{1}{x^{K-1}} \frac{1}{qx + pz_2 - z_3} \frac{dx}{x - z_1\xi(w + \lambda(1 - qx - pz_2))}},$$

where D_α *denotes the circle with center at the origin and with radius* $|\alpha|$*,* $\left|\frac{pz_2 - z_3}{q}\right| < |\alpha| < |r_1|$*,* r_1 *is the root with the smallest absolute value of*

$$x - z_1\xi\big(w + \lambda(1 - qx - pz_2)\big) = 0. \tag{14}$$

Proof: According to Theorem 1 the joint transform B_n, S_n, L_n^c, and L_n^o for the M/PH/1/K queue can be reduced as follows: (due to the Poisson arrivals we have that $f_n = 1$ and the vector e is of dimension one, i.e., $e = 1$ in Theorem 1),

$$\mathbb{E}\big[e^{-wB_n} z_1^{S_n} z_2^{L_n^c} z_3^{L_n^o}\big] = z_1 e_n \otimes s\mathbf{Q}_K^{-1} e_1^T \otimes S^o, \tag{15}$$

where $\mathbf{Q}_K$ in this case is a K-by-K tridiagonal block matrix with upper diagonal blocks equal to $\mathbf{E}_0 = -q\lambda\mathbf{I}$, i-th diagonal blocks equal to $\mathbf{E}_1 = w\mathbf{I} + \lambda(1 - pz_2)\mathbf{I} - \mathbf{S}$, $i = 1, \cdots, K-1$, and K-th diagonal block equal to $\mathbf{E}_1^* = w\mathbf{I} + \lambda(1 - z_3)\mathbf{I} - \mathbf{S}$, and lower-diagonal blocks equal to $\mathbf{E}_2 = -z_1 S^o s$. Let $u = (u_1, \cdots, u_K) := e_n \otimes s\mathbf{Q}_K^{-1}$. Note that the entries of the row vector u are in their turns a row vectors of dimension M_s and are all functions of w, z_1, z_2, and z_3. Then (15) in terms of u rewrites

$$\mathbb{E}\big[e^{-wB_n} z_1^{S_n} z_2^{L_n^c} z_3^{L_n^o}\big] = z_1 u_1 S^o. \tag{16}$$

The definition of u gives that $u\mathbf{Q}_K = e_n \otimes s$. Developing the latter equation yields

$$\mathbf{1}_{\{i \geq 2\}} u_{i-1}\mathbf{E}_0 + u_i\big[\mathbf{1}_{\{i \leq K-1\}}\mathbf{E}_1 + \mathbf{1}_{\{i=K\}}\mathbf{E}_1^*\big] + \mathbf{1}_{\{i \leq K-1\}} u_{i+1}\mathbf{E}_2 = \mathbf{1}_{\{i=n\}} s, \tag{17}$$

where $i = 1, \cdots, K$. Since u_1 is analytic we deduce from (17) that u_i, $i = 2, \cdots, K$, are analytic. Multiplying (17) by x^i and summing it over i we find that

$$\sum_{i=1}^{K} u_i x^i = \big(u_1\mathbf{E}_2 + x^K u_K(x\mathbf{E}_0 + \mathbf{E}_1 - \mathbf{E}_1^*) + x^n s\big)\Big(x\mathbf{E}_0 + \mathbf{E}_1 + \frac{1}{x}\mathbf{E}_2\Big)^{-1}$$

$$= \big(z_1 u_1 S^o s - x^n s + \lambda x^K(qx + pz_2 - z_3)u_K\big)\Big(\mathbf{S} - \rho\mathbf{I} + \frac{z_1}{x}S^o s\Big)^{-1}, \quad (18)$$

where $\rho := w + \lambda(1 - qx - pz_2)$. Let $\mathbf{S}_* := \mathbf{S} - \rho\mathbf{I}$. Note that under the condition $Re[\rho] \geq 0$ the matrix $\mathbf{S}_*$ is nonsingular. Hence, the Sherman-Morrison formula, see, e.g., [3, Fact 2.14.2, p. 67], yields that

$$\Big(\mathbf{S}_* + \frac{z_1}{x}S^o s\Big)^{-1} = \mathbf{S}_*^{-1} - \frac{z_1}{x + z_1 s\mathbf{S}_*^{-1}S^o}\mathbf{S}_*^{-1}S^o s\mathbf{S}_*^{-1}. \quad (19)$$

The multiplication to the right of (18) with the column vector S^o and (19) give

$$\sum_{i=1}^{K} u_i x^i S^o = \frac{x}{x + z_1 s\mathbf{S}_*^{-1}S^o}\big(z_1 u_1 S^o s - x^n s + \lambda x^K(qx + pz_2 - z_3)u_K\big)\mathbf{S}_*^{-1}S^o, \quad (20)$$

From (2) we know that $s\mathbf{S}_*^{-1}S^o = -\xi(\rho)$ and $\mathbf{S}_*^{-1}S^o = -\big(\xi^1(\rho), \cdots, \xi^{Ms}(\rho)\big)^T$, where $\xi^i(\rho) = e_i(\rho\mathbf{I} - \mathbf{S})^{-1}S^o$. Therefore, $\xi(\rho) = s(\xi^1(\rho), \cdots, \xi^{Ms}(\rho)\big)^T$ is a linear combination of $\xi^i(\rho)$, $i = 1, \cdots, M_s$. Inserting $s\mathbf{S}_*^{-1}S^o$ and $\mathbf{S}_*^{-1}S^o$ into (20) yields

$$\sum_{i=1}^{K} u_i x^i S^o = \frac{-x}{x - z_1\xi(\rho)}\Big[(z_1 u_1 S^o - x^n)\xi(\rho) + \lambda x^K(qx + pz_2 - z_3)\sum_{j=1}^{M_s} u_{Kj}\xi^j(\rho)\Big], \quad (21)$$

where $u_K = (u_{K1}, \cdots, u_{KM_s})$. We recall that $u_i S^o$ is a joint transform function. For this reason, the l.h.s. of (21) should be analytical for any finite x. This gives that the singular points, roots of $x - z_1\xi(\rho)$, on the r.h.s. of (21) are removable.

Lemma 1 and the analyticity of $\sum_{i=1}^{K} u_i x^i S^o$ gives that

$$z_1 u_1 S^o \xi(\rho_i) + \lambda r_i^K(qr_i + pz_2 - z_3)\sum_{j=1}^{M_s} u_{Kj}\xi^j(\rho_i) = r_i^n\xi(\rho_i), \quad i = 1, \cdots, M_s + 1, \quad (22)$$

where $\rho_i := w + \lambda(1 - qr_i - pz_2)$. The system of equations in (22) has $M_s + 1$ equations with $M_s + 1$ unknowns which are $z_1 u_1 S^o, u_{K1}, \cdots, u_{KM_s}$. Using Cramer's rule we find that

$$\mathbb{E}\big[e^{-wB_n} z_1^{S_n} z_2^{L_n^c} z_3^{L_n^o}\big] = z_1 u_1 S^o = \frac{\det(\mathbf{M}_1)}{\det(\mathbf{M})}, \quad (23)$$

where $\det(\mathbf{M})$ is the determinant of the $(M_s + 1)$-by-$(M_s + 1)$ matrix $\mathbf{M}$ with i-th row equal to $\big(\xi(\rho_i)/[\lambda r_i^K\ (qr_i + pz_2 - z_3)], \xi^1(\rho_i), \cdots, \xi^{M_s}(\rho_i)\big)$, $i = 1, \cdots, M_s + 1$, and $\mathbf{M}_1$ is obtained from $\mathbf{M}$ by replacing its first column by

$$\Big(\frac{\xi(\rho_1)}{\lambda r_1^{K-n}(qr_1 + pz_2 - z_3)}, \cdots, \frac{\xi(\rho_{M_s+1})}{\lambda r_{M_s+1}^{K-n}(qr_{M_s+1} + pz_2 - z_3)}\Big)^T.$$

