Stokes-Dirichlet/Neuman problems and complex analysis

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Abstract

On a bounded and simply connected open set $\mathbb{G} \subset \mathbb{R}^2 \cong \mathbb{C}$, with a sufficiently smooth boundary $\partial\mathbb{G}$, the following boundary value problem for a pair $\{\varphi, \chi\}$ of *analytic* functions is studied:

$$\begin{cases} \varphi\,,\, \chi\,:\, \mathbb{G} \to \mathbb{C}\,, \quad \text{both analytic,} \\ \big[\, z\overline{\varphi'} \pm \varphi + \overline{\chi'}\,\big]\Big|_{\partial\mathbb{G}} = G \in \mathbb{L}_2(\partial\mathbb{G}), \end{cases} \tag{0.1}$$

Multiplication by i transforms the +version into the −version.

Necessary and sufficient conditions on G for solvability and also results on the behaviour of the solution near $\partial\mathbb{G}$ are found.

The original motivation for this study is to provide a sound mathematical link between 2D Stokes boundary value problems and 2D free boundary evolution equations of Hopper type, cf. **[H]**, with 'arbitrary Hamiltonian' , cf. **[G]**. During this, the interesting (and for the author unexpected) fact came up that *both* the Dirichlet *and* the Neumann Problem for the 2D-Stokes equations can be reduced to the problem (0.1). Full details of all this are in the underlying note. A brief overview now follows.

On $\mathbb{G} \subset \mathbb{R}^2 \cong \mathbb{C}$, the stationary behaviour of a *pressure-velocity flow pair* $\{p, \underline{v}\}$, where $p : \mathbb{G} \to \mathbb{R}$ and $\underline{v} : \mathbb{G} \to \mathbb{R}^2$, can often be modelled by Stokes' equations

$$\begin{cases} \nabla \cdot \mathcal{T} = \underline{0} \\ \nabla \cdot \underline{v} = 0 \end{cases}, \qquad \text{with stress matrix} \qquad \mathcal{T} = -p\,\mathcal{I} + \Big[\frac{\mathrm{d}\underline{v}}{\mathrm{d}\underline{x}}\Big] + \Big[\frac{\mathrm{d}\underline{v}}{\mathrm{d}\underline{x}}\Big]^\top. \tag{0.2}$$

Only Cartesian coordinates will be employed!

It is classical folklore, scattered in the litterature, that there exists a *bi-harmonic potential pair* $\psi, \phi : \mathbb{G} \to \mathbb{R}$, (the stream function and Airy function, respectively), such that, cf. (1.3),

$$\underline{v} = \nabla \times (\psi\, \underline{e}_3),\,, \qquad \mathcal{T} = 2\Big[(\mathrm{D}^2\phi) - (\Delta\phi)\mathcal{I}\Big]. \tag{0.3}$$

Consistency in $\mathcal{T}$ requires that ϕ and ψ are related: For $z = x + \mathrm{i}y \in \mathbb{G}$ one necessarily has, cf. Appendix B,

$$\phi(\underline{x}) + \mathrm{i}\psi(\underline{x}) = \overline{z}\varphi(z) + \chi(z)\,, \qquad \text{with analytic} \quad \varphi, \chi\,:\, \mathbb{G} \to \mathbb{C}\,. \tag{0.4}$$

Also this is classical folklore. For a strongly related approach in the field of 'elasticity' cf. **[E]** and **[M]** Ch 4. In the Appendices to this note full details are presented on $\psi, \phi, \varphi, \chi$ and on the kinematic expressions derived from them. For a *full set* of the latter see (1.5).

By means of the *analytic potentials* φ, χ we investigate boundary value problems for Stokes' equations with respective boundary conditions:

$$\textbf{Stokes-Dirichlet: } \underline{v}\Big|_{\partial\mathbb{G}} \in \mathbb{L}_2(\partial\mathbb{G}), \qquad \textbf{Stokes-Neumann: } \mathcal{T}\underline{n}\Big|_{\partial\mathbb{G}} \in \mathbb{H}^{-1}(\partial\mathbb{G}). \tag{0.5}$$

As it turns out *both problems* can be reduced to (0.1). By means of a conformal mapping the problem (0.1) is then transformed to an integral operator equation on the unit circle.

Contents

For convenience nothing new is claimed here! JdG November 2010

1 Generalities on Stokes' Equations in $\mathbb{R}^2$

On a bounded simply connected open domain $\mathbb{G} \subset \mathbb{R}^2$, $\underline{0} \in \mathbb{G}$, we consider the set of Stokes equations

$$\begin{aligned} \frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2} - \frac{\partial p}{\partial x} = 0 \\ \frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial y^2} - \frac{\partial p}{\partial y} = 0 \\ \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0 \end{aligned} \tag{1.1}$$

Alternative formulations are

$$\left\{ \begin{array}{c} \Delta \underline{v} - \nabla p = \underline{0} \\ \nabla \cdot \underline{v} = 0 \end{array} \right. \qquad \left\{ \begin{array}{c} \nabla \cdot \mathcal{T} = \underline{0} \\ \nabla \cdot \underline{v} = 0 \end{array} \right. \qquad \left\{ \begin{array}{c} \partial_i \mathcal{T}_{ij} = 0 \\ \partial_i v_i = 0 \end{array} \right. , \tag{1.2}$$

with

$$\mathcal{T} = -p\,\mathcal{I} + [\frac{\mathrm{d}\underline{v}}{\mathrm{d}\underline{x}}] + [\frac{\mathrm{d}\underline{v}}{\mathrm{d}\underline{x}}]^\top \qquad \text{and} \qquad \mathcal{T}_{ij} = -p\,\delta_{ij} + \partial_j v_i + \partial_i v_j \ .$$

The boundary $\partial\mathbb{G}$ of $\mathbb{G}$ is supposed to admit a positively oriented arclength parametrization $s \mapsto \underline{x}(s)$, $0 \le s < L$ with bounded (generalized) derivative $s \mapsto \dot{\underline{x}}(s)$. Besides the unit *tangent vector* $s \mapsto \underline{t}(\underline{x}(s)) = \dot{\underline{x}}(s) = \mathrm{kol}[\dot{x}(s), \dot{y}(s)]$ we also need the *outside normal* $s \mapsto \underline{n}(\underline{x}(s)) = \mathrm{kol}[\,\dot{y}(s), -\dot{x}(s)]$.

The next theorem contains some classical results regarding the general solution of Stokes' equations without regarding boundary conditions.

Theorem 1.1 (Classical results)

• *If $\underline{x} \mapsto p(\underline{x})$, $\underline{v}(\underline{x})$ solves (1.1), (1.2) on $\mathbb{G}$, then there exist a 'stream function' $\underline{x} \mapsto \psi(\underline{x})$ and an 'Airy function' $\underline{x} \mapsto \phi(\underline{x})$ on $\mathbb{G}$, with $\Delta\Delta\phi = 0$, $\Delta\Delta\psi = 0$, such that*

$$\underline{v} = \left[\begin{array}{c} \partial_y \psi \\ -\partial_x \psi \end{array} \right], \qquad p = \Delta\phi\,, \qquad \mathcal{T} = 2 \left[\begin{array}{cc} -\partial_y\partial_y\phi & \partial_x\partial_y\phi \\ \partial_x\partial_y\phi & -\partial_x\partial_x\phi \end{array} \right], \tag{1.3}$$

and the function $z = x + \mathrm{i}y \mapsto \Delta\phi(\underline{x}) + \mathrm{i}\Delta\psi(\underline{x})$ being analytic.
Here ψ is unique up to a constant and ϕ is unique up to a polynomial of 1st degree.

• *The pair of biharmonic functions $\phi\,,\psi$ cannot be chosen arbitrarily. There has to exist a pair of analytic functions $z \mapsto \varphi(z)\,, \chi(z)$ on $\mathbb{G}$, such that*

$$\phi(\underline{x}) + \mathrm{i}\psi(\underline{x}) = \overline{z}\varphi(z) + \chi(z)\,, \qquad z = x + \mathrm{i}y \in \mathbb{G}\,, \tag{1.4}$$

• *All solutions of Stokes' equations have such* holomorphic representation.

• *Let $s \mapsto z(s) \in \overline{G}$ be a curve with arclength parametrization s. Differentiation along such a curve is denoted $\frac{\mathrm{d}}{\mathrm{d}s}$. We write $\frac{\mathrm{d}z}{\mathrm{d}s} = \dot{z}$. The ordered pair $\{\underline{n}\,, \dot{\underline{x}}\} = \{-\mathrm{i}\dot{z}\,, \dot{z}\}$ is* *meant to be a positively oriented orthonormal system in* $\mathbb{R}^2$. *We have*

$$\begin{aligned} v_1 + \mathrm{i}v_2 &= -\varphi + z\overline{\varphi'} + \overline{\chi'} & p &= -\tfrac{1}{2}\big(\mathcal{T}_{11} + \mathcal{T}_{22}\big) = 4\,\mathrm{Re}\,\varphi' \\ \underline{v}\cdot\underline{n} &= \frac{\mathrm{d}}{\mathrm{d}s}\,\mathrm{Im}\,(\overline{z}\varphi + \chi) & \mathrm{rot}\,\underline{v} &= \partial_x v_2 - \partial_y v_1 = -4\,\mathrm{Im}\,\varphi' \\ \underline{v}\cdot\underline{\dot{x}} &= \frac{\mathrm{d}}{\mathrm{d}s}\,\mathrm{Re}\,(\overline{z}\varphi + \chi) - 2\,\mathrm{Re}\,(\overline{\varphi}\dot{z}) & \mathcal{T}_{22} - \mathcal{T}_{11} + 2\,\mathrm{i}\mathcal{T}_{12} &= -4(\overline{z}\varphi'' + \chi'') \\ \mathcal{T}\cdot\underline{n} &= 2\,\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s}(\varphi + z\overline{\varphi'} + \overline{\chi'}) & \mathcal{T}\cdot\underline{\dot{x}} &= 2\frac{\mathrm{d}}{\mathrm{d}s}\{z\overline{\varphi'} + \overline{\chi'} - 4\,\mathrm{Re}\,\varphi\} \end{aligned} \tag{1.5}$$

• *If the pair* $\{\varphi\,,\,\chi\}$ *is replaced by the pair* $\{\varphi+\alpha\,,\,\chi+\overline{\alpha}z+\beta\}$, *with* $\alpha\,,\,\beta\in\mathbb{C}$, *the same solution is represented.*
The holomorphic representation of a solution by $\{\varphi\,,\,\chi\}$ *is unique if one additionally requires that for some fixed* $\underline{a}\in\mathbb{G}$ *one has* $\varphi(\underline{a})=\chi(\underline{a})=0$. *We usually take* $\underline{a}=\underline{0}$
• *In this way the 'Euclidean motion' solution*

$$p(\underline{x}) = E\,, \qquad \underline{v}(\underline{x}) = A\begin{bmatrix}1\\0\end{bmatrix} + B\begin{bmatrix}0\\1\end{bmatrix} + C\begin{bmatrix}-y\\x\end{bmatrix}, \qquad A\,,B\,,C\,,E\in\mathbb{R}\,. \tag{1.6}$$

has the unique holomorphic representation

$$\varphi(z) = \frac{1}{4}(E - 2\,\mathrm{i}C)z \qquad \chi(z) = (A - \,\mathrm{i}B)z\,. \tag{1.7}$$

Proof For a detailed mathematical proof of those classical results + some addenda see Appendix B. ■

2 Boundary Value Problems and their Uniqueness

The **Stokes-Dirichlet problem** is formulated as follows

$$\begin{cases} \Delta\underline{v} - \nabla p = \underline{0} & ,\ \ \underline{x}\in\mathbb{G} \\ \nabla\cdot\underline{v}(\underline{x}) = 0 & ,\ \ \underline{x}\in\mathbb{G} \\ \underline{v}(\underline{x}) = \underline{g}(\underline{x}) & ,\ \ \underline{x}\in\partial\mathbb{G} \\ p(\underline{0}) = B & ,\ \ B\in\mathbb{R}\,. \end{cases} . \tag{2.1}$$

On the prescribed *boundary velocity field* $s\mapsto\underline{g}(\underline{x}(s)) = V_1(s)\underline{n}(\underline{x}(s)) + V_2(s)\underline{t}(\underline{x}(s))\in\mathbb{R}^2$ we put

$$\textbf{Condition on }\ \underline{g}\ : \qquad \bullet \int_0^L V_1(s)\,\mathrm{d}s = 0 \tag{2.2}$$

This condition is necessary in order to be consistent with $\nabla\cdot\underline{v}(\underline{x})=0,\ \underline{x}\in\mathbb{G}$.
Keep in mind that $V_1\,,\,V_2$ are *not* the cartesian components of $\underline{g}$.