The Laplace expansion of the determinant along the first column of $\mathbf{M}$ and $\mathbf{M}_1$ gives that

$$\begin{aligned}\mathbb{E}\left[e^{-wB_n}z_1^{S_n}z_2^{L_n^c}z_3^{L_n^o}\right] &= \frac{\sum_{i=1}^{M_s+1}\frac{\xi(\rho_i)(-1)^{i+1}}{\lambda r_i^{K-n}(qr_i+pz_2-z_3)}\det\big(\mathbf{M}_1(i,1)\big)}{\sum_{i=1}^{M_s+1}\frac{\xi(\rho_i)(-1)^{i+1}}{\lambda r_i^{K}(qr_i+pz_2-z_3)}\det\big(\mathbf{M}(i,1)\big)}\\ &= \frac{\sum_{i=1}^{M_s+1}\frac{(-1)^{i}}{r_i^{K-1-n}(qr_i+pz_2-z_3)}\det\big(\mathbf{M}(i,1)\big)}{\sum_{i=1}^{M_s+1}\frac{(-1)^{i}}{r_i^{K-1}(qr_i+pz_2-z_3)}\det\big(\mathbf{M}(i,1)\big)}, \end{aligned} \tag{24}$$

where $\mathbf{M}(i,1)$ is the M_s-by-M_s matrix that results by deleting the i-th row and the first column of $\mathbf{M}$, and the second equality follows from $\xi(\rho_i) = r_i/z_1$ and $\mathbf{M}_1(i,1) = \mathbf{M}(i,1)$.

Let D_α denote the circle with center at the origin and with radius equal to $|\alpha|$. Assume that $\left|\frac{pz_2-z_3}{q}\right| < |\alpha| < |r_1|$. Let us define $f_i(x) \sim_i g_i(x)$ if $f_i(x)/g_i(x) = h(x)$ that is independent of i. Therefore, for $K-1-n \geq 1$ or $q > 0$ the following equality holds

$$\frac{\sum_{i=1}^{M_s+1}\frac{(-1)^{i}}{r_i^{K-1-n}(qr_i+pz_2-z_3)}\det\big(\mathbf{M}(i,1)\big)}{\sum_{i=1}^{M_s+1}\frac{(-1)^{i}}{r_i^{K-1}(qr_i+pz_2-z_3)}\det\big(\mathbf{M}(i,1)\big)} = \frac{\frac{1}{2\pi\mathbf{i}}\int_{D_\alpha}\frac{1}{x^{K-1-n}}\frac{1}{qx+pz_2-z_3}\frac{dx}{x-z_1\xi(w+\lambda(1-qx-pz_2))}}{\frac{1}{2\pi\mathbf{i}}\int_{D_\alpha}\frac{1}{x^{K-1}}\frac{1}{qx+pz_2-z_3}\frac{dx}{x-z_1\xi(w+\lambda(1-qx-pz_2))}}, \tag{25}$$

if and only if

$$\mathrm{Res}_{r_i}\frac{1}{x-z_1\xi\big(w+\lambda(1-qx-pz_2)\big)} \sim_i (-1)^i\det\big(\mathbf{M}(i,1)\big), \tag{26}$$

where $\mathrm{Res}_a f(z)$ is the residue of the complex function $f(z)$ at point a. In the following we shall prove condition (26).

Since the service times have a phase-type distribution, $\xi(w)$ is a rational function with denominator, $Q(w)$, of degree M_s and numerator of degree $< M_s$. Note that by Lemma 1 the roots of $x - z_1\xi\big(w+\lambda(1-qx-pz_2)\big)$ are distinct. Therefore, we deduce that

$$\begin{aligned}\mathrm{Res}_{r_i}\frac{1}{x-z_1\xi\big(w+\lambda(1-qx-pz_2)\big)} &= \frac{Q\big(w+\lambda(1-qr_i-pz_2)\big)}{(-\lambda q)^{M_s}\prod_{j=1,j\neq i}^{M_s+1}(r_i-r_j)}\\ &= \frac{Q(\rho_i)}{\prod_{j=1,j\neq i}^{M_s+1}(\rho_i-\rho_j)}.\end{aligned}$$

$\mathbf{M}(i,1)$ is an M_s-by-M_s matrix of j-th row equal to $\big(\xi^1(\rho_j),\cdots,\xi^{M_s}(\rho_j)\big)$ for $j = 1,\cdots,M_s+1$ and $j \neq i$. We have that (see the appendix for the proof)

$$\det\big(\mathbf{M}(i,1)\big) = C\frac{\prod_{j=1,j\neq i}^{M_s}\prod_{k=j+1,k\neq i}^{M_s+1}(\rho_k-\rho_j)}{\prod_{j=1,j\neq i}^{M_s+1}Q(\rho_j)}, \tag{27}$$

where C is a constant, see the Appendix. It is easily checked that

$$\prod_{j=1,j\neq i}^{M_s}\prod_{k=j+1,k\neq i}^{M_s+1}(\rho_k-\rho_i)=(-1)^{i-1}\frac{\prod_{j=1}^{M_s}\prod_{k=j+1}^{M_s+1}(\rho_k-\rho_j)}{\prod_{j=1,j\neq i}^{M_s+1}(\rho_j-\rho_i)}. \tag{28}$$

Substituting the last equation into (27) yields

$$\begin{aligned}\det\big(\mathbf{M}(i,1)\big) &= C(-1)^{i-1}\frac{\prod_{j=1}^{M_s}\prod_{k=j+1}^{M_s+1}(\rho_k-\rho_j)}{\prod_{j=1}^{M_s+1}Q(\rho_j)}\times\frac{Q(\rho_i)}{\prod_{j=1,j\neq i}^{M_s+1}(\rho_j-\rho_i)}\\ &\sim_i (-1)^i\frac{Q(\rho_i)}{\prod_{j=1,j\neq i}^{M_s+1}(\rho_i-\rho_j)}.\end{aligned}$$

The latter equation yields (26) which completes the proof. □

Remark 2. *We emphasize that Proposition 2 extends the result of Rosenlund* [12] *on the M/G/1/K in two ways. First, it gives the four variate joint transform of* B_n, S_n, L_n^c, *and* L_n^o, *for the case when* $n>1$. *Second, it allows the dropping of customers even when the queue is not full.*

2.2. **M/PH/1/K queue under threshold policy.** Let $m\in\{1,\dots,K\}$ denote the threshold of the M/PH/1/K queue length. According to the threshold policy if the queue length at time t is i the inter-arrival times and service times are then defined as follows. For $i\le m-1$, we have that $-\mathbf{F}_i=F_i^o f=\lambda_0$, $\mathbf{S}_i=\mathbf{S}_0$, $s_i=s$, and $p_i=p_0$. For $m\le i\le K-1$, we have that $-\mathbf{F}_i=F_i^o f=\lambda_1$, $\mathbf{S}_i=\mathbf{S}_1$ and $s_i=s$, and $p_i=p_1$ and $p_K=1$.

Let $\xi_i(w)=s(w\mathbf{I}-\mathbf{S}_i)^{-1}S_i^o=P_i(w)/Q_i(w)$, $i=0,1$, denote the LST of the service times when the queue length is below the threshold or above it. Moreover, we let $\xi_i^l(w)=e_l(w\mathbf{I}-\mathbf{S}_i)^{-1}S_i^o=P_i^l(w)/Q_i^l(w)$, $i=0,1$. Note that since $Q_0(w)$ is the common denominator of $\xi_0^l(w)$ we have that $\xi_0^l(w)=P_0^l(w)/Q_0(w)$ is a rational function where $P_0^l(w)$ is polynomial of degree $<M_s$. Let $\mathbf{C}_0$ denote the matrix with (j,l)-entry equal to the coefficient of w^{j-1} of the polynomial $P_0^l(w)$. In the following, we shall assume that the matrix $\mathbf{C}_0$ is invertible. Note that the Erlang, hyper-exponential, and Coxian distribution satisfy the latter assumption.

Lemma 2. *The function* $x-z_1\xi_i\big(w+\lambda(1-q_ix-p_iz_2)\big)$, $i=0,1$, *has* M_s+1 *distinct non-null roots* $r_{1i},\cdots,r_{(M_s+1)i}$, *such that* $0<|r_{1i}|<\cdots<|r_{(M_s+1)i}|$.