Theorem 2.1 (Uniqueness of the Stokes-Dirichlet problem)
Consider the Stokes-Dirichlet problem (2.1). Suppose $\underline{0} \in \mathbb{G}$.
- *If* $\underline{g} = \underline{0}$, $B = 0$, *then* $\underline{v}(\underline{x}) = \underline{0}$, $p(\underline{x}) = 0$, $\underline{x} \in \mathbb{G}$.
- *For given* $\underline{g} \in \mathbb{L}_2(\partial\mathbb{G}; \mathbb{R}^2)$, $B \in \mathbb{R}$ *there is at most one solution pair* $\{\underline{v}, p\}$ *with (unique) holomorphic representation* $\{\varphi, \chi\}$, *if one, in addition to* $\varphi(0) = \chi(0) = 0$, *requires.*

$$\operatorname{Re}\varphi'(0) = \frac{1}{4}B \in \mathbb{R}\,. \tag{2.3}$$

Proof
- On $\partial\mathbb{G}$ we suppose

$$\underline{v} = \begin{bmatrix} \partial_y\psi \\ -\partial_x\psi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So we have to investigate the set of solutions of

$$\Delta\Delta\psi(\underline{x}) = 0\,, \quad \underline{x} \in \mathbb{G}, \qquad \nabla\psi(\underline{x}) = \underline{0}, \quad \underline{x} \in \partial\mathbb{G}.$$

It follows that $\dfrac{\partial}{\partial\underline{n}}\psi = \dfrac{\partial}{\partial\underline{t}}\psi = 0$ at $\partial\mathbb{G}$. So $\psi = C \in \mathbb{R}$ is constant at $\partial\mathbb{G}$. We take $\psi = 0$ at $\partial\mathbb{G}$.
With Green II

$$0 = \int_{\mathbb{G}} \psi(\underline{x})\Delta\Delta\psi(\underline{x})\,\mathrm{d}\underline{x} = \int_{\partial\mathbb{G}} \psi\frac{\partial}{\partial\underline{n}}\Delta\psi\,\mathrm{d}s - \int_{\partial\mathbb{G}} (\frac{\partial}{\partial\underline{n}}\psi)\Delta\psi\,\mathrm{d}s + \int_{\mathbb{G}} |\Delta\psi|^2\,\mathrm{d}\underline{x} =$$

$$= C\int_{G} \Delta\Delta\psi\,\mathrm{d}\underline{x} + \int_{\mathbb{G}} |\Delta\psi|^2\,\mathrm{d}\underline{x}\,.$$

it now follows that $\Delta\psi = 0$. Hence, the stream function $\psi = C$. So the velocity $\underline{v} = \underline{0}$. The 'consistency conditions' (B.2) tell us that the Airy function ϕ has to satisfy $\partial_x\partial_y\phi = 0$ and $\partial_x\partial_x\phi - \partial_y\partial_y\phi = 0$. Therefore it has the form $\phi(\underline{x}) = \frac{1}{2}B\underline{x}^\top\underline{x} + \underline{b}^\top\underline{x} + c$. So the pressure $p = \Delta\phi$ can only be a constant. The condition $p(\underline{0}) = 0$ forces this constant to be 0.
- If there are 2 solutions they differ by the zero solution just found. ■

Now we come to the **Stokes-Neumann problem**, which is formulated as follows

$$\begin{cases} \nabla\cdot\mathcal{T}(\underline{x}) = \underline{0} & , \quad \underline{x} \in \mathbb{G} \\ \nabla\cdot\underline{v}(\underline{x}) = 0 & , \quad \underline{x} \in \mathbb{G} \\ \mathcal{T}(\underline{x})\cdot\underline{n}(\underline{x}) = \underline{f}(\underline{x}) & , \quad \underline{x} \in \partial\mathbb{G} \end{cases}. \tag{2.4}$$

On the prescribed *boundary stress field* $\underline{x} \mapsto \underline{f}(\underline{x}) \in \mathbb{R}^2$ we put

$$\textbf{Conditions on } \underline{f}\ : \quad \begin{array}{l} \bullet\ \ \underline{f}(\underline{x}(s)) = \dfrac{\mathrm{d}}{\mathrm{d}s}\{K_1(s)\underline{n}(\underline{x}(s)) + K_2(s)\underline{t}(\underline{x}(s))\}\,, \\ \bullet\ \ \int_{\partial\mathbb{G}} K_1(s)\,\mathrm{d}s = \underline{0}\,, \end{array} \tag{2.5}$$

These nicely correspond to equilibrium of forces and momenta, respectively,

$$\int_{\partial\mathbb{G}} \underline{f}(\underline{x}(s))\,\mathrm{d}s = \underline{0}\,, \qquad \int_{\partial\mathbb{G}} \underline{x}(s)\times\underline{f}(\underline{x}(s))\,\mathrm{d}s = \underline{0}.$$

Indeed, if we denote the force at $\underline{x}(s)\in\partial\mathbb{G}$ by $\alpha(s)\underline{n}(\underline{x}(s))+\beta(s)\underline{t}(\underline{x}(s))$, the condition of equilibrium of forces says $\int_{\partial\mathbb{G}}\alpha\underline{n}+\beta\underline{t}\ \mathrm{d}s=\underline{0}$. Therefore we can write

$$\alpha(s)\underline{n}(\underline{x}(s))+\beta(s)\underline{t}(\underline{x}(s)) = \frac{\mathrm{d}}{\mathrm{d}s}\{K_1(s)\underline{n}(\underline{x}(s))+K_2(s)\underline{t}(\underline{x}(s))\}.$$

Further, the condition of equilibrium of momenta says $\int_{\partial\mathbb{G}}\underline{x}\times\frac{\mathrm{d}}{\mathrm{d}s}\{K_1\underline{n}+K_2\underline{t}\}\,\mathrm{d}s=\underline{0}$. This means

$$\underline{0} = \int_{\partial\mathbb{G}}\frac{\mathrm{d}}{\mathrm{d}s}\{\underline{x}\times(K_1\underline{n}+K_2\underline{t})\}\,\mathrm{d}s = \int_{\partial\mathbb{G}}\underline{t}\times\{K_1\underline{n}+K_2\underline{t}\}\,\mathrm{d}s.$$

Which says $\underline{e}_3\int_{\partial\mathbb{G}}K_1\,\mathrm{d}s=\underline{0}$. [1]

To (2.5) we could add the **optional condition**

$$\bullet\quad \int_{\partial\mathbb{G}}\{K_1(s)\underline{n}(s)+K_2(s)\underline{t}(s)\}\,\mathrm{d}s = \underline{0}\,, \tag{2.6}$$

because adding a constant vectorfield to $K_1\underline{n}+K_2\underline{t}$ does not alter $\underline{f}$. We don't. For subtleties regarding this possibility, see the end of this section.
Example: The special choice $K_1=0$, $K_2=\kappa=$ constant, models surface tension at the boundary. Then $\underline{f}=-\kappa\underline{n}$. Keep in mind that $\underline{n}$ is the outside normal!

Theorem 2.2 (Uniqueness of the Stokes-Neumann problem)
Consider the Stokes-Neumann problem (2.4). Suppose $\underline{0}\in\mathbb{G}$.
- *If $\underline{f}=\underline{0}$, the set of solutions is given by the Euclidean motions (1.6) with $p=E=0$.*
- *For any given $\underline{f}\in\mathbb{L}_2(\partial\mathbb{G};\mathbb{R}^2)$ and any given $\underline{v}(\underline{0})=\underline{v}_0\in\mathbb{R}^2$, there is at most one solution with (unique) holomorphic representation $\{\varphi,\chi\}$ if one, in addition to $\varphi(0)=\chi(0)=0$, requires*

$$\operatorname{Im}\varphi'(0)=\mu\in\mathbb{R}\,, \qquad \chi'(0)=v_0\in\mathbb{C}. \tag{2.7}$$

Proof
- On $\partial\mathbb{G}$ we suppose

$$\mathcal{T}\cdot\underline{n} = -2\frac{\mathrm{d}}{\mathrm{d}s}\begin{bmatrix}\partial_y\phi\\ -\partial_x\phi\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}.$$

So we have to investigate the set of solutions of

$$\Delta\Delta\phi(\underline{x})=0\,,\quad \underline{x}\in\mathbb{G}, \qquad \nabla\phi(\underline{x})=\underline{a}=\text{constant},\quad \underline{x}\in\partial\mathbb{G}.$$

[1] JdG thanks Dr. A.A.F. van de Ven for clearing up this point

Consider $\tilde{\phi}(\underline{x}) = \phi(\underline{x}) - \underline{a}^\top \underline{x}$, which satisfies

$$\Delta\Delta\tilde{\phi}(\underline{x}) = 0\,, \quad \underline{x} \in \mathbb{G}, \qquad \nabla\tilde{\phi}(\underline{x}) = \underline{0}, \quad \underline{x} \in \partial\mathbb{G}.$$

This implies $\dfrac{\mathrm{d}}{\mathrm{d}s}\tilde{\phi}(\underline{x}(s)) = 0\,,$ at $\underline{x}(s) \in \partial\mathbb{G}$. Hence $\tilde{\phi}(\underline{x}) = \alpha = \text{constant}\,,$ at $\underline{x}(s) \in \partial\mathbb{G}$.
Introduce $\hat{\phi}(\underline{x}) = \phi(\underline{x}) - \underline{a}^\top \underline{x} - \alpha$, which satisfies

$$\Delta\Delta\hat{\phi}(\underline{x}) = 0\,, \quad \underline{x} \in \mathbb{G}, \qquad \frac{\partial}{\partial \underline{n}}\hat{\phi}(\underline{x}) = \underline{0}, \quad \hat{\phi}(\underline{x}) = 0, \quad \underline{x} \in \partial\mathbb{G}.$$

From $0 = \int_{\mathbb{G}} \hat{\phi}(\underline{x})\Delta\Delta\hat{\phi}(\underline{x})\,\mathrm{d}\underline{x}$ and Green II it now follows that $\hat{\phi} = 0$ and therefore the Airy function is of the form $\phi(\underline{x}) = \underline{a}^\top \underline{x} + \alpha$. The 'consistency conditions' (B.2) tell us that the stream function ψ has to satisfy $\partial_x\partial_y\psi = 0$ and $\partial_x\partial_x\psi - \partial_y\partial_y\psi = 0$. Therefore it has the form $\psi(\underline{x}) = \frac{1}{2}C\underline{x}^\top \underline{x} + \underline{b}^\top \underline{x} + c$.
As a consequence the homogeneous Stokes-Neumann problem is solved by all Euclidean motion solutions (1.6), represented by (1.7) with $E = 0$.
• If there are 2 solutions they differ by a solution represented by (2.7) which is reduced to 0 because of $\operatorname{Im}\varphi'(0) = 0\,, \chi'(0) = 0$. ■

Lemma 2.3
Let φ, χ : $\mathbb{G} \to \mathbb{C}$ be analytic with $\varphi(0) = \chi(0) = 0$.
Suppose that $z \mapsto \varphi(z)$ and $z \mapsto \overline{z}\varphi'(z) + \chi'(z)$ both extend to a continuous function on $\overline{\mathbb{G}}$.
• *If $\operatorname{Re}\varphi'(0) = 0$ and for all s*

$$z(s)\overline{\varphi'(z(s))} - \varphi(z(s)) + \overline{\chi'(z(s))} = C, \quad z(s) \in \partial\mathbb{G}\,, \tag{2.8}$$

with $C \in \mathbb{C}$ a constant.
Then $\varphi(z) = 0$, identically on $\mathbb{G}$ and $\chi(z) = \overline{C}z$.
• *If $\operatorname{Im}\varphi'(0) = 0$ and for all s*

$$z(s)\overline{\varphi'(z(s))} + \varphi(z(s)) + \overline{\chi'(z(s))} = D, \quad z(s) \in \partial\mathbb{G}\,, \tag{2.9}$$

with $D \in \mathbb{C}$ a constant.
Then $\varphi(z) = 0$, identically on $\mathbb{G}$, and $\chi(z) = \overline{D}z$.

Proof
• First suppose C=0 and consider the pair $\{\varphi, \chi\}$ as a holomorphic representation of the solution of Stokes' equations. Then, according to Theorem 2.1, $v_1 + \mathrm{i}v_2$ and p vanish identically on $\mathbb{G}$. Therefore $z\overline{\varphi'} - \varphi + \overline{\chi'} = 0$, identically on $\mathbb{G}$. Taking the derivative $\dfrac{\partial}{\partial z}$ leads to $\operatorname{Im}\varphi' = 0$ on $\mathbb{G}$. So $\varphi(z) = Az$, with $A \in \mathbb{R}$. Because $\operatorname{Re}\varphi'(0) = 0$ we necessarily have $A = 0$. Then from (2.8) also χ' has to be 0. Hence χ is constant. With the condition $\chi(0) = 0$ it follows that $\chi = 0$ on $\mathbb{G}$.

Finally, if $C \neq 0$, the only solution pair can be $\varphi(z) = 0$, $\chi(z) = \overline{C}z$ on $\mathbb{G}$.
• Two proofs are presented.
First take $C = \mathrm{i}D$ in (2.8) and multiply both sides by $-\mathrm{i}$. We get back (2.9), with φ, χ replaced by $\mathrm{i}\varphi$, $\mathrm{i}\chi$. Now the first result can be applied.
For the second proof consider the pair $\{\varphi, \chi\}$ as a holomorphic representation of the solution of Stokes' equations. We find at $\partial\mathbb{G}$

$$\mathcal{T}\underline{n}(s) = 2\,\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s}\Big(z(s)\overline{\varphi'(z(s))} + \varphi(z(s)) + \overline{\chi'(z(s))}\Big) = 2\,\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s}D = 0.$$

According to the uniqueness result in Theorem (2.2) we necessarily have $\varphi(z) = -\frac{\mathrm{i}}{2}Cz$, $\chi(z) = (A - \mathrm{i}B)z$, $A, B, C \in \mathbb{R}$. Then $\operatorname{Im}\varphi'(0) = 0$ implies $C = 0$. Finally, with (2.9), $A - \mathrm{i}B = E$. ∎

Concluding this section we look at the Stokes-Neumann problem in terms of φ, χ.
So we want to find analytic $\varphi, \chi : \mathbb{G} \to \mathbb{C}$, such that at the boundary $\partial\mathbb{G}$

$$\mathcal{T}\underline{n}(s) = 2\,\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s}\Big(z(s)\overline{\varphi'(z(s))} + \varphi(z(s)) + \overline{\chi'(z(s))}\Big) = -\,\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s}\{K(s)\dot{z}(s)\}. \tag{2.10}$$

Here $K(s) = K_1(s) + \mathrm{i}K_2(s)$, cf. (2.5).
Note that (2.10) does not alter if φ is replaced by $\varphi - \frac{\mathrm{i}}{2}Cz + C_1$ and χ by $\chi + (A - \mathrm{i}B)z + C_2$, with constants $A, B, C \in \mathbb{R}$ and $C_1, C_2 \in \mathbb{C}$.
Now in identity (2.10) we 'cancel' the $\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s}$ and with Lemma 2.10 we acquire uniqueness for the system

$$\begin{cases} z(s)\overline{\varphi'(z(s))} + \varphi(z(s)) + \overline{\chi'(z(s))} = -\frac{1}{2}K(s)\dot{z}(s), \quad z(s) \in \partial\mathbb{G}, \\ \varphi(0) = \chi(0) = 0, \quad \operatorname{Im}\varphi'(0) = 0. \end{cases} \tag{2.11}$$

There is a subtlety here! [2] If we add a constant $E \in \mathbb{C}$ to the right hand side in (2.11) the (unique if it exists) solution $\chi(z)$ becomes $\chi(z) + \overline{E}z$, a uniform rectilinear motion is added to the solution of Stokes' equations. It we kept to the 'optional' condition (2.6), it would forbid adding such E and leads us into consistency troubles. A requirement of type $\chi'(a) = 0$ at a suitable point $a \in \mathbb{G}$ could possibly 'save' the optional condition. At this point however we are quite content with the achieved uniqueness for problem (2.11).