Proof. By analogy with the proof of Lemma 1. □

We are now ready to state our first result.

Proposition 3 (M/PH/1/K under Threshold Policy)**.** *The joint transform of* B_1, S_1, L_1^c,*and* L_1^o *in the M/PH/1/K queue operating under the threshold policy is given by*

$$\mathbb{E}\left[e^{-wB_1}z_1^{S_1}z_2^{L_1^c}z_3^{L_1^o}\right] = \frac{\sum_{i=1}^{M_s+1}\frac{z_1-\beta(i)}{r_{i1}^{m-2}}\frac{Q_0(\rho_{i0})}{\prod_{j=1,j\neq i}^{M_s+1}(\rho_{i0}-\rho_{j0})}}{\sum_{i=1}^{M_s+1}\frac{z_1-\beta(i)}{r_{i1}^{m-1}}\frac{Q_0(\rho_{i0})}{\prod_{j=1,j\neq i}^{M_s+1}(\rho_{i0}-\rho_{j0})}}, \tag{29}$$

where, r_{i0}, $i = 1, \cdots, M_s + 1$, are the roots of

$$x - z_1\xi_0\big(w + \lambda_0(1 - q_0x - p_0z_2)\big) = 0,$$

$\rho_{i0} = w + \lambda_0(1 - q_0r_{i0} - p_0z_2)$, *$Q_0(w)$ is the denominator of $\xi_0(w)$,*

$$\beta(i) = \sum_{l=1}^{M_s} v(l) \sum_{m=1,m\neq i}^{M_s+1} Q_0(\rho_{m0}) \sum_{k=1}^{M_s} c_0(l,k)v_0(k,m),$$

where $c_0(l,k)$ is the (l,k)-entry of $\mathbf{C}_0^{-1}$,

$$v_0(k,m) = \frac{(-1)^{k-1}}{\prod_{l=1,l\neq m}^{M_s} u_l - u_m} \sum u_{m_1 0} \times \cdots \times u_{m_{M_s-k}0}, \quad k,m = 1,\cdots,M_s \tag{30}$$

where $1 \leq m_1 < \cdots < m_{M_s-k} \leq M_s$, $m_1, \cdots, m_{M_s-k} \neq k$, and $(u_1,\cdots,u_{M_s}) = (\rho_{10},\cdots,\rho_{(i-1)0},\rho_{(i+1)0},\cdots,\rho_{(M_s+1)0})$ (for $k = M_s$, $\sum u_{m_1 0} \times \cdots \times u_{m_{M_s-k}0} := 1$), and

$$v(l) = z_1\frac{\frac{1}{2\pi\mathbf{i}}\int_{D_{\alpha_1}}\frac{1}{x^{K-m}}\frac{\xi_1^l(w+\lambda(1-q_1x-p_1z_2))}{q_1x+p_1z_2-z_3}\frac{dx}{x-z_1\xi_1(w+\lambda(1-q_1x-p_1z_2))}}{\frac{1}{2\pi\mathbf{i}}\int_{D_{\alpha_1}}\frac{1}{x^{K-m}}\frac{1}{q_1x+p_1z_2-z_3}\frac{dx}{x-z_1\xi_1(w+\lambda(1-q_1x-p_1z_2))}}, \tag{31}$$

where D_{α_1} denotes the circle with center at the origin and with radius $|\alpha_1|$, $\frac{|p_1z_2-z_3|}{q_1} < |\alpha_1| < |r_{11}|$, r_{11} is the root with the smallest absolute value of

$$x - z_1\xi_1\big(w + \lambda_1(1 - q_1x - p_1z_2)\big) = 0.$$

Proof: By analogy with Proposition 2 the joint transform B_1, S_1, L_1^c, and L_1^o for the M/PH/1/K queue can written as follows:

$$\mathbb{E}\left[e^{-wB_1}z_1^{S_1}z_2^{L_1^c}z_3^{L_1^o}\right] = z_1e_1 \otimes s\mathbf{Q}_K^{-1}e_1^T \otimes S_0^o, \tag{32}$$

where in this case $\mathbf{Q}_K$ has the following canonical form

$$\mathbf{Q}_K = \left(\begin{array}{c|c}\mathbf{F}_{00} & \mathbf{F}_{01}\\ \hline \mathbf{F}_{10} & \mathbf{F}_{11}\end{array}\right).$$

The matrix $\mathbf{F}_{ll}$, $l = 0, 1$, is a block tridiagonal matrix with upper diagonal blocks equal to $\mathbf{E}_{0l} = -q_l\lambda_l\mathbf{I}$, diagonal blocks equal to $\mathbf{E}_{1l} = w\mathbf{I} + \lambda_l(1 - p_lz_2)\mathbf{I} - \mathbf{S}_l$ and lower-diagonal blocks equal to $\mathbf{E}_{2l} = -z_1S_l^os$. Note that $\mathbf{F}_{00}$ is an $(m-1)$-by-$(m-1)$ block matrix and $\mathbf{F}_{11}$ is an $(K-m+1)$-by-$(K-m+1)$ block matrix. Moreover, the $(K-m+1, K-m+1)$-block entry of $\mathbf{F}_{11}$ is equal to $\mathbf{E}_{11}^* = w\mathbf{I}+\lambda_1(1-z_3)\mathbf{I}-\mathbf{S}_1$. The matrix $\mathbf{F}_{01}$ is a block matrix with all its blocks equal to the zero matrix except the $(m-1,1)$-block that is $\mathbf{E}_{00} = -q_0\lambda_0\mathbf{I}$. Finally, the matrix $\mathbf{F}_{10}$ is a block matrix with all blocks equal to the zero matrix except the $(1,m-1)$-block that is $\mathbf{E}_{21} = -z_1S_1^os$.

Equations (12) and (32) yield that

$$\begin{aligned}\mathbb{E}\big[e^{-wB_1}z_1^{S_1}z_2^{L_1^c}z_3^{L_1^o}\big] &= z_1e_1\otimes s\big(\mathbf{F}_{00}-\mathbf{F}_{01}\mathbf{F}_{11}^{-1}\mathbf{F}_{10}\big)^{-1}e_1^T\otimes S_0^o \\ &= z_1e_1\otimes s\big(\mathbf{F}_{00}-q_0\lambda_0z_1\mathbf{F}_{11}^{-1}(1,1)\mathbf{S}_1^osU^TU\big)^{-1}e_1^T\otimes S_0^o, \quad (33)\end{aligned}$$

where U is a row vector of blocks with all entries equal to zero except the last that is $\mathbf{I}$ and $\mathbf{F}_{11}^{-1}(1,1)$ is the $(1,1)$-block entry of $\mathbf{F}_{11}^{-1}$.

We shall now derive an expression for $z_1\mathbf{F}_{11}^{-1}(1,1)\mathbf{S}_1^o$. First, observe that $\mathbf{F}_{11}$ has the same structure as the matrix $\mathbf{Q}_K$ in (15) with K replaced by $K-m+1$, λ by λ_1, $\mathbf{S}$ by $\mathbf{S}_1$, and S^os by S_1^os. Second, note that $z_1\mathbf{F}_{11}^{-1}(1,1)\mathbf{S}_1^o$ is a column vector with size M_s and with j-th entry, referred to as $v(j)$, that reads

$$v(l) = z_1e_1\otimes e_l(\mathbf{F}_{11})^{-1}e_1^T\otimes S_1^o, \quad j=1,\cdots,M_s. \quad (34)$$

Therefore, by analogy with the proof of Proposition 2 we find that $v(l)$ satisfies (31).

Note that $\mathbf{F}_{00}$ has the same structure as the matrix $\mathbf{Q}_K$ in (15) with $K=m-1$, $\mathbf{E}_0=\mathbf{E}_{00}$, $\mathbf{E}_1=\mathbf{E}_{10}$, $\mathbf{E}_2=\mathbf{E}_{20}$, and $\mathbf{E}_1^*=\mathbf{E}_{10}-q_0\lambda_0vsU^TU$. Moreover, (33) has the same form as (15). By analogy with the proof of Proposition 2 we find that

$$\begin{aligned}\sum_{i=1}^{K}o_ix^iS_0^o &= \frac{-x}{x-z_1\xi_0(\rho)}\Big[(z_1u_1S_0^o-x)\xi_0(\rho)+\lambda_0q_0x^{m-1}\sum_{j=1}^{M_s}o_{m-1j}\Big(x\xi_0^j(\rho) \\ &\quad -v(j)\xi_0(\rho)\Big)\Big],\end{aligned}$$

where $o=(o_1,\cdots,o_{m-1}) := e_1\otimes s\big(\mathbf{F}_{00}-q_0\lambda_0vsU^TU\big)^{-1}$, $o_{m-1}=(o_{m-11},\cdots,o_{m-1M_s})$, and $\rho=w+\lambda_0(1-q_0x-p_0z_2)$. Let us denote r_{i0}, $i=0,\cdots,M_s+1$, the roots of $x-z_1\xi_0(w+\lambda_0(1-q_0x-p_0z_2))$. The analyticity of $\sum_{i=1}^{K}o_ix^iS_0^o$ gives that

$$z_1o_1S_0^o\xi_0(\rho_{i0})+\lambda_0q_0r_{i0}^{m-1}\sum_{j=1}^{M_s}o_{m-1j}\big(r_{i0}\xi_0^j(\rho_{i0})-v(j)\xi_0(\rho_{i0})\big)=r_{i0}\xi_0(\rho_{i0}),$$

where $i=1,\cdots,M_s+1$ and $\rho_i=w+\lambda_0(1-q_0x-p_0z_2)$. Cramer's rule yields that

$$\begin{aligned}\mathbb{E}\big[e^{-wB_1}z_1^{S_1}z_2^{L_1^c}z_3^{L_1^o}\big] &= z_1o_1S_0^o=\frac{\sum_{i=1}^{M_s+1}\frac{\xi_0(\rho_{i0})(-1)^i}{r_{i1}^{m-1}}\det\big(\mathbf{M}_0(i)\big)}{\sum_{i=1}^{M_s+1}\frac{\xi_0(\rho_{i0})(-1)^i}{r_{i0}^{m}}\det\big(\mathbf{M}_0(i)\big)} \\ &= \frac{\sum_{i=1}^{M_s+1}\frac{(-1)^i}{r_{i1}^{m-2}}\det\big(\mathbf{M}_0(i)\big)}{\sum_{i=1}^{M_s+1}\frac{(-1)^i}{r_{i0}^{m-1}}\det\big(\mathbf{M}_0(i)\big)}, \quad (35)\end{aligned}$$

where $\mathbf{M}_0(i)$ is an M_s-by-M_s matrix with j-th row, $j=1,\cdots,M_s+1$ and $j\neq i$, equal to $\big(\xi_0^1(\rho_{j0})-v(1)/z_1,\cdots,\xi_0^{M_s}(\rho_{j0})-v(M_s)/z_1\big)$. It is easily seen that $\mathbf{M}_0(i)$ can be decomposed as follows

$$\mathbf{M}_0(i)=\mathbf{V}(i)-\frac{1}{z_1}e^Tv,$$

where $\mathbf{V}(i)$ is an M_s-by-M_s matrix with j-th row, $j = 1, \cdots, M_s+1$ and $j \neq i$, equal to $\big(\xi_0^1(\rho_{j0}), \cdots, \xi_0^{M_s}(\rho_{j0})\big)$. Since $\xi_0^l(w) = P_0^l(w)/Q_0(w)$, $l = 1, \cdots, M_s$, are rational functions with common denominator $Q_0(w)$ the matrix $\mathbf{V}(i)$ can be decomposed as follows

$$\mathbf{V}(i) = \mathbf{D}(i)\mathbf{V}_0(i)\mathbf{C}_0,$$

where $\mathbf{D}(i)$ is an M_s-by-M_s diagonal matrix with j-th diagonal element, $j = 1, \cdots, M_s+1$ and $j \neq i$, equal to $1/Q_0(\rho_{j0})$, $\mathbf{V}_0(i)$ is a Vandermonde matrix with j-th row, $j = 1, \cdots, M_s+1$ and $j \neq i$, equal to $\big(1, \rho_{j0}, \cdots, \rho_{j0}^{M_s-1}\big)$, and $\mathbf{C}_0$ is a matrix with (j,l)-entry equal to the coefficient of w^{j-1} of the polynomial $P_0^l(w)$. By Sylvester's determinant we have that

$$\begin{aligned}\det\big(\mathbf{M}_0(i)\big) &= \frac{1}{z_1}\det\big(\mathbf{V}(i)\big)\big(z_1 - v\mathbf{V}(i)^{-1}e^T\big)\\ &= \frac{1}{z_1}\det\big(\mathbf{V}(i)\big)\big(z_1 - v\mathbf{C}_0^{-1}\mathbf{V}_0(i)^{-1}\mathbf{D}(i)^{-1}e^T\big)\\ &= \frac{1}{z_1}\det\big(\mathbf{V}(i)\big)\big(z_1 - v\mathbf{C}_0^{-1}\mathbf{V}_0(i)^{-1}d\big), \qquad (36)\end{aligned}$$

where d is a column vector of dimension M_s with $d(j)$, j-th entry $j = 1, \cdots, M_s+1$ and $j \neq i$, equal to $Q_0(\rho_{j0})$. By analogy with the Appendix we find that

$$\begin{aligned}\det\big(\mathbf{V}_0(i)\big) &= \det\big(\mathbf{C}_0\big)\frac{\prod_{j=1,j\neq i}^{M_s}\prod_{k=j+1,k\neq i}^{M_s+1}(\rho_{k0}-\rho_{j0})}{\prod_{j=1,j\neq i}^{M_s+1}Q_0(\rho_{j0})}\\ &= \det\big(\mathbf{C}_0\big)(-1)^{M_s+i-1}\frac{\prod_{j=1}^{M_s}\prod_{k=j+1}^{M_s+1}(\rho_{k0}-\rho_{j0})}{\prod_{j=1,j\neq i}^{M_s+1}(\rho_{i0}-\rho_{j0})}\frac{Q_0(\rho_{i0})}{\prod_{j=1}^{M_s+1}Q_0(\rho_{j0})}, \qquad (37)\end{aligned}$$

where the last equality follows from (28). Let $v_0(k,l)$ denote the (k,l)-entry of $\mathbf{V}_0(i)^{-1}$ which is of Vandermonde type. Note that the inverse of a Vandermonde matrix is known in closed form, see e.g. [8]. We deduce from [8] that the values of $v_0(k,l)$ given in (30). Let us denote $c_0(i,j)$ the (i,j)-entry of $\mathbf{C}_0^{-1}$ it then follows right away that

$$v\mathbf{C}_0^{-1}\mathbf{V}_0(i)^{-1}d = \sum_{l=1}^{M_s} v(l)\sum_{m=1}^{M_s} Q_0(\rho_{m0})\sum_{k=1}^{M_s} c_0(l,k)v_0(k,m).$$

Substituting the last equation into (35) gives (29), which completes the proof. □

2.3. **PH/M/1/K Queue.** For the level independent PH/M/1/K we have that $-\mathbf{S}_i = S_i^o s_i = \mu$, $i = 1, \cdots, K$, $\mathbf{F}_i = \mathbf{F}$ and $F_i^o f_i = F^o f$, $i = 1, \cdots, K$. Let $\phi(w) = f(w\mathbf{I} - \mathbf{F})^{-1}F^o$ denote the LST of the inter-arrival times. Moreover, we assume that $q_i = q$, $i = 1, \cdots, K-1$, and $q_K = 0$.

Lemma 3. *The function* $x - (q + xpz_2)\phi\big(w + \mu(1 - z_1x)\big)$ *has* M_a+1 *distinct non-null roots* $o_1, \cdots, o_{M_a+1}$, *such that* $0 < |o_1| < |o_2| < \cdots < |o_{M_a+1}|$.