[2] This sublety arose and was cleared up in a 'discussion on the *constants*' with Nasrin Arab.

3 A Basic Existence Result

On a simply connected open domain $\mathbb{G}$, $0 \in \mathbb{G}$ with 'sufficiently smooth' boundary $\partial\mathbb{G}$ and prescribed $F = F_1 + \mathrm{i}F_2 : \partial\mathbb{G} \to \mathbb{C}$ we want to show the existence of analytic φ, $\chi : \mathbb{G} \to \mathbb{C}$

$$\begin{cases} z(s)\overline{\varphi'(z(s))} + \varphi(z(s)) + \overline{\chi'(z(s))} = F(s)\dot{z}(s), \quad z(s) \in \partial\mathbb{G}, \\ \varphi(0) = \chi(0) = 0, \quad \operatorname{Im}\varphi'(0) = 0. \end{cases} \tag{3.1}$$

In this equation, instead of $+\varphi(z(s))$ also $-\varphi(z(s))$ can be taken. As we have seen, this is just a matter of redefining the unknown functions by a factor i. We keep to the +sign in this section.
Multiply both sides of (3.1) by $\dot{\bar{z}}$, then

$$\frac{\mathrm{d}}{\mathrm{d}s}\big(z(s)\overline{\varphi(z(s))} + \overline{\chi(z(s))} \,\big) + 2\,\mathrm{i}\,\mathrm{Im}\,\{(\varphi(z(s))\dot{\bar{z}}(s)\} = F(s). \tag{3.2}$$

Integration along $\partial\mathbb{G}$ of the real part of this identity leads to the necessary condition $\int_{\partial\mathbb{G}} F_1(s)\,\mathrm{d}s = 0$, for solvability. This nicely corresponds to the conditions (2.5), casu quo (2.2).

At this point the unique **conformal bijection**

$$\Omega : \mathbb{D} \to \mathbb{G}, \quad \zeta \mapsto \Omega(\zeta), \qquad \Omega(0) = 0,\ \Omega'(0) > 0, \tag{3.3}$$

is introduced from the open unit disk $\mathbb{D}$ in the $\zeta-$plane into the complex $z = x + \mathrm{i}y-$plane. Note that, if $\partial\mathbb{G}$ happens to be a Jordan curve with a Hölder continuous derivative, then Ω extends to a bijective $\mathscr{C}^{1;\alpha}$-map $\Omega : \overline{\mathbb{D}} \to \overline{\mathbb{G}}$, cf. **[P]** Thm 3.6, p49.
Corresponding to the usual parametrisation $\theta \to e^{\mathrm{i}\theta}$, $0 \le \theta < 2\pi$ of $\partial\mathbb{D} = \mathbb{S}^1$ we define $\theta \mapsto s(\theta)$ by $z(s(\theta)) = \Omega(e^{\mathrm{i}\theta})$.
Finally the new unknown functions

$$\Phi(\zeta) = \varphi(\Omega(\zeta)), \qquad \mathcal{X}(\zeta) = \chi(\Omega(\zeta)), \tag{3.4}$$

are introduced. Then, with

$$\partial_\theta\,\Phi(e^{\mathrm{i}\theta}) = \Phi'(e^{\mathrm{i}\theta})\,\mathrm{i}e^{\mathrm{i}\theta} = \varphi'(\Omega(e^{\mathrm{i}\theta}))\partial_\theta\Omega(e^{\mathrm{i}\theta}) = \varphi'(\Omega(e^{\mathrm{i}\theta}))\Omega'(e^{\mathrm{i}\theta})\,\mathrm{i}e^{\mathrm{i}\theta},$$

(3.1) can be rewritten, along $\partial\mathbb{D}$, as

$$\begin{cases} \Omega(\zeta)(\partial_\theta\overline{\Phi(\zeta)} + (\partial_\theta\overline{\Omega(\zeta)}\Phi(\zeta) + \partial_\theta\overline{\mathcal{X}(\zeta)} = \big|\partial_\theta\Omega(\zeta)\big|F(s(\theta))\,, \quad \zeta = e^{\mathrm{i}\theta}, \\ \Phi(0) = \mathcal{X}(0) = 0, \quad \operatorname{Im}\Phi'(0) = 0. \end{cases} \tag{3.5}$$

The first line can be rewritten

$$\partial_\theta\big[\,\Omega(\zeta)\overline{\Phi(\zeta)} + \overline{\mathcal{X}(\zeta)}\,\big] + 2\,\mathrm{i}\,\mathrm{Im}\,\big[\,(\partial_\theta\overline{\Omega(\zeta)})\Phi(\zeta)\,\big] = \big|\partial_\theta\Omega(\zeta)\big|F(s(\theta))\,, \quad \zeta = e^{\mathrm{i}\theta}. \tag{3.6}$$

Integration of the real part of this identity leads once more to the necessary condition $\int_0^{2\pi} F_1(s(\theta))\dfrac{\mathrm{d}s(\theta)}{\mathrm{d}\theta}\,\mathrm{d}\theta = 0$, for solvability.
We start the investigation of (3.5) with a Lemma

Lemma 3.1
Let $f \ : \ \mathbb{D} \to \mathbb{C}$ *be analytic with* $f(0) = 0$.
Split in real and imaginary parts $f(\zeta) = f_1(\zeta) + \mathrm{i} f_2(\zeta)$.
We have
1. $\theta \mapsto f_1(e^{\mathrm{i}\theta}) \ \in \ \mathbb{L}_2\big(\mathbb{S}^1; \mathbb{R}\big)$ *if and only if* $\theta \mapsto f_2(e^{\mathrm{i}\theta}) \ \in \ \mathbb{L}_2\big(\mathbb{S}^1; \mathbb{R}\big)$.
2. *The mapping* $\mathsf{J} \ : \ \mathbb{L}_2\big(\mathbb{S}^1; \mathbb{R}; \{1\}^{\perp}\big) \to \mathbb{L}_2\big(\mathbb{S}^1; \mathbb{R}; \{1\}^{\perp}\big), \quad f_1 \mapsto \mathsf{J} f_1 = f_2$ *, is orthogonal and* $\mathsf{J}\mathsf{J}^{\star} = -\mathsf{J} = \mathsf{J}^{-1}$, $\mathsf{J}^2 = -\mathsf{I}$, $\mathsf{J} \cos n\theta = \sin n\theta$, $\mathsf{J} \sin n\theta = -\cos n\theta, \ n \in \mathbb{N}$.
3. *The operator* J *is represented by the principal value integral*

$$\mathsf{J} f_1(\theta) = f_2(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cot\big(\frac{1}{2}(\theta - \theta_1)\big) f_1(\theta_1)\, \mathrm{d}\theta_1. \tag{3.7}$$

4. $\partial_\theta \mathsf{J} = \mathsf{J} \partial_\theta, \quad \partial_\theta f_1(e^{\mathrm{i}\theta}) + \mathrm{i}\partial_\theta f_2(e^{\mathrm{i}\theta}) = \mathrm{i}(\zeta \partial_\zeta f)(e^{\mathrm{i}\theta})$.
5. *Product formula for* $f, \, g \ : \ \mathbb{D} \to \mathbb{C}$, *both* $\mathbb{C}$*-analytic*

$$\mathsf{J}(f_1 g_1) = \mathsf{J}\big((\mathsf{J} f_1)(\mathsf{J} g_1)\big) + (\mathsf{J} f_1) g_1 + f_1 (\mathsf{J} g_1).$$

Proof See Appendix A sub 11. ■

We now come to the main theorem of this section

Theorem 3.2 (Basic Existence Result)
Let $F_1 \, , F_2 \, : \partial\mathbb{G} \to \mathbb{R}$ *be given.*
Suppose the conformal mapping $\Omega \ : \ \mathbb{D} \to \mathbb{G} \subset \mathbb{C}$ *to be such that*

a. $\theta \mapsto \big|\partial_\theta \Omega(e^{\mathrm{i}\theta})\big| F_1(s(\theta)) \ \in \ \mathbb{L}_2\big(\mathbb{S}^1; \mathbb{R}; \{1\}^{\perp}\big)$.

b. $\theta \mapsto \big|\partial_\theta \Omega(e^{\mathrm{i}\theta})\big| F_2(s(\theta)) \ \in \ \mathbb{L}_2\big(\mathbb{S}^1; \mathbb{R}\big)$.

c. $\theta \mapsto \big|\partial_\theta \Omega(e^{\mathrm{i}\theta})\big|$ *and* $\theta \mapsto \big|\partial_\theta \Omega(e^{\mathrm{i}\theta})\big|^{-1}$ *are bounded on* $\mathbb{S}^1$.

d. $\theta \mapsto \big|\partial_\theta \partial_\theta \Omega(e^{\mathrm{i}\theta})\big|$ *is bounded on* $\mathbb{S}^1$.

Then there exist unique $\Phi, \mathcal{X} \ : \ \partial\mathbb{D} \to \mathbb{C}$, *with properties*

$$\begin{aligned} &\bullet\ \theta \mapsto \Phi(e^{\mathrm{i}\theta}) \ \in \mathbb{L}_2(\mathbb{S}; \mathbb{C}), \qquad \theta \mapsto \mathcal{X}(e^{\mathrm{i}\theta}) \ \in \mathbb{L}_2(\mathbb{S}; \mathbb{C}) \, , \\ &\bullet\ \Phi \, , \, \mathcal{X} \ \ \textit{extend to} \ \ \Phi \, , \, \mathcal{X} \ : \ \overline{\mathbb{D}} \to \mathbb{C} \, , \ \textit{which are analytic on } \mathbb{D} \, . \end{aligned} \tag{3.8}$$

and which satisfy

$$\begin{cases} \Omega(\zeta)(\partial_\theta \overline{\Phi(\zeta)} + (\partial_\theta \overline{\Omega(\zeta)} \Phi(\zeta) + \partial_\theta \overline{\mathcal{X}(\zeta)} = \big|\partial_\theta \Omega(\zeta)\big| F(s(\theta)) \, , \quad \zeta = e^{\mathrm{i}\theta}, \\ \Phi(0) = \mathcal{X}(0) = 0, \quad \operatorname{Im} \Phi'(0) = 0. \end{cases} \tag{3.9}$$

If, instead of condition **d.**, *we require the Hölder condition*

e. $\theta \mapsto \Omega(e^{\mathrm{i}\theta}) \in \mathscr{C}^{1;\alpha}(\mathbb{S}^1)$, *for some* $0 < \alpha < 1$,

the theorem holds as well.

Proof We proceed in 6 steps.
I. Split (3.5), (3.6) in real and imaginary parts at $\partial\mathbb{G}$

$$\left\{ \begin{array}{rcl} \partial_\theta \operatorname{Re}\big[\,\overline{\Omega}\,\Phi\,\big] + \partial_\theta \mathcal{X}_1 & = & |\Omega'|F_1 \\ -\partial_\theta \operatorname{Im}\big[\,\overline{\Omega}\,\Phi\,\big] + 2\operatorname{Im}\big[\,(\partial_\theta\overline{\Omega})\,\Phi\,\big] - \partial_\theta\mathcal{X}_2 & = & |\Omega'|F_2 \end{array} \right. \tag{3.10}$$

By the way, note that the pair $\mathcal{X} = 0,\ \Phi = -\,\mathrm{i}\Omega$ satisfies this set of equations if $F_1 = F_2 = 0$. However it does NOT satisfy our condition $\operatorname{Im}\Phi'(0) = 0$.
We now eliminate $\mathcal{X}_2$ by applying J to the 1st line and add it to the 2nd.

$$\left\{ \begin{array}{rcl} \partial_\theta \operatorname{Re}\big[\,\overline{\Omega}\,\Phi\,\big] + \partial_\theta \mathcal{X}_1 & = & |\Omega'|F_1 \\ \partial_\theta\big\{\mathsf{J}\operatorname{Re}\big[\,\overline{\Omega}\,\Phi\,\big] - \operatorname{Im}\big[\,\overline{\Omega}\,\Phi\,\big]\big\} + 2\operatorname{Im}\big[\,(\partial_\theta\overline{\Omega})\,\Phi\,\big] & = & \big\{\mathsf{J}\big(\,|\Omega'|F_1\,\big) + |\Omega'|F_2\big\} \end{array} \right. \tag{3.11}$$

From now on the factors Ω_1, Ω_2, $\partial_\theta\Omega_1 = \dot{\Omega}_1$, $\partial_\theta\Omega_2 = \dot{\Omega}_2$, are to be considered as multiplication operators. Because of the analyticic extendibility requirement we put, cf. Lemma 3.1, $\Phi = \Phi_1 + \mathrm{i}\mathsf{J}\Phi_1$, etc. Thus the 2nd equation becomes an operator equation for Φ_1 only. Using the product formula of Lemma 3.1, which gives us

$$\mathsf{J}\big((\mathsf{J}\Omega_1)(\mathsf{J}\Phi_1)\big) = \mathsf{J}(\Omega_1\Phi_1) - (\mathsf{J}\Omega_1)\Phi_1 - \Omega_1(\mathsf{J}\Phi_1), \tag{3.12}$$

combined with the 2nd line of (3.11), we find the operator equation

$$\partial_\theta\Big(\big[\mathsf{J}\Omega_1 - \Omega_1\mathsf{J}\big]\Phi_1\Big) + \big[\dot{\Omega}_1\mathsf{J} - \dot{\Omega}_2\big]\Phi_1 = \frac{1}{2}\Big[\mathsf{J}\big(|\Omega'|F_1\big) + |\Omega'|F_2\Big]. \tag{3.13}$$

So we have to study the operators on the left hand side of (3.13).
II. First notice that the operator

$$\mathsf{L}:\ \mathbb{L}_2\big(\mathbb{S}^1;\mathbb{R};\{1,\sin\theta\}^\perp\big) \to \mathbb{L}_2\big(\mathbb{S}^1;\mathbb{R}\big)\ :\quad \Phi_1 \ \mapsto\ \mathsf{L}\Phi_1 = \big[\dot{\Omega}_1\mathsf{J} - \dot{\Omega}_2\big]\Phi_1\,,$$

is a bijection. Indeed, on $\mathbb{S}^1$ investigate

$$\big[\dot{\Omega}_1\mathsf{J} - \dot{\Omega}_2\big]\Phi_1 = \operatorname{Im}\{\dot{\overline{\Omega}}\Phi\} = \operatorname{Re}\{-\,\mathrm{i}\dot{\overline{\Omega}}\Phi\} = R\ \in \mathbb{L}_2(\mathbb{S}^1).$$

Divide by $|\dot{\Omega}|^2$, then on $\mathbb{S}^1$,

$$\operatorname{Re}\frac{\Phi}{\mathrm{i}\dot{\Omega}} = \frac{R}{|\dot{\Omega}|^2} = S(\theta) + \overline{S(\theta)},$$

where S is uniquely written as the *complex* Fourier expansion (of a $\mathbb{R}$-valued function)

$$S(\theta) = \sum_{\ell=0}^{\infty} s_\ell\, e^{\mathrm{i}\ell\theta}\,, \qquad \text{with}\ \ s_\ell \in \mathbb{C}\,,\ s_0 \in \mathbb{R}\,.$$

After analytic extension into D we write

$$-\operatorname{Re}\frac{\Phi(\zeta)}{\zeta\Omega'(\zeta)} = S(\zeta) + S^{\dagger}(\zeta)\,, \qquad \text{for } \zeta = e^{\,\mathrm{i}\theta},$$

from which $\Phi(\zeta) = -2\zeta\Omega'(\zeta)S(\zeta) + \mathrm{i}\alpha\zeta\Omega'(\zeta)$ for $|\zeta| < 1$ and $\alpha \in \mathbb{R}$, follows.
Since $\Phi'(0) \in \mathbb{R}$ is required, only $\alpha = 0$ is acceptible. The $\mathbb{L}_2$-properties follow from the (supposed) boundedness of Ω' and $(\,\Omega'\,)^{-1}$ on $\mathbb{S}^1$.

III. Together with (3.7) the operator

$$\mathsf{K}:\ \mathbb{L}_2\big(\mathbb{S}^1;\mathbb{R}\,;\{1,\sin\theta\}^{\perp}\big) \to \mathbb{L}_2\big(\mathbb{S}^1;\mathbb{R}\,\big)\ :\quad \Phi_1 \ \mapsto\ \mathsf{K}\Phi_1 = \partial_\theta\Big(\big[J\Omega_1 - \Omega_1 J\big]\,\Phi_1\Big)\,,$$

can be written, with some trigonometry,

$$\mathsf{K}\Phi_1(\theta) = -\frac{1}{2\pi}\,\partial_\theta\int_{-\pi}^{\pi}\cot\big(\frac{\theta-\theta_1}{2}\big)\big\{\Omega_1(e^{\,\mathrm{i}\theta}) - \Omega_1(e^{\,\mathrm{i}\theta_1})\big\}\,\Phi_1(\theta_1)\,\mathrm{d}\theta_1 \ =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin(\theta-\theta_1)}{1-\cos(\theta-\theta_1)}\Big[\frac{\Omega_1(e^{\,\mathrm{i}\theta}) - \Omega_1(e^{\,\mathrm{i}\theta_1})}{\sin(\theta-\theta_1)} - \partial_\theta\Omega_1(e^{\,\mathrm{i}\theta})\Big]\Phi_1(\theta_1)\,\mathrm{d}\theta_1 \qquad (3.14)$$

Then condition **d.**, together with L'Hôpital's rule, imply that K is Hilbert-Schmidt.
If there were $\Phi_1 \in \mathbb{L}_2\big(\mathbb{S}^1;\mathbb{R}\,;\{1,\sin\theta\}^{\perp}\big)$, $\Phi_1 \neq 0$, with $(\mathsf{K}+\mathsf{L})\Phi_1 = 0$, we could introduce $\mathcal{X}_1 = -\operatorname{Re}\big[\,\overline{\Omega}\,\Phi\,\big] + \gamma$, with constant $\gamma = \operatorname{Re}\int_{-\pi}^{\pi}\big[\,\overline{\Omega}(e^{\,\mathrm{i}\theta})\,\Phi(e^{\,\mathrm{i}\theta})\,\big]\,\mathrm{d}\theta$. Note that such Φ_1 is necessarily continuous !!
The nonzero pair $\{\,\Phi_1 + \mathrm{i}\mathsf{J}\Phi_1,\ \mathcal{X}_1 + \mathrm{i}\mathsf{J}\mathcal{X}_1\,\}$ then leads to a non-zero solution pair $\{\varphi, \chi\}$ of (2.11), with $K = 0$, which contradicts the uniqueness result of Lemma 2.3 . So $\mathsf{K}+\mathsf{L}$ is injective.
Since $\mathsf{K}+\mathsf{L}$ is a compact perturbation of the bijection L, which has index 0, the problem $(\mathsf{K}+\mathsf{L})\Phi_1 = R$ is uniquely solvable for any $R \in \mathbb{L}_2\big(\mathbb{S}^1;\mathbb{R}\,\big)$. For the 'index theory' see, e.g., **[GGK]**.

IV. Substitute the found Φ_1 with $\mathsf{J}\Phi_1$ in the first equation of (3.11). Its righthand side $-\frac{1}{2}|\partial_\theta\Omega|K_1$ can be written as a derivative. With the requirement $\mathcal{X}(0) = 0$, it leads to a unique $\mathcal{X}$.

V. Split the operator $\mathsf{K} = \mathsf{K}_\varepsilon + \mathsf{K}_{\pi-\varepsilon}$, $0 < \varepsilon < \pi$. On the square $[-\pi,\pi]\times[-\pi,\pi]$, and inside the strip $|\theta-\theta_1| < \varepsilon$, the kernel of K_ε takes the values of the kernel of K. Outside this strip it is taken to be 0. So

$$\mathsf{K}_\varepsilon\Phi_1(\theta) = \frac{1}{2\pi}\int_{\max\{-\pi,\theta-\varepsilon\}}^{\min\{\pi,\theta+\varepsilon\}}\mathcal{K}(\theta\,,\,\theta_1)\,\Phi_1(\theta_1)\,\mathrm{d}\theta_1\,,$$

with $\mathcal{K}$ the kernel of (3.14).

Note that the 'remains' $\mathsf{K}_{\pi-\varepsilon}$ is Hilbert-Schmidt.
We now show that for some $C > 0$ we have $\|\mathsf{K}_\varepsilon\| \leq C\varepsilon^{\min\{\alpha \frac{1}{2}\}}$.
The Mean Value Theorem applied to

$$x \quad \mapsto \quad \Omega_1(e^{\,\mathrm{i}x}) + \frac{\Omega_1(e^{\,\mathrm{i}\theta_1}) - \Omega_1(e^{\,\mathrm{i}\theta})}{\sin(\theta-\theta_1)} \sin(x-\theta_1), \quad \text{on interval} \quad [\theta_1, \theta] \text{ or } [\theta, \theta_1],$$

provides us with

$$\frac{\Omega_1(e^{\,\mathrm{i}\theta}) - \Omega_1(e^{\,\mathrm{i}\theta_1})}{\sin(\theta-\theta_1)} = \frac{\partial_\theta \Omega_1(e^{\,\mathrm{i}\xi})}{\cos(\xi-\theta_1)}\,, \qquad \text{for some } \xi \text{ in between } \theta\,, \theta_1\,.$$

We now split K_ε in a 'bounded kernel part' and a 'singular kernel part'

$$\mathsf{K}_\varepsilon = \mathsf{K}_{\varepsilon,B} + \mathsf{K}_{\varepsilon,S}.$$

For some ξ in between $\theta\,, \theta_1$,

$$\mathsf{K}_{\varepsilon,B}\Phi_1(\theta) = \int_{\max\{-\pi,\theta-\varepsilon\}}^{\min\{\pi,\theta+\varepsilon\}} \frac{\sin(\theta-\theta_1)}{1-\cos(\theta-\theta_1)} \frac{1-\cos(\xi-\theta_1)}{\cos(\xi-\theta_1)}\, \partial_\theta\Omega_1(e^{\,\mathrm{i}\xi})\, \Phi_1(\theta_1)\, \mathrm{d}\theta_1\,.$$

and

$$\mathsf{K}_{\varepsilon,S}\Phi_1(\theta) = \int_{\max\{-\pi,\theta-\varepsilon\}}^{\min\{\pi,\theta+\varepsilon\}} \frac{\sin(\theta-\theta_1)}{1-\cos(\theta-\theta_1)} \Big[\partial_\theta\Omega_1(e^{\,\mathrm{i}\xi}) - \partial_\theta\Omega_1(e^{\,\mathrm{i}\theta})\Big] \Phi_1(\theta_1)\, \mathrm{d}\theta_1\,.$$

Since the kernel of $\mathsf{K}_{\varepsilon,B}$ is bounded we find $\|\mathsf{K}_{\varepsilon,B}\| < C_1\sqrt{\varepsilon}$, for some $C_1 > 0$.
Next, by means of the required Hölder condition the kernel of $\mathsf{K}_{\varepsilon,S}$ is estimated

$$\frac{|\sin(\theta-\theta_1)|}{1-\cos(\theta-\theta_1)} \Big|\partial_\theta\Omega_1(e^{\,\mathrm{i}\xi}) - \partial_\theta\Omega_1(e^{\,\mathrm{i}\theta})\Big| \leq C_2 \frac{|\xi-\theta|^\alpha}{|\theta_1-\theta|} \leq C_2|\theta_1-\theta|^{\alpha-1},$$

on $[-\pi, \pi]$. It now follows

$$|\mathsf{K}_{\varepsilon,S}\Phi_1(\theta)|^2 \leq C_3 \int_{\max\{-\pi,\theta-\varepsilon\}}^{\min\{\pi,\theta+\varepsilon\}} |\theta-\theta_2|^{\alpha-1} \mathrm{d}\theta_2 \,\cdot\, \int_{\max\{-\pi,\theta-\varepsilon\}}^{\min\{\pi,\theta+\varepsilon\}} |\theta-\theta_1|^{\alpha-1} |\Phi_1(\theta_1)|^2 \mathrm{d}\theta_1\,.$$

The first integral is is a function of θ bounded by $\leq \frac{2}{\alpha}\varepsilon^\alpha$.
Finally, after a change of variables,

$$\int_{-\pi}^{\pi} |\mathsf{K}_{\varepsilon,S}\Phi_1(\theta)|^2\, \mathrm{d}\theta \;\leq\; C_3(\frac{2}{\alpha}\varepsilon^\alpha)^2 \int_{-\pi}^{\pi} |\Phi_1(\theta)|^2\, \mathrm{d}\theta\,,$$

which says

$$\|\mathsf{K}_{\varepsilon,S}\| \;\leq\; \sqrt{C_3}\,\frac{2}{\alpha}\varepsilon^\alpha\,.$$

VI. (3.13) can now be written

$$\mathsf{K}_{\pi-\varepsilon}\Phi_1 + \big(\mathsf{K}_\varepsilon + \mathsf{L}\big)\Phi_1 = \frac{1}{2}\Big[\mathsf{J}\big(|\Omega'|F_1\big) + |\Omega'|F_2\Big]. \tag{3.15}$$

For ε sufficiently small the second operator is still a bijection. The operator $\mathsf{K}+\mathsf{L}$ is a compact perturbation of this bijection. Therefore the argument of **III.** applies again. ■

Notation

- For given $\Theta : \overline{\mathbb{D}} \to \mathbb{C}$ we introduce the restriction to a circle

$$\Theta\Big|_r \;:\; \partial\mathbb{D} \to \mathbb{C} \;:\; \theta \mapsto \Theta(re^{\,\mathrm{i}\theta})\,, \qquad 0 < r \le 1.$$