Proof. By analogy with Lemma 1. □

Proposition 4 (PH/M/1/K Queue)**.** *The joint transform of B_n, S_n, L_n^o, and L_n^c for the PH/M/1/K queue with $p > 0$ and $n = 1, \cdots, K$ is given by*

$$\mathbb{E}\big[e^{-wB_n} z_1^{S_n} z_2^{L_n^c} z_3^{L_n^o}\big] = \big((w+\mu)(1-pz_2) - q\mu z_1\big)\left(R + f(\alpha) + \frac{g(\alpha)I(\alpha)}{h(\alpha)}\right), \quad (38)$$

where, $|\alpha| \in \mathbb{C}$ with $\frac{q}{p|z_2|} < |\alpha| < |o_1|$ and o_1 is the root with the smallest absolute value of

$$x - (q + z_2 px)\phi\big(w + \mu(1 - z_1 x)\big) = 0, \quad (39)$$

and where

$$f(\alpha) = \frac{1}{2\pi\mathbf{i}} \int_{D_\alpha} \frac{1}{x^{n-1}} \frac{1}{q+pz_2x} \frac{1}{w+\mu(1-z_1x)} \frac{dx}{x - (q+pz_2x)\phi\big(w+\mu(1-z_1x)\big)}, \quad (40)$$

$$g(\alpha) = \frac{1}{2\pi\mathbf{i}} \int_{D_\alpha} \frac{1}{x^{n-1}} \frac{1}{q+pz_2x} \frac{dx}{x - (q+pz_2x)\phi\big(w+\mu(1-z_1x)\big)}, \quad (41)$$

$$h(\alpha) = \frac{1}{2\pi\mathbf{i}} \int_{D_\alpha} \frac{q+(pz_2-z_3)x}{x^K(q+pz_2x)} \frac{dx}{x - (q+pxz_2)\phi\big(w+\mu(1-z_1x)\big)}, \quad (42)$$

$$I(\alpha) = \frac{1}{2\pi\mathbf{i}} \int_{D_\alpha} \frac{q+(pz_2-z_3)x}{x^K(q+pz_2x)} \frac{1}{w+\mu(1-z_1x)} \frac{dx}{x - (q+pxz_2)\phi\big(w+\mu(1-z_1x)\big)}, \quad (43)$$

D_α denotes the circle with center at the origin and with radius to $|\alpha|$, and finally

$$R = -\frac{(\mu z_1)^n}{(w+\mu)^{n-1}} \frac{1}{q\mu z_1 + p(w+\mu)z_2} \frac{1}{(w+\mu)(1-pz_2) - q\mu z_1}. \quad (44)$$

Proof. According to Theorem 1 the joint transform B_n, S_n, L_n^c, and L_n^o in this case can be written as follows: (due to the exponential service times we have that $s_n = 1$ and $S_1^o = \mu$ in Theorem 1),

$$\mathbb{E}\big[e^{-wB_n} z_1^{S_n} z_2^{L_n^c} z_3^{L_n^o}\big] = \mu z_1 e_n \otimes f\mathbf{Q}_K^{-1} e_1^T \otimes e, \quad (45)$$

where $\mathbf{Q}_K$ in this case is a K-by-K tridiagonal block matrix with upper diagonal blocks equal to $\mathbf{E}_0 = -qF^of$, i-th diagonal blocks equal to $\mathbf{E}_1 = (w+\mu)\mathbf{I} - \mathbf{F} - pz_2F^of$, $i = 1, \cdots, K-1$, and K-th diagonal block equal to $\mathbf{E}_1^* = (w+\mu)\mathbf{I} - \mathbf{F} - z_3F^of$, and lower-diagonal blocks equal to $\mathbf{E}_2 = -z_1\mu\mathbf{I}$. Let $u = (u_1, \cdots, u_K) := e_n \otimes f\mathbf{Q}_K^{-1}$. Note that the entries of the row vector u are in their turn row vectors of dimension M_a and are all functions of w, z_1, z_2, and z_3. Eq. (45) in terms of u rewrites

$$\mathbb{E}\big[e^{-wB_n} z_1^{S_n} z_2^{L_n^c} z_3^{L_n^o}\big] = \mu z_1 u_1 e^T = \mu z_1 \sum_{i=1}^{M_a} u_{1j}. \quad (46)$$

By analogy with the proof in Proposition 2 we find that

$$\begin{aligned}\sum_{i=1}^{K} u_i x^i &= \big(u_1\mathbf{E}_2 + x^K u_K(x\mathbf{E}_0 + \mathbf{E}_1 - \mathbf{E}_1^*) + x^n f\big)\Big(x\mathbf{E}_0 + \mathbf{E}_1 + \frac{1}{x}\mathbf{E}_2\Big)^{-1} \\ &= \big(\mu z_1 u_1 - x^n f + x^K(qx + pz_2 - z_3)u_K F^o f\big)\Big(\mathbf{F} - \theta\mathbf{I} + (qx + pz_2)F^o f\Big)^{-1},\end{aligned}$$

where $\theta := w + \mu(1 - z_1/x)$. Let $\mathbf{F}_* := \mathbf{F} - \theta\mathbf{I}$. Note that under the condition $Re[\theta] \geq 0$ the matrix $\mathbf{F}_*$ is nonsingular. Hence, the Sherman-Morrison formula, see, e.g., [3, Fact 2.14.2, p. 67], yields

$$\Big(\mathbf{F}_* + (qx + pz_2)F^o f\Big)^{-1} = \mathbf{F}_*^{-1} - \frac{qx + pz_2}{1 + (qx + pz_2)t\mathbf{F}_*^{-1}F^o}\mathbf{F}_*^{-1}F^o f\mathbf{F}_*^{-1}. \quad (47)$$

Multiplying to the right of $\sum_{i=1}^{K} u_i x^i$ with the column vector F^o and using (47) gives

$$\sum_{i=1}^{K} u_i x^i F^o = \frac{1}{1 + (qx + pz_2)f\mathbf{F}_*^{-1}F^o}\big(\mu_1 z_1 u_1 - x^n f + x^K(qx + pz_2 - z_3)u_K F^o f\big)\mathbf{F}_*^{-1}F^o. \quad (48)$$

From (1) we have that $f\mathbf{F}_*^{-1}F^o = -\phi(\theta)$ and $\mathbf{F}_*^{-1}F^o = -\big(\phi^1(\theta), \cdots, \phi^{M_a}(\theta)\big)^T$, where $\phi^i(\theta) = e_i(\theta\mathbf{I} - \mathbf{F})^{-1}F^o$. Therefore, $\phi(\theta) = f\big(\phi^1(\theta), \cdots, \phi^{M_a}(\theta)\big)^T$ is a linear combination of $\phi^i(\theta)$, $i = 1, \cdots, M_a$. Inserting $f\mathbf{F}_*^{-1}F^o$ and $\mathbf{F}_*^{-1}F^o$ into (48) yields

$$\sum_{i=1}^{K} u_i x^i F^o = -\frac{x^K(qx + pz_2 - z_3)\phi(\theta)u_K F^o + \mu_1 z_1 \sum_{j=1}^{M_a} u_{1j}\phi^j(\theta) - x^n\phi(\theta)}{1 - (qx + pz_2)\phi(\theta)}, \quad (49)$$

where $u_K = (u_{11}, \cdots, u_{1M_a})$. Note that $u_i F^o$ is a joint transform function. For this reason, the l.h.s. of (49) is analytical for any finite x and the poles on the r.h.s. of (49) should be removable. Note that the roots of $1 - (qx + pz_2)\phi\big(w + \mu(1 - z_1/x)\big)$ are equal to the inverse of the roots of $x - (q + xpz_2)\phi\big(w + \mu(1 - z_1 x)\big)$.