- For $g \in \mathbb{L}_2(\mathbb{S};\mathbb{C})$ the (complex) Fourier expansion $g(\theta) = \sum_{\ell=-\infty}^{\infty} g_\ell e^{\,\mathrm{i}\ell\theta}$ is split in a positive and negative part, respectively,

$$g^+(\theta) = \sum_{\ell=1}^{\infty} g_\ell e^{\,\mathrm{i}\ell\theta} \quad \text{and} \quad g^-(\theta) = \sum_{k=0}^{\infty} g_{-k} e^{-\,\mathrm{i}k\theta}.$$

The previous Theorem implies $\Phi\Big|_r \to \Phi\Big|_1,\ \mathcal{X}\Big|_r \to \mathcal{X}\Big|_1$ in $\mathbb{L}_2(\mathbb{S};\mathbb{C})$ as $r\uparrow 1$.
It follows, since $\theta \mapsto \Omega(e^{\,\mathrm{i}\theta})$ is supposed to be continuously differentiable,

$$\begin{aligned} &\bullet\ \big[\overline{\Omega}\Phi + \mathcal{X}\big]\Big|_r \longrightarrow \big[\overline{\Omega}\Phi + \mathcal{X}\big]\Big|_1, \quad \text{in} \quad \mathbb{L}_2(\mathbb{S};\mathbb{C}), \quad \text{as} \quad r\uparrow 1, \\ &\bullet\ \partial_\theta\big[\overline{\Omega}\Phi + \mathcal{X}\big]\Big|_r \longrightarrow \partial_\theta\big[\overline{\Omega}\Phi + \mathcal{X}\big]\Big|_1, \quad \text{in} \quad \mathbb{H}^{-1}(\mathbb{S};\mathbb{C}), \quad \text{as} \quad r\uparrow 1, \end{aligned} \tag{3.16}$$

However, since $\partial_\theta\big[\overline{\Omega}\Phi + \mathcal{X}\big]\Big|_1 \in \mathbb{L}_2(\mathbb{S};\mathbb{C})$, cf. (3.10), we expect the latter convergence also to be in $\mathbb{L}_2(\mathbb{S};\mathbb{C})$. There is a simple proof for this if we assume some extra smoothness on Ω.

Theorem 3.3 (Behaviour near the Boundary 1)

a. *Assume that the sequence of Fourier coefficients* $\{n \mapsto 2n\Omega_n\} \in \ell_1(\mathbb{N})$ *, then the solution* $\Phi, \mathcal{X}$ *of Theorem 3.2 enjoys the properties*

$$\partial_\theta\big[\overline{\Omega}\Phi + \mathcal{X}\big]\Big|_r \longrightarrow \partial_\theta\big[\overline{\Omega}\Phi + \mathcal{X}\big]\Big|_1, \quad \textit{in} \quad \mathbb{L}_2(\mathbb{S};\mathbb{C}), \quad \textit{as} \quad r\uparrow 1, \tag{3.17}$$

$$\partial_\theta\big[\overline{\Omega}\Phi\big]^-\Big|_r \longrightarrow \partial_\theta\big[\overline{\Omega}\Phi\big]^-\Big|_1, \quad \textit{in} \quad \mathbb{L}_2(\mathbb{S};\mathbb{C}), \quad \textit{as} \quad r\uparrow 1. \tag{3.18}$$

b. *Condition* **a.** *is satisfied if* $\{\theta \mapsto \Omega(e^{\,\mathrm{i}\theta})\} \in \mathbb{H}^{\frac{3}{2}+\alpha}(\mathbb{S};\mathbb{C}) \cap \mathscr{C}^{1;\alpha}\big(\mathbb{S}^1\big)$ *, with* $\alpha > 0$ *. E.g. if* $\{\theta \mapsto \Omega(e^{\,\mathrm{i}\theta})\} \in \mathscr{C}^2\big(\mathbb{S}^1\big)$ *.*

Proof

- The Fourier expansion of $-\,\mathrm{i}\partial_\theta\big[\overline{\Omega}\Phi+\mathcal{X}\big]\Big|_r\quad 0<r\le 1$, reads

$$-\,\mathrm{i}\partial_\theta\Big[\Big(\sum_{n=1}^\infty r^n\overline{\Omega}_n\,\mathrm{e}^{-\,\mathrm{i}n\theta}\Big)\Big(\sum_{m=1}^\infty r^m\Phi_m\,\mathrm{e}^{\mathrm{i}m\theta}\Big)+\Big(\sum_{k=1}^\infty r^k\mathcal{X}_k\,\mathrm{e}^{\mathrm{i}k\theta}\Big)\Big]\;=$$

$$=\sum_{k=1}^\infty k\,\{r^k\mathcal{X}_k+\sum_{m-n=k,\,n\ge1,\,m\ge1}r^{n+m}\overline{\Omega}_n\Phi_m\}\,\mathrm{e}^{\mathrm{i}k\theta}-\sum_{\ell=0}^\infty\ell\,\{\sum_{n-m=\ell,\,n\ge1,\,m\ge1}r^{n+m}\overline{\Omega}_n\Phi_m\}\,\mathrm{e}^{-\,\mathrm{i}\ell\theta}\;=$$

$$=\sum_{k=1}^\infty k\,\{r^k\mathcal{X}_k+\sum_{n=1}^\infty r^{2n+k}\,\overline{\Omega}_n\Phi_{n+k}\}\,\mathrm{e}^{\mathrm{i}k\theta}-\sum_{\ell=0}^\infty\ell\,\{\sum_{m=1}^\infty r^{2m+\ell}\,\overline{\Omega}_{m+\ell}\Phi_m\}\,\mathrm{e}^{-\,\mathrm{i}\ell\theta}\,.$$

From the previous we know that, for $r=1$, the coefficient sequences $k\{\cdot\}$ and $\ell\{\cdot\}$ are both in $\ell_2(\mathbb{N})$. Because of analyticity this is also true for $0<r<1$. We have to show that no 'discontinuity' occurs at $r=1$.
The positive and negative parts of the coefficient sequences of

$$-\,\mathrm{i}\partial_\theta\Big\{\,\big[\overline{\Omega}\Phi+\mathcal{X}\big]\Big|_1-\big[\overline{\Omega}\Phi+\mathcal{X}\big]\Big|_r\,\Big\}$$

are, respectively,

$$k\;\mapsto\;k\Big\{(1-r^k)\mathcal{X}_k+\sum_{n=1}^\infty(1-r^{2n+k})\overline{\Omega}_n\Phi_{n+k}\Big\}\,,\qquad \ell\;\mapsto\;-\,\ell\Big\{\sum_{m=1}^\infty(1-r^{2m+\ell})\overline{\Omega}_{m+\ell}\Phi_m\Big\}.$$

We have to show that both tend to 0 in $\ell_2(\mathbb{N})$, as $r\uparrow 1$.
We use the identity

$$(1-r^k)\frac{1-r^{2n+k}}{1-r^k}=(1-r^k)\Big\{1+\frac{r^k}{1+r+\cdots+r^{k-1}}(1+r+\cdots+r^{2n-1})\Big\},$$

and the fact that

$$\frac{r^k}{1+r+\cdots+r^{k-1}}\;\uparrow\;\frac{1}{k}\qquad\text{as}\qquad r\uparrow 1.$$

- The 'positive' sequence can be split

$$k\;\mapsto\;(1-r^k)\Big\{\,k\big\{\mathcal{X}_k+\sum_{n=1}^\infty\overline{\Omega}_n\Phi_{n+k}\big\}+k\frac{r^k}{1+r+\cdots+r^{k-1}}\sum_{n=1}^\infty(1+r+\cdots+r^{2n-1})\overline{\Omega}_n\Phi_{n+k}\Big\}. \tag{3.19}$$

The sequence $k\;\mapsto\;k\big\{\mathcal{X}_k+\sum_{n=1}^\infty\overline{\Omega}_n\Phi_{n+k}\big\}$ is ℓ_2 because of (3.10). We are ready if we can show that the operators

$$\{\,\Phi_k\,\}\quad\mapsto\quad\{\sum_{n=1}^\infty(1+r+\cdots+r^{2n-1})\overline{\Omega}_n\Phi_{n+k}\}\,, \tag{3.20}$$

are uniformly bounded (as ℓ_2-operators) on the interval $0 < r \le 1$.
If it happens that $\{n \ \mapsto \ 2n\Omega_n\} \in \ell_1(\mathbb{N})$ we estimate

$$\sum_{k=1}^{\infty} \big| \sum_{n=1}^{\infty} 2n\overline{\Omega}_n \Phi_{n+k} \big|^2 \ \le\ \sum_{k=1}^{\infty} \{ \big| \sum_{m=1}^{\infty} 2m|\Omega_m| \} \{ \sum_{n=1}^{\infty} 2n|\Omega_n|\,|\Phi_{n+k}|^2 \} \ \le\ \Big(\sum_{m=1}^{\infty} 2m|\Omega_m| \Big)^2 \sum_{k=1}^{\infty} |\Phi_k|^2 \,.$$

It follows that the 'positive' sequence tends to 0 if $r \uparrow 1$.

- The 'negative' sequence $\ell \mapsto -\ell\big\{ \sum_{m=1}^{\infty} (1 - r^{2m+\ell})\overline{\Omega}_{m+\ell}\Phi_m \big\}$ can be written

$$\ell \ \mapsto \ -(1-r^\ell)\Big\{ \ell\{ \sum_{m=1}^{\infty} \overline{\Omega}_{m+\ell}\Phi_m \} + \ell \frac{r^\ell}{1 + r + \cdots + r^{\ell-1}} \sum_{m=1}^{\infty} (1 + r + \cdots + r^{2m-1})\overline{\Omega}_{m+\ell}\Phi_m \Big\} \,. \tag{3.21}$$

With a similar estimate as before it turns out that also this $\ell_2(\mathbb{N})$ sequence tends to 0 if $r \uparrow 1$.
- For the last statement in the theorem note that the coefficients $\mathcal{X}_k$ do not occur in the 'negative' sequence. ■

The natural question arises whether the results of the previous theorem could also be obtained if only $\{\theta \ \mapsto \ \Omega(e^{\,\mathrm{i}\theta})\} \in \mathscr{C}^{1;\alpha}\big(\mathbb{S}^1\big)$, with $\alpha > 0$, is assumed. I got half way by invoking a theorem on Fourier multipliers which map periodic Hölder spaces into themselves. [3]

Theorem 3.4 (Behaviour near the Boundary 2)
Assume that $\{\theta \ \mapsto \ \Omega(e^{\,\mathrm{i}\theta})\} \in \mathscr{C}^{1;\alpha}\big(\mathbb{S}^1\big)$ *, with* $\alpha > 0$ *, then*

$$\partial_\theta \big[\overline{\Omega}\Phi + \mathcal{X}\big]^+\Big|_r \ \longrightarrow\ \partial_\theta \big[\overline{\Omega}\Phi + \mathcal{X}\big]^+\Big|_1 , \quad in \quad \mathbb{L}_2(\mathbb{S};\mathbb{C}), \quad as \quad r \uparrow 1, \tag{3.22}$$

Proof
We 'only' have to show that the the operators (3.20) are still uniformly bounded (as ℓ_2-operators) on the interval $0 < r \le 1$ under the weaker condition.
Consider the 'multiplication operator expression'

$$\Big(\sum_{n=1}^{\infty} (1{+}r{+}\cdots{+}r^{2n-1})\overline{\Omega}_n \, \mathrm{e}^{-\,\mathrm{i}n\theta} \Big) \Big(\sum_{m=1}^{\infty} \Phi_m \, \mathrm{e}^{\,\mathrm{i}m\theta} \Big) \ =$$

$$= \sum_{k=1}^{\infty} \Big\{ \sum_{m-n=k,\, n\ge1,\, m\ge1} (1 + r + \cdots + r^{2n-1})\overline{\Omega}_n\Phi_m \Big\} \, \mathrm{e}^{\,\mathrm{i}k\theta} +$$

[3] JdG thanks Dr. G. Prokert for advice and references.

$$+\sum_{\ell=0}^{\infty}\Big\{\sum_{n-m=\ell,\,n\geq1,\,m\geq1}(1+r+\cdots+r^{2n-1})\overline{\Omega}_n\Phi_m\Big\}\,\mathrm{e}^{-\,\mathrm{i}\ell\theta}\;=$$

$$=\sum_{k=1}^{\infty}\Big\{\sum_{n=1}^{\infty}(1+r+\cdots+r^{2n-1})\,\overline{\Omega}_n\Phi_{n+k}\Big\}\,\mathrm{e}^{\mathrm{i}k\theta}+\sum_{\ell=0}^{\infty}\Big\{\sum_{m=1}^{\infty}(1+r+\cdots+r^{2(m+\ell)-1})\,\overline{\Omega}_{m+\ell}\Phi_m\Big\}\,\mathrm{e}^{-\,\mathrm{i}\ell\theta}\,.$$

If the very first sum in this expression represents a bounded function, uniformly in $0<r\leq1$, we are ready. According to our assumption, the function

$$\Big\{\,\theta\;\mapsto\;\sum_{n=1}^{\infty}n\overline{\Omega}_n\,\mathrm{e}^{-\,\mathrm{i}n\theta}\Big\}\;\in\mathscr{C}^{\alpha}\big(\mathbb{S}^1\big)\,.$$

This remains so if the respective Fourier coefficients are multiplied by $\dfrac{1+r+\cdots+r^{2n-1}}{n}$, because they satisfy the conditions (1.2)-(1.3) in **[AB]**. ■