Lemma 3 and the analyticity of $\sum_{i=1}^{K} u_i x^i F^o$ gives that

$$\frac{q + (pz_2 - z_3)o_i}{o_i^{K+1}}\phi(\theta_i)u_K F^o + \mu_1 z_1 \sum_{j=1}^{M_a} u_{1j}\phi^j(\theta_i) = \frac{1}{o_i^n}\phi(\theta_i), \; i = 1, \cdots, M_a + 1, \quad (50)$$

where $\theta_i := w + \mu(1 - z_1 o_i)$. The system of equations in (50) has $M_a + 1$ equations with $M_a + 1$ unknowns which are $u_K F^o, u_{11}, \cdots, u_{1M_a}$. Using Cramer's rule we find that

$$\mathbb{E}\big[e^{-wB_n} z_1^{S_n} z_2^{L_n^c} z_3^{L_n^o}\big] = \mu z_1 u_1 e^T = \mu z_1 \sum_{j=1}^{M_a} u_{1j} = -\frac{\det(\mathbf{H})}{\det(\mathbf{K})}, \quad (51)$$

where $\det(\mathbf{K})$ is the determinant of the matrix $\mathbf{K}$ with i-th row equal to $\big(\frac{q+(pz_2-z_3)o_i}{o_i^{K+1}} \cdot\phi(\theta_i),\phi^1(\theta_i),\cdots,\phi^{M_a}(\theta_i)\big)$, $i=1,\cdots,M_a+1$, and $\mathbf{H}$ is an (M_a+2)-by-(M_a+2) matrix with i-th row, $i=1,\cdots,M_a+1$, equal to $\big(\frac{q+(pz_2-z_3)o_i}{o_i^{K+1}}\phi(\theta_i),\phi^1(\theta_i),\cdots,\phi^{M_a}(\theta_i), \frac{1}{o_i^n}\phi(\theta_i)\big)$ and (M_a+2)-th row equal to $(0,1,\cdots,1,0)$.

The Laplace expansion of the determinant along the first column of $\mathbf{K}$ and $\mathbf{H}$ gives that

$$\begin{aligned}\mathbb{E}\big[e^{-wB_n}z_1^{S_n}z_2^{L_n^c}z_3^{L_n^o}\big] &= -\frac{\sum_{i=1}^{M_a+1}\frac{q+(pz_2-z_3)o_i}{o_i^{K+1}}\phi(\theta_i)(-1)^{i+1}\det\big(\mathbf{H}(i,1)\big)}{\sum_{i=1}^{M_a+1}\frac{q+(pz_2-z_3)o_i}{o_i^{K+1}}\phi(\theta_i)(-1)^{i+1}\det\big(\mathbf{K}(i,1)\big)}\\ &= -\frac{\sum_{i=1}^{M_a+1}\frac{1}{o_i^K}\frac{q+(pz_2-z_3)o_i}{q+pz_2o_i}(-1)^{i+1}\det\big(\mathbf{H}(i,1)\big)}{\sum_{i=1}^{M_a+1}\frac{1}{o_i^K}\frac{q+(pz_2-z_3)o_i}{q+pz_2o_i}(-1)^{i+1}\det\big(\mathbf{K}(i,1)\big)},\end{aligned} \tag{52}$$

where a matrix $\mathbf{X}(i,1)$ is obtained by deleting the i-th row and the first column of the matrix $\mathbf{X}$, and the second equality follows from $\phi(\theta_i)=o_i/(q+pz_2o_i)$.

Note that $\phi(w)$ is a rational function with denominator, $Q_\phi(w)$, of degree $d=M_a$ and numerator of degree $< d$. By analogy with the determinant of $\mathbf{M}(i,1)$ given in (27) and (28) we find that

$$\begin{aligned}\det\big(\mathbf{K}(i,1)\big) &= C_k\frac{\prod_{j=1,j\neq i}^{M_a}\prod_{k=j+1,k\neq i}^{M_a+1}(\theta_k-\theta_j)}{\prod_{j=1,j\neq i}^{M_a+1}Q_\phi(\theta_j)}\\ &= C_k(-1)^{i-1}\frac{\prod_{j=1}^{M_a}\prod_{k=j+1}^{M_a+1}(\theta_k-\theta_j)}{\prod_{j=1}^{M_a+1}Q_\phi(\theta_j)}\frac{Q_\phi(\theta_i)}{\prod_{j=1,j\neq i}^{M_a+1}(\theta_j-\theta_i)}\\ &= C_k(-1)^{M_a+i-1}\frac{\prod_{j=1}^{M_a}\prod_{k=j+1}^{M_a+1}(\theta_k-\theta_j)}{\prod_{j=1}^{M_a+1}Q_\phi(\theta_j)}\\ &\quad\times\text{Res}_{o_i}\frac{1}{x-(q+xpz_2)\phi\big(w+\mu(1-z_1x)\big)},\end{aligned} \tag{53}$$

where C_k is a constant that is a function of the polynomials parameters of the numerators of $\phi^i(w)$, $i=1,\cdots,M_a$. Let $\alpha\in\mathbb{C}$ with $q/|pz_2|<|\alpha|<|o_1|$. We find that

$$\sum_{i=1}^{M_a+1}\frac{1}{o_i^K}\frac{q+(pz_2-z_3)o_i}{q+pz_2o_i}(-1)^{i+1}\det\big(\mathbf{K}(i,1)\big)=C_k(-1)^{M_a}\frac{\prod_{j=1}^{M_a}\prod_{k=j+1}^{M_a+1}(\theta_k-\theta_j)}{\prod_{j=1}^{M_a+1}Q_\phi(\theta_j)}\big(-h(\alpha)\big), \tag{54}$$

where $h(\alpha)$ is given in(42). Note that the minus sign that is next to $h(\alpha)$ is due to the fact that the sum of all residues of the function

$$\frac{q+(pz_2-z_3)x}{x^K(q+pz_2x)}\frac{1}{x-(q+pxz_2)\phi\big(w+\mu(1-z_1x)\big)},$$

including the residue at infinity which is equal to zero ($K\geq 1$), is zero. We shall refer to the latter property of complex functions as the Inside-Outside property.

The expansion of the determinant of $\mathbf{H}(i,1)$ along the last column yields

$$\det\big(\mathbf{H}(i,1)\big) = \sum_{j=1,j\neq i}^{M_a+1} \frac{1}{o_j^{n-1}} \frac{(-1)^{M_a+j+1}}{q+pz_2o_j}\det\big(\mathbf{J}\big), \tag{55}$$

where $\mathbf{J}$ is obtained by deleting the j-th row and the last column of the matrix $\mathbf{H}(i,1)$. It is easily seen that $\mathbf{J}$ is an M_a-by-M_a matrix with the l-th row equal to $(\phi^1(\theta_l),\cdots,\phi^{M_a}(\theta_l))$, $l=1,\cdots,M_a+1$ and $l\neq i,j$, and the last row is equal to e. By analogy with the determinant of $\mathbf{M}(i,1)$ we find that

$$\begin{aligned}
\det\big(\mathbf{J}\big) &= \frac{C_J}{Q_\phi(0)}\prod_{l=1,l\neq i,j}^{M_a+1}\frac{\theta_l}{Q_\phi(\theta_l)}\prod_{l=1,l\neq i,j}^{M_a}\prod_{k=l+1,k\neq i,j}^{M_a+1}(\theta_k-\theta_l)\\
&= \frac{C_J}{Q_\phi(0)}\prod_{l=1,l\neq i,j}^{M_a+1}\frac{\theta_l}{Q_\phi(\theta_l)}(-1)^{i+j-1}\frac{\prod_{l=1}^{M_a}\prod_{k=l+1}^{M_a+1}(\theta_k-\theta_l)}{\prod_{l=1,l\neq i}^{M_a+1}(\theta_l-\theta_i)\prod_{l=1,l\neq i,j}^{M_a+1}(\theta_l-\theta_j)}\\
&= \frac{C_J(-1)^{i+j-1}}{Q_\phi(0)}\prod_{l=1}^{M_a+1}\theta_l\frac{\prod_{l=1}^{M_a}\prod_{k=l+1}^{M_a+1}(\theta_k-\theta_l)}{\prod_{l=1}^{M_a+1}Q_\phi(\theta_l)}\frac{Q_\phi(\theta_i)}{\theta_i\prod_{l=1,l\neq i}^{M_a+1}(\theta_l-\theta_i)}\\
&\quad \frac{Q_\phi(\theta_j)}{\theta_j\prod_{l=1,l\neq i,j}^{M_a+1}(\theta_l-\theta_j)},
\end{aligned}$$

where $Q_\phi(0)$ is due to the last row of $\det\big(\mathbf{J}\big)$ which is equal to $e=(1,\cdots,1)=\big(P_\phi^1(0)/Q_\phi^1(0),\cdots,P_\phi^{M_a}(0)/Q_\phi^{M_a}(0)\big)$. It follows from the definitions of the matrices $\mathbf{J}$ and $\mathbf{K}$ that $C_J=C_k$. We note that