Additional Remark As for the 'negative' part $\partial_\theta\big[\overline{\Omega}\Phi+\mathcal{X}\big]^-\Big|_r=\partial_\theta\big[\overline{\Omega}\Phi\big]^-\Big|_r$, we should be able to prove, cf. (3.21), that from $\big\{\ell\;\rightarrow\;\sum_{m=1}^{\infty}\overline{\Omega}_{m+\ell}\,m\Phi_m\big\}\;\in\ell_2$ it follows that also $\big\{\ell\;\rightarrow\;\sum_{m=1}^{\infty}\overline{\Omega}_{m+\ell}\,(1+r+\cdots+r^{2m-1})\Phi_m\big\}\;\in\ell_2$, and uniformly bounded, for $0<r\leq1$. Let us see how far we get. The second sum in (3.21) can be split

$$\sum_{m=1}^{\infty}(1+r+\cdots+r^{2(m+\ell)-1})\overline{\Omega}_{m+\ell}\Phi_m\;-\;(1+r+\cdots+r^{2\ell-1})\sum_{m=1}^{\infty}\overline{\Omega}_{m+\ell}\,r^{2m}\Phi_m\,. \tag{3.23}$$

The first term presents no trouble. It is 'multiplication by a bounded function', as in the previous proof. For the second term we would like to show uniform boundedness for

$$\ell\sum_{m=1}^{\infty}\overline{\Omega}_{m+\ell}\,r^{2m}\Phi_m=\sum_{m=1}^{\infty}(m+\ell)\overline{\Omega}_{m+\ell}\,r^{2m}\Phi_m-\sum_{m=1}^{\infty}\overline{\Omega}_{m+\ell}\,r^{2m}m\Phi_m\,.$$

Here the first term comes from multiplication by $\overline{\Omega'}$, which is supposed to be continuous on $\overline{D}$. The second term finally confronts us with the question whether from $\big\{\ell\;\rightarrow\;\sum_{m=1}^{\infty}\overline{\Omega}_{m+\ell}\,m\Phi_m\big\}\;\in\ell_2$ it follows that

$$\forall\,\varepsilon>0\;\;\exists\,N\in\mathbb{N}\;\;\forall\;0<r\leq1\;\;:\quad\sum_{\ell=N}^{\infty}\Big|\sum_{m=1}^{\infty}\overline{\Omega}_{m+\ell}\,r^{2m}m\Phi_m\Big|^2\;<\varepsilon\;. \tag{3.24}$$

I could not prove this!

4 Results on Stokes Boundary Value Problems

In this section we formulate our results for simply connected domains $\mathbb{G} \subset \mathbb{R}^2 \sim \mathbb{C}$ with boundary $\partial\mathbb{G}$ and $0 \in \mathbb{G}$. The boundary is supposed to be an arclength parametrized Jordan curve with a Hölder continuous and positively oriented tangent vector $s \mapsto \dot{\underline{x}}(s) = \dot{z}(s)$. Let, as before, $\Omega : \mathbb{D} \to \mathbb{G}$ denote the unique conformal mapping with $\Omega(0) = 0$ and $\Omega'(0) > 0$. Again $\theta \mapsto s(\theta)$ is defined by $\Omega(e^{\mathrm{i}\theta}) = s(\theta)$, $0 \leq \theta < 2\pi$.

The following two theorems are immediate consequences of the preceding sections. Looking at the smoothness assumptions of the preceding theorems, it is clear that the $\mathbb{H}^2$-condition on the boundary $\partial\mathbb{G}$ in the next theorem can be somewhat relaxed.

Theorem 4.1 (Stokes-Dirichlet)
Consider the Stokes-Dirichlet problem (2.1) with boundary $\{s \mapsto \underline{x}(s)\} \in \mathbb{H}^2(\partial\mathbb{G})$.
The prescribed boundary velocity field is given by

$$s \mapsto \underline{g}(\underline{x}(s)) = V_1(s)\underline{n}(\underline{x}(s)) + V_2(s)\underline{t}(\underline{x}(s)) = -\mathrm{i}\big(V_1(s) + \mathrm{i}V_2(s)\big)\dot{z}(s) = -\mathrm{i}\,V(s)\dot{z}(s) \in \mathbb{L}_2(\partial\mathbb{G}),$$

where $\int_{\partial\mathbb{G}} V_1(s)\,\mathrm{d}s = 0$.
There exist unique **analytic** $\varphi, \chi : \mathbb{G} \to \mathbb{C}$, *with* $\varphi(0) = \chi(0) = \operatorname{Re}\varphi'(0) = 0$, *and* $\varphi\big|_{\partial\mathbb{G}}, \chi\big|_{\partial\mathbb{G}} \in \mathbb{L}_2(\partial\mathbb{G})$, *such that*

$$z(s)\overline{\varphi'(z(s))} - \varphi(z(s)) + \overline{\chi'(z(s))} = -\mathrm{i}V(s)\dot{z}(s), \quad z(s) \in \partial\mathbb{G}.$$

We have

- $\varphi(\Omega(re^{\mathrm{i}\theta})) \to \varphi\big|_{\partial\mathbb{G}}(s(\theta))$ *and* $\chi(\Omega(re^{\mathrm{i}\theta})) \to \chi\big|_{\partial\mathbb{G}}(s(\theta))$,
  in $\mathbb{L}_2(\mathbb{S})$*-sense, as* $r \uparrow 1$.
- $\big[v_1(z) + \mathrm{i}v_2(z)\big]\Big|_{z=\Omega(re^{\mathrm{i}\theta})} = \big[z\overline{\varphi'(z)} - \varphi(z) + \overline{\chi'(z)}\big]\Big|_{z=\Omega(re^{\mathrm{i}\theta})} \to \underline{g}(\underline{x}(s(\theta)))$,
  in $\mathbb{L}_2(\mathbb{S})$*-sense, as* $r \uparrow 1$.
- *The normal stress at* $\partial\mathbb{G}$ *is well defined (as a* $\mathbb{H}^{-1}$*-limit) and given by*
  $$\big(\mathcal{T} \cdot \underline{n}\big)(\underline{x}(s)) = 2\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s}\underline{g}(\underline{x}(s) + 4\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s}\varphi(z(s)) \in \mathbb{H}^{-1}(\partial\mathbb{G}).$$

■

Theorem 4.2 (Stokes-Neumann)
Consider the Stokes-Neumann problem (2.4) with boundary $\{s \mapsto \underline{x}(s)\} \in \mathbb{H}^2(\partial\mathbb{G})$.
The prescribed boundary stress field

$$s \mapsto \underline{f}(\underline{x}(s)) = \mathcal{T}(\underline{x}(s)) \cdot \underline{n}(\underline{x}(s)) = 2\,\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s}\Big(z(s)\overline{\varphi'(z(s))} + \varphi(z(s)) + \overline{\chi'(z(s))}\Big) =$$

$$= -\,\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s}\{K(s)\dot{z}(s)\} \in \mathbb{H}^{-1}(\partial\mathbb{G}),$$

whith $s \mapsto K(s) = K_1(s) + \mathrm{i}K_2(s) \in \mathbb{L}_2(\partial\mathbb{G})$, *and* $\int_{\partial\mathbb{G}} K_1(s)\,\mathrm{d}s = 0$.
There exist unique **analytic** $\varphi, \chi : \mathbb{G} \to \mathbb{C}$, *with* $\varphi(0) = \chi(0) = \operatorname{Im}\varphi'(0) = 0$, *and* $\varphi\big|_{\partial\mathbb{G}}, \chi\big|_{\partial\mathbb{G}} \in \mathbb{L}_2(\partial\mathbb{G})$, *such that*

$$z(s)\overline{\varphi'(z(s))} + \varphi(z(s)) + \overline{\chi'(z(s))} = -\frac{1}{2}K(s)\dot{z}(s), \quad z(s) \in \partial\mathbb{G}.$$

We have

- $\varphi(\Omega(re^{\mathrm{i}\theta})) \to \varphi\big|_{\partial\mathbb{G}}(s(\theta))$ *and* $\chi(\Omega(re^{\mathrm{i}\theta})) \to \chi\big|_{\partial\mathbb{G}}(s(\theta))$,
  in $\mathbb{L}_2(\mathbb{S})$*-sense, as* $r \uparrow 1$.
- $\big[z\overline{\varphi'(z)} + \varphi(z) + \overline{\chi'(z)}\big]\Big|_{z=\Omega(re^{\mathrm{i}\theta})} \to \underline{g}(\underline{x}(s(\theta)))$, *in* $\mathbb{L}_2(\mathbb{S})$*-sense, as* $r \uparrow 1$.
- $\big(\mathcal{T}\cdot\underline{n}(z)\big)\Big|_{z=\Omega(re^{\mathrm{i}\theta})} \to -\,\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s}\{K(s)\dot{z}(s)\}\Big|_{s=s(\theta)}$ *in* $\mathbb{H}^{-1}(\mathbb{S})$*-sense, as* $r \uparrow 1$.
- *The velocity field at* $\partial\mathbb{G}$ *is well defined (as a* $\mathbb{L}_2$*-limit) and given by*
  $v_1(z(s)) + \mathrm{i}v_2(z(s)) = -\frac{1}{2}K(s)\dot{z}(s) - 2\varphi(z(s)) \in \mathbb{L}_2(\partial\mathbb{G})$.

■

Of special interest in the context of free boundary value problems are solutions of the Stokes-Neumann problems with $K_1 = 0$. In **[H]**, taking $K_1 = 0$, $K_2 = \kappa =$ constant, (surface tension), Hopper derives an ingenious equation for the time evolution of the domain $\mathbb{G}$. This *Hopper equation* is a non-linear time evolution equation for the conformal map $\Omega(\cdot\, t) : \mathbb{D} \to \mathbb{G}$. In a series of papers, following **[H]**, Hopper shows that his equation has several classes of exact solutions $\zeta \mapsto \Omega(\zeta, t)$, which are polynomial or rational in ζ. For more of those see also **[K]**.
In **[G]** it has been shown that already $K_1 = 0$, $K_2 = K_2(\Omega, t)$ is enough for this phenomenon to happen. Reason enough for looking at the structure of the solution if $K_1 = 0$. Then the analytic φ and χ are in a special relation to each other:

- Suppose $\dfrac{\mathrm{d}}{\mathrm{d}s}\,\mathrm{Re}\left(\overline{z(s)}\,\varphi(z(s)) + \chi(z(s))\right|_{z(s)\in\partial\mathbb{G}} = 0$ and $\chi : \mathbb{G} \to \mathbb{C}$ being given, then $\mathrm{Re}\{\dfrac{\varphi}{z}\}\Big|_{\partial\mathbb{G}} = \dfrac{C - \mathrm{Re}\,\chi}{\overline{z}z}\Big|_{\partial\mathbb{G}}$, with $C \in \mathbb{R}$ any constant. Hence, cf. (A.9),

$$\varphi(\Omega(\zeta)) = \frac{\Omega(\zeta)}{2\pi}\int_0^{2\pi} \frac{C - \mathrm{Re}\,\chi(\Omega(e^{\,\mathrm{i}\theta}))}{|\Omega(e^{\,\mathrm{i}\theta})|^2}\;\frac{e^{\,\mathrm{i}\theta}+\zeta}{e^{\,\mathrm{i}\theta}-\zeta}\,\mathrm{d}\theta\,, \quad |\zeta| < 1. \tag{4.1}$$

  It is straightforward that $\varphi(0) = 0\,,\ \mathrm{Im}\,\varphi'(0) = 0$, in this case.

- Suppose $\dfrac{\mathrm{d}}{\mathrm{d}s}\,\mathrm{Re}\left(\overline{z(s)}\,\varphi(z(s)) + \chi(z(s))\right|_{z(s)\in\partial\mathbb{G}} = 0$ and $\varphi : \mathbb{G} \to \mathbb{C}$ being given, then $\mathrm{Re}\,\{\chi\}\Big|_{\partial\mathbb{G}} = C - \mathrm{Re}\left[\overline{z}\varphi\right]\Big|_{\partial\mathbb{G}}$, with $C \in \mathbb{R}$. Hence, cf. (A.9),

$$\chi(\Omega(\zeta)) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left[\,C - \overline{\Omega(e^{\,\mathrm{i}\theta})}\varphi(\Omega(e^{\,\mathrm{i}\theta}))\right]\;\frac{e^{\,\mathrm{i}\theta}+\zeta}{e^{\,\mathrm{i}\theta}-\zeta}\,\mathrm{d}\theta\,, \quad |\zeta| < 1. \tag{4.2}$$

  Take $C = \frac{1}{2\pi}\int_0^{2\pi}\left[\,\overline{\Omega(e^{\,\mathrm{i}\theta})}\varphi(\Omega(e^{\,\mathrm{i}\theta}))\,\right]\mathrm{d}\theta\,,$ then $\chi(0) = 0$.

We conclude with a theorem on some unusual (non physical?) boundary value problems for Stokes' equations. The proof is based on the fact that an analytic function om $\mathbb{G}$ is, up to a constant, fixed by its real (or imaginary) part at the boundary $\partial\mathbb{G}$, on the simple connectedness assumption on $\mathbb{G}$ and on table (1.5).