$$\begin{aligned}
\prod_{l=1}^{M_a+1}\theta_l &= (\mu z_1)^{M_a+1}\prod_{l=1}^{M_a+1}\Big(\frac{w+\mu}{\mu z_1}-o_l\Big)\\
&= (\mu z_1)^{M_a+1}\frac{\frac{w+\mu}{\mu z_1}Q_\phi(0)-(q+pz_2\frac{w+\mu}{\mu z_1})P_\phi(0)}{(-\mu z_1)^{M_a}}\\
&= (-1)^{M_a}Q_\phi(0)\big[(w+\mu)(1-pz_2)-q\mu z_1\big],
\end{aligned}$$

where the second equality follows from the fact that o_l, $l=1,\cdots,M_a+1$, are the roots of $x-(q+xpz_2)\phi\big(w+\mu(1-z_1x)\big)$ and $\phi(w)=P_\phi(w)/Q_\phi(w)$, and the last from $\phi(0)=1$. Inserting $\det\big(\mathbf{J}\big)$ and $\prod_{l=1}^{M_a+1}\theta_l$ into (55) yields

$$\begin{aligned}
\det\big(\mathbf{H}(i,1)\big) &= \sum_{j=1,j\neq i}^{M_a+1}\frac{1}{o_j^{n-1}}\frac{(-1)^{M_a+j+1}}{q+pz_2o_j}\det\big(\mathbf{J}\big)\\
&= C_J(-1)^i\big[(w+\mu)(1-pz_2)-q\mu z_1\big]\frac{\prod_{l=1}^{M_a}\prod_{k=l+1}^{M_a+1}(\theta_k-\theta_l)}{\prod_{l=1}^{M_a+1}Q_\phi(\theta_l)}\\
&\quad \frac{Q_\phi(\theta_i)}{\theta_i\prod_{l=1,l\neq i}^{M_a+1}(\theta_l-\theta_i)}\sum_{j=1,j\neq i}^{M_a+1}\frac{1}{o_j^{n-1}}\frac{1}{q+pz_2o_j}\frac{Q_\phi(\theta_j)}{\theta_j\prod_{l=1,l\neq i,j}^{M_a+1}(\theta_l-\theta_j)}.
\end{aligned} \tag{56}$$

Note that, for $p > 0$ and $n = 1, \dots, K$, we have that

$$\begin{aligned}
&\sum_{j=1,j\neq i}^{M_a+1} \frac{1}{o_j^{n-1}} \frac{1}{q+pz_2o_j} \frac{Q_\phi(\theta_j)}{\theta_j \prod_{l=1,l\neq i,j}^{M_a+1}(\theta_l-\theta_j)} \\
&= (-1)^{M_a} \sum_{j=1}^{M_a+1} \frac{1}{o_j^{n-1}} \frac{1}{q+pz_2o_j} \frac{(\theta_i-\theta_j)Q_\phi(\theta_j)}{\theta_j \prod_{l=1,l\neq j}^{M_a+1}(\theta_j-\theta_l)} \\
&= (-1)^{M_a} \Bigg[\theta_i \sum_{j=1}^{M_a+1} \frac{1}{o_j^{n-1}} \frac{1}{q+pz_2o_j} \frac{1}{\theta_j} \mathrm{Res}_{o_j} \frac{1}{x-(q+pz_2x)\phi\big(w+\mu(1-z_1x)\big)} \\
&\quad - \sum_{j=1}^{M_a+1} \frac{1}{o_j^{n-1}} \frac{1}{q+pz_2o_j} \mathrm{Res}_{o_j} \frac{1}{x-(q+pz_2x)\phi\big(w+\mu(1-z_1x)\big)}\Bigg] \\
&= (-1)^{M_a+1}\big(\theta_i(f(\alpha)+R)+g(\alpha)\big),
\end{aligned}$$

where the last equality follows for $p > 0$ from the Inside-Outside property of the integrands of $f(\alpha)$ and $g(\alpha)$ given in (40) and (41),

$$\begin{aligned}
R &= \mathrm{Res}_{\frac{w+\mu}{\mu z_1}} \frac{1}{x^{n-1}} \frac{1}{q+pz_2x} \frac{1}{w+\mu(1-z_1x)} \frac{1}{x-(q+pz_2x)\phi\big(w+\mu(1-z_1x)\big)} \\
&= -\frac{(\mu z_1)^n}{(w+\mu)^{n-1}} \frac{1}{q\mu z_1+p(w+\mu)z_2} \frac{1}{(w+\mu)(1-pz_2)-q\mu z_1}. \qquad (57)
\end{aligned}$$

Substituting (53) and (56) into (51) yields

$$\mathbb{E}\big[e^{-wB_n} z_1^{S_n} z_2^{L_n^c} z_3^{L_n^o}\big] = \big((w+\mu)(1-pz_2)-q\mu z_1\big)\left(R+f(\alpha)+\frac{g(\alpha)I(\alpha)}{h(\alpha)}\right),$$

where $I(\alpha)$ is given in (43), which completes the proof. □

Remark 3. *We emphasize that Proposition 2 extends the result of Rosenlund* [13] *on the G/M/1/K in two ways. First, it gives the four variate joint transform of B_n, S_n, L_n^c, and L_n^o, for the case when $n \geq 1$. Second, it allows the dropping of customers even when the queue is not full. Note that in the particular case with $n = 1$ and $p = 1 - q = 0$, we have that $f(\alpha) = 0$, $g(\alpha) = 1$, and $R = -1/\big(w + \mu(1 - z_1)\big)$. Inserting these values into (38) yields that*

$$\begin{aligned}
\mathbb{E}\big[e^{-wB_n} z_1^{S_n} z_2^{L_n^c} z_3^{L_n^o}\big] &= \frac{\mu z_1 \sum_{i=1}^{M_a+1} \frac{1-z_3o_i}{o_i^K} \frac{1-\phi(w+\mu(1-z_1o_i))}{w+\mu(1-z_1o_i)} \frac{Q_\phi(\theta_i)}{\prod_{l=1,l\neq i}^{M_a+1}\theta_l-\theta_i}}{\sum_{i=1}^{M_a+1} \frac{1}{o_i^K} \frac{Q_\phi(\theta_i)}{\prod_{l=1,l\neq i}^{M_a+1}\theta_l-\theta_i}} \\
&= \frac{\int_{D_\alpha} \frac{\mu z_1(1-z_3x)}{x^K} \frac{1-\phi(w+\mu(1-z_1x))}{w+\mu(1-z_1x)} \frac{dx}{x-\phi(w+\mu(1-z_1x))}}{\int_{D_\alpha} \frac{1-z_3x}{x^K} \frac{dx}{x-\phi(w+\mu(1-z_1x))}}.
\end{aligned}$$

We note that the last equation is in agreement with (11) in [13].

3. Discussion: non-distinct roots

Until now we have assumed that the roots of the equations in (14) and (39) are distinct. We shall now relax these assumptions and show that the results in Propositions 2 and 4 still hold. In the following, we shall focus on extending the result in Proposition 2. Similarly this can be done for Proposition 4.

Let consider that $r_{i+l} = r_i + l\epsilon$, $\epsilon > 0$, $i \in \{1, \ldots, M_s + 1\}$ and $l = 0, \ldots, L-1$, and take the limit in our final result for $\epsilon \to 0$. This means that r_i is a root of multiplicity L. In order to show that the results in Proposition 2 hold in this case, it is readily seen that one needs to prove that

$$
\begin{aligned}
&Res_{r_i} \frac{1}{x^{K-1}} \frac{1}{qx + pz_2 - z_3} \frac{1}{x - z_1\xi(w + \lambda(1 - qx - pz_2))} \\
&\qquad = \lim_{\epsilon \to 0} \sum_{l=0}^{L-1} \frac{1}{r_{i+l}^{K-1}} \frac{1}{qr_{i+l} + pz_2 - z_3} \frac{Q(\rho_{i+l})}{\prod_{j=1, j\neq i+l}^{M_s+1}(\rho_{i+l} - \rho_j)}. \quad (58)
\end{aligned}
$$

First, note that when r_i is a root of multiplicity L the complex residue reads

$$
\begin{aligned}
&Res_{r_i} \frac{1}{x^{K-1}} \frac{1}{qx + pz_2 - z_3} \frac{1}{x - z_1\xi(w + \lambda(1 - qx - pz_2))} \\
&= \frac{1}{(L-1)!} \frac{d^{L-1}}{dx^{L-1}} \left(\frac{1}{x^{K-1}(qx + pz_2 - z_3)} \frac{(x - r_i)^L}{x - z_1\xi(w + \lambda(1 - qx - pz_2))} \right)\Bigg|_{x=r_i} \\
&= \frac{1}{(-\lambda q)^{L-1}(L-1)!} \frac{d^{L-1}}{dx^{L-1}} \left(\frac{1}{x^{K-1}(qx + pz_2 - z_3)} \frac{Q(\rho)}{\prod_{j=1, j\neq i, \cdots, i+L-1}^{M_s+1}(\rho - \rho_j)} \right)\Bigg|_{x=r_i} \\
&= \frac{1}{(-\lambda q)^{L-1}(L-1)!} \lim_{\epsilon \to 0} \frac{1}{\epsilon^{L-1}} \sum_{l=1}^{L-1} \left(\frac{\binom{L-1}{l}(-1)^{L-1-l}}{(r_i + l\epsilon)^{K-1}(q(r_i + l\epsilon) + pz_2 - z_3)} \right. \\
&\quad \left. \times \frac{Q(\rho_i - \lambda q l \epsilon)}{\prod_{j=1, j\neq i, \cdots, i+L-1}^{M_s+1}(\rho_i - \lambda q l \epsilon - \rho_j)} \right), \quad (59)
\end{aligned}
$$

where $\rho = w + \lambda(1 - qx - pz_2)$, $\rho_i = w + \lambda(1 - qr_i - pz_2)$, and the last equality follows from the following identity for the analytical function $f(x)$ around x_0:

$$
\frac{d^n}{dx^n} f(x)\Bigg|_{x_0} = \lim_{\epsilon \to 0} \frac{1}{\epsilon^n} \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} f\big(x_0 + i\epsilon\big).
$$

Note that the previous equation follows right away using the Taylor series of $f(x_0 + i\epsilon)$ around x_0 and the binomial series of $(x-1)^n$ and its derivatives.

The r.h.s. of (58) rewrites

$$\begin{aligned}
&\lim_{\epsilon\to 0}\sum_{l=0}^{L-1}\frac{1}{r_{i+l}^{K-1}}\frac{1}{qr_{i+l}+pz_2-z_3}\frac{Q(\rho_{i+l})}{\prod_{j=1,j\neq i+l}^{M_s+1}(\rho_{i+l}-\rho_j)}\\
&= \lim_{\epsilon\to 0}\sum_{l=0}^{L-1}\frac{1}{(r_i+l\epsilon)^{K-1}}\frac{1}{q(r_i+l\epsilon)+pz_2-z_3}\frac{Q(\rho_i-\lambda ql\epsilon)}{\prod_{j=1,j\neq i+l}^{M_s+1}(\rho_i-\lambda ql\epsilon-\rho_j)}, \quad (60)
\end{aligned}$$

where,

$$\begin{aligned}
\frac{Q(\rho_i+l\epsilon_0)}{\prod_{j=1,j\neq i+l}^{M_s+1}(\rho_i+l\epsilon_0-\rho_j)} &= \frac{(-1)^{L-1-l}Q(\rho_i+l\epsilon_0)}{\epsilon_0^{L-1}l!(L-1-l)!\prod_{j=1,l\neq 0,\cdots,L-1}^{M_s+1}(\rho_i+l\epsilon_0-\rho_j)}\\
&= \frac{\begin{pmatrix}L-1\\ l\end{pmatrix}}{(L-1)!}\frac{(-1)^{L-1-l}Q(\rho_i+l\epsilon_0)}{\epsilon_0^{L-1}\prod_{j=1,l\neq 0,\cdots,L-1}^{M_s+1}(\rho_i+l\epsilon_0-\rho_j)},
\end{aligned}$$

with $\epsilon_0=-\lambda q\epsilon$. Inserting the last equation into (60) yields that the r.h.s. and l.h.s. of (58) are equal, which completes the proof.

Appendix

$\mathbf{M}(i,1)$ is an M_s-by-M_s matrix of j-th row equal to $\big(\xi^1(\rho_j),\cdots,\xi^{M_s}(\rho_j)\big)$ for $j=1,\cdots,M_s+1$ and $j\neq i$. Note that $\xi(\rho)$ is a linear combination of $\xi^1(\rho),\cdots,\xi^{Ms}(\rho)$, and it is a rational function with denominator, $Q(\rho)$, of degree $d=M_s$ and numerator of degree $<d$. Moreover, $\xi^i(\rho)$, $i=1,\cdots,M_s$, are also rational functions with denominator of degree $d_i\leq M_s$ and numerator of degree $<d_i$.

Lemma 4.

$$\det\big(\mathbf{M}(i,1)\big)=C\frac{\prod_{j=1,j\neq i}^{M_s}\prod_{k=j+1,k\neq i}^{M_s+1}(\rho_k-\rho_j)}{\prod_{j=1,j\neq i}^{M_s+1}Q(\rho_j)}, \quad (61)$$

where C is a constant.

Proof. Note that $Q(\rho)$ is the common denominator of $\xi^i(\rho)$, $i=1,\cdots,M_s$, which yields

$$\det\big(\mathbf{M}(i,1)\big)=\det\big(\mathbf{P}(\rho_{\mathbf{1}},\ldots,\rho_{\mathbf{M_s+1}})\big)\prod_{j=1,j\neq i}^{M_s+1}\frac{1}{Q(\rho_j)},$$

where

$$\mathbf{P}(\rho_{\mathbf{1}},\ldots,\rho_{\mathbf{M_s+1}})=\begin{pmatrix}P^1(\rho_1) & \ldots & P^{M_s}(\rho_1)\\ \vdots & & \vdots\\ P^1(\rho_{i-1}) & \ldots & P^{M_s}(\rho_{i-1})\\ P^1(\rho_{i+1}) & \ldots & P^{M_s}(\rho_{i+1})\\ \vdots & & \vdots\\ P^1(\rho_{M_s+1}) & \ldots & P^{M_s}(\rho_{M_s+1})\end{pmatrix},$$

where $P^i(\rho)$, $i = 1, \cdots, M_s$, are polynomials of degree $< M_s$. Note that $P^i(\rho) \neq P^j(\rho)$, $i \neq j$. Therefore, we deduce that

$$\det\big(\mathbf{P}(\rho_{\mathbf{1}}, \ldots, \rho_{\mathbf{M_s+1}})\big) = C\mathbf{V}(\rho_1, \ldots, \rho_{M_s+1}), \tag{62}$$

where C is the determinant of the matrix with (j,l)-entry equal to the coefficient of w^{j-1} of $P^l(w)$, and $\mathbf{V}(\rho_1, \ldots, \rho_{M_s+1})$ is the Vandermonde matrix:

$$\mathbf{V}(\rho_1, \ldots, \rho_{M_s+1}) = \begin{pmatrix} 1 & \rho_1 & \ldots & (\rho_1)^{M_s} \\ \vdots & & & \vdots \\ 1 & \rho_{i-1} & \ldots & (\rho_{i-1})^{M_s} \\ 1 & \rho_{i+1} & \ldots & (\rho_{i+1})^{M_s} \\ \vdots & & & \vdots \\ 1 & \rho_{M_s+1} & \ldots & (\rho_{M_s+1})^{M_s} \end{pmatrix},$$

It is well known that, see, e.g., [5],

$$\det\big(\mathbf{V}(\rho_1, \ldots, \rho_{M_s+1})\big) = \prod_{j=1, j\neq i}^{M_s} \prod_{k=j+1, k\neq i}^{M_s+1} (\rho_k - \rho_j).$$

Substituting the latter equation into (62) completes the proof. □

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