Theorem 4.3
Let $\mathbb{G} \subset \mathbb{R}^2$ be bounded and simply connected.
Suppose $\partial\mathbb{G}$ has a $\mathbb{H}^1$ arclength parametrization.
For any of the function pairs $\{\,p\,,\ \underline{v}\cdot\underline{n}\,\}\,,\{\,p\,,\ \underline{v}\cdot\dot{\underline{x}}\,\}\,,\{\,\mathrm{rot}\,\underline{v}\,,\ \underline{v}\cdot\underline{n}\,\}\,,\{\,\mathrm{rot}\,\underline{v}\,,\ \underline{v}\cdot\dot{\underline{x}}\,\}\,,$ prescribed at the boundary and all in $\mathbb{L}_2(\partial\mathbb{G})$, there is a unique pressure-velocity flow pair *$\{p, \underline{v}\}$, which solves Stokes' equations. From within, the boundary values are approached in $\mathbb{L}_2$-sense in the way described before.*

■

A APPENDIX: Complex Analysis revisited

1. We identify $\mathbb{R}^2$ and $\mathbb{C}$ by means of the bijection

$$\underline{x} = \begin{bmatrix} x \\ y \end{bmatrix} \mapsto z = x + \mathrm{i}y.$$

2. Multiplication by i, or by any fixed complex number, complex conjugation, taking real or imaginary parts

$$z \mapsto \mathrm{i}z, \qquad z \mapsto \overline{z}, \qquad , z \mapsto \operatorname{Re} z, \qquad z \mapsto \operatorname{Im} z,$$

will often be considered as $\mathbb{R}$-linear mappings in $\mathbb{R}^2$.

3. Functions

$$F: \quad \mathbb{C} \to \mathbb{C}: \quad z = x + \mathrm{i}y \mapsto F(z) = F(x + \mathrm{i}y) = \operatorname{Re} F(z) + \mathrm{i} \operatorname{Im} F(z),$$

possibly local and not necessarily analytic, are identified with, or correspond to

$$\underline{F}: \quad \mathbb{R}^2 \to \mathbb{R}^2: \quad \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} F_1(x,y) \\ F_2(x,y) \end{bmatrix} = \begin{bmatrix} \operatorname{Re} F(x + \mathrm{i}y) \\ \operatorname{Im} F(x + \mathrm{i}y) \end{bmatrix},$$

and vice versa. Such functions will sometimes be considered as vector fields. In a context of cartesian coordinates no confusion arises.

4. We have the usual (commuting) vector partial differentiation operators

$$\partial_z = \frac{1}{2}(\partial_x - \mathrm{i}\partial_y), \quad \partial_{\overline{z}} = \frac{1}{2}(\partial_x + \mathrm{i}\partial_y), \qquad \text{hence} \quad \partial_x = \partial_z + \partial_{\overline{z}}, \quad \partial_y = \mathrm{i}(\partial_z - \partial_{\overline{z}}) \quad \text{(A.1)}$$

Note that for the componentwise Laplacian acting on $\underline{F}$, we have

$$\Delta \underline{F} = 4\partial_{\overline{z}} \partial_z \underline{F}. \qquad \text{(A.2)}$$

It follows that if one has $\partial_{\overline{z}} \underline{F} = \underline{0}$ or/and $\partial_z \underline{F} = \underline{0}$, then, componentwise, $\Delta \underline{F} = \underline{0}$.
Which says that $\underline{F}$ is a *stack of 2 harmonic functions*.
Of importance is also the complex representation of *Euler operator*

$$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = z\frac{\partial}{\partial z} + \overline{z}\frac{\partial}{\partial \overline{z}}. \qquad \text{(A.3)}$$

5. If $\partial_{\overline{z}} \underline{F} = \underline{0}$ we say that $\underline{F}$ $(= F)$ is *analytic*. If $\partial_z \underline{F} = \underline{0}$ we say that $\underline{F}$ $(= F)$ is *anti-analytic*.

This nicely corresponds to the respective *Cauchy-Riemann* and *anti-Cauchy-Riemann* relations

$$\text{C.R.}: \left\{ \begin{array}{rcl} \partial_x \operatorname{Re} F - \partial_y \operatorname{Im} F &=& 0 \\ \partial_y \operatorname{Re} F + \partial_x \operatorname{Im} F &=& 0 \end{array} \right. , \qquad \text{a.C.R.}: \left\{ \begin{array}{rcl} \partial_x \operatorname{Re} F + \partial_y \operatorname{Im} F &=& 0 \\ \partial_y \operatorname{Re} F - \partial_x \operatorname{Im} F &=& 0 \end{array} \right. . \tag{A.4}$$

Note that analyticity of $z \mapsto F(z)$ implies anti-analyticity of $z \mapsto \overline{F(z)}$ and vice versa.

6. If a stack $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} F_1(x,y) \\ F_2(x,y) \end{bmatrix}$ of two harmonic functions corresponds to an analytic function $z \mapsto F(z)$, we say that F_2 is a *harmonic conjugate* of F_1. From (A.4) it is clear that a harmonic conjugate is unique up to a constant.
If on a simply connected domain $G \subset \mathbb{R}^2$, with $\underline{0} \in G$, a harmonic function $\underline{x} \mapsto F_1(\underline{x}) \in \mathbb{R}$ is given, a harmonic conjugate is constructed by

$$\underline{x} \;\mapsto\; F_2(\underline{x}) = \int_{\underline{0}}^{\underline{x}} \{-\partial_y F_1(\underline{x}(s))\dot{x} + \partial_x F_1(\underline{x}(s))\dot{y}\}\,\mathrm{d}s. \tag{A.5}$$

The result does not depend on the path of integration $s \mapsto \underline{x}(s)$, since the vectorfield $\underline{x} \mapsto \begin{bmatrix} -\partial_y F_1(\underline{x}) \\ \partial_x F_1(\underline{x}) \end{bmatrix}$ is obviously *conservative.*

7. If on a connected domain $\mathbb{G} \subset \mathbb{R}^2$, with $\underline{0} \in \mathbb{G}$, a stack $\underline{x} \mapsto \begin{bmatrix} F_1(\underline{x}) \\ F_2(\underline{x}) \end{bmatrix}$ is harmonic, i.e. $\Delta \underline{F} = 0$, it corresponds to an analytic function $z \mapsto F(z)$ on $\mathbb{G}$ if *one* of the C.R.-relations is satisfied all over $\mathbb{G}$ and the *other* C.R.-relation is satisfied at *one point*, say $z = 0$. Indeed, suppose the second C.R.-relation is satisfied all over $\mathbb{G}$. Then
$\partial_x(\partial_x F_1 - \partial_y F_2) = -\partial_y(\partial_y F_1 + \partial_x F_2) = 0$ and $\partial_y(\partial_x F_1 - \partial_y F_2) = \partial_x(\partial_y F_1 + \partial_x F_2) = 0$.
Therefore $\partial_x F_1 - \partial_y F_2 = \text{constant} = 0$.

8. Next we gather some useful expressions for the commutation relations between ∂_x, ∂_y, Δ and the projections Re, Im. All to be applied to smooth $\mathbb{C}$-valued functions on domains in $\mathbb{C}$.

$$\begin{array}{rcl} \partial_x \operatorname{Re} &=& \operatorname{Re}\partial_x = \operatorname{Re}(\partial_z + \partial_{\overline{z}}) \\ \partial_y \operatorname{Re} &=& \operatorname{Re}\partial_y = -\operatorname{Im}(\partial_z - \partial_{\overline{z}}) \\ \Delta \operatorname{Re} &=& \operatorname{Re}\Delta = 4\operatorname{Re}\partial_z\partial_{\overline{z}} \end{array} \qquad \begin{array}{rcl} \partial_x \operatorname{Im} &=& \operatorname{Im}\partial_x = \operatorname{Im}(\partial_z + \partial_{\overline{z}}) \\ \partial_y \operatorname{Im} &=& \operatorname{Im}\partial_y = \operatorname{Re}(\partial_z - \partial_{\overline{z}}) \\ \Delta \operatorname{Im} &=& \operatorname{Im}\Delta = 4\operatorname{Im}\partial_z\partial_{\overline{z}} \end{array} \tag{A.6}$$

9. On a simply connected domain $\mathbb{G} \subset \mathbb{R}^2$, with $\underline{0} \in \mathbb{G}$ we consider a *biharmonic* function $\underline{x} \mapsto \phi(\underline{x})$. This means $\Delta\Delta\phi = 0$. The claim is that there exist analytic $\varphi, \chi \;:\; \mathbb{G} \to \mathbb{C}$, such that

$$\phi(\underline{x}) = \operatorname{Re}\left(\overline{z}\varphi(z) + \chi(z)\right), \quad z = x + \mathrm{i}y. \tag{A.7}$$

To show this, note first that $\Delta\phi$ is harmonic on $\mathbb{G}$. So there is an analytic ψ on $\mathbb{G}$ such that $\Delta\phi = \operatorname{Re}\psi$. Introduce the analytic function $z \mapsto \varphi(z) = \frac{1}{4}\int_0^z \psi(\zeta)\mathrm{d}\zeta$. Then $4\varphi'(z) = \psi(z)$.

We now have $\Delta\big(\phi(\underline{x}) - \operatorname{Re}(\overline{z}\varphi(z))\big) = 0$. So $\phi - \operatorname{Re}(\overline{z}\varphi)$ is harmonic on G and there exists analytic χ on $\mathbb{G}$ such that

$$\phi(\underline{x}) - \operatorname{Re}(\overline{z}\varphi(z)) = \operatorname{Re}\chi(z), \quad z = x + \mathrm{i}y.$$

This proves the claim.

11. Let $\mathbb{L}_2(\mathbb{S}^1)$ denote the standard real Hilbert space on the unit circle $\mathbb{S}^1 \subset \mathbb{C}$. Let $\tilde{f}_1 \in \mathbb{L}_2(\mathbb{S}^1)$. For $\tilde{f}_1$ we will employ the Fourier expansion convention

$$\tilde{f}_1(\theta) = a_0 + \sum_{n=1}^{\infty}\{a_n \cos(n\theta) - b_n \sin(n\theta)\}.$$

Extend $\tilde{f}_1$ to a harmonic function f_1 on the unit disk $\mathbb{D} \subset \mathbb{C}$ by solving the Dirichlet problem. Let f_2, the harmonic conjugate of f_1, be fixed by taking $f_2(\underline{0}) = 0$. Let $\tilde{f}_2$ denote the limit to the boundary $\mathbb{S}^1$ of $\mathbb{D}$. Then

$$\tilde{f}_2(\theta) = \sum_{n=1}^{\infty}\{b_n \cos(n\theta) + a_n \sin(n\theta)\}.$$

All this can be seen by taking real and imaginary parts from the power series expansion of $f_1 + \mathrm{i}f_2$ up to the boundary $\mathbb{S}^1$

$$\tilde{f}_1(\theta) + \mathrm{i}\tilde{f}_2(\theta) = f_1(e^{\mathrm{i}\theta}) + \mathrm{i}f_2(e^{\mathrm{i}\theta}) = \sum_{n=0}^{\infty}(a_n + \mathrm{i}b_n)e^{\mathrm{i}n\theta}, \qquad b_0 = 0.$$

Let further $\mathbb{L}_2\big(\mathbb{S}^1; \mathbb{R}; \perp\{1\}\big)$ denote the linear subspace of all $\tilde{g} \in \mathbb{L}_2(\mathbb{S}^1)$ with $\int_0^{2\pi} \tilde{g}(\theta)\, \mathrm{d}\theta = 0$.
The operator

$$\mathsf{J} : \ \mathbb{L}_2\big(\mathbb{S}^1; \mathbb{R}; \perp\{1\}\big) \ : \quad \tilde{f}_1 \mapsto \mathsf{J}\tilde{f}_1 = \tilde{f}_2,$$

is orthogonal and skew-symmetric:

$$\mathsf{J}^{\star} = -\mathsf{J} = \mathsf{J}^{-1}, \ \mathsf{J}^2 = -\mathsf{I}. \tag{A.8}$$

Note that $\mathsf{J}\{\operatorname{Re}(a_n + \mathrm{i}b_n)e^{\mathrm{i}n\theta}\} = \operatorname{Re}\{-\mathrm{i}(a_n + \mathrm{i}b_n)e^{\mathrm{i}n\theta}\}$.

- The operator $\mathsf{N} : \ \mathbb{L}_2\big(\mathbb{S}^1; \mathbb{R}; \perp\{1\}\big) \to \mathbb{L}_2\big(\mathbb{S}^1; \mathbb{R}; \perp\{1\}\big)$ is defined by

$$\mathsf{N}f_1 = \sum_{n=1}^{\infty} n\{b_n \cos(n\theta) + a_n \sin(n\theta)\}.$$

We have $\mathsf{N}^{\star} = N, \ \mathsf{J}\partial_\theta = \partial_\theta\mathsf{J} = \mathsf{N}$ and therefore $\partial_\theta = -\mathsf{N}\mathsf{J}$.

- For analytic functions $z \mapsto f(z)$ on the unit disk $\mathbb{D}$ we will consider a splitting in real Fourier series on $\mathbb{S}^1$. We put

$$f(e^{\mathrm{i}\theta}) = \sum_{n=1}^{\infty}(a_n + \mathrm{i}b_n)e^{\mathrm{i}n\theta} = f_1(e^{\mathrm{i}\theta}) + \mathrm{i}f_2(e^{\mathrm{i}\theta}) = f_1(e^{\mathrm{i}\theta}) + \mathrm{i}Jf_1(e^{\mathrm{i}\theta}).$$

• **Proof of Lemma 1.4**
The operator J defined by

$$\mathsf{J}\{a_n \cos(n\theta) - b_n \sin(n\theta)\} = b_n \cos(n\theta) + a_n \sin(n\theta), \quad n = 1, 2, 3, \ldots ,$$

can be represented as

$$\mathsf{J} f_1(\theta) = \lim_{r \uparrow 1} \frac{1}{\pi} \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} r^n \sin\big(n(\theta - \theta_1)\big) f(\theta_1)\, \mathrm{d}\theta_1,$$

as can easily be checked term by term. Calculate

$$\sum_{n=1}^{\infty} r^n \sin(n\alpha) = \operatorname{Im} \sum_{n=1}^{\infty} (r e^{\,\mathrm{i}\alpha})^n = \frac{r \sin(\alpha)}{1 + r^2 - 2r\cos(\alpha)} = \frac{2r \sin(\frac{1}{2}\alpha)\cos(\frac{1}{2}\alpha)}{(1-r)^2 + 4r\sin^2(\frac{1}{2}\alpha)} \underset{r\uparrow 1}{\longrightarrow} \frac{1}{2}\cot(\frac{1}{2}\alpha).$$

Therefore

$$\mathsf{J} f_1(\theta) = \lim_{r \uparrow 1} \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{2r \sin\big(\frac{1}{2}(\theta - \theta_1)\big) \cos\big(\frac{1}{2}(\theta - \theta_1)\big)}{(1-r)^2 + 4r \sin^2\big(\frac{1}{2}(\theta - \theta_1)\big)}\, f_1(\theta_1)\, \mathrm{d}\theta_1.$$

Since the kernel is 2π-periodic and odd in $(\theta - \theta_1)$, the result follows. ∎

12. Corollary For analytic $F : \mathbb{D} \to \mathbb{C}$, $\operatorname{Im} F'(0) = 0$, we have the presentation

$$F(\zeta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{Re} F(e^{\,\mathrm{i}\theta})\, \frac{e^{\,\mathrm{i}\theta} + \zeta}{e^{\,\mathrm{i}\theta} - \zeta}\, \mathrm{d}\theta\,, \quad |\zeta| < 1. \tag{A.9}$$

Note that taking the real part leads to the Poisson formula.

B APPENDIX: Details on Stokes' equations

Proof of Theorem 1.1

• Suppose that the pair $\underline{v}\,, p$ is a solution on some domain $\mathbb{G}$. Since $\nabla \cdot \underline{v} = 0$, there exists a 'stream function' ψ such that $\underline{v} = \begin{bmatrix} \partial_y \psi \\ -\partial_x \psi \end{bmatrix}$, where ψ is fixed up to a constant.
Similarly, since $\nabla \cdot \mathcal{T} = \underline{0}$, it follow that, for suitable functions f, g we are allowed to write $\mathcal{T} = 2\begin{bmatrix} \partial_y f & \partial_y g \\ -\partial_x f & -\partial_x g \end{bmatrix}$. Because of symmetry $\partial_x f + \partial_y g = 0$. Hence $\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} -\partial_y \phi \\ \partial_x \phi \end{bmatrix}$, for suitable ϕ, the 'Airy function'. It follows that we are allowed to write

$$\mathcal{T} = 2 \begin{bmatrix} -\partial_y \partial_y \phi & \partial_x \partial_y \phi \\ \partial_x \partial_y \phi & -\partial_x \partial_x \phi \end{bmatrix}.$$

Note that ϕ is fixed up to a polynomial of 1st degree.

In order to show analyticity of $x + \mathrm{i}y \;\mapsto\; \Delta\phi(\underline{x}) + \mathrm{i}\Delta\psi(\underline{x})$ calculate and find equal to 0

$$(\partial_x+\mathrm{i}\partial_y)(\Delta\phi+\mathrm{i}\Delta\psi) = \{\partial_x(\Delta\phi)-\partial_y\Delta\psi\}+\mathrm{i}\{\partial_y(\Delta\phi)+\partial_x\Delta\psi\} = \{\partial_x p-\Delta v_1\}+\mathrm{i}\{\partial_y p-\Delta v_2\} = 0,$$

because of Stokes' equations. As a consequence ϕ, ψ are bi-harmonic.

- Because of bi-harmonicity there are analytic functions f_1, f_2, g_1, g_2 on $\mathbb{G}$ such that, cf. (A.7),

$$\phi = \mathrm{Re}\,(\overline{z}f_1 + g_1) \qquad\qquad \psi = \mathrm{Im}\,(\overline{z}f_2 + g_2),$$

From the C.R.-relations and (A.6) we get

$$\left.\begin{array}{lcr} \partial_x\Delta\phi = \partial_y\Delta\psi & \Rightarrow & \mathrm{Re}\, f_1'' = \mathrm{Re}\, f_2''\,, \\ \partial_y\Delta\phi = -\partial_x\Delta\psi & \Rightarrow & -\,\mathrm{Im}\, f_1'' = -\,\mathrm{Im}\, f_2'' \end{array}\right\} \;\Rightarrow f_1'' = f_2''\,. \tag{B.1}$$

Next, consistency of the stress matrix requires

$$\mathcal{T} = 2\left[\begin{array}{cc} -\partial_y\partial_y\phi & \partial_x\partial_y\phi \\ \partial_x\partial_y\phi & -\partial_x\partial_x\phi \end{array}\right] = \left[\begin{array}{cc} -\Delta\phi + 2\partial_x\partial_y\psi & -\partial_x\partial_x\psi + \partial_y\partial_y\psi \\ -\partial_x\partial_x\psi + \partial_y\partial_y\psi & -\Delta\phi - 2\partial_x\partial_y\psi \end{array}\right].$$

This requires

$$\partial_x\partial_x\phi - \partial_y\partial_y\phi = 2\partial_x\partial_y\psi\,, \qquad 2\partial_x\partial_y\phi = -\partial_x\partial_x\psi + \partial_y\partial_y\psi\,. \tag{B.2}$$

Calculate, cf. (A.6),

$$\begin{array}{rcl rcl} \partial_x\phi &=& \mathrm{Re}\,(\overline{z}f_1' + g_1' + f_1) & \partial_x\psi &=& \mathrm{Im}\,(\overline{z}f_2' + g_2' + f_2) \\ \partial_y\phi &=& -\,\mathrm{Im}\,(\overline{z}f_1' + g_1' - f_1) & \partial_y\psi &=& \mathrm{Re}\,(\overline{z}f_2' + g_2' - f_2) \\ \partial_x\partial_y\phi &=& -\,\mathrm{Im}\,(\overline{z}f_1'' + g_1'' - f_1' + f_1') & \partial_x\partial_y\psi &=& \mathrm{Re}\,(\overline{z}f_2'' + g_2'' + f_2' - f_2') \\ \partial_x\partial_x\phi &=& \mathrm{Re}\,(\overline{z}f_1'' + g_1'' + f_1' + f_1') & \partial_x\partial_x\psi &=& \mathrm{Im}\,(\overline{z}f_2'' + g_2'' + f_2' + f_2') \\ \partial_y\partial_y\phi &=& -\,\mathrm{Re}\,(\overline{z}f_1'' + g_1'' - f_1' - f_1') & \partial_y\partial_y\psi &=& -\,\mathrm{Im}\,(\overline{z}f_2'' + g_2'' - f_2' - f_2') \end{array} \tag{B.3}$$

Substitution of (B.3) in (B.2) leads, together with (B.1) to $\quad g_1'' = g_2''$.
We find

$$\psi(x,y) = \mathrm{Im}\,\{\overline{z}f_2(z)+g_2(z)\}, \quad \phi(x,y) = \mathrm{Re}\,\{\overline{z}\big(f_2(z)+\alpha z+\beta\big)+g_2(z)+\gamma z+\delta\},\ \ \alpha,\beta,\gamma,\delta \in \mathbb{C}.$$

Define $\varphi(z) = f_2(z) + (\,\mathrm{Re}\,\alpha)z$ and $\chi(z) = g_2(z)$, then

$$\psi(x,y) = \mathrm{Im}\,\{\overline{z}\varphi(z) + \chi(z)\}, \qquad \phi(x,y) = \mathrm{Re}\,\{\overline{z}\varphi(z) + \chi(z)\} + \mathrm{Re}\,\{\beta\overline{z} + \gamma z + \delta\}.$$

- If we just throw away the second term in the expression for ϕ, the stress matrix $\mathcal{T}$ is not altered. The only freedom left is a constant added to φ. We are left with

$$\psi(x,y) = \mathrm{Im}\,\{\overline{z}\varphi(z) + \chi(z)\}, \qquad\qquad \phi(x,y) = \mathrm{Re}\,\{\overline{z}\varphi(z) + \chi(z)\}. \tag{B.4}$$

• Finally we check the formulae for the kinematic and dynamic quantities, cf. (B.3),

$$\begin{aligned}
v_1 + \mathrm{i}v_2 &= \partial_y\psi - \mathrm{i}\partial_x\psi = \partial_y \operatorname{Im}(\overline{z}\varphi + \chi) - \mathrm{i}\partial_x \operatorname{Im}(\overline{z}\varphi + \chi) = \\
&= \operatorname{Re}(\partial_z - \partial_{\overline{z}})\overline{z}\varphi + \chi) - \mathrm{i}\operatorname{Im}(\partial_z + \partial_{\overline{z}})\overline{z}\varphi + \chi) = \\
&= \operatorname{Re}(\overline{z}\varphi' + \chi' - \varphi) - \mathrm{i}\operatorname{Im}(\overline{z}\varphi' + \chi' + \varphi) = \\
&= \overline{\overline{z}\varphi' + \chi'} - \varphi = -\varphi + z\overline{\varphi'} + \overline{\chi'}. \\
\partial_x v_2 - \partial_y v_1 &= \operatorname{Im}(\partial_x - \mathrm{i}\partial_y)(v_1 + \mathrm{i}v_2) = 2\operatorname{Im}\partial_z(-\varphi + z\overline{\varphi'} + \overline{\chi'}) = -4\operatorname{Im}\varphi'. \\
\mathcal{T}_{11} + \mathcal{T}_{22} &= -2p = -2\Delta\phi = -2\Delta\operatorname{Re}(\overline{z}\varphi + \chi) = \\
&= -8\operatorname{Re}\partial_z\partial_{\overline{z}}(\overline{z}\varphi + \chi) = -8\operatorname{Re}\varphi'. \\
\mathcal{T}_{22} - \mathcal{T}_{11} + 2\,\mathrm{i}\mathcal{T}_{12} &= -2\partial_x\partial_x\phi + 2\partial_y\partial_y\phi + 4\,\mathrm{i}2\partial_x\partial_y\phi = \\
&= -2\operatorname{Re}(\overline{z}\varphi'' + \chi'' + 2\varphi') - 2\operatorname{Re}(\overline{z}\varphi'' + \chi'' - 2\varphi') - 4\,\mathrm{i}\operatorname{Im}(\overline{z}\varphi'' + \chi'') = \\
&= -4\operatorname{Re}(\overline{z}\varphi'' + \chi'') - 4\,\mathrm{i}\operatorname{Im}(\overline{z}\varphi'' + \chi'') = -4(\overline{z}\varphi'' + \chi''). \\
\underline{v}\cdot\underline{n} &= \operatorname{Re}\{(v_1 - \mathrm{i}v_2)\cdot -\mathrm{i}\dot{z}\} = \operatorname{Im}\{(v_1 - \mathrm{i}v_2)\dot{z}\} = \\
&= \operatorname{Im}\{(-\overline{\varphi} + \overline{z}\varphi' + \chi')\dot{z}\} = \operatorname{Im}\{\frac{\mathrm{d}}{\mathrm{d}s}(\overline{z}\varphi + \chi) - \varphi\dot{\overline{z}} - \overline{\varphi}\dot{z}\} = \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\operatorname{Im}(\overline{z}\varphi + \chi). \\
\mathcal{T}\cdot\underline{n} &= 2\begin{bmatrix} -\partial_y\partial_y\phi & \partial_x\partial_y\phi \\ \partial_x\partial_y\phi & -\partial_x\partial_x\phi \end{bmatrix}\begin{bmatrix} \dot{y} \\ -\dot{x} \end{bmatrix} = -2\frac{\mathrm{d}}{\mathrm{d}s}\begin{bmatrix} \partial_y\phi \\ -\partial_x\phi \end{bmatrix} = \\
&= -2\frac{\mathrm{d}}{\mathrm{d}s}\begin{bmatrix} \partial_y\operatorname{Re}(\overline{z}\varphi + \chi) \\ -\partial_x\operatorname{Re}(\overline{z}\varphi + \chi) \end{bmatrix} = \\
&= 2\frac{\mathrm{d}}{\mathrm{d}s}\{\operatorname{Im}(\overline{z}\varphi' + \chi' - \varphi) + \mathrm{i}\operatorname{Re}(\overline{z}\varphi' + \chi' + \varphi)\} = \\
&= 2\,\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s}\{z\overline{\varphi'} + \overline{\chi'} + \varphi\}. \\
\mathcal{T}\cdot\underline{\dot{x}} &= 2\frac{\mathrm{d}}{\mathrm{d}s}\{z\overline{\varphi'} + \overline{\chi'} - 4\operatorname{Re}\varphi\}.
\end{aligned}$$

• If we put $\varphi_1(z) = \varphi(z) + A$ and $\chi_1(z) = \chi(z) + \overline{A}z + C$, with $A\,,C \in \mathbb{C}$ we still find the same expressions for $v_1\,,v_2\,,p$. Note also that the corresponding altered stream function $\psi_1(\underline{x}) = \psi(\underline{x}) + \operatorname{Im}(\overline{z}A + \overline{A}z + B) = \psi(\underline{x}) + \operatorname{Im}B$ and the Airy function $\phi_1(\underline{x}) = \phi_(\underline{x}) + \operatorname{Re}(\overline{z}A + \overline{A}z + B)$ show, respectively, an added constant and an added 1st degree polynomial which don't alter the velocity and the stress tensor.

Conclusion If for some fixed $\underline{a}$ in the fluid domain we additionally require $\varphi(\underline{a}) = \chi(\underline{a}) = 0$, there is precisely one pair $\{\varphi\,,\chi\}$ that describes a solution of the Stokes equations. ■

